


QUANTUM STOCHASTIC PROCESSES

L. Accardi

Contents

1.) Quantum Stochastic Processes
2.) The local algebras associated to a stochastic process
3.) Markov processes and dilations
4.) Perturbations of semi-groups: the Feynman-Kac formula
5.) Perturbations of stochastic processes
6.) The Wigner-Weisskopf atom
1. Quantum stochastic processes.

Let \( \mathcal{A} \) be a \( \sigma \)-algebra with identity (usually it will be a C*- or a W*-algebra). A quantum stochastic process over \( \mathcal{A} \) indexed by \( \mathbb{R} \) is defined by a triple \( (\mathcal{A}, (J_t)_{t \in \mathbb{R}}, \varphi) \) where

- \( \mathcal{A} \) is a \( \sigma \)-algebra with identity.
- \( J_t : \mathcal{A} \rightarrow \mathcal{A} \) is an embedding (\( t \in \mathbb{R} \)).
- \( \varphi \) is a state on \( \mathcal{A} \).

**Example 1.** Classical real valued stochastic processes.

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space and let \( X_t : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R} \) (\( t \in \mathbb{R} \)) be a real valued stochastic process. By choosing

- \( \mathcal{A} = L^0(\Omega, \mathcal{F}, \mathbb{P}) \),
- \( J_t : f \in \mathcal{A} \mapsto J_t(f) = f \cdot X_t = f(X_t) \) (\( t \in \mathbb{R} \))
- \( \varphi(a) = f \mathbb{E}[f^2] ; a \in \mathcal{A} \),

the triple \((\mathcal{A}, (J_t)_{t \in \mathbb{R}}, \varphi)\) is a quantum stochastic process in the sense defined above. Conversely, one easily sees that to a given quantum stochastic process \((\mathcal{A}, (J_t)_{t \in \mathbb{R}}, \varphi)\) such that \( \mathcal{A} \) is an abelian C*-algebra, one can associate a classical stochastic process, characterized up to stochastic equivalence, by the property of having the same finite dimensional correlation functions as the initial one. Thus, since the quantum stochastic processes include the classical ones, in the following we shall only speak of stochastic processes.

**Example 2.** (A "small" quantum system interacting with a "larger" one).

Let \( \mathcal{H}_0 \) and \( \mathcal{F} \) be two Hilbert spaces. One might regard \( \mathcal{H}_0 \) as the quantum state space of a "small system" interacting with an "extended system" with state space \( \mathcal{F} \) (a typical situation is \( \mathcal{H}_0 = L^2 \) : \( \mathcal{F} = \) some Fock space); in this case \( \mathcal{H}_0 \otimes \mathcal{F} \) will be the state space of the "composite system". The evolution of the "composite system" is given by a 1-parameter group \((\mathcal{U}_t)_{t \in \mathbb{R}}\) of unitary operators on \( \mathcal{H}_0 \otimes \mathcal{F} \):

\[
\mathcal{U}_t \in \mathcal{B}(\mathcal{H}_0 \otimes \mathcal{F}) \Rightarrow \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{B}(\mathcal{F})
\]

and there is a natural embedding \( j_0 : \mathcal{B}(\mathcal{H}_0) \hookrightarrow \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{B}(\mathcal{F}) \) (1.2)

denoting, for each \( t \in \mathbb{R} \) and \( a \in \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{B}(\mathcal{F}) \):

\[
u_t(a) = \mathcal{U}_t \cdot a \cdot \mathcal{U}_t^*
\]

(1.3)

one can define, for each \( t \in \mathbb{R} \), the embedding:

\[
J_t : b \in \mathcal{B}(\mathcal{H}_0) \hookrightarrow J_t(b) = b \otimes 1 \in \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{B}(\mathcal{F})
\]

(1.4)

Usually a state \( \varphi \) on \( \mathcal{B}(\mathcal{H}_0 \otimes \mathcal{F}) \) is given (\( \varphi = 0 \) and \( \varphi(1_{\mathcal{H}_0} \otimes 1_{\mathcal{F}}) = 1 \)) and, if we are interested only in the time evolution of the "small system", then all the interesting physical quantities can be expressed in terms of the correlation functions:

\[
\varphi(J_t(1_{\mathcal{H}_0}) \cdots J_{t_n}(1_{\mathcal{H}_0}))
\]

(1.5)

where \( b_j \in \mathcal{B}(\mathcal{H}_0) \) (\( j = 1, \ldots, n \)) and \( t_1, \ldots, t_n \) are real numbers which need not to be neither time-ordered nor mutually different. Choosing \( \mathcal{A} = \mathcal{B}(\mathcal{H}_0), \mathcal{A} = \mathcal{B}(\mathcal{H}_0 \otimes \mathcal{F}) \), and \((J_t)_{t \in \mathbb{R}}\) as in (1.5), one obtains a quantum stochastic process in the sense defined above.

