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Evolution of hypersurfaces by curvature functions

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Introduction

In this thesis we study the evolution of some hypersurfaces by certain geometric flows. The speed of these flows has direction normal to the surface at every point and its modulus depends on the principal curvatures of the surface.

We consider a smooth orientable $n-$dimensional manifold M immersed in $\mathbb{R}^{n+1}$. Let $\mathbf{F}_0 : M \to \mathbb{R}^{n+1}$ be the initial surface and $\mathbf{F} : M \times [0,T) \to \mathbb{R}^{n+1}$ the one parameter family defined by

$$\begin{cases} \frac{\partial \mathbf{F}}{\partial t}(\mathbf{p},t) = & -\mathcal{S}(\mathcal{W})\,\boldsymbol{\nu}(\mathbf{F}(\mathbf{p},t)) \\ \mathbf{F}(\mathbf{p},0) = & \mathbf{F}_0(\mathbf{p}) \end{cases}$$

where $\boldsymbol{\nu}$ is the outer normal vector of $M_t = \mathbf{F}(\cdot,t)$ at point $\mathbf{p}$, $\mathcal{W}$ is the Weingarten operator and the speed $\mathcal{S}$ is a symmetric function of the principal curvatures.

In the first chapter we recall some general properties of this class of flows, such as the short time existence and regularity of solutions. Then we compute the evolution equations for the most important geometric quantities of the surfaces M_t generalizing formulas given in [1], and derive the evolution equations for any homogeneous function of principal curvatures.

The aim of the second chapter is to prove some convexity estimates for closed, mean convex surfaces evolving with a speed given by $\mathcal{S} = \frac{\bar{H}}{\log \bar{H}}$ where $\bar{H}$ is the sum of the mean curvature H and a constant $H_0 = e^{2n(n-1)}$. We are able to prove that for all $\eta > 0$ there exists c_η such that the scalar curvature R satisfies the estimate

$$-R \leq 2\eta H^2 + c_\eta$$

as long as the flow exists. This estimate implies that if the evolving surfaces are rescaled near a singularity, any limit of the rescalings has nonnegative scalar curvature. Analogous estimates have been obtained in the literature for mean curvature flow and Ricci flow and are important to define a surgery procedure. Here we consider a speed which is a nonhomogeneous function of the mean curvature and has a nice behavior for what concerns these properties. In fact, although our proof follows partly the method of [16] for mean curvature flow, it uses only the maximum principle without using the integral estimates and the Sobolev inequality.

The third chapter is dedicated to the study of a closed, convex initial manifold moving by powers of scalar curvature: $\mathcal{S} = R^p$, with $p > 1/2$. For this choice of powers the speed is a homogeneous function of the principal curvatures of degree strictly greater than one. We show that if the initial surface satisfies a pinching estimate of the form $\frac{R}{H^2} > c(n,p)$, then the surface remains convex during the flow and shrinks to a point in finite time. Moreover, using a standard rescaling argument, we show that the shape of the evolving surfaces approaches the one of a sphere. Results of this kind are well known for flows whose speed is given by homogeneous functions of degree one of principal curvatures: $\mathcal{S} = H$ (see [13]), $\mathcal{S} = \sqrt{R}$ (see [8]) and $\mathcal{S} = \sqrt[n]{K}$, K being the Gauss curvature (see [7]), and the generalization to any monotone function of degree one ([1] and [4]). Higher degree functions have been used as speed in dimension two ([2] and [21]) and for $n \geq 2$ the flow with $\mathcal{S} = H^k$ has been studied in [22] and [23]. In our case, $\mathcal{S} = R^p$, the study of the regularity of solutions and of the convergence of the rescalings requires some specific results for fully nonlinear equations ([3]) and for degenerate parabolic equations ([9]) because the homogeneity degree is strictly greater than one. On the other hand the higher homogeneity degree allows to prove the convergence to a round point using only the maximum principle and avoiding again the integral estimates. Finally we also construct an example of a non convex surface forming a neckpinch singularity.

In the last chapter we consider the case where the initial surface is an entire graph over $\mathbb{R}^n$ instead of a compact surface. After giving a proof of the maximum principle for graphs, we show that an evolving graph remains a graph generalizing the result in [10] for mean curvature flow to a wider class of geometric flows. In particular, if the speed of the flow is $\mathcal{S} = R^p$ with $p \geq 1/2$, we prove a long time existence result under the assumption that the initial surface has at most linear growth at infinity and satisfies the pinching condition $\frac{R}{H^2} \geq c(n,p)$ where $c(n,p)$ is the same used in the compact case. Such an assumption is somehow restrictive in the case of an entire graph, but it is satisfied by certain convex functions with linear growth such as $u_0(\mathbf{x}) = \sqrt{1 + |\mathbf{x}|^2}$. We also give some explicit examples of rotationally symmetric translating solutions for this flow.

I wish to thank my supervisor Prof. Carlo Sinestrari for his careful guide and his precious advice.

Chapter 1

General setting

In this chapter we introduce the notation and the definitions of the geometric quantities associated to immersed surfaces, and we describe the class of flows which will be considered in this thesis. We recall some basic results and derive evolution equations which hold for any flow of this class.

More precisely, we state the short time existence and the regularity theorems for the solutions of the fully non linear parabolic equations that describe the general initial value problem. Furthermore, we compute the evolution equations for the most important geometric quantities, expressing them in a compact notation as the one used in [1]. Then we find the evolution equation for a general homogeneous function of the principal curvatures. This formula is useful, for instance, to find a monotone function for a given flow. In particular, it will be applied to the symmetric polynomials of principal curvatures, and their quotients.

At the end of the chapter we recall the maximum principle for parabolic equations on compact surfaces because it will be applied often in the following.

1.1 Preliminaries

Let M be an $n-$dimensional smooth, orientable manifold without boundary and $\mathbf{F} : M \to \mathbb{R}^{n+1}$ a smooth immersion in Euclidean space.

We denote by g the metric on $\mathbf{F}(M)$ induced by the standard scalar product $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^{n+1}$

$$g : T\mathbf{F}(M) \times T\mathbf{F}(M) \to \mathbb{R}.$$

Given local coordinates

$$\begin{aligned} \varphi : \qquad \mathbb{R}^n &\to M \\ (x_1, ..., x_n) &\mapsto \varphi(x_1, ..., x_n) \end{aligned}$$

we denote $\mathbf{p} = \varphi(\mathbf{x})$, and write

$$g_{ij}(\mathbf{p}) = \left\langle \frac{\partial \mathbf{F}}{\partial x_i}(\mathbf{p}), \frac{\partial \mathbf{F}}{\partial x_j}(\mathbf{p}) \right\rangle.$$

The inverse matrix of g is denoted by g^* and its elements $g^{ij} = \{g_{ij}\}^{-1}$ are also used to arise the indices in the Einstein summation convention.

Since $\mathbf{F}(M)$ is orientable, there exists an outer normal vector field $\boldsymbol{\nu}$ on M and we consider the second fundamental form

$$\mathcal{II}(\mathbf{p}) : T_{\mathbf{p}}\mathbf{F}(M) \times T_{\mathbf{p}}\mathbf{F}(M) \to \mathbb{R}$$

whose elements are

$$h_{ij}(\mathbf{p}) = \left\langle \frac{\partial \mathbf{F}}{\partial x_i}(\mathbf{p}), \frac{\partial \boldsymbol{\nu}}{\partial x_j}(\mathbf{p}) \right\rangle = -\left\langle \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j}(\mathbf{p}), \boldsymbol{\nu}(\mathbf{p}) \right\rangle. \tag{1.1}$$

The Weingarten map $\mathcal{W} : T_{\mathbf{p}}\mathbf{F}(M) \to T_{\mathbf{p}}\mathbf{F}(M)$ is characterized by the relation

$$\mathcal{II}(u, w) = g(\mathcal{W}(u), w) \quad \forall u, w \in T_{\mathbf{p}}\mathbf{F}(M)$$

and its elements are $h_i^j = h_{ik} g^{kj}$ where we used the Einstein summation convention.

In what follows the eigenvalues of the Weingarten map will be referred as principal curvatures and denoted as $(\lambda_1, ..., \lambda_n)$ with $\lambda_1 \leq ... \leq \lambda_n$.

Notation 1.1 *Given a symmetric function of principal curvatures $s(\lambda_1, ..., \lambda_n)$, the correspond-*

ing function depending on the elements of the Weingarten map will be denoted by

$$\mathcal{S}(\mathcal{W}) = s(\lambda_1, ..., \lambda_n).$$

The most important symmetric functions of principal curvatures are the elementary symmetric polynomials: for all $k = 1, ..., n$

$$s_k = \sum_{1 \leq i_1 < ... < i_k \leq n} \lambda_{i_1} ... \lambda_{i_k}$$

is the $k-$th symmetric polynomial, while $s_0 = 1$ by definition. In particular from now on we use the following notation for the mean curvature

$$H = s_1 = Tr(\mathcal{W}),$$

the squared norm of the second fundamental form

$$|A|^2 = Tr\left(\mathcal{W}\mathcal{W}^T\right)$$

and the scalar curvature

$$R = 2s_2 = H^2 - |A|^2.$$

If the surface $\mathbf{F}(M)$ can be considered, at least locally, as a graph

$$\mathbf{F}(\mathbf{p}) = (x_1, ..., x_n, u(x_1, ..., x_n)) =: (\mathbf{x}, u(\mathbf{x}))$$

then we have

$$g_{ij}(\mathbf{p}) = \delta_{ij} + D_i u(\mathbf{x}) D_j u(\mathbf{x})$$

and

$$g^{ij}(\mathbf{p}) = \delta^{ij} - \frac{D_i u(\mathbf{x}) D_j u(\mathbf{x})}{1 + |Du(\mathbf{x})|^2}.$$

Moreover if the outer normal vector points below

$$\boldsymbol{\nu}(\mathbf{p})=\frac{(Du(\mathbf{x}),-1)}{\sqrt{1+|Du(\mathbf{x})|^2}} \tag{1.2}$$

and the elements of the second fundamental form and the Christoffel symbols satisfy:

$$h_{ij}(\mathbf{p})=\frac{D^2_{ij}u(\mathbf{x})}{\sqrt{1+|Du(\mathbf{x})|^2}} \quad \text{and} \quad \Gamma^k_{ij}(\mathbf{p})=\frac{D^2_{ij}u(\mathbf{x})\,D_k u(\mathbf{x})}{1+|Du(\mathbf{x})|^2}. \tag{1.3}$$

1.2 Geometric flows

Let us consider $\mathbf{F}_0 : M \to \mathbb{R}^{n+1}$ a smooth immersion of an orientable hypersurface as described before, with $n \geq 2$. Let

$$\mathbf{F} : M \times [0,T) \to \mathbb{R}^{n+1}$$

be the one parameter family of immersions defined by the following initial value problem:

$$\begin{cases} \frac{\partial \mathbf{F}}{\partial t}(\mathbf{p},t) = & -\mathcal{S}(\mathcal{W})\,\boldsymbol{\nu}(\mathbf{F}(\mathbf{p},t)) \\ \mathbf{F}(\mathbf{p},0) = & \mathbf{F}_0(\mathbf{p}). \end{cases} \tag{1.4}$$

Notation 1.2 *For all $t \in [0,T)$ we denote by $\mathbf{F}_t = \mathbf{F}(\cdot,t)$ and by M_t the images of $\mathbf{F}_t$. In what follows we will consider on every surface M_t the quantities defined in the previous section without referring explicitly to their dependence on points and time.*

We assume that for all $\mathbf{y} \in M_0$ the speed $\mathcal{S}$ is a smooth function of the Weingarten map $\mathcal{W}$ of $T_{\mathbf{p}}M_0$ corresponding to a symmetric function of principal curvatures

$$s(\lambda_1(\mathbf{F}(\mathbf{p},t)),...,\lambda_n(\mathbf{F}(\mathbf{p},t))).$$

The condition required in the following short time existence theorem completes the description of the class of flows we are going to study.

Theorem 1.3 (Short time existence) *If M_0 is a closed hypersurface, then the initial value problem (1.4) has a unique smooth solution for a short time interval $[0,T)$ provided that for* *any* $\mathbf{y} \in M_0$ *the function* s *associated to* $\mathcal{S}$ *satisfies*

$$\frac{\partial}{\partial \lambda_i} s\left(\lambda_1(\mathbf{y}), ..., \lambda_n(\mathbf{y})\right) > 0 \quad \forall i = 1, ..., n.$$

Proof. See Theorem 3.1 in [14]. ■

To prove the regularity of the solutions to the initial value problem (1.4) we rely on the following theorem.

Theorem 1.4 *Let* Ω *be a domain in* $\mathbb{R}^n$*. Let* $u \in C^4(\Omega \times [0, T))$ *satisfying*

$$\frac{\partial u}{\partial t} = G\left(D^2 u, Du\right)$$

where G *is a* C^2 *function. Denoting by* $\dot{G}$ *the derivative with respect to* $D^2 u$*, assume that there exist two constants* δ *and* Δ *such that* $0 < \delta I \leq \dot{G} \leq \Delta I$ *hold and assume to have an estimate on the* C^2 *norm of* u *i. e.*

$$\sup_{\Omega \times [0,T)} \left(\left|D^2 u\right| + |Du|\right) \leq k_1 < \infty \qquad \text{and} \qquad \sup_{\Omega \times [0,T)} |\partial_t u| \leq k_2 < \infty.$$

If $\ddot{G}^{pq,rs} M_{pq} M_{rs} \leq 0$ *for all matrices* M_{pq} *such that* $\dot{G}^{pq} M_{pq} = 0$*, then in any relatively compact set* $\Omega' \subset \Omega$ *and for* $\tau \in (0, T)$ *we have an estimate for the parabolic* $C^{2,\alpha}$ *norm of* u*:* $\|u\|_{C^{2,\alpha}(\Omega' \times [\tau, T))} \leq k_3$ *where* k_3 *depends on* δ*,* Δ*,* k_1*,* k_2*,* τ*, dist* $(\Omega', \partial\Omega)$ *and the bounds on* $\ddot{G}$*.*

Proof. See Theorem 6 in [3]. Let us only mention that the proof is based on estimates for fully nonlinear elliptic and parabolic equations obtained by Caffarelli [6] and by Krylov-Safonov [18]. ■

The assumption on the second derivatives of G in the previous theorem clearly holds if G is concave with respect to the first argument, which is a standard assumption in the theory of fully nonlinear parabolic equations. However, it is satisfied also if G is a composition of a concave function with a monotone one. We can apply the theorem to deduce the following result, which gives a criterion ensuring that the evolving surfaces M_t stay smooth as long as the principal curvatures stay all bounded.

Corollary 1.5 (Long time existence) *Let the surfaces M_t be the images of solutions to the initial value problem (1.4) for all $t \in [0,T)$. Suppose that, when the M_t's are locally written as the graph of a function $u(x,t)$, the equation (1.4) takes the form*

$$\frac{\partial u}{\partial t} = G\left(D^2u, Du\right) = a\left(Du\right)\phi\left[L\left(D^2u, Du\right)\right]$$

with $a(Du) > 0$, ϕ increasing in L and L concave in D^2u. If all the principal curvatures of M_t are uniformly bounded in $[0,T)$, then all derivatives of the curvatures are also uniformly bounded and M_t converges to a smooth surface M_T. As a consequence the flow (1.4) exists as long as the principal curvatures are all bounded.

Proof. Let us take any point $(\mathbf{p}_0, t_0) \in M \times [0,T)$ and consider the evolving surfaces

$$M_t \cap \left(\left(B_{2\varepsilon}\left(\mathbf{p}_0\right) \cap T_{\mathbf{p}_0}M_{t_0}\right) \times \mathbb{R}\cdot\boldsymbol{\nu}\left(\mathbf{p}_0, t_0\right)\right)$$

as the graph of a function $u(\mathbf{x},t)$ with $\mathbf{x} \in \Omega = B_{2\varepsilon}(\mathbf{p}_0) \cap T_{\mathbf{p}_0}M_{t_0}$ and $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$:

$$\mathbf{F}_t\left(\mathbf{p}, t\right) = \left(x_1, ..., x_n, u\left(x_1, ..., x_n, t\right)\right).$$

Note that, since the principal curvatures of M_t are uniformly bounded, we can choose a uniform $\varepsilon > 0$ such that the above can be done for all $\mathbf{p}_0 \in M$ and $t_0 \in [\varepsilon, T-\varepsilon)$ and such that the C^2 norm of u is bounded by the same constant for the graph representation around any point. Here we use also the correspondence between the second fundamental form h_{ij} of M_t and the Hessian $D^2_{ij}u$ of u shown in the previous section.

Once written $G\left(D^2u, Du\right)$ in the required form we want to deduce the $C^{2,\alpha}$ estimate for the norm of u by the previous theorem. Note that $\dot{G}^{pq}M_{pq} = a(Du)\,\phi'\dot{L}^{pq}M_{pq}$ and

$$\ddot{G}^{pq,rs}M_{pq}M_{rs} = a\left(Du\right)\left[\phi''\dot{L}^{pq} + \phi'\ddot{L}^{pq,rs}\right]M_{pq}M_{rs}$$

where $a(Du)$ is independent of D^2u and positive. For all matrices M_{pq} such that $\dot{G}^{pq}M_{pq} = 0$ we have either $\dot{L}^{pq}M_{pq} = 0$ or $\phi' = 0$. Since ϕ is increasing $\phi' = 0$ implies $\phi'' = 0$ and the concavity of L, $\ddot{L}^{pq,rs}M_{pq}M_{rs} \leq 0$, for all matrices M_{pq} is equivalent to $\ddot{G}^{pq,rs}M_{pq}M_{rs} \leq 0$ for all matrices M_{pq} such that $\dot{G}^{pq} M_{pq} = 0$.

