CENTRAL EXTENSION OF VIRASORO TYPE SUBALGEBRAS OF THE ZAMOLODCHIKOV-$w_\infty$ LIE ALGEBRA

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It is known that the centerless Zamolodchikov–$w_\infty$ $\ast$–Lie algebra of conformal field theory does not admit nontrivial central extensions, but the Witt $\ast$–Lie algebra, which is a sub–algebra of $w_\infty$, admits a nontrivial central extension: the Virasoro algebra. Therefore the following question naturally arises: are there other natural sub–algebras of $w_\infty$ which admit nontrivial central extensions other than the Virasoro one? We show that for certain infinite dimensional closed subalgebras of $w_\infty$, which are natural generalizations of the Witt algebra the answer is negative.

Keywords: Witt algebra; Virasoro algebra; Central extensions

1. Introduction

The centerless Virasoro (or Witt)-Zamolodchikov-$w_\infty$ $\ast$–Lie algebra (cf.3–6) is the infinite dimensional $\ast$–Lie algebra, with generators

\[ \{ \hat{B}_k^n : n \in \mathbb{N}, n \geq 2, k \in \mathbb{Z} \} \quad (1) \]

commutation relations

\[ [\hat{B}_k^n, \hat{B}_K^N] = (k(n - 1) - K(n - 1)) \hat{B}_{k+K}^{n+N-2} \quad (2) \]

and involution
The central extensions of the $w_\infty$ algebra have been widely studied in the physical literature. In particular, Bakas proved in Ref. 3 that the $w_\infty^*$–Lie algebra does not admit non-trivial central extensions. That was done by showing that, after a suitable contraction which yields the $w_\infty$ commutation relations, the central terms appearing in the algebra $W_\infty$, which is defined as a $N \to \infty$ limit of the Zamolodchikov type Lie algebras $W_N$, vanish. A direct proof of the triviality of all central extensions of $w_\infty$ based on the cocycle definition of a central extension and avoiding the ambiguities that arise from passing to the (non-unique) limit of $W_N$, was given in Ref. 1.

The $*$–Lie sub–algebra of the $w_\infty$ algebra, generated by the family \{$\hat{B}_n^k : k \in \mathbb{Z}$\} is the Witt algebra which admits the Virasoro non trivial central extension

$$[\hat{B}_m^2, \hat{B}_n^2] = (m - n) \hat{B}_{m+n}^2 + \delta_{m+n,0} m (m^2 - 1) E$$

where traditionally $E = c/12$ where $c \in \mathbb{C}$ is the “central charge”.

In this paper we examine whether certain infinite dimensional closed subalgebras of $w_\infty$, which are natural generalizations of the Witt algebra, can also be non-trivially centrally extended.

2. Closed subalgebras of $w_\infty$

In this section we investigate the structure of the Lie sub–algebras of $w_\infty$. More precisely, we investigate which subsets of the generators of $w_\infty$ are such that the Lie algebra (resp. $*$–Lie algebra) generated by them, is a proper sub–algebra of $w_\infty$. To this goal notice that, if $\hat{S}$ is any subset of the generators of $w_\infty$, then there exists a unique partition \{$\hat{S}_+, \hat{S}_0, \hat{S}_-$\} of $\hat{S}$ defined by

$$\hat{S}_+ := \{ \hat{B}_n^k \in \hat{S} : k > 0 \}$$
$$\hat{S}_0 := \{ \hat{B}_n^k \in \hat{S} : k = 0 \}$$
$$\hat{S}_- := \{ \hat{B}_n^k \in \hat{S} : k < 0 \}$$
From (3) we know that a generator $\hat{B}_k^n$ is self-adjoint if and only if $k = 0$. Therefore $S_0$ is a self-adjoint set. Moreover the set $\hat{S}$ generates a $*$-sub-algebra if and only if $(\hat{S}_+)^* = \hat{S}_-$. Denote $\hat{L}(\hat{S})$ the Lie sub-algebra of $w_\infty$ generated by $\hat{S}$. From (2), we see that the sets $\hat{S}_+, \hat{S}_-$ generate Lie sub-algebras $\hat{L}(\hat{S}_+), \hat{L}(\hat{S}_-)$ of $\hat{L}(\hat{S})$, while $S_0$ generates a Lie $*$-sub-algebra $L(S_0)$. Denote $\mathbb{N}_{\geq 2} := \{n \in \mathbb{N} : n \geq 2\}$. The map $B_k^n \mapsto (n,k) \in \mathbb{N}_{\geq 2} \times \mathbb{Z}$ defines a one-to-one correspondence between the set of generators (1) and the set $\mathbb{N}_{\geq 2} \times \mathbb{Z}$. Therefore the sub-set of generators $\hat{S}$ will be in one-to-one correspondence with a subset $S \subseteq \mathbb{N}_{\geq 2} \times \mathbb{Z}$. The images of the subsets $S_\varepsilon$ where $\varepsilon \in \{+,0,-\}$ under this correspondence will be denoted by $S_\varepsilon$.

