MARKOVIAN KMS-STATES FOR ONE-DIMENSIONAL SPIN CHAINS

LUIGI ACCARDI
Centro Interdipartimentale V. Volterra,
Università di Tor Vergata Roma II, I-00133 Rome, Italy

VOLKMAR LIEBSCHER
GSF-National Research Centre for Environment and Health,
Institute for Biomathematics and Biometry,
Ingolstädter Landstr. 1, D-85764 Neuherberg, Germany

Received 11 November 1998
Revised 28 September 1999

We characterize a class of quantum Markov states in terms of a locality property of their modular automorphism group or, equivalently, of their $\varphi$-conditional expectations and we give an explicit description of the structure of these states. This study is meant as a starting point for the investigation of the structure of Markovian KMS-states of quantum spin chains as well as of multidimensional quantum spin lattices.

1. Introduction

Quantum Markov chains were introduced in Refs. 1 and 6 and since then several progresses have been made in their applications to physical models. In particular, starting from Ref. 20 a subclass of quantum Markov chains, also called finitely correlated states, was shown to coincide with the so-called valence bond states introduced in the late '80s as an affirmative example of the Hadane conjecture on antiferromagnetic Heisenberg models with integer spin. More recently the same class of states was shown to coincide with the class of the so-called spin ladder models which possess the split property.19 In another direction, the quantum Markov chains are currently used as trial states in Hartree–Fock approximations of solid state models, for example the Heisenberg model or the fixed points of the density matrix renormalization group (DMRG).17,23

Several progresses have also been made in the problem of clarifying the mathematical structure of quantum Markov chains.5,10,11 In particular, Park was able to compute explicitly their entropy,22 Fannes, Nachtergaele and Werner clarified

*E-mail: liebsche@gsf.de

645
the ergodic structure of an important subclass of these states and gave necessary and sufficient conditions for a quantum Markov state to be pure\textsuperscript{14–16} and Matsui\textsuperscript{18} characterized them as zero energy states of nearest neighbor Hamiltonians on one-dimensional spin lattices.

This paper goes in some sense in the complementary direction with respect to the above-mentioned Matsui’s result, i.e. we look for characterizations of quantum Markov chains as equilibrium states, defined in terms of the Kubo–Martin–Schwinger (KMS) condition. More precisely, following Ref. 4, we characterize a class of quantum Markov states in terms of a locality property of their modular automorphism group or, equivalently of their \( \varphi \)-conditional expectations. Starting from this property we are able to give a full structure theorem for the corresponding class of Markovian states.

We restrict our discussion to one-dimensional lattices, but our technique extends, with minor modifications, to multidimensional quantum spin lattices as well as to continuous time Markov processes. Thus the present results can therefore be considered as a starting point for a definition and a structure theory of quantum Markov fields.

2. Basic Notions and Notations

Throughout this paper \( \mathcal{A} \) will be the \( C^* \)-algebra \( \bigotimes_{n \in G} M_d \), obtained as infinite tensor product of the finite-dimensional matrix algebra \( M_d, d \in \mathbb{N}, d \geq 2 \), cf. Sec. 1.22 of Ref. 25. We consider only the cases \( G = \mathbb{N} \) or \( G = \mathbb{Z} \). Thus for all \( n \in G \) there are injective unital \(*\)-homomorphisms \( i_n : M_d \to \mathcal{A} \) such that \( i_n(b) \) and \( i_m(b') \) commute for different \( n, m \in G \). Denote for all \( \mathcal{F} \subseteq G \) by \( \mathcal{A}_\mathcal{F} = C^*(\{i_n(b) : n \in \mathcal{F}, b \in M_d\}) \) the \( C^* \)-subalgebra generated by \( \{i_n(b) : n \in \mathcal{F}, b \in M_d\} \). Further we denote \( n| := \{\ldots, n\} \cap G \) and \( [n, m| := \{n, \ldots, m\} \).

We follow Ref. 4 for the definition of Markov states on \( \mathcal{A} \):

**Definition 2.1.** Let there be given a triple \( (\mathcal{A}_i, \mathcal{A}_b, \mathcal{A}_o) \) (\( i = \text{inside}, o = \text{outside}, b = \text{boundary} \)) of commuting \( C^* \)-subalgebras of \( \mathcal{A} \) (a localization) and denote \( \mathcal{A}_{ib} = C^*(\mathcal{A}_i \cup \mathcal{A}_b) \) and similarly for \( \mathcal{A}_{bi}, \mathcal{A}_{ob} \). A quasi-conditional expectation with respect to the triple \( (\mathcal{A}_i, \mathcal{A}_b, \mathcal{A}_o) \) is a completely positive unit preserving map \( \mathcal{E} : \mathcal{A}_{ob} \to \mathcal{A}_{ob} \) such that \( \text{Fix} \mathcal{E} \supseteq \mathcal{A}_o \), i.e.

\[
\mathcal{E}(a_o) = a_o, \quad a_o \in \mathcal{A}_o.
\]  

An equivalent formulation gives the following:

**Lemma 2.1.** \( \mathcal{E} \) is a quasi-conditional expectation iff

\[
\mathcal{E}(a_o a_b a_i) = a_o \mathcal{E}(a_b a_i), \quad a_i \in \mathcal{A}_i, a_b \in \mathcal{A}_b, a_o \in \mathcal{A}_o.
\]  

**Proof.** In fact, by polarization the equation

\[
\mathcal{E}(a^* a) = \mathcal{E}(a^*) \mathcal{E}(a)
\]
implies
\[ \mathcal{E}(a^* y) = \mathcal{E}(a^*) \mathcal{E}(y), \quad (y \in \mathcal{A}) \]
and the fact that \( \mathcal{A}_o \) is a *-algebra completes the proof. \( \square \)

3. Markovian States on \( \bigotimes_{n \in \mathbb{N}} \mathcal{M}_d \)

First let \( G = \mathbb{N} \). We call a state \( \varphi \) on \( \mathcal{A} \) locally faithful if it is faithful on each \( \mathcal{A}_n \), \( n \in \mathbb{N} \). Then, for each \( n \in \mathbb{N} \) there exists the \( \varphi \)-conditional expectation \( \mathcal{E}_{n+1}^\varphi : \mathcal{A}_{n+1} \rightarrow \mathcal{A}_n \), defined as in Ref. 3. The following facts were proven in Ref. 4.

Theorem 3.1. For a state \( \varphi \) on \( \mathcal{A} \) which is locally faithful, the following statements are equivalent:

(a) For all \( n \in \mathbb{N} \), the \( \varphi \)-conditional expectation \( \mathcal{E}_{n+1}^\varphi := \mathcal{E}_{n+1}^\varphi, n \in \mathbb{N} \) leaves the algebra \( \mathcal{A}_{n-1} \) pointwise invariant.