Finally the regularity of u can be proved by a standard iterative argument using parabolic Schauder estimates. ■

Note that if $\mathbf{F}_t(\mathbf{p}, t) = (\mathbf{x}, u(\mathbf{x}, t))$, then $\frac{\partial}{\partial t}\mathbf{F} = \frac{\partial u}{\partial t}\mathbf{e}_{n+1}$, hence by formula (1.2) we have

$$\left\langle \frac{\partial}{\partial t}\mathbf{F}, \boldsymbol{\nu} \right\rangle = \frac{\partial u}{\partial t} \langle \mathbf{e}_{n+1}, \boldsymbol{\nu} \rangle = -\frac{\partial u}{\partial t} \frac{1}{\sqrt{1 + |Du(\mathbf{x})|^2}}$$

and

$$\frac{\partial u}{\partial t} = -\sqrt{1 + |Du(\mathbf{x})|^2} \left\langle \frac{\partial}{\partial t}\mathbf{F}, \boldsymbol{\nu} \right\rangle = \mathcal{S}(\mathcal{W}) \sqrt{1 + |Du(\mathbf{x})|^2}. \tag{1.5}$$

Remark 1.6 *We can apply Corollary 1.5 with* $a(Du) = \sqrt{1 + |Du(\mathbf{x})|^2}$ *provided, after writing* $\mathcal{W}$ *in terms of* Du, D^2u *using formulas (1.3), we have* $\mathcal{S}(\mathcal{W}) = \phi\left[L\left(D^2u, Du\right)\right]$ *for some functions* ϕ *and* L *satisfying the assumptions of the corollary. This can be done if the speed* $\mathcal{S}$ *comes from a function* s *of the principal curvatures that can be written as*

$$s(\lambda_1, ..., \lambda_n) = \phi\left[l(\lambda_1, ..., \lambda_n)\right].$$

where ϕ *is strictly increasing and* l *is concave and satisfies the following condition*

$$\frac{\dot{l}^k - \dot{l}^j}{\lambda_k - \lambda_j} \leq 0 \quad \textit{for all } k \neq j. \tag{1.6}$$

Proof. The matrix $W = \left\{h^i_j\right\}$ that represents the Weingarten operator is obtained as product of the matrix $G^{-1} = \left\{g^{il}\right\}$ with the matrix of the second fundamental form $A = \{h_{lj}\}$, where G^{-1} and A are both symmetric. Though W is not necessarily symmetric, it is conjugate to a symmetric matrix B that, hence, has the same eigenvalues $(\lambda_1, ..., \lambda_n)$ of W. In fact there exists an invertible matrix K such that $G^{-1} = K^T K$, thus $W = K^T K A$ and

$$\left(K^T\right)^{-1} W K^T = K A K^T = B.$$

Now we can apply Corollary 5.2 of [5] to the symmetric function B: if $l(\lambda_1, ..., \lambda_n)$ is a concave function and satisfies the inequality (1.6), then the function $L(B)$ is concave.

Moreover the function $L(A)$ is concave because it is given by the composition of a linear application:

$$B \to (K)^{-1} B \left(K^T\right)^{-1} = A$$

and the the concave function $L(B)$. ■

Notation 1.7 *In what follows* $Hess_\nabla$ *will denote the second tensorial derivative as a* 2*-covariant tensor. If it is contracted by the standard metric* g *we have the standard Laplace Beltrami operator: for any tensor* T *we write*

$$g^* \left(Hess_\nabla T\right) = g^{ij} \nabla_i \nabla_j T = \Delta T.$$

More in general, if $\{m_{ij}\}$ *is a metric we denote by* $\mathcal{M}$ *the tensor* $m^i_j = mg^*$ *and by*

$$\Delta_m T = \mathcal{M} g^* \left(Hess_\nabla T\right) = m^{ij} \nabla_i \nabla_j T.$$

The same notation is extended to contract other 2-covariant tensors:

Notation 1.8 *Given a symmetric bilinear form* b_{ij} *we denote by* $\langle T_i, V_i \rangle_b = T_i b^{ij} V_j$ *and by* $|T_i|^2_b = \langle T_i, T_i \rangle_b$*. If the tensors* T *and* V *have more than one index, then the bilinear form* b_{ij} *is used only to contract the index* i*.*

Once established the existence of solutions to the given flow (1.4) and their regularity, at least for a short time, we can compute some useful evolution equations.

Proposition 1.9 *If for all* $t \in [0, T)$ *the functions* $\mathbf{F}_t : M \longrightarrow \mathbb{R}^{n+1}$ *are smooth immersions satisfying the evolution equation (1.4), then the metric* g*, its inverse* g^**, the measure* μ *and the outer normal* $\boldsymbol{\nu}$*, evolve according to the following equations:*

$$\begin{aligned} \frac{\partial g}{\partial t} &= -2\mathcal{S}\mathcal{I}\mathcal{I} \\ \frac{\partial g^*}{\partial t} &= 2\mathcal{S} g^* \mathcal{I}\mathcal{I} g^* \\ \frac{\partial \mu}{\partial t} &= -\mathcal{S} H \mu \\ \frac{\partial \boldsymbol{\nu}}{\partial t} &= \nabla \mathcal{S}. \end{aligned}$$

Moreover for the second fundamental form $\mathcal{II}$, the Weingarten map $\mathcal{W}$ and the speed $\mathcal{S}$ we have

$$\begin{aligned}\frac{\partial \mathcal{II}}{\partial t} &= Hess_{\nabla}\mathcal{S} - \mathcal{S}\mathcal{II}^2 \\ \frac{\partial \mathcal{W}}{\partial t} &= g^* Hess_{\nabla}\mathcal{S} + \mathcal{S}\mathcal{W}^2 \\ \frac{\partial \mathcal{S}}{\partial t} &= \dot{\mathcal{S}} g^* \left(Hess_{\nabla}\mathcal{S}\right) + \mathcal{S}\dot{\mathcal{S}} g^* \left(\mathcal{II}^2\right) \end{aligned} \tag{1.7}$$

where $\mathcal{II}^2 = \mathcal{II} g^* \mathcal{II}$ and the point denotes the derivative with respect to $\mathcal{W}$.

Proof. Fixed a frame $\left\{\frac{\partial \mathbf{F}}{\partial x_1}, ..., \frac{\partial \mathbf{F}}{\partial x_n}\right\}$ on the tangent space $T_{\mathbf{p}}M_t$ and the outer normal vector $\boldsymbol{\nu}$, we compute the evolution equations for the elements of the metric and the second fundamental form. For all $i, j = 1, ..., n$

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} &= \frac{\partial}{\partial t}\left\langle \frac{\partial \mathbf{F}}{\partial x_i}, \frac{\partial \mathbf{F}}{\partial x_j}\right\rangle = -2\left\langle \frac{\partial\left(\mathcal{S}\boldsymbol{\nu}\right)}{\partial x_i}, \frac{\partial \mathbf{F}}{\partial x_j}\right\rangle \\ &= -2\left\langle \frac{\partial \mathcal{S}}{\partial x_i}\boldsymbol{\nu}, \frac{\partial \mathbf{F}}{\partial x_j}\right\rangle - 2\mathcal{S}\left\langle \frac{\partial \boldsymbol{\nu}}{\partial x_i}, \frac{\partial \mathbf{F}}{\partial x_j}\right\rangle = -2\mathcal{S}h_{ij}\end{aligned}$$

because $\boldsymbol{\nu}$ is orthogonal to the tangent space $T_{\mathbf{p}}M$.

To prove the equation for the elements of g^* note that

$$0 = \frac{\partial \delta^i_l}{\partial t} = \frac{\partial\left(g_{lk}g^{ki}\right)}{\partial t} = \frac{\partial g_{lk}}{\partial t}g^{ki} + g_{lk}\frac{\partial g^{ki}}{\partial t}$$

and

$$\frac{\partial g^{ij}}{\partial t} = g^{jl}g_{lk}\frac{\partial g^{ik}}{\partial t} = -g^{jl}\frac{\partial g_{lk}}{\partial t}g^{ki},$$

hence in compact notation

$$\frac{\partial g^*}{\partial t} = -g^*\frac{\partial g}{\partial t}g^* = 2\mathcal{S}g^*\mathcal{II}g^*.$$

Being the norm of $\boldsymbol{\nu}$ fixed in time then $\frac{\partial \boldsymbol{\nu}}{\partial t} \in T_{\mathbf{p}}M$, hence

$$\begin{aligned}
\frac{\partial \boldsymbol{\nu}}{\partial t} &= \left\langle \frac{\partial \boldsymbol{\nu}}{\partial t}, \frac{\partial \mathbf{F}}{\partial x_i} \right\rangle \frac{\partial \mathbf{F}}{\partial x_j} g^{ij} = -\left\langle \boldsymbol{\nu}, \frac{\partial}{\partial t} \frac{\partial \mathbf{F}}{\partial x_i} \right\rangle \frac{\partial \mathbf{F}}{\partial x_j} g^{ij} \\
&= \left\langle \boldsymbol{\nu}, \frac{\partial \left(\mathcal{S}\boldsymbol{\nu}\right)}{\partial x_i} \right\rangle \frac{\partial \mathbf{F}}{\partial x_j} g^{ij} \\
&= \frac{\partial \mathcal{S}}{\partial x_i} \frac{\partial \mathbf{F}}{\partial x_j} g^{ij} + \mathcal{S} \left\langle \boldsymbol{\nu}, \frac{\partial \boldsymbol{\nu}}{\partial x_i} \right\rangle \frac{\partial \mathbf{F}}{\partial x_j} g^{ij} \\
&= \frac{\partial \mathcal{S}}{\partial x_i} \frac{\partial \mathbf{F}}{\partial x_j} g^{ij} = \nabla \mathcal{S}.
\end{aligned}$$

For the elements of $\mathcal{II}$ we have

$$\begin{aligned}
\frac{\partial h_{ij}}{\partial t} &= -\frac{\partial}{\partial t} \left\langle \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j}, \boldsymbol{\nu} \right\rangle = -\left\langle \frac{\partial}{\partial t} \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j}, \boldsymbol{\nu} \right\rangle - \left\langle \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j}, \frac{\partial \boldsymbol{\nu}}{\partial t} \right\rangle \\
&= \left\langle \frac{\partial^2 \left(\mathcal{S}\boldsymbol{\nu}\right)}{\partial x_i \partial x_j}, \boldsymbol{\nu} \right\rangle - \left\langle \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j}, \frac{\partial \mathcal{S}}{\partial x_l} \frac{\partial \mathbf{F}}{\partial x_k} g^{lk} \right\rangle,
\end{aligned}$$

let us observe that

$$\frac{\partial \boldsymbol{\nu}}{\partial x_j} = h_{jl} g^{lk} \frac{\partial \mathbf{F}}{\partial x_k} \qquad \text{and} \qquad \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j} = \Gamma^m_{ij} \frac{\partial \mathbf{F}}{\partial x_m} - h_{ij} \boldsymbol{\nu}, \tag{1.8}$$

then the equation becomes

$$\begin{aligned}
\frac{\partial h_{ij}}{\partial t} &= \frac{\partial^2 \mathcal{S}}{\partial x_i \partial x_j} + 2 \frac{\partial \mathcal{S}}{\partial x_i} \left\langle \frac{\partial \boldsymbol{\nu}}{\partial x_j}, \boldsymbol{\nu} \right\rangle + \mathcal{S} \left\langle \frac{\partial}{\partial x_i} \left(h_{jl} g^{lk} \frac{\partial \mathbf{F}}{\partial x_k} \right), \boldsymbol{\nu} \right\rangle \\
&\quad - \left\langle \Gamma^m_{ij} \frac{\partial \mathbf{F}}{\partial x_m} - h_{ij} \boldsymbol{\nu}, \frac{\partial \mathcal{S}}{\partial x_l} \frac{\partial \mathbf{F}}{\partial x_k} g^{lk} \right\rangle \\
&= \frac{\partial^2 \mathcal{S}}{\partial x_i \partial x_j} + \mathcal{S} h_{jl} g^{lk} \left\langle \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_k}, \boldsymbol{\nu} \right\rangle - \Gamma^m_{ij} \frac{\partial \mathcal{S}}{\partial x_l} g^{lk} \left\langle \frac{\partial \mathbf{F}}{\partial x_m}, \frac{\partial \mathbf{F}}{\partial x_k} \right\rangle \\
&= \frac{\partial^2 \mathcal{S}}{\partial x_i \partial x_j} + \mathcal{S} h_{jl} g^{lk} \left\langle \Gamma^m_{ik} \frac{\partial \mathbf{F}}{\partial x_m} - h_{ik} \boldsymbol{\nu}, \boldsymbol{\nu} \right\rangle - \Gamma^m_{ij} \frac{\partial \mathcal{S}}{\partial x_l} g^{lk} g_{mk} \\
&= \frac{\partial^2 \mathcal{S}}{\partial x_i \partial x_j} - \Gamma^l_{ij} \frac{\partial \mathcal{S}}{\partial x_l} - \mathcal{S} h_{jl} g^{lk} h_{ki} = \nabla_i \nabla_j \mathcal{S} - \mathcal{S} \left(\mathcal{II}^2 \right)_{ij}.
\end{aligned}$$

To compute the evolution equation of μ, we consider the evolution of the element of measure $d\mu := \sqrt{\det g} d\mathbf{x}$:

$$\begin{aligned} \frac{\partial}{\partial t} d\mu &= \frac{\partial}{\partial t}\sqrt{\det g} d\mathbf{x} = \frac{1}{2\sqrt{\det g}}\frac{\partial}{\partial t}\left(\det g\right) d\mathbf{x} \\ &= g^*\left(\frac{\partial g}{\partial t}\right)\frac{\det g}{2\sqrt{\det g}} d\mathbf{x} = -\mathcal{S} g^*\left(\mathcal{II}\right)\sqrt{\det g} d\mathbf{x}, \end{aligned}$$

the equation in the statement follows from $g^*\left(\mathcal{II}\right) = Tr\left(\mathcal{W}\right) = H$.

Finally recalling that $\mathcal{W} = g^*\mathcal{II}$ we have

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial t} &= g^*\frac{\partial \mathcal{II}}{\partial t} + \frac{\partial g^*}{\partial t}\mathcal{II} \\ &= g^* Hess_{\nabla}\mathcal{S} - \mathcal{S} g^*\mathcal{II}^2 + 2\mathcal{S} g^*\mathcal{II} g^*\mathcal{II}, \end{aligned}$$

the equivalence with the formula in the statement holds by definition:

$$g^*\mathcal{II}^2 = g^*\mathcal{II} g^*\mathcal{II} = \mathcal{W}\mathcal{W} = \mathcal{W}^2.$$

The equation for $\mathcal{S}$ follows immediately

$$\frac{\partial \mathcal{S}}{\partial t} = \dot{\mathcal{S}}\frac{\partial \mathcal{W}}{\partial t} = \dot{\mathcal{S}} g^*\left(Hess_{\nabla}\mathcal{S}\right) + \mathcal{S}\dot{\mathcal{S}} g^*\left(\mathcal{II}^2\right).$$

■

It is useful to express the evolution equations for the second fundamental form and the Weingarten map in such a way that the time derivative and the second order operator are applied to the same function.

