Non-generators in extensions of infinitary algebras

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Abstract. Contrary to the finitary case, the set $\Gamma(\mathbf{A})$ of all the non-generators of an infinitary algebra $\mathbf{A}$ is not necessarily a subalgebra of $\mathbf{A}$. We show that the phenomenon is ubiquitous: every algebra with at least one infinitary operation can be embedded into some algebra $\mathbf{B}$ such that $\Gamma(\mathbf{B})$ is not a subalgebra of $\mathbf{B}$.

As far as expansions are concerned, there are examples of infinite algebras $\mathbf{A}$ such that in every expansion $\mathbf{B}$ of $\mathbf{A}$ the set $\Gamma(\mathbf{B})$ is a subalgebra of $\mathbf{B}$. However, under relatively weak assumptions on $\mathbf{A}$, it is possible to get some expansion $\mathbf{B}$ of $\mathbf{A}$ such that $\Gamma(\mathbf{B})$ fails to be a subalgebra of $\mathbf{B}$.

1. Introduction

It is well-known that in a finitary algebra $\mathbf{A}$ the set $\Gamma(\mathbf{A})$ of all the non-generators is the intersection of all the maximal proper subalgebras of $\mathbf{A}$, hence $\Gamma(\mathbf{A})$ is a subalgebra of $\mathbf{A}$. Hansoul [H] proved that in the infinitary case $\Gamma(\mathbf{A})$ is not necessarily a subalgebra of $\mathbf{A}$; however Hansoul's proof is indirect and does not provide an explicit counterexample. Besides providing an explicit counterexample, we show that every algebra $\mathbf{A}$ with at least one infinitary operation can be embedded into some algebra $\mathbf{B}$ such that $\Gamma(\mathbf{B})$ is not a subalgebra of $\mathbf{B}$. Moreover, assuming the Axiom of Choice, $\mathbf{B}$ can be obtained in such a way that $B \setminus A$ is countable.

Let us now consider expansions instead of extensions, namely, we add operations rather than elements. We find conditions ensuring that some algebra $\mathbf{A}$ has (or has not) an expansion $\mathbf{B}$ such that $\Gamma(\mathbf{B})$ fails to be a subalgebra of $\mathbf{B}$.

2. Preliminaries

We now recall the basic notions. See [Gr] for more details and unexplained notions. An *algebraic structure*, *algebra*, for short, is a set endowed with a family of operations and, possibly, constants. We allow *infinitary* operations, that is, operations depending on an infinite number of arguments. In detail, to every operation symbol f there is associated a possibly infinite set I. In each

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algebra whose type contains f, the symbol f is interpreted by some function $f_{\mathbf{A}} : A^I \to A$. As usual, when no risk of ambiguity is possible, we shall drop subscripts. A *finitary* algebra is an algebra having only finitary (= not infinitary) operations, that is, the I's above are always finite. The case of finitary algebras is the most studied in the literature [Gr].

Countable means either finite or denumerable. An algebra $\mathbf{A}$ is *countable* (*finite*) if its domain A is countable (finite). For convenience, we allow algebras with empty domain (when the type contains no constant). If $\mathbf{A}$ is an algebra and $X \subseteq A$, then $\langle X \rangle$ denotes the *subalgebra of* $\mathbf{A}$ *generated by* X, that is, the intersection of all the subalgebras of $\mathbf{A}$ which contain X. It is easily seen that $\langle X \rangle$ is indeed a subalgebra of $\mathbf{A}$ (by abuse of notation, here we do not distinguish between a subalgebra and its domain!) As usual, $\langle X, a \rangle$ is an abbreviation for $\langle X \cup \{a\} \rangle$.

If $\mathbf{A}$ is an algebra and $a \in A$, the element a is a *non-generator (of* $\mathbf{A}$*)* if, for every $X \subseteq A$, $\langle X, a \rangle = A$ implies $\langle X \rangle = A$. Otherwise, a is called a *relative generator*. Thus a is a relative generator if there is $X \subseteq A$ such that $\langle X, a \rangle = A$ but $\langle X \rangle \neq A$. As a stronger case of the latter notion, a is *indispensable* if a belongs to every generating set, equivalently, if $A \setminus \{a\}$ is a subalgebra. Originally considered in groups, non-generators have been subsequently studied in various special algebraic structures, as well as in the general universal setting. See, e. g., [BS, J, KV] for more details and references.

In the finitary case the set of all the non-generators of some algebra $\mathbf{A}$ is the domain for a subalgebra. This can be proved by observing that in the finitary case the set of non-generators is the intersection of all the maximal (proper) subalgebras of $\mathbf{A}$. The argument uses some algebraic properties of the lattice of subalgebras of a finitary algebra [J], does not work for infinitary algebras [H] and needs the axiom of choice [RR, Form AL 9]. We first present a more direct argument with a somewhat broader range of applicability. The argument is easy, but it has been probably overlooked by some authors.

