A ROLE OF SINGLETONS IN QUANTUM CENTRAL LIMIT THEOREMS

LUIGI ACCARDI, YUKIHIRO HASHIMOTO AND NOBUAKI OBATA

Abstract. A role of singletons in quantum central limit theorems is studied. A common feature of quantum central limit distributions, the singleton condition which guarantees the symmetry of the limit distributions, is revisited in the category of discrete groups and monoids. Introducing a general notion of quantum independence, the singleton independence which include the singleton condition as an extremal case, we clarify the role of singletons and investigate the mechanism of arising non-symmetric limit distributions.

Introduction

In recent years the notion of statistical independence has been extensively studied in the context of algebraic probability theory with wide applications and we have caught a glimpse of a rich world spreading beyond the classical probability theory. For example, the free independence due to Voiculescu (see e.g., [20]) gives rise to the Wigner semi-circle law as a central limit distribution and is applied to a study of random matrices, of quantum electrodynamics and so forth. There have appeared so far several different approaches to the notion of independence in order to unify the diversity; see e.g., [5], [7], [9], [10], [15], [18]. Having observed the common feature that the limit distributions obtained in these works are always symmetric, we propose in [2] the idea of singleton condition which guarantees the symmetry of the limit distributions. In this paper we shall revisit this condition with some examples arising from discrete
groups and monoids (semigroups with unit). It is proved that the limit distributions obtained from these algebraic structures lie between the Gaussian and the Wigner semi-circle laws in the sense that the moments are bounded by those of them. This is an extension of the result in [13].

On the other hand, there is an interesting example of a limit distribution obtained by means of a certain rescaled limit of the Haagerup states on the free group. The computation was done explicitly and the Ullman family of distributions is obtained [12]. Thus, in a different limit procedure non-symmetric distributions appear. Motivated by this new phenomenon, we introduced the concept of singleton independence in the previous paper [3]. In this paper we clarify the role of singletons and investigate the mechanism of surviving odd moments, or equivalently, of arising a non-symmetric limit distribution. It is noticeable that our singleton independence bears an analytical feature, i.e., is expressed in terms of inequalities. As Bożejko pointed out recently, our results are related to the \( \psi \)-independence of Bożejko and Speicher [7] (see also [6]) and a careful study of the relation will create an application to orthogonal polynomials [1]. Thus the notion of statistical independence in algebraic probability is expected to bring a new interaction with harmonic analysis on discrete graphs or on discrete algebraic structures such as monoids, hypergroups, etc. Partial results are found in [14], [17] and a study in this direction is now in progress.

1. Singleton condition

Let \((\mathcal{A}, \varphi)\) be an algebraic probability space, that is, \(\mathcal{A}\) is a \(*\)-algebra with unit 1 and \(\varphi\) is a state, i.e., a positive linear function on \(\mathcal{A}\). Here by positivity we mean that

\[
\varphi \left( \sum_{i=1}^{n} a_i^* a_i \right) \geq 0
\]

for any choice of \(n \geq 1\) and \(a_1, \ldots, a_n \in \mathcal{A}\). For analytic argument we need the following

DEFINITION 1.1. A family of sequences \(\{b^{(j)}_n\}_{n=1}^{\infty}, j \in J\), where \(b^{(j)}_n \in \mathcal{A}\), is said to satisfy the condition of boundedness of the mixed momenta if for each \(k \geq 1\) there exists a positive constant \(\nu_k \geq 0\).
such that

\[(1.1) \quad \left| \varphi \left( b_{n_1}^{(j_1)} \cdots b_{n_k}^{(j_k)} \right) \right| \leq \nu_k \]

for any choice of \(j_1, \ldots, j_k \in J\) and \(n_1, \ldots, n_k \in \mathbb{N}\).

Now we come to

**DEFINITION 1.2.** A family of sequences \(\{b^{(j)} = (b_n^{(j)})_{n=1}^{\infty} ; j \in J\}\), where \(b_n^{(j)} \in \mathcal{A}\) with mean zero, is said to satisfy the singleton condition (with respect to \(\varphi\)) if

\[(1.2) \quad \varphi \left( b_{n_1}^{(j_1)} \cdots b_{n_k}^{(j_k)} \right) = 0 \]

for any choice of \(k \geq 1, j_1, \ldots, j_k \in J\), and \(n_1, \ldots, n_k \in \mathbb{N}\) with a certain \(n_s\) being different from all other ones.

Given a sequence \(b = (b_n)_{n=1}^{\infty} \subset \mathcal{A}\) we put

\[S_N(b) = \sum_{n=1}^{N} b_n.\]

By a combinatorial argument modeled after von Waldenfels (see e.g., [10]), we come to the following

