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# Spatial Duopoly under Uniform Delivered Pricing when Firms Avoid Turning Customers Away

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**Abstract:** This paper studies a spatial duopoly under uniform delivered pricing when firms do not ration the supply of the good, thus extending to a spatial context the analysis of oligopolistic markets with no rationing. The paper shows the existence of the equilibrium in prices under different tie-breaking rules (TBR) and compare the features of the equilibria found under these rules, thereby allowing to highlight the importance of the choice of the TBR in studying these models. When consumers buy from the nearest firm in case of equal prices (*efficient* TBR*)*, any symmetric price pair within a given range is a Nash equilibrium, with each firm serving exactly half of the market line. If demand in each local market is equally split between the firms charging the same price (*random* TBR), the only equilibrium price is the one that gives zero profits to each firm. The degree of competitiveness of the market crucially depends on the TBR. Under the *efficient* TBR, all (but one) price equilibria deliver positive profits to both firms. Under the *random* TBR, the market outcome is very competitive in that firms make zero profits. None of the equilibria found under any tie-breaking rule are allocatively efficient.

*Keywords: Spatial duopoly, uniform delivered price, rationing.*

## 1. Introduction

This paper studies the equilibrium of a spatial duopoly under uniform delivered pricing when firms do not ration the supply of the good. Under this pricing policy, the firm sets the same price to all the customers and delivers the good, bearing the transportation cost. The no

* This paper is based on my DPhil thesis at the University of York.

rationing assumption implies that each firm does not refuse to supply the good to any customer willing to buy from it.

The main contributions of this paper are the following. First, it proves the existence of the equilibrium under different tie-breaking rules, i.e. rules for the resolution of the conflict between firms when they charge equal prices in the same market. The paper allows to compare the features of the equilibria found under the different tie breaking rules, thereby allowing to highlight the importance of the choice of the tie-breaking rules in studying UDP models. Secondly, it extends to a spatial context the analysis of oligopolistic markets when rationing is not possible.

The empirical relevance of UDP policies is witnessed by some early studies (Greenhut, 1981; Phlips, 1983) and from the widespread casual observation of firms adopting this pricing policy (like, for instance, utilities, mail ordering firms, furniture and appliance stores and, in a aspatial context, insurance companies, etc). Also, from a theoretical point of view, the importance of this pricing policy has been demonstrated by Kats and Thisse (1993), which show that UDP is the equilibrium pricing strategy when firms choose first *location*, then *pricing policy* and eventually *price*.

In spite of this, UDP models have not been extensively studied in the spatial oligopoly literature, mainly because of problems of existence of equilibrium even more serious than in other spatial models. In the case of a homogenous duopoly, Beckmann and Thisse (1986) shows that no price equilibrium can exist when consumers buy from the nearest firm in case of a price tie. A similar result is obtained de Palma et al. (1986), under the hypothesis that every consumer has the probability one half of buying from each firm in case of a price tie.

Several strategies have been used in the literature to overcome this problem. Some author have assumed that the products sold by the different firms are heterogeneous, although different approaches to product heterogeneity have been taken in the literature. (see e.g. de Palma *et al.*, 1987, Anderson *et al.*, 1992b and De Fraja and Norman, 1993)[1]. Others have

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[1] Some authors assume that consumers' demand is given by a logit function, with consumers buying some fraction of the goods from either firms (see e.g. De Palma *et al.*, 1987, and Anderson *et al.*, 1992b). Other assume that consumers view the goods supplied by the

studied the price equilibrium in mixed strategies (Kats and Thisse, 1993). In a unpublished earlier version of this work (Kats and Thisse, 1989), Kats and Thisse have taken a different route and analysed a UDP duopoly model under the assumption that firms are either forced or decide on their own to supply all the customers who ask them. Assuming that customers buy from the nearest firm in case of a price tie, they show that there always exists an equilibrium in price strategies.

This latter approach anticipated a more recent interest that the oligopoly literature has shown towards the effects of the absence of rationing on the properties of the oligopolistic markets. Several justification have been provided in the literature for the no rationing assumption. Indeed, firms may be prevented from rationing by regulatory requirements to satisfy all the demand. These are quite common for network utilities – i.e. firm operating in the (now competitive) domestic electricity markets in the UK are required to publish their (uniform) prices and are not allowed to refuse to supply any customer in the region – and insurance companies – i.e. German car insurers are legally required to accept all customers for third party liability insurance (Wambach, 1999) –. Also, firms may not want to ration since turning down customers may be costly to the firm in term of goodwill, reputation or offence caused (Dixon, 1990, and the Operational Research and inventory models cited therein). Finally, Baye and Morgan (2002) argues that many price setting oligopoly environments have the feature to award the whole production on a winner-take-all basis to the lowest price firm.

Following closely the approach undertaken by Kats and Thisse, this paper studies a single-stage spatial duopoly pricing game where consumers have linear downward sloping demand and firms set UDP prices and cannot ration the supply. Assuming no rationing in a spatial linear market under UDP is equivalent to assuming that when one firms sells the good at a given price to a consumer located along the market line, it has to supply the good at the same price to any consumers willing to buy from it. Assuming that firms are symmetrically located along the market line and that locations are exogenous, this paper studies the price

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different firms as imperfect substitutes and demand varies continuously as prices vary (De Fraja and Norman, 1993).

equilibrium in pure strategies under two different tie-breaking rules.[2] The first tie-breaking rule is such that, when both firms charge the same price at the same location, this location is totally supplied by the closest firm; in the rest of the paper I refer to it as the *efficient* tie-breaking rule. The second tie-breaking rule is that, in case of matching prices, total demand in each local market is equally shared between the two firms; in the rest of the paper I refer to it as the *random* tie-breaking rule. The equilibria under the different tie-breaking rule are characterised under similar market assumptions so to allow to compare the different outcomes.[3]

The paper finds that a price equilibrium always exists in a spatial duopoly under UDP when firms cannot ration the supply. While the existence of the price equilibrium has already been shown in a similar set up by Kats and Thisse (1989) under the *efficient* tie-breaking rule in contrast with the non existence result of Beckmann and Thisse (1986), this paper characterises the equilibrium also for the case of the *random* tie-breaking rule. In this respect, this paper might be seen as bearing the same relationship to de Palma et al. (1986) as does Kats and Thisse (1989) to Beckmann and Thisse (1986).

The nature of the equilibria are deeply different according the tie-breaking rule adopted, pointing out the relevance of the choice of the tie-breaking rule in analysing these models. Under the *efficient* tie-breaking rule, any symmetric price pair within a given range is a Nash equilibrium, with each firm serving exactly half of the market line. This range is such that the equilibrium prices are neither to low to cause negative profits nor too high to give firms an incentive to undercut the rival and serve the whole market line. Under the *random* tie-breaking rule, there is only one equilibrium price, where both firms make zero profits. As firms equally share each local market when setting the same price, undercutting is profitable at all price but the one that brings about zero profits. Then, the only equilibrium price is the

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[2] These two tie-breaking rules were firstly employed in a UDP model by Gronberg and Meyer (1981).