For the sake of readability, if g is an infinitary operation with arguments indexed by some set I, we shall write $g(\dots, a_i, \dots)$ in place of $g((a_i)_{i \in I})$. Sometimes we need to fix some special element of I, call it 0, and then we shall write $g(a_0, \dots, a_i, \dots)$ when we need to know the argument of g at place 0. In any case, we do not make any special assumption on the set I, in particular, we do not necessarily assume that I is countable, well-ordered, etc.

Proposition 2.1. *Suppose that* $\mathbf{A}$ *is a possibly infinitary algebra and* $a_1, \dots, a_n \in A$ *is a finite set of non-generators.*

If f *is a finitary* n*-ary operation of* $\mathbf{A}$*, then* $f(a_1, \dots, a_n)$ *is a non-generator.*

More generally, if g *is a possibly infinitary operation of* $\mathbf{A}$ *and* $(b_i)_{i \in I}$ *is a sequence of elements chosen from the finite set* $\{a_1, \dots, a_n\}$ *(hence repetitions in the sequence* $(b_i)_{i \in I}$ *are allowed), then* $g(\dots, b_i, \dots)$ *is a non-generator.*

Proof. Assume $\langle X, f(a_1, \dots, a_n) \rangle = A$, thus $\langle X, a_1, \dots, a_n \rangle = A$, since $f(a_1, \dots, a_n) \in \langle a_1, \dots, a_n \rangle$. Since a_n is a non-generator, then $\langle X, a_1, \dots, a_{n-1} \rangle =$

A and so on, in a finite number of steps we get $\langle X \rangle = A$. The last statement is proved in the same way. $\square$

If $\mathbf{A}$ and $\mathbf{B}$ are algebras with the same domain, $\mathbf{B}$ is an *expansion* of $\mathbf{A}$ if $\mathbf{B}$ has (possibly) more operations than $\mathbf{A}$, that is, the case $\mathbf{A} = \mathbf{B}$ is included. If $\mathbf{B}$ is an expansion of $\mathbf{A}$, then $\mathbf{A}$ is said to be a *reduct* of $\mathbf{B}$. Notice that the notion of expansion is distinct from *extension*. An algebra $\mathbf{B}$ is an extension of $\mathbf{A}$ just in case $\mathbf{A}$ is a subalgebra of $\mathbf{B}$. For short, in an extension we add elements, in an expansion we add operations.

If $\mathbf{A}$ is an infinitary algebra, the *finitary reduct* $\mathbf{A}^{fr}$ of $\mathbf{A}$ is the reduct of $\mathbf{A}$ in which only the finitary operations are considered.

Corollary 2.2. *If $\mathbf{A}$ is a (possibly infinitary) algebra, then the set of the non-generators of $\mathbf{A}$ is a subalgebra of the finitary reduct $\mathbf{A}^{fr}$ of $\mathbf{A}$.*

In particular, if $\mathbf{A}$ is a finitary algebra, then the set of non-generators of $\mathbf{A}$ is a subalgebra of $\mathbf{A}$.

The first statement in Corollary 2.2 does not follow from the second statement. It might happen that some element a is a non-generator in $\mathbf{A}$, but a is a relative generator in the finitary reduct of $\mathbf{A}$. Just consider an algebra $\mathbf{A}$ with only infinitary operations. The finitary reduct $\mathbf{A}^{fr}$ is then a set without operations, thus every element of $\mathbf{A}^{fr}$ is a relative generator, actually, an indispensable element. However, some element of the original infinitary algebra $\mathbf{A}$ might be a non-generator.

In the infinitary case the set of non-generators is not necessarily a subalgebra. However, there is a significant case in which this happens. See clause (1) in the next theorem, taken from [L].

Theorem 2.3. [L] *(1) In every complete semilattice the set of non-generators is a complete subsemilattice and is the intersection of all the maximal proper complete subsemilattices.*

(2) There is a complete lattice such that the set of non-generators is a complete sublattice, but it is not the intersection of all the maximal proper complete sublattices.

(3) There is a complete lattice such that the set of non-generators is not a complete sublattice.

In passing, we exploit the special feature of (complete) semilattices which provides the reason why 2.3(1) holds. Indeed, the proof of 2.3(1) from [L] shows that the first statement in the next proposition holds for complete semilattices.

Proposition 2.4. *Suppose that $\mathbf{A}$ is a possibly infinitary algebra such that each element of $\mathbf{A}$ is either indispensable or a non-generator. Then in $\mathbf{A}$ the set of non-generators is the intersection of all the maximal proper subalgebras.*

Proof. It is elementary to see that a non-generator belongs to every maximal proper subalgebra. On the other hand, an element a is indispensable if and only if $A \setminus \{a\}$ is a subalgebra (necessarily, proper maximal) of $\mathbf{A}$. The assumptions then imply that there is no other maximal proper subalgebra, thus the set of non-generators is the intersection of all the maximal proper subalgebras. □

3. Non-generators in extensions

Proposition 3.1. *There is an algebra* $\mathbf{B}$ *with a single operation depending on countably many arguments and such that the set of all the non-generators in* $\mathbf{B}$ *fails to be a subalgebra of* $\mathbf{B}$.