(b) For each \( n \in \mathbb{N} \), there exists a completely positive unit preserving map \( \mathcal{E}_{n+1}^\varphi : \mathcal{A}_{n+1} \rightarrow \mathcal{A}_n \) such that:
\[ \varphi_n \circ \mathcal{E}_{n+1}^\varphi = \varphi_{n+1}, \]
where \( \varphi_n \) denotes the restriction of \( \varphi \) on \( \mathcal{A}_n \), and \( \mathcal{A}_{n-1} \subset \text{Fix} \mathcal{E}_{n+1}^\varphi \).

(c) For any \( n \in \mathbb{N} \)
\[ \sigma_{t}^{n+1} | \mathcal{A}_{n-1} = \sigma_{t}^{n} | \mathcal{A}_{n-1}, \]
where \( \sigma_{t}^{n+1}, \sigma_{t}^{n} \) denote the modular automorphism groups associated to the pairs \( (\mathcal{A}_{n+1}, \varphi_{n+1}) \) and \( (\mathcal{A}_n, \varphi_n) \), respectively.

Therefore, we can make the following definition which also suits the states which are not locally faithful.

Definition 3.1. A state \( \varphi \) on \( \mathcal{A} \) is called Markovian if for each \( n \in \mathbb{N} \) it is invariant under a map \( \mathcal{E}_{n+1}^\varphi : \mathcal{A}_{n+1} \rightarrow \mathcal{A}_n \) which is a quasi-conditional expectation with respect to the localization \( (\mathcal{A}_{n+1}, \mathcal{A}_n, \mathcal{A}_{n-1}) \).

If \( \varphi \) is a Markovian state on \( \mathcal{A} \) and \( n \in \mathbb{N} \), the limit of the ergodic averages of \( \mathcal{E}_{n+1}^\varphi \) always exists and is a completely positive unit preserving map \( \mathcal{E}_{n+1}^\varphi : \mathcal{A}_{n+1} \rightarrow \mathcal{A}_n \). Moreover, \( \mathcal{E}_{n+1}^\varphi \) projects onto
\[ \text{Fix} \mathcal{E}_{n+1}^\varphi = \text{Fix} \mathcal{E}_{n+1}^\varphi \supseteq \mathcal{A}_{n-1} \]
and
\[ \varphi_n \circ \mathcal{E}_{n+1}^\varphi = \varphi_{n+1}. \]
Thus \( \varphi \) is Markovian with respect to the collection \( (\mathcal{E}_{n+1}^\varphi)_{n \in \mathbb{N}} \) iff it is Markovian with respect to \( (\mathcal{E}_{n+1}^\varphi)_{n \in \mathbb{N}} \). If \( \varphi \) is locally faithful, we know\(^3\) that \( \text{Fix} \mathcal{E}_{n+1}^\varphi \) is a *-algebra and \( \mathcal{E}_{n+1}^\varphi \) is an Umegaki conditional expectation. For \( d = 2 \), this is
true for all such completely positive unit preserving maps, see Lemma 6.1. Suppose in addition that $\varphi$ is stationary, i.e.

$$\varphi \circ \theta = \varphi,$$

where $\theta$ is the shift on $A$. By translation invariance there is an Umegaki conditional expectation $E : M_2 \otimes M_d \to M_d$ with

$$\mathcal{E}^{\infty}_{n+1, n+1} = i_n \circ E \circ (i^{-1}_n \otimes i^{-1}_{n+1}) | A_{n, n+1}$$

for all $n \in \mathbb{N}$.

We will follow this scheme, but drop the assumption that $\varphi$ is locally faithful and stationary. Just assume that $\varphi$ is a Markovian state with the property that all $\mathcal{E}^{\infty}_{n+1, n+1}$, $n \in \mathbb{N}$ satisfy (6) for some Umegaki conditional expectation $E$, i.e. the fixed points of $\mathcal{E}^{\infty}_{n+1, n+1}$ form a $*$-algebra. Let $B_0$ be the range of $E$ and let $P_1, \ldots, P_n$ be minimal projections in the center of $B_0$ such that

$$\sum_j P_j = 1$$

and $P_j B_0 = P_j B_0 P_j$ is a factor. Then $B_0$ can be realized as a matrix algebra on the space

$$C^d = : \mathcal{H} = \bigoplus_j \mathcal{H}_j, \quad \mathcal{H}_j = P_j \mathcal{H}$$

and $P_j B_0 = P_j B_0 P_j$ is a subfactor of $B(\mathcal{H}_j)$. Therefore, $\mathcal{H}_j = \mathcal{H}_{j0} \otimes \mathcal{H}_{j1}$ and

$$P_j B_0 P_j = P_j B(\mathcal{H}_{j0}) \otimes 1_{\mathcal{H}_{j1}} P_j.$$

From this we see

$$B'_0 = \bigoplus_j P_j 1_{\mathcal{H}_{j0}} \otimes B(\mathcal{H}_{j1}) P_j$$

and

$$B_0 \vee E'_0 = (B_0 \cap E'_0)' = \bigoplus_j P_j M_d P_j.$$

Clearly, for each $j$, the following holds

$$P_j M_d P_j \cong P_j B_0 P_j \otimes P_j E'_0 P_j.$$

We will fix this identification in the following. The conditional expectation property of $\mathcal{E}^{\infty}_{n+1, n+1}$ is reflected by the fact that $E$ is a conditional expectation, if we identify $B_0$ with $B_0 \otimes 1 \subset M_d \otimes M_d$.

**Lemma 3.1.** Any conditional expectation $E$ from $M_2 \otimes M_d$ onto $B_0 \otimes 1 \sim B_0$ has the form

$$E(x) = \sum_j P_j \Phi_j (P_j x P_j) P_j,$$
where $\Phi_j : \mathcal{B}(\mathcal{H}_{j0}) \otimes \mathcal{B}(\mathcal{H}_{j1}) \otimes M_d \mapsto \mathcal{B}(\mathcal{H}_{j0}) \otimes 1_{j1}$ is the Umegaki conditional expectation

$$
\Phi_j(b_{j0} \otimes b_{j1} \otimes b) = b_{j0}\phi_j(b_{j1} \otimes b) \otimes 1
$$

for states $\phi_j$ on $\mathcal{B}(\mathcal{H}_{j1}) \otimes M_d$.

**Proof.** We know that $B_0$ is mapped into itself by the conditional expectation $E_P$,

$$
E_P(x) = \sum_j P_j x P_j, \quad x \in M_d \otimes M_d
$$

which is the unique conditional expectation onto $B_0 \vee B_0$. Since $P_k \in \text{Fix } E$, it follows that $P_k E(x) = E(P_k x P_k) = P_k E(P_k x P_k) P_k$ which implies by (7)

$$
E(x) = \sum_j P_j E(P_j x P_j) P_j.
$$

Denoting $E_j = E(P_j \cdot P_j)$, we see that $E$ is a conditional expectation iff all the $E_j$'s are conditional expectations from $\mathcal{B}(\mathcal{H}_j) \otimes M_d$ onto $\mathcal{B}(\mathcal{H}_{j0}) \otimes 1_{j1} \otimes 1_{M_d}$. Now we get from the conditional expectation property

$$
E_j(a \otimes b \otimes c) = a \otimes 1 \otimes 1 E_j(1 \otimes b \otimes c) = E_j(1 \otimes b \otimes c) a \otimes 1 \otimes 1, \quad a \in \mathcal{B}(\mathcal{H}_{j0}).
$$

Since $E_j(1 \otimes b \otimes c)$ commutes with all $a \otimes 1 \otimes 1$, it must be a scalar. Therefore, there is a state $\phi_j$ on $\mathcal{B}(\mathcal{H}_{j1}) \otimes M_d$ with $E_j(1 \otimes b \otimes c) = \phi_j(b \otimes c)$. This shows $E_j = \phi_j$ and completes the proof.