[3] Differently from this paper, Kats and Thisse study a circular spatial model where firms' locations are not necessarily symmetric and consumers have unit demand. Also, they only study the model under the *efficient* tie-breaking rule.

one that gives zero profits to each firm. Since consumers are assumed to a have downward sloping demand, the paper is also able to provide a welfare comparisons of the equilibria obtained under the different tie-breaking rules. For low values of the transportation cost, consumers' surplus is higher under the *random* tie-breaking rule, which implies that consumers are better off selecting *random*ly which firm they patronise than adopting the so-called socially optimal behaviour. On the other hand, social welfare is always higher under the *efficient* tie-breaking rule. This is partly due to the zero profits obtained by the firms under the other tie-breaking rule.

The paper can also be seen as the extension to a spatial context of the analysis of oligopolistic markets without rationing, that so far has been mostly undertaken in a aspatial context. In this respect, this paper is very closed related to Dastidar (1995) which studies a Bertrand oligopoly with increasing return to scale and shows that, when demand is equally shared between the lowest price firms, there exists a continuum of price equilibria, all (but one) with positive profits.

Notice that one of the key questions addressed by this literature is indeed the degree of competitiveness of these markets. Harrington (1989) analyse a Bertrand oligopoly with constant returns and shows that a unique reasonable outcome of the game exists which is not the competitive solution but which is nevertheless approximately competitive. Dastidar (1995) shows that the market outcome is very different from the standard Bertrand paradox and shows (almost) always positive profits to the firms. Baye and Morgan (2002) study a homogenous oligopoly where, in case of a price tie, the supply is undertaken only by one firm *random*ly selected between the lowest price firms; they provide the necessary and sufficient conditions for a zero profits equilibrium to emerge and illustrate how deviations from these conditions lead to positive profits outcomes. With respect to this issue of the competitiveness of these markets, this paper illustrates that in a spatial context the degree of competitiveness (as measured by firms' profits) crucially depends on the tie-breaking rule which is in force in the market. Under the *efficient* tie-breaking rule, all (but one) any price equilibria deliver positive profits to both firms. Under the *random* tie-breaking rule, the market outcome is very competitive in that firms make zero profits. None of the equilibria found under any tie-breaking rule are allocatively efficient.

The structure of the paper is as follows. The model is described in section 2: sections 3 and 4 characterise the equilibria of the game under the different tie-breaking rules. Section 5 makes some normative judgements on the different market arrangements. Some concluding remarks are given in Section 6. All the proofs are relegated to the Appendix.

# 2. The model

I assume a spatial linear market of unit length in which competition in prices between two profit-maximising firms takes place at each point on the market line. The two firms, which are referred to as firm 0 and firm 1, produce perfectly homogeneous goods. Consumers are evenly distributed over the line. At each location along the line, consumers have linear demand given by $q = 1 - p$. Consumers' density is normalised to 1.

The pricing policy adopted by both firms is uniform delivered pricing: the same price is charged to all customers, irrespective of their location, and firms deliver at their cost the good to customers' locations. Each firm produces with constant (and identical) marginal and average cost that, without further loss of generality, is normalised to zero. Transportation cost (denoted by $c$) is assumed to be linearly increasing with quantity and distance. Transport is under firms' control and no arbitrage can take place among consumers.

I assume that firms cannot ration the supply of the good to any customers willing to buy from it. This implies that, once a price has been set by one of the two firms, all the customers can buy at that price from that firm. From this assumption, it follows that if a firm sets a price lower than the rival, it may end up serving all the customers along the unit line.

If the two firms charge the same price, two different rules on the resolution of the conflict over markets are studied:

*efficient* tie-breaking rule: the first tie-breaking rule is such that, in case of both firms charging the same price at the same location, the market is supplied by the closest firm, that is by the firm which bears a lower transportation cost in serving that location. In the rest of the paper, this tie-breaking rule is referred to as the *efficient* tie-breaking rule.

*random* tie-breaking rule: the second tie-breaking rule is such that, in case of both firms charging the same price at the same location, total demand in each local market is equally shared between the two firms. This tie-breaking rule is referred to as *random* in the rest of the paper.

These rules are usually interpreted in the literature as originating from different behaviour on the consumers' side. The *random* tie-breaking rule may be the result of customers selecting *random*ly the firm from which to buy; then, if assigning an equal probability to buying from each firm, each local market is equally shared between the firms supplying that market (at least in expected terms). As for the *efficient* tie-breaking rule, it is assumed that consumers buy from the nearest firm (see e.g. Lederer and Hurter, 1986, and MacLeod *et al.*, 1988). This behaviour is usually defined as socially optimal because, given the quantities exchanged and the locations of the two firms, it minimises the total transportation cost. Since the choice of patronising only the nearest firm originates directly from the consumers, this also explains the apparent contradiction of the firms being unable to ration but, at the same time, serving only some of the consumers.[4]

I model the strategic interaction between the two firms as a single-stage game where locations are fixed and each firm simply chooses the price $p_i \in [0, 1]$ where $i = 0, 1$. For the sake of simplicity, I restrict my attention to the case where firms locations are symmetric. Then, denoting with $x_0$ and $x_1$ the locations of firm 0 and 1 respectively without loss of generality, I can restrict the locations to be such that $x_0 \in [0, ½]$ and $x_1 \in [½, 1]$ without any further loss of generality. Given the nature of the game, the equilibrium concept is Nash equilibrium.

In case of the *efficient* tie-breaking rule, firm 0's profits are given by

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[4] Following Gronberg and Meyer (1981), another possible interpretation of the tie-breaking rules makes them dependent on firms' behaviour. The *efficient* tie-breaking rule can be interpreted as the result of a collusive behaviour between the two firms over the locations they serve, provided that all demand at each location is satisfied. Collusion implies that firms agree to share markets at any location so that each firm serves exclusively the locations where it has a comparative advantage in terms of costs with respect to the rival. On the contrary, the *random* tie-breaking rule can be interpreted as the result of the firms being not able or not allowed to reach the collusive agreement on the locations each firm has to serve exclusively. In this case, firms split each local market. Notice that, also in this case, the apparent contradiction of the model under the *efficient* tie-breaking rule between firms unable to ration and consumers buying only from the nearest firm is easily reconciled. Indeed, when firms collude over the locations they serve, they never refuse to supply at any locations since – differently from the UDP model with rationing – there is neither unsatisfied demand nor demand that could be positively satisfied by the rival firm.