Proof. Let $B = \mathbb{N} \cup \{\infty\}$, where $\infty \notin \mathbb{N}$. On B consider the infinitary operation f depending on countably many arguments defined by

$$f(b_0, b_1, b_2, \dots) = \begin{cases} b_0 & \text{if the set } S = \{\, b_n \mid n \in \mathbb{N} \,\} \text{ is finite and } \infty \notin S, \\ 0 & \text{if } b_0 = b_1 = \infty, \\ n+1 & \text{if } b_0 = \infty \text{ and } b_1 = n, \\ \infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Because of the second and third clauses $\langle \infty \rangle = B$. Because of the first clause, if $S \subseteq \mathbb{N}$ and S is finite, then $\langle S \rangle = S$. On the other hand, if S is infinite, then, because of the fourth clause, $\langle S \rangle \supseteq \langle \infty \rangle = B$.

Hence, for $X \subseteq B$, we have $\langle X \rangle = B$ if and only if either $\infty \in X$, or X is infinite. This implies that every element of $\mathbb{N}$ is a non-generator and that ∞ is a relative generator. However, $\mathbb{N}$ is not a subalgebra of $\mathbf{B}$, due to the fourth clause. □

Another proof of Proposition 3.1 (with a quite different counterexample) can be found in [L].

We now show that every algebra with at least one infinitary operation can be extended to an algebra in which the set of all the non-generators fails to be a subalgebra.

For simplicity, we shall assume that, for every type and every operation f in the type, we have chosen one argument which shall always indicated as the first argument in expressions like $f(b_0, \dots, b_i, \dots)$. Technically, the following definition is dependent on the above choices; however, we shall later show that, with a bit more effort, a proof for Theorem 3.4 below can be given without performing any special choice of this kind.

Definition 3.2. If $\mathbf{A}$ and $\mathbf{B}$ are two possibly infinitary algebras of the same type without constants and $A \cap B = \emptyset$, let the *union* $\mathbf{C} = \mathbf{A} \cup \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ be defined as follows. The domain of $\mathbf{C}$ is $C = A \cup B$ and, for every operation

f in the type, the interpretation of f on $\mathbf{C}$ is defined by

$$f_{\mathbf{C}}(c_0, \dots, c_i, \dots) = \begin{cases} f_{\mathbf{A}}(c_0, \dots, c_i, \dots) & \text{if } c_0, \dots, c_i, \dots \in A, \\ f_{\mathbf{B}}(c_0, \dots, c_i, \dots) & \text{if } c_0, \dots, c_i, \dots \in B, \\ c_0 & \text{otherwise.} \end{cases} \tag{3.2}$$

In a bit more general situation, we may allow exactly one between $\mathbf{A}$ and $\mathbf{B}$, say, $\mathbf{A}$, to have the type expanded with further constants. If this is case, constants are interpreted in $\mathbf{C}$ by the same elements as in $\mathbf{A}$. Thus in this case $\mathbf{C}$ and $\mathbf{A}$ have the same type, an expansion of the type of $\mathbf{B}$.

Lemma 3.3. *Under the above assumptions and definitions,* $\mathbf{A}$ *is a subalgebra of* $\mathbf{C}$*, and* $\mathbf{B}$ *is a subalgebra of the appropriate reduct of* $\mathbf{C}$*. Moreover, for every* $X \subseteq C$

(i) X *is a subalgebra of* $\mathbf{C}$ *if and only if both* $X \cap A$ *is a subalgebra of* $\mathbf{A}$ *and* $X \cap B$ *is a subalgebra of* $\mathbf{B}$*.*
(ii) $\langle X \rangle_{\mathbf{C}} = \langle X \cap A \rangle_{\mathbf{A}} \cup \langle X \cap B \rangle_{\mathbf{B}}$*.*
(iii) An element $a \in A$ *is a non-generator (indispensable) in* $\mathbf{A}$ *if and only if* a *is a non-generator (indispensable) in* $\mathbf{C}$*. The same holds for elements of* $\mathbf{B}$*.*

Theorem 3.4. *If* $\mathbf{A}$ *is an algebra with at least one infinitary operation, then* $\mathbf{A}$ *can be extended to some algebra* $\mathbf{C}$ *such that the set of non-generators of* $\mathbf{C}$ *fails to be a subalgebra of* $\mathbf{C}$*.*

Proof. Let $\mathbf{A}$ be an algebra with the infinitary operation f. If f depends on countably many arguments, expand the algebra $\mathbf{B}$ from Proposition 3.1 by adding a trivial operation $g_{\mathbf{B}}$ for every operation of $\mathbf{A}$ distinct from f. We set $g_{\mathbf{B}}(b_0, \dots, b_i, \dots) = b_0$. If f in $\mathbf{A}$ depends on uncountably many arguments, choose a countably infinite subset J of the arguments and define f in $\mathbf{B}$ in a way similar to (3.1), in such a way that $f_{\mathbf{B}}$ depends only on the arguments in J. The proof of Proposition 3.1 carries over even in this situation, thus the set of non-generators of $\mathbf{B}$, as defined here, is not a subalgebra of $\mathbf{B}$.

