A NOTE ON MEYER' S NOTE

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Contents

1 NOTATIONS AND STATEMENT OF THE PROBLEM

Let us denote

- $\Gamma(L^2(\mathbf{R}_+)$ the Boson Fock space over the one-particle space $L^2(\mathbf{R}_+)$
- $\mathcal{E} = \{\psi(f) : f \in L^2(\mathbf{R}_+)\}$ the set of exponential vectors in $\Gamma(L^2(\mathbf{R}_+)$.
- $\Phi = \psi(0)$ the vacuum state in $\Gamma(L^2(\mathbf{R}_+)$.
- $\Gamma(\chi_{[0,t]})$ the orthogonal projector defined by

$$\Gamma(\chi_{[0,t]})\psi(f) = \psi(\chi_{[0,t]}f)$$

- $\Phi_{t]} := \Gamma(\chi_{[0,t]})\Phi \quad ; \quad \Phi_{[t} := \Gamma(\chi_{[t,\infty)})\Phi$
- W(f) $f \in L^2(\mathbf{R}_+)$ the Weyl operator characterized by the property

$$W(f)\psi(g) = e^{-\frac{\|f\|^2}{2} - <f,g>}\psi(f+h)$$

- A , A^+ , N the annihilation, creation and number (or gauge or conservation) fields defined, on $\mathcal{E}$ by the relations:

$$A(f)\psi(g) =< f, g > \psi(g)$$

$$A^+(f)\psi(g) = \frac{d}{dt} |_{t=0} \psi(g + tf)$$

$$N_t\psi(g) = \frac{d}{ds} |_{s=0} \psi(e^{s\chi_{[0,t]}}g)$$

we write N(s,t) for $N_t - N_s$. The W(f) are unitary operators on $\mathcal{H}$ satisfying

the CCR

$$W(f)W(g) = e^{-\frac{\|f\|^2}{2} - <f,g>}\psi(f+h)$$

- $f_{t]} = \chi_{[0,t]}f \quad ; \quad f_{[t} = \chi_{[t,\infty]}f$
- H_o a complex Hilbert space , called the initial space.
- $\mathcal{H} = H_o \otimes \Gamma(L^2(\mathbf{R}_+)$
- $\mathcal{H}_{t]} = H_o \otimes \Gamma(L^2([0,t]) \otimes \Phi_{[t}$
- $\mathcal{B} = \mathcal{B}(\mathcal{H}) = \mathcal{B}\Big(H_o \otimes \Gamma(L^2(\mathbf{R}_+)\Big)$
- $\mathcal{B}_{t]} = \mathcal{B}(H_o \otimes \Gamma(L^2([0,t])) \otimes 1_{t]}$
- $\mathcal{B}_{[t} = \mathcal{B}(1_{H_o} \otimes 1_{t]} \otimes \Gamma(L^2([t,\infty))$
- θ_t the shift on $L^2(\mathbf{R}_+)$.

- $\sigma_t = \Gamma(\theta_t)$ the shift on $\Gamma(L^2(\mathbf{R}_+)$

- $u_t^o = \imath_o \otimes \sigma_t$ the free time shift on $\mathcal{B}$. where $\imath_o$ is the identity map on $\mathcal{B}(H_o)$. The objects described above provide a simple and, for a certain

class of models, canonical example of a quantum Markov process (in fact also of a quantum independent increment process in the sense of [2]) and the Feynman-Kac formula allows to perturb such structures by means of unitary cocycles (for the free time shift) giving rise to new quantum processes [1]. In particular, the generator L of the quantum Markovian semigroup canonically associated to the new process is the sum of the generator L_o of the semigroup associated to the original process and of an additional perturbative piece, denoted L_I. The problem with the above class of quantum Markov processes is that the free time shift u_t^o acts trivially on the initial algebra and therefore the corresponding semigroup is zero so that, as remarked in [1], in this case one is in fact dealing with FK perturbations of the identity semigroup. As a consequence of this one looses one of the most attractive analytical advantages of the classical FK formula, namely the possibility of dealing with perturbations L_I so singular that the operator $L_o + L_I$ is not well defined (a typical example is the possibility of giving a meaning, via the FK formula, to the formal generator $-\triangle + V$ where $\triangle$ is the Laplacian on $\mathbf{R}^n$ and V is a highly singular potential). By analogy with the classical case we would like to have a time shift v_t^o wich shifts also the initial random variables (observables) and not only the increments. Moreover, in order to be able to apply the quantum FK perturbation technique, the free time shift should be such that the structure of the associated unitary cocycles should be determined quite explicitly, which is rarely the case if this time shift is itself a Feynman-Kac perturbation of u_t^o. This problem was posed by A. Meyer during the Obervolfach meeting and in the following I want to outline a possible general scheme for a solution and illustrate it with an example.