We want to study the following problem: which subset of $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ corresponds to those generators (1) which belong to $\hat{L}(\hat{S})$ (resp. $\hat{L}(\hat{S}_\varepsilon)$, $\varepsilon \in \{+,0,-\}$)? This sub-set will be denoted by $L(S)$ (resp. $L(S_\varepsilon)$, $\varepsilon \in \{0,-\}$).

The answer to this question, for a generic $\hat{S}$, is very difficult therefore we begin to analyze a simpler problem, namely: Can we construct interesting families of subsets $\hat{S} \subseteq \mathbb{N}_{\geq 2} \times \mathbb{Z}$ with the property that the linear span of such a subset is a proper Lie $*$-sub-algebra of $w_\infty$? Notice that, if $B_k^n, B_{k+K}^n \in \hat{S}$, then from (2) one sees that, if $k(N-1) - K(n-1) \neq 0$ then the generator $B_{k+K}^{n+N-2} \in \hat{L}(\hat{S})$.

Moreover, the set $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ is an associative semi-group under the composition law

$$(n,k) \cdot (N,K) := (n + N - 2, k + K) \quad (5)$$

In fact, it is the product of the semi-group $\mathbb{N}_{\geq 2}$ with composition law

$$n \cdot N := n + N - 2 \quad (6)$$

and the (semi-) group $\mathbb{Z}$ with the usual addition. Thus the set $L(S)$ will be contained in the sub-semi-group of $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ generated by $S$. Conversely, if $S_0$ is any sub-semi-group of $\mathbb{N}_{\geq 2} \times \mathbb{Z}$, then the linear span of $S_0 := \{B_k^n : (n,k) \in S_0\}$ is a Lie–sub-algebra of $w_\infty$ and it is a Lie $*$-sub-algebra if and only if $S_0$ is a self-adjoint subset under the involution

$$(n,k) \in \mathbb{N}_{\geq 2} \times \mathbb{Z} \mapsto (n,-k) \in \mathbb{N}_{\geq 2} \times \mathbb{Z}$$

For this reason, it is interesting to study the sub-semi-groups of $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ under the composition law (Ref. 5). An interesting class of these semigroups are those of the form
\[ S = S_1 \times S_2 \quad (7) \]

where \( S_1 \) is a sub–semi–group of \( \mathbb{N}_{\geq 2} \) with composition law (6) and \( S_2 \) a sub–semi–group of \( \mathbb{Z} \). The composition law (6) has an identity, given by the number 2, which is in \( \mathbb{N}_{\geq 2} \). Hence \( \{2\} \times \mathbb{Z} \) is a self–adjoint sub–semi–group of \( \mathbb{N}_{\geq 2} \times \mathbb{Z} \). Therefore the linear span of the set \( \{B_k^2 : k \in \mathbb{Z}\} \) is a Lie *–sub–algebra of \( w_\infty \) which is precisely the Witt (or centerless Virasoro) algebra.

Notice that \( \{2\} \) is the only finite sub–semi–group of \( \mathbb{N}_{\geq 2} \). In fact if \( S \) is such a semigroup and \( n \in S \), then \( \forall \nu \in \mathbb{N} \cup \{0\} \)

\[ n + \ldots + n \quad (\nu \text{ times}) = \nu n - 2(\nu - 1) = \nu(n - 2) + 2 \in S \quad (8) \]

and, for varying \( \nu \), this is a finite set if and only if \( n = 2 \). Notice also that the sub–semi–group of \( \mathbb{N}_{\geq 2} \) generated by the single element \( n \in \mathbb{N}_{\geq 2} \) is the set of elements of the form (8) for \( \nu \in \mathbb{N} \cup \{0\} \). Denoting by \( S_n \) this semi–group, one has that \( S_n \times \mathbb{Z} \) is a self–adjoint sub–semi–group of \( \mathbb{N}_{\geq 2} \times \mathbb{Z} \). Therefore \( \forall n \in \mathbb{N}_{\geq 2} \) the linear span of the set

\[ \{B_k^{\nu(n-2)+2} : \nu \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}\} \]

is a closed Lie *–sub–algebra of \( w_\infty \). Letting \( \nu = n - 2 \geq 0 \) and (for fixed \( \nu \))

\[ W^n_k := B_k^{n+N+2} \]

we arrive at the following definition.

**Definition 2.1.** For any natural integer \( N \geq 0 \) we denote \( w_N \) the *–Lie subalgebra of \( w_\infty \) defined by

\[ w_N := \text{span} \{W^n_k : n \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}\} \]

with Lie brackets (inherited from \( w_\infty \))

\[ [W^n_k, W^l_m] = ((k l - m n) N + (k - m)) W^{n+l}_{k+m} \quad (9) \]

For \( N = 0 \), \( w_0 \) is the Witt algebra.
The question of the existence of non–trivial central extensions of \( w_\mathbb{N} \) is the subject of this paper.

