Control of quantum stochastic differential equations

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## Contents

1. Quantum Stochastic Calculus .................................................. 3
2. Matrix Elements and Iteration Schemes ...................................... 7
3. Error Analysis ....................................................................... 13
4. Applications to quantum stochastic control ............................... 17

References .............................................................................. 19
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Abstract. We review the basic features of the quantum stochastic calculus. Iteration schemes for the computation of the matrix elements of solutions of unitary quantum stochastic evolutions and associated quantum flows are provided along with a basic error analysis of the convergence of the iteration schemes. The application of quantum stochastic calculus to the solution of the quantum version of the quadratic cost control problem is described.

1. Quantum Stochastic Calculus

Let \( B_t = \{B_t(\omega)/\omega \in \Omega\}, t \geq 0, \) be one-dimensional Brownian motion. Integration with respect to \( B_t \) was defined by Itô in [28]. A basic result of the theory is that stochastic integral equations of the form

\[
X_t = X_0 + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dB_s
\]

(1.1)

can be interpreted as stochastic differential equations of the form

\[
dX_t = b(t, X_t) \, dt + \sigma(t, X_t) \, dB_t
\]

(1.2)

where differentials are handled with the use of Itô’s formula

\[
(dB_t)^2 = dt, \quad dB_t \, dt = dt \, dB_t = (dt)^2 = 0
\]

(1.3)

In [27], Hudson and Parthasarathy obtained a Fock space representation of Brownian motion and Poisson process.

Definition 1. The Boson Fock space \( \Gamma = \Gamma(L^2(\mathbb{R}_+, \mathbb{C})) \) over \( L^2(\mathbb{R}_+, \mathbb{C}) \) is the Hilbert space completion of the linear span of the exponential vectors \( \psi(f) \) under the inner product

\[
< \psi(f), \psi(g) > = e^{<f,g>}
\]

(1.4)

where \( f, g \in L^2(\mathbb{R}_+, \mathbb{C}) \) and \( <f,g> = \int_0^{+\infty} \bar{f}(s) \, g(s) \, ds \) where, here and in what follows, \( \bar{z} \) denotes the complex conjugate of \( z \in \mathbb{C} \).
The annihilation, creation and conservation operators \( A(f), A^\dagger(f) \) and \( \Lambda(F) \) respectively, are defined on the exponential vectors \( \psi(g) \) of \( \Gamma \) as follows.

**Definition 2.**

\[
\begin{align*}
A_t \psi(g) &= \int_0^t g(s) \, ds \, \psi(g) \\
A^\dagger_t \psi(g) &= \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \psi(g + \epsilon X_{[0,t]}) \\
\Lambda_t \psi(g) &= \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \psi(e^{\epsilon X_{[0,t]}} g)
\end{align*}
\]

The basic quantum stochastic differentials \( dA_t, dA^\dagger_t, \) and \( d\Lambda_t \) are defined as follows.

**Definition 3.**

\[
\begin{align*}
dA_t &= A_{t+dt} - A_t \\
dA^\dagger_t &= A^\dagger_{t+dt} - A^\dagger_t \\
d\Lambda_t &= \Lambda_{t+dt} - \Lambda_t
\end{align*}
\]

The fundamental result which connects classical with quantum stochastics is that the processes \( B_t \) and \( P_t \) defined by

\[
\begin{align*}
B_t &= A_t + A^\dagger_t \\
P_t &= \Lambda_t + \sqrt{\lambda}(A_t + A^\dagger_t) + \lambda t
\end{align*}
\]

are identified, through their statistical properties e.g their vacuum characteristic functionals

\[
\begin{align*}
< \psi(0), e^{isB_t} \psi(0) > &= e^{-\frac{s^2}{2}t} \quad (1.13) \\
< \psi(0), e^{isP_t} \psi(0) > &= e^{\lambda(e^{is} - 1)t} \quad (1.14)
\end{align*}
\]

with Brownian motion and Poisson process of intensity \( \lambda \) respectively.

Hudson and Parthasarathy defined stochastic integration with respect to the noise differentials of Definition 3 and obtained the Itô multiplication table.
Within the framework of Hudson-Parthasarathy Quantum Stochastic Calculus, classical quantum mechanical evolution equations take the form

\[ dU_t = -\left( \left( iH + \frac{1}{2} L^*L \right) dt + L^*W dA_t - L dA_t^\dagger + (1 - W) d\Lambda_t \right) U_t \] (1.15)

\[ U_0 = 1 \]

where, for each \( t \geq 0 \), \( U_t \) is a unitary operator defined on the tensor product \( \mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbb{C})) \) of a system Hilbert space \( \mathcal{H} \) and the noise (or reservoir) Fock space \( \Gamma \). Here \( H, L, W \) are in \( \mathcal{B}(\mathcal{H}) \), the space of bounded linear operators on \( \mathcal{H} \), with \( W \) unitary and \( H \) self-adjoint. Notice that for \( L = W = -1 \) equation (1.15) reduces to a classical SDE of the form (1.2). Here and in what follows we identify time-independent, bounded, system space operators \( X \) with their ampliation \( X \otimes 1 \) to \( \mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathbb{C})) \).