**Remark 1.** Both in examples 1.) and 2.) one could have chosen a smaller algebra \( \mathcal{A} \) - for example the norm (in \( L^0(\Omega, \mathcal{F}, \mathbb{P}) \) or in \( \mathcal{B}(\mathcal{H} \otimes \mathcal{F}) \)) closure of the \( \sigma \)-algebra generated by the family \((J_t(\mathcal{A}) ; t \in \mathbb{R})\). In general, if \( \mathcal{A} \) is generated, algebraically or topologically, by the family \((J_t(\mathcal{A}) ; t \in \mathbb{R})\), we say that the stochastic process \((\mathcal{A}, (J_t)_{t \in \mathbb{R}}, \varphi)\) is minimal. In the following, unless explicitly stated, by "stochastic process" we will mean "minimal stochastic process".
Remark 2.) The occurrence of not necessarily time-ordered correlation functions in (1.6) arises naturally, for example in the computation of moments of observables of the form
\[ \sum_{\kappa=1}^{n} \gamma_{\kappa}(h_{\kappa}) ; s_{1} < ... < s_{n} ; h_{1}, ..., h_{n} \in \mathcal{A}(E) \]
Usually some commutation or anti-commutation relations (arising for example from Einstein causality) are available, and one is reduced to time-ordered correlations. Finally, by polarization and eventually choosing some \( b_{j} \) equal to 1, one verifies that the correlations (1.6) are uniquely determined by the so called correlation kernels:
\[ \gamma_{t_{1}, ..., t_{n}}(b_{1}, ..., b_{n}) = \mathcal{F}(j_{t_{1}}(b_{1}) ... j_{t_{n}}(b_{n}))(b) \]
(1.7)
\( (b_{j} \in \mathcal{A} ; t_{j} \in \mathbb{R} ; j = 1, ..., n) \). In [3] an abstract characterization of the correlation kernels is given, and it is shown that any family of correlation kernels defines (uniquely up to stochastic equivalence) a stochastic process.

2.) The local algebras associated to a stochastic process

Given a stochastic process \( \{ \mathcal{A}_{t} ; (j_{t})_{t} \in \mathbb{R} \} \) over a \( \sigma \)-algebra with identity \( \mathcal{B} \), one can define, for each sub-set \( I \subseteq \mathbb{R} \), the algebra \( \mathcal{A}_{I} = \bigvee_{t \in I} j_{t}(\mathcal{B}) \) (2.1)
where the right-hand side of (2.1) denotes the algebra generated by the set \( \{ j_{t}(\mathcal{B}) : t \in I \} \) (we leave unspecified the topology under which this algebra is closed; this will be clear, case by case, from the context). We will use the notations:
\[ \mathcal{A} = \bigvee_{t \in \mathbb{R}} j_{t}(\mathcal{B}) \]
(2.2)
\[ \mathcal{A}_{t} = \bigvee_{s \in \mathbb{R}} j_{s}(\mathcal{B}) \]
(2.3)
\[ \mathcal{A}_{t} - j_{t}(\mathcal{B}) \]
(2.4)
Clearly:
\[ s \leq t \rightarrow \mathcal{A}_{s} \subseteq \mathcal{A}_{t} \]
(2.5)
A family \( \{ \mathcal{A}_{s} \}_{s \in \mathbb{R}} \) of sub-algebras of \( \mathcal{A} \), satisfying (2.5), is called a filtration. Given a family \( \mathcal{F} \) of sub-sets of \( \mathbb{R} \) a family \( \{ \mathcal{A}_{I} \} \) of sub-algebras of \( \mathcal{A} \) satisfying:
\[ I \subseteq J \rightarrow \mathcal{A}_{J} \subseteq \mathcal{A}_{I} \]
(2.6)
is called a family of local sub-algebras of \( \mathcal{A} \) or simply a localization on \( \mathcal{A} \) based \( \mathcal{F} \).

Example. In the case of a classical stochastic process \( (X_{t})_{t} \in \mathbb{R} \) cf. the Example (1.) in Section (1.), the local algebras \( \mathcal{A}_{I} \) \( (I \subseteq \mathbb{R}) \) are sub-algebras of \( L^{\infty}(\mathcal{B}, \mathcal{F}_{I}, \mathbb{P}) \), where \( \mathcal{F}_{I} \) is the \( \sigma \)-algebra generated by the random variables \( (X_{t})_{t} \in \mathbb{I} \).

Given a family \( \{ \mathcal{A}_{I} \}_{I \subseteq \mathbb{R}} \) of local algebras \( \subseteq \mathcal{A} \) a 1-parameter group of automorphisms (sometimes endomorphisms) of \( \mathcal{A} \) is called a shift (with respect to that localization) if:
\[ u_{t} \mathcal{A}_{I} = \mathcal{A}_{I + t} ; \forall t \in \mathbb{R} ; I \subseteq \mathbb{R} ; \text{ (covariance)} \]
(2.7)
for any \( I \subseteq \mathbb{R} \) and \( t \in \mathbb{R} \). If the localization \( \{ \mathcal{A}_{I} \} \) is defined by a stochastic process through (2.1), then (2.7) is equivalent to:
\[ u_{t} \mathcal{J}_{s} = \mathcal{J}_{s + t} ; \forall s, t \in \mathbb{R} \]
(2.8)

Example. For a classical stochastic process \( (X_{t})_{t} \), one has:
\[ j_{t}(f) = f(X_{t}) \in L^{\infty}(\mathcal{B}, \mathcal{F}_{I}, \mathbb{P}) \]
(2.9)
\[ u_{t} f(X_{s}) = f(X_{s + t}) ; s, t \in \mathbb{R} \]
(2.10)