Notice that \( w_\mathbb{N} \) is a direct generalization of the Witt algebra \( w_0 \). Furthermore, notice that the Witt algebra is the vector space generated by the generators of the form \( \hat{B}_k^{\varphi(k)} : k \in \mathbb{Z} \) where \( \varphi \) is the constant function \( \varphi(k) = 2 \), \( \forall k \in \mathbb{Z} \).

One may wonder if there exist other functions \( \varphi : \mathbb{Z} \rightarrow \mathbb{N} \geq 2 \) with this property. The following Lemma shows that this is not the case.

**Lemma 2.1.** Let \( \varphi : \mathbb{Z} \rightarrow \mathbb{N} \geq 2 \) be a function such that the linear span of \( \{ \hat{B}_k^{\varphi(k)} : k \in \mathbb{Z} \} \) is a \( * \)-Lie algebra. Then \( \varphi \) is the constant function \( \varphi(k) = 2 \), \( \forall k \in \mathbb{Z} \).

**Proof.** The condition \( (\hat{B}_k^{\varphi(k)})^* = \hat{B}_k^{\varphi(-k)} \) for all \( k \in \mathbb{N} \) implies that \( \varphi(k) = \varphi(-k) \). This, together with the condition

\[
[\hat{B}_k^{\varphi(k)}, \hat{B}_{-k}^{\varphi(-k)}] = 2k(\varphi(k) - 1)B_0^{\varphi(k)+\varphi(-k)}; \quad \forall k \in \mathbb{Z}
\]

gives that, \( \forall k \in \mathbb{Z} \)

\[
\varphi(0) = \varphi(k) + \varphi(k) = 2\varphi(k) - 2 \Leftrightarrow 2\varphi(k) = \varphi(0) + 2 \Leftrightarrow \varphi(k) = \frac{1}{2} \varphi(0) + 1
\]

But then the condition

\[
[\hat{B}_0^{\varphi(0)}, \hat{B}_k^{\varphi(k)}] = -k(\varphi(0) - 1)\hat{B}_k^{\varphi(k)+\varphi(0)}
\]

gives that

\[
\varphi(k) + \varphi(0) = \varphi(k) \Leftrightarrow \varphi(0) = 2
\]

Therefore \( \forall k \in \mathbb{Z} \), \( \varphi(k) = \frac{1}{2} \varphi(0) + 1 = 2 \).

A class of examples not of product type, i.e. defined by semi–groups not of the form (7), might be built as follows. Suppose that \( [\hat{B}_k^n, \hat{B}_{k'}^{n'}] = 0 \), \( [\hat{B}_k^n, \hat{B}_{k'}^m] \neq 0 \), and \( [\hat{B}_{k'}^{n'}, \hat{B}_{k''}^m] \neq 0 \). Then the \( * \)-algebra generated by \( \{ \hat{B}_k^n, \hat{B}_{k'}^{n'}, \hat{B}_{k''}^m \} \) should not be of product type.
3. Abelian sub–algebras of $\omega_{\infty}$

Lemma 3.1. Any subset of the set

$$A_0 := \{ \hat{B}_n^0 : n \in \mathbb{N}_{\geq 2} \} \quad (10)$$

consists of commuting self–adjoint generators. The set (10) is a maximal set with this property and generates a maximal Abelian $*$–sub–algebra of $\omega_{\infty}$.

**Proof.** The commutativity of the set (10) is clear from (2). The same identity shows that if $X \in \omega_{\infty}$, then $\forall n \in \mathbb{N}_{\geq 2}$, $[\hat{B}_n^0, X]$ is a linear combination of the (linearly independent) generators of the form $\hat{B}_k^0$ with $k \neq 0$. Therefore either $X \in A_0$ or $X$ cannot commute with $A_0$. This proves maximality. That $A_0$ is a $*$–sub–algebra follows from the fact that the generators are self–adjoint. \hfill \square

Lemma 3.2. If a subset $\hat{S}$ of generators of the form (1) contains an element of the form $\hat{B}_n^0$, then $\hat{S}$ can be a commutative subset if and only if

$$\hat{B}_k^m \in \hat{S} \Rightarrow k = 0 \quad (11)$$

**Proof.** From Lemma 3.1 we know that (11) is a sufficient condition for commutativity of $\hat{S}$. Let us prove that, under the conditions of the Lemma, it is also necessary. Suppose that $\hat{B}_k^m \in \hat{S}$ and that $k \neq 0$. Then (2) implies that $0 = [\hat{B}_0^m, \hat{B}_k^N] = k(m-1)\hat{B}_k^{n+m-2}$. Since by assumption $m, n \geq 2$ and $\hat{B}_k^{n+m-2} \neq 0$, it follows that $k = 0$, against the assumption. \hfill \square

Lemma 3.3. Two generators $\hat{B}_k^0$, $\hat{B}_K^N$ with $k, K \neq 0$, commute if and only if $\text{sgn} (k) = \text{sgn} (K) =: \pm$ and there exist $p, q \in \mathbb{N} \cup \{ 0 \}$ mutually prime, such that, for some $k', K' \geq 1$: $(n, k) = (1 + qk', \pm pk')$ and $(n, K) = (1 + qK', \pm pK')$.