2 A POSSIBLE SCHEME FOR A SOLUTION

Let us look for a time shift v_t^o of the form

$$v_t^o = j_t \otimes \sigma_t : \mathcal{B} = \mathcal{B}(\mathcal{H}) = \mathcal{B}\Big(H_o \otimes \Gamma(L^2(\mathbf{R}_+)\Big) \cong \mathcal{B}(H_o) \otimes \mathcal{B}\Big(\Gamma(L^2(\mathbf{R}_+)\Big) \longrightarrow \mathcal{B} \tag{1}$$

where

$$j_t : \mathcal{B}(H_o) \cong \mathcal{B}(H_o) \otimes 1 \longrightarrow \mathcal{B}_{t]} \tag{2}$$

is a $*$-homomorphism. For all $a_o \in \mathcal{B}(H_o)$, $b \in \mathcal{B}(\Gamma(L^2(\mathbf{R}_+))$ one has

$$v_s^o v_t^o(a_o \otimes b) = j_s \otimes \sigma_s(j_t(a_o) \otimes \sigma_t(b)) = (j_s \otimes \sigma_s)(j_t(a_o)) \cdot (j_s \otimes \sigma_s)(1_o \otimes \sigma_t(b)) \tag{3}$$

$$= (j_s \otimes \sigma_s)(j_t(a_o)) \cdot (1_{t]} \otimes \sigma_{s+t}(b))$$

and since we want v_t^o to be a 1-parameter semi-group of *-endomorphisms, it follows that, for any $a_o \in \mathcal{B}(H_o)$, $b \in \mathcal{B}(\Gamma(L^2(\mathbf{R}_+))$ the right hand side of (3) must be equal to

$$v_{s+t}^o(a_o \otimes b) = j_{s+t}(a_o) \otimes \sigma_{s+t}(b) \tag{4}$$

Thus v_s^o will be a 1-parameter semi-group if and only if

$$(j_s \otimes \sigma_s)(j_t(a_o)) = j_{s+t}(a_o) \qquad \forall a_o \in \mathcal{B}(H_o) \tag{5}$$

Here we give an example of a j_t satisfying condition (5) above. First shall

we give the expression of j_t in unbounded form and then shall write the corresponding bounded form.
In the notations of Section (1) choose $H_o = L^2(\mathbf{R})$ with a_o, a_o^+ denoting the usual annihilation and creation operators. Define

$$j_t(a_o^\epsilon) = a_o^\epsilon + X_{[0,t]} \qquad ; \qquad a_o^\epsilon = a_o \text{ or } a_o^+ \tag{6}$$

with $(s,t) \mapsto X_{[0,t]}$ a σ-homogeneous normal additive process, i.e.

$$X_{[0,t]} = X_{[0,t]}^+ \hat{\in} 1_{H_o} \otimes \mathcal{B}\Big(\Gamma(L^2([0,t])\Big) \qquad ; \qquad [X_{[0,t]}, X_{[0,t]}^+] = 0 \tag{7}$$

$$X_{[r,s]} + X_{[s,t]} = X_{[r,t]} \qquad ; \qquad r < s < t \tag{8}$$

$$\sigma_r(X_{[s,t]}) = X_{[s+r,t+r]} \tag{9}$$

$$[X_{[r,s]}, X_{[u,t]}] = 0 \qquad if\ (u,t) \cap (r,s) = \emptyset \tag{10}$$

where $[.,.]$ denotes, as usual, the commutator. In bounded form and under the additional assumption that $X_{[0,t]}$ is self-adjoint, j_t can be defined on the Weyl operators on H_o by:

$$j_t(W_o(z)) = j_t(\exp i(za_o^+ + z^+ a_o)) = \exp i(zj_t(a_o^+) + z^+ j_t(a_o)) = \tag{11}$$

$$= e^{i(za_o^+ + z^+ a_o) + (2Rez)X_{[0,t]})} = W_o(z)e^{i(2Rez)X_{[0,t]}}$$

An example of $X_{[0,t]}$ sastisfying the required conditions is the momentum operator $P(\chi_{[0,t]})$. Another example is $X_{[0,t]} = W_t - W_o$, which gives the usual free shift in Wiener space (not only in the increment space cf. Meyer ' s contribution to these proceedings). Other examples could be obtained using the position or number processes or mixtures of them i.e., Weyl shifts of the form:

$$j_t(W_o(z)) = W_o(z)W(z\chi_{[0,t]}; e^{iz\chi_{[0,t]}}) \tag{12}$$

(cf. the remark at the end of this note).