**LEMMA 1.3.** [2] Assume that \(\{b^{(j)} = (b_n^{(j)})_{n=1}^{\infty} ; j \in J\}\), where \(b_n^{(j)} \in \mathcal{A}\) with mean \(\varphi(b_n^{(j)}) = 0\), satisfies the condition of boundedness of the mixed momenta. Then, for any \(\alpha > 0\) it holds that

\[(1.3) \quad \lim_{N \to \infty} \varphi \left( \frac{S_N(b^{(1)})}{N^\alpha} \cdot \frac{S_N(b^{(2)})}{N^\alpha} \cdots \frac{S_N(b^{(k)})}{N^\alpha} \right) = \sum_{\alpha k \leq \rho \leq k} \sum_{\sigma: \{1, \ldots, \rho\} \to \{1, \ldots, \rho\} \text{ surjective}} \sum_{\varphi: \{1, \ldots, n\} \to \{1, \ldots, n\} \text{ order-preserving}} \varphi \left( b_{\sigma(1)}^{(1)} \cdots b_{\sigma(k)}^{(k)} \right) \]

in the sense that one limit exists if and only if the other does and the limits coincide. Moreover, assume that the singleton condition is satisfied. Then

\[(1.4) \quad \lim_{N \to \infty} \varphi \left( \frac{S_N(b^{(1)})}{N^\alpha} \cdot \frac{S_N(b^{(2)})}{N^\alpha} \cdots \frac{S_N(b^{(k)})}{N^\alpha} \right) = 0\]
takes place if \( \alpha > 1/2 \) or if \( \alpha = 1/2 \) and \( k \) is odd. If \( \alpha = 1/2 \) and \( k = 2n \) is even,

\[
\lim_{N \to \infty} \varphi \left( \frac{S_N(b^{(1)})}{N^{1/2}} \cdot \frac{S_N(b^{(2)})}{N^{1/2}} \cdots \frac{S_N(b^{(2n)})}{N^{1/2}} \right)
\]

(1.5) \[
= \lim_{N \to \infty} N^{-n} \sum_{\pi: \{1, \ldots, 2n\} \to \{1, \ldots, N\}} \sum_{\sigma: \{1, \ldots, n\} \to \{1, \ldots, N\}} \varphi \left( b_{\sigma \pi(1)}^{(1)} \cdots b_{\sigma \pi(2n)}^{(2n)} \right),
\]

and the Gaussian bound takes place:

\[
\limsup_{N \to \infty} \left| \varphi \left( \frac{S_N(b^{(1)})}{N^{1/2}} \cdot \frac{S_N(b^{(2)})}{N^{1/2}} \cdots \frac{S_N(b^{(2n)})}{N^{1/2}} \right) \right| \leq \frac{(2n)!}{2^n n!} \nu_{2n},
\]

where \( \nu_k \) appears in (1.1).

The existence of the limit such as (1.5) can be discussed in terms of the entangled ergodic theorem [3]; however, little is known about the convergence, in this connection see [16].

2. Minimal generators of a discrete group

A symmetric random walk on a discrete group provides a geometric interpretation of the singleton condition. Let \( G \) be a discrete group and \( (\pi, \mathcal{H} = \ell^2(G)) \) the left regular representation:

\[
\pi(g)\xi(h) = \xi(g^{-1}h), \quad g, h \in G, \quad \xi \in \mathcal{H}.
\]

Let \( \delta_g = \chi_{\{g\}} \) be the characteristic function of \( \{g\} \) and define a state \( \varphi \) on \( \mathcal{B}(\mathcal{H}) \) by

\[
\varphi(a) = \langle a\delta_e, \delta_e \rangle,
\]

where \( e \) is the unit of \( G \). Let \( \mathcal{A} \) be the \( * \)-algebra generated by \( \{\pi(g) ; g \in G\} \) and consider the algebraic probability space \( (\mathcal{A}, \varphi) \).

**Definition 2.1.** Let \( G \) be a discrete group and \( \Sigma \subset G \) a subset which generates \( G \), i.e., \( G = \langle \Sigma \rangle \equiv \{ s_1^{e_1} \cdots s_n^{e_n} ; s_i \in \Sigma, e_i \in \{\pm 1\}, n \geq 0 \} \). Then \( \Sigma \) is called minimal if no proper subset of \( \Sigma \) generates the whole \( G \).

**Theorem 2.2.** Let \( G \) be a discrete group equipped with countably infinite generators \( \Sigma = \{g_1, g_2, \ldots \} \) and consider the sequence of algebraic random variables \( a_n = \pi(g_n) \). If \( \Sigma \) is minimal, \( \{(a_n)_{n=1}^\infty, (a_n^*)_{n=1}^\infty \} \) satisfies the singleton condition with respect to \( \varphi \).
Proof. Consider a product
\[ u = a_{i_1}^{\epsilon_1} \cdots a_{i_s}^{\epsilon_s} \cdots a_{i_n}^{\epsilon_n}, \quad \epsilon_k \in \{+1, *, -1\} \]
and assume that an index \( i_s \) appears only once. It is sufficient to prove that \( \varphi(u) = 0 \). For each \( k \) take \( g_{i_k} \in \Sigma \) such that \( a_{i_k} = \pi(g_{i_k}) \). Since \( \pi \) is a unitary representation, we need only to show that \( g_{i_1}^{\eta_1} \cdots g_{i_n}^{\eta_n} \neq e \), where \( \eta_k = +1 \) or \( = -1 \) according as \( \epsilon_k = +1 \) or \( = * \). If \( g_{i_1}^{\eta_1} \cdots g_{i_n}^{\eta_n} = e \) happens, \( g_i \) can be expressed as a product of others and hence \( G = \langle \Sigma \setminus \{g_i\} \rangle \). This is contradiction by the minimality of \( \Sigma \). \( \square \)

Notations and assumptions being the same as in Theorem 2.2, we consider real quantum random variables:
\[ b_n = a_n + a_n^{*}. \]
Then \( \phi(b_n) = 0 \) and \( \phi(b_n^2) = 2 + 2\delta_e(g_n^2) \). For simplicity assume that \( g_n^2 \neq e \) for any \( g_n \in \Sigma \). Then, in connection with central limit theorems we are interested in the asymptotic behavior of
\[ \lim_{N \to \infty} \frac{b_1 + b_2 + \cdots + b_N}{\sqrt{2N}}. \]
Here we mention the following result on bounds of the momenta.