Let $\mathbf{C} = \mathbf{A} \cup \mathbf{B}$, as in Definition 3.2. By Lemma 3.3(i) and (iii) the set of the non-generators of $\mathbf{C}$ is not a subalgebra of $\mathbf{C}$. □

Remark 3.5. Given some algebra $\mathbf{A}$, the algebra $\mathbf{C}$ constructed in Theorem 3.4 has the same type of $\mathbf{A}$, but might be very different from $\mathbf{A}$. If we have some special class $\mathcal{K}$ of algebras in mind, it is not necessarily the case that $\mathbf{C}$ belongs to $\mathcal{K}$. It is an open problem whether there is some construction which produces an algebra in $\mathcal{K}$ satisfying the conclusion of Theorem 3.4, of course, assuming suitable closure properties of $\mathcal{K}$.

3.1. A choiceless proof of Theorem 3.4. In the proof of Theorem 3.4 we have used the Axiom of Choice (AC) twice. We now show that the use of AC can be avoided.

We have used a consequence of AC in the proof of 3.4 by assuming that if I is some infinite set, then we can find a countably infinite $J \subseteq I$. We now modify the example provided in Proposition 3.1 in such a way that B can be taken to be equipotent to some arbitrary, possibly not well-orderable, infinite set. As usual in a choiceless setting, a set S is *finite* if S can be put in a bijective correspondence with a natural number, *infinite* otherwise.

Lemma 3.6. *Suppose that I is an infinite set.*

There is an algebra $\mathbf{B}$ such that B is in a bijective correspondence with the set $I \cup \{\infty\}$ ($\infty \notin I$), $\mathbf{B}$ has only one operation f, f depends on I-many arguments and the set of all the non-generators in $\mathbf{B}$ fails to be a subalgebra of $\mathbf{B}$.

Proof. Let $B = \{ b_i \mid i \in I \} \cup \{\infty\}$, where the b_i's are chosen arbitrarily in such a way that $i \neq j$ implies $b_i \neq b_j$ and ∞ is distinct from every b_i. In fact, we could have already taken $B = I \cup \{\infty\}$, but this might lead to notational confusion.

Pick some special element $\bar{b} \in B \setminus \{\infty\}$ (we do not need AC in order to perform this). Define an operation f on B depending on the set I of arguments by

$$
\begin{aligned}
&f(\infty, \dots, \infty, \dots, \infty) = \bar{b}, && \\
&f(\bar{b}, \dots, \bar{b}, \infty, \bar{b}, \dots, \bar{b}) = b_i && \text{with a single occurrence of } \infty \text{ at place } i, \\
&f(\dots, d_i, \dots) = \bar{b} && \text{if } S = \{ d_i \mid i \in I \} \text{ is finite and } \infty \notin S, \\
&f(\dots, d_i, \dots) = \infty && \text{in all the remaining cases.}
\end{aligned}
$$

By the first two clauses, $\langle \infty \rangle = B$. Hence, as in the proof of Proposition 3.1, if $X \subseteq B$, then $\langle X \rangle = B$ if and only if either X is infinite or $\infty \in X$ (if X is infinite, we do not need AC to generate ∞. Put b_i at place i if $b_i \in X$, otherwise put ∞ at place i, then the outcome of f is ∞, by the fourth clause). This implies that the set Γ of the non-generators of $\mathbf{B}$ is $B \setminus \{\infty\}$. However, Γ is not a subalgebra of $\mathbf{B}$, since $f(\dots, b_i, \dots) = \infty$. □

In the proof of Theorem 3.4 we have also used AC when $\mathbf{A}$ has infinitely many operations, since we have chosen some specific argument for every operation. We can do without AC by adding just an element to the union of $\mathbf{A}$ and $\mathbf{B}$.