3 THE SEMI-GROUP ASSOCIATED TO THE CHOICE $X_{[0,t]} = P(\chi_{[0,t]})$

The semi-group P_o^t , associated to the "free" evolution v_t^o is

$$P_o^t = E_{o]}(v_t^o(a_o)) \qquad ; \quad a_o \in \mathcal{B}(H_o) \tag{13}$$

$$E_{o]} = \imath_o \otimes < \Phi, (\ . \ )\Phi > : \ \mathcal{B}(H_o) \otimes \mathcal{B} \longrightarrow \mathcal{B}(H_o) \cong \mathcal{B}(H_o) \otimes 1 \tag{14}$$

In our case, choosing $b = W_o(z)$ $(z \in \mathbf{C})$ and $X_{[0,t]} = P(\chi_{[0,t]})$, one finds

$$P_o^t(W_o(z)) = E_{o]}(v_t^o(W_o(z))) = E_{o]}\Big(W_o(z)e^{i(2Rez)P(\chi_{[0,t]})}\Big) = W_o(z)e^{-2(Rez)^2 t} \tag{15}$$

Hence the Weyl operators are in the domain of the generator of P_o^t and one has:

$$P_o^t = \exp tL \tag{16}$$

with

$$L(W_o(z)) = -2(Rez)^2 W_o(z) \qquad ; \qquad z \in \mathbf{C}) \tag{17}$$

The explicit form of the generator can be obtained with the following semi-heuristic, considerations:

$$W_o(z) = \exp i(za_o^+ + z^+ a_o) \tag{18}$$

therefore

$$\frac{\partial}{\partial a_o^+} W_o(z) = (iz)W_o(z) \qquad ; \qquad \frac{\partial}{\partial a_o} W_o(z) = (iz^+)W_o(z)$$

hence

$$\Big(\frac{\partial}{\partial a_o^+}+\frac{\partial}{\partial a_o}\Big)W_o(z)=i(2Rez)W_o(z)$$

and therefore

$$\Big[\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial a_o^+}+\frac{\partial}{\partial a_o}\Big)\Big]^2=L \tag{19}$$

Now, from

$$[a_o\ ,\ a_o^+]=1 \tag{20}$$

we deduce

$$[a_o\ ,\ .\ ]=\frac{\partial}{\partial a_o^+}\qquad ;\qquad [a_o^+\ ,\ .\ ]=-\frac{\partial}{\partial a_o} \tag{21}$$

In conclusion

$$L=\Big[\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial a_o^+}+\frac{\partial}{\partial a_o}\Big)\Big]^2=\Big[\frac{1}{\sqrt{2}}\Big([a_o\ ,\ .\ ]+[a_o^+\ ,\ .\ ]\Big)\Big]^2=\frac{1}{2}[a_o-a_o^+\ ,\ .\ ]^2 \tag{22}$$

and since $\frac{1}{\sqrt{2}}[a_o-a_o^+\ ,\ .\ ]$

$$L=-[p\ ,\ .\ ]^2=-[p\ ,\ [p\ ,\ .\ ]\ ] \tag{23}$$

So the free semigroup is a quasifree semigroup of the type considered by Lindblad in [4]. Notice moreover that, if f is a smooth function and M_f is multiplication by f in $L^2(\mathbf{R})$, then with the identification $p=\frac{1}{i}\frac{\partial}{\partial x}$ one has

$$[p,M_f]=\frac{1}{i}M_{(\frac{\partial}{\partial x}f)} \tag{24}$$

hence

$$-[p\ ,\ [p\ ,\ M_f\ ]\ ]=M_{(\frac{\partial^2}{\partial x^2}f)} \tag{25}$$

which gives the right answer when we restrict our attention to the classical Wiener process.