Corollary 1.10 *The following evolution equations hold:*

$$\begin{aligned} \frac{\partial \mathcal{II}}{\partial t} &= \dot{\mathcal{S}} g^*\left(Hess_{\nabla}\mathcal{II}\right) + \ddot{\mathcal{S}}\left(\nabla\mathcal{W}, \nabla\mathcal{W}\right) \\ & -\dot{\mathcal{S}} g^*\left(\mathcal{II}\right)\mathcal{II}^2 + \dot{\mathcal{S}} g^*\left(\mathcal{II}^2\right)\mathcal{II} - \mathcal{S}\mathcal{II}^2 \end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial t} &= \dot{\mathcal{S}}g^{*}\left(Hess_{\nabla}\mathcal{W}\right)+g^{*}\ddot{\mathcal{S}}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right)\\ &\quad -\dot{\mathcal{S}}g^{*}\left(\mathcal{II}\right)\mathcal{W}^{2}+\dot{\mathcal{S}}g^{*}\left(\mathcal{II}^{2}\right)\mathcal{W}+\mathcal{S}\mathcal{W}^{2}.\end{aligned}$$

Proof. Let us write the evolution equation for $\mathcal{II}$ in the more precise index notation: for any $i,j,k,l=1,...,n$

$$\frac{\partial h_{ij}}{\partial t}=\nabla_{i}\nabla_{j}\mathcal{S}-\mathcal{S}\left(\mathcal{II}^{2}\right)_{ij}$$

where

$$\begin{aligned}\nabla_{i}\nabla_{j}\mathcal{S} &= \nabla_{i}\left(\dot{\mathcal{S}}^{l}_{r}\nabla_{j}h^{r}_{l}\right)=\dot{\mathcal{S}}^{l}_{r}\nabla_{i}\nabla_{j}h^{r}_{l}+\ddot{\mathcal{S}}^{ql}_{pr}\nabla_{i}h^{p}_{q}\nabla_{j}h^{r}_{l}\\ &= \dot{\mathcal{S}}^{l}_{r}g^{kr}\nabla_{i}\nabla_{j}h_{lk}+\ddot{\mathcal{S}}^{ql}_{pr}g^{ps}\nabla_{i}h_{sq}g^{kr}\nabla_{j}h_{lk}.\end{aligned}$$

Now we use Ricci identity for exchanging the second derivative to obtain

$$\begin{aligned}\nabla_{i}\nabla_{j}h_{lk} &= \nabla_{i}\nabla_{l}h_{jk}\\ &= \nabla_{l}\nabla_{i}h_{jk}+\left(h_{ik}h_{lr}-h_{ir}h_{lk}\right)h^{r}_{j}+\left(h_{ij}h_{lr}-h_{ir}h_{lj}\right)h^{r}_{k}\\ &= \nabla_{l}\nabla_{k}h_{ij}+h_{ik}\left(\mathcal{II}^{2}\right)_{lj}-h_{lk}\left(\mathcal{II}^{2}\right)_{ij}+h_{ij}\left(\mathcal{II}^{2}\right)_{lk}-h_{lj}\left(\mathcal{II}^{2}\right)_{ik},\end{aligned}$$

thus

$$\begin{aligned}\nabla_{i}\nabla_{j}\mathcal{S} &= \dot{\mathcal{S}}^{l}_{r}g^{kr}\nabla_{l}\nabla_{k}h_{ij}+\ddot{\mathcal{S}}^{ql}_{pr}g^{ps}\nabla_{i}h_{sq}g^{kr}\nabla_{j}h_{lk}+\dot{\mathcal{S}}^{l}_{r}g^{kr}h_{ik}\left(\mathcal{II}^{2}\right)_{lj}\\ &\quad -\dot{\mathcal{S}}^{l}_{r}g^{kr}h_{lk}\left(\mathcal{II}^{2}\right)_{ij}+\dot{\mathcal{S}}^{l}_{r}g^{kr}h_{ij}\left(\mathcal{II}^{2}\right)_{lk}-\dot{\mathcal{S}}^{l}_{r}g^{kr}h_{lj}\left(\mathcal{II}^{2}\right)_{ik}\end{aligned}\tag{1.9}$$

Hence in compact notation we have

$$\begin{aligned}Hess_{\nabla}\mathcal{S} &= \dot{\mathcal{S}}g^{*}\left(Hess_{\nabla}\mathcal{II}\right)+\ddot{\mathcal{S}}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right)+\dot{\mathcal{S}}g^{*}\left(\mathcal{II},\mathcal{II}^{2}\right)\\ &\quad -\dot{\mathcal{S}}g^{*}\left(\mathcal{II}\right)\mathcal{II}^{2}+\dot{\mathcal{S}}g^{*}\left(\mathcal{II}^{2}\right)\mathcal{II}-\dot{\mathcal{S}}g^{*}\left(\mathcal{II},\mathcal{II}^{2}\right)\end{aligned}$$

and the statement follows:

$$\begin{aligned}\frac{\partial \mathcal{II}}{\partial t} &= Hess_{\nabla}\mathcal{S} - \mathcal{S}\mathcal{II}^2 = \dot{\mathcal{S}}g^*\left(Hess_{\nabla}\mathcal{II}\right) + \ddot{\mathcal{S}}\left(\nabla\mathcal{W}, \nabla\mathcal{W}\right) \\ &\quad -\dot{\mathcal{S}}g^*\left(\mathcal{II}\right)\mathcal{II}^2 + \dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)\mathcal{II} - \mathcal{S}\mathcal{II}^2.\end{aligned}$$

The equation for $\mathcal{W}$ is obtained in the same way: arising indices in formula (1.9) we have

$$\begin{aligned}\nabla^i\nabla_j\mathcal{S} &= \dot{\mathcal{S}}_r^l g^{kr}\nabla_l\nabla_k h_j^i + \ddot{\mathcal{S}}_{pr}^{ql} g^{ps}\nabla^i h_{sq} g^{kr}\nabla_j h_{lk} + \dot{\mathcal{S}}_r^l g^{kr} h_k^i \left(\mathcal{II}^2\right)_{lj} \\ &\quad -\dot{\mathcal{S}}_r^l g^{kr} h_{lk}\left(\mathcal{II}^2\right)_j^i + \dot{\mathcal{S}}_r^l g^{kr} h_j^i \left(\mathcal{II}^2\right)_{lk} - \dot{\mathcal{S}}_r^l g^{kr} h_{lj}\left(\mathcal{II}^2\right)_k^i\end{aligned}$$

hence

$$\begin{aligned}g^* Hess_{\nabla}\mathcal{S} &= \dot{\mathcal{S}}g^*\left(Hess_{\nabla}\mathcal{W}\right) + g^*\ddot{\mathcal{S}}\left(\nabla\mathcal{W}, \nabla\mathcal{W}\right) + \dot{\mathcal{S}}g^*\left(\mathcal{W}, \mathcal{II}^2\right) \\ &\quad -\dot{\mathcal{S}}g^*\left(\mathcal{II}\right)\mathcal{W}^2 + \dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)\mathcal{W} - \dot{\mathcal{S}}g^*\left(\mathcal{W}, \mathcal{II}^2\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial t} &= g^* Hess_{\nabla}\mathcal{S} + \mathcal{S}\mathcal{W}^2 = \dot{\mathcal{S}}g^*\left(Hess_{\nabla}\mathcal{W}\right) + g^*\ddot{\mathcal{S}}\left(\nabla\mathcal{W}, \nabla\mathcal{W}\right) \\ &\quad -\dot{\mathcal{S}}g^*\left(\mathcal{II}\right)\mathcal{W}^2 + \dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)\mathcal{W} + \mathcal{S}\mathcal{W}^2.\end{aligned}$$

■

In the next lemma we derive the evolution equation of the function $\frac{\mathcal{S}}{2\langle \mathbf{F}, \boldsymbol{\nu}\rangle - r}$ that will be useful in the following chapters.

Lemma 1.11 *If* $v := \frac{\mathcal{S}}{2\langle \mathbf{F}, \boldsymbol{\nu}\rangle - r}$*, then the following evolution equation holds*

$$\frac{\partial v}{\partial t} = \dot{\mathcal{S}}g^*\left(Hessv\right) + 4\frac{\dot{\mathcal{S}}g^*\left(\nabla v, \nabla\left\langle \mathbf{F}, \boldsymbol{\nu}\right\rangle\right)}{2\left\langle \mathbf{F}, \boldsymbol{\nu}\right\rangle - r} + 2v^2 + \frac{2v\dot{\mathcal{S}}g^*\left(\mathcal{II}\right)}{2\left\langle \mathbf{F}, \boldsymbol{\nu}\right\rangle - r} - \frac{r\dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)v}{2\left\langle \mathbf{F}, \boldsymbol{\nu}\right\rangle - r}$$

If the speed $\mathcal{S}$ *is a homogeneous function of the Weingarten map* $\mathcal{W}$ *of degree* β *we have*

$$\frac{\partial v}{\partial t} = \dot{\mathcal{S}}g^*\left(Hessv\right) + 4\frac{\dot{\mathcal{S}}g^*\left(\nabla v, \nabla\left\langle \mathbf{F}, \boldsymbol{\nu}\right\rangle\right)}{2\left\langle \mathbf{F}, \boldsymbol{\nu}\right\rangle - r} + 2\left(\beta+1\right)v^2 - \frac{r\dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)v}{2\left\langle \mathbf{F}, \boldsymbol{\nu}\right\rangle - r}. \tag{1.10}$$

Proof. In order to find the evolution equation for v it is necessary to compute the evolution equation for the scalar product $\langle \mathbf{F}, \boldsymbol{\nu} \rangle$. From the definition of the flow follows

$$\frac{\partial}{\partial t} \langle \mathbf{F}, \boldsymbol{\nu} \rangle = \left\langle \frac{\partial}{\partial t}\mathbf{F}, \boldsymbol{\nu} \right\rangle + \left\langle \mathbf{F}, \frac{\partial}{\partial t}\boldsymbol{\nu} \right\rangle = -\mathcal{S} + \langle \mathbf{F}, \nabla \mathcal{S} \rangle .$$

Then, recalling formulas in (1.8), we compute the first and second derivatives of $\langle \mathbf{F}, \boldsymbol{\nu} \rangle$:

$$\nabla_j \langle \mathbf{F}, \boldsymbol{\nu} \rangle = \frac{\partial}{\partial x_j} \langle \mathbf{F}, \boldsymbol{\nu} \rangle = \left\langle \frac{\partial \mathbf{F}}{\partial x_j}, \boldsymbol{\nu} \right\rangle + \left\langle \mathbf{F}, \frac{\partial \boldsymbol{\nu}}{\partial x_j} \right\rangle = h_j^k \left\langle \mathbf{F}, \frac{\partial \mathbf{F}}{\partial x_k} \right\rangle ,$$

and

$$\begin{aligned}
\nabla_i \nabla_j \langle \mathbf{F}, \boldsymbol{\nu} \rangle &= \frac{\partial}{\partial x_i} \left(\nabla_j \langle \mathbf{F}, \boldsymbol{\nu} \rangle \right) - \Gamma_{ij}^m \nabla_m \langle \mathbf{F}, \boldsymbol{\nu} \rangle \\
&= \frac{\partial h_j^k}{\partial x_i} \left\langle \mathbf{F}, \frac{\partial \mathbf{F}}{\partial x_k} \right\rangle + h_j^k g_{ik} + h_j^k \left\langle \mathbf{F}, \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_k} \right\rangle - \Gamma_{ij}^m h_m^k \left\langle \mathbf{F}, \frac{\partial \mathbf{F}}{\partial x_k} \right\rangle \\
&= \left(\frac{\partial h_j^k}{\partial x_i} - \Gamma_{ij}^m h_m^k \right) \left\langle \mathbf{F}, \frac{\partial \mathbf{F}}{\partial x_k} \right\rangle + h_{ij} + h_j^k \Gamma_{ik}^m \left\langle \mathbf{F}, \frac{\partial \mathbf{F}}{\partial x_m} \right\rangle - h_{ik} h_j^k \langle \mathbf{F}, \boldsymbol{\nu} \rangle \\
&= \left(\frac{\partial h_j^k}{\partial x_i} - \Gamma_{ij}^m h_m^k + h_j^m \Gamma_{im}^k \right) \left\langle \mathbf{F}, \frac{\partial \mathbf{F}}{\partial x_k} \right\rangle + h_{ij} - h_{ik} h_j^k \langle \mathbf{F}, \boldsymbol{\nu} \rangle \\
&= \left\langle \mathbf{F}, \nabla_i h_j^k \right\rangle + h_{ij} - h_{ik} h_j^k \langle \mathbf{F}, \boldsymbol{\nu} \rangle .
\end{aligned}$$

Hence

$$Hess_{\nabla} \langle \mathbf{F}, \boldsymbol{\nu} \rangle = \langle \mathbf{F}, \nabla \mathcal{W} \rangle + \mathcal{II} - \mathcal{II}^2 \langle \mathbf{F}, \boldsymbol{\nu} \rangle$$

and thus, since $\nabla \mathcal{S} = \dot{\mathcal{S}} \nabla \mathcal{W}$, it follows

$$\begin{aligned}
\frac{\partial}{\partial t} \langle \mathbf{F}, \boldsymbol{\nu} \rangle &= \dot{\mathcal{S}} g^* \left(Hess_{\nabla} \langle \mathbf{F}, \boldsymbol{\nu} \rangle \right) - \mathcal{S} + \langle \mathbf{F}, \nabla \mathcal{S} \rangle - \dot{\mathcal{S}} g^* \langle \mathbf{F}, \nabla \mathcal{W} \rangle - \dot{\mathcal{S}} g^* \left(\mathcal{II} \right) + \dot{\mathcal{S}} g^* \left(\mathcal{II}^2 \right) \langle \mathbf{F}, \boldsymbol{\nu} \rangle \\
&= \dot{\mathcal{S}} g^* \left(Hess_{\nabla} \langle \mathbf{F}, \boldsymbol{\nu} \rangle \right) - \mathcal{S} - \dot{\mathcal{S}} g^* \left(\mathcal{II} \right) + \dot{\mathcal{S}} g^* \left(\mathcal{II}^2 \right) \langle \mathbf{F}, \boldsymbol{\nu} \rangle ,
\end{aligned}$$

in particular if the speed $\mathcal{S}$ is a homogeneous function of the Weingarten map $\mathcal{W}$ of degree β we have

$$\frac{\partial}{\partial t} \langle \mathbf{F}, \boldsymbol{\nu} \rangle = \dot{\mathcal{S}} g^* \left(Hess_{\nabla} \langle \mathbf{F}, \boldsymbol{\nu} \rangle \right) - (\beta + 1) \mathcal{S} + \dot{\mathcal{S}} g^* \left(\mathcal{II}^2 \right) \langle \mathbf{F}, \boldsymbol{\nu} \rangle .$$

The evolution equation for v then follows by the equation for $\mathcal{S}$ (1.7):

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{1}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r}\frac{\partial \mathcal{S}}{\partial t} - \frac{2\mathcal{S}}{(2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r)^2}\frac{\partial \langle \mathbf{F},\boldsymbol{\nu}\rangle}{\partial t} \\ &= \frac{1}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r}\Big[\dot{\mathcal{S}}g^*(Hess_\nabla \mathcal{S}) + \dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)\mathcal{S} \\ &\quad -2v\dot{\mathcal{S}}g^*(Hess_\nabla \langle \mathbf{F},\boldsymbol{\nu}\rangle) + 2v\mathcal{S} + 2v\dot{\mathcal{S}}g^*(\mathcal{II}) - \frac{2\langle \mathbf{F},\boldsymbol{\nu}\rangle}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r}\dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)\mathcal{S}\Big]\end{aligned}$$

We also have to compute

$$\nabla v = \frac{\nabla \mathcal{S}}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r} - \frac{2\mathcal{S}}{(2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r)^2}\nabla \langle \mathbf{F},\boldsymbol{\nu}\rangle$$

and

$$\begin{aligned}\dot{\mathcal{S}}g^*(Hess_\nabla v) &= \frac{\dot{\mathcal{S}}g^*(Hess_\nabla \mathcal{S})}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r} - 4\frac{\dot{\mathcal{S}}g^*(\nabla \mathcal{S},\nabla \langle \mathbf{F},\boldsymbol{\nu}\rangle)}{(2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r)^2} \\ &\quad +8\frac{\mathcal{S}\dot{\mathcal{S}}g^*(\nabla \langle \mathbf{F},\boldsymbol{\nu}\rangle,\nabla \langle \mathbf{F},\boldsymbol{\nu}\rangle)}{(2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r)^3} - \frac{2\mathcal{S}\dot{\mathcal{S}}g^*(Hess_\nabla \langle \mathbf{F},\boldsymbol{\nu}\rangle)}{(2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r)^2} \\ &= \frac{\dot{\mathcal{S}}g^*(Hess_\nabla \mathcal{S})}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r} - 4\frac{\dot{\mathcal{S}}g^*(\nabla v,\nabla \langle \mathbf{F},\boldsymbol{\nu}\rangle)}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r} - \frac{2v\dot{\mathcal{S}}g^*(Hess_\nabla \langle \mathbf{F},\boldsymbol{\nu}\rangle)}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r}.\end{aligned}$$

Then the evolution equation for v becomes

$$\begin{aligned}\frac{\partial v}{\partial t} &= \dot{\mathcal{S}}g^*(Hessv) + 4\frac{\dot{\mathcal{S}}g^*(\nabla v,\nabla \langle \mathbf{F},\boldsymbol{\nu}\rangle)}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r} \\ &\quad +\frac{1}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r}\left[2v\mathcal{S} + 2v\dot{\mathcal{S}}g^*(\mathcal{II}) - \frac{r}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r}\mathcal{S}\dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)\right] \\ &= \dot{\mathcal{S}}g^*(Hessv) + 4\frac{\dot{\mathcal{S}}g^*(\nabla v,\nabla \langle \mathbf{F},\boldsymbol{\nu}\rangle)}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r} + 2v^2 + \frac{2v\dot{\mathcal{S}}g^*(\mathcal{II})}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r} - \frac{rv\dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)}{2\langle \mathbf{F},\boldsymbol{\nu}\rangle - r}\end{aligned}$$

this proves the first part of the statement, while the second part follows by Euler's theorem $\dot{\mathcal{S}}g^*(\mathcal{II}) = \beta\mathcal{S}$. ■

In the next theorem we derive the evolution equation of any homogeneous function of the Weingarten map $\mathcal{W}$ defined on an evolving surface. The statement shows the symmetric role played by the function and the speed. The evolution equation (1.7) for the speed itself is also immediately verified.