Choiceless proof of Theorem 3.4. Pick some infinitary operation $f_{\mathbf{A}}$ in $\mathbf{A}$ and construct an operation $f_{\mathbf{B}}$ on B as in the proof of Lemma 3.6. We shall construct an algebra $\mathbf{C}$ on $A \cup B \cup \{c\}$, where we can suppose that $A \cap B = \emptyset$ and $c \notin A \cup B$. Expand $\mathbf{B}$ in an appropriate way and join all the operations on $\mathbf{A}$ and $\mathbf{B}$ by setting to c all the otherwise undefined values. By the proof of Lemma 3.6, all the elements of $B \setminus \{\infty\}$ are non generators in $\mathbf{C}$, too, however $B \setminus \{\infty\}$ is not a subalgebra (we are essentially using a result analogue to Lemma 3.3, but notice that, as it stands, Lemma 3.3 does not hold in the present situation; the cases when $X \cap A = \emptyset$ or $X \cap B = \emptyset$ should be treated in a special way). $\square$

4. Non-generators in expansions

Clearly, the assumption that the algebra $\mathbf{A}$ in Theorem 3.4 has an infinitary operation is necessary. Indeed, in every finitary algebra the set Γ of non-generators is the intersection of the maximal proper subalgebras, hence Γ is a subalgebra itself.

However, it follows trivially from Theorem 3.4 that, for every algebra $\mathbf{A}$, there is some extension $\mathbf{B}$ of some expansion $\mathbf{A}^+$ of $\mathbf{A}$ such that in $\mathbf{B}$ the set of non-generators fails to be a subalgebra. Can we do by expansions alone, that is, without extending the domain of the algebra? The answer is obviously no, in general, since every proper subalgebra of a finite algebra can be extended to a maximal proper subalgebra, and then the classical argument shows that in this case, too, the set of non-generators is the intersection of the maximal proper subalgebras. As another counterexample, if some algebra $\mathbf{A}$ is generated by the set of its constants, then every element of $\mathbf{A}$ is a non-generator, and this fact still holds in any expansion of $\mathbf{A}$,

In the following remark we shall see that there are similar examples even for infinite algebras without constants; however, under quite weak hypotheses it is possible to expand some algebra (without extending it) in such a way that the non-generators fail to constitute a subalgebra. See Proposition 4.3 below.

Remark 4.1. (a) We first notice that

(*) If some algebra $\mathbf{A}$ has an element b such that $\langle b \rangle = A$ and b is indispensable, then the set Γ of non-generators of $\mathbf{A}$ is $A \setminus \{b\}$, hence Γ is a subalgebra of $\mathbf{A}$.

Indeed, b fails to be a non-generator, since it is indispensable. On the other hand, if $a \in A \setminus \{b\}$ and $\langle X, a \rangle = A$, then $b \in X$, since b is indispensable, and hence, by assumption, $\langle X \rangle \supseteq \langle b \rangle = A$.

(b) Now consider the algebra $\mathbf{A}$ over $\mathbb{Z} \cup \{b\}$, where b is any new element not in $\mathbb{Z}$. The algebra $\mathbf{A}$ has two unary operations f and h which are interpreted, respectively, as the successor and predecessor functions on $\mathbb{Z}$, and are such that $f(b) = h(b) = 0$. We claim that

(**) In any expansion $\mathbf{A}^+$ of $\mathbf{A}$ the set of non-generators of $\mathbf{A}^+$ is a subalgebra of $\mathbf{A}^+$.

Indeed, $\langle b \rangle_{\mathbf{A}} = A$, a fortiori, $\langle b \rangle_{\mathbf{A}^+} = A$ in any expansion $\mathbf{A}^+$ of $\mathbf{A}$. If b remains indispensable in $\mathbf{A}^+$, then, by (*), the set of non-generators of $\mathbf{A}^+$ is $A \setminus \{b\}$, hence it is a subalgebra. Otherwise, b is not indispensable in $\mathbf{A}^+$, hence $f(a_1, \dots, a_i, \dots) = b$, for some operation f and $a_1, \dots, a_i, \dots \in A \setminus \{b\}$.

Let $a \in A \setminus \{b\}$. Since $\langle a \rangle_{\mathbf{A}} = A \setminus \{b\}$, then $\langle a \rangle_{\mathbf{A}^+} \supseteq A \setminus \{b\}$, since $\mathbf{A}^+$ is an expansion of $\mathbf{A}$. Since $f(a_1, a_2, \dots) = b$, then $\langle a \rangle_{\mathbf{A}^+} = A$. We have showed that $\langle a \rangle_{\mathbf{A}^+} = A$, for every $a \in A$.

If $\mathbf{A}^+$ has no constant in its type, then all the elements of $\mathbf{A}^+$ are relative generators, since $\langle \emptyset \rangle = \emptyset$ but $\langle \emptyset, a \rangle = \langle a \rangle = A$. Thus the set of non-generators is a subalgebra, recalling the convention that we consider an empty set as a subalgebra. If $\mathbf{A}^+$ has some constant c, then $\langle c \rangle = A$, but $c \in \langle X \rangle$, for every $X \subseteq A$, including the case $X = \emptyset$, hence $\langle X \rangle = A$, for every $X \subseteq A$. This trivially implies that every element of $\mathbf{A}^+$ is a non-generator, hence non-generators form a subalgebra in this case, as well.