**Proof.** We have that

$$0 = [\hat{B}_k^m, \hat{B}_K^N] = (k(N-1) - K(n-1))\hat{B}_k^{n+N-2}$$

Since $\hat{B}_k^{n+N-2} \neq 0$, this is equivalent to $k(N-1) - K(n-1) = 0$. Since $N, n \geq 2$, this is possible if and only if $k$ and $K$ have the same sign. In this case the condition is equivalent to
where $p$ and $q$ are mutually prime natural integers and the $\pm$ sign is the common sign of $k$ and $K$. This means that $k = \pm pk'$, $n - 1 = qk'$ and $K = \pm pK'$, $N - 1 = qK'$ where the sign $\pm$ is the same in both cases and $k', K' \geq 1$. This is equivalent to the statement in the Lemma.

**Definition 3.1.** A half–line in $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ is a subset either of the form

$$H_{\varepsilon, p, q} := \{(1 + qk, \varepsilon pk) : k \in \mathbb{N} \cup \{0\}\}$$

where $\varepsilon \in \{\pm 1\}$ and $q, p \in \mathbb{N} \cup \{0\}$ are mutually prime, or of the form

$$H_{1,0,q} := \{(1 + qk,0) : k \in \mathbb{N} \cup \{0\}\}$$

**Theorem 3.1.** Each of the three sets of indices $H_{1,0,1} = \{(1 + k,0) : k \in \mathbb{N} \cup \{0\}\}$, $H_{+,1,1} = \{(1 + k,k) : k \in \mathbb{N} \cup \{0\}\}$ and $H_{-,1,1} = \{(1 + k,-k) : k \in \mathbb{N} \cup \{0\}\}$ defines a maximal family of mutually commuting generators.

**Proof.** We know from Lemma 3.1 that $H_{1,0,1}$ is a mutually commuting family. The same is true for $H_{+,1,1}$ and $H_{-,1,1}$ because of Lemma 3.3. Now let $\hat{S}$ be a mutually commuting family of generators (1). If $\hat{S}$ contains a generator of the form $\hat{B}_n^0$, for some $n \in \mathbb{N}_{\geq 2}$, from Lemma 3.2 we know that $\hat{S} \subseteq H_{1,0,1}$. If this is not the case, then from Lemma 3.3 we know that $\hat{S}$ is contained in some half–line $H_{\varepsilon, p, q}$ in $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ with $p \neq 0$. But all half–lines of this type, with $\varepsilon = +1$ (resp. $\varepsilon = -1$), are contained in $H_{+,1,1}$ (resp. $H_{-,1,1}$) and this implies the statement.

Notice that, of the three families listed in Theorem 3.1, only $H_{1,0,1}$ generates a $*$–sub–algebra.

### 4. Basic facts on central extensions of Lie algebras

If $L$ and $\tilde{L}$ are two complex Lie algebras, we say that $\tilde{L}$ is a one-dimensional central extension of $L$ with central element $E$ if there is a Lie algebra exact sequence $0 \rightarrow CE \rightarrow \tilde{L} \rightarrow L \rightarrow 0$ where $CE$ is the one-dimensional trivial Lie algebra and the image of $CE$ is contained in the center $\text{Cent}(L)$ of $\tilde{L}$ i.e.,
where \([\cdot, \cdot]_\tilde{L}\) are the Lie brackets in \(\tilde{L}\). For *-Lie algebras we also require that the central element \(E\) is self-adjoint, i.e.

\[(E)^* = E\]  \hfill (12)

A 2-cocycle on \(L\) is a bilinear form \(\phi : L \times L \mapsto \mathbb{C}\) on \(L\) satisfying, for all \(l_1, l_2 \in L\), the skew-symmetry condition

\[\phi(l_1, l_2) = -\phi(l_2, l_1)\]

(in particular \(\phi(l, l) = 0\) for all \(l \in L\)) and the 2-cocycle identity:

\[\phi([l_1, l_2]_L, l_3) + \phi([l_2, l_3]_L, l_1) + \phi([l_3, l_1]_L, l_2) = 0\]  \hfill (13)