$$\Pi_0 = \begin{cases} \Pi_0^-(p_0, p_1) & \text{when } p_0 < p_1 \\ \Pi_0^=(p_0, p_1) & \text{when } p_0 = p_1 \\ \Pi_0^+(p_0, p_1) & \text{when } p_0 > p_1 \end{cases} \quad (1)$$

where the superscripts –, = and + are respectively used to intuitively distinguish between the possible cases of firm 0 charging a price lower, equal or higher than the price set by the rival. In each of these cases, we have that

$$\Pi_0^-(p_0,p_1) \equiv \int_0^1 (1-p_0)(p_0 - c|a - x_0|)da\,, \quad (2)$$

$$\Pi_0^=(p_0,p_1) \equiv \int_0^{½} (1-p_0)(p_0 - c|a - x_0|)da \quad (3)$$

and

$$\Pi_0^+(p_0, p_1) \equiv 0\,. \quad (4)$$

In case of the *random* tie-breaking rule, firm 0's profits are given by

$$\tilde{\Pi}_0 = \begin{cases} \tilde{\Pi}_0^-(p_0, p_1) & \text{when } p_0 < p_1 \\ \tilde{\Pi}_0^=(p_0, p_1) & \text{when } p_0 = p_1 \\ \tilde{\Pi}_0^+(p_0, p_1) & \text{when } p_0 > p_1 \end{cases} \quad (5)$$

where the superscript have identical meaning as above and where

$$\tilde{\Pi}_0^-(p_0,p_1) \equiv \int_0^1 (1-p_0)(p_0 - c|a - x_0|)da\,, \quad (6)$$

$$\tilde{\Pi}_0^=(p_0,p_1) \equiv \int_0^1 \tfrac{1}{2}(1-p_0)(p_0 - c|a - x_0|)da \quad (7)$$

and

$$\tilde{\Pi}_0^+(p_0,p_1) \equiv 0 \quad (8)$$

Similar formulae apply for firm 1.

## 3. The equilibrium under the *efficient* tie-breaking rule

This section characterises the equilibrium of the single-stage price game under the *efficient* tie-breaking rule. I recall here that this implies that when the two firms set the same price, firms end up serving only those locations where they have a comparative advantage over the rival in term of transportation cost. Let $x_0 = x$ and $x_1 = 1 - x$, where $x$ gives the distance from firm 0's location (respectively, firm 1's) to the left (right) end of the market line.

It is useful to start the analysis of this case by introducing the following definitions:

**Definition 1.** For $i, j = 0, 1$ and $i \neq j$,

*i*) $p^m \equiv \text{argmax}\left\{\Pi_i^{-}(\cdot)\right\}$. Formally,

$$p^m \equiv \frac{1}{2}\left[1 + c\left(\frac{1}{2} - x + x^2\right)\right]; \quad (9)$$

*ii*) $\underline{p}$ is the smallest solution w. r. to $p$ to $\Pi_i^{=}(p_i, p_j) = 0$. Formally,

$$\underline{p} \equiv c\left(\frac{1}{4} - x + 2x^2\right); \quad (10)$$

*iii*) $v_1$ is the smallest solution w. r. to $p$ to $\Pi_i^{=}(p_i, p_j) = \Pi_i^{-}(p_i, p_j)$. Formally,

$$v_1 \equiv c\left(\frac{3}{4} - x\right); (11)$$

*iv*) $v_2$ is the largest solution w. r. to $p$ to $\Pi_i^{=}(p_i, p_j) = \Pi_i^{-}(p_i^m, p_j)$ when $v_2 \in \Re^+$ and $p_i^m \leq v_2$ and is equal to 1 otherwise. Formally,

$$v_2 \equiv \begin{cases} \frac{1}{2} + c\left(\frac{1}{8} - \frac{1}{2}x + x^2\right) + \frac{1}{8}\kappa & \text{when } c \in \left[\frac{1}{1 - x - x^2}, \frac{2}{1 - 2x + 2x^2}\right] \\ 1 & \text{otherwise} \end{cases} \quad (12)$$

where $\kappa \equiv \sqrt{-32c^2x^2 - 32cx + 24c^2x - 16 + 24c - 7c^2 + 32c^2x^4}$.

Notice that, in the definitions given above, subscripts are omitted since the symmetry of firms' locations makes the different critical prices identical across firms.

I turn now to illustrate the different definitions given above. First, $p^m$ give the price each firm would charge if it were a monopolist; this is also the best price any firm could charge when it undercuts the rival, provided that the rival sets a sufficiently high price. Also, the price $\underline{p}$ gives the minimum price which allows each firm non negative profits when it matches the price set by the rival. Moreover, $v_1$ gives the price for which firm *i* is indifferent between matching the price set by the rival or undercutting it. Finally, $v_2$ is the highest price which gives the firm that matches the rival the same profits it would gain if it operates as a monopolist. Notice indeed that, for some values of the transportation cost, a firm would not necessarily prefer to be a monopolist since this would imply to serve the whole market line and may instead make higher profits sharing the market with the rival. Under these cost conditions, if firm *j* set a price (slightly) above $v_2$, firm *i* would prefer to undercut the rival by charging the monopoly price; on the other hand, if firm *j* set a price (slightly) below $v_2$, firm *i* best reply would be to match the rival.

As to the last definition, two pathological cases may occur when the transportation cost is either very small or very large. When $c$ is small, a price like $v_2$ does not exist since the monopoly price, whenever possible, gives each firm higher profits than those obtained by matching the rival. On the other hand, when $c$ is large, the monopoly price gives always negative profits and each firm can obtain non negative profits only if it matches the rival and shares the market with it. In both these particular cases, I assume the price $v_2$ to be equal to 1.

It is now possible to state the following result.

**Proposition 1.** Assume $c \leq \frac{4}{1-4x+8x^2}$. Let $\overline{p} \equiv \min\{v_1, v_2\}$**.** Then, any price pair $p_0$ and $p_1$ such that $p_0 = p_1$ and $p_0, p_1 \in [\underline{p}, \overline{p}]$ is a Nash equilibrium of the game under the *efficient* tie-breaking rule.

**Figure 1**

The result given in the Proposition may be intuitively illustrated with the aid of Figure 1 where, for different values of the transportation cost, firm *i*'s profits are plotted against its own price under the two cases of this firm matching the price chosen by the rival or undercutting it.[5] Identical pictures could be drawn for firm $j$. In any panels of the Figure the critical price levels identified in Definition 1 are also drawn, since they play a crucial role in the equilibrium of the game.