The quantum stochastic differential equation satisfied by the quantum flow

\[ j_t(X) = U_t^*XU_t \] (1.16)

where \( X \) is a bounded system space operator, is

\[ dj_t(X)(1.47) j_t \left( i[H, X] - \frac{1}{2} (L^*LX + XL^*L - 2L^*XL) \right) dt + j_t ([L^*, X] W) dA_t + j_t (W^*[X, L]) dA_t^\dagger + j_t (W^*XW - X) d\Lambda_t \]

\[ j_0(X) = X, \quad t \in [0, T] \]

The commutation relations associated with the operator processes \( A_t, A_t^\dagger \) are the Canonical (or Heisenberg) Commutation Relations (CCR), namely

\[ [A_t, A_t^\dagger] = t I \] (1.18)

Classical and quantum stochastic calculi were unified by Accardi, Lu, and Volovich in [17] within the framework of the white noise theory of
Denoting the basic white noise functionals by $a_t$ and $a_t^\dagger$, they showed that the stochastic differentials of the Hudson-Parthasarathy processes of [27] can be written as

\begin{align}
(1.19) \quad &dA_t = a_t \, dt \\
(1.20) \quad &dA_t^\dagger = a_t^\dagger \, dt \\
(1.21) \quad &d\Lambda_t = a_t a_t^\dagger \, dt
\end{align}

and Hudson-Parthasarathy stochastic differential equations are reduced to white noise equations. This unification started a whole new theory corresponding to quantum stochastic processes given by powers of the white noise functionals. The results for the first such nonlinear extension, the square of white noise, can be summarized as follows.

Let $U(sl(2; \mathbb{R}))$ denote the universal enveloping algebra of $sl(2; \mathbb{R})$ with generators $B^\dagger$, $M$, $B^-$ satisfying the commutation relations

\begin{equation}
(1.22) \quad [B^-, B^\dagger] = M, \quad [M, B^\dagger] = 2B^\dagger, \quad [M, B^-] = -2B^-
\end{equation}

with involution

\begin{equation}
(1.23) \quad (B^-)^* = B^\dagger, \quad M^* = M
\end{equation}

After renormalization (cf. [17]), the square of white noise stochastic differentials

\begin{align}
(1.24) \quad &dB_t^- = a_t^2 \, dt \\
(1.25) \quad &dB_t^\dagger = a_t^{\dagger 2} \, dt \\
(1.26) \quad &d\Lambda_t = a_t a_t^\dagger \, dt
\end{align}

can be defined on a Fock space. It was proved by Accardi-Skeide in [20] that the Fock space suitable for representing the square of white noise processes is the Finite Difference Fock space developed by Boukas-Feinsilver in [22] based on the Finite Difference Lie algebra of Feinsilver (cf. [26]).

The unitarity of solutions problem for “square of white noise” evolutions was open for several years. Preliminary work was done by Accardi, Hida, Boukas, and Kuo in [1], [4], [5],[13], [15]. In [8] Accardi and Boukas used the Boson Fock space representation of the square of white noise processes obtained by Accardi-Frantz-Skeide in [14], to show that square of white noise unitary evolutions satisfy quantum SDE of the type...
$$dU_t = \left( -\frac{1}{2}(D\mathcal{L})_t D_- + iH \right) dt + dA_t(D_-) + dA_t^\dagger(-l(W)D_-) + d\mathcal{L}_t(W - I) \right) U_t$$

$$U_0 = 1$$

formulated on the module $\mathcal{B}(\mathcal{H}_S) \otimes \Gamma(\mathcal{K})$, where $\mathcal{H}_S$ is a system Hilbert space, $\mathcal{K} = l_2(\mathbb{N})$ and $\Gamma(\mathcal{K})$ denotes the Fock space over $\mathcal{K}$ (see [14] for notation and details).

Applications of quantum stochastic calculus to the control of quantum evolution and Langevin equations (quantum flows) can be found in [7], [9], [10], [21], [23], [24], [25].

2. Matrix Elements and Iteration Schemes

The fundamental theorems of the Hudson-Partasarathy quantum stochastic calculus give formulas for expressing the matrix elements of quantum stochastic integrals in terms of ordinary Riemann-Lebesgue integrals.

**Theorem 1.** Let

\begin{equation}
(2.1) \quad M(t) = \int_0^t E(s) \, d\Lambda(s) + F(s) \, dA(s) + G(s) \, dA^\dagger(s) + H(s) \, ds
\end{equation}

where $E$, $F$, $G$, $H$ are (in general) time dependent adapted processes.

Let also $u \otimes \psi(f)$ and $v \otimes \psi(g)$ be in the exponential domain of $\mathcal{H} \otimes \Gamma$. Then

\begin{equation}
\langle u \otimes \psi(f), M(t) v \otimes \psi(g) \rangle = \int_0^t E(s) g(s) F(s) + F(s) G(s) + H(s) \rangle v \otimes \psi(g) > ds
\end{equation}

**Proof.** See theorem 4.1 of [27]

\[ \square \]