One-dimensional central extensions of \(L\) are classified by 2-cocycles in the sense that \(\tilde{L}\) is a central extension of \(L\) if and only if, as vector space, it is the direct sum

\[\tilde{L} = M \oplus \mathbb{C}E\]

where \(M\) is a Lie algebra isomorphic to \(L\), and there exists a 2-cocycle on \(L\) such that, for all \(l_1, l_2 \in L\), the Lie brackets in \(\tilde{L}\) are given by

\[[l_1, l_2]_{\tilde{L}} = [l_1, l_2]_L + \phi(l_1, l_2) E\]  \hfill (14)

where, in the right hand sides of (14), \(L\) is identified to \(L \oplus \{0\} \subseteq L \oplus \mathbb{C}E\), and \(\phi : L \times L \mapsto \mathbb{C}\) is a 2-cocycle on \(L\),

\[[l_1, l_2]_{\Lambda} = [l_1, l_2]_L + \phi(l_1, l_2) E\]

where \([\cdot, \cdot]_L\) are the Lie brackets in \(L\). A central extension is trivial if the corresponding 2-cocycle \(\phi\) is uniquely determined by a linear function \(f : L \mapsto \mathbb{C}\) through the identity

\[\phi(l_1, l_2) = f([l_1, l_2]_L), \quad \forall l_1, l_2 \in L\]  \hfill (15)
Such a 2-cocycle is called a 2-coboundary, or a trivial 2-cocycle. Two extensions are called equivalent if each of them is a trivial extension of the other. This is the case if and only if the difference of the corresponding 2-cocycles is a trivial cocycle. A central extension $\tilde{L}$ of $L$ is called universal whenever there exists a homomorphism from $\tilde{L}$ to any other central extension of $L$. A Lie algebra $L$ possesses a universal central extension if and only if $L$ is perfect (i.e. $L = [L, L]$). In this case, the universal central extension of $L$ is unique up to isomorphism.

Notice that the 2-cocycle identity (13) implies that, if $l_c \in \text{Cent}(L)$ is an element of the center of $L$, then

$$\phi([l_1, l_2], l_c) = 0 \quad ; \quad \forall l_1, l_2 \in L$$

i.e. $l_c$ is $\phi$–orthogonal to the derived set $[L, L]$ of $L$. Similarly (15) implies that a necessary condition for the 2-cocycle $\phi$ to be trivial is that the center of $L$ is $\phi$–orthogonal to the whole algebra $L$. Because of (14) this is equivalent to say that the center of $L$ is mapped into the center of $\tilde{L}$. Therefore a sufficient condition for a 2-cocycle $\phi$ on $L$ to be non trivial is that there exist $l_c \in \text{Cent}(L)$ and $x \in L \setminus [L, L]$ such that

$$\phi(x, l_c) \neq 0$$

This practical rule is useful for Lie algebras $L$ with a large derivative $[L, L]$.

5. Central extensions of $w_N$

Throughout this section we assume that $w_N$ is a central extension of $w_N$, where $N > 0$ is fixed. For $N = 0$, the Witt algebra $w_0$ admits the well-known non-trivial Virasoro central extension

$$[W^0_{k,l}, W^0_{m,n}] = (k - m) W^0_{k+m} + \delta_{k+m,0} m (m^2 - 1) E$$

We denote by $c(n, k; l, m)$ the value assumed by the corresponding 2–cocycle on the pair of generators $(W^n_k, W^l_m)$, i.e.:

$$c(n, k; l, m) := \phi(W^n_k, W^l_m) \in \mathbb{C}$$

(16)

$$[W^n_k, W^l_m] = ((k l - mn) N + (k - m)) W^{n+l}_{k+m} + c(n, k; l, m) E$$
The skew-symmetry of $\phi$ and the adjointness condition (12) imply respectively that:

\[ c(n, k; l, m) = -c(l, m; n, k) \]  \hspace{1cm} (17)

\[ c(n, k; l, m) = -c(n, -k; l, -m) \]  \hspace{1cm} (18)

If at least one of $n, l$ is negative we set

\[ c(n, k; l, m) = 0 \]  \hspace{1cm} (19)

**Lemma 5.1.**  The derived set of the $w_N$ *–Lie algebra is itself.

**Proof.** From (9) we see that the derived set of the $w_N$ *–Lie algebra is

\[ \text{Der}(W_N) := \{ W_{n+l}^{m+k+m} : (kl - mN + (k - m) \neq 0, n, l \in \mathbb{N} \cup \{0\}, k, m \in \mathbb{Z} \} \]

Choosing $(n, k) = (0, 0)$ we see that $\text{Der}(W_N)$ contains the generators of the form $W_m^l$ with $l \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$. Choosing $n = 0$ and $(k, m) = (1, -1)$ we see that $\text{Der}(W_N)$ also contains the generators of the form $W_0^l$ such that $lN + 2 \neq 0$ which is always true for all $l \in \mathbb{N} \cup \{0\}$. \hfill $\Box$