The intuitive illustration of the nature of the equilibria described in the Proposition goes as follows. Consider first the case of $c$ being positive but lower than $\frac{1}{1-x-x^2}$, as it illustrated in Panel a). Notice that, in this case, $\underline{p} < v_1 < v_2$. Proposition 1 argues that any pair of prices between $\underline{p}$ and $v_1$ is a Nash equilibrium. Indeed, if firm $j$ charges a price within this interval, the picture clearly illustrates that the best reply for firm $i$ is to match this price, since undercutting would give rise to lower profits and charging a even lower price or a price higher than $v_1$ would grant zero profits instead. Consider now the case depicted in Panel b), which occurs when $c \in \left[\frac{1}{1-x-x^2}, \frac{4}{3-4x}\right]$. The ranking between the critical $p$'s is now such that $\underline{p} < v_2 < v_1$. Hence, any pair of prices between $\underline{p}$ and $v_2$ is a Nash equilibrium. Now, if firm $j$ charges a price between $v_2$ and $v_1$, although firm $i$ prefers to match this price rather than undercutting it by a small amount, the best response for firm $i$ is to undercut the price set by firm $j$ by a large amount and charge the monopoly price. The case depicted in Panel c) is very similar to the previous case, the only difference being that firm $i$ prefers to match than undercutting for all prices set by the rival greater than $\underline{p}$ but smaller than 1. Finally, Panel d) illustrates the case of $c$ being between $\frac{2}{1-2x+2x^2}$ and $\frac{4}{1-4x+8x^2}$. The peculiarity of this case is that firm $i$ would never make positive profits if it were a monopolist. Hence, whenever firm $j$ charges a price that allows firm $i$ to gain non-negative profits by matching it, this is also the best reply.

It is also easy to see that the price pairs with the illustrated properties in Proposition 1 are the only Nash equilibria of the game under analysis. While the reader is referred to the proof for

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[5] Clearly, $\Pi_i^=$ is drawn only as function of $p_i$ under the hypothesis that the same price is also charged by the rival; similarly, $\Pi_i^-$ is drawn under the hypothesis that firm $j$ charges a price higher than the one charged by firm $i$.

the (rather intuitive though) motivation of the impossibility of asymmetric equilibria, I only deal here with the impossibility of symmetric Nash equilibrium price pairs outside the interval between $\underline{p}$ and $\overline{p}$. Indeed, if the candidate Nash equilibrium prices were smaller than $\underline{p}$, both firms would make negative profits: any firm would then be better off charging a higher price so to be driven out of the market and make zero profits. On the other hand, an equilibrium with both prices above $\overline{p}$ cannot exists. When $\overline{p}$ is equal to $v_1$, either firms would be better off by undercutting the rival either by a small amount or by a large amount and charge the monopoly price, depending on the relationship between these candidate equilibrium prices and the monopoly price. When instead $\overline{p}$ is equal to $v_2$, either firm would be better off by charging the monopoly price.

Lastly, it is easy to verify that, provided that the transportation cost is small enough, there always exists a continuum of price pairs that are Nash equilibria. This is equivalent to saying that $\underline{p}$ is typically greater than $\overline{p}$. The only case in which there exists only one Nash equilibrium price pair is when both firms are located right in the middle of the market line, since in this case $\underline{p} = \overline{p}$ (where $\underline{p} = \overline{p}$ is in this case necessarily equal to $v_1$).

Notice that this result does not critically depend on the linearity of cost and demand functions, which are only used to simplify the exposition and make possible the welfare comparisons provided in section 5, but simply requires that demand at each locations is downward sloping and that transportation cost increases with distance.

## 4. The equilibrium under the *random* tie-breaking rule

This section characterises the equilibrium of the price game under the *random* tie-breaking rule. Recall that this implies that when the two firms set the same price, half of the demand at each location along the market line is expected to be addressed to each firm. This may occur because, as firms are identical, customers *random*ise their choice and buy from either firm with the same probability. Similarly as before, I set $x_0 = x$ and $x_1 = 1 - x$, where again $x$ gives the distance from firm 0's location (respectively, firm 1's) to the left (right) end of the market line.

Now, I can state

**Proposition 2.** Assume $c < \dfrac{2}{1-2x+2x^2}$. Let $\widetilde{p}_0$ and $\widetilde{p}_1$ be the Nash equilibrium price pair under the *random* tie-breaking rule. Then, there always exists one and only one equilibrium price pair $\widetilde{p}_0$ and $\widetilde{p}_1$ such that

$$\widetilde{p}_i = c\left(\frac{1}{2} - x + x^2\right) \qquad \text{for } i = 0, 1. \quad (13)$$

Note that the restriction imposed in the Proposition to the values of the transportation cost ensures that firms obtain nonnegative profits in equilibrium for any pair of symmetric locations.

The main reason of the result reported in the Proposition is the following. Because of the nature of the expression for profits in (5), profits obtained when undercutting the rival and serving all the market line are always twice as much the profits obtained by matching the rival's price. This implies that each firm always finds profitable to shave any market price except the price that gives zero profit. As a result, the only possible equilibrium shows the typical feature of the standard Bertrand duopoly, with both firms simply breaking even. However, the market equilibrium does not show any efficiency properties typical of the Bertrand setting. This is because the firms simply average out the transportation cost across markets. While prices are equal *on average* to marginal cost of provision (marginal cost, here set to zero, plus transportation cost), in any local market the equilibrium price differs from the actual total marginal cost.

## 5. Welfare comparisons

This section discusses the normative properties of the equilibria obtained under the different tie-breaking rules under analysis. To this purpose, it is necessary to single out a price pair amongst the set of equilibrium price pairs which are found under the *efficient* tie-breaking rule. The Pareto optimality of the equilibrium price pair that gives the firms the highest level of joint profits makes it a focal point on which it seems reasonable to concentrate the analysis. This price comes as the solution of the following problem

$$\begin{aligned}&\max_p \int_0^{1/2} (1-p)(p-c|a-x_0|)da + \int_{1/2}^{1} (1-p)(p-c|a'-x_1|)da' \\ &\text{s.t. } p \in [\underline{p}, \overline{p}]\end{aligned} \tag{14}$$

Note that because of the symmetry of cost structures and locations, the price which solves (14) is also the price which gives to each firm the highest profits among the equilibrium prices. The formal characterisation of the joint profit maximising Nash equilibrium price pair is given in the following Proposition.

**Proposition 3**. Let $c < \dfrac{4}{1-4x+8x^2}$. Let $\hat{p}_0$ and $\hat{p}_1$ be the Nash equilibrium price pair that gives the highest joint profits to the two firms under the *efficient* tie-breaking rule. Then,

$$\hat{p}_i = \begin{cases} c\left(\dfrac{3}{4}-x\right) & \text{when } c \in \left(0, \dfrac{4}{5-4x-8x^2}\right] \\ \dfrac{1}{2}+c\left(\dfrac{1}{8}-\dfrac{1}{2}x-x^2\right) & \text{when } c \in \left[\dfrac{4}{5-4x-8x^2}, \dfrac{4}{1-4x+8x^2}\right) \end{cases} \quad \text{for } i = 0, 1 \tag{15}$$

When the value of transportation cost is high enough, the Nash equilibrium price pair that gives the highest profits is simply made out of the two prices that maximise the profits of each firm given that it serves the consumers located in its half of the market line. However, for low values of the transportation cost, the unconstrained single firm profit maximising price is not an equilibrium. Thus, for such values of *c*, the price which delivers the highest profits among the set of equilibrium prices is the upper boundary of the range of equilibrium prices.