Now we want to return to a Markovian state $\varphi$. It is standard to see\(^4\) that

$$
\varphi(b_0 \otimes \cdots \otimes b_n) = \varphi(b_0 \otimes \cdots \otimes b_{n-1} \otimes \mathcal{E}(b_n \otimes 1))
$$

$$
= \varphi_0(\mathcal{E}(b_0 \otimes \mathcal{E}(b_1 \otimes \cdots \otimes \mathcal{E}(b_{n-1} \otimes \mathcal{E}(b_n \otimes 1))))),
$$

i.e. $\varphi$ is a quantum Markov chain in the sense of Refs. 1 and 6. Thus the knowledge of the initial state $\varphi_0$ (a state on $M_d$) and of $\mathcal{E}$ uniquely determines the structure of a Markov state.

**Lemma 3.2.** If $\varphi$ satisfies (11), then with the conditional expectation $E_P$ defined by (10):

$$
\varphi(b_0 \otimes \cdots \otimes b_n) = \varphi(E_P(b_0) \otimes \cdots \otimes E_P(b_n))
$$

and setting $E_j(x) = P_j x P_j$:

$$
\varphi(b_0 \otimes \cdots \otimes b_n) = \sum_{j_1, \ldots, j_n} \varphi(\mathcal{E}_{j_1}(b_0) \cdots \mathcal{E}_{j_n}(b_n)).
$$
**Proof.** In the proof of Lemma 2.1 we saw that $\varphi(E_P(a \otimes b)) = E(\varphi(E_P(b_0) \otimes \cdots \otimes E_P(b_n))) = \varphi(\varphi(E_P(b_0) \otimes \cdots \otimes E_P(b_n) \otimes E(b_{n-1} \otimes E_P(b_n)))$ $= \varphi(\varphi(b_0 \otimes \cdots \otimes E_P(b_n)))$

$= \varphi(b_0 \otimes \cdots \otimes b_n)$.

Since $E_P = \sum_j E_j$, this proves the second formula. □

Now we set

$$P_{j_0 \ldots j_n} := \varphi(E_{j_0}(1) \cdots E_{j_n}(1)) = \varphi(P_{j_0} \otimes \cdots \otimes P_{j_n}).$$

Further, we define maps $E_j : P_j M_d P_j \otimes P_j B_0 P_j' \leftrightarrow P_j B_0 P_j$ through

$$E_j((a \otimes b)) = E_j(a \otimes E_j(b)).$$

Due to (12) and Markovianity, we obtain

$$\varphi(b_0 \otimes \cdots \otimes b_n) = \sum_{j_0, \ldots, j_{n+1}} \varphi_0(E_{j_0-j_1}(b_0 \otimes \cdots \otimes E_{j_{n+1}}(b_0 \otimes P_{j_{n+1}}) \cdots)).$$

First, we look at the center of $b_0$. Set $\pi_{j'} := \varphi_j(P_j \otimes P_j')$ and $\pi_j := \varphi_0(P_j)$. Then one has

$$\varphi(P_{j_0}^{(0)} \cdots P_{j_n}^{(n)} P_{j_{n+1}}^{(n+1)}) = \varphi(P_0^{(0)} \cdots E^{(\infty)}(P_{j_n} \otimes P_{j_{n+1}})).$$

By definition,

$$E^{(\infty)}(P_{j_n} \otimes P_{j_{n+1}}) = P_{j_n} \varphi_{j_n}(P_{j_n} \otimes P_{j_{n+1}})P_{j_n} = \varphi_{j_n}(1 \otimes P_{j_{n+1}})P_{j_n}.$$

We set $p_{j'} = \varphi_j(P_j \otimes P_j')$ and obtain

$$\varphi(P_{j_0}^{(0)} \cdots P_{j_n}^{(n)} P_{j_{n+1}}^{(n+1)}) = \varphi(P_0^{(0)} \cdots P_{j_0}^{(n)} \cdot \varphi_{j_0}(1 \otimes P_{j_{n+1}}) \cdots P_{j_n}^{(n)} P_{j_{n+1}}^{(n+1)} \cdots P_{j_{n+1}}^{(n+1)}

Denoting $\pi_j = \varphi_0(P_j)$ we obtain

$$\varphi(P_{j_0}^{(0)} \cdots P_{j_{n+1}}^{(n+1)}) = \pi_{j_0} \cdots \pi_{j_{n+1}} \cdots \pi_{j_{n+1}}.$$

**Corollary 3.1.** For all indices $j_0, \ldots, j_n$

$$P_{j_0 \ldots j_n} = \pi_{j_0} \cdots \pi_{j_{n-1}} \cdots \pi_{j_{n+1}}.$$

**Proof.** It is easy to see that

$$E_{j'}(P_j \otimes P_j') = E_{j'}(P_j \otimes P_j') = P_j E_{j'}(P_j \otimes P_j') = P_j \pi_{j'}.$$ A simple induction completes the proof. □
\[ (a \otimes b) = \mathcal{E}(a \otimes b). \] This, \[ \mu(\mathcal{E}_n, \mathcal{E}(P_{j_1} \otimes \mathcal{E}_n(b_n))) = \mathcal{E}(b_n \otimes 1)) = (b_n \otimes \mathcal{E}_n(1)). \]

\( \blacksquare \)

**Remark 3.1.** Observe that the right-hand side in (13) is equal to
\[
\text{Prob}(X_0 = j_0, X_1 = j_1, \ldots, X_n = j_n, X_{n+1} = j_{n+1}),
\]
where \((X_n)\) is the homogeneous Markov process on \(\{1, \ldots, n\}\) (or equivalently, on the atoms of the center of \(\text{Fix} \mathcal{E}\) with initial distribution \(P(X_0 = j) = \pi_j\) and transition probabilities \(\pi_{jj'} = \phi_j(1 \otimes P_{j'})\)). We denote the law of this process by \(\mu\).