A stochastic process \( \{ \mathcal{A}_{t} , (j_{t})_{t} \} \) on \( \mathcal{B} \) is called stationary, if there exists a shift \( (u_{t}) \) on \( \mathcal{A} \) (i.e. a 1-parameter automorphism group of \( \mathcal{A} \) satisfying (2.8)) such that:
\[ \mathcal{F} \cdot u_{t} = \mathcal{F} ; \forall t \in \mathbb{R} \]
(2.11)
Recall that a conditional expectation from \( \mathcal{A} \) onto a sub-algebra \( \mathcal{C} \) is a linear map \( E : \mathcal{A} \rightarrow \mathcal{C} \) satisfying:
\[ E(1) = 1 ; E(ca) = cE(a) ; \forall a \in \mathcal{A} ; \forall c \in \mathcal{C} \]
(2.12)
Sometimes in classical stochastic processes - always for a natural choice of the local algebras \( \{ \mathcal{A}_{I} \} \) for any local algebra \( \mathcal{A}_{I} \) \( (I \subseteq \mathbb{R}) \) there exists a conditional expectation \( E_{I} : \mathcal{A} \rightarrow \mathcal{A}_{I} \) such that:
\[ \mathcal{F} \cdot E_{I} = \mathcal{F} \]
(2.13)
i.e. compatible with the state \( \varphi \). The family \((E_t)\) satisfies
\[
\mathfrak{L}_{\mathcal{J}} \Rightarrow E_t \cdot E_s \leq E_{ts} \quad (\text{projectivity})
\tag{2.14}
\]
and if the state \( \varphi \) is shift-invariant, then:
\[
\frac{u}{u} \cdot E_t = E_{ts} \cdot \frac{u}{u} \tag{2.15}
\]
Any family \((E_t)\) of surjective conditional expectations \(E_t : \mathcal{A} \twoheadrightarrow \mathcal{A}_t\)
will be called projective if it satisfies (2.14) and covariant if it satisfies (2.15). In particular, in case of a filtration (a localization based on the "past half-lines" \((-\infty, t) : t \in \mathbb{R})\) conditions (2.14) and (2.15) become:
\[
\begin{align*}
\mathfrak{s} \in t \Rightarrow & E_s \cdot E_t = E_{st} \\
& E_{st} = E_{st} \\
& u \cdot E_t = E_{st} \cdot u
\end{align*}
\tag{2.16, 2.17}
\]

3.) Markov processes and dilations.

A markovian structure on a \( \mathfrak{w} \)-algebra \( \mathcal{A} \) is defined by the assignment of:
- a "past-filtration" \( (\mathcal{A}_t)_{t \in \mathbb{R}} \) on \( \mathcal{A} \);
- a "future-filtration" \( (\mathcal{A}_t')_{t \in \mathbb{R}} \) on \( \mathcal{A} \);
- for each \( t \in \mathbb{R} \) - an "algebra at time \( t \)" \( \mathcal{A}_t \) such that:
\[
\mathcal{A}_t \subseteq \mathcal{A}_t' \cap \mathcal{A}_t''
\tag{3.1}
\]
- A projective system of conditional expectations \( E_t : \mathcal{A} \twoheadrightarrow \mathcal{A}_t \)
i.e.:
\[
\mathfrak{s} \in t \Rightarrow E_s \cdot E_t = E_{st} \\
\text{enjoying the Markov property:}
\tag{3.2}
\]
\[
E_t((\mathcal{A}_t) \subseteq \mathcal{A}_t) : \forall \mathfrak{r} \in \mathbb{R}
\tag{3.3}
\]
If the localization \((\mathcal{A}_t), (\mathcal{A}_t'), (\mathcal{A}_t'')\) admits a shift \( u_t \), i.e.
\[
u \cdot \mathcal{A}_t = \mathcal{A}_{st+t} ; \quad u_t \cdot \mathcal{A}_t = \mathcal{A}_{st} ; \quad u_t \cdot \mathcal{A}_t = \mathcal{A}_{st-t}
\tag{3.4}
\]
and if the family \((E_t)\) of conditional expectations is covariant, i.e.
\[
u \cdot E_s = E_{st} \cdot u_t
\tag{3.5}
\] then we speak of a covariant markovian structure.

Example Let \( X_t : (\Omega, \mathcal{F}, P) \to \mathbb{R} (t \in \mathbb{R}) \) be a classical Markov process; let \( \mathcal{F}_t, \mathcal{F}_t', \mathcal{F}_t'' \) be respectively the past, present and the future \( \mathfrak{w} \)-algebras; denote
\[
\mathcal{A} = \mathcal{L}_w(\Omega, \mathcal{F}, P) ; \mathcal{A}_t = \mathcal{L}_w(\Omega, \mathcal{F}_t, P)
\]
and let \( E_t = E(\cdot | \mathcal{F}_t) \) be the \( P \)-conditional expectation on \( \mathcal{F}_t \). Clearly these objects define a markovian structure on \( \mathcal{A} \) - a covariant markovian structure if the process \( (X_t) \) is stationary.

The connection between markovian structures and semi-groups is made precise by the following

Theorem (3.1) Let \((E_t)_{t \in \mathbb{R}}\) be a family of maps (not necessarily conditional expectations) satisfying conditions (3.2) - projectivity - and (3.5) - covariance. Denote \( \mathcal{A}_0 \) the vector space generated by the family:
\[
\{E_{t}, u \cdot E_{s} \cdot \mathcal{A}_0 : t \in \mathbb{R}
\tag{3.6}
\]
(\( \mathcal{A}_0 \) - \( \mathcal{A}_0 \) in the markovian case), and define
\[
P_t = E_{t} \cdot \mathcal{A}_0 \cdot \mathcal{A}_0
\tag{3.7}
\]
then the family \((P_t)\) is a semi-group of \( \mathcal{A}_0 \) into itself.

Proof. For \( a \in \mathcal{A}_0 \) and \( s, t \in \mathbb{R} \) one has:
\[
P_t \cdot P_s (a) = E_{t} \cdot u \cdot E_{s} \cdot u \cdot E_{t} (a) = E_{t} \cdot u \cdot E_{s-t} (a) = E_{t} \cdot u \cdot E_{s-t} (a) = P_{t-s} (a)
\]
If the maps \((E_t)\) are completely positive, identity preserving, (e.g. conditional expectations) then the semi-group \((P_t)\) is completely positive identity preserving. Any such a semi-group will be called a markovian semi-group on \( \mathcal{A}_0 \). If \( \mathcal{A}_0 \) is a non-commutative algebra, one also speaks of a quantum dynamical semi-group.