**Theorem 2.** Let

\begin{equation}
(2.3) \quad M(t) = \int_0^t E(s) \, d\Lambda(s) + F(s) \, dA(s) + G(s) \, dA^\dagger(s) + H(s) \, ds
\end{equation}

and

\begin{equation}
(2.4) \quad M'(t) = \int_0^t E'(s) \, d\Lambda(s) + F'(s) \, dA(s) + G'(s) \, dA^\dagger(s) + H'(s) \, ds
\end{equation}

where $E$, $F$, $G$, $H$, $E'$, $F'$, $G'$, $H'$ are (in general) time dependent adapted processes. Let also $u \otimes \psi(f)$ and $v \otimes \psi(g)$ be in the exponential domain of $\mathcal{H} \otimes \Gamma$. Then
\[ <M(t) u \otimes \psi(f), M'(t) v \otimes \psi(g)> = \int_0^t <M(s) u \otimes \psi(f), \left( \bar{f}(s) g(s) E'(s) + g(s) F'(s) + \bar{g}(s) G'(s) + H'(s) \right) v \otimes \psi(g) > + <\left( \bar{g}(s) f(s) E(s) + f(s) F(s) + \bar{g}(s) G(s) + H(s) \right) u \otimes \psi(f), M'(s) v \otimes \psi(g)> + <\left( f(s) E(s) + G(s) \right) u \otimes \psi(f), \left( g(s) E'(s) + G'(s) \right) v \otimes \psi(g)> \] ds

Proof. See theorem 4.3 of [27] \( \square \)

We are interested in defining iteration schemes which can be used to compute the matrix elements

\[ < u \otimes \psi(f), U_t v \otimes \psi(g) >, \quad < u \otimes \psi(f), j_t(X) v \otimes \psi(g) > \]

and the corresponding probability amplitudes

\[ \langle u \otimes \psi(f), U_t v \otimes \psi(g) \rangle^2, \quad \langle u \otimes \psi(f), j_t(X) v \otimes \psi(g) \rangle^2 \]

related to the quantum flow (1.16) and the Hudson-Parthasarathy stochastic differential equation

\[ dU_t = \left( K dt + B dA_t + C dA_t^\dagger + D d\Lambda_t \right) U_t \]

with initial condition

\[ U_0 = I \]

where \( t \in [0, T] \) for some \( T > 0 \), and \( K, B, C, D \) are bounded system space operators of the form appearing in (1.15), i.e

\[ K = -\left( iH + \frac{1}{2} L^* L \right) \]
\[ B = -L^* W \]
\[ C = L \]
\[ D = W - 1 \]

Equations (2.8) and (2.9) have the integral form

\[ U_t = I + \int_0^t K U_s dA_s + B U_s dA_s^\dagger + C U_s dA_s^\dagger + D U_s d\Lambda_s \]
defined (cf. [27], Proposition 7.1) as the \([0, T]\)-uniform limit of the sequence \(U_n = \{U_{n,t} / t \geq 0\}\) defined recursively on the exponential domain of \(H \otimes \Gamma\) by

\[
U_{0,t} = I
\]

and, for \(n \geq 1\),

\[
U_{n,t} = I + \int_0^t K U_{n-1,s} ds + B U_{n-1,s} dA_s + C U_{n-1,s} dA_s^t + D U_{n-1,s} \Lambda_s
\]

By Theorem 1, the matrix elements of (2.16) are given, for \(n \geq 1\), by the recursion scheme

\[
\langle (2.17) \psi(f), U_{n,t} v \otimes \psi(g) \rangle = \langle u \otimes \psi(f), v \otimes \psi(g) \rangle + \int_0^t \{ \mathcal{M}(s) g(s) < u \otimes \psi(f), D U_{n-1,s} v \otimes \psi(g) > + g(s) < u \otimes \psi(f), B U_{n-1,s} v \otimes \psi(g) > + \mathcal{F}(s) < u \otimes \psi(f), C U_{n-1,s} v \otimes \psi(g) > + < u \otimes \psi(f), K U_{n-1,s} v \otimes \psi(g) > \} ds
\]

Letting

\[
(2.18) u_{D^*} = D^* u \\
(2.19) u_{B^*} = B^* u \\
(2.20) u_{C^*} = C^* u \\
(2.21) u_{K^*} = K^* u
\]

we can rewrite iteration scheme (2.17) as

**Iteration Scheme 1. (Unitary Evolutions)**

\[
\langle (2.22) \psi(f), U_{n,t} v \otimes \psi(g) \rangle = \langle u \otimes \psi(f), v \otimes \psi(g) \rangle + \int_0^t \{ \mathcal{M}(s) g(s) < u_{D^*} \otimes \psi(f), U_{n-1,s} v \otimes \psi(g) > + g(s) < u_{B^*} \otimes \psi(f), U_{n-1,s} v \otimes \psi(g) > + \mathcal{F}(s) < u_{C^*} \otimes \psi(f), U_{n-1,s} v \otimes \psi(g) > + < u_{K^*} \otimes \psi(f), U_{n-1,s} v \otimes \psi(g) > \} ds
\]

with

\[
(2.23) < u \otimes \psi(f), U_{0,t} v \otimes \psi(g) > = \langle u \otimes \psi(f), v \otimes \psi(g) \rangle
\]