I can now proceed to compare the equilibria under the different tie-breaking rule in terms of their welfare properties. Clearly, it has to be emphasised that the results of this comparison heavily rests on equilibrium selection under the *efficient* tie-breaking rule. The level of welfare generated by the different market arrangements is compared using the traditional measures given by the aggregate consumers' surplus, industry profits and social welfare as given by the unweighted sum of the two previous terms. Notice that, since demand originates from consumers with quasi-linear preferences, aggregate consumers' surplus is an exact measure of their utility.

The results of the welfare analysis are given in the following Proposition.

**Proposition 4.** Assume $c < \dfrac{2}{1-2x+2x^2}$. Let $S$, $\Pi$ and $W$ ($\widetilde{S}, \widetilde{\Pi}$ and $\widetilde{W}$) be the equilibrium aggregate consumers' surplus, total firms' profit and social welfare under the *efficient* (*random*) tie-breaking rule. Then, when the firms under the *efficient* tie-breaking rule set the joint profit maximising price amongst the set of equilibrium price pair, $S < \widetilde{S}$ when $c < \dfrac{4}{3-4x}$, $\widetilde{S} < S$ otherwise, and $\widetilde{\Pi} < \Pi$ and $\widetilde{W} < W$ always.

First note that the maximum allowed level for $c$ is the minimum within the two maximum levels of the transportation cost for which an equilibrium exists under the two tie-breaking rules. In other words, the limitation on $c$ ensures that are taken into consideration only those values of the cost parameter for which an equilibrium exists under both tie-breaking rules.

The Proposition illustrates that consumers are better off under the *random* tie-breaking rule when the value of the transportation cost is low enough. This contrasts with the usual definition of the *efficient* tie-breaking rule being the 'socially optimal one'. The obvious reason for this definition is that consumers' behaviour under the *efficient* rule is socially optimal as it minimises total transportation cost for given prices and location. However, differently from what implied by the mentioned commonly used definition, it is found here that consumers may be better off if they buy from a *random*ly selected firm instead of buying from the nearest firm.

In general, which one of the two equilibria obtained under the different tie-breaking rules is preferred by the consumers depends on the joint result of two different effects. A first effect is due to the different way in which competition between firms takes place when the different tie-breaking rules are assumed. Under the *efficient* tie-breaking rule, firms serve a given market area. Then, the price they choose is the one that (among the equilibrium prices) gives them the highest profits given the market area they supply. On the other hand, under the *random* tie-breaking rule, competition is much fiercer as firms are caught under the traditional Bertrand paradox; they undercut each other over the entire market line down to the price where both firms make zero profits. The other effect regards the level of total transportation cost borne by the firms and paid for by the consumers through prices. In this respect, it is clear that when consumers buy from the closest firm, firms pay an overall total transportation cost lower than when they serve also consumers at remote locations. The total result of the two effects is that, when the transportation cost is low enough, the competitive effect prevails and the equilibrium price is lower under the *random* rule. The opposite holds for high enough values of $c$.

From the other comparisons provided in Proposition 4, it is clear that the equilibrium under the *efficient* tie-breaking rule is always preferred both in terms of aggregate of firms' profits and overall aggregate social welfare. This latter result indicates that the higher profits under the *efficient* tie-breaking rule are able to outplay the lower consumers' surplus that can result under this *random* tie-breaking rule when $c$ is high.

# 6. Conclusions

This paper has studied the equilibrium of a spatial duopoly under uniform delivered pricing when firms do not ration the supply of the good. Differently from most other papers which only analyse UDP oligopolies only under one tie-breaking rule, this paper has studied and compared the market equilibria which emerge under the two most commonly used tie-breaking rules, the *random* and the *efficient* tie-breaking rules.

The paper has highlighted that the nature of the equilibria and its welfare properties differ profoundly according to adopted tie-breaking rule. Firstly, under the *efficient* tie-breaking rule there exists a continuum of price equilibria while under the *random* tie-breaking rule there exists only one price equilibrium. Secondly, under the *efficient* tie-breaking rule both firms always make positive profits while profits are always equal to zero for both firms under the *random* tie-breaking rule. Finally, consumers' preferences over the different market arrangements crucially depends on the level of the transportation cost and on the equilibrium price selected by the firms amongst the many equilibrium prices.

These results are of interest to the literature in spatial oligopoly in that they make a contribution to the understanding of markets where firms charge uniform delivered price. Also, the paper has extended to a spatial context the analysis of oligopolistic markets when rationing is not possible that so far has been mostly undertaken in a aspatial context. The main question usually raised by this literature is on the degree of competitiveness of these markets. This paper has shown that, in a spatial context with firms charging uniform delivered prices, the answer crucially depends on the tie-breaking rule which is used in the market. When a price tie is solved using the *random* tie-breaking rule, the market outcome is very competitive in that firms make zero profits. On the other hand, under the *efficient* tie-breaking rule, all (bot one) price equilibria deliver positive profits to both firms; provided that the transportation cost is high enough, one of the possible equilibria is with both firms charging the price they would charge if they were monopolist in their half of the market line. None of the equilibria found under any tie-breaking rule is allocatively efficient.

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**APPENDIX**

This Appendix gives the proofs of Propositions 1 and 2, together with a sketch of the proof of Proposition 5. The proof of Proposition 3 and 4 are omitted here since they makes a simple (but tedious) use of standard constrained maximisation techniques: further details are available from the author upon request.