In the following we shall only consider the markovian case, i.e.
\[
\mathcal{A}_0 = \mathfrak{A}_0. \quad \text{Thus, denoting } \mathfrak{A} = \mathcal{A}_0 \text{ and } j_0 = \text{the identity embedding } \mathcal{A}_0 \hookrightarrow \mathfrak{A}, \text{ one obtains the commutative diagramme:}
\tag{3.8}
\]
\[
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{t} & \mathfrak{A} \\
\downarrow{j_0} & & \downarrow{j_0} \\
\mathfrak{A} & \xrightarrow{t} & \mathfrak{A}
\end{array}
\]
where $j_0^{-1}$ denotes the left inverse of $j_0$.

**Definition (3.2)** Let $\mathcal{B}$ be a $C^*$-algebra (with identity) and $(\mathcal{P}_t^+)$ a markovian semi-group on $\mathcal{B}$, a $C^*$-dilation of the pair $(\mathcal{B}, (\mathcal{P}_t^+))$ is a quadruple $\{A, j_0, E_0, (u_t^0)\}$ making commutative the diagramme (3.8) and such that $j_0: \mathcal{B} \rightarrow A$ is a $C^*$-embedding: $(u_t^0)$ is a 1-parameter automorphisms group of $A$; $E_0: \mathcal{B} \rightarrow \mathcal{A}$ is a norm-one projection satisfying:

$$E_0 \cdot u_t^0 (j_0 (\mathcal{B})) \subseteq j_0 (\mathcal{B}); \forall t \geq 0$$

(3.9)

If, moreover, denoting $\mathcal{A}[t] = \mathcal{A} \cap \mathcal{A}[t]$ one has:

$$E_0 \cdot u_t^0 \cdot E_0 \cdot u_t^{-1} | \mathcal{A}[t] = E_0 | \mathcal{A}[t] ; \forall t \geq 0$$

(3.10)

then we speak of a (covariant) markovian dilation of $(\mathcal{B}, (\mathcal{P}_t^+))$.

Finally if there exists a state (weight) $\varphi$ on $\mathcal{A}$ satisfying:

$$\varphi \cdot E_0 = \varphi ; \varphi \cdot u_t^0 = \varphi ; \forall t \geq 0$$

(3.11)

then we speak of a stationary markovian dilation of $(\mathcal{B}, (\mathcal{P}_t^+))$.

**Remark** One easily sees that there is a one-to-one correspondence between covariant markovian dilations of $(\mathcal{B}, (\mathcal{P}_t^+))$ and covariant markovian structures (as defined at the beginning of this section) with $\mathcal{A} \subseteq \mathcal{B}$ and $E_0, u_t^0, j_0, \mathcal{P}_t^+$.

A beautiful classification theory of dilations of completely positive semi-groups has been developed by B. Kummerer and W. Schroder. In the classical case, i.e. when $\mathcal{B}$ is abelian we know that:

i) any markovian semi-group $(\mathcal{P}_t^+)$ on $\mathcal{B}$ has a covariant markovian dilation (obtained through the well known Daniell-Kolmogorov construction).

ii) $(\mathcal{P}_t^+)$ has a stationary markovian dilation if and only if there exists a state (weight) $\varphi_0$ on $\mathcal{B}$ such that

$$\varphi_0 = \varphi_0 \cdot \mathcal{P}_t^+$$

(3.12)

In the quantum case the situation is more complicated and only recently R. Hudson and K.R. Parthasarathy [6] have shown that the statement (i) holds; while A. Frigerio [5] (cf. also the paper by A. Frigerio and V. Gorini [4]) has found the correct quantum analogue of the

statement (ii).

In the following sections we will describe the main technical tools through which the solution of the above mentioned problems has been achieved.

4) Perturbations of semi-group: the Feynman-Kac formula.

Let $\{A, (\mathcal{A}_t^0), (\mathcal{A}_t^1), (\mathcal{D}_t^0), (\mathcal{D}_t^1), (E_t^0), (E_t^1)\}$ be a given covariant markovian structure, and let be given a covariant family of local $\mathcal{C}^*$-algebra $(\mathcal{A}_s^t)$ ($s \leq t$, $s, t \in \mathbb{R}$) such that

$$\mathcal{A}_s^t \subseteq \mathcal{A}_s^0 \cap \mathcal{A}_t^1$$

(4.1)

A markovian cocycle (with respect to the structure defined above) is a 1-parameter family $(M_{s,t})_{s \leq t}$ of elements of $\mathcal{A}$ such that:

$$M_{s,t} \in \mathcal{A}_s^t \cap \mathcal{A}_t^0 ; \forall s \leq t ; (\text{markovianity})$$

(4.2)

$$M_{s+r,s} = M_{s+r,s} \cdot M_{s,s} ; \forall s \leq t$$

(4.3)

Denoting, for $s \leq t$, $M_{s,t} = u_{t-s} (M_{0,s})$, then the two parameter family $(M_{s,t})_{s \leq t}$ is such that:

$$M_{s,t} \in \mathcal{A}_s^t \cap \mathcal{A}_t^0$$

(4.4)

$$M_{r,s} M_{s,t} = M_{r,t} ; \forall r \leq s \leq t$$

(4.5)

$$u_t (M_{r,s}) = M_{r+s,t} ; \forall r \leq s \leq t$$

(4.6)

and the three conditions above are those which, in classical probability theory, define the so-called multiplicative functionals associated to a given family $\{\mathcal{F}_s^t\}$ of $\sigma$-algebras. Typical examples are given by:

$$\mathcal{A} = L^2 (\Omega, \mathcal{F}, P) ; (\Omega, \mathcal{F}, P) \text{ - a Wiener probability space; } (\mathcal{G}_{s,t})_{s \leq t} \text{ - a real valued Wiener process;}$$

$$M_{s,t} = \exp \frac{1}{2} \int_s^t V(\omega) dr + \int_s^t S(\omega) dW$$

(4.7)

with $V, S : \mathbb{R} \rightarrow \mathbb{R}$ sufficiently regular functions.