Combining the remark after equation (15), with Lemma 5.1 one deduces that, in any central extension of $W_N$, the central element is mapped to the central element of the extension so that, for any $l \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{Z}$

\[ c(0, 0; l, m) = 0 \]  \hspace{1cm} (20)

**Lemma 5.2.** On the $w_N$ generators $W_m^l$, for the family $\{c(n, k; l, m)\}$ defined by (16), the 2–cocycle identity (13) is equivalent to

\[ ((k_1 n_2 - k_2 n_1) N + (k_1 - k_2)) c(n_1 + n_2, k_1 + k_2; n_3, k_3) \]
\[ +((k_2 n_3 - k_3 n_2) N + (k_2 - k_3)) c(n_2 + n_3, k_2 + k_3; n_1, k_1) \]
\[ +((k_3 n_1 - k_1 n_3) N + (k_3 - k_1)) c(n_3 + n_1, k_3 + k_1; n_2, k_2) = 0 \]  \hspace{1cm} (21)

Conversely any family $\{c(n, k; l, m)\}$ satisfying (21) defines, through (16), a 2–cocycle on $w_N$. 
Proof. For all $n_i, k_i$, where $i = 1, 2, 3$, making use of (17) we have

$$0 = \phi([W^{n_1}_{k_1}, W^{n_2}_{k_2}], W^{n_3}_{k_3}) + \phi([W^{n_2}_{k_2}, W^{n_3}_{k_3}], W^{n_1}_{k_1}) + \phi([W^{n_3}_{k_3}, W^{n_1}_{k_1}], W^{n_2}_{k_2})$$

$$= ((k_1 n_2 - k_2 n_1) N + (k_1 - k_3)) \phi(W^{n_1+n_2}_{k_1+k_2}, W^{n_3}_{k_3})$$

$$+ ((k_2 n_3 - k_3 n_2) N + (k_2 - k_3)) \phi(W^{n_2+n_3}_{k_2+k_3}, W^{n_1}_{k_1})$$

$$+ ((k_3 n_1 - k_1 n_3) N + (k_3 - k_1)) \phi(W^{n_3+n_1}_{k_3+k_1}, W^{n_2}_{k_2})$$

$$= ((k_1 n_2 - k_2 n_1) N + (k_1 - k_3)) c(n_1 + n_2, k_1 + k_2, n_3, k_3)$$

$$+ ((k_2 n_3 - k_3 n_2) N + (k_2 - k_3)) c(n_2 + n_3, k_2 + k_3, n_1, k_1)$$

$$+ ((k_3 n_1 - k_1 n_3) N + (k_3 - k_1)) c(n_3 + n_1, k_3 + k_1, n_2, k_2)$$

The converse is clear due to the linear independence of the generators. 

We notice that the sum of the first and third (resp. second and fourth) arguments in the three 2-cocycle values $c(n_2 + n_3, k_2 + k_3; n_1, k_1)$, $c(n_1 + n_2, k_1 + k_2; n_3, k_3)$ and $c(n_3 + n_1, k_3 + k_1; n_2, k_2)$ appearing in (21) is equal to $n_1 + n_2 + n_3$ (resp. $k_1 + k_2 + k_3$). We are thus led to the following definition.

Definition 5.1. Given natural integers $n_1, n_2, n_3 \geq 0$ and $k_1, k_2, k_3 \in \mathbb{Z}$, define $S \in \mathbb{N} \cup \{0\}$ and $M \in \mathbb{Z}$ by:

$$S := n_1 + n_2 + n_3 \quad ; \quad M := k_1 + k_2 + k_3$$

and

$$\psi_{S,M}(n_i, k_i) := c(S - n_i, M - k_i; n_i, k_i) ; \quad i \in \{1, 2, 3\}$$

(22)
Corollary 5.1. The skew-symmetry condition (17) becomes

\[ \psi_{S,M}(n_i, k_i) = -\psi_{S,M}(S - n_i, M - k_i) \]

and (21) is equivalent to

\[
((k_1 n_2 - k_2 n_1) N + (k_1 - k_2)) c(S - n_3, M - k_3; n_3, k_3) \\
+ ((k_2 n_3 - k_3 n_2) N + (k_2 - k_3)) c(S - n_1, M - k_1; n_1, k_1) \\
+ ((k_3 n_1 - k_1 n_3) N + (k_3 - k_1)) c(S - n_2, M - k_2; n_2, k_2) = 0
\]

or in \(\psi\)-form

\[
((k_1 n_2 - k_2 n_1) N + (k_1 - k_2)) \psi_{S,M}(n_3, k_3) \\
+ ((k_2 n_3 - k_3 n_2) N + (k_2 - k_3)) \psi_{S,M}(n_1, k_1) \\
+ ((k_3 n_1 - k_1 n_3) N + (k_3 - k_1)) \psi_{S,M}(n_2, k_2) = 0
\]

Proof. The proof follows directly from Definition 5.1.