The limit form of (2.22) as \(n \to +\infty\) is
Iteration Scheme 2. (Time Iteration of Unitary Evolutions)

\[\langle 2.25 \rangle \psi(f), U_{t_n} v \otimes \psi(g) \rangle = \langle u \otimes \psi(f), U_{t_n-1} v \otimes \psi(g) \rangle + \int_{t_{n-1}}^{t_n} \{ \mathcal{T}(s)g(s) < u_{B^*} \otimes \psi(f), U_s v \otimes \psi(g) > + g(s) < u_{K^*} \otimes \psi(f), U_s v \otimes \psi(g) > \} ds \]

which, by subtraction of the cases \( t = t_n \) and \( t = t_{n-1} \) where \( t_{n-1} \leq t_n \), implies

\[\langle 2.26 \rangle \psi(f), U_{t_n} v \otimes \psi(g) \rangle = \langle u \otimes \psi(f), U_{t_n-1} v \otimes \psi(g) \rangle + \int_{t_{n-1}}^{t_n} \{ \mathcal{T}(s)g(s) < u_{B^*} \otimes \psi(f), U_s v \otimes \psi(g) > + g(s) < u_{K^*} \otimes \psi(f), U_s v \otimes \psi(g) > \} ds \]

or

The integral form of the quantum flow equation (1.17) is

\[\langle 2.27 \rangle \psi(X) = X + \int_0^t j_s(\hat{K}) ds + j_s(\hat{B}) dA_s + j_s(\hat{C}) dA_s^\dagger + j_s(\hat{D}) d\Lambda_s \]

where

\[\langle 2.28 \rangle \hat{K} = i[H, X] - \frac{1}{2}(L^* LX + XL^* L - 2L^* XL) \]

\[\langle 2.29 \rangle \hat{B} = [L^*, X] W \]

\[\langle 2.30 \rangle \hat{C} = W^*[X, L] \]

\[\langle 2.31 \rangle \hat{D} = W^* X W - X \]

The corresponding iteration scheme is

\[\langle 2.32 \rangle j_{0,t}(X) = X \]

and for \( n \geq 1 \)

\[\langle X.33 \rangle j_{n,t}(X) = \int_0^t j_{n-1,s}(\hat{K}) ds + j_{n-1,s}(\hat{B}) dA_s + j_{n-1,s}(\hat{C}) dA_s^\dagger + j_{n-1,s}(\hat{D}) d\Lambda_s \]

The matrix element form of the iteration scheme (2.32) and (2.33) is
\[ <u \otimes \psi(f), j_{n,t}(X)v \otimes \psi(g) > = <u \otimes \psi(f), Xv \otimes \psi(g) > + \]
\[ <u \otimes \psi(f), \left( \int_0^t j_{n-1,s}(\hat{K}) ds + j_{n-1,s}(\hat{B}) dA_s + j_{n-1,s}(\hat{C}) dA_s^t + j_{n-1,s}(\hat{D}) dA_s \right) v \otimes \psi(g) > \]

which by Theorem 1 yields

**Iteration Scheme 3.** *(General Quantum Flows)*

\[ <u \otimes \psi(f), j_{1,t}(X)v \otimes \psi(g) > = <u \otimes \psi(f), Xv \otimes \psi(g) > + \int_0^t \{ \mathcal{J}(s)g(s) < u \otimes \psi(f), j_{n-1,s}(\hat{B})v \otimes \psi(g) > + \mathcal{J}(s) < u \otimes \psi(f), j_{n-1,s}(\hat{C})v \otimes \psi(g) > + < u \otimes \psi(f), j_{n-1,t}(\hat{K})v \otimes \psi(g) > \} ds \]

Notice that for \( n = 1 \) (2.35) becomes

\[ (2.36) \otimes \psi(f), j_{1,t}(X)v \otimes \psi(g) > \]

\[ = < u \otimes \psi(f), Xv \otimes \psi(g) > + < u \otimes \psi(f), \left( \int_0^t \hat{K} ds + \hat{B} dA_s + \hat{C} dA_s^t + \hat{D} dA_s \right) v \otimes \psi(g) > \]

\[ = < u \otimes \psi(f), Xv \otimes \psi(g) > + \int_0^t \{ \mathcal{J}(s)g(s) < u \otimes \psi(f), \hat{D} v \otimes \psi(g) > + g(s) < u \otimes \psi(f), \hat{B} > \mathcal{J}(s) < u \otimes \psi(f), \hat{C} v \otimes \psi(g) > + < u \otimes \psi(f), \hat{K} v \otimes \psi(g) > \} ds \]

\[ = < u \otimes \psi(f), Xv \otimes \psi(g) > + \int_0^t \{ \mathcal{J}(s)g(s) < u \otimes \psi(f), \hat{D} v \otimes \psi(g) > + g(s) < u \otimes \psi(f), \hat{B} > \mathcal{J}(s) < u \otimes \psi(f), \hat{C} v \otimes \psi(g) > + < u \otimes \psi(f), \hat{K} v \otimes \psi(g) > \} ds \]

and so, letting \( u_{X^*} = X^* u \), we have

\[ <(2.37)\psi(f), j_{1,t}(X)v \otimes \psi(g) > = < u_{X^*} \otimes \psi(f), v \otimes \psi(g) > + \int_0^t \mathcal{J}(s)g(s) ds < u \otimes \psi(f), v \otimes \psi(g) > + \int_0^t g(s) ds < u \otimes \psi(f), v \otimes \psi(g) > + \int_0^t \mathcal{J}(s) ds < u \otimes \psi(f), v \otimes \psi(g) > + \int_0^t ds < u \otimes \psi(f), v \otimes \psi(g) > \]

The general theory of quantum flows, in the context of Hudson-Partasarathy calculus, can be found in [29]. We now consider flows \( \{ j_t(X) \mid t \geq 0 \} \) of the standard quantum mechanical form

\[ (2.38) \quad j_t(X) = U_t^* X U_t \]

where \( U_t \) is , for each \( t \geq 0 \), a unitary operator.