**Lemma 3.3.** For a fixed sequence \((j_n) =: \omega\), there exists a state \(\varphi_\omega\) on
\[
\bigotimes_{n \in \mathbb{N}} P_{j_n} B_0 P_{j_n} \cong \bigotimes_{n \in \mathbb{N}} P_{j_n} B_0 P_{j_n} \otimes P_{j_n} B_0' P_{j_n}
\]
\[
\cong P_{j_0} B_0 P_{j_0} \otimes \bigotimes_{n \in \mathbb{N}} P_{j_n} B_0' P_{j_n} \otimes P_{j_n} B_0 P_{j_n+1}. \tag{14}
\]
such that, for all \(n \in \mathbb{N}, b_0, \ldots, b_n \in M_d:\)
\[
\varphi(\mathcal{E}_0(b_0) \otimes \cdots \otimes \mathcal{E}_n(b_n) \otimes P_{j_n+1} \otimes \cdots) = p_{j_0 \cdots j_n j_{n+1}} \varphi_\omega(\mathcal{E}_0(b_0) \otimes \cdots \otimes \mathcal{E}_n(b_n) \otimes P_{j_n+1} \otimes \cdots). \tag{15}
\]

**Proof.** Clearly, for all \(n\) there is a state \(\varphi^0_n\) which fulfills (15) for all \(b_0, \ldots, b_n \in M_d\).

We need only to prove compatibility. Observe
\[
\varphi(\mathcal{E}_0(b_0) \otimes \cdots \otimes \mathcal{E}_n(b_n) \otimes P_{j_n+1} \otimes P_{j_{n+2}})
\]
\[
= \varphi(\mathcal{E}_0(b_0) \otimes \cdots \otimes \mathcal{E}_n(b_n) \otimes \mathcal{E}(P_{j_n+1} \otimes P_{j_{n+2}}) \otimes \cdots)
\]
\[
= \varphi(\mathcal{E}_0(b_0) \otimes \cdots \otimes \mathcal{E}_n(b_n) \otimes P_{j_{n+1}} \otimes 1 \cdots) \phi_{j_{n+1}}(P_{j_{n+1}} \otimes P_{j_{n+2}})
\]
\[
= \varphi(\mathcal{E}_0(b_0) \otimes \cdots \otimes \mathcal{E}_n(b_n) \otimes P_{j_{n+1}} \otimes 1 \cdots) \varphi^0_{j_{n+1} j_{n+2}}.
\]

Together with \(p_{j_0 \cdots j_n j_{n+2}} \varphi^0_{j_{n+1} j_{n+2}} = p_{j_0 \cdots j_{n+1} j_{n+2}}\), this shows compatibility of the states \((\varphi^0_n)\) if \(p_{j_0 \cdots j_{n+1} j_{n+2}} > 0\). But this is true for \(\mu\)-a.a. \(\omega = (j_n)\). \(\blacksquare\)

**Remark 3.2.** Observe that (12) is not enough to construct the states \(\varphi_\omega\). We need in (15) at least one additional \(P_{j_{n+1}}\) on the right. This reflects the quantum Markovian structure of \(\varphi\).

For \(\pi_{jj'} > 0\), we define the states \(\eta_{jj'}\) on \(P_{j} B_0' P_{j'} \otimes P_{j'} B_0 P_{j'}\),
\[
\eta_{jj'}(b_j' \otimes b_{j'}) = \frac{\phi_j(b_j' \otimes b_{j'})}{\pi_{jj'}}, \quad b_j, b_{j'} \in P_{j} B_0' P_{j}, b_{j'}, b_{j'}' \in P_{j'} B_0 P_{j'}.
\]
Similarly, for \(\pi_j > 0\), we define the states \(\eta_j\) by
\[
\eta_j(b_j) = \frac{\varphi_\omega(b_j)}{\pi_j}, \quad b_j \in P_{j} B_0' P_{j}.
\]
Lemma 3.4. In the representation (14),

$$
\varphi_\omega = \eta_{j_0} \bigotimes_{n \in \mathbb{N}} \eta_{j_n,j_{n+1}}
$$

(16)

for \( \mu \)-a.e. \( \omega = (j_n) \).

**Proof.** Fix \( n \in \mathbb{N}, (j_n) = \omega \) such that \( p_{j_0,...,j_{n+1}} > 0 \) and \( b_k \in P_{j_k} B_0 P_{j_k}, b'_k \in P_{j_k} B'_0 P_{j_k}, k = 0, \ldots, n \). From (11) we obtain

\[
\begin{align*}
p_{j_0,...,j_{n+1}} \varphi_\omega(b_0 \otimes b'_0 \otimes \cdots \otimes b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes b'_n) \\
&= \varphi(b_0 \otimes b'_0 \otimes \cdots \otimes b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes b'_n \otimes P_{j_{n+1}}) \\
&= \varphi(b_0 \otimes b'_0 \otimes \cdots \otimes b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes b'_n \otimes \mathcal{E}(b_n \otimes b'_n \otimes P_{j_{n+1}})) \\
&= \varphi(b_0 \otimes b'_0 \otimes \cdots \otimes b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes 1) \varphi_{j_n,j_{n+1}}(b'_n \otimes P_{j_{n+1}}) \\
&= \varphi(b_0 \otimes b'_0 \otimes \cdots \otimes b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes 1) \eta_{j_n,j_{n+1}}(b'_n \otimes P_{j_{n+1}}) \pi_{j_n,j_{n+1}} \\
&\vdots \\
&= \eta_{j_0}(b_0) \eta_{j_0,j_1}(b'_0 \otimes b_1) \cdots \eta_{j_n,j_{n+1}}(b'_n \otimes 1) \pi_{j_n,j_1} \pi_{j_n,j_{n+1}} \\
&= \eta_{j_0,...,j_{n+1}}(b_0) \eta_{j_0,j_1}(b'_0 \otimes b_1) \cdots \eta_{j_n,j_{n+1}}(b'_n \otimes 1).
\end{align*}
\]

Thus, for all such \( \omega \), (16) is valid. But this set of \( \omega \) has \( \mu \)-measure 1 and the proof is complete. \( \square \)

Summarizing, we obtain

**Theorem 3.2.** Let \( \varphi \) be a Markovian state with stationary completely positive unit preserving maps \( E_{n+1,\eta[j]} \) related to \( \mathcal{E} \) which is an Umegaki conditional expectation onto \( B_0 \) with minimal central projections \( (P_{j})_j \). Define \( \pi_j, \pi_{j,j'}, \eta_j, \eta_{j,j'} \) as above. Further, let \( \mu \) be the law of the classical Markov process with initial distribution \( (\pi_j)_j \) and transition probabilities \( (\pi_{j,j'})_{j,j'} \). Then

$$
\varphi = \int \mathbb{P}(d\omega) \varphi_\omega
$$

(17)

in the sense that for all \( n \in \mathbb{N} \)

$$
\begin{align*}
\varphi(b_0 \otimes \cdots \otimes b_n \otimes 1 \otimes \cdots) \\
= \sum_{j_0,\ldots,j_{n+1}} p_{j_0,...,j_{n+1}} \varphi_{j_0,...,j_{n+1}}(E_{j_0}(b_0) \otimes \cdots \otimes E_{j_n}(b_n) \otimes P_{j_{n+1}} \otimes \cdots).
\end{align*}
$$