Proposition 5.1. For any \(\lambda \in \mathbb{R}\) the family \(\{c(n, k; l, m)\}\), defined by

\[
c(n, k; l, m) := \delta_{k+m,0} \lambda k
\]

defines, through (16), a 2–cocycle on \(w_N\).

Proof. Condition (17) is verified by inspection and (18) follows from the fact that \(\lambda\) is real. We want to prove that (24) this is satisfied by the family \(\{c(n, k; l, m)\}\), defined by (25). Direct substitution shows that, if the family \(\{c(n, k; l, m)\}\) is defined by (25), then \(\psi_{S,M}\), defined by (22), satisfies (24). Moreover, \(\psi_{S,M}(n_i, k_i) = \delta_{M,0} \lambda k_i\) implies that \(c(S - n_i, M - k_i; n_i, k_i) = \delta_{M,0} \lambda k_i\). For \(i = 1\) we get \(c(S - n_1, M - k_1; n_1, k_1) = \delta_{M,0} \lambda k_1\) which for \(n_3 = 0\) becomes \(c(n_2, k_2 + k_3; n_1, k_1) = \delta_{M,0} \lambda k_1\). Letting \(k_2 + k_3 := K\) we have that

\[ c(n_2, K; n_1, k_1) = \delta_{k_1+K,0} \lambda k_1 \]

i.e. \(c(n, k; l, m) = \delta_{k+m,0} \lambda k\).

\[
\square
\]

Proposition 5.2. The central extension
\[ [W^n_k, W^l_m] = ((k - m) N + (k - m)) W^{n+l}_{k+m} + \delta_{k+m,0} \lambda k E \]

of \( w_N \) is trivial.

**Proof.** We look for a linear complex-valued function \( f \) defined on \( w_N \) such that

\[ f ([W^n_k, W^l_m]) = \delta_{k+m,0} \lambda k \]

(26)

By the \( w_N \) commutation relations (9) and the linearity of \( f \), equation (Ref. 26) is equivalent to

\[ ((k - m) N + (k - m)) f (W^{n+l}_{k+m}) = \delta_{k+m,0} \lambda k \]

(27)

For \( k + m \neq 0 \) this is equivalent to

\[ f (W^{n+l}_{2}) = 0 ; \forall x \in \mathbb{Z} \setminus \{0\} \]

(28)

For \( k + m = 0 \Leftrightarrow m = -k \) (27) is equivalent to

\[ ((k + n) N + 2k) f (W^{n+l}_{0}) = k \lambda \Leftrightarrow ((l + n) N + 2) f (W^{n+l}_{0}) = \lambda \]

\[ \Leftrightarrow f (W^{n+l}_{0}) = \frac{\lambda}{(l + n) N + 2} \]

and this, together with (28) uniquely defines a linear functional \( f \) with the required property. Therefore the central extension of \( w_N \) is trivial. \( \square \)

**Lemma 5.3.** Let \( z \in \mathbb{C} \). If \( z = 2 \bar{z} \) then \( z = 0 \).

**Proof.** If \( z = x + iy, x, y \in \mathbb{R} \), then \( z = 2 \bar{z} \) implies that \( x = 2x \) and \( y = -2y \). Therefore \( x = y = 0 \). \( \square \)

**Lemma 5.4.** In the notation of Definition 5.1, let \( S \in \mathbb{N} \cup \{0\}, M = 0 \) and \( N > 0 \). Then:

(i) \( \psi_{S,0}(0,1) = c(S, -1; 0, 1) = 0 \)

(ii) For all \( k \in \mathbb{Z} \), \( \psi_{S,0}(0, -k) = c(S, k; 0, -k) = 0 \)

(iii) For all \( n \geq 0 \) and \( k \in \mathbb{Z} \), \( c(S - n, k; n, -k) = 0 \)

Notice that (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i).
Proof. (i) For $n = S - n_1, n_3 = 0, k_1 = 0, k_2 = -1$ and $k_3 = 1$, (21) yields

\[
(n_1 N + 1) c(S, -1; 0, 1) = (S - n_1) N + 2 \]

which for $n_1 = S$, since by (19) $c(-S, -1; 2 S, 1) = 0$, after dividing out $(S N + 1)$, yields with the use of (17) and (18)

\[
c(S, -1; 0, 1) = 2 c(0, -1; S, 1) = -2 c(S, 1; 0, -1) = 2 c(S, -1; 0, 1)
\]

which, by Lemma 5.3, implies that $c(S, -1; 0, 1) = 0$.