**Proposition 1.** Let \( X \) be a bounded system space operator, let \( U_t \) and \( U_{n,t} \) be for each \( t \in [0, T] \) and \( n \geq 1 \) as in (2.14) and (2.16) respectively, and let \( U_t^* \) and \( U_{n,t}^* \) be their adjoints. If
$$j_t(X) = U_t^* X U_t$$

and

$$j_{n,t}(X) = U_{n,t}^* X U_{n,t}$$

then

$$\lim_{n \to \infty} < u \otimes \psi(f), j_{n,t}(X) \psi \otimes \psi(g) > = < u \otimes \psi(f), j_t(X) \psi \otimes \psi(g) >$$

for all $u \otimes \psi(f)$ and $\psi \otimes \psi(g)$ in the exponential domain of $\mathcal{H} \otimes \Gamma$. Convergence is uniform on $[0, T]$.

**Proof.**

$$| < u \otimes \psi(f), j_t(X) \psi \otimes \psi(g) > - < u \otimes \psi(f), j_{n,t}(X) \psi \otimes \psi(g) > |$$

$$= | < u \otimes \psi(f), (j_t(X) - j_{n,t}(X)) \psi \otimes \psi(g) > |$$

$$= | < u \otimes \psi(f), (U_t^* X U_t - U_{n,t}^* X U_{n,t}) \psi \otimes \psi(g) > |$$

$$\leq | < u \otimes \psi(f), (U_t^* - U_{n,t}^*) X U_t \psi \otimes \psi(g) > | + | < u \otimes \psi(f), U_{n,t}^* X (U_t - U_{n,t}) \psi \otimes \psi(g) > |$$

$$= | < (U_t - U_{n,t}) u \otimes \psi(f), X U_t \psi \otimes \psi(g) > | + | < U_{n,t} u \otimes \psi(f), X (U_t - U_{n,t}) \psi \otimes \psi(g) > |$$

$$\leq \| (U_t - U_{n,t}) u \otimes \psi(f) \| \| X \| \| U_t \| \| \psi \otimes \psi(g) \| + \| U_{n,t} u \otimes \psi(f) \| \| X \| \| (U_t - U_{n,t}) \psi \otimes \psi(g) \|$$

$$\leq \| (U_t - U_{n,t}) u \otimes \psi(f) \| \| X \| \| \psi \otimes \psi(g) \| + \| U_{n,t} u \otimes \psi(f) \| \| X \| \| (U_t - U_{n,t}) \psi \otimes \psi(g) \|$$

since $\| U_t \| = 1$. Since $U_{n,t}$ converges to $U_t$ on the exponential domain of $\mathcal{H} \otimes \Gamma$ uniformly with respect to $t$ and $\| U_{n,t} u \otimes \psi(f) \|$ is bounded, it follows that

$$| < u \otimes \psi(f), j_{n,t}(X) \psi \otimes \psi(g) > - < u \otimes \psi(f), j_t(X) \psi \otimes \psi(g) > | \to 0$$

as $n \to +\infty$.

The iteration scheme for the matrix element associated with (2.40) is obtained, with the use of Theorems 1 and 2 as follows:

$$< u \otimes \psi(f), j_{n,t}(X) \psi \otimes \psi(g) > = < U_{n,t} u \otimes \psi(f), X U_{k,t} \psi \otimes \psi(g) >$$

$$= < \left( I + \int_0^t K U_{n-1,s} ds + B U_{n-1,s} dA_s + C U_{n-1,s} dA^\dagger_s + D U_{n-1,s} d\Lambda_s \right) u \otimes \psi(f), X \left( I + \int_0^t K U_{k-1,s} ds + B U_{k-1,s} dA_s + C U_{k-1,s} dA^\dagger_s + D U_{k-1,s} d\Lambda_s \right) v \otimes \psi(g) >$$
Iteration Scheme 4. (Quantum Mechanical Flows) For $n, k \geq 1$

\[
\langle u \otimes \psi(f), U_{n,t}^{*} X U_{k,s} v \otimes \psi(g) \rangle = \langle u \otimes \psi(f), (X v) \otimes \psi(g) \rangle + \int_{0}^{t} \langle \tilde{f}(s) g(s) X D + (s) X B + \tilde{f}(s) X C + X K \rangle U_{k-1,s} v \otimes \psi(g) > + \int_{0}^{t} \langle (s) f(s) X D + f(s) X B + \tilde{g}(s) X C + X K \rangle U_{k-1,s} v \otimes \psi(g) > + \int_{0}^{t} \langle g(s) f(s) X D + f(s) B + \tilde{g}(s) C + K \rangle U_{n-1,s} v \otimes \psi(g) > + \int_{0}^{t} \langle g(s) f(s) X D + (s) X B + \tilde{g}(s) X C + X K \rangle U_{n-1,s} v \otimes \psi(g) > \]

and using (2.16) we obtain

Notice that for $n = 0$ or $k = 0$ (2.43) reduces to (2.22).