Theorem 1.12 *If P is a homogeneous function of degree α of the Weingarten map $\mathcal{W}$, then the evolution equation of P by the flow (1.4) is the following*

$$\begin{aligned}\frac{\partial P}{\partial t} &= \dot{\mathcal{S}}g^*\left(Hess_\nabla P\right) - \dot{\mathcal{S}}g^*\ddot{P}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right) + \dot{P}g^*\ddot{\mathcal{S}}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right)\\ &\quad -\dot{\mathcal{S}}g^*\left(\mathcal{II}\right)\dot{P}\left(\mathcal{W}^2\right) + \mathcal{S}\dot{P}\left(\mathcal{W}^2\right) + \alpha\dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)P.\end{aligned}$$

Furthermore if $\mathcal{S}$ a homogeneous function of the Weingarten map $\mathcal{W}$ of degree β, then

$$\begin{aligned}\frac{\partial P}{\partial t} &= \dot{\mathcal{S}}g^*\left(Hess_\nabla P\right) - \dot{\mathcal{S}}g^*\ddot{P}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right) + \dot{P}g^*\ddot{\mathcal{S}}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right)\\ &\quad -\left(\beta-1\right)\mathcal{S}\dot{P}\left(\mathcal{W}^2\right) + \alpha\dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)P.\end{aligned}$$

Proof. Let us observe that

$$Hess_\nabla P = \nabla\left(\dot{P}\nabla\mathcal{W}\right) = \dot{P}Hess_\nabla\mathcal{W} + \ddot{P}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right).$$

Hence the evolution equation for P is

$$\begin{aligned}\frac{\partial P}{\partial t} &= \dot{P}\frac{\partial\mathcal{W}}{\partial t} = \dot{P}\dot{\mathcal{S}}g^*\left(Hess_\nabla\mathcal{W}\right) + \dot{P}g^*\ddot{\mathcal{S}}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right)\\ &\quad -\dot{\mathcal{S}}g^*\left(\mathcal{II}\right)\dot{P}\left(\mathcal{W}^2\right) + \dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)\dot{P}\left(\mathcal{W}\right) + \mathcal{S}\dot{P}\left(\mathcal{W}^2\right)\\ &= \dot{\mathcal{S}}g^*\left(Hess_\nabla P\right) - \dot{\mathcal{S}}g^*\ddot{P}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right) + \dot{P}g^*\ddot{\mathcal{S}}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right)\\ &\quad -\dot{\mathcal{S}}g^*\left(\mathcal{II}\right)\dot{P}\left(\mathcal{W}^2\right) + \alpha\dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right)P + \mathcal{S}\dot{P}\left(\mathcal{W}^2\right),\end{aligned}$$

where we used Euler's theorem $\dot{P}\mathcal{W} = \alpha P$. The same theorem is used to find the second expression: if $\mathcal{S}$ is homogeneous of degree β then $\dot{\mathcal{S}}g^*\left(\mathcal{II}\right) = \beta\mathcal{S}$. ∎

Example 1.13 *If we consider a flow whose speed depends only on the mean curvature: $\mathcal{S} = \phi\left(H\right)$, then the evolution equation for a homogeneous function P of degree α of the Weingarten map $\mathcal{W}$ is*

$$\begin{aligned}\frac{\partial P}{\partial t} &= \dot{\phi}\Delta P - \dot{\phi}\ddot{P}\left\langle\nabla\mathcal{W},\nabla\mathcal{W}\right\rangle + \dot{P}g^*\ddot{\phi}\left(\nabla H,\nabla H\right)\\ &\quad -\dot{\phi}H\dot{P}\left(\mathcal{W}^2\right) + \phi\left(H\right)\dot{P}\left(\mathcal{W}^2\right) + \alpha\dot{\phi}\left|A\right|^2 P.\end{aligned}$$

Typical homogeneous functions to which apply the previous theorem are the symmetric polynomials and their quotients, in fact in the following chapters we will deal mainly with this type of functions.

Definition 1.14 *We define Γ_k as the region where the first k symmetric polynomials of the principal curvatures are strictly positive:*

$$\Gamma_k = \{(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n \ s.\ t.\ s_l > 0 \quad \forall l = 0, ..., k\}.$$

For $k = 1, ..., n$ and $(\lambda_1, ..., \lambda_n) \in \Gamma_{k-1}$ we denote by Q_k the quotients $Q_k = \frac{s_k}{s_{k-1}}$ and by $P_k = \frac{Q_k}{H}$.

Corollary 1.15 *Let us consider the evolving surfaces M_t such that at any point the principal curvatures $(\lambda_1, ..., \lambda_n) \in \Gamma_{k-1}$. Then the quotients Q_k and P_k are homogeneous function of $\mathcal{W}$ respectively of degree one and zero, their evolution equations are*

$$\begin{aligned}\frac{\partial Q_k}{\partial t} &= \dot{\mathcal{S}} g^* (Hess_\nabla Q_k) - \dot{\mathcal{S}} g^* \ddot{Q}_k (\nabla \mathcal{W}, \nabla \mathcal{W}) + \dot{Q}_k g^* \ddot{\mathcal{S}} (\nabla \mathcal{W}, \nabla \mathcal{W}) \\ &\quad - \dot{\mathcal{S}} g^* (\mathcal{II}) \dot{Q}_k \left(\mathcal{W}^2\right) + \mathcal{S} \dot{Q}_k \left(\mathcal{W}^2\right) + \dot{\mathcal{S}} g^* \left(\mathcal{II}^2\right) Q_k\end{aligned}$$

and

$$\begin{aligned}\frac{\partial P_k}{\partial t} &= \dot{\mathcal{S}} g^* (Hess_\nabla P_k) - \frac{1}{H} \dot{\mathcal{S}} g^* \ddot{Q}_k (\nabla \mathcal{W}, \nabla \mathcal{W}) + \frac{2}{H} \dot{\mathcal{S}} g^* (\nabla P_k, \nabla H) \\ &\quad + \dot{P}_k g^* \ddot{\mathcal{S}} (\nabla \mathcal{W}, \nabla \mathcal{W}) - \dot{\mathcal{S}} g^* (\mathcal{II}) \dot{P}_k \left(\mathcal{W}^2\right) + \mathcal{S} \dot{P}_k \left(\mathcal{W}^2\right).\end{aligned}$$

Proof. The equation for Q_k is an immediate application of the theorem, whereas the one for P_k follows from the equality

$$\begin{aligned}\ddot{P}_k (\nabla \mathcal{W}, \nabla \mathcal{W}) &= \frac{1}{H} \ddot{Q}_k (\nabla \mathcal{W}, \nabla \mathcal{W}) - \frac{2}{H^2} \dot{Q}_k (\nabla \mathcal{W}, \nabla H) + 2 \frac{Q_k}{H^3} |\nabla H|^2 \\ &= \frac{1}{H} \ddot{Q}_k (\nabla \mathcal{W}, \nabla \mathcal{W}) - \frac{2}{H} \dot{P}_k (\nabla \mathcal{W}, \nabla H) \\ &= \frac{1}{H} \ddot{Q}_k (\nabla \mathcal{W}, \nabla \mathcal{W}) - \frac{2}{H} (\nabla P_k, \nabla H).\end{aligned}$$

■

Here we recall the statement of the maximum principle for function defined on a compact surface because it will be used very often in chapter 2 and 3 in this form.

Theorem 1.16 (Maximum principle for compact surfaces) *Let M be a closed manifold and $f : M \times [0,T) \to \mathbb{R}$ be a smooth function. If*

$$\frac{\partial f}{\partial t} \leq \dot{\mathcal{S}} g^* \left(Hess_{\nabla} f \right) + \langle \mathbf{a}, \nabla f \rangle$$

where $\dot{\mathcal{S}}$ is positive definite and $\mathbf{a}$ is a smooth vector field, then

$$\sup_{M_t} f \leq \sup_{M_0} f \quad \forall t \in [0,T] \, .$$

Proof. We can use the same argument shown in [20] for a function defined in a domain in $\mathbb{R}^n$; this case is even easier because M has no boundary. ■

Chapter 2

Convexity estimates

The term *convexity estimate* has been used in the literature on mean curvature flow to denote a priori estimates satisfied by suitable functions of the curvatures showing that the convexity properties of the evolving surface improve when a singularity is formed. A typical example is the following result, proved in [15].

Theorem 2.1 *Let M_0 be a closed n-dimensional surface immersed in $\mathbb{R}^{n+1}$ and let M_t, with $t \in [0,T)$ be its evolution by the mean curvature flow (the one with normal speed given by $\mathcal{S} = H$). Suppose that M_0 has positive mean curvature. Then M_t has positive mean curvature for all t. In addition, for every $\eta > 0$ there exists $C_\eta > 0$ such that the estimate*

$$-s_2 \geq \eta H^2 + C_\eta$$

holds everywhere on M_t, for all $t \in [0,T)$.

Such an estimate implies that any limit of rescalings of the evolving surfaces near a singularity has nonnegative scalar curvature. More powerful estimates have been proved in [16], showing that the rescalings are convex. It should be noticed that these results exhibit strong analogies with the so-called *Hamilton-Ivey estimate* in the three-dimensional Ricci flow, which states that the negative sectional curvatures become negligible with respect to the positive ones when a singularity is approached.

It is natural to expect that some kind of convexity estimates hold not only for the mean curvature flow, but for other geometric flows of the form (1.4) as well. In this chapter we prove an analogue of Theorem 2.1

Theorem 2.2 *Let $\mathbf{F}_0 : M \to \mathbb{R}^{n+1}$ a smooth immersion of a closed hypersurface with $n \geq 2$ and let $\mathbf{F} : M \times [0,T) \to \mathbb{R}^{n+1}$ the one parameter family of immersions defined by*

$$\begin{cases} \frac{\partial \mathbf{F}}{\partial t}(\mathbf{p},t) = & -\frac{\bar{H}}{\log \bar{H}}\boldsymbol{\nu}(\mathbf{F}(\mathbf{p},t)) \\ \mathbf{F}(\mathbf{p},0) = & \mathbf{F}_0(\mathbf{p}) \end{cases} \tag{2.1}$$

where $\bar{H} = H + H_0$ and $H_0 = e^{2n(n-1)}$. If M_0 has positive mean curvature, then $M_t = \mathbf{F}(M,t)$ has positive mean curvature for all $t \in [0,T)$. Furthermore for every $\eta > 0$ there exists $c_\eta > 0$ such that the scalar curvature satisfies the estimate

$$-R \geq 2\eta H^2 + c_\eta(M_0)$$

everywhere on M_t, for all $t \in [0,T)$.

The reason for this choice of speed is that the corresponding flow has in some sense better properties than the mean curvature flow. In fact, we are able to obtain convexity estimates for this flow by using only the maximum principle instead of the integral estimates used in [15].

Like in the case of the mean curvature flow, a central role in the proof is played by the quotient $f = -\frac{1}{2}\frac{R}{H^2}$. We first prove that f is bounded from above, obtaining the estimate

$$|A|^2 - H^2 \leq 2c_2 H^2.$$

Then we consider a suitable function $f_{\sigma,\eta}$ associated to f, and prove that it is bounded from above too. This will imply that for all $\eta > 0$ there exists a constant $c_4(\eta)$ such that

$$-R \leq 2\eta H^2 + c_4(\eta).$$

Let the surfaces M_t's to be rescaled near the singular time in such a way that the mean curvature is uniformly bounded, if we consider a limit $\tilde{M}$ of these rescalings, then on $\tilde{M}$ the previous inequality becomes $-\tilde{R} \leq 2\eta \tilde{H}^2$. The convexity estimate is then obtained letting $\eta \to 0$.

2.1 Preliminary results

As in the statement of the main theorem (Theorem 2.2), we consider a closed hypersurface M_0 of $\mathbb{R}^{n+1}$ evolving by the flow (1.4) with speed

$$\mathcal{S} = \frac{\bar{H}}{\log \bar{H}}$$

where $\bar{H} = H + H_0$ and $H_0 = e^{2n(n-1)}$.

Note that $\mathcal{S}$ is a function of the mean curvature, hence we can compute the derivatives of $\mathcal{S}$ with respect to the Weingarten map as in Example 1.13 with $\phi(H) = \frac{H+H_0}{\log(H+H_0)}$:

$$\frac{\partial \mathcal{S}}{\partial h^i_j} = \phi' \delta^j_i = \frac{1}{\left(\log \bar{H}\right)^2}\left(\log \bar{H} - \bar{H}\frac{1}{\bar{H}}\right)\delta^j_i = \frac{\log \bar{H} - 1}{\left(\log \bar{H}\right)^2}\delta^j_i$$

that can be written as $\dot{\mathcal{S}} = \frac{\log \bar{H} - 1}{\left(\log \bar{H}\right)^2} Id$ and

$$\begin{aligned}
\frac{\partial^2 \mathcal{S}}{\partial h^l_k \partial h^i_j}\delta^k_l \delta^j_i &= \phi'' \delta^k_l \delta^j_i = \left[\frac{1}{\left(\log \bar{H}\right)^2} - 2\frac{\log \bar{H} - 1}{\left(\log \bar{H}\right)^3}\right]\frac{d\left(\log \bar{H}\right)}{dH}\delta^k_l \delta^j_i \\
&= -\frac{\log \bar{H} - 2}{\bar{H}\left(\log \bar{H}\right)^3}\delta^k_l \delta^j_i. \qquad (2.2)
\end{aligned}$$

As a consequence of the evolution equation for H, the next lemma shows that the hypothesis on M_0 is preserved along the flow:

Lemma 2.3 *Let us consider the flow (2.1). If the initial surface M_0 has positive mean curvature everywhere, then there exists a unique smooth solution to the flow at least for a short time and the minimum $\min_{M_t} H$ is nondecreasing in time.*

Proof. The short time existence of the flow is guaranteed by Theorem 1.3 because

$$\log \bar{H} > \log H_0 > 1.$$

As for the second part, it suffices to apply the maximum principle (Theorem 1.16) to the evolution equation for H given by Theorem 1.12:

$$\begin{aligned}\frac{\partial H}{\partial t} &= \frac{\log \bar{H}-1}{\left(\log \bar{H}\right)^2}\Delta H - \frac{\log \bar{H}-2}{\bar{H}\left(\log \bar{H}\right)^3}\left|\nabla H\right|^2 \\ &\quad -\frac{\log \bar{H}-1}{\left(\log \bar{H}\right)^2} H\left|A\right|^2 + \frac{\bar{H}}{\log \bar{H}}\left|A\right|^2 + \frac{\log \bar{H}-1}{\left(\log \bar{H}\right)^2}\left|A\right|^2 H \\ &= \frac{\log \bar{H}-1}{\left(\log \bar{H}\right)^2}\Delta H - \frac{\log \bar{H}-2}{\bar{H}\left(\log \bar{H}\right)^3}\left|\nabla H\right|^2 + \frac{\bar{H}}{\log \bar{H}}\left|A\right|^2. \end{aligned} \tag{2.3}$$

■

The existence of the flow is guaranteed by Corollary 1.5 as long as all the principal curvatures are bounded. In what follows we prove that the mean curvature H diverges in finite time, hence at least one of the principal curvatures blows up and the surface degenerates.

Proposition 2.4 *If M_0 has positive mean curvature, then the solution to the problem (2.1) develops a singularity in finite time.*

Proof. It is know that on every surface the inequality $|A|^2 \geq \frac{1}{n}H^2$ holds, and that $\frac{\bar{H}}{\log \bar{H}} \geq 1$, hence we can estimate the last term of the evolution equation for H (2.3)

$$\frac{\partial H}{\partial t} \geq \frac{\log \bar{H}-1}{\left(\log \bar{H}\right)^2}\Delta H - \frac{\log \bar{H}-2}{\bar{H}\left(\log \bar{H}\right)^3}\left|\nabla H\right|^2 + \frac{1}{n}H^2.$$

Let us define the function $\varphi(t) = \min_{M_t} H$. The evolution equation for φ then satisfies the estimate $\frac{d\varphi}{dt} \geq \frac{1}{n}\varphi^2$ in a weak sense. We deduce that $\frac{dt}{d\varphi} \leq n\varphi^{-2}$, hence the maximum time interval of existence is finite because of the convergence of the improper integral:

$$t(\varphi) \leq n\int_{\varphi_0}^{\infty} \varphi^{-2} d\varphi < \infty.$$

■

In all what follows c_k, will denote constants depending only on the dimension n and the initial manifold M_0.

The following two lemmas are necessary to estimate the first order term in the evolution equation for $f = -\frac{s_2}{H^2}$.

Lemma 2.5 *If $H > 0$ and $s_2 \leq -c_1 H^2 < 0$ then, once denoted the principal curvatures in increasing order, we have*

$$\alpha(c_1) \leq \frac{|\lambda_1|}{\lambda_n} < n-1$$

where

$$\alpha(c_1) = \frac{4c_1 + n - \sqrt{n(8c_1+n)}}{4c_1(n-1)}.$$

Proof. Since $s_2 < 0$ there is at least one negative curvature, then $\lambda_1 < 0$. On the other hand we have

$$0 < H \leq \lambda_1 + (n-1)\lambda_n,$$

hence $\frac{|\lambda_1|}{\lambda_n} < n-1$.