**Theorem (4.1)** Let $(M_{s,t})_{s \leq t}$ be a markovian cocycle and define, for $t > 0$

$$\mathcal{P}_t^+ (a) = E_0 [M_{s,t} u_{t-s} (a) M_{0,t}^+] ; a_0 \in \mathcal{A}_0$$

(4.8)

It follows that $(\mathcal{P}_t^+)$ is a semi-group $\mathcal{A}_0 \rightarrow \mathcal{A}_0$. 


Proof. For \( a \in \mathcal{A} \) and \( s, t \in \mathbb{R} \), one has:
\[
P^t P_s(a) = E_0(t,s) u^{(s)a} E_0(s,t) u^{(t)a} (M_{0} + \mathcal{N}^+_s) + \mathcal{N}^+_t =
\]
\[
E_0(t,s) u^{(s)a} E_0(s,t) u^{(t)a} M_{0} + E_0(t,s) u^{(s)a} E_0(s,t) \mathcal{N}^+_t =
\]
\[
E_0(t,s) u^{(s)t} E_0(s,t) u^{(t)a} \mathcal{N}^+_t =
\]
\[
E_0(t,s) u^{(s)t} E_0(s,t) u^{(t)a} \mathcal{N}^+_t =
\]
\[
E_0(t,s) u^{(s)t} E_0(s,t) u^{(t)a} \mathcal{N}^+_t = P^t P_s(a).
\]
Any semi-group \( (P^t) \) defined as above, will be called a Feynman-Kac perturbation of the semi-group \( P^t = E_0(t,s) u^{(s)t} \), \( t > 0 \).

Formula (4.8) will be referred to as the Feynman-Kac formula. This formula generalizes several known constructions:

1.) The classical Feynman-Kac formula. This is obtained by choosing, in the notations of formula (4.7):
\[
M_{0} = \exp \left( \frac{t}{2} \int_{0}^{t} W(s) \, ds \right)
\]
(4.9)
where \( W \) is a suitably regular function (e.g. measurable bounded below).

2.) The non-interaction representation. This is obtained by choosing the markovian structure to be trivial (i.e. all the local algebras are equal to \( \mathcal{A} \) and \( E_0 \) is the identity map on \( \mathcal{A} \)), and the cocycle \( M_{0} = \mathcal{U}_{0,t} \) to be unitary. In this case, writing \( \mathcal{U}_{t,s} \) instead of \( P^t \) the Feynman-Kac formula becomes:
\[
\mathcal{U}_{t,s}(a) = \mathcal{U}_{0,t} u^{(s)a} \mathcal{U}_{0,s} u^{(t)a} ; \quad a \in \mathcal{A}
\]
(4.10)
The cocycle property then assures that \( (\mathcal{U}_{t,s}) \) is a 1-parameter automorphisms group of \( \mathcal{A} \) (cf. the proof of Theorem (4.1), with all the conditional expectations equal to the identity). The pair \( \{ \mathcal{U}_{0,t}, \mathcal{U}_{0,s} \} \) where \( \mathcal{U}_{0,t} \) is a 1-parameter automorphisms group and \( \mathcal{U}_{0,s} \) is a unitary (markovian) \( (\mathcal{U}_{0,t}) \)-cocycle is called an interaction representation for the 1-parameter automorphisms group \( (\mathcal{U}_{0,t}) \) defined by (4.10). The connection with the notion of interaction representation usually met in physics is given by the following formal considerations: let \( \mathcal{U}_{0,t} \) be of the form:
\[
\mathcal{U}_{0,t}(a) = \mathcal{U}_{0,t}^{(s)} a \mathcal{U}_{0,t}^{(t)} ; \quad a \in \mathcal{A}
\]
(4.11)
with \( \mathcal{U}_{0,t}^{(s)} = \exp iH_{0}^{s} \) a unitary in \( \mathcal{A} \) and let \( H_{1} \in \mathcal{A} \) be a self-adjoint operator. Define
\[
H_{1}(t) = \mathcal{U}_{0,t}^{(t)} H_{1} \mathcal{U}_{0,t}^{(t)} ; \quad t \in \mathbb{R}
\]
(4.12)
and let \( \mathcal{U}_{0,t} \) be defined by:
\[
\mathcal{U}_{0,t} = \mathcal{U}_{0,t}^{(s)} H_{1} ; \quad \mathcal{U}_{0,t}^{(s)} = \mathcal{U}_{0,s}^{-1}
\]
(4.13)
then \( \{ \mathcal{U}_{0,t} \} \) is a unitary \( (\mathcal{U}_{0,t}) \)-cocycle (markovian in an appropriate localization) and
\[
\mathcal{U}_{t,s} = \mathcal{U}_{0,t} \mathcal{U}_{0,s}^{-1} ;
\]
is a 1-parameter unitary group in \( \mathcal{A} \) satisfying the formal equation
\[
\frac{d}{dt} \mathcal{U}_{t,s} = i \mathcal{U}_{t,s} \left[ H_{1} + H_{2} \right]
\]
(4.14)
In many concrete examples either \( H_{2} + H_{1} \) or \( H_{1}(t) \) are not well defined as operators so that equation (4.13) or (4.14) has no rigorous meaning. But we will see that in many cases it is still possible to define, using quantum stochastic calculus, a markovian cocycle \( \{ \mathcal{U}_{0,t} \} \) and a 1-parameter unitary group \( \{ \mathcal{U}_{t,s} \} \) having all the properties of the formal solutions of the equations (4.13) and (4.14) (cf. Section (6.) in the following).