(18)

Moreover,

$$
\varphi_{j_0,...,j_{n+1}} = \eta_{j_0} \bigotimes_{n=0}^{\infty} \eta_{j_n,j_{n+1}}.
$$

(19)
Conversely, fit $B_0$, a probability distribution $(\pi_j)_j$, transition probabilities $(p_{ij})_{ij}$, states $\eta_j$ on $B(\mathcal{H}_j)$ for each $j$ with $\pi_j > 0$ and states $\eta_{ji,j'}$ on $B(\mathcal{H}_j \otimes \mathcal{H}_{j'})$ for all $j, j'$ with $\pi_{jj'} > 0$. Then (17) and (19) define a state $\varphi$ on $A$ which is Markovian. In the notations of Lemma 3.1, the structure of a corresponding Umegaki conditional expectation $E$ is determined by

$$\phi_j(b' \otimes b) = \phi_j(b' \otimes E_P(b)).$$

(20)

($E_P$ is given by (10) and

$$\varphi_j(b' \otimes E_P(c \otimes d')) = \sum_{j'} \pi_{jj'} \eta_{j,j'}(b' \otimes c) \pi_{j',j''} \eta_{j',j''} (d' \otimes 1)$$

Remark 3.3. Please note that the symbol $\int_\Theta$ is not a usual direct integral because we need in (15) an additional $P_{j+1}$ to be present. In other words, it is like a “Markovian” direct integral.

Remark 3.4. One can easily apply the above technique also to the inhomogeneous case. In fact, we did something similar for the “fiber” state $\varphi_\omega$.

Proof. The structure of Markovian states is determined by the above considerations. Thus we prove here only that states $\varphi$ determined by (18) are Markovian. Clearly, this equation determines a state $\varphi$ on $A$. Due to (20) it is enough to check for $\mu$-a.e. $(j_n)$

$$\varphi((b_0 \otimes b'_0) \otimes \cdots \otimes (b_n \otimes b'_n)) \otimes 1 \cdots$$

$$= \varphi((b_0 \otimes b'_0) \otimes \cdots \otimes E((b_{n-1} \otimes b'_{n-1}) \otimes (b_n \otimes b'_n)) \otimes 1 \cdots$$

for all $b_j \in P_{j}B_{j}P_{j}$, $b'_j \in P_{j}B_{j}P_{j}$. Then

$$\varphi((b_0 \otimes b'_0) \otimes \cdots \otimes E((b_{n-1} \otimes b'_{n-1}) \otimes (b_n \otimes b'_n)) \otimes 1 \cdots$$

$$= \sum_{j} \varphi((b_0 \otimes b'_0) \otimes \cdots \otimes b_{n-1} \otimes 1)\eta_{j_n,j_{n-1}}(b'_{n-1} \otimes b_n)\eta_{j_{n-1}}(b'_n \otimes P_j)$$

$$= \sum_{j} b_{j_0,j_{-1},j,n}(b_0)\eta_{j_0,j_1}(b'_0 \otimes b_1)\cdots \eta_{j_{n-2},j_{n-1}}(b'_{n-2} \otimes b_{n-1})\eta_{j_{n-1}}(b'_n \otimes P_j)$$

$$\times (b'_{n-1} \otimes b_n)\eta_{j_n,j}(b'_n \otimes P_j)$$

$$= \sum_{j} \varphi((b_0 \otimes b'_0) \otimes \cdots \otimes (b_n \otimes b'_n) \otimes 1 \cdots)$$

$$= \varphi((b_0 \otimes b'_0) \otimes \cdots \otimes (b_n \otimes b'_n) \otimes 1 \cdots).$$

This completes the proof.

We will shortly present some examples.
Example 3.1. Assume that $B_0$ is maximal Abelian, i.e. $B_0 = \bigoplus_j C_P$, all $\mathcal{H}_j \cong C$. This leads to $\mathcal{A}_0 = C$ and $\varphi_\infty$ being trivial. But to each $P_j$ there belongs a natural state $\varphi_j : a \mapsto \text{tr} P_j a$. So (18) translates into

$$\varphi(b_0 \otimes \cdots \otimes b_n) = \int \mu(d(j_n)) \varphi_{j_n}(b_0) \cdots \varphi_{j_n}(b_n).$$

Thus $\varphi$ is a mixture of ordinary product states under a Markov process. Such states were considered in Ref. 5 and a limit theorem provided even a continuous time analogue.

Example 3.2. If $B_0$ is a factor, the Markov process $(X_n)$ is constant, i.e. $\mu$ is trivial. Thus $\varphi = \eta \otimes \otimes_{n=0}^{\infty} \bar{\eta}$, where $\bar{\eta}$ is a state on $B'_0 \otimes B_0$. There are other recent works\textsuperscript{14} suggesting one to think of such states as quantum Markov states. We want to note that such states also play a role in the construction of valence bond solid (VBS) states.\textsuperscript{15}

In particular, if $B_0 = C1$ or $B_0 = M_d$, the Markovian states are just ordinary product states.

Example 3.3. If $d = 2$, there are only three possible types of algebras $B_0$: $B_0 = C1$, $B_0 = M_2$ and maximal Abelian algebras. In view of Lemma 6.1, all Markovian states are either product states or come from a classical Markov process. This seems to be a partial reverse of the transfer matrix principle which relates to any classical spin system a quantum spin system of lower dimension.\textsuperscript{24}

We can look for several criteria to be fulfilled for the state $\varphi$ defined by (17) and (19) with the various ingredients. The following results are straightforward, we omit the proofs.

Lemma 3.5. $\varphi$ is locally faithful iff

(a) $\pi_j > 0$ for all $j$,
(b) for all $j, j'$, it holds $\pi_{jj'} > 0$,
(c) for all $j$, the state $\eta_j$ is faithful and
(d) for all $j, j'$ the state $\eta_{jj'}$ is faithful.

Remark 3.5. In fact, $\mathcal{B}_\infty$ is faithful iff (b) and (d) are satisfied.