**Proposition 2.3.** Let \( G \) be a discrete group with minimal generators \( \Sigma = \{g_1, g_2, \cdots \} \). Assume that \( g_n^2 \neq e \) for any \( g_n \in \Sigma \). Then for the sequence of real quantum random variables \( b_n = \pi(g_n) + \pi(g_n)^{*} \) we have

\[ \limsup_{N \to \infty} \varphi \left( \left( \frac{b_1 + b_2 + \cdots + b_N}{\sqrt{2N}} \right)^{2n} \right) \leq \frac{(2n)!}{2^n n!} \tag{2.1} \]

and

\[ \liminf_{N \to \infty} \varphi \left( \left( \frac{b_1 + b_2 + \cdots + b_N}{\sqrt{2N}} \right)^{2n} \right) \geq \frac{(2n)!}{(n + 1)! n!}. \tag{2.2} \]

The Gaussian bound (2.1) follows from (1.6) in Lemma 1.3 and is achieved, for example, by the free abelian group with countably infinite generators. For the proof of the Wigner bound (2.2) we rely on the universal property of free groups. By using the canonical homomorphism \( p : (F_\infty, \{f_i\}) \to (G, \Sigma) \) with \( p(f_i) = g_i \), where \( F_\infty \) is the free group generated by \( \{f_i\} \), we may estimate the number of the return paths,
and hence the momenta, see [13] for details. The right-hand side of (2.2) coincides with the moments of the Wigner semi-circle law:

\[
\frac{1}{2\pi} \int_{-2}^{2} t^{2n} \sqrt{4 - t^2} \, dt = \frac{(2n)!}{n!(n + 1)!},
\]

which is obtained from the symmetric random walk on a free group, see e.g., [19].

3. Minimal generators of a discrete monoid

The discussion in the previous section can be extended to a discrete monoid (semigroup with unit). Let \( G \) be a monoid with unit \( e \), that is, \( ge = eg = g \) for any \( g \in G \). Let \( \mathcal{H} = \ell^2(G) \) and for \( \xi \in \mathcal{H} \) put

\[
\pi(g)\xi(h) = \xi(gh), \quad g, h \in G.
\]

For boundedness of \( \pi(g) \) we need notation. For \( g \in G \) and \( S \subset G \) we put

\[
R(g \to S) = \{ x \in G; gx \in S \}.
\]

If \( S = \{ h \} \) we write simply \( R(g \to h) \) for \( R(g \to \{ h \}) \).

**Lemma 3.1.** \( \pi(g) \in \mathcal{B}(\mathcal{H}) \) if and only if \( \sup_{h \in G} |R(g \to h)| < \infty \), where \( | \cdot | \) denotes the cardinality. In that case

(3.1) \[ \| \pi(g) \|^2 = \sup_{h \in G} |R(g \to h)|. \]

**Proof.** By definition we have

\[
\| \pi(g)\xi \|^2 = \sum_{x \in G} |\pi(g)\xi(x)|^2 = \sum_{x \in G} |\xi(gx)|^2
\]

\[
= \sum_{h \in G} |\{ x \in G; gx = h \}||\xi(h)||^2 = \sum_{h \in G} |R(g \to h)||\xi(h)||^2.
\]

Hence \( \| \pi(g) \|^2 \leq \sup_{h \in G} |R(g \to h)| \). Equality (3.1) is examined by taking \( \xi = \delta_x, x \in G \).

From now on we assume that a monoid \( G \) under consideration satisfies the condition:

\[
\sup_{h \in G} |R(g \to h)| < \infty \quad \text{for any } g \in G.
\]

Let \( \mathcal{A} \) be a *-algebra generated by \( \{ \pi(g); g \in G \} \subset \mathcal{B}(\mathcal{H}) \) and let \( \varphi \) be the state defined by \( \varphi(a) = \langle a\delta_e, \delta_e \rangle, a \in \mathcal{A} \).
LEMMA 3.2. For $S \subset G$ it holds that

\[(3.2) \quad \pi(g)\chi_S = \chi_{R(g \rightarrow S)},\]

\[(3.3) \quad \pi(g)^*\chi_S(x) = |S \cap R(g \rightarrow x)|.\]

In particular,

\[(3.4) \quad \pi(g)^*\delta_h = \delta_{gh}.\]

Proof. In view of

\[\pi(g)\chi_S(x) = \chi_S(gx) = \begin{cases} 1, & gx \in S, \\ 0, & \text{otherwise}, \end{cases}\]

we come to (3.2). Next, by definition we have

\[\pi(g)^*\chi_S(x) = \langle \pi(g)^*\chi_S, \delta_x \rangle = \langle \chi_S, \pi(g)\delta_x \rangle \]

\[= \sum_{h \in S} \pi(g)\delta_x(h) = \sum_{h \in S} \delta_x(gh) = |S \cap R(g \rightarrow x)|,\]

which proves (3.3). Setting $S = \{h\}$ in (3.3), we obtain

\[\pi(g)^*\delta_h(x) = |\{h\} \cap R(g \rightarrow x)| = \begin{cases} 1, & gh = x, \\ 0, & \text{otherwise}, \end{cases}\]

from which (3.4) follows immediately. \qed

For a subset $S \subset G$ we denote by $\langle S \rangle$ the smallest submonoid of $G$ which contains $S$ and $e$. For a monoid $G$ we consider the condition:

(A) $R(a \rightarrow b) \subset \langle a, b \rangle$ for any $a, b \in G$.