3.) Perturbations of the identity semi-group. Consider a markovian structure as in the beginning of this section, and let \( \mathcal{A} \) be of the form:
\[
\mathcal{A} = \mathcal{B}(H_{0}) \otimes \mathcal{A} \otimes \mathcal{B}(H_{1}) \otimes \mathcal{B}(F)
\]
(4.15)
where \( H_{0} \) and \( F \) are complex separable Hilbert spaces. Assume that the shift \( (\mathcal{U}_{0,t}) \) as the form:
\[
\mathcal{U}_{0,t} = \mathcal{U}_{0,t}^{(s,t)} \mathcal{U}_{0,s}^{(t)}
\]
(4.16)
where \( \mathcal{U}_{0,s} \) is the identity map on \( \mathcal{B}(H_{0}) \) and \( \mathcal{U}_{0,t}^{(s,t)} \) is a 1-parameter automorphisms group of \( \mathcal{B}(F) \). In this case the semi-group \( \mathcal{P}_{0} = \mathcal{U}_{0,t} \) of the identity semi-group on \( \mathcal{A} \otimes \mathcal{B}(H_{0}) \otimes 1 \), and its Feynman-Kac perturbation with respect to a unitary markovian cocycle \( \{ \mathcal{U}_{0,t} \} \) has the form:
\[
P_{0}^{t}(a) = E_0(t,s) u^{(t)s} u^{(t)a} u^{(t)s}
\]
(4.17)
A semi-group of this form will be called a Feynman-Kac perturbation of the identity semi-group.

Hilbert space. Any markovian semi-group on $\mathcal{B}(H)$ admitting a Lindblad generator has a covariant markovian dilation which is a Feynman-Kac perturbation of the identity semi-group.

5.) Perturbation of stochastic process

In the proceeding section we have shown that any markovian cocycle gives rise to a perturbation of a markovian semi-group. In this section we show that any unitary markovian cocycle gives rise to a perturbation of a covariant markovian structure which is still a covariant markovian structure. This is a purely quantum-probabilistic phenomenon, since in the abelian case unitary markovian cocycles give rise only to trivial (i.e. identity) perturbations.

Let $\mathcal{A}_t$, $(\mathcal{A}_t)$, $(\mathcal{A}_s)_t$, $(\mathcal{A}_{s,t})$, $(\mathcal{A}_{s,t})$, be as in Section (5); let $(U_{0,t})$ be a unitary markovian cocycle, and define

$$u_t(s) = U_{0,t} \cdot u_0^0(a) \cdot U_{0,t}^{-1} : s \in \mathcal{A}$$  \hspace{1cm} (5.1)

Then $(u_t)$ is a $t$-parameter automorphism group of $\mathcal{A}$ and defining:

$$\mathcal{A}_t = \mathcal{A}_0 : \mathcal{A} = u_t(\mathcal{A}_0) \subseteq \mathcal{A}_{0,t}$$  \hspace{1cm} (5.2)

one easily verifies that for each $a \in \mathcal{A}$:

$$u_t \cdot E_a(a) = E_{u_t(a)} : u_t(a)$$  \hspace{1cm} (5.3)

thus the family $(E_t)$ is also covariant for the evolution $(u_t)$ defined by (5.1).

Define now, for $t \geq 0$:

$$\mathcal{B}_t = \bigvee_{s \leq t} u_s(\mathcal{A}_s) = \bigvee_{s \leq t} u_s(\mathcal{A}_0)$$  \hspace{1cm} (5.4)

and similarly for $\mathcal{B}_s$. Remark that:

$$\mathcal{B}_t \subseteq U_{0,t} \cdot \mathcal{A} \cdot U_{0,t}^{-1}$$  \hspace{1cm} (5.5)

whence, due to the markov property of $(E_t)$:

$$E_{t+s} \cdot \mathcal{B}_{t+s} \subseteq U_{0,t+s} \cdot \mathcal{B}_{t+s} \subseteq U_{0,t+s} \cdot \mathcal{A} \cdot U_{0,t+s}^{-1}$$

$$E_{t+s} \cdot \mathcal{A} \cdot U_{0,t+s}^{-1} = u_{t+s}(\mathcal{A}_0) = \mathcal{B}_{t+s}$$  \hspace{1cm} (5.6)

Thus $(\mathcal{B}_t)$ is markovian also with respect to the localization $(\mathcal{B}_s)$,

$$(\mathcal{B}_t), (\mathcal{A}_t)$$ or equivalently, defining:

$$\mathcal{B} = \bigvee_{t \in \mathbb{R}^+} u_t(\mathcal{A})$$  \hspace{1cm} (5.7)

the family $(\mathcal{B}, (\mathcal{B}_t), (\mathcal{B}_s)_t, (\mathcal{B}_{s,t}), (\mathcal{B}_{s,t})_t)$ is still a covariant markovian structure. In particular, for any state $\varphi_0$ on $B_0 = \mathcal{A}_0$, defining $\gamma = \varphi_0 \cdot E_0$ (state on $\mathcal{B}$), $J_0 = \text{identity embedding}$

$\mathcal{B}_0 \rightarrow \mathcal{B} : J_t = u_t^{-1} \cdot J_0 \cdot u_t$ ($t \geq 0$), the triple $(\mathcal{B}, (J_t)_t, \gamma)$ is a markovian stochastic process over $\mathcal{B}_0$, in the sense defined at the beginning of Section (1.). As shown by A. Frigerio and V. Gorini [4], [5], in the case of boson dilutions the process will be stationary if and only if the associated semi-group satisfies a detailed balance conditions. More generally, in the framework of local algebras, it can be shown that the stationarity of the process is related to the behaviour of the semi-group under appropriate "time-reflections" (cf. [1], [2]).