The arguments in the above remark give a proof of the following proposition.

Proposition 4.2. *Suppose that* $\mathbf{A}$ *is an algebra with an indispensable element* b *such that* $\langle b \rangle = A$*. Suppose further that* $\langle a \rangle = A \setminus \{b\}$*, for every* $a \in A \setminus \{b\}$*.*

Then in every expansion $\mathbf{A}^+$ *of* $\mathbf{A}$ *the set of non-generators of* $\mathbf{A}^+$ *is a subalgebra of* $\mathbf{A}^+$*.*

We now give conditions under which some algebra can be actually expanded in such a way that non-generators fail to form a subalgebra.

Proposition 4.3. *Suppose that* $\mathbf{A}$ *is an algebra without constants and* Δ *is a proper nonempty subalgebra of* $\mathbf{A}$ *such that, whenever* $a \in X \subseteq \Delta$ *and* $\langle X, a \rangle = \Delta$*, then* $\langle X \rangle = \Delta$*.*

Then $\mathbf{A}$ *can be expanded to some algebra* $\mathbf{A}^+$ *such that in* $\mathbf{A}^+$ *the set of the non-generators fails to be a subalgebra.*

Proof. Add to $\mathbf{A}$ a unary operation f_c, for each $c \in A$, defined by

$$f_c(b) = \begin{cases} c & \text{if } b \notin \Delta; \\ b & \text{if } b \in \Delta. \end{cases}$$

Pick some $r \in A \setminus \Delta$; this is possible since Δ is assumed to be a proper subalgebra of $\mathbf{A}$. On A define an infinitary operation g depending on Δ-many arguments by

$$g(b_0, \dots, b_i, \dots) = \begin{cases} r & \text{if } \{\, b_i \mid b_i \text{ among the arguments of } g \,\} \supseteq \Delta; \\ b_0 & \text{otherwise.} \end{cases}$$

In words, g acts as the "first projection", unless every element of Δ appears among the arguments of g, in which case the outcome of g is r. Let $\mathbf{A}^+$ be the expansion of $\mathbf{A}$ obtained by adding the operation g, as well as all the unary operations f_c, for $c \in A$.

Because of the f_c's, if $b \notin \Delta$, then $\langle b \rangle_{\mathbf{A}^+} = A$, hence b is a non-generator, since $\mathbf{A}$, and hence $\mathbf{A}^+$, have no constant, thus $\langle \emptyset \rangle = \emptyset$, while $\langle \emptyset, b \rangle = A$.

We now claim that if $a \in \Delta$, then a is a non-generator in $\mathbf{A}^+$. Indeed, suppose that $\langle X, a \rangle_{\mathbf{A}^+} = A$. If $X \not\subseteq \Delta$, then $b \in X$, for some $b \notin \Delta$, hence $\langle X \rangle_{\mathbf{A}^+} \supseteq \langle b \rangle_{\mathbf{A}^+} = A$, by a comment above. Hence we can suppose that $X \subseteq \Delta$. Since Δ is a subalgebra of $\mathbf{A}$, then $\langle X, a \rangle_{\mathbf{A}} \subseteq \Delta$. Since $\langle X, a \rangle_{\mathbf{A}^+} = A$, instead, then it is necessary to apply the new operations in $\mathbf{A}^+$. The operations f_c have no use to "go outside Γ", hence we need to resort to g. We can apply g

in a nontrivial way only if $\langle X, a\rangle_{\mathbf{A}} = \Delta$, but then $\langle X\rangle_{\mathbf{A}} = \Delta$, by assumption, hence we can apply g anyway, getting $r \in \langle X\rangle_{\mathbf{A}^+}$, and then $\langle X\rangle_{\mathbf{A}^+} = A$, since $\langle r\rangle_{\mathbf{A}^+} = A$.

We have showed that Γ is the set of the non-generators of $\mathbf{A}^+$; however, Γ is not a subalgebra of $\mathbf{A}^+$, because of g. □

For example, Proposition 4.3 can be applied to every algebra with domain $\mathbb{Z} \cup X$, with $X \not\subseteq \mathbb{Z}$ and a unary operation f which is defined as the successor function on $\mathbb{Z}$ and arbitrarily otherwise. This is in contrast with the example in Remark 4.1(b).

5. Further remarks

In the next proposition we state some results showing that an algebra can be extended in order to obtain a lot of non-generators in the extension. We shall present proofs elsewhere.