This is equivalent to the following

(A') $R(a \rightarrow S) \subset \langle a, S \rangle$ for any $a \in G$ and $S \subset G$.

Under condition (A), for any $a, b \in G$ the solutions to the equation $ax = b$ belong to $\langle a, b \rangle$ whenever they exist. Note that, in general, an element of a monoid may have more than one inverse.

LEMMA 3.3. Let $G$ be a monoid satisfying (A). Then, for any $g_1, \cdots, g_n \in G$ and $\epsilon_1, \cdots, \epsilon_n \in \{+1, *\}$ we have

\[(3.5) \quad \text{supp} \, \pi(g_1)^{\epsilon_1} \cdots \pi(g_n)^{\epsilon_n}\delta_c \subset \langle g_1, g_2, \cdots, g_n \rangle.\]
**Proof.** We prove the assertion by induction on \( n \). For \( n = 1 \) the statement is rather obvious. In fact, by (3.2) we see that \( \pi(g_1)\delta_e = \chi_{R(g_1 \to e)} \), and hence by condition (A) we have

\[
\text{supp } \pi(g_1)\delta_e = R(g_1 \to e) \subset \langle g_1 \rangle.
\]

On the other hand, by (3.4) we have \( \pi(g_1)^*\delta_e = \delta_{g_1} \) and

\[
\text{supp } \pi(g_1)^*\delta_e = \{g_1\} \subset \langle g_1 \rangle.
\]

Thus (3.5) is valid for \( n = 1 \).

Suppose next that the statement is valid up to \( n - 1 \). Write

\[
\pi(g_1)^{r_1}\pi(g_2)^{r_2} \cdots \pi(g_n)^{r_n}\delta_e = \pi(g_1)^{r_1} \left( \pi(g_2)^{r_2} \cdots \pi(g_n)^{r_n}\delta_e \right) \equiv \pi(g_1)^{r_1}\Psi.
\]

By the assumption of induction \( W \equiv \text{supp } \Psi \subset \langle g_2, \cdots, g_n \rangle \). If \( W = \emptyset \), \( \Psi = 0 \) and (3.5) is obvious. Suppose that \( W \neq \emptyset \). Then

\[
\Psi = \sum_{h \in W} c_h \delta_h.
\]

Now consider \( \pi(g_1)\Psi \). In view of (3.2) and condition (A) we see that

\[
\text{supp } \pi(g_1)\Psi \subset \bigcup_{h \in W} R(g_1 \to h) \subset \bigcup_{h \in W} \langle g_1, h \rangle \subset \langle g_1, W \rangle \subset \langle g_1, g_2, \cdots, g_n \rangle.
\]

As for \( \pi(g_1)^*\Psi \), by (3.4) we obtain

\[
\pi(g_1)^*\Psi = \sum_{h \in W} c_h \delta_{g_1 h}
\]

and hence

\[
\text{supp } \pi(g_1)^*\Psi \subset g_1 W \subset \langle g_1, g_2, \cdots, g_n \rangle.
\]

Thus (3.5) is also valid for \( n \). \( \square \)

During the above proof we have seen that

\[
\pi(g_1)^{r_1} \cdots \pi(g_n)^{r_n}\delta_e = 0 \quad \text{or} \quad \sum_{w \in W} c_w \delta_w,
\]

where \( \emptyset \neq W \subset \langle g_1, g_2, \cdots, g_n \rangle \) and \( c_w \in \mathbb{N} \).

**Definition 3.4.** Let \( G \) be a monoid. We say that a subset \( \Sigma \subset G \) is a set of generators if \( \langle \Sigma \rangle = G \). A set of generators \( \Sigma \) is called minimal if \( B \langle A \rangle \cap \langle A \rangle = \emptyset \) for any pair of non-empty subsets \( A, B \subset \Sigma \) with \( A \cap B = \emptyset \).

A set of generators \( \Sigma \) is minimal if and only if for any \( s \in \Sigma \) and \( b \in \langle \Sigma \setminus \{s\} \rangle \), the equation \( sx = b \) has no solution in \( \langle \Sigma \setminus \{s\} \rangle \).
THEOREM 3.5. Let $G$ be a monoid satisfying condition (A), and let $\Sigma \subset G$ be a countable infinite set of generators $\Sigma = \{g_1, g_2, \cdots \}$. Define a sequence of quantum random variables by $a_j = \pi(g_j)$. If $\Sigma$ is minimal, then $\{(a_j)_{j=1}^\infty, (a^*_j)_{j=1}^\infty\}$ satisfies the singleton condition.