3. Error Analysis

Proposition 2. Let $\epsilon > 0$, and let $U_t$ and $U_{n,t}$, where $0 \leq t \leq T < +\infty$, be defined respectively by (2.14) and (2.16). Then for all $u \otimes \psi(f)$
and $v \otimes \psi(g)$ in the exponential domain of $\mathcal{H} \otimes \Gamma$, with $g$ locally bounded and $u, v \neq 0$

\[(3.1) \langle u \otimes \psi(f), U_t v \otimes \psi(g) \rangle - \langle u \otimes \psi(f), U_{n,t} v \otimes \psi(g) \rangle \mid < \epsilon \]

for all $t \in [0, T]$ provided that

\[(3.2) \sqrt{\frac{\lambda^{n+1} T^{n+1}}{(n+1)!}} < \frac{\epsilon}{\|u\| \|v\| e \|f\|_2} e \|g\|_2} \]

where $\epsilon > 0$ is the required degree of accuracy and

\[(3.3) \|f\|^2 = \int_0^{+\infty} |f(s)|^2 \, ds \]
\[(3.4) \|g\|^2 = \int_0^{+\infty} |g(s)|^2 \, ds \]
\[(3.5) \lambda = 6 \alpha(T)^2 e^T M \]
\[(3.6) M = \max (\|K\|, \|B\|, \|C\|, \|D\|) \]
\[(3.7) \alpha(T) = \sup_{0 \leq s \leq T} \max (\|g(s)\|^2, |g(s)|, 1) \]

**Proof.**

\[
\mid \mid \langle u \otimes \psi(f), U_t v \otimes \psi(g) \rangle - \langle u \otimes \psi(f), U_{n,t} v \otimes \psi(g) \rangle \mid ^2 \\
= \mid \mid \langle u \otimes \psi(f), (U_t - U_{n,t}) v \otimes \psi(g) \rangle \mid ^2 \leq \|u \otimes \psi(f)\|^2 \|(U_t - U_{n,t}) v \otimes \psi(g)\|^2 \\
= \|u \otimes \psi(f)\|^2 \| \{ \int_0^t K (U_{s_1} - U_{n-1,s_1}) \, ds_1 + B (U_{s_1} - U_{n-1,s_1}) \, dA_{s_1} + C (U_{s_1} - U_{n-1,s_1}) \, dA_{s_1}^1 \\
+ D (U_{s_1} - U_{n-1,s_1}) \, dA_{s_1} \} v \otimes \psi(g)\|^2 \\
\]

which by Corollary 1 and Theorem 4.4 of [27] is
\begin{align*}
\leq & \ 6 \alpha(T)^2 \|u \otimes \psi(f)\|^2 \int_0^T e^{t-s_1} \left\{ \|K (U_{s_1} - U_{n-1,s_1}) v \otimes \psi(g)\|^2 \\
+ & \ ||B (U_{s_1} - U_{n-1,s_1}) v \otimes \psi(g)\|^2 + ||C (U_{s_1} - U_{n-1,s_1}) v \otimes \psi(g)\|^2 \\
+ & \ ||D (U_{s_1} - U_{n-1,s_1}) v \otimes \psi(g)\|^2 \right\} \, ds_1 \\
\leq & \ 6 \alpha(T)^2 \max \left\{ \|K\|, \|B\|, \|C\|, \|D\| \right\}^2 \|u \otimes \psi(f)\|^2 \int_0^T e^{t-s_1} \|u \otimes \psi(g)\|^2 \, ds_1 \\
= & \ \lambda \|u \otimes \psi(f)\|^2 \int_0^T \|U_{n-1,s_1} v \otimes \psi(g)\|^2 \, ds_1 \\
\leq & \ \lambda^2 \|u \otimes \psi(f)\|^2 \int_0^T \int_0^{s_1} \|U_{s_2} - U_{n-2,s_2} v \otimes \psi(g)\|^2 \, ds_2 \, ds_1 \\
\leq & \ \lambda^3 \|u \otimes \psi(f)\|^2 \int_0^T \int_0^{s_1} \int_0^{s_2} \|U_{s_3} - U_{n-3,s_3} v \otimes \psi(g)\|^2 \, ds_3 \, ds_2 \, ds_1 \\
\vdots \\
\leq & \ \lambda^n \|u \otimes \psi(f)\|^2 \int_0^T \int_0^{s_1} \int_0^{s_2} \ldots \int_0^{s_n-1} \|U_{s_n} - U_{0,s_n} v \otimes \psi(g)\|^2 \, ds_n \ldots ds_3 \, ds_2 \, ds_1 \\
\text{which using} \\
U_{s_n} & = 1 + \int_0^{s_n} KU_{s_{n+1}} \, ds_{n+1} + BU_{s_{n+1}} \, dA_{s_{n+1}} + C U_{s_{n+1}} \, dA_{s_{n+1}}^i + DU_{s_{n+1}} \, dA_{s_{n+1}} \\
U_{0,s_n} & = 1 \\
\text{and the unitarity of } U_{s_n+1} \text{, becomes} \\
\leq & \ \lambda^{n+1} \|u \otimes \psi(f)\|^2 \int_0^T \int_0^{s_1} \int_0^{s_2} \ldots \int_0^{s_n-1} \int_0^{s_n} \|U_{s_{n+1}} v \otimes \psi(g)\|^2 \, ds_{n+1} \ldots ds_3 \, ds_2 \, ds_1 \\
= & \ \lambda^{n+1} \|u \otimes \psi(f)\|^2 \int_0^T \int_0^{s_1} \int_0^{s_2} \ldots \int_0^{s_n-1} \int_0^{s_n} \|v \otimes \psi(g)\|^2 \, ds_{n+1} \ldots ds_3 \, ds_2 \, ds_1 \\
= & \ \|u \otimes \psi(f)\|^2 \|v \otimes \psi(g)\|^2 \lambda^{n+1} \int_0^T \int_0^{s_1} \int_0^{s_2} \ldots \int_0^{s_n-1} ds_{n+1} \ldots ds_3 \, ds_2 \, ds_1 \\
= & \ \|u \|^2 \|v\|^2 e^{l/f} e^{g/2} \lambda^{n+1} \frac{T_{n+1}}{(n+1)!} \\
= & \ \|u\|^2 \|v\|^2 e^{l/f} e^{g/2} \lambda^{n+1} \frac{T_{n+1}}{(n+1)!} \\
\text{which is less than } \epsilon^2 \text{ provided that } (3.2) \text{ is satisfied.} \end{align*}
Corollary 1. In the notation of Proposition 2