For the opposite estimate, let us consider an n-tuple $\left(\bar{\lambda}_1, ..., \bar{\lambda}_n\right)(c_1)$ satisfying $s_2 \leq -c_1 H^2$ and denote $\frac{|\bar{\lambda}_1|}{\bar{\lambda}_n} = \mu$. Note that for this n-tuple

$$H \geq -(n-1)\left|\bar{\lambda}_1\right| + \bar{\lambda}_n = \left[-(n-1)\mu + 1\right]\bar{\lambda}_n.$$

If $\mu \geq \frac{1}{n-1}$ then the statement is proved because $\alpha(c_1) < \frac{1}{n-1}$, otherwise we have

$$\bar{\lambda}_n \leq \frac{1}{1-(n-1)\mu} H \qquad \text{and} \qquad \bar{\lambda}_1 \geq -\frac{\mu}{1-(n-1)\mu} H.$$

Computing the second symmetric polynomial

$$s_2\left(\bar{\lambda}_1, ..., \bar{\lambda}_n\right) = \sum_{i<j} \bar{\lambda}_i \bar{\lambda}_j \geq \frac{n(n-1)}{2} \bar{\lambda}_1 \bar{\lambda}_n \geq -\frac{\mu n(n-1)}{2\left[1-(n-1)\mu\right]^2} H^2$$

and using the hypothesis $s_2 \leq -c_1 H^2$, we deduce

$$c_1 \leq -\frac{s_2}{H^2} \leq \frac{\mu n(n-1)}{2\left[1-(n-1)\mu\right]^2}.$$

This leads to an estimate on μ:

$$2c_1\left[\mu(n-1)\right]^2 - (n+4c_1)\mu(n-1) + 2c_1 \leq 0$$

that implies

$$\mu(n-1) \geq \frac{4c_1 + n - \sqrt{n(8c_1+n)}}{4c_1},$$

thus we have

$$\frac{|\bar{\lambda}_1|}{\bar{\lambda}_n} \geq \frac{4c_1 + n - \sqrt{n(8c_1+n)}}{4c_1(n-1)} =: \alpha(c_1)$$

and the statement follows. ■

Let us observe that $\alpha(c_1)$ is monotonically increasing for positive arguments, moreover

$$\alpha(c_1) \to 0 \text{ as } c_1 \to 0 \qquad \text{and} \qquad \alpha(c_1) \to \frac{1}{n-1} \text{ as } c_1 \to \infty.$$

For the formulation of the next lemma we recall the notation introduced in chapter 1:

$$|H\nabla A - A\nabla H|^2 = \langle H\nabla_i h_{jk} - h_{ij}\nabla_k H, H\nabla_i h_{jk} - h_{ij}\nabla_k H\rangle$$

and

$$|\nabla H|^2_{|A|^2 g - Hh} = \left(|A|^2 g^{ij} - Hh^{ij}\right)\nabla_i H \nabla_j H.$$

Lemma 2.6 *If $H > 0$ and $P_2 = \frac{s_2}{H^2} \leq -c_1$ then there exists a constant $\varepsilon > 0$ such that*

$$|H\nabla A - A\nabla H|^2 - \varepsilon(c_1)|\nabla H|^2_{|A|^2 g - Hh} \geq 0,$$

more precisely we can take

$$\varepsilon(c_1) = \frac{4c_1 + n - \sqrt{n(8c_1+n)}}{2\left[4c_1 + n - \sqrt{n(8c_1+n)} + 4c_1(n-1)^2\right]}.$$

Proof. By Codazzi equation and Schwarz inequality we infer the following estimate

$$\begin{aligned} 2\langle H\nabla_i h_{jk}, h_{ij}\nabla_k H\rangle &= \langle H\nabla_i h_{jk} + H\nabla_k h_{ij}, h_{ij}\nabla_k H\rangle \\ &= \langle H\nabla_i h_{jk}, h_{ij}\nabla_k H + h_{jk}\nabla_i H\rangle \\ &\leq H^2|\nabla A|^2 + \frac{1}{4}|h_{ij}\nabla_k H + h_{jk}\nabla_i H|^2 \\ &= H^2|\nabla A|^2 + \frac{1}{2}|A|^2|\nabla H|^2 + \frac{1}{2}\langle h_{ij}\nabla_k H, h_{jk}\nabla_i H\rangle. \end{aligned}$$

Hence we have

$$\begin{aligned} |H\nabla A - A\nabla H|^2 &= H^2 |\nabla A|^2 + |A|^2 |\nabla H|^2 - 2\langle H\nabla_i h_{jk}, h_{ij}\nabla_k H\rangle \\ &\geq \frac{1}{2}|A|^2 |\nabla H|^2 - \frac{1}{2}\langle h_{ij}\nabla_k H, h_{jk}\nabla_i H\rangle . \end{aligned}$$

If we chose suitable coordinates such that the matrix $W = \left\{h^i_j\right\}$ is diagonal at a given point, then its entries are the principal curvatures and the difference $|H\nabla A - A\nabla H|^2 - \varepsilon |\nabla H|^2_{|A|^2 g - Hh}$ can be estimated by

$$\begin{aligned} & |H\nabla A - A\nabla H|^2 - \varepsilon |\nabla H|^2_{|A|^2 g - Hh} \\ \geq\ & \frac{1}{2}|A|^2 |\nabla H|^2 - \frac{1}{2}\sum_{i=1}^{n} \lambda_i^2 (\nabla_i H)^2 - \varepsilon |A|^2 |\nabla H|^2 + \varepsilon H \sum_{i=1}^{n} \lambda_i (\nabla_i H)^2 \\ =\ & \sum_{1=1}^{n} \left[\left(\frac{1}{2} - \varepsilon\right)\left(|A|^2 - \lambda_i^2\right) + \varepsilon\lambda_i (H - \lambda_i)\right](\nabla_i H)^2 . \end{aligned}$$

Assume that $\varepsilon < \frac{1}{2}$ and note that for $\lambda_i = 0$, or $\lambda_i > 0$ and $H - \lambda_i > 0$ the coefficient of $(\nabla_i H)^2$ is positive. In the remaining cases we apply the previous lemma. For i such that λ_i is negative

$$\begin{aligned} \left(\frac{1}{2} - \varepsilon\right)\left(|A|^2 - \lambda_i^2\right) + \varepsilon\lambda_i (H - \lambda_i) &\geq \\ \left(\frac{1}{2} - \varepsilon\right)\left(|A|^2 - \lambda_1^2\right) - \varepsilon |\lambda_1| (H + |\lambda_1|) &\geq \left(\frac{1}{2} - \varepsilon\right)\lambda_n^2 - \varepsilon (n-1) |\lambda_1| \lambda_n \\ &\geq \lambda_n^2 \left[\frac{1}{2} - \varepsilon - \varepsilon (n-1)^2\right], \end{aligned}$$

for i such that λ_i is positive and $H - \lambda_i$ is negative

$$\begin{aligned} \left(\frac{1}{2} - \varepsilon\right)\left(|A|^2 - \lambda_i^2\right) + \varepsilon\lambda_i (H - \lambda_i) &\geq \\ \left(\frac{1}{2} - \varepsilon\right)\left(|A|^2 - \lambda_n^2\right) + \varepsilon\lambda_n (H - \lambda_n) &\geq \left(\frac{1}{2} - \varepsilon\right)\lambda_1^2 - \varepsilon (n-1) |\lambda_1| \lambda_n \\ &\geq |\lambda_1| \lambda_n \left[\left(\frac{1}{2} - \varepsilon\right)\frac{|\lambda_1|}{\lambda_n} - \varepsilon (n-1)\right] \\ &\geq |\lambda_1| \lambda_n \left[\left(\frac{1}{2} - \varepsilon\right)\alpha (c_1) - \varepsilon (n-1)\right]. \end{aligned}$$

Hence the statement is verified for

$$\varepsilon\left(\alpha\left(c_1\right)\right)=\min\left\{\frac{1}{2\left[(n-1)^2+1\right]},\frac{\alpha}{2\left[\alpha+(n-1)\right]}\right\}$$

where α is defined in Lemma 2.5. Note that

$$\lim_{c_1\to\infty}\frac{\alpha}{2\left[\alpha+(n-1)\right]}=\frac{1}{2\left[(n-1)^2+1\right]},$$

and that the monotonicity of α implies $\varepsilon\left(\alpha\left(c_1\right)\right)=\frac{\alpha(c_1)}{2[\alpha(c_1)+(n-1)]}$. ■

2.2 Estimate on the scalar curvature

Now we aim to prove that the function $f=-\frac{s_2}{H^2}$ is uniformly bounded. In this way we will be able to prove that for all times

$$0\leq|A|^2-H^2\leq 2c_2H^2,$$

and that all principal curvatures are bounded provided that H is bounded.

Since at any point of the initial surface $\bar{H}>e^{2n(n-1)}$, then we have

$$\frac{\log\bar{H}-2}{\left(\log\bar{H}-1\right)\log\bar{H}}<\frac{1}{\log\bar{H}}\leq\frac{1}{2n\left(n-1\right)}\leq\frac{1}{2\left[(n-1)^2+1\right]};$$

hence it is always possible to choose a constant c_1 large enough such that the following estimate holds on M_0

$$\frac{\log\bar{H}-2}{\left(\log\bar{H}-1\right)\log\bar{H}}\leq\varepsilon\left(c_1\right)$$

where $\varepsilon\left(c_1\right)\leq\frac{1}{2\left[(n-1)^2+1\right]}$ is defined in Lemma 2.6. Furthermore note that, since the minimum of H is increasing in time, the estimate is satisfied with the same constant c_1 on all the evolving surfaces.

Proposition 2.7 *If the estimate $H>0$ holds on the initial surface M_0, then the function $f=-P_2=-\frac{s_2}{H^2}$ is bounded from above by a constant c_2 as long as the flow exists.*

Proof. The evolution equation for f is obtained by Corollary 1.15 replacing expressions at page 24

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \dot{\mathcal{S}}g^*\left(Hess_\nabla f\right) + \frac{1}{H}\dot{\mathcal{S}}g^*\ddot{Q}_2\left(\nabla\mathcal{W},\nabla\mathcal{W}\right) + \frac{2}{H}\dot{\mathcal{S}}g^*\left(\nabla f,\nabla H\right) \\
&\quad + \dot{f}g^*\ddot{\mathcal{S}}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right) - \dot{\mathcal{S}}g^*\left(\mathcal{II}\right)\dot{f}\left(\mathcal{W}^2\right) + \mathcal{S}\dot{f}\left(\mathcal{W}^2\right) \\
&= \frac{\log\bar{H}-1}{\left(\log\bar{H}\right)^2}\Delta f + \frac{\log\bar{H}-1}{H\left(\log\bar{H}\right)^2}\ddot{Q}_2\left\langle\nabla\mathcal{W},\nabla\mathcal{W}\right\rangle + \frac{2\left(\log\bar{H}-1\right)}{H\left(\log\bar{H}\right)^2}\left\langle\nabla f,\nabla H\right\rangle \\
&\quad - \frac{\log\bar{H}-2}{\bar{H}\left(\log\bar{H}\right)^3}\dot{f}g^*\left\langle\nabla H,\nabla H\right\rangle - \frac{\left(\log\bar{H}-1\right)H}{\left(\log\bar{H}\right)^2}\dot{f}\left(\mathcal{W}^2\right) + \frac{\bar{H}}{\log\bar{H}}\dot{f}\left(\mathcal{W}^2\right).
\end{aligned}$$

Recalling that $s_2 = \frac{1}{2}R = \frac{1}{2}\left(H^2 - |A|^2\right)$, we compute its derivatives

$$\frac{\partial s_2}{\partial h^i_j} = H\frac{\partial H}{\partial h^i_j} - h^r_s\frac{\partial h^s_r}{\partial h^i_j} = H\delta^j_i - h^r_s\delta^j_r\delta^s_i = H\delta^j_i - h^j_i,$$

that we write as $\dot{s}_2 = HId - hg^*$, and

$$\frac{\partial^2 s_2}{\partial h^l_k\partial h^i_j} = \frac{\partial}{\partial h^l_k}\left(H\delta^j_i - h^j_i\right) = \frac{\partial H}{\partial h^l_k}\delta^j_i - \frac{\partial h^j_i}{\partial h^l_k} = \delta^k_l\delta^j_i - \delta^k_i\delta^j_l.$$

Thus

$$\begin{aligned}
\ddot{Q}_2\left\langle\nabla\mathcal{W},\nabla\mathcal{W}\right\rangle &= \frac{\ddot{s}_2}{H}\left\langle\nabla\mathcal{W},\nabla\mathcal{W}\right\rangle - 2\frac{1}{H^2}\dot{s}_2\otimes Id\left\langle\nabla\mathcal{W},\nabla\mathcal{W}\right\rangle + 2\frac{s_2}{H^3}Id\otimes Id\left\langle\nabla\mathcal{W},\nabla\mathcal{W}\right\rangle \\
&= \frac{1}{H}\left(\delta^k_l\delta^j_i - \delta^k_i\delta^j_l\right)\left\langle\nabla h^i_j,\nabla h^l_k\right\rangle \\
&\quad - \frac{2}{H^2}\left(H\delta^j_i - h^j_i\right)\left\langle\nabla h^i_j,\nabla H\right\rangle + \frac{R}{H^3}\left|\nabla H\right|^2 \\
&= \frac{1}{H}\left|\nabla H\right|^2 - \frac{1}{H}\left|\nabla A\right|^2 - \frac{2}{H}\left|\nabla H\right|^2 \\
&\quad + \frac{2}{H^2}\left\langle A\nabla A,\nabla H\right\rangle + \frac{1}{H}\left|\nabla H\right|^2 - \frac{|A|^2}{H^3}\left|\nabla H\right|^2 \\
&= -\frac{1}{H^3}\left[H^2\left|\nabla A\right|^2 + |A|^2\left|\nabla H\right|^2 - 2\left\langle A\nabla H,H\nabla A\right\rangle\right] \\
&= -\frac{1}{H^3}\left|H\nabla A - A\nabla H\right|^2.
\end{aligned}$$

Furthermore we have

$$\dot{f} = -\frac{\dot{s}_2}{H^2} + 2\frac{s_2}{H^3}Id = -\frac{1}{H}Id + \frac{1}{H^2}hg^* + \frac{R}{H^3}Id = -\frac{|A|^2}{H^3}Id + \frac{1}{H^2}hg^*$$

and

$$\dot{f}\left(\mathcal{W}^2\right) = -\frac{|A|^2}{H^3}Id\left(\mathcal{W}^2\right) + \frac{1}{H^2}hg^*\left(\mathcal{W}^2\right) = -\frac{|A|^4 - HC}{H^3}$$

where C is the trace of $\mathcal{W}^3$.

Hence f evolves according to the equation

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\log \bar{H} - 1}{\left(\log \bar{H}\right)^2}\Delta f - \frac{\log \bar{H} - 1}{H^4\left(\log \bar{H}\right)^2}\left|H\nabla A - A\nabla H\right|^2 + \frac{2\left(\log \bar{H} - 1\right)}{H\left(\log \bar{H}\right)^2}\left\langle \nabla f, \nabla H\right\rangle \\
&\quad + \frac{\log \bar{H} - 2}{\bar{H}H^3\left(\log \bar{H}\right)^3}\left|\nabla H\right|^2_{|A|^2 g - Hh} - \frac{\left(\bar{H} - H\right)\log \bar{H} + H}{\left(\log \bar{H}\right)^2}\frac{|A|^4 - HC}{H^3} \\
&= \frac{\log \bar{H} - 1}{\left(\log \bar{H}\right)^2}\Delta f + \frac{2\left(\log \bar{H} - 1\right)}{H\left(\log \bar{H}\right)^2}\left\langle \nabla f, \nabla H\right\rangle \qquad (2.4) \\
&\quad - \frac{\log \bar{H} - 1}{H^4\left(\log \bar{H}\right)^2}\left[\left|H\nabla A - A\nabla H\right|^2 - \frac{H\left(\log \bar{H} - 2\right)}{\bar{H}\left(\log \bar{H} - 1\right)\log \bar{H}}\left|\nabla H\right|^2_{|A|^2 g - Hh}\right] \\
&\quad - \frac{\left(\bar{H} - H\right)\log \bar{H} + H}{\left(\log \bar{H}\right)^2}\frac{|A|^4 - HC}{H^3}.
\end{aligned}$$

We can assume that the maximum of f is positive, then Lemma 3.2 in [15] assures that the last term of the equation for $\frac{\partial f}{\partial t}$ is negative: $f = \frac{1}{2}\frac{|A|^2 - H^2}{H^2} \geq \frac{1}{2}\eta$ implies $|A|^4 - HC \geq \frac{1}{2}\eta\,|A|^2 H^2$. Moreover if we choose the constant c_1 as described before we have

$$\frac{H\left(\log \bar{H} - 2\right)}{\bar{H}\left(\log \bar{H} - 1\right)\log \bar{H}} < \frac{\log \bar{H} - 2}{\left(\log \bar{H} - 1\right)\log \bar{H}} \leq \varepsilon\left(c_1\right),$$

hence if $f \geq c_1$ where it attains a maximum then Lemma 2.6 gives

$$\frac{\partial f}{\partial t} \leq \frac{\log \bar{H} - 1}{\left(\log \bar{H}\right)^2}\Delta f + \frac{2\left(\log \bar{H} - 1\right)}{H\left(\log \bar{H}\right)^2}\left\langle \nabla f, \nabla H\right\rangle.$$

By an application of the maximum principle we deduce that $f \leq c_2 := \max\left\{\max_{M_0} f, c_1\right\}$ for all times $t \in [0, T)$. ■

In order to prove that the negative part of the scalar curvature vanishes near a singularity, we need to improve the result obtained by the boundedness of f: showing that also the function

$$f_{\sigma,\eta} := \frac{|A|^2 - (1+\eta) H^2}{H^2} \left(\log \bar{H}\right)^{\sigma}$$

is bounded from above for all times, we find the following crucial estimate.

Theorem 2.8 *If on the initial surface M_0 we have $H > 0$, then for any $\eta > 0$ there exists $c_4(\eta)$ such that the estimate*

$$|A|^2 \leq (1+2\eta) H^2 + c_4(\eta)$$

holds as long as the flow exists.