Proof. Consider the product

$$x = a^{\epsilon_1}_{i_1} \cdots a^{\epsilon_j}_{j_1} a^*_{a^{\eta_1}_{j_1}} \cdots a^*_{a^{\eta_m}_{j_m}}, \quad \epsilon_k, \epsilon, \eta_k \in \{+1, *\},$$

and assume that $s \neq i_1, \cdots, i_j, j_1, \cdots, j_m$. Since the argument is similar, we may assume $\epsilon = *$. Then

$$\varphi(x) = (a^*_s a^{\eta_1}_{j_1} \cdots a^{\eta_m}_{j_m} \delta_0, a^{\epsilon_1}_{i_1} \cdots a^{\epsilon_j}_{j_1} \delta_\epsilon).$$

From Lemma 3.3 it follows that both $\text{supp} (a^{\epsilon_1}_{i_1} \cdots a^{\epsilon_j}_{j_1} \delta_\epsilon)$ and $\text{supp} (a^{\epsilon_1}_{i_1} \cdots a^{\epsilon_j}_{j_1} \delta_\epsilon)$ are contained in $\langle g_{i_1}, \cdots, g_{i_j}, j_1, \cdots, j_m \rangle \subset \langle \Sigma \setminus \{g_s\} \rangle$. Hence

$$\text{supp} (a^{\epsilon_1}_{i_1} \cdots a^{\epsilon_j}_{j_1} \delta_\epsilon) \subset \langle \Sigma \setminus \{g_s\} \rangle,$$

and

$$\text{supp} (a^{\epsilon_1}_{i_1} \cdots a^{\epsilon_j}_{j_1} \delta_\epsilon) \subset g_s \langle \Sigma \setminus \{g_s\} \rangle.$$

Then, by the minimality of $\Sigma$ we see that $\varphi(x) = 0$ for (3.6) is the inner product of two functions with disjoint supports. 

Notations and assumptions being the same as in Theorem 3.5, we consider real quantum random variables:

$$b_n = \pi(g_n) + \pi(g_n)^* = a_n + a^*_n.$$

Then, $\varphi(b_n) = 0$ and $\varphi(b_n^2) = 1 + 2 \delta_\epsilon(g_n^2) + |R(g_n \to e)|$. For simplicity we assume that $g_n^2 \neq e$ and $R(g_n \to e) = 0$ for all $n \geq 1$. We are then interested in

$$\lim_{N \to \infty} \frac{b_1 + b_2 + \cdots + b_N}{\sqrt{N}}.$$

The following result is compared with Proposition 2.3.

PROPOSITION 3.6. Notations and assumptions being the same as above, we have

$$\limsup_{N \to \infty} \varphi \left( \left( \frac{b_1 + b_2 + \cdots + b_N}{\sqrt{N}} \right)^{2n} \right) \leq \frac{(2n)!}{2^{2n} n!}$$

and

$$\liminf_{N \to \infty} \varphi \left( \left( \frac{b_1 + b_2 + \cdots + b_N}{\sqrt{N}} \right)^{2n} \right) \geq \frac{(2n)!}{(n + 1)n!}.$$
Proof. We need only to show the Wigner bound (3.8). To this end it is sufficient to show that \( \varphi(x) \geq 1 \) for any \( x = a_{i_1} \ldots a_{i_{2n}} \) such that \( (i_1, \ldots, i_{2n}) \) forms a non-crossing pair-partition with \( \epsilon_p = +1 \) and \( \epsilon_q = * \) for each pair \( (i_p, i_q) \) with \( p < q \). In fact, one has

\[
\pi(g)\pi(g)\delta_z = \pi(g)\delta_{g2} = \chi_{R(g \rightarrow g2)} = \delta_z + \sum_{w \in R(g \rightarrow g2) \setminus \{z\}} \delta_w,
\]

whence \( \varphi(x) \geq 1 \) by induction. \( \square \)

4. Singleton independence

We need some combinatorial notion. For \( \alpha = (j, \epsilon) \in \mathbb{N} \times \{+1, *\} \) we put

\[
\alpha^* = \begin{cases} (j, *), & \text{if } \epsilon = +1, \\ (j, +1), & \text{if } \epsilon = *.
\end{cases}
\]

DEFINITION 4.1. Consider a finite sequence \( \alpha_1 \ldots \alpha_m \), where \( \alpha_p \in \mathbb{N} \times \{+1, *\} \). Then \( \alpha_s \) is called a singleton if \( \alpha_s \neq \alpha^*_s \) for any \( k \neq s \). A singleton \( \alpha_s \) is called outer if \( \alpha_p \neq \alpha^*_q \) for any \( p < s < q \), and is called inner if \( \alpha_p = \alpha^*_q \) for some \( p < s < q \).

For example, consider the product \( \alpha_1 \alpha_2 \alpha_3 \alpha_2 \). The second \( \alpha_2 \) is an inner singleton and the forth \( \alpha_3 \) and the last \( \alpha_2 \) are outer singletons. Notice that both \( \alpha_2 \)'s are singletons though appearing twice.