\[ | \langle 3.8 \rangle u \otimes \psi(f), U_t v \otimes \psi(g) \rangle - | \langle u \otimes \psi(f), U_{n,t} v \otimes \psi(g) \rangle | < \epsilon \]

for all \( t \in [0, T] \) and \( u, v, f, g \) with \( u, v \neq 0 \), provided that

\[ (3.9) \quad \sqrt{\frac{\lambda^{n+1} T^{n+1}}{(n+1)!}} < \frac{\epsilon}{\|u\| e^{1/2} \|v\| e^{1/2}} \]

Proof. The proof follows by applying the triangle inequality to (3.1). \( \square \)

Proposition 3. In the notation of Proposition 1

\[ | \langle 3.10 \rangle \psi(f), j_{n,t}(X) v \otimes \psi(g) \rangle - | \langle u \otimes \psi(f), j_t(X) v \otimes \psi(g) \rangle | < \epsilon \]

for all \( t \in [0, T] \) and \( u, v, f, g \) with \( u, v \neq 0 \), provided that

\[ (3.11) \quad \sqrt{\frac{\lambda^{n+1} T^{n+1}}{(n+1)!}} + 1 < \frac{\lambda^{1/2} T^{1/2}}{\|u\|^{1/2} \|v\|^{1/2} e^{1/2} \|a\|^{1/2} \|X\|^{1/2}} \]

Proof.

\[ | \langle u \otimes \psi(f), j_{n,t}(X) v \otimes \psi(g) \rangle | \leq \|U_t - U_{n,t}\| \|v \otimes \psi(g)\| + \|U_t - U_{n,t}\| \|u \otimes \psi(f)\| \|X\| \|U_t - U_{n,t}\| \|v \otimes \psi(g)\| \]

and using, as in the proof of Proposition 2,

\[ \|(U_t - U_{n,t}) a \otimes \psi(b)\|^2 \leq \|a \otimes \psi(b)\|^2 \frac{\lambda^{n+1} T^{n+1}}{(n+1)!} \]

we obtain

\[ | \langle u \otimes \psi(f), j_{n,t}(X) v \otimes \psi(g) \rangle | \leq \|u \otimes \psi(f)\| \|v \otimes \psi(g)\| \|X\| \left( 2 \sqrt{\frac{\lambda^{n+1} T^{n+1}}{(n+1)!}} + \frac{\lambda^{n+1} T^{n+1}}{(n+1)!} \right) \]

\[ \leq \left( \sqrt{\frac{\lambda^{n+1} T^{n+1}}{(n+1)!}} + 1 \right)^2 \|u\| \|v\| e^{1/2} \|a\|^{1/2} \|X\| < \epsilon \]

from which (3.11) follows. \( \square \)
Corollary 2. In the notation of Proposition 3

\[ \| \| < u(3.12) f, j_{n,t}(X) v \otimes \psi(g) \| - \| < u \otimes \psi(f), j_{n,t}(X) v \otimes \psi(g) \| < \epsilon \]

for all \( t \in [0, T] \) and \( u, v, f, g \) with \( u, v \neq 0 \), provided that

\[ \sqrt{\lambda_{n+1} T^{n+1} (n+1)!} + 1 < \frac{\epsilon^{1/2}}{\|u\|^{1/2} \|v\|^{1/2} e^{\|u\|^{1/2} (n+1)! + 1}} \]

(3.13)

Proof. The proof follows by applying the triangle inequality to (3.10).

\[ \square \]

4. Applications to quantum stochastic control

The quadratic cost control problem of classical stochastic control theory was extended to the quantum stochastic framework by L. Accardi and A. Boukas in [7], [9], [10], [21], [23], [24], [25].