6.) The Wigner-Weiskopf atom

In this section I will outline some results obtained in collaboration with D. Applebaum and which will be published elsewhere. For the description of the Wigner-Weiskopf model we follow the exposition given by W. von Waldenfels in [9] and we also refer to this paper for a more complete discussion of the physical limits of this approximation. In its simplest version the model describes a 2-levels atom in interaction with an electro-magnetic field. In the "rotating wave approximation" the system is described on the Hilbert space

$$\mathcal{H} = C^2 \otimes \mathcal{F}(\mathcal{H}^A) \cong C^2 \otimes \bigoplus_{\lambda \in \Lambda} \mathcal{F}_\lambda$$  \hspace{1cm} (6.1)

where $\Lambda$ is a finite set (indexing the frequencies of the EM field), $|\Lambda|$ denotes the cardinality of $\Lambda$ and, for each $\lambda \in \Lambda$, $\mathcal{F}_\lambda \cong \mathcal{F}(\mathcal{C})$ is the Fock space over the Hilbert space $\mathcal{C}$ (with scalar product $\langle u, v \rangle = \langle uv \rangle$; $u, v \in \mathcal{C}$). On each space $\mathcal{F}_\lambda$ the creation and annihilation operators $B^\dagger_\lambda, B_\lambda$ are defined in the usual way and they satisfy the commutation relations:

$$[B_\lambda, B^\dagger_{\lambda'}] = \delta_{\lambda, \lambda'}, \quad [B_\lambda, B_\lambda] = [B^\dagger_\lambda, B^\dagger_\lambda] = 0$$  \hspace{1cm} (6.2)

Introducing the spin matrices:

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \sigma_z = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$  \hspace{1cm} (6.3)

The hamiltonian of the system in the rotating wave approximation is:
This means that one substitutes for \( B_{\lambda}(t) \) and \( B_{\lambda}^+(t) \) two operators \( F(t) \), \( F^+(t) \) satisfying :
\[
[F(t), F(s)] = [F^+(t), F^+(s)] = 0 \quad (6.4)
\]
\[\langle F(t), F^+(s) \rangle = \delta(t-s) \quad (6.5)\]
and on the algebra generated by the family \( \{F(t), F^+(t)\} \) one introduces the quasi-free state characterized by
\[
\delta(F(t) \cdot F(s)) = \delta(F^+(t) \cdot F^+(s)) = 0 \quad (6.6)
\]
\[\delta(F(t) \cdot F^+(s)) = (1 + 0) \cdot \delta(t-s) \quad (6.7)\]
With these approximations the equation for the unitary cocycle becomes
\[
\frac{d}{dt} U(t) = -iU(t) \cdot H(t) \quad ; \quad U(0) = 1 \quad (6.8)
\]
\[H(t) = \sigma \cdot \Omega(t) + \sigma \cdot \Omega^+(t) \quad (6.9)\]
Equation (6.8) is purely formal because, due to (6.6), (6.7) and (6.9), \( H(t) \) is not a well defined operator but an operator valued distribution.
In analogy with the classical procedure von Waldenfels [9] introduced to methods for the solution of equation (6.8) :

I.) The "Stratonovich method", corresponding to the "singular coupling limit method" in the physical literature, consisting in three steps : 

i) regularize the covariance with the substitution, in (6.5) and (6.7) 
\[\delta(t-s) \rightarrow K(t-s) \] for some smooth function \( K(t) \).

ii) solve the corresponding ordinary differential equation, finding a cocycle \( U_c(0,t) \). 

iii) determine the limit of \( U_c(0,t) \) - and of the associated process (Section 5) as \( t \rightarrow 0 \) and \( K(t-s) \rightarrow \delta(t-s) \).

II.) The "multiplicative Ito integral method", (corresponding to the approximation methods in classical probability) in which – instead of the covariance – you regularize the fields. This can be done in several ways. In [9] one considers for each fixed \( T \in \mathbb{R}_+ \) a partition 
\[\{0 = t_0 < t_1 < \ldots < t_n = T\} \] of the interval \([0,T]\) and introduces the piecewise constant fields :
\[
F(t) = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} F(t) dt = F(t \in [t_k, t_{k+1}]) : t_k \leq t < t_{k+1} \]
One then solves the ordinary differential equation :
\[ \frac{d}{dt} U(t) = -dU(t) + H(t) \]

and studies the limit of \( U(t) \) (and of the corresponding process) as
\[ |\omega| = \max_k \left( t_k + \frac{1}{k} \right) \to 0. \]

For the Wigner-Weiskopf model the existence of the limiting cocycle (and of the corresponding process) was established by von Waldenfels [9] in both cases (I.) and (II.). A third possibility, is to interpret (6.8) as a quantum stochastic differential equation and use the results of R. Hudson and K.R. Parthasarathy [6] to establish the existence, uniqueness and unitarity of the cocycle \( U(t) \).

Namely, one considers the Hilbert space
\[ \Gamma(L^2(\mathbb{R}^2, dt)) \otimes \Gamma(L^2(\mathbb{R}^2, dt)^*) = \mathcal{H}, \]
where \( \Gamma(H) \) denotes the (boson) Fock space of \( H \) and \( H^* \) denotes the conjugate Hilbert space of \( H \). On this Hilbert space one considers the representation of the CCR with creation and annihilation operators given by:
\[
\begin{align*}
\Gamma(t) &= \sqrt{\cosh h^2} \Theta_{x,t}^* \Theta_x^* \Theta_{x,t}^* \\
\Gamma^*(t) &= \sqrt{\cosh h^2} \Theta_{x,t}^* \Theta_x^* \Theta_{x,t}^* \\
\end{align*}
\]
where \( a(\cdot) \) and \( a^*(\cdot) \) are the annihilation and creation operators over
\[ \Gamma(\mathcal{H}), \]
and by definition, \( \gamma = 2 \operatorname{Re} \beta \), and:
\[
\cosh h^2 = \frac{1}{1 - \exp(-\omega / \sqrt{\beta})}; \quad \sinh h^2 = \frac{\exp(-\omega / \sqrt{\beta})}{1 - \exp(-\omega / \sqrt{\beta})} = 0.
\]