Proposition 5.1. *(1) If* $\mathbf{A}$ *is a countable algebra with at least one operation of arity* ≥ 2*, then* $\mathbf{A}$ *can be extended to some algebra* $\mathbf{B}$ *over the set* $A \cup \{\infty\}$ *(*$\infty \notin A$*), in such a way that* A *is the set of all the non-generators of* $\mathbf{B}$ *and* ∞ *is indispensable.*

(2) Every algebra $\mathbf{A}$ *with at least one operation can be extended to some algebra* $\mathbf{B}$ *in such a way that* $B \setminus A \neq \emptyset$ *and in* $\mathbf{B}$*: (a) every element of* A *is a non-generator and (b) every element of* $B \setminus A$ *is indispensable. Henceforth, by Proposition 2.4, in* $\mathbf{B}$ *the set of non-generators is the intersection of all the maximal proper subalgebras.*

(3) Every algebra $\mathbf{A}$ *with at least one operation can be extended to some algebra* $\mathbf{C}$ *such that every* $c \in C$ *is a non-generator in* $\mathbf{C}$.

Remark 5.2. (a) Contrary to Proposition 5.1, in particular, contrary to Clause (3), it is not always the case that an algebra can be extended to an algebra in which all the elements are relative generators. Indeed, if the type of some algebra $\mathbf{A}$ has a constant c, then (the interpretation) of c is a non-generator, hence c remains a non-generator in any extension (and in any expansion) of $\mathbf{A}$.

(b) However, if $\mathbf{A}$ has no constant, then $\mathbf{A}$ can be expanded to an algebra in which all the elements are relative generators. Just add to $\mathbf{A}$ all possible unary operations. Then $\langle a\rangle = \langle a, \emptyset\rangle = A$ in the expansion, but $\langle \emptyset\rangle = \emptyset$, since there is no constant.

(c) The situation with indispensable elements is different. If $\mathbf{B}$ is either an extension or an expansion of $\mathbf{A}$, $a \in A$ and a is indispensable in $\mathbf{B}$, then $B \setminus \{a\}$ is a subalgebra of $\mathbf{B}$, hence $A \setminus \{a\}$ is a subalgebra of $\mathbf{A}$, thus a is indispensable in $\mathbf{A}$, as well.

In conclusion, if $\mathbf{A}$ has some element a which is not indispensable, then we cannot extend or expand $\mathbf{A}$ in such a way that a becomes indispensable.

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6. Appendix

In this appendix we give a proof of Proposition 5.1.

Proposition 6.1. *Suppose that $\mathbf{A}$ is a countable (possibly infinitary) algebra with at least one operation of arity ≥ 2.*

Then $\mathbf{A}$ can be extended to some algebra $\mathbf{B}$ over the set $A \cup \{\infty\}$ ($\infty \notin A$), in such a way that A is the set of all the non-generators of $\mathbf{B}$ and ∞ is indispensable.

Proof. We give the proof in the case when $A = \{a_0, a_1, \dots\}$ is countably infinite and $\mathbf{A}$ has at least one binary operation $f_{\mathbf{A}}$. The general case is proved by an inessential modification. Notice that the proposition is trivially true when $\mathbf{A}$ is an empty algebra.

Choose some new element $\infty \notin A$. Define $f_{\mathbf{B}}$ on $B = A \cup \{\infty\}$ by

$$f_{\mathbf{B}}(b,c) = \begin{cases} f_{\mathbf{A}}(b,c) & \text{if } b, c \in A, \\ a_0 & \text{if } b = c = \infty, \\ a_{n+1} & \text{if } b = \infty,\ c = a_n, \\ \text{arbitrarily} & \text{in all the other cases} \end{cases}$$

For every other operation g of $\mathbf{A}$, let $g_{\mathbf{B}}(b_1, b_2, \dots) = g_{\mathbf{A}}(b_1, b_2, \dots)$ if $b_1, b_2, \dots \in A$, and define $g_{\mathbf{B}}(b_1, b_2, \dots)$ in an arbitrary way if some b_i is ∞.

Since $\mathbf{A}$ turns out to be a subalgebra of $\mathbf{B}$, then ∞ is indispensable in $\mathbf{B}$. Moreover, $\langle \infty \rangle_{\mathbf{B}} = B$, because of the definition of $f_{\mathbf{B}}$. From the above two facts we get that, for every $X \subseteq B$, $\langle X \rangle_{\mathbf{B}} = B$ if and only if $\infty \in X$. This implies that every $a \in A$ is a non-generator in $\mathbf{B}$. □

Proposition 6.1 bears a vague resemblance to Alexandroff construction of topological compactifications. It is not clear whether this observation can be inserted in some rigorous setting.

There are many possible variations on Proposition 6.1. For example, if $\mathbf{A}$ has at least $|A|$-many unary operations, choose injectively one such operation f for each $a \in A$ and set $f(\infty) = a$ in $\mathbf{B}$. Then the argument of Proposition 6.1 carries over. In words, we do not need a binary operation when we have enough unary operations. Of course, as we mentioned, some operation is necessary: in an algebra with no operation every element is indispensable.

We now present still another variation on Proposition 6.1, to the effect that it is enough to have just one unary operation, if we extend $\mathbf{A}$ with more than one new element.