Lemma 3.6. $\varphi$ is stationary iff

(a) $\mu$ is stationary, i.e. $\sum_j \pi_j \pi_{jj'} = \pi_{jj'}$ and
(b) we have for all $j'$

$$\sum_j \pi_j \pi_{jj'} \eta_{jj'}(P_j \otimes b) = \pi_{jj'} \eta_{jj'}(b), \quad b \in P_j B_0 P_{j'}.$$ \hspace{1cm} (21)

Consequently, the stationary Markovian state is unique iff the invariant distribution $(\pi_j)$ is unique.
\[ B_0 = \bigoplus_j C P_j, \forall \mathcal{H}_j \subseteq C. \]

\[ P_j \text{ there belongs a natural} \]

\[ \psi_j \to \psi_j. \]

for a Markov process. Such \( \psi_j \) is provided even a continuous

\[ (X_n) \text{ is constant}, \mu \text{ is} \]

on \( \mathcal{B}_0 \). There are other recent

\[ \text{from Markov states. We want} \]

\[ \text{a description of valence bond solid} \]

\[ \text{valence states are just ordinary} \]

\[ \text{algebras} \mathcal{B}_0: \mathcal{B}_0 = C_1, \]

\[ \text{from 6.1, all Markovian states} \]

\[ \text{markov process. This seems to} \]

\[ \text{which relates to any classical} \]

\[ \text{the state} \varphi \text{ defined by (17) and (19)} \]

\[ \text{results are straightforward, we} \]

\[ \text{satisfied.} \]

\[ \varphi = \pi_j \varphi_j. \]

\[ \text{iff the invariant distribution} \]

\[ \text{Proof. Clearly,} \varphi \text{ is stationary only if} (X_n) \text{ is stationary. Moreover,} \]

\[ \varphi(b \otimes (b_1 \otimes b'_1) \otimes \cdots \otimes (b_n \otimes b'_n) \otimes 1) \]

\[ = \sum_j \varphi(P_j \otimes (b_1 \otimes b'_1) \otimes \cdots \otimes (b_n \otimes b'_n) \otimes 1) \]

\[ = \sum_j \pi_j \pi_{j_{1 \ldots j_n}} \eta_{j_{1 \ldots j_n}} (1 \otimes b_1) \eta_{j_{1 \ldots j_n}} (b'_1 \otimes b_2) \cdots \sum_j \pi_{j_{1 \ldots j_n}} \eta_{j_{1 \ldots j_n}} (b'_n \otimes 1). \]

On the other hand,

\[ \varphi(b_1 \otimes (b'_1 \otimes \cdots \otimes (b_n \otimes b'_n) \otimes 1) \]

\[ = \pi_j \pi_{j_{1 \ldots j_n}} \pi_{j_{1 \ldots j_n}} (b_1 \otimes b_2) \cdots \sum_j \pi_{j_{1 \ldots j_n}} \eta_{j_{1 \ldots j_n}} (b'_n \otimes 1). \]

This shows (21). The proof of sufficiency is straightforward.

Moreover, if the invariant distribution \( \pi_j \) is unique, then (21) determines all the \( \varphi^0 \) which are essential for determining the Markovian state. This completes the proof.

\[ \square \]

An interesting problem seems to be whether a state \( \varphi \) defined by (17) and (19) is really left-invariant under the original completely positive unit preserving maps \( \mathcal{E}_{n+1,n} \) or equivalently, \( \mathcal{E}_{n+1,n}^\infty \).

**Theorem 3.3.** Suppose the completely positive unit preserving maps \( \mathcal{E}_{n+1,n}^\infty \), \( n \in \mathbb{N} \) come from one completely positive unit preserving map for which \( \mathcal{E} \) is an Umegaki conditional expectation given by (9). From this description, construct \( \pi_{ij} \) and \( \eta_{ij} \).

Then a state \( \varphi \) is invariant under all \( \mathcal{E}_{n+1,n}^\infty \), \( n \in \mathbb{N} \), if

(a) \( \text{It has a representation (17) where} \mu \text{ is the law of the Markov process} \)

(b) \( (X_n) \text{ never visits points} j \text{ for which there is a pair} (j', j'') \text{ with} \pi_{jj'} \pi_{j'j''} > 0 \text{ associated with and there exist} b, c, c' \text{ such that} \)

\[ \frac{\phi_{j'}(b \otimes b_{j'} \otimes c \otimes (c' \otimes P'_{j''}))}{\pi_{jj'}} \neq \frac{\phi_{j''}(b \otimes b_{j} \otimes c \otimes (c' \otimes P'_{j''}))}{\pi_{j''}}. \]

**Proof.** We already showed that conditions (17) and (19) are necessary for a state to be invariant. So, to prove the necessity of (19), we can go to the fibers and have to characterize the fiber states \( \varphi_{jj'} \), which are invariant under the maps \( \mathcal{E}_{n}^\infty \) given by

\[ \mathcal{E}_{n}^\infty (b_0 \otimes \cdots \otimes b_n \otimes b_{n+1} \otimes \cdots) = b_0 \otimes \cdots \otimes b_{n-1} \otimes \mathcal{E}_{j_{n+1}}^\infty (b_n \otimes b_{n+1}) \otimes \cdots, \]

where \( \mathcal{E}_{j_{n+1}}^\infty \) is defined by

\[ \mathcal{E}_{j_{n+1}}^\infty ((b_j \otimes b'_j) \otimes (b_j \otimes b'_j)) = b_j \phi_{j_{n+1}} (b'_j \otimes b_j \otimes b'_j). \]
and the state $\tilde{\phi}_{j,j'}$ on $P_j E'_0 P_j \otimes P_{j'} M_{0} P_{j'}$ is given by

$$
\tilde{\phi}_{j,j'}(b' \otimes b) = \frac{\phi_{j}(b' \otimes b)}{\pi_{j,j'}}.
$$

We obtain for any invariant state $\varphi$ for $\mu$-a.a. fibers

$$
\varphi((b_0 \otimes b'_0) \otimes \cdots \otimes \varphi_{\infty,j_{n-1},j_n}((b_{n-1} \otimes b'_{n-1}) \otimes (b_n \otimes b'_n)))
= \varphi((b_0 \otimes b'_0) \otimes \cdots \otimes \varphi_{\infty,j_{n-1},j_n}((b_{n-1} \otimes b'_{n-1}) \otimes \varphi_{\infty,j_{n},j_{n+1}}((b_{n} \otimes b'_n) \otimes 1))).
$$

It follows

$$
\varphi((b_0 \otimes b'_0) \otimes \cdots \otimes b_{n-1} \otimes \cdots) (\tilde{\phi}_{j_{n-1},j_{n}}(b'_{n-1} \otimes (b_n \otimes b'_n))
- \tilde{\phi}_{j_{n-1},j_{n}}(b'_{n-1} \otimes b_n) \tilde{\phi}_{j_{n},j_{n+1}}(b'_n \otimes P_{j_{n+1}})) = 0.
$$

This shows that the term in parentheses has to vanish for $\mu$-a.a. $(j_n)$. This is equivalent to (b).