Proof. Let us note that

$$f_{\sigma,\eta} = \left[\left(\frac{|A|^2}{H^2} - 1\right) - \eta\right] \left(\log \bar{H}\right)^{\sigma} = (2f - \eta) \left(\log \bar{H}\right)^{\sigma},$$

then the evolution equation for $f_{\sigma,\eta}$ follows by formulas (2.4) and (2.3):

$$\begin{aligned}
\frac{\partial f_{\sigma,\eta}}{\partial t} &= 2\left(\log \bar{H}\right)^{\sigma} \frac{\partial f}{\partial t} + f_{\sigma,\eta} \frac{\sigma}{\bar{H}\log \bar{H}} \frac{\partial \bar{H}}{\partial t} \\
&= 2\left(\log \bar{H}\right)^{\sigma} \left\{ \frac{\log \bar{H} - 1}{\left(\log \bar{H}\right)^2} \Delta f + \frac{2\left(\log \bar{H} - 1\right)}{H\left(\log \bar{H}\right)^2} \langle \nabla f, \nabla H \rangle \right. \\
&\quad - \frac{\log \bar{H} - 1}{H^4 \left(\log \bar{H}\right)^2} \left[|H\nabla A - A\nabla H|^2 - \frac{H\left(\log \bar{H} - 2\right)}{\bar{H}\left(\log \bar{H} - 1\right)\log \bar{H}} |\nabla H|^2_{|A|^2 g - Hh} \right] \\
&\quad \left. - \frac{\left(\bar{H} - H\right)\log \bar{H} + H}{\left(\log \bar{H}\right)^2} \frac{|A|^4 - HC}{H^3} \right\} \\
&\quad + f_{\sigma,\eta} \frac{\sigma}{\bar{H}\log \bar{H}} \left[\frac{\log \bar{H} - 1}{\left(\log \bar{H}\right)^2} \Delta H - \frac{\log \bar{H} - 2}{\bar{H}\left(\log \bar{H}\right)^3} |\nabla H|^2 + \frac{\bar{H}}{\log \bar{H}} |A|^2 \right]
\end{aligned}$$

$$
\begin{aligned}
= \;& 2\left(\log \bar{H}\right)^{\sigma} \frac{\log \bar{H}-1}{\left(\log \bar{H}\right)^{2}} \Delta f+\sigma f_{\sigma, \eta} \frac{\log \bar{H}-1}{\bar{H}\left(\log \bar{H}\right)^{3}} \Delta H+2\left(\log \bar{H}\right)^{\sigma} \frac{2\left(\log \bar{H}-1\right)}{H\left(\log \bar{H}\right)^{2}}\langle\nabla f, \nabla H\rangle \\
& -2\left(\log \bar{H}\right)^{\sigma} \frac{\log \bar{H}-1}{H^{4}\left(\log \bar{H}\right)^{2}}\left[|H \nabla A-A \nabla H|^{2}-\frac{H\left(\log \bar{H}-2\right)}{\bar{H}\left(\log \bar{H}-1\right) \log \bar{H}}|\nabla H|_{|A|^{2} g-H h}^{2}\right] \\
& -\sigma f_{\sigma, \eta} \frac{\log \bar{H}-2}{\bar{H}^{2}\left(\log \bar{H}\right)^{4}}|\nabla H|^{2} \\
& +\sigma f_{\sigma, \eta} \frac{1}{\left(\log \bar{H}\right)^{2}}|A|^{2}-2\left(\log \bar{H}\right)^{\sigma} \frac{\left(\bar{H}-H\right) \log \bar{H}+H}{\left(\log \bar{H}\right)^{2}} \frac{|A|^{4}-H C}{H^{3}} .
\end{aligned}
$$

It is useful to compute the gradient and the Laplace operator of $f_{\sigma,\eta}$ in order to replace them in the equation above: as for the gradient we have

$$
\nabla f_{\sigma, \eta}=2\left(\log \bar{H}\right)^{\sigma} \nabla f+\sigma \frac{f_{\sigma, \eta}}{\bar{H}\left(\log \bar{H}\right)} \nabla H,
$$

thus

$$
\frac{2\left(\log \bar{H}\right)^{\sigma-1}}{\bar{H}} \nabla f=\frac{1}{\bar{H}\left(\log \bar{H}\right)} \nabla f_{\sigma, \eta}-\sigma \frac{f_{\sigma, \eta}}{\bar{H}^{2}\left(\log \bar{H}\right)^{2}} \nabla H,
$$

while the Laplacian of $f_{\sigma,\eta}$ is

$$
\begin{aligned}
\Delta f_{\sigma, \eta} = \;& 2 \sigma \frac{\left(\log \bar{H}\right)^{\sigma-1}}{\bar{H}}\langle\nabla f, \nabla H\rangle+2\left(\log \bar{H}\right)^{\sigma} \Delta f+\sigma \frac{1}{\bar{H}\left(\log \bar{H}\right)}\langle\nabla f_{\sigma, \eta}, \nabla H\rangle \\
& -\sigma \frac{f_{\sigma, \eta}}{\bar{H}^{2}\left(\log \bar{H}\right)}|\nabla H|^{2}-\sigma \frac{f_{\sigma, \eta}}{\bar{H}^{2}\left(\log \bar{H}\right)^{2}}|\nabla H|^{2}+\sigma \frac{f_{\sigma, \eta}}{\bar{H}\left(\log \bar{H}\right)} \Delta H \\
= \;& 2\left(\log \bar{H}\right)^{\sigma} \Delta f+\sigma \frac{f_{\sigma, \eta}}{\bar{H}\left(\log \bar{H}\right)} \Delta H+2 \sigma \frac{1}{\bar{H}\left(\log \bar{H}\right)}\langle\nabla f_{\sigma, \eta}, \nabla H\rangle \\
& -\sigma^{2} \frac{f_{\sigma, \eta}}{\bar{H}^{2}\left(\log \bar{H}\right)^{2}}|\nabla H|^{2}-\sigma f_{\sigma, \eta} \frac{\log \bar{H}+1}{\bar{H}^{2}\left(\log \bar{H}\right)^{2}}|\nabla H|^{2} .
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\frac{\partial f_{\sigma,\eta}}{\partial t} &= \frac{\log \bar{H}-1}{\left(\log \bar{H}\right)^2}\Delta f_{\sigma,\eta} - 2\sigma \frac{\log \bar{H}-1}{\bar{H}\left(\log \bar{H}\right)^3}\left\langle \nabla f_{\sigma,\eta}, \nabla H\right\rangle \\
&+\sigma^2 f_{\sigma,\eta}\frac{\log \bar{H}-1}{\bar{H}^2\left(\log \bar{H}\right)^4}\left|\nabla H\right|^2 + \sigma f_{\sigma,\eta}\frac{\left(\log \bar{H}\right)^2-1}{\bar{H}^2\left(\log \bar{H}\right)^4}\left|\nabla H\right|^2 \\
&+2\frac{\log \bar{H}-1}{H\left(\log \bar{H}\right)^2}\left\langle \nabla f_{\sigma,\eta}, \nabla H\right\rangle - 2\sigma f_{\sigma,\eta}\frac{\log \bar{H}-1}{H\bar{H}\left(\log \bar{H}\right)^3}\left|\nabla H\right|^2 - \sigma f_{\sigma,\eta}\frac{\log \bar{H}-2}{\bar{H}^2\left(\log \bar{H}\right)^4}\left|\nabla H\right|^2 \\
&-2\left(\log \bar{H}\right)^{\sigma-2}\frac{\log \bar{H}-1}{H^4}\left[\left|H\nabla A - A\nabla H\right|^2 - \frac{H\left(\log \bar{H}-2\right)}{\bar{H}\left(\log \bar{H}-1\right)\log \bar{H}}\left|\nabla H\right|^2_{|A|^2 g - Hh}\right] \\
&+\sigma f_{\sigma,\eta}\frac{1}{\left(\log \bar{H}\right)^2}|A|^2 - 2\left(\log \bar{H}\right)^{\sigma}\frac{\left(\bar{H}-H\right)\log \bar{H}+H}{\left(\log \bar{H}\right)^2}\frac{|A|^4-HC}{H^3}
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{\partial f_{\sigma,\eta}}{\partial t} &= \frac{\log \bar{H}-1}{\left(\log \bar{H}\right)^2}\Delta f_{\sigma,\eta} + 2\left(\log \bar{H} - \sigma\frac{H}{\bar{H}}\right)\frac{\log \bar{H}-1}{H\left(\log \bar{H}\right)^3}\left\langle \nabla f_{\sigma,\eta}, \nabla H\right\rangle \\
&-\sigma f_{\sigma,\eta}\left[\frac{\log \bar{H}-2}{\log \bar{H}-1} + 2\frac{\bar{H}}{H}\log \bar{H} - \log \bar{H} - 1 - \sigma\right]\frac{\log \bar{H}-1}{\bar{H}^2\left(\log \bar{H}\right)^4}\left|\nabla H\right|^2 \\
&-2\left(\log \bar{H}\right)^{\sigma}\frac{\log \bar{H}-1}{H^4\left(\log \bar{H}\right)^2}\left[\left|H\nabla A - A\nabla H\right|^2 - \frac{H\left(\log \bar{H}-2\right)}{\bar{H}\left(\log \bar{H}-1\right)\log \bar{H}}\left|\nabla H\right|^2_{|A|^2 g - Hh}\right] \\
&-\left[\frac{2\left(|A|^4-HC\right)}{H^2|A|^2}\frac{\left(\bar{H}-H\right)\log \bar{H}+H}{H} - \sigma\left(2f-\eta\right)\right]|A|^2\left(\log \bar{H}\right)^{\sigma-2}.
\end{aligned}
$$

Now we assume that the maximum of $f_{\sigma,\eta}$ on $M\times[0,T)$ is attained at $(\mathbf{p}_0, t_0)$ with $t_0>0$, and that the maximum is positive: otherwise the boundedness of $f_{\sigma,\eta}$ from above is obvious.

Let us define $H^*(\eta)$ as the minimum value of $\bar{H}>H_0$ such that for any $\bar{H}\geq H^*(\eta)$ we have

$$
\frac{H\left(\log \bar{H}-2\right)}{\bar{H}\left(\log \bar{H}-1\right)\log \bar{H}} \leq \varepsilon\left(\frac{1}{2}\eta\right) = \frac{2\eta+n-\sqrt{n\left(4\eta+n\right)}}{2\left[2\eta+n-\sqrt{n\left(4\eta+n\right)}+2\eta\left(n-1\right)^2\right]},
$$

where ε is defined as in Lemma 2.6, and recall that the previous proposition gives $f\leq c_2$.

If $H_0<\bar{H}(\mathbf{p}_0,t_0)\leq H^*(\eta)$ then $f_{\sigma,\eta}$ is bounded by

$$
f_{\sigma,\eta} \leq \left(2c_2-\eta\right)\left(\log \bar{H}\right)^{\sigma} < \left(2c_2-\eta\right)\left(\log H^*(\eta)\right)^{\sigma}.
$$

Consider then the case of $\bar{H}(\mathbf{p}_0, t_0) > H^*(\eta)$. The gradient terms in the third line of the evolution equation for $f_{\sigma,\eta}$ are estimated by Lemma 2.6, while the second line is negative for $\sigma \leq 3$ because $H^*(\eta) > H_0 = e^{2n(n-1)}$ implies

$$2\frac{\bar{H}}{H}\log\bar{H} - \log\bar{H} - 1 > \log\bar{H} - 1 > 2n(n-1) - 1 \geq 3.$$

Finally note that $f_{\sigma,\eta} \geq 0$ is equivalent to $f \geq \frac{1}{2}\eta$, hence, applying again Lemma 3.2 in [15], we have the inequality

$$|A|^4 - HC \geq \frac{1}{2}\eta\,|A|^2 H^2.$$

Then we can choose $\sigma = \sigma(\eta) \leq 3$ such that

$$\sigma \leq \frac{\eta}{2c_2 - \eta} \leq \frac{2\left(|A|^4 - HC\right)}{|A|^2 H^2 (2f - \eta)}$$

to obtain $\frac{\partial f_{\sigma,\eta}}{\partial t}(\mathbf{p}_0, t_0) \leq 0$ and deduce by contradiction that $\max f_{\sigma,\eta} \leq \max_{M_0} f_{\sigma,\eta}$.

It follows that the function $f_{\sigma,\eta}$ is bounded for all times by a constant depending on η

$$f_{\sigma,\eta} = \frac{|A|^2 - (1+\eta) H^2 \left(\log\bar{H}\right)^{\sigma}}{H^2} \leq c_3(\eta),$$

hence

$$|A|^2 \leq (1+\eta) H^2 + c_3(\eta) H^2 \left(\log\bar{H}\right)^{-\sigma}.$$

Since $\bar{H} > H$ we have $|A|^2 < (1+2\eta) H^2 + c_4(\eta)$ for any η. ∎

The previous estimate concludes the proof of Theorem 2.2.

As a consequence of this result we prove in the next corollary that near a singularity of M_t the negative part of the scalar curvature tends to zero.

Corollary 2.9 *If we rescale the evolving surfaces near a singularity, then any smooth limit of the rescalings has nonnegative scalar curvature.*

Proof. Let us consider a sequence of points and times $\{(\mathbf{p}_k, t_k)\}_k$ such that $\max_{M_{t_k}} H$ is attained at point $\mathbf{p}_k$. Multiplying by $H_k := H(\mathbf{p}_k, t_k)$ we define a rescaling $\tilde{M}_{t_k}$ of M_{t_k}.

On the rescaled surfaces $\tilde{M}_{t_k}$ we have $\tilde{H}_k \leq 1$ by definition and $\left|\tilde{A}\right|_k^2$ bounded from above thank to the previous proposition:

$$\left|\tilde{A}\right|_k^2 \leq (1+2\eta)\,\tilde{H}_k^2 + \frac{c_4(\eta)}{H_k^2}.$$

If there exists a smooth limiting surface $\tilde{M}_\infty$, the estimate

$$\left|\tilde{A}\right|^2 \leq (1+2\eta)\,\tilde{H}^2$$

holds on $\tilde{M}_\infty$ for all $\eta > 0$. Since η is arbitrarily chosen, we consider the limit for $\eta \to 0$ and obtain $-\tilde{R} = \left|\tilde{A}\right|^2 - \tilde{H}^2 \leq 0$. ■

Chapter 3

Convex surfaces evolving by powers of scalar curvature

In this chapter we study the evolution of a convex hypersurface by powers of scalar curvature. The theorem we are going to prove is the following:

Theorem 3.1 *Let $\mathbf{F}_0 : M \longrightarrow \mathbb{R}^{n+1}$ be a smooth immersion of a closed, strictly convex hypersurface of $\mathbb{R}^{n+1}$ with $n \geq 2$. The following initial value problem*

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{F}(\mathbf{p},t) = -R^p(\mathbf{p},t)\,\boldsymbol{\nu}(\mathbf{p},t), \quad t \geq 0 \\ \mathbf{F}(\cdot,0) = \mathbf{F}_0(M) \end{cases} \tag{3.1}$$

where the power p is supposed to be $p > 1/2$, has a unique smooth solution for a short time interval.

If the initial surface M_0 satisfies a pinching estimate on the principal curvatures $\frac{R}{H^2} > c(n,p)$ for a suitable constant $c(n,p)$, then the surfaces $M_t = \mathbf{F}(\cdot,t)$ are convex for all times and converge to a single point in a finite time T.

Moreover the rescaled immersions given by $\tilde{\mathbf{F}}(\mathbf{p},t) = (c'(T-t))^{-\frac{1}{2p+1}}\mathbf{F}(\mathbf{p},t)$ converge exponentially in the C^∞ topology to a smooth embedding $\tilde{\mathbf{F}}_\infty$, whose image is a sphere, as the rescaled time parameter

$$\tau = -\frac{1}{c'}\ln\left(1-\frac{t}{T}\right) \to \infty.$$

The key idea of the proof consists of showing that the function

$$f = \frac{|A|^2}{H^2} - \frac{1}{n} = \frac{1}{n} \sum_{i<j} (\lambda_i - \lambda_j)^2$$

defined on M_t decreases and tends uniformly to zero as $t \to T$.

Since $\frac{R}{H^2} = \frac{n-1}{n} - f$, then $\frac{R}{H^2} > c$ if and only if $f < \gamma = \frac{n-1}{n} - c$, hence, for the monotonicity of f, the pinching estimate on the principal curvatures of the evolving surfaces is preserved along the flow.

Furthermore let us observe that the only surface such that $f \equiv 0$ is the sphere. Then the convergence of f to zero on the limiting surface is essential to prove that the shape of the M_t's approaches that of a sphere.

We recall that an analogue theorem has been proved by B. Chow in the case of $p = \frac{1}{2}$ (see [8]) with pinching condition $\frac{R}{H^2} > \frac{n-2}{n-1}$. We can compare this condition with the assumption of the above theorem: the constant γ, defined in Theorem 3.7 for $\frac{1}{2} < p \leq 1$ is

$$\gamma = \min \left\{ \frac{1}{n(n-1)}, \frac{4(n-1)}{n(n+2)^2} \right\},$$

and thus

$$c = \begin{cases} \frac{(n+4)(n-1)}{(n+2)^2} & \text{for } n < 4 \\ \frac{n-2}{n-1} & \text{for } n \geq 4. \end{cases}$$

Hence, even if in dimension 2 and 3 we need a stronger condition, for all $n \geq 4$ our assumption is equivalent to the one required in the case $p = \frac{1}{2}$.

Furthermore note that in dimension $n = 2$ the scalar curvature coincides with the Gauss curvature for surfaces and some estimate can be improved, in fact the results shown in [7] and [2], corresponding respectively to the case $p = \frac{1}{2}$ and $p = 1$, are proved under the only hypothesis of convexity.

3.1 Evolution equations

In this section first we prove the short time existence of the flow (3.1), then we use the formulas derived in the first chapter to compute the evolution equations necessary in the following.

Theorem 3.2 *The initial value problem (3.1) has a unique solution for some short time interval* $[0,T)$.