Proof of Proposition 1

First of all, notice that, when $x \in [0, ½]$, it is easy to prove the following inequalities that will be used extensively in the rest of the proof:

$$\frac{1}{1-x-x^2} < \frac{4}{3-4x} < \frac{2}{1-2x+2x^2} < \frac{4}{1-4x+8x^2} \quad \text{(A.1)}$$

I turn now to illustrate the ranking between $\underline{p}$, $v_1$ and $v_2$ under the different values of the transportation cost $c$. When $c \in \left(0, \frac{1}{1-x-x^2}\right]$, it is possible to show that $\underline{p} < v_1 < v_2$. While it is immediate to show that $\underline{p} < v_1$, recall that, from Definition 1, $v_2 = 1$. It turns out that $v_1 <$ 1 when $c < \frac{4}{3-4x_0}$, which holds true because of (A.1). When $c \in \left[\frac{1}{1-x-x^2}, \frac{4}{1-4x+8x^2}\right)$, it is possible to show that $\underline{p} < v_2 \leq v_1$. I first concentrate on the ranking between $\underline{p}$ and $v_2$: let $z_1 \equiv \underline{p} - v_2$ and recall that $\kappa_1 \equiv \sqrt{-32c^2x^2 - 32cx + 24c^2x - 16 + 24c - 7c^2 + 32c^2x^4}$. Consider first the case of $c \in \left[\frac{1}{1-x-x^2}, \frac{2}{1-2x+2x^2}\right]$, which implies that $v_2 < 1$. Then, $z_1 = -\frac{1}{2}cx + \frac{1}{8}c + cx^2 - \frac{1}{2} - \frac{1}{8}\kappa_1$. Let now $z_1' = -\frac{1}{2}cx + \frac{1}{8}c + cx^2 - \frac{1}{2}$, which is equal to 0 when $c = \frac{4}{1-4x+8x^2}$. Since $\frac{dz_1'}{dc} > 0$, $z_1' < 0$ when $c < \frac{4}{1-4x+8x^2}$ which always holds true; hence, *a fortiori*, $z_1 < 0$. When $c \in \left[\frac{2}{1-2x+2x^2}, \frac{4}{1-4x+8x^2}\right]$, by definition $v_2 = 1$. Now, it is easy to establish that $\underline{p} < 1$ whenever $c < \frac{4}{1-4x+8x^2}$.

Now I illustrate some properties of functions $\Pi_i^-(\cdot)$ and $\Pi_i^=(\cdot)$ which will be useful in the rest of the proof:

[F.1] $\Pi_i^-(\cdot)$ is a concave second degree functions of $p$ with one of the roots always equal to 1; the other root is smaller than 1 provided that $c < \frac{2}{1-2x+2x^2}$;

[F.2] $\Pi_i^=(\cdot)$ is a concave second degree functions of $p$ with one of the roots always equal to 1; the other root is smaller than 1 provided that $c < \frac{4}{1-4x+8x^2}$;

[F.3] $\Pi_i^=(\cdot) > \Pi_i^-(\cdot)$ for any $p_i \in [0, v_1]$.

The proof then proceeds along the following steps:

i) an equilibrium must be a single price equilibrium. By contradiction, assume that an equilibrium price pair $p_i$ and $p_j$ such that $p_i \neq p_j$ exists, for $i, j = 0, 1$ and $i \neq j$. Without loss of generality, let $p_i < p_j$. Denote now by $p_i^0$ the smallest solution w. r. to $p_i$ to $\Pi_i^-(p_i, p_j) = 0$. Notice that, because of symmetry amongst the firms, $p_i^0 = p_j^0$ so that it is possible to drop the subscripts $i$ or $j$ and simply write $p^0$. Now, if $p_i < p_j < p^0$ or $p_i < p^0 < p_j$, because of [F.1], $\Pi_i^-(p_i, p_j) < 0 = \Pi_i^+(p_j + \varepsilon, p_j)$, which makes impossible for $p_i$ and $p_j$ to be a Nash equilibrium. Also because of [F.1], if $p^0 < p_i < p_j$, $\Pi_i^-(p_i + \varepsilon, p_j) > \Pi_i^-(p_i, p_j)$ and/or $\Pi_j^-(p_i - \varepsilon, p_i) > \Pi_j^+(p_j, p_i) = 0$ with a positive and whatever small ε, which again contradicts the initial hypothesis on $p_i$ and $p_j$ being a Nash equilibrium pair.

ii) any pair of prices $p_i$ and $p_j$ such that $p_i = p_j$ and $p_i, p_j \in \lfloor \underline{p}, \overline{p} \rfloor$ is a Nash equilibrium, for $i, j = 0, 1$ and $i \neq j$. Letting $p_i = p_j = p$, this is equivalent to state that, when firm $j$ charges $p \in \lfloor \underline{p}, \overline{p} \rfloor$, the following conditions must hold: a.1) $\Pi_i^=(p, p) \geq \Pi_i^-(p', p)$ for any $p' < p$; a.2) $\Pi_i^=(p, p) \geq \Pi_i^+(p', p)$ for any $p' > p$, and b) $\Pi_i^=(p, p) \geq 0$, for $i, j = 0, 1$ and $i \neq j$. First of all, notice that, because of (4), condition a.2) is identical to b) and can then be neglected. Condition b) always holds true, since, by the way itself $\underline{p}$ is defined and by [F.2], $\Pi_i^-(\cdot) \geq 0$ for any $p \in \lfloor \underline{p}, 1 \rfloor$. As to condition a.1), assume first that $c \in \left(0, \dfrac{1}{1 - x - x^2}\right]$, so that $\underline{p} < v_1 < v_2$. For any $p \in \lfloor \underline{p}, v_1 \rfloor$, by [F.3] we have that $\Pi_i^=(p, p) \geq \Pi_i^-(p - \varepsilon, p)$ for ε positive and whatever small and for $i = 0, 1$. Assume now that $c > \dfrac{1}{1 - x - x^2}$, so that $\underline{p} < v_2 \leq v_1$. When $c \in \left[\dfrac{1}{1 - x - x^2}, \dfrac{2}{1 - 2x + 2x^2}\right]$, for any $p \in \lfloor \underline{p}, v_2 \rfloor$ by [F.3] we have that $\Pi_i^=(p, p) \geq \Pi_i^-(p - \varepsilon, p)$ for any positive ε; when $c \in \left[\dfrac{2}{1 - 2x + 2x^2}, \dfrac{4}{1 - 4x + 8x^2}\right]$, for any $p \in \lfloor \underline{p}, 1 \rfloor$ by [F.3] we have again that $\Pi_i^=(p, p) \geq \Pi_i^-(p - \varepsilon, p) = 0$ for any positive and whatever small ε.

iii) any pair of prices $p_i$ and $p_j$ such that $p_i = p_j$ and $p_i, p_j \notin \lfloor \underline{p}, \overline{p} \rfloor$ cannot be an equilibrium, for $i, j = 0, 1$ and $i \neq j$. Let $p_i = p_j = p$. When $p < \underline{p}$, this cannot be an equilibrium since, by

[F.2] and for any admissible value of $c$, $\Pi_i^{=}(p, p) < 0$ which contradicts condition b) in ii). When $p >$ $\underline{p}$, different cases may occur according to the value of $c$. When $c \in \left(0, \frac{1}{1-x-x^2}\right]$, firm $i$ has an incentive to undercut the rival since $\Pi_i^{-}(p-\varepsilon, p) > \Pi_i^{=}(p, p)$, with ε positive and whatever small, which contradicts condition a.1 in ii). When $c \in \left[\frac{1}{1-x-x^2}, \frac{2}{1-2x+2x^2}\right]$, firm $i$ prefers to charge the monopoly price, which is smaller than the price set by the rival, since $\Pi_i^{-}(p^m, p) > \Pi_i^{=}(p, p)$, which again contradicts condition a.1 in ii). Finally, when $c \in \left[\frac{2}{1-2x+2x^2}, \frac{4}{1-4x+8x^2}\right]$, both firms make negative profits in equilibrium, since $\Pi_i^{=}(p, p) = 0$, which contradicts condition b) in ii).