(ii) For $n = S, n_3 = 0, n_2 = 0, k_1 = k, k_2 = 1, k_3 = -(k + 1)$, letting $a_k := c(S, k; 0, -k)$, since by (i) $a_{-1} = 0$, (21) yields

\[
(k - S N - 1) a_{k+1} = (k + 2) a_k
\]

which implies that $a_k = 0$ for all $k$.

(iii) For $k_1 = k, k_2 = -k, k_3 = 0, n_1 = S - n, n_2 = 0$ and $n_3 = n$, after dividing out $k \neq 0$ and using $c(S, k; 0, -k) = 0$, (21) yields

\[
c(S - n, k; n, -k) = -\frac{(S - n) N + 2}{n N + 1} c(S - n, 0; n, 0)
\]

for all $k \neq 0$. Similarly, for $k_1 = -k \neq 0, k_2 = k, k_3 = 0, n_1 = 0, n_2 = n$ and $n_3 = S - n$, (21) yields
\[ c(S - n, k; n, -k) = -\frac{nN + 2}{(S-n)N+1} c(S - n, 0; n, 0) \]

for all \( k \neq 0 \). Thus

\[ \frac{(S - n)N + 2}{nN + 1} c(S - n, 0; n, 0) = \frac{nN + 2}{(S-n)N+1} c(S - n, 0; n, 0) \]  \hspace{1cm} (31)

If \( S = 2n \) then \( c(S - n, 0; n, 0) = c(n, 0; n, 0) = 0 \) by (17). If \( S \neq 2n \) then \( c(S - n, 0; n, 0) = 0 \) by (31).

**Proposition 5.3.** Let \( S \in \mathbb{N} \cup \{0\} \) and \( M \in \mathbb{Z} \). In the notation of Definition 5.1, all non-trivial 2–cocycles \( \psi_{S,M}(n, k) \) on \( w_N \) are given by

\[ \psi_{S,M}(n, k) = \delta_{S,0} \delta_{M,0} k (k^2 - 1) \]

**Proof.** Case (i): \( S = 0 \). Then \( n_1 + n_2 + n_3 = 0 \) and so \( n_1 = n_2 = n_3 = 0 \) which means that we are reduced to the standard Witt-Virasoro case \( W_0^k = \hat{B}_2^k \). Therefore, the only non-trivial cocycle is

\[ \psi_{S,M}(n, k) = \psi_{0,M}(n, k) = \delta_{M,0} k (k^2 - 1) \]

Case (ii): \( S \neq 0 \) and \( M \neq 0 \). For \( n_3 = k_3 = 0 \), using \( c(n_2, k_2; n_1, k_1) = -c(n_1, k_1; n_2, k_2) \), \( n_1 + n_2 = S \) and \( k_1 + k_2 = M \), (21) yields

\[ (k_1 (n_2 N + 1) - k_2 (n_1 N + 1)) c(S, M; 0, 0) - (k_2 + k_1) c(n_1, k_1; n_2, k_2) = 0 \]

which, letting \( n_2 = n \), \( k_2 = k \), \( n_1 = S - n \) and \( k_1 = M - k \), implies that

\[ \psi_{S,M}(n, k) = c(S - n, M - k; n, k) \]

\[ = ( (M - k) (n N + 1) - k ((S - n) N + 1) ) c(S, M; 0, 0) = 0 \]

by (20).

Case (iii): \( S \neq 0 \) and \( M = 0 \). For \( k_3 = 0, n_1 = 0, k_1 \neq 0 \), using Lemma 5.4 (ii) and (iii), (24) yields
and the result follows by the arbitrariness of $n_2$ and $k_2$.

The next corollary shows that there are no non-trivial central extensions of $w_N$ other than the Virasoro one.

**Corollary 5.2.** The non-trivial central extensions of $w_N$ are given by

$$[W^m_n, W^l_l] = (n + m - 2) W_{k+m} + \delta_{n,0} \delta_{l,0} \delta_{k+m,0} m (m^2 - 1) E$$

Thus only the Virasoro sector of $w_N$ can be extended in a non-trivial way.

**Proof.** By Proposition 5.3, in the notation of Definition 5.1,

$$\psi_{S,M}(n_1, k_1) = c(S - n_1, M - k_1; n_1, k_1) = \delta_{S,0} \delta_{M,0} k_1 (k_1^2 - 1)$$

i.e.,

$$c(n_2 + n_3, k_2 + k_3; n_1, k_1) = \delta_{n_1 + n_2 + n_3,0} \delta_{k_1 + k_2 + k_3,0} k_1 (k_1^2 - 1)$$

which, letting $n_3 = k_3 = 0$, $n_1 = n$, $k_1 = k$, $n_2 = l$ and $k_2 = m$ implies that

$$c(n, k; l, m) = \delta_{n+l,0} \delta_{k+m,0} m (m^2 - 1) = \delta_{n,0} \delta_{l,0} \delta_{k+m,0} m (m^2 - 1)$$

References