Given a sequence \( \{g_j\}_{j=1}^{\infty} \) of elements of \( \ast \)-algebra \( \mathcal{A} \), for \( \alpha \in \mathbb{N} \times \{+1, *\} \) we put

\[
g_\alpha = \begin{cases} g_j & \text{if } \alpha = (j, +1) \\ g^*_j & \text{if } \alpha = (j, *).
\end{cases}
\]

DEFINITION 4.2. Let \( \mathcal{A} \) be a \( \ast \)-algebra and let \( \{\varphi_\gamma; 0 \leq \gamma \leq \overline{\gamma}\} \) be a family of states on \( \mathcal{A} \). \( \overline{\gamma} > 0 \). Let \( \{g_j\}_{j=1}^{\infty} \) be a sequence of elements of \( \mathcal{A} \) such that \( \varphi_\gamma (g_\alpha) = \gamma \) for all \( \alpha \in \mathbb{N} \times \{+1, *\} \). Then the sequence \( \{g_j\} \) is called singleton independent with respect to \( \varphi_\gamma \) if for any \( k \geq 1 \) there exists \( M_k \geq 0 \) such that

\[
|\varphi_\gamma (g_{a_1} \cdots g_{a_k})| \leq \gamma M_k |\varphi_\gamma (g_{a_1} \cdots g_{a_k} |),
\]

(4.1)
for any choice of $\alpha_1, \ldots, \alpha_k$ with some $\alpha_s$ being a singleton for $\alpha_1 \cdots \alpha_k$. Here $\tilde{g}_\alpha$ stands for the omission. (The case of $\gamma = 0$ is related to the singleton condition, see Definition 1.2.)

By repeated application of (4.1) we come to

\begin{equation}
|\varphi_{\gamma}(g_{a_1} \cdots g_{a_k})| \leq \gamma^s \tilde{M}_k |\varphi_{\gamma}(g_{\beta_1} \cdots g_{\beta_{k-s}})|,
\end{equation}

for any choice of $g_{a_1}, \ldots, g_{a_k}$ with $s$ singletons, where $\tilde{M}_k = M_k M_{k-1} \cdots M_1$ and $\beta_1 \cdots \beta_{k-s}$ is obtained from $\alpha_1 \cdots \alpha_k$ by removing the $s$ singletons.

In the following we assume that the boundedness condition (1.1) is fulfilled for any $\varphi_{\gamma}$ uniformly in $\gamma$. Namely, for each $k \geq 1$ there exists $C_k \geq 0$ such that

\begin{equation}
|\varphi_{\gamma}(g_{a_1} \cdots g_{a_k})| \leq C_k
\end{equation}

for any choice of $g_{a_1}, \ldots, g_{a_k}$ and $0 \leq \gamma \leq \overline{\gamma}$.

When $\varphi_{\gamma}$ is fixed we write $\tilde{g}_{\alpha} = g_{\alpha} - \gamma$ so that $\varphi_{\gamma}(\tilde{g}_{\alpha}) = 0$. Put

$$S^\epsilon_N = \sum_{j=1}^{N} \tilde{g}^\epsilon_j, \quad \epsilon \in \{+1, \ast\}.$$

Throughout we fix $k \geq 1$ and $\epsilon_1, \ldots, \epsilon_k \in \{+1, \ast\}$ and consider the product

\begin{equation}
S^\epsilon_{N_1} \cdots S^\epsilon_{N_k} = \sum_{j_1, \ldots, j_k=1}^{N} \tilde{g}_{j_1}^{\epsilon_1} \cdots \tilde{g}_{j_k}^{\epsilon_k} = \sum_{\alpha \in A_N} \tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k},
\end{equation}

where $A_N = A_N(\epsilon_1, \ldots, \epsilon_k)$ is the set of maps $\alpha : \{1, \cdots, k\} \rightarrow \{1, \cdots, N\} \times \{+1, \ast\}$ such that the second component of $\alpha_l$ coincides with the given $\epsilon_l$, $1 \leq l \leq k$. Let $\pi : \{1, \cdots, N\} \times \{+1, \ast\} \rightarrow \{1, \cdots, N\}$ be the projection defined by $\pi(j, \epsilon) = j$ and put $\overline{\alpha} = \pi \circ \alpha$.

Each $\alpha \in A_N$ determines a partition of $\{1, \cdots, k\}$ by the inverse image of $\overline{\alpha}$. Thereby the sum (4.4) over $A_N$ is divided according to the cardinality of the inverse image of $\overline{\alpha}$. Let $\mathcal{P}_{k,p}$ be the collection of partitions of $\{1, \cdots, k\}$ into a disjoint union of $p$ non-empty subsets. For $(S_1, \ldots, S_p) \in \mathcal{P}_{k,p}$ we denote by $[S_1, \cdots, S_p]$ the set of $\alpha \in A_N$ such that $\overline{\alpha}$ is constant on each $S_j$ and takes different values on different $S_j$'s.
With these notations we have

\[
\varphi_{\lambda/\sqrt{N}} \left( \frac{S_N^1 \cdots S_N^k}{\sqrt{N}} \right)
\]

(4.5) \[
= N^{-k/2} \sum_{\alpha \in A_N} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k})
\]

\[
= N^{-k/2} \sum_{p=1}^{k} \sum_{(S_1, \cdots, S_p) \in \mathcal{P}_{k,p}} \sum_{\alpha \in [S_1, \cdots, S_p]} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k})
\]

and the large \( N \) asymptotics is in question.