To prove sufficiency, it is enough to show that for $\mu$-a.a. $\omega$ the states $\varphi_\omega$ are invariant under the maps $\varphi_{\infty,\omega}$, $n \in \mathbb{N}$, described above. By the calculations before Theorem 3.2, the following must hold

$$
\eta_{j,j'}(b' \otimes c) = \tilde{\phi}_{j,j'}(b' \otimes c \otimes 1).
$$

Now we find

$$
\varphi((b_0 \otimes b'_0) \otimes \cdots \otimes \varphi_{\infty,j_{n-1},j_n}((b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes b'_n)))
= \varphi((b_0 \otimes b'_0) \otimes \cdots \otimes b_{n-1} \otimes 1) \tilde{\phi}_{j_{n},j_{n+1}}(b'_n \otimes b_n \otimes b'_n).
$$

From the assumption we know that $\mu$-a.a. $\omega = (j_n)$ fulfill for all $n \in \mathbb{N}$, $n \geq 1$

$$
\tilde{\phi}_{j_{n-1},j_{n}}(b' \otimes c \otimes c') = \tilde{\phi}_{j_{n-1},j_{n}}(b' \otimes c \otimes 1) \tilde{\phi}_{j_{n},j_{n+1}}(c \otimes 1 \otimes 1).
$$

Thus we can conclude from the definition of $\varphi_\omega$

$$
\varphi((b_0 \otimes b'_0) \otimes \cdots \otimes \varphi_{\infty,j_{n-1},j_n}((b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes b'_n)))
= \varphi((b_0 \otimes b'_0) \otimes \cdots \otimes b_{n-1} \otimes 1) \varphi_{j_{n-1},j_{n}}(b_{n-1} \otimes b_n \otimes 1) \tilde{\phi}_{j_{n},j_{n+1}}(b'_n \otimes 1 \otimes 1)
= \varphi((b_0 \otimes b'_0) \otimes \cdots \otimes b_{n-1} \otimes 1) \varphi_{j_{n-1},j_{n}}(b_{n-1} \otimes b_n) \varphi_{j_{n},j_{n+1}}(b'_n \otimes 1)
= \varphi_{j_{0}}(b_0) \varphi_{j_{n},j_{n+1}}(b'_0 \otimes 1) \cdots \varphi_{j_{n-2},j_{n-1}}(b'_{n-2} \otimes b_{n-1}) \varphi_{j_{n-1},j_{n}}(b'_{n-1} \otimes b_n)
\times \varphi_{j_{n},j_{n+1}}(b'_n \otimes 1)
= \varphi((b_0 \otimes b'_0) \otimes \cdots \otimes b_{n-1} \otimes b'_{n-1} \otimes b_n \otimes b'_n).
$$

This completes the proof. \(\square\)
4. Relations to Potentials

In analogy to the classical case, we want to relate a Markovian state to a potential. Fix a locally faithful state \( \varphi \) on \( A \) (chosen as in the previous section). We define self-adjoint operators \( (h_n), h_n \in A_n \) by

\[
\rho_n = e^{-ih_n}, \quad n \in \mathbb{N}
\]

if \( \rho_n \) is the density matrix of \( \varphi_n \). The following is known from Theorem 4.2 of Ref. 4.

**Proposition 4.1.** For a locally faithful state \( \varphi \) on \( A \), the following statements are equivalent:

(a) \( \varphi \) is a Markovian state.
(b) For all \( n \in \mathbb{N} \)

\[
e^{-i(t h_n)(t h_{n+1})} \in A_{[n,n+1]}, \quad t \in \mathbb{R}.
\]

(c) For each \( n \in \mathbb{N} \)

\[
e^{-1/(2h_n)}e^{1/(2h_{n+1})} \in A_{[n,n+1]}.
\]

**Remark 4.1.** In Ref. 4 a sequence \((h_n)_{n \in \mathbb{N}}\) satisfying (23) was called an **Ising potential**. Note that not all Ising potentials define a Markovian state because some compatibility conditions have to be fulfilled. Also the characterizations of Ising potential from Theorem 3.2 of Ref. 4 is too indirect for the construction of examples.

Now, we want to go one step forward in this direction and look at an intrinsic property of \((h_n)\) assuring that \( \varphi \) is Markovian.

**Proposition 4.2.** \( \varphi \) is a Markovian state if and only if for all \( n \in \mathbb{N} \) there is some \( h_n \in A_n \) such that \( h_{n+1} - h_n \in A_{[n,n+1]}, h_n \) commutes with \( h_n \) and \( h_{n+1} \) commutes with \( h_n - h_n \).

**Proof.** Suppose that \( \varphi \) is Markovian. So there are the ingredients \( \mu, \eta_j, \eta_{jj'} \). We know from Lemma 3.5 that \( p_{j_0, \ldots, j_n} > 0 \) for all \( j_0, \ldots, j_n \) and all \( \eta_j, \eta_{jj'} \) are faithful. So we relate to the latter state potentials \( h_j \in P_j B_0 P_j, h_{jj'} \in P_j B_0 P_j \otimes P_j B_0 P_j \).

Further, there are also (faithful) states

\[
\tilde{\eta}_j(b) = \sum_{j'} \pi_{jj'} \eta_{jj'}(b \otimes P_{j'}), \quad b \in P_j B_0 P_j
\]

with potentials \( \tilde{h}_j \in P_j B_0 P_j \). Then, using Theorem 3.2 we find by easy calculation

\[
h_n = \sum_{j_0, \ldots, j_n} \mathcal{E}_{j_0, \ldots, j_n} (-\ln(p_{j_0, \ldots, j_n})1 + i\eta(h'_{j_0}) + i\eta(h_{j_0, j_1})
\]

\[
+ \cdots + i\eta(-1_n((h'_{j_{n-1}, j_n}) + i\eta(h_{j_n}))
\]

with

\[
\mathcal{E}_{j_0, \ldots, j_n}(b) = P_{j_0} \otimes \cdots \otimes P_{j_n} b P_{j_0} \otimes \cdots \otimes P_{j_n}
\]
and the short hand \( i_{m,n} = i_m \otimes i_n \). Setting \( h_n = \sum_j i_n(\tilde{h}_j) \) we derive easily the assertion.

Conversely, we obtain
\[
e^{-ith_n}e^{ith_{n+1}} = e^{-ith_n}e^{-i((h_n-h_n)+h_n)} = e^{-ith_n}e^{i((h_{n+1}-h_n)+h_n)} \in A_{n,n+1}.
\]

By (23) this implies the Markovianity of \( \varphi \) and the proof is complete. □

**Corollary 4.1.** Suppose \( \varphi \) is a locally faithful Markovian state on \( A \). Then it is a KMS state with respect to the one-parameter automorphism group \( \sigma_t \):
\[
\sigma_t(a) = \lim_{n \to \infty} e^{-ith_{0,n}} e^{ith_{0,n}}, \quad a \in A, t \in \mathbb{R}.
\]

In particular, \( \varphi \) has a faithful extension to \( \pi_\varphi(A)' \), where \( \pi_\varphi \) is the GNS-representation of \( \varphi \).

**Proof.** The first part follows from the remark after the proof of Theorem 4.2 of Ref. 4. The second part is an application of Corollary 5.3.9 of Ref. 9. □

5. Markovian States on \( \bigotimes \mathbb{Z} M_d \)

Now we want to deal with Markovian states on the full chain. The definition now reads as

**Definition 5.1.** We call a state \( \varphi \) on a Markovian state if it is for all \( k < n \in \mathbb{Z} \) invariant under a map \( \mathcal{E}_{[k,n+1],[k,n]} : A_{[k,n+1]} \to A_{[k,n]} \) which is a quasi-conditional expectation with respect to the localization \( (A_{n+1}, A_n, A_{[k,n]}) \).