In the case of first order white noise it was shown that if \( U = \{U_t / t \geq 0\} \) is a stochastic process satisfying on a finite interval \([0, T]\) the quantum stochastic differential equation

\[ dU_t = (F U_t + u_t) dt + \Psi U_t dA_t + \Phi U_t dA_t^\dagger + Z U_t d\Lambda_t, \quad U_0 = 1 \]  

(4.1)

then the performance functional

\[ Q_\xi,T(u) = \int_0^T \left[ < U_t \xi, X^2 U_t \xi > + < u_t \xi, u_t \xi > \right] dt - < u_T \xi, U_T \xi > \]  

(4.2)

satisfies

\[ \min Q_\xi,T(u) = < \xi, \Pi \xi > \]  

(4.3)

where the minimum is taken over all processes of the form \( u_t = -\Pi U_t, \xi \) is an arbitrary vector in the exponential domain of the tensor product of the system Hilbert space and the BosonFock space over \( L^2([0, +\infty), \mathbb{C}) \), and \( \Pi \) is the solution of the Algebraic Riccati Equation

\[ \Pi F + F^* \Pi + \Phi^* \Pi \Phi - \Pi^2 + X^2 = 0 \]  

(4.4)

with the additional conditions

\[ \Pi \Psi + \Phi^* \Pi + \Phi^* \Pi Z = 0 \]

(4.5)

\[ \Pi Z + Z^* \Pi + Z^* \Pi Z = 0 \]  

(4.6)
Using this we have proved (ref. [9], [10]) that if $X$ is a bounded self-adjoint system operator such that the pair $(i H, X)$ is stabilizable, then the quadratic performance functional

\[
J_{\xi,T}(L, W) = \int_0^T \left[ \|j_t(X)\xi\|^2 + \frac{1}{4}\|j_t(L^* L)\xi\|^2 \right] dt + \frac{1}{2}\|j_T(L)\xi\|^2
\]

associated with the quantum stochastic flow \{\(j_t(X) = U_t^* X U_t / t \geq 0\)\} satisfying

\[
(4.8) \quad (H)(X) = j_t(i[H, X] - \frac{1}{2}(L^* L X + X L^* L - 2L^* XL)) dt + j_t([L^*, X] W) dA_t + j_t(W^* [X, L]) dA_t^\dagger + j_t(W^* X W - X) d\Lambda_t, \quad j_0(X) = X
\]

where $U = \{U_t / t \geq 0\}$ is the solution of

\[
dU_t = -[i H + \frac{1}{2} L^* L] dt + L^* W dA_t - L dA_t^\dagger + (1 - W) d\Lambda_t U_t, \quad U_0 = 1
\]

is minimized by choosing

\[
(4.10) \quad L = \sqrt{2} \Pi^{1/2} W_1
\]
\[
(4.11) \quad W = W_2
\]

where $\Pi$ is the solution of the Algebraic Riccati Equation

\[
(4.12) \quad i [H, \Pi] + \Pi^2 + X^2 = 0
\]

and $W_1, W_2$ are bounded unitary system operators commuting with $\Pi$. Moreover

\[
(4.13) \quad \min_{L, W} J_{\xi,T}(L, W) = \langle \xi, \Pi \xi \rangle
\]

In the case of quantum stochastic differential equations driven by the square of white noise processes, we have shown (ref. [10]) that if $X$ is a bounded self-adjoint system operator such that the pair $(i H, X)$ is stabilizable then the performance functional

\[
J_{\xi,T}(D, W) = \int_0^T \left[ \|j_t(X)\xi\|^2 + \frac{1}{4}\|j_t((D^* |D^*))\xi\|^2 \right] dt + \frac{1}{2} < \xi, j_T((D^* |D^*))\xi >
\]

associated with the quantum flow \{\(j_t(X) = U_t^* X U_t / t \geq 0\)\}, where $U = \{U_t / t \geq 0\}$ is the solution of the quantum stochastic differential equation
\[ dU_t = \left( \left( \frac{1}{2} D^* D^* + iH \right) dt + dA_t(D_-) + dA_t^\dagger(-r(W)D_-) + dL_t(W - I) \right) \]
\[ U_0 = 1 \]
is minimized by choosing

\[ D_- = \sum_n D_{-n} \otimes e_n \tag{4.16} \]

and

\[ W = \sum_{\alpha,\beta,\gamma} W_{\alpha,\beta,\gamma} \otimes \rho^+(B^{+\alpha} M^\beta B^{-\gamma}) \tag{4.17} \]

so that

\[ \frac{1}{2} (D^* D^*) = \left( \frac{1}{2} \sum_n D_{-n} D_{-n}^* \right) \otimes 1 = \Pi \tag{4.18} \]

and

\[ [H, D_{-n, m}] = [D_{-n}, D_{-m}^*] = [D_{-n}, W_{\alpha,\beta,\gamma}] = [D_{-n}, W_{\alpha,\beta,\gamma}^*] = 0 \tag{4.19} \]

for all \( n, m, \alpha, \beta, \gamma \), where \( \Pi \) is the positive self-adjoint solution of the Algebraic Riccati equation

\[ i [H, \Pi] + \Pi^2 + X^2 = 0 \tag{4.20} \]

Moreover

\[ \min_{D_- W} J_{\xi,T}(D_-, W) = \langle \xi, \Pi \xi \rangle \tag{4.21} \]

References


L. Accardi, A. Boukas, *Control of elementary quantum flows*, Proceedings of the 5th IFAC symposium on nonlinear control systems, July 4-6, 2001, St. Petersburg, Russia.