If \( \alpha \in [S_1, \cdots, S_p] \) and \( S_1 = \{l\} \), then \( \alpha_l \equiv (j_l, c_l) \) is a singleton in the sequence \( \alpha_1 \cdots \alpha_k \). For the index \( j_l \) appears only once in \( j_l, \cdots, j_k \).

**Lemma 4.3.** For \( 0 \leq s \leq k \) let \( \mathcal{P}_{k,p}^s \) denote the set of partitions \( (S_1, \cdots, S_p) \in \mathcal{P}_{k,p} \) with \( s \) singletons, i.e., \( |\{i; |S_i| = 1\}| = s \). Then it holds that \( p \leq (k + s)/2 \). Moreover, if \( p < (k + s)/2 \) then

\[
\lim_{N \to \infty} N^{-k/2} \sum_{(S_1, \cdots, S_p)} \sum_{\alpha \in [S_1, \cdots, S_p]} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}) = 0.
\]

**Proof.** For \( (S_1, \cdots, S_p) \in \mathcal{P}_{k,p}^s \) we have

\[
k = \sum_{j=1}^{p} |S_j| = \sum_{|S_j| \geq 2} |S_j| + s \geq 2(p - s) + s = 2p - s.
\]

Then, in view of (4.2) and (4.3), we see that

\[
N^{-k/2} \sum_{(S_1, \cdots, S_p) \in \mathcal{P}_{k,p}^s} \sum_{\alpha \in [S_1, \cdots, S_p]} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k})
\leq N^{-k/2} \mathcal{P}_{k,p}^s \frac{N^p}{p!} \tilde{M}_k C_{k-s} \left( \frac{\lambda}{\sqrt{N}} \right)^s
= \tilde{M}_k C_{k-s} \frac{\lambda^s}{p!} N^{-(k+s)/2}.
\]

The last quantity goes to 0 as \( N \to \infty \) if \( p < (k + s)/2 \), thereby (4.6) follows.

It follows from Lemma 4.3 that the non-trivial contribution to the limit of (4.5) comes from those partitions \( (S_1, \cdots, S_p) \in \mathcal{P}_{k,p}^s \) satisfying \( p = (k + s)/2 \), that is, \( k = 2p - s \). In that case, \( 1 \leq |S_j| \leq 2 \) for all \( j \).

In fact, if \( |S_1| \geq 3 \), we have

\[
k \geq 3 + \sum_{j \geq 2 \mid S_j \geq 2} |S_j| + s \geq 3 + 2(p - s - 1) + s = 2p - s + 1,
\]
which is incompatible with \( k = 2p - s \). Taking this into account, let \( A_N' \) denote the set of \( \alpha \in A_N \) which determines a pair-partition with singletons, i.e., the corresponding partition \((S_1, \ldots, S_s, T_1, \ldots, T_t)\) of \( \{1, 2, \ldots, k\} \) satisfies:

\[
|S_j| = 1, \quad |T_j| = 2, \quad s + 2t = k, \quad s \geq 0, \quad t \geq 0.
\]

Then

\[
\lim_{N \to \infty} \varphi_{\lambda/\sqrt{N}} \left( \frac{S_1^a}{\sqrt{N}} \cdots \frac{S_s^a}{\sqrt{N}} \right) = \lim_{N \to \infty} N^{-k/2} \sum_{\alpha \in A_N'} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}).
\]

Let \( \alpha \in A_N' \) and \((S_1, \ldots, S_s, T_1, \ldots, T_t)\) the corresponding partition as in (4.7). Then \( \alpha \in [S_1, \ldots, S_s, T_1, \ldots, T_t] \). Going back to (4.2), we have

\[

\sum_{\alpha \in [S_1, \ldots, S_s, T_1, \ldots, T_t]} \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}) \right| \leq \tilde{M}_k \left( \frac{\lambda}{\sqrt{N}} \right)^s \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\beta_1} \cdots \tilde{g}_{\beta_{2t}}) \right|,
\]

where \((\beta_1, \ldots, \beta_{2t})\) is obtained from \((\alpha_1, \ldots, \alpha_k)\) by removing the singletons. Then

\[
N^{-k/2} \sum_{\alpha \in [S_1, \ldots, S_s, T_1, \ldots, T_t]} \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}) \right| \leq N^{-k/2} \frac{N^s}{s!} \frac{N^t}{t!} \tilde{M}_k \left( \frac{\lambda}{\sqrt{N}} \right)^s \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\beta_1} \cdots \tilde{g}_{\beta_{2t}}) \right|
\]

\[
= \frac{\lambda^s}{s! t!} \tilde{M}_k \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\beta_1} \cdots \tilde{g}_{\beta_{2t}}) \right|.
\]

In general, a pair-partition \((T_1, \ldots, T_t)\) of \( \{1, 2, \ldots, 2t\} \) is called negligible if there exists \( C \geq 0 \) such that

\[
\left| \varphi_{\gamma}(\tilde{g}_{\beta_1} \cdots \tilde{g}_{\beta_{2t}}) \right| \leq C \gamma
\]

holds for any \( \beta : \{1, 2, \ldots, 2t\} \to N \times \{+1, *\} \) such that \( \beta \in [T_1, \ldots, T_t] \). We say that \( \alpha \in A_N' \) is negligible if the pair-partition \( \beta \) determined by \( \alpha \) as above is negligible. In that case (4.10) becomes

\[
N^{-k/2} \sum_{\alpha \in [S_1, \ldots, S_s, T_1, \ldots, T_t]} \left| \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}) \right| \leq \frac{\lambda^s}{s! t!} \tilde{M}_k C \frac{\lambda}{\sqrt{N}},
\]

which goes to 0 as \( N \to \infty \).