Finally, from [F.2] it is evident that it is necessary to assume that $c \leq \frac{4}{1-4x+8x^2}$ to grant that the firms obtain nonnegative profits in equilibrium.

❐

**Proof of Proposition 2**

This proof is very similar in its nature to the standard proof of the Nash equilibrium in a symmetric Bertrand duopoly.

First notice that the function $\tilde{\Pi}_i^{-}(\cdot)$ is a concave second degree equation, whose roots are 1 $c\left(\frac{1}{2} - x + x^2\right)$; let $p^0 \equiv c\left(\frac{1}{2} - x + x^2\right)$. Notice that, because of the symmetry between the two firms, it is possible to drop the subscripts $i$ and $j$. When $c < \frac{2}{1-2x+2x^2}$, $\tilde{\Pi}_i^{-}(\cdot) \geq 0$ and, since $\tilde{\Pi}_i^{=}(\cdot) = ½\tilde{\Pi}_i^{-}(\cdot)$, $\tilde{\Pi}_i^{=}(\cdot) \leq \tilde{\Pi}_i^{-}(\cdot)$ for any $p \in [p^0, 1]$.

Then,

i) an equilibrium must be a single price equilibrium. By contradiction, assume that an equilibrium price pair $p_i$ and $p_j$ such that $p_i \neq p_j$ exists, for $i, j = 0, 1$ and $i \neq j$. Without loss of generality, let $p_i < p_j$. Now, if $p_i < p_j < p^0$ or $p_i < p^0 < p_j$, because of the relationship between $\tilde{\Pi}_i^{=}(\cdot)$and $\tilde{\Pi}_i^{-}(\cdot)$ and because of (8), $\tilde{\Pi}_i^{-}(p_i, p_j) < \tilde{\Pi}_i^{+}(p_j + \varepsilon, p_j) = 0$, which makes impossible for $p_i$ and $p_j$ to be a Nash equilibrium. Also, if $p^0 < p_i < p_j$, $\tilde{\Pi}_i^{-}(p_i + \varepsilon, p_j) > \tilde{\Pi}_i^{-}(p_i, p_j)$ and/or

$\widetilde{\Pi}_j^-(p_i - \varepsilon, p_i) > \widetilde{\Pi}_j^+(p_j, p_i) = 0$ with a positive and whatever small ε, which again contradicts the initial hypothesis on $p_i$ and $p_j$ being a Nash equilibrium pair.

ii) the pair $\widetilde{p}_i$ and $\widetilde{p}_j$ is a Nash equilibrium. When firm *j* charges $\widetilde{p}_j$, firm *i*'s optimal response is to charge $\widetilde{p}_i = \widetilde{p}_j$ since a) if firm *i* sets $p_i < \widetilde{p}_i$ it makes lower (and negative) profits since $0 = \widetilde{\Pi}_i^=(\widetilde{p}_i, \widetilde{p}_j) \geq \widetilde{\Pi}_i^-(p_i, \widetilde{p}_j)$; b) if firm *i* sets $p_i > \widetilde{p}_i$ it makes equal profits than if it matches the rival since $0 = \widetilde{\Pi}_i^=(\widetilde{p}_i, \widetilde{p}_j) = \widetilde{\Pi}_i^+(p_i, \widetilde{p}_j) = 0$. An identical argument applies to firm *j*.

iii) the pair $\widetilde{p}_i$ and $\widetilde{p}_j$ is the unique possible Nash equilibrium. By contradiction, assume that an equilibrium exists with prices $p_i$ and $p_j$, where $p_i = p_j$. If $p_i > \widetilde{p}_i$, $\widetilde{\Pi}_i^=(p_i, p_j) < \widetilde{\Pi}_i^+(p_i + \varepsilon, p_j) = 0$, with ε is positive and sufficiently small. If $p_i < \widetilde{p}_i$, then $\widetilde{\Pi}_i^=(p_i, p_j) < \widetilde{\Pi}_i^-(p_i - \varepsilon, p_j)$ (with ε > 0 and sufficiently small) or $\widetilde{\Pi}_i^=(p_i, p_j) < \widetilde{\Pi}_i^-(p_i^M, p_j)$ (with $p_i^M$ being the optimal monopoly price), depending on whether $p_i$ is smaller or larger than $p_i^M$.

❒

**Proof of Proposition 5**

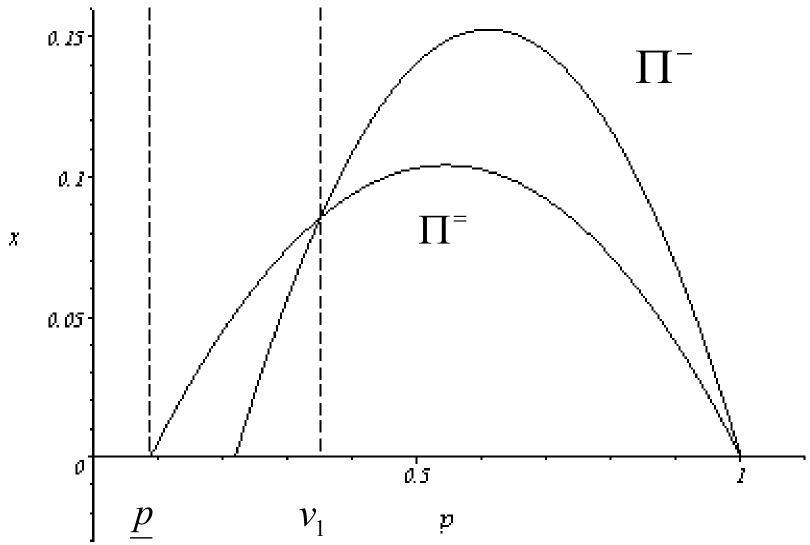
The proof of the result involves the simple but tedious study of the welfare functions evaluated at equilibrium prices, which are given in Table 1. Further details are available from the author upon request.