4 v_t^o- Markovian cocycles : an example

Consider the stochastic differential equation (SDE)

$$dU=\Big((L_o+X_{[0,t]})dA^+-(L_o^++X_{[0,t]})dA+Zdt\Big)U \tag{26}$$

the unitarity condition (using the Fock Ito table for dA , dA^+) is

$$Z = iH - \frac{1}{2} \mid L_o^+ + X_{[0,t]} \mid^2 \tag{27}$$

with H self-adjoint. By shifting the equation (26), with H = 0 , with the free shift v_s^o, we obtain the equation for $v_s^o(U_t)$ namely

$$dv_s^o(U_t) = \Big((L_o+X_{[0,t+s]})dA_s^+(t)-(L_o^+ +X_{[0,t+s]})dA_s(t)-\frac{1}{2} \mid L_o^+ +X_{[0,t+s]} \mid^2 dt\Big)v_s^o(U_t) \tag{28}$$

where we have used the notation

$$dA_s(t) = A(s+t+dt) - A(s+t) \tag{29}$$

which means that, by definition

$$\int_0^T Y_{t+s} dA_s(t) := \int_s^{s+T} Y_t dA(t) \tag{30}$$

Now, written in integral form, the equations (26) and (28) look respectively like

$$U_t = 1+\int_0^t \Big((L_o+X_{[0,r]})dA^+(r)-(L_o^+ +X_{[0,r]})dA(r)-\frac{1}{2} \mid L_o^+ +X_{[0,r]} \mid^2 dr\Big)U_r \tag{31}$$

$$v_s^o(U_t) = 1+\int_0^t \Big((L_o+X_{[0,s+r]})dA^+(s+r)-(L_o^+ +X_{[0,s+r]})dA(s+r)-\frac{1}{2} \mid L_o^+ +X_{[0,s+r]} \mid^2 dr\Big)v_s^o(U_r) \tag{32}$$

So that $v_s^o(U_t) \cdot U_s$ and U_{s+t} satisfy the same SDE (in the t-variable) with the same initial condition i.e. U_s at t = 0. Therefore, if the $X_{[0,t]}$-process is regular enough to assure the existence and uniqueness of the solutions of the above SDE, it will follow that

$$v_s^o(U_t) \cdot U_s = U_{s+t} \tag{33}$$

which means that U_t is a v_t^o - Markovian cocycle. The formal unitarity of follows from (27) and in many interesting cases the unitarity can be effectively proved. Having the unitary cocycle, we can apply the FK perturbation scheme to the free semigroup associated to v_t^o. Denoting $\mathcal{L}_o$ the generator of this semigroup, a simple calculation shows that the formal generator of the perturbed semigroup

$$P^t(b_o) := E_{o]}\Big(U_t^* \cdot v_t^o(b_o) \cdot U_t\Big) \tag{34}$$

will be

$$\mathcal{L}_o + L_o^+ b + bL_o + L_o^+ bL_o \tag{35}$$

If the operators $\mathcal{L}_o$, L_o are unbounded, the expression (35) will not be in general a well defined operator. However, for $X_{[0,t]}$ as in Section (2), the operators $L_o + X_{[0,t]}$; $L_o^+ + X_{[0,t]}$ are always well defined and therefore equation (1) makes sense and in some cases the conditions for the existence of a solution of this equation are much weaker than those which allow to realize $\mathcal{L}_o + L_o^+ b + bL_o + L_o^+ bL_o$ as a well defined operator.
Applying the considerations above to the additive functional $X_{[0,t]} = P(\chi_{[0,t]})$, for which the regularity problems mentioned above can be solved with standard techniques, one can produce singular perturbations of the noncommutative Laplacian in full analogy with the classical case. The physical meaning

of the operator $X_{[0,t]}$ in (26) can be understood in terms of Barchielli' s analysis [3]: $X_{[0,t]}$ is the input field (the number process in Barchielli 's paper) which interacts with an apparatus described by the operators L_o, Z in (26). The free evolution of the system is given by the time shift v_t^o and equation (26) describes the interaction cocycle giving the evolution of the observables of the coupled system (input field + apparatus) according to the scheme proposed in [1]. The advantage of the present approach with respect to [3] is that, due to the v_t^o-cocycle property of the solution of (26), the interacting evolution $x \mapsto v_t^o(U_t^+ x U_t)$ will now be a 1-parameter automorphism group, in agreement with the basic principles of quantum physics.

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