In conclusion, we state the following
THEOREM 4.4. Let $\mathcal{A}$ be a $*$-algebra with a family of states $\{\varphi_\gamma; 0 \leq \gamma \leq \bar{\gamma}\}, \bar{\gamma} > 0$. Let $\{g_j\}_{j=1}^\infty$ be a sequence of elements of $\mathcal{A}$ satisfying the singleton independence (4.1) and the uniform boundedness (4.3). Then for any $k \geq 1$ and $\epsilon_1, \cdots, \epsilon_k \in \{+1, *\}$ we have
\[(4.12)\]
$$\lim_{N \to \infty} \varphi_{\lambda/\sqrt{N}} \left(\frac{S_{N_1}}{\sqrt{N}} \cdots \frac{S_{N_k}}{\sqrt{N}}\right) = \lim_{N \to \infty} N^{-k/2} \sum_{\alpha \in \mathcal{A}_N^N} \varphi_{\lambda/\sqrt{N}}(\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k}),$$
where $A_N^N$ denotes the set of $\alpha \in A_N$ determining a non-negligible pair-partition with singletons.

5. Haagerup states on the free group

Let $F_\infty$ be the free group generated by $\{g_1, g_2, \cdots\}$. Each $x \in F_\infty$, $x \neq e$, admits a unique expression of the form:
$$x = g_{\alpha_1} \cdots g_{\alpha_n}, \quad \alpha_i \neq \alpha_{i+1}, \quad 1 \leq i \leq n - 1,$$
where $g_\alpha = g_j^\epsilon$ for $\alpha = (j, \epsilon) \in \mathbb{N} \times \{\pm 1\}$. In that case $n$ is called the length of $x$ and we write $|x| = n$. For a general theory of length functions see e.g., [4], [8]. Let $\mathcal{A}$ be the group $*$-algebra associated with $F_\infty$, where $g_j^\epsilon = g_j^{-1}$. For each $0 \leq \gamma \leq 1$ there exists a state $\varphi_\gamma$ on $\mathcal{A}$ uniquely determined by
$$\varphi_\gamma(x) = \gamma^{|x|}, \quad x \in F_\infty.$$This $\varphi_\gamma$ is called the Haagerup state, see [11].

The two sequences $\{(g_j)_{j=1}^\infty, (g_j^{-1})_{j=1}^\infty\}$ satisfy the singleton condition (cf. Section 1) with respect to the Haagerup state $\varphi_\gamma$ only when $\gamma = 0$; while the singleton independence and the uniform boundedness (cf. Section 4) hold. In fact, the idea of the singleton independence was motivated by the Haagerup states. In this concrete case (4.12) is computed explicitly, where a pair-partition with singletons is negligible if there appears a crossing pair.

DEFINITION 5.1. Assume that a product $\tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_n}$ contains $s \geq 0$ inner singletons and no outer singletons. Let $\alpha_{j_1}, \cdots, \alpha_{j_s}$ be the suffices which correspond the singletons and denote the rest by $\beta_1, \cdots, \beta_{k-s}$ in order. We say that the product satisfies the condition (NCI) if $g_{\beta_1} \cdots g_{\beta_{k-s}} = e$. For any $k \geq 1$ and $\epsilon_1, \cdots, \epsilon_k \in \{+1, *\}$ let
A role of singletons in quantum central limit theorems

NCI_k(s; \epsilon_1, \cdots, \epsilon_k) be the set of equivalence classes of products \( \tilde{g}_{\alpha_1} \cdots \tilde{g}_{\alpha_k} \) which consist of \((k - s)/2\) non-crossing pairs and of \(s\) inner singletons.

**Theorem 5.2.** Let \(k \geq 1\) and \(\epsilon_1, \cdots, \epsilon_k \in \{+1, \ast\}\). For the Haagerup states \(\{\varphi_\gamma\}\) on the free group \(F_\infty\) it holds that

\[
\lim_{N \to \infty} \varphi_{\lambda/\sqrt{N}} \left( \frac{S_N^{\epsilon_1}}{\sqrt{N}} \cdots \frac{S_N^{\epsilon_k}}{\sqrt{N}} \right) = \sum_{\epsilon = 0}^{k-2} (-\lambda)^\epsilon \cdot |NCI_k(s; \epsilon_1, \cdots, \epsilon_k)|.
\]

For further study concerning the above result see [3]. Another examples of the singleton independence are known from the unitary representations of free groups.

**Acknowledgements.** The authors are grateful to Professor M. Bożejko for stimulating conversation. The second named author expresses his sincere thanks to Professor D. M. Chung and Dr. U. C. Ji for their hospitality during the SPA conference at Taejon in February 1998.

**References**


Luigi Accardi, Yukihiro Hashimoto, and Nobuaki Obata
Graduate School of Polymathematics
Nagoya University
Nagoya 464-8602, Japan

Luigi Accardi
Centro Vito Volterra
Università di Roma "Tor Vergata"
Roma 00133, Italia