Proof. Let $\lambda_1 \leq ... \leq \lambda_n$ be the principal curvatures of M_t at a point $\mathbf{y}$, i.e. the eigenvalues of the second fundamental form at $\mathbf{y}$. Computing $\frac{\partial R^p}{\partial \lambda_k}$ we have for any k

$$\begin{aligned}
\frac{\partial R^p}{\partial \lambda_k} &= pR^{p-1}\frac{\partial}{\partial \lambda_k}\left[\sum_{i,j=1}^{n}\lambda_i\lambda_j - \sum_{i=1}^{n}\lambda_i^2\right] \\
&= pR^{p-1}\left[2\sum_{i,j=1}^{n}\frac{\partial \lambda_i}{\partial \lambda_k}\lambda_j - 2\sum_{i=1}^{n}\lambda_i\frac{\partial \lambda_i}{\partial \lambda_k}\right] \\
&= 2pR^{p-1}\left[\sum_{i,j=1}^{n}\delta_k^i\lambda_j - \sum_{i=1}^{n}\lambda_i\delta_k^i\right] \\
&= 2pR^{p-1}\left(H-\lambda_k\right).
\end{aligned}$$

Then Theorem 1.3 gives the short time existence as a consequence of the convexity of M_0. ∎

Definition 3.3 *We define the metric* $m_{ij} = Hg_{ij} - h_{ij}$ *such that* $\dot{R} = 2mg^*$.

The evolution equations found in respectively in Proposition 1.9 and Theorem 1.12 are written explicitly here for this flow in the following proposition setting $\mathcal{S} = R^p$.

Proposition 3.4 *If for any* $t \in [0,T)$ *the functions* $F_t : M \longrightarrow \mathbb{R}^{n+1}$ *are smooth immersions satisfying the evolution equation (3.1), then the following equations hold*

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= -2R^p h_{ij} \\
\frac{\partial d\mu}{\partial t} &= -R^p H d\mu \\
\frac{\partial \boldsymbol{\nu}}{\partial t} &= \nabla R^p \\
\frac{\partial R^p}{\partial t} &= 2pR^{p-1}\left\{\Delta_m R^p + \left(H\left|A\right|^2 - C\right)R^p\right\}.
\end{aligned} \tag{3.2}$$

Proposition 3.5 *Let* P *be a generic homogeneous function of degree* α *of the Weingarten map*

$\mathcal{W}$ defined on the surface M_t evolving by (3.1). The evolution equation of P is

$$\begin{aligned}\frac{\partial P}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m P - mg^*g^*\ddot{P}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right)\\ &+\dot{P}g^*\left(\nabla H,\nabla H\right) - \dot{P}g^*\left(\nabla A,\nabla A\right) + \frac{p-1}{2}\frac{1}{R}\dot{P}g^*\left(\nabla R,\nabla R\right)\\ &-\frac{2p-1}{2p}R\dot{P}\left(\mathcal{W}^2\right) + \alpha\left(H\left|A\right|^2 - C\right)P\Big\}.\end{aligned}$$

Proof. We compute the derivatives of the speed $\mathcal{S} = R^p = \left(H^2 - |A|^2\right)^p$ with respect to the Weingarten map:

$$\frac{\partial R^p}{\partial h^i_j} = pR^{p-1}\frac{\partial R}{\partial h^i_j} = 2pR^{p-1}\left(H\delta^j_i - h^j_i\right),$$

i. e. $\dot{\mathcal{S}} = 2pR^{p-1}mg^*$, and

$$\begin{aligned}\frac{\partial^2 R^p}{\partial h^l_k\partial h^i_j} &= 2pR^{p-1}\frac{\partial}{\partial h^l_k}\left(H\delta^j_i - h^j_i\right) + 2p\left(p-1\right)R^{p-2}m^j_i\frac{\partial R}{\partial h^l_k}\\ &= 2pR^{p-1}\left[\delta^k_l\delta^j_i - \delta^k_i\delta^j_l + 2\left(p-1\right)\frac{1}{R}m^j_i m^k_l\right].\end{aligned}$$

Applying the second formula in Theorem 1.12 with $\beta = 2p$ we have

$$\begin{aligned}\frac{\partial P}{\partial t} &= 2pR^{p-1}\Big\{mg^*g^*\left(Hess_\nabla P\right) - mg^*g^*\ddot{P}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right)\\ &+\dot{P}g^*\left[\left(\nabla H,\nabla H\right) - \left(\nabla A,\nabla A\right) + \frac{p-1}{2}\frac{1}{R}\left(\nabla R,\nabla R\right)\right]\\ &-\frac{2p-1}{2p}R\dot{P}\left(\mathcal{W}^2\right) + \alpha mg^*\left(\mathcal{W}^2\right)P\Big\},\end{aligned}$$

then the statement follows using Notation 1.7 and computing

$$mg^*\left(\mathcal{W}^2\right) = \left(HId - hg^*\right)\left(\mathcal{W}^2\right) = H\left|A\right|^2 - C$$

where C is again the trace of $\mathcal{W}^3$. ■

Corollary 3.6 *We have the following evolution equations:*

$$\begin{aligned}
\frac{\partial H}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m H + |\nabla H|^2 - |\nabla A|^2 + \frac{p-1}{2}\frac{1}{R}|\nabla R|^2 \\
&\quad -\frac{2p-1}{2p}R|A|^2 + \left(H|A|^2 - C\right)H\Big\} \\
\frac{\partial H^2}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m H^2 + 2|\nabla_i H|_h^2 - 2|\nabla_i A|_{Hg}^2 + (p-1)\frac{H}{R}|\nabla R|^2 \\
&\quad -\frac{2p-1}{p}R|A|^2 H + 2\left(H|A|^2 - C\right)H^2\Big\} \\
\frac{\partial |A|^2}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m |A|^2 + 2|\nabla_i H|_h^2 - 2|\nabla_i A|_{Hg}^2 \\
&\quad +(p-1)\frac{1}{R}|\nabla_i R|_h^2 - \frac{2p-1}{p}RC + 2\left(H|A|^2 - C\right)|A|^2\Big\} \\
\frac{\partial R}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m R + (p-1)\frac{1}{R}|\nabla_i R|_m^2 + \frac{1}{p}\left(H|A|^2 - C\right)R\Big\}
\end{aligned} \tag{3.3}$$

Proof. The evolution equation for H is an immediate consequence of Proposition 3.5. The evolution for H^2 and $|A|^2$ follow from the equalities

$$\begin{aligned}
&-2mg^*g^*(\nabla H,\nabla H) + 2Hg^*g^*(\nabla H,\nabla H) - 2Hg^*g^*(\nabla A,\nabla A) \\
=& -2|\nabla_i H|_m^2 + 2H|\nabla H|^2 - 2H|\nabla A|^2 \\
=& \ 2|\nabla_i H|_h^2 - 2H|\nabla A|^2
\end{aligned}$$

and

$$\begin{aligned}
&= -2mg^*g^*(\nabla A,\nabla A) + 2hg^*g^*(\nabla H,\nabla H) - 2hg^*g^*(\nabla A,\nabla A) \\
&= -2|\nabla_i A|_m^2 + 2|\nabla_i H|_h^2 - 2|\nabla_i A|_h^2 \\
&= -2H|\nabla A|^2 + 2|\nabla_i H|_h^2.
\end{aligned}$$

The equation for R is obtained as the difference of the previous ones. ■

3.2 Pinching of curvatures

Let us introduce the function $f := \frac{|A|^2}{H^2} - \frac{1}{n}$. Then Lemma 2.3 i) of [13] establishes the relation

$$0 \leq fH^2 = \frac{1}{n} \sum_{i<j} (\lambda_i - \lambda_j)^2 .$$

Hence f gives a measure of how much the principal curvatures differ from each other. We aim to prove that, under suitable hypotheses on the initial surface, the maximum of f is nonincreasing in time. Then we can estimate f on $M \times [0, T)$ by its maximum value on M_0.

Furthermore we prove that f tends to zero as $t \to T$, at least at those points where the mean curvature diverges. This is called the pinching of curvatures because it implies that all principal curvatures tend to have the same value.

This section is devoted to the proof of the following theorem.

Theorem 3.7 *Let us define*

$$\gamma_1 = \frac{1}{n\,(n-1)} \qquad \text{and} \qquad \gamma_2 = \begin{cases} \frac{4(n-1)}{n(n+2)^2} & \text{for } \frac{1}{2} < p \leq 1 \\ \frac{4(n-1)}{n(n+2)^2} \left(\frac{1}{2p-1}\right)^2 & \text{for } p > 1. \end{cases}$$

If on M_0 we have $f < \gamma := \min\{\gamma_1, \gamma_2\}$, then there exist two constants $\sigma > 0$ and $c_1 > 0$ such that $f \leq c_1 H^{-\sigma}$ i.e.

$$|A|^2 - \frac{1}{n} H^2 \leq c_1 H^{2-\sigma} \qquad \forall t \in [0, T) .$$

As announced, the first step of the proof consists of showing the monotonicity of the function f applying the maximum principle (Theorem 1.16) to its evolution equation. For technical reasons in the following lemma we give two different formulas: one suited for $\frac{1}{2} < p \leq 1$, and an other one for $p > 1$.

Lemma 3.8 *The function f satisfies the following evolution equations*

$$\begin{aligned} \frac{\partial f}{\partial t} \;&=\; 2pR^{p-1} \Big\{ \Delta_m f - 4\,(1-p)\,\frac{R}{H^5}\,|H\nabla A - A\nabla H|^2 + (1-p)\,\frac{H}{R}\,|\nabla_i f|^2_{|A|^2 g - Hh} \\ &\quad + 4p\frac{1}{H}\,\langle \nabla_i f, \nabla_i H\rangle_m - 2\,(2p-1)\,\frac{R}{H^4}\left[|\nabla_i A|^2_{Hg} - |\nabla_i H|^2_h + \frac{1}{2p} H \left(HC - |A|^4 \right) \right] \Big\} \end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m f-(p-1)\frac{H}{R}\left|\nabla_i f\right|^2_{|A|^2g-Hh}+4\frac{1}{H}\left\langle \nabla_i f,\nabla_i H\right\rangle_m \\
&\quad +4(p-1)\frac{1}{H^2}\left\langle \nabla_i f,\nabla_i H\right\rangle_{|A|^2g-Hh}-2\frac{R}{H^4}\left(\left|\nabla_i A\right|^2_{Hg}-\left|\nabla_i H\right|^2_h\right) \\
&\quad -4(p-1)\frac{R}{H^5}\left|\nabla_i H\right|^2_{|A|^2g-Hh}-\frac{2p-1}{p}\frac{R}{H^3}\left(HC-|A|^4\right)\Big\}.
\end{aligned}
\tag{3.5}
$$

We will use the first one in the case $\frac{1}{2}<p\leq 1$ *and the second one for* $p>1$.

Proof. The evolution equation for f is given by Proposition 3.5 replacing

$$
\dot f^p_q=-\frac{2}{H^3}\left(|A|^2\delta^p_q-Hh^p_q\right)
$$

and

$$
\begin{aligned}
\ddot f^{pl}_{qk} &= -\frac{2}{H^3}\left(2h^l_k\delta^p_q-\delta^l_k h^p_q-H\delta^p_k\delta^l_q\right)+\frac{6}{H^4}\delta^l_k\left(|A|^2\delta^p_q-Hh^p_q\right) \\
&= -\frac{2}{H^3}\left(h^l_k\delta^p_q-H\delta^p_k\delta^l_q\right)-\frac{3}{H}\delta^l_k\dot f^p_q.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m f-mg^*g^*\ddot f^{pl}_{qk}\left(\nabla h^q_p,\nabla h^k_l\right)+\dot f g^*\left(\nabla H,\nabla H\right) \\
&\quad -\dot f g^*\left(\nabla A,\nabla A\right)+\frac{p-1}{2}\frac{1}{R}\dot f g^*\left(\nabla R,\nabla R\right)-\frac{2p-1}{2p}R\dot f\left(\mathcal{W}^2\right)\Big\} \\
&= 2pR^{p-1}\Big\{\Delta_m f+\frac{2}{H^3}\left\langle \nabla_i H,A\nabla_i A\right\rangle_m-\frac{2}{H^2}\left|\nabla_i A\right|^2_m \\
&\quad +\frac{3}{H}\left\langle \nabla_i f,\nabla_i H\right\rangle_m-\frac{2}{H^3}\left|\nabla_i H\right|^2_{|A|^2g-Hh}+\frac{2}{H^3}\left|\nabla_i A\right|^2_{|A|^2g-Hh} \\
&\quad -(p-1)\frac{1}{RH^3}\left|\nabla_i R\right|^2_{|A|^2g-Hh}+\frac{2p-1}{p}\frac{R}{H^3}\left(|A|^4-HC\right)\Big\}.
\end{aligned}
$$

Since we aim to apply the maximum principle, we need to write the equation above in the form of Theorem 1.16. Let us observe that

$$
|A|^2g-Hh=H\left(Hg-h\right)-Rg
$$

and that

$$\nabla R = \nabla\left(\frac{R}{H^2}H^2\right) = -H^2\nabla f + 2\frac{R}{H}\nabla H,$$

then

$$(\nabla R, \nabla R) = H^4(\nabla f, \nabla f) - 4RH(\nabla f, \nabla H) + 4\frac{R^2}{H^2}(\nabla H, \nabla H). \tag{3.6}$$

Thus the equation for f becomes

$$\begin{aligned}\frac{\partial f}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m f + \frac{2}{H^3}\langle\nabla_i H, A\nabla_i A\rangle_m + \frac{3}{H}\langle\nabla_i f, \nabla_i H\rangle_m \\ &\quad -\frac{2}{H^3}|\nabla_i H|^2_{|A|^2 g - Hh} - 2\frac{R}{H^3}|\nabla A|^2 - (p-1)\frac{H}{R}|\nabla_i f|^2_{|A|^2 g - Hh} \\ &\quad +4(p-1)\frac{1}{H^2}\langle\nabla_i f, \nabla_i H\rangle_{|A|^2 g - Hh} - 4(p-1)\frac{R}{H^5}|\nabla_i H|^2_{|A|^2 g - Hh} \\ &\quad +\frac{2p-1}{p}\frac{R}{H^3}\left(|A|^4 - HC\right)\Big\}.\end{aligned}$$

To obtain equation (3.5) it suffices to apply the following equality

$$\frac{2}{H^3}\langle\nabla_i H, A\nabla_i A\rangle_m = \frac{1}{H}\langle\nabla_i f, \nabla_i H\rangle_m + 2\frac{|A|^2}{H^4}|\nabla H|^2_m,$$

hence for $p \geq 1$ we write the equation for f as

$$\begin{aligned}\frac{\partial f}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m f + 4\frac{1}{H}\langle\nabla_i f, \nabla_i H\rangle_m + 2\frac{R}{H^4}|\nabla_i H|^2_h - 2\frac{R}{H^3}|\nabla A|^2 \\ &\quad -(p-1)\frac{H}{R}|\nabla_i f|^2_{|A|^2 g - Hh} + 4(p-1)\frac{1}{H^2}\langle\nabla_i f, \nabla_i H\rangle_{|A|^2 g - Hh} \\ &\quad -4(p-1)\frac{R}{H^5}|\nabla_i H|^2_{|A|^2 g - Hh} - \frac{2p-1}{p}\frac{R}{H^3}\left(HC - |A|^4\right)\Big\}.\end{aligned}$$

Now equation (3.4) can be found by a rearrangement of the first order terms:

$$\begin{aligned}&4\frac{1}{H}\langle\nabla_i f, \nabla_i H\rangle_m + 4(p-1)\frac{1}{H^2}\langle\nabla_i f, \nabla_i H\rangle_{|A|^2 g - Hh} \\ &-2\frac{R}{H^4}\left(|\nabla_i A|^2_{Hg} - |\nabla_i H|^2_h\right) - 4(p-1)\frac{R}{H^5}|\nabla_i H|^2_{|A|^2 g - Hh}\end{aligned}$$

$$
\begin{aligned}
&= 4\frac{1}{H}\langle\nabla_i f,\nabla_i H\rangle_m + 4(p-1)\frac{1}{H}\langle\nabla_i f,\nabla_i H\rangle_m - 4(p-1)\frac{R}{H^2}\langle\nabla f,\nabla H\rangle \\
&\quad -2(2p-1)\frac{R}{H^4}\left(|\nabla_i A|^2_{Hg} - |\nabla_i H|^2_h\right) - 4(1-p)\frac{R}{H^4}\left(|\nabla_i A|^2_{Hg} - |\nabla_i H|^2_h\right) \\
&\quad +4(1-p)\frac{R}{H^5}|A|^2|\nabla H|^2 - 4(1-p)\frac{R}{H^4}|\nabla_i H|^2_h \\
&= 4p\frac{1}{H}\langle\nabla_i f,\nabla_i H\rangle_m - 2(2p-1)\frac{R}{H^4}\left(|\nabla_i A|^2_{Hg} - |\nabla_i H|^2_h\right) \\
&\quad +4(1-p)\frac{R}{H^5}\left[H^3\langle\nabla f,\nabla H\rangle - H^2|\nabla A|^2 + |A|^2|\nabla H|^2\right].
\end{aligned}
$$

Hence formula (3.4) for $\frac{1}{2} < p < 1$ follows from

$$H^3\langle\nabla f,\nabla H\rangle = 2\langle A\nabla H, H\nabla A\rangle - 2|A|^2|\nabla H|^2$$

and

$$
\begin{aligned}
|H\nabla A - A\nabla H|^2 &= H^2|\nabla A|^2 - 2\langle A\nabla H, H\nabla A\rangle + |A|^2|\nabla H|^2 \\
&= H^2|\nabla A|^2 - H^3\langle\nabla f,\nabla H\rangle - |A|^2|\nabla H|^2. \qquad (3.7)
\end{aligned}
$$

■

The next proposition is crucial to estimate some terms in the evolution equation for f.