❒

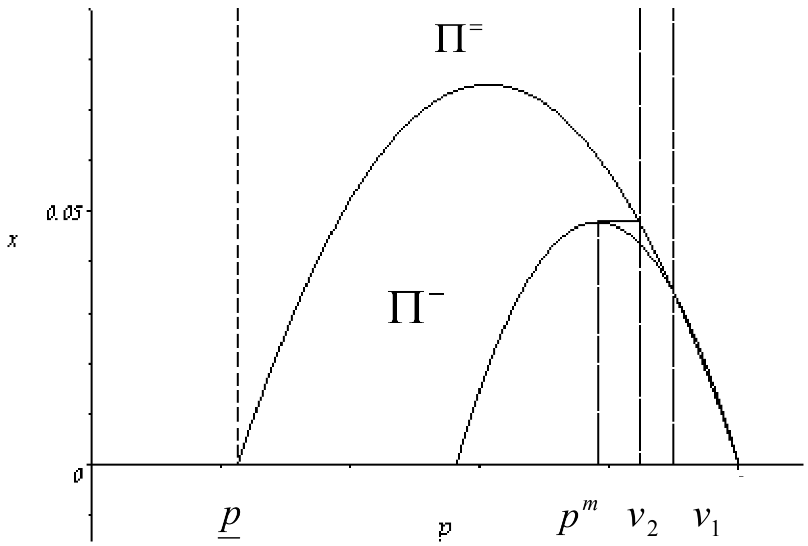
| **Table 1** |
| --- |

Table 1 - Equilibrium welfare measures.

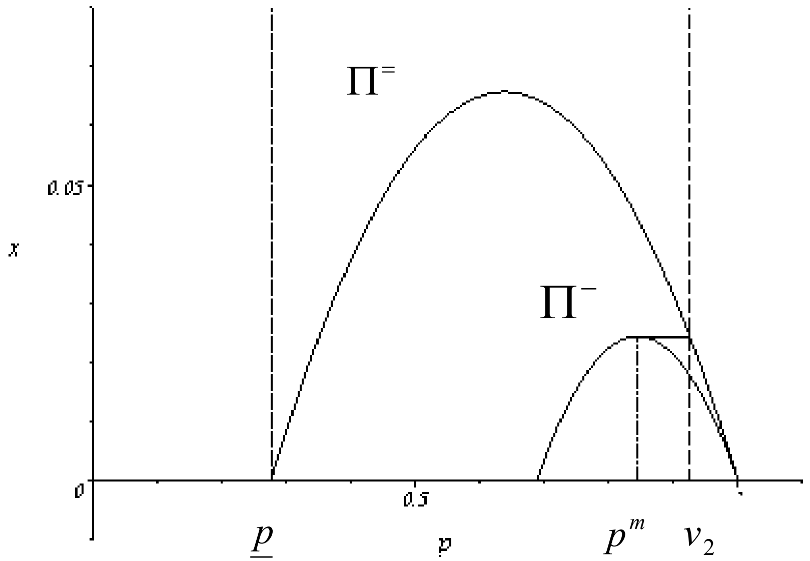
| *Efficient tie-breaking rule* | |
|---|---|
| $S = \begin{cases} \frac{1}{32}(4cx - 3c + 4)^2 \\ \frac{1}{128}(8cx^2 - 4cx + c - 4)^2 \end{cases}$ | when $c \in \left(0, \frac{4}{5 - 4x - 8x^2}\right]$ <br> when $c \in \left[\frac{4}{5 - 4x - 8x^2}, \frac{4}{1 - 4x + 8x^2}\right)$ |
| $\Pi = \begin{cases} \frac{1}{8}c(4x^2 - 1)(3c - 4cx - 4) \\ \frac{1}{64}(8cx^2 - 4cx + c - 4)^2 \end{cases}$ | when $c \in \left(0, \frac{4}{5 - 4x - 8x^2}\right]$ <br> when $c \in \left[\frac{4}{5 - 4x - 8x^2}, \frac{4}{1 - 4x + 8x^2}\right)$ |
| $W = \begin{cases} \frac{1}{32}(4cx - 3c + 4)(4 + c + 4cx - 16cx^2) \\ \frac{3}{128}(8cx^2 - 4cx + c - 4)^2 \end{cases}$ | when $c \in \left(0, \frac{4}{5 - 4x - 8x^2}\right]$ <br> when $c \in \left[\frac{4}{5 - 4x - 8x^2}, \frac{4}{1 - 4x + 8x^2}\right)$ |
| *Random tie-breaking rule* | |
| $\tilde{S} = \frac{1}{8}(2cx^2 - 2cx + c - 2)^2$ | when $c \in \left(0, \frac{2}{2x^2 - 2x + 1}\right]$ |
| $\tilde{\Pi} = 0$ | when $c \in \left(0, \frac{2}{2x^2 - 2x + 1}\right]$ |
| $\tilde{W} = \frac{1}{8}(2cx^2 - 2cx + c - 2)^2$ | when $c \in \left(0, \frac{2}{2x^2 - 2x + 1}\right]$ |



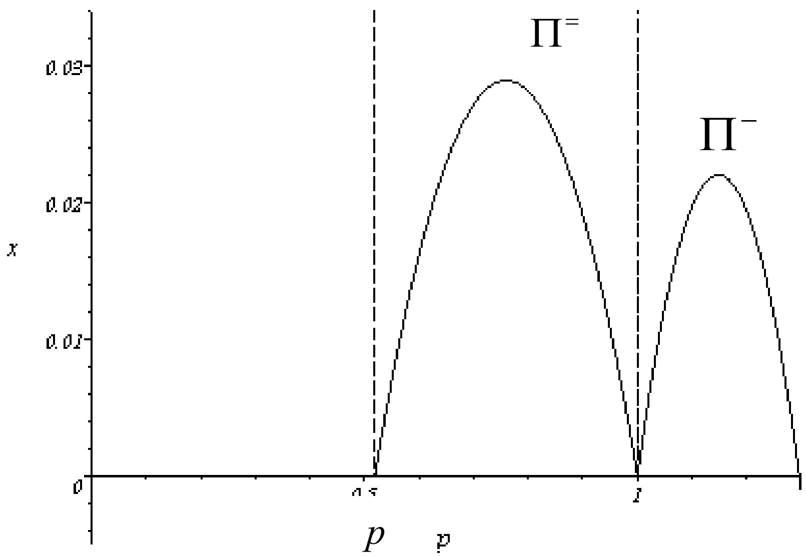

Panel a): $c \in (1, 1/(1\text{-}x\text{-}x^2)]$




Panel b) $c \in (1/(1\text{-}x\text{-}x^2), 4/(3\text{-}4x)]$




Panel c): $c \in (4/(3\text{-}4x), 2/(1\text{-}2x\text{+}2x^2)]$




Panel d) $c \in (2/(1\text{-}2x\text{+}2x^2), 4/(1\text{-}4x\text{+}8x^2)]$

Figure 1 - $\Pi^-$(.) and $\Pi^=$(.) under different parametric conditions