We will restrict to the case of locally faithful states. Again, the completely positive unit preserving maps should be stationary. A next problem is to construct a suitable Umegaki conditional expectation \( \mathcal{E}_\infty : M_d \otimes M_d \to M_d \) like in the case of a half-chain.

**Lemma 5.1.** Suppose \( \varphi \) is locally faithful. Then for each \( n \in \mathbb{Z} \) there exists an Umegaki conditional expectation \( \mathcal{E}_\infty : M_d \otimes M_d \to M_d \) such that every state \( \varphi \) invariant under all \( \mathcal{E}_{[k,n+1],[k,n]}^\varphi \) for \( n > k \) \( \in \mathbb{Z} \) is invariant under the lifting of \( \mathcal{E}_\infty^\varphi \) to \( A_{n+1} \) too. Furthermore, \( A_{n-1} \otimes \text{Fix}(\mathcal{E}_\infty) \subset \text{Fix}(\mathcal{E}_{[k,n+1],[k,n]}^\varphi) \) for all \( Z \ni k \leq n \).

**Proof.** From Ref. 3 we get for all \( k < n \) the Umegaki conditional expectation \( \mathcal{E}_\infty^k \) which projects onto \( \text{Fix}(\mathcal{E}_{[k,n+1],[k,n]}^\varphi) \). Moreover, we know that \( A_{[k,n+1]} \) is mapped into \( A_{[k,n]} \) by \( \mathcal{E}_\infty^{k-1} \). So \( \mathcal{E}_\infty^{k-1} \) is another Umegaki conditional expectation leaving \( \varphi \) invariant. Therefore, the results of Ref. 3 imply that
\[
\mathcal{E}_\infty^{k-1} \circ \mathcal{E}_\infty^k = \mathcal{E}_\infty^k \circ \mathcal{E}_\infty^{k-1} = \mathcal{E}_\infty^{k-1}.
\]

Now, putting
\[
\mathcal{E}_{\infty} = \lim_{k \to -\infty} \mathcal{E}_\infty^k \circ (i_n \otimes i_{n+1})
\]
we obtain a completely positive unit preserving map which has all the announced properties. \hfill \square

In the following, we will assume \(E_n^\infty = E \circ (i_n \otimes i_{n+1})\) for all \(n \in \mathbb{Z}\) and some Umegaki conditional expectation \(E\). We can also drop the assumption that \(\varphi\) is locally faithful. Clearly, we can repeat the whole computations from the half-line chain. Thus

**Theorem 5.1.** Let \(\varphi\) be a Markovian state with stationary completely positive unit preserving maps \(E_n^\infty\) such that \(E\) is an Umegaki conditional expectation onto \(B_0\) with minimal central projections \((P_j)_j\). Define \(\pi_{jj'}\), \(\eta_{j,j'}\) as above. Further, let \(\mu\) be the law of the classical Markov process \((X_n)\) with transition probabilities \((p_{jj'})_{j,j'}\). Then \((17)\) holds in the sense that

\[
\varphi(\cdots \otimes b_k \otimes \cdots \otimes b_n) = \int \mu(d\theta) \varphi_{\omega}(\cdots \otimes E_{j_k}(b_k) \otimes \cdots \otimes E_{j_1}(b_n) \otimes \cdots) ,
\]

where the states \(\varphi_\omega\) on \(A_\omega = \bigotimes_{n \in \mathbb{Z}} B(H_{j_n})\) are determined as

\[
\varphi_{j_n} = \bigotimes_{n \in \mathbb{Z}} \eta_{j_n, j_{n+1}} .
\]

Conversely, fix \(B_0\), transition probabilities \((\pi_{jj'})_{j,j'}\), and states \(\eta_{j,j'}\) on \(B(H_{j_1} \otimes H_{j_0})\) for all \(j,j' \) with \(q_{jj'} > 0\). Then \((17)\) and \((25)\) define a state \(\varphi\) on \(A\) which is Markovian. In the sense of Lemma 3.1, the structure of a corresponding Umegaki conditional expectation \(E\) is determined by \((20)\).

**Remark 5.1.** A close look at \((24)\) and \((25)\) shows that the structure of Markovian states is invariant under time reversal, mapping \(A_n\) into \(A_{-n}\). This happens in analogy to classical processes, for which the backward and forward Markov properties are equivalent. In the quantum case this is quite unexpected, as there is a considerable asymmetry in the definition of quasi-conditionnal expectations.

So it remains to look at the uniqueness of Markovian states in this more general framework.

**Corollary 5.1.** Suppose \((\pi_{jj'})\) is aperiodic. Then all \(E\) Markovian states are stationary.

If \((\pi_{jj'})\) has period \(p\), any Markovian state is periodic and the dimension of the set of Markovian states is at most \(p\).

**Proof.** It is straightforward, all results depend only on the possible \((\pi_{jj'})_{j,j'}\) Markov processes. \hfill \square

6. A Special Property for \(d = 2\)

There remains the following question. How much restrictive is the assumption that \(E\) is an Umegaki conditional expectation?
Lemma 6.1. Suppose $\mathcal{E}$ is a completely positive unit preserving map on $M_2$ which is also a projection: $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$. Then it is an Umegaki conditional expectation, i.e. its range is a $\sigma$-algebra.

Proof. We assume the contrary. Clearly, $1 \in \text{Fix} \mathcal{E}$. Moreover, there should be another $a \in M_2$ with $a = \mathcal{E}(a)$ not being a multiple of $1$. Without loss of generality we may assume $a$ to be self-adjoint. Thus it has two-point spectrum. Shifting $a$ by a multiple of $1$ and scaling appropriately we achieve that $a$ is a projection. Say, $a = (1 + \sigma_x)/2$, where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. Because we assumed that $\text{Fix} \mathcal{E}$ is not a $\sigma$-algebra there is yet another $b \in \text{Fix} \mathcal{E}$ which is not in the linear hull of $1$ and $a$. Again, we may assume that $b$ is self-adjoint. Using linear algebra we may force $b = \sigma_y$, say. But $\sigma_x^2 = \sigma_y^2 = 1 \in \text{Fix} \mathcal{E}$ and as Lemma 2.1, Eq. (3) shows

$$
\sigma_x = i\sigma_x \mathcal{E}(\sigma_x) = i\mathcal{E}(\sigma_x) \mathcal{E}(\sigma_y) = i\mathcal{E}(\sigma_x \sigma_y) = \mathcal{E}(\sigma_z).
$$

Consequently, $\sigma_z \in \text{Fix} \mathcal{E}$ which forces $\text{Fix} \mathcal{E} = M_2$. This contradiction completes the proof. \hfill \Box

References

Moreover, there should be $\alpha$. Without loss of generality \(2\alpha\) is a projection. Say, \(2\alpha = e_1 + e_2 + e_3\). Because we assumed that \(e_2\) is not in the linear hull of \(e_1, e_3\). Using linear algebra we have

\[
\mathbb{E}(\sigma_2) = \mathbb{E}(e_2) = 0
\]

This contradiction completes the proof.

---