With these notations the unitary (markovian) cocycle \( U_t \) is defined as the solution of the quantum stochastic differential equation
\[
dU_t = U_t \left(-i \sigma @ D(t) - i \sigma @ D^*(t) - \gamma h \left( \cosh h^2 \sigma @ \sigma \otimes \Theta_x^* \Theta_x^* \right) \right) dt
\]
\[
\text{Denoting} \ F \ 	ext{the conditional expectation characterized by:}
\[
\begin{align*}
F: \mathcal{B}(\mathcal{H}) &\to \mathcal{B}(\mathcal{H}) \\
F(x @ \sigma) &\to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\Gamma(L^2(\mathbb{R}^2)) \otimes \Gamma(L^2(\mathbb{R}^2)^*)) \\
&\to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})
\end{align*}
\]

where \( \mathcal{B}(\mathcal{H}) \) (resp. \( \mathcal{H} \)) denotes the Fock vacuum in \( \Gamma(L^2(\mathbb{R}^2)) \) (resp. \( \Gamma(L^2(\mathbb{R}^2)^*) \)) and applying the theory outlined in Section (4), one obtains a semi-group on \( \mathcal{B}(\mathcal{H}) = (2 \times 2 \text{ matrices}) \) via the prescription:
\[
x \in \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes (x @ 1 @ 1) @ (x @ 1 @ 1) \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})
\]
whose generator is:
\[
L(x) = -\frac{1}{2} \cosh h^2 \gamma(\sigma @ x @ \sigma @ x) + \cosh h^2 \gamma(\sigma @ x @ \sigma @ x) + \\
-\frac{1}{2} \sinh h^2 \gamma(\sigma @ x @ \sigma @ x) + \sinh h^2 \gamma(\sigma @ x @ \sigma @ x)
\]
\((x \in \mathcal{B}(\mathcal{H}))\). Referring the algebra of \( 2 \times 2 \) complex matrices; \( \mathcal{B}(\mathcal{H}) \) to the standard bias, we find for \( L \) the matrix:
\[
\begin{pmatrix}
-\gamma^2 & \gamma(\sigma @ 1) & 0 & 0 \\
\gamma^2 & -\gamma(\sigma @ 1) & 0 & 0 \\
0 & 0 & \mathcal{B}(\mathcal{H}) & 0 \\
0 & 0 & 0 & \mathcal{B}(\mathcal{H})
\end{pmatrix}
\]
which is exactly the formula found by von Waldenfels via the "multiplicative Ito method" [9] (in his notations \( \gamma = 2 \operatorname{Re} \beta \)). To obtain the formula found by von Waldenfels via the "Stratonovich method" instead of (6.10) one has to look for the solution of the quantum stochastic differential equation:
\[
dU_t = U_t \left(-i \sigma @ D(t) - i \sigma @ D^*(t) - \gamma h \left( \cosh h^2 \sigma @ \sigma \otimes \Theta_x^* \Theta_x^* \right) \right) dt - \frac{1}{2} \gamma h \left( \cosh h^2 \sigma @ \sigma \otimes \Theta_x^* \Theta_x^* \right) dt
\]

where, in von Waldenfels notations: \( \gamma = 2 \operatorname{Re} \beta \), \( \beta = 2 \operatorname{Im} \beta \). The connection between the multiplicative Ito (i.e. singular coupling) method and quantum stochastic differential equations was suggested by Frigerio and Gorini [4] and the explicit form of the semi-group obtained in the Wigner-Weiskopf model in the "multiplicative Ito" case (i.e. corresponding to equation (10)) has been independently obtained by H. Maassen [8].

**REFERENCES**


ABSOLUTELY CONTINUOUS IN Variant MEASURES FOR SOME MAPS OF THE CIRCLE

P. M. Blecher and M. V. Jakobson

1. Statement of results. We consider the two-parameter family of maps on the circle

\[ f_{q,\omega} : x \mapsto x + 2\pi \cdot \sin 2\pi x, \quad x \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \]

and we find a set \( M = \{ (q, \omega) \} \) of positive Lebesgue measure such that \( (q, \omega) \in M \) implies the stochastic behaviour of \( f_{q,\omega} \). We present analytical and numerical results which describe the structure of \( M \) as follows.

There exists a sequence of points \( A_k = (q_k, \omega_k) \), \( k \in \mathbb{N} \), converging to the limit \( A_\infty = (q_\infty, \omega_\infty) \), where \( q_\infty = 1,169701... \), \( \omega_\infty = q_\infty/2\pi \), satisfying

Theorem 1. For any \( k \) there exists a set \( \mathcal{M}_k \subset \mathbb{R}^2 \) of positive Lebesgue measure, such that \( A_k \) is a density point of \( \mathcal{M}_k \), and if \( (q, \omega) \in \mathcal{M}_k \) then the map \( f_{q,\omega} : S^1 \to S^1 \) has an absolutely continuous invariant probability measure \( \mu_{q,\omega} \). The map \( f_{q,\omega} \) cyclically permutes \( k \) adjacent intervals \( I_1, I_2, \ldots, I_k \), \( I_i \subset [0, k-1] \), \( k \cdot I_i = S^1 \). The support of \( \mu_{q,\omega} \) consists of \( k \) intervals \( \xi(i) \subset \xi(i+1) \) of equal measure. For any \( i \) the map \( f_{q,\omega}^i \) is an exact endomorphism on the measure space \( (S^1, \mathcal{M}_k, \mu_{q,\omega}) \), and its natural extension is a Bernoulli automorphism.

In order to prove Theorem 1 for a given \( k \) it suffices to verify some conditions of non-degeneracy, see Sect. 3. For \( k=1 \) these conditions are verified analytically. For \( 2 \leq k \leq 7 \) they were verified with the help of a computer.