From now on, *operation* means a (possibly infinitary) operation of arity at least 1 (sometimes constants are considered as nullary operations).

Proposition 6.2. *Every algebra $\mathbf{A}$ with at least one operation (possibly infinitary) can be extended to some algebra $\mathbf{B}$ in such a way that $B \setminus A \neq \emptyset$ and in $\mathbf{B}$ every element of A is a non-generator and every element of $B \setminus A$ is indispensable. Henceforth, by Proposition 2.4, in $\mathbf{B}$ the set of non-generators is the intersection of all the maximal proper subalgebras.*

Proof. If $\mathbf{A}$ is an empty algebra, take $\mathbf{B}$ a one-element extension and the conclusion holds. Thus we can assume $A \neq \emptyset$.

Let $B = A \cup \{ \infty_a \mid a \in A \}$, where each ∞_a is a new element not already in A and $\infty_a \neq \infty_c$, for $a \neq c \in A$. Pick some operation f and let $f_{\mathbf{B}}(\infty_a, \infty_a, \dots) = a$, for every $a \in A$. Define $f_{\mathbf{B}}$ for the remaining values, as well as every other operation on $\mathbf{B}$ arbitrarily, but letting them always take values on A and, of course, let them agree on A with the values given by the structure on $\mathbf{A}$. Then in $\mathbf{B}$ every element of $B \setminus A$ is indispensable, since it does not belong to images of the operations. Thus if $X \subseteq B$ and $\langle X \rangle_{\mathbf{B}} = B$, then $X \supseteq B \setminus A$. On the other hand, $\langle B \setminus A \rangle_{\mathbf{B}} = B$ because of the definition of $f_{\mathbf{B}}$, hence every element of A is a non-generator. □

We can iterate the process used in Propositions 6.1 and 6.2 and then take some limit on the unions, to get an extension in which all the elements are non-generators. We present the version dealing with Proposition 6.2 and leave to the reader the details for the case analogue to Proposition 6.1.

Corollary 6.3. *Every algebra $\mathbf{A}$ with at least one operation (possibly infinitary) can be extended to some algebra $\mathbf{C}$ such that every $c \in C$ is a non-generator in $\mathbf{C}$.*

Proof. Given $\mathbf{A}$, let $\mathbf{A}_1$ be the algebra constructed in Proposition 6.2 and called $\mathbf{B}$ there. Construct an algebra $\mathbf{A}_2$ by taking $\mathbf{A} = \mathbf{A}_1$ in Proposition 6.2 and letting $\mathbf{A}_2$ be the corresponding algebra $\mathbf{B}$. Iterate the procedure in order to construct a sequence $(\mathbf{A}_n)_{n \in \mathbb{N}}$ such that $\mathbf{A}_n$ is a subalgebra of $\mathbf{A}_{n+1}$, for $n \in \mathbb{N}$ (take $\mathbf{A}_0 = \mathbf{A}$).

We shall define an algebra $\mathbf{C}$ over the set-theoretical union $\bigcup_{n \in \mathbb{N}} A_n$. If f is a finitary operation, then $f_{\mathbf{C}}$ is defined uniquely on C by the requirement that each $\mathbf{A}_n$ be a subalgebra of $\mathbf{C}$. The same applies if we evaluate some infinitary operation f when all the arguments of f lie in some A_n. Otherwise, take the outcome of f to be an arbitrary element of $\mathbf{A}_0$ (in fact, the following argument works if we take the outcome of f to be an arbitrary element of $\mathbf{A}_n$, where n is the smallest index such that some argument of f belongs to $\mathbf{A}_n$).

Let c belong to C; we want to show that c is a non-generator in $\mathbf{C}$. Since $C = \bigcup_{n\in\mathbb{N}} A_n$, there is some n such that $c \in \mathbf{A}_n$. Choose n minimal. Suppose that $X \subseteq C$ and $\langle X, c\rangle = C$. Because of the way we have constructed $\mathbf{B}$ in Proposition 6.2 and by the above clause concerning infinitary operations, if we evaluate some operation f when c is among its arguments, then $f(\dots, c, \dots, c_i, \dots)$ belongs to A_n. It follows that $\langle X\rangle \supseteq C \setminus A_n$. Again by the construction in Proposition 6.2, $\langle A_{n+1}\rangle \supseteq A_n$, hence $\langle X\rangle \supseteq C$. We have proved that c is a non-generator in $\mathbf{C}$. □

Remark 6.4. (a) The assumption that $\mathbf{A}$ has a least an operation in Corollary 6.3 is necessary. As already pointed out, in an algebra without operations all the elements are indispensable.

(b) The algebra $\mathbf{C}$ obtained in Corollary 6.3 is necessarily infinite. A finite algebra has proper maximal subalgebras. Moreover, as already mentioned, every non-generator belongs to every maximal proper subalgebra; no finitary assumption is necessary in order to prove this statement.

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