Proposition 3.9 *The following assertions hold on any surface M_t:*

i) If $f < \frac{1}{n(n-1)}$ then there exists $\eta > 0$ such that $\lambda_1 > \eta H$, hence M_t is strictly convex.

ii) If $f \leq \frac{4(n-1)}{n(n+2)^2}$ then $H|\nabla A|^2 - |\nabla_i H|^2_h \geq 0$.

iii) For $p > 1$, if $f \leq \frac{4(n-1)}{n(n+2)^2}\left(\frac{1-\varepsilon}{2p-1}\right)^2$ then

$$H|\nabla A|^2 - |\nabla_i H|^2_h \geq \frac{2(n-1)}{n(n+2)}\frac{2(p-1)+\varepsilon}{2p-1}H|\nabla H|^2.$$

Proof. We begin writing H and $|A|^2$ as polynomial functions of the λ_i's

$$H = \sum_{i=1}^{n}\lambda_i \quad \text{and} \quad |A|^2 = \sum_{i=1}^{n}\lambda_i^2,$$

and isolating one eigenvalue, say λ_k:

$$H = H' + \lambda_k \qquad \text{and} \qquad |A|^2 = \left|A'\right|^2 + \lambda_k^2.$$

It follows

$$\begin{aligned}
fH^2 &= |A|^2 - \frac{1}{n}H^2 = \left|A'\right|^2 + \lambda_k^2 - \frac{1}{n}H'^2 - \frac{2}{n}H'\lambda_k - \frac{1}{n}\lambda_k^2 \\
&= \left|A'\right|^2 - \frac{1}{n-1}H'^2 + \frac{1}{n\,(n-1)}H'^2 + \frac{n-1}{n}\lambda_k^2 - \frac{2}{n}H'\lambda_k \\
&= \left(\left|A'\right|^2 - \frac{1}{n-1}H'^2\right) + \frac{n}{n-1}\left(\frac{n-1}{n}\lambda_k - \frac{1}{n}H'\right)^2 \\
&\geq \frac{n}{n-1}\left(\frac{n-1}{n}\lambda_k - \frac{1}{n}H'\right)^2,
\end{aligned}$$

thus, denoted by $\gamma^* = \sup_{M_t} f$, the condition $f \leq \gamma^*$ implies

$$\begin{aligned}
\gamma^* H^2 &\geq \frac{n}{n-1}\left(\frac{n-1}{n}\lambda_k - \frac{1}{n}H'\right)^2 = \frac{n}{n-1}\left(\lambda_k - \frac{1}{n}H\right)^2 \\
\sqrt{\gamma^*}H &\geq \sqrt{\frac{n}{n-1}}\left|\lambda_k - \frac{1}{n}H\right| \\
\sqrt{\frac{n-1}{n}\gamma^*}H &\geq \left|\lambda_k - \frac{1}{n}H\right|.
\end{aligned}$$

In order to prove the first part we set $\lambda_k = \lambda_1$ and require $\gamma^* < \frac{1}{n(n-1)}$, then

$$\sqrt{\frac{n-1}{n}\gamma^*}H > \frac{1}{n}H - \lambda_1$$

implies

$$\lambda_1 > \frac{1}{n} - \sqrt{\frac{n-1}{n}\gamma^*}H > 0. \tag{3.8}$$

For the proof of assertions ii) and iii) we take $\lambda_k = \lambda_n$

$$\begin{aligned}
\sqrt{\frac{n-1}{n}\gamma^*}H &\geq \lambda_n - \frac{1}{n}H \\
\lambda_n &\leq \left(\sqrt{\frac{n-1}{n}\gamma^*} + \frac{1}{n}\right)H.
\end{aligned} \tag{3.9}$$

Note that Lemma 2.2 of [13] gives $|\nabla A|^2 \geq \frac{3}{n+2}|\nabla H|^2$ and $h^{ij}\nabla_i H \nabla_j H \leq \lambda_n |\nabla H|^2$, then

$$\begin{aligned} H|\nabla A|^2 - |\nabla_i H|_h^2 &\geq \frac{3}{n+2} H |\nabla H|^2 - \lambda_n |\nabla H|^2 \\ &\geq \left(\frac{2(n-1)}{n(n+2)} - \sqrt{\frac{n-1}{n}\gamma^*} \right) H |\nabla H|^2 . \end{aligned}$$

Hence $\gamma^* < \frac{4(n-1)}{n(n+2)^2}$ implies $H|\nabla A|^2 - |\nabla_i H|_h^2 > 0$ and in particular for $p > 1$, if

$$\gamma^* < \frac{4(n-1)}{n(n+2)^2} \left(\frac{1-\varepsilon}{2p-1} \right)^2$$

is verified, we find

$$H|\nabla A|^2 - |\nabla_i H|_h^2 > \frac{2(n-1)}{n(n+2)} \left(1 - \frac{1-\varepsilon}{2p-1} \right) H |\nabla H|^2 .$$

■

The next corollary establishes the monotonicity of f.

Corollary 3.10 *Under the hypotheses of Theorem 3.7 the estimate* $f < \gamma$*, or equivalently* $\frac{R}{H^2} > c = \frac{n-1}{n} - \gamma$*, holds as long as the solution to problem (3.1) exists.*

Proof. If we have $f < \gamma$ on M_0, then M_0 is strictly convex, as established in Proposition 3.9 i), with $h_{ij} \geq \eta H g_{ij}$ and Lemma 2.3 of [13] gives

$$HC - |A|^4 \geq n\eta^2 f H^4 \geq 0. \tag{3.10}$$

For $\frac{1}{2} < p \leq 1$ we apply Proposition 3.9 ii) to the equation (3.4) and obtain

$$\frac{\partial f}{\partial t} \leq 2pR^{p-1} \left\{ \Delta_m f + (1-p)\frac{H}{R} |\nabla_i f|^2_{|A|^2 g - Hh} + 4p \frac{1}{H} \langle \nabla_i f, \nabla_i H \rangle_m \right\} .$$

Hence by maximum principle we can deduce that $\max_{M_t} f \leq \max_{M_0} f < \gamma$ for all times.

Now consider the case $p > 1$. Note that assuming $f < \gamma$ one can estimate from below the eigenvalues of the matrix $m^i_j = H\delta^i_j - h^i_j$ and the elements of the bilinear form $|A|^2 g_{ij} - Hh_{ij}$.

For the matrix m inequality (3.9) implies

$$H-\lambda_j \geq H-\lambda_n \geq \left(\frac{n-1}{n}-\sqrt{\frac{n-1}{n}\gamma}\right) H, \tag{3.11}$$

hence it follows

$$\begin{aligned} |\cdot|^2_{|A|^2 g - Hh} &= H\,|\cdot|^2_m - R\,|\cdot|^2 \\ &\geq \left(\frac{n-1}{n}-\sqrt{\frac{n-1}{n}\gamma}-\frac{R}{H^2}\right) H^2\,|\cdot|^2 . \end{aligned} \tag{3.12}$$

Moreover, being

$$\frac{R}{H^2} = 1-\frac{|A|^2}{H^2} = \frac{n-1}{n} - f \leq \frac{n-1}{n} \tag{3.13}$$

we infer

$$|\cdot|^2_{|A|^2 g - Hh} \geq -\sqrt{\frac{n-1}{n}\gamma} H^2\,|\cdot|^2 . \tag{3.14}$$

These considerations lead to the following estimate for the evolution equation of f (3.5) in the case $p > 1$: if $f < \gamma$ then

$$\begin{aligned} \frac{\partial f}{\partial t} \leq\ & 2pR^{p-1}\left\{\Delta_m f - (p-1)\frac{H}{R}\,|\nabla_i f|^2_{|A|^2 g - Hh} + 4\frac{1}{H}\langle \nabla_i f, \nabla_i H\rangle_m \right. \\ & +4(p-1)\frac{1}{H^2}\langle \nabla_i f, \nabla_i H\rangle_{|A|^2 g - Hh} - 2\left(\frac{2(n-1)}{n(n+2)}-\sqrt{\frac{n-1}{n}\gamma}\right)\frac{R}{H^3}\,|\nabla H|^2 \\ & \left. +4(p-1)\sqrt{\frac{n-1}{n}\gamma}\frac{R}{H^3}\,|\nabla H|^2 - \frac{2p-1}{p}\frac{R}{H^3}\left(HC-|A|^4\right)\right\} \\ =\ & 2pR^{p-1}\left\{\Delta_m f - (p-1)\frac{H}{R}\,|\nabla_i f|^2_{|A|^2 g - Hh} + 4\frac{1}{H}\langle \nabla_i f, \nabla_i H\rangle_m \right. \\ & \left. +4(p-1)\frac{1}{H^2}\langle \nabla_i f, \nabla_i H\rangle_{|A|^2 g - Hh} - 2\left(\frac{2(n-1)}{n(n+2)}-(2p-1)\sqrt{\frac{n-1}{n}\gamma}\right)\frac{R}{H^3}\,|\nabla H|^2\right\} \\ \leq\ & 2pR^{p-1}\left\{\Delta_m f - (p-1)\frac{H}{R}\,|\nabla_i f|^2_{|A|^2 g - Hh} + 4\frac{1}{H}\langle \nabla_i f, \nabla_i H\rangle_m \right. \\ & \left. +4(p-1)\frac{1}{H^2}\langle \nabla_i f, \nabla_i H\rangle_{|A|^2 g - Hh}\right\} . \end{aligned}$$

Also in this case the maximum principle implies that f is nonincreasing and the estimate $f < \gamma$ holds on all the existence time interval of the solution of (3.1). ∎

Now we can prove Theorem 3.7 to conclude this section.

Proof of Theorem 3.7. We derive the evolution equation for $f_\sigma := fH^\sigma$ with $\sigma > 0$ and prove its monotonicity.

We start computing

$$\dot{H}^\sigma = \sigma H^{\sigma-1} Id \quad \text{and} \quad \ddot{H}^\sigma = \sigma(\sigma-1) H^{\sigma-2} Id \otimes Id$$

to find the evolution equation for H^σ using Proposition 3.5:

$$\begin{aligned}\frac{\partial H^\sigma}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m H^\sigma - \sigma(\sigma-1) H^{\sigma-2}|\nabla_i H|_m^2 + \sigma H^{\sigma-1}|\nabla H|^2 - \sigma H^{\sigma-1}|\nabla A|^2 \\ &\quad + \frac{p-1}{2}\sigma\frac{H^{\sigma-1}}{R}|\nabla R|^2 - \frac{2p-1}{2p}\sigma H^{\sigma-1}R|A|^2 + \sigma\left(H|A|^2 - C\right)H^\sigma\Big\},\end{aligned}$$

then formula (3.6) gives

$$\begin{aligned}\frac{\partial H^\sigma}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m H^\sigma + \sigma(1-\sigma) H^{\sigma-2}|\nabla_i H|_m^2 + \sigma H^{\sigma-1}|\nabla H|^2 \\ &\quad -\sigma H^{\sigma-1}|\nabla A|^2 + \frac{p-1}{2}\sigma\frac{H^{\sigma+3}}{R}|\nabla f|^2 - 2(p-1)\sigma H^\sigma\langle\nabla f, \nabla H\rangle \\ &\quad + 2(p-1)\sigma H^{\sigma-1}\frac{R}{H^2}|\nabla H|^2 - \frac{2p-1}{2p}\sigma H^{\sigma-1}R|A|^2 + \sigma\left(H|A|^2 - C\right)H^\sigma\Big\}.\end{aligned} \tag{3.15}$$

In particular for $\frac{1}{2} < p \leq 1$ we use formula (3.7) to have

$$\begin{aligned}& \sigma H^{\sigma-1}|\nabla H|^2 - \sigma H^{\sigma-1}|\nabla A|^2 \\ & -2(p-1)\sigma H^\sigma\langle\nabla f, \nabla H\rangle + 2(p-1)\sigma H^{\sigma-1}\frac{R}{H^2}|\nabla H|^2 \\ =\; & \sigma H^{\sigma-1}|\nabla H|^2 - \sigma H^{\sigma-1}|\nabla A|^2 + 2(p-1)\sigma H^{\sigma-3}|H\nabla A - A\nabla H|^2 \\ & -2(p-1)\sigma H^{\sigma-1}|\nabla A|^2 + 2(p-1)\sigma H^{\sigma-3}|A|^2|\nabla H|^2 + 2(p-1)\sigma H^{\sigma-1}\frac{R}{H^2}|\nabla H|^2 \\ =\; & -2(1-p)\sigma H^{\sigma-3}|H\nabla A - A\nabla H|^2 - (2p-1)\sigma H^{\sigma-1}\left(|\nabla A|^2 - |\nabla H|^2\right),\end{aligned}$$

hence

$$\begin{aligned}
\frac{\partial H^{\sigma}}{\partial t} &= 2pR^{p-1}\Big\{\Delta_m H^{\sigma} - 2(1-p)\sigma H^{\sigma-3}\left|H\nabla A - A\nabla H\right|^2 \\
&\quad -\frac{1-p}{2}\sigma\frac{H^{\sigma+3}}{R}\left|\nabla f\right|^2 + \sigma(1-\sigma)H^{\sigma-2}\left|\nabla_i H\right|_m^2 \\
&\quad -(2p-1)\sigma H^{\sigma-1}\left[\left|\nabla A\right|^2 - \left|\nabla H\right|^2 + \frac{1}{2p}R\left|A\right|^2\right] + \sigma\left(H\left|A\right|^2 - C\right)H^{\sigma}\Big\}.
\end{aligned}$$

Now combining the previous formula with equation (3.4) we have

$$\begin{aligned}
\frac{\partial f_{\sigma}}{\partial t} &= 2pR^{p-1}\left\{\Delta_m f_{\sigma} - 2\sigma H^{\sigma-1}\left\langle\nabla_i f, \nabla_i H\right\rangle_m\right. \\
&\quad -4(1-p)RH^{\sigma-5}\left|H\nabla A - A\nabla H\right|^2 - 2(1-p)\sigma f H^{\sigma-3}\left|H\nabla A - A\nabla H\right|^2 \\
&\quad +(1-p)\frac{H^{\sigma+1}}{R}\left|\nabla_i f\right|^2_{|A|^2 g - Hh} - \frac{1-p}{2}\sigma f\frac{H^{\sigma+3}}{R}\left|\nabla f\right|^2 + 4pH^{\sigma-1}\left\langle\nabla_i f, \nabla_i H\right\rangle_m \\
&\quad +(1-\sigma)\sigma f H^{\sigma-2}\left|\nabla_i H\right|_m^2 - 2(2p-1)H^{\sigma-2}\frac{R}{H^2}\left(\left|\nabla_i A\right|^2_{Hg} - \left|\nabla_i H\right|_h^2\right) \\
&\quad -(2p-1)\sigma f H^{\sigma-1}\left(\left|\nabla A\right|^2 - \left|\nabla H\right|^2\right) \\
&\quad \left.-\frac{2p-1}{p}H^{\sigma-3}R\left[\left(HC - \left|A\right|^4\right) + \frac{1}{2}\sigma f H^2\left|A\right|^2\right] + \sigma\left(H\left|A\right|^2 - C\right)f_{\sigma}\right\}.
\end{aligned}$$

In order to express everything in terms of f_{σ}, observe that

$$\nabla f = \nabla\left(f_{\sigma}H^{-\sigma}\right) = H^{-\sigma}\nabla f_{\sigma} - \sigma f H^{-1}\nabla H,$$

$$\left|\nabla f\right|^2 = H^{-2\sigma}\left|\nabla f_{\sigma}\right|^2 - 2\sigma f H^{-1-\sigma}\left\langle\nabla f_{\sigma}, \nabla H\right\rangle + \sigma^2 f^2 H^{-2}\left|\nabla H\right|^2 \tag{3.16}$$

and $\left\langle\nabla f, \nabla H\right\rangle = H^{-\sigma}\left\langle\nabla f_{\sigma}, \nabla H\right\rangle - \sigma f H^{-1}\left|\nabla H\right|^2$.

The following identities lead to a new formulation of the evolution equation for f_{σ}. Replacing

$$\begin{aligned}
&-2\sigma H^{\sigma-1}\left\langle\nabla_i f, \nabla_i H\right\rangle_m + (1-p)\frac{H^{\sigma+1}}{R}\left|\nabla_i f\right|^2_{|A|^2 g - Hh} + 4pH^{\sigma-1}\left\langle\nabla_i f, \nabla_i H\right\rangle_m \\
= \; &(1-p)\frac{H^{1-\sigma}}{R}\left|\nabla_i f_{\sigma}\right|^2_{|A|^2 g - Hh} \\
&-2(1-p)\sigma f\frac{1}{R}\left\langle\nabla_i f_{\sigma}, \nabla_i H\right\rangle_{|A|^2 g - Hh} + (1-p)\sigma^2 f f_{\sigma}\frac{1}{RH}\left|\nabla_i H\right|^2_{|A|^2 g - Hh} \\
&+2(2p-\sigma)\frac{1}{H}\left\langle\nabla_i f_{\sigma}, \nabla_i H\right\rangle_m - 2(2p-\sigma)\sigma f_{\sigma}\frac{1}{H^2}\left|\nabla_i H\right|_m^2
\end{aligned}$$

and

$$\begin{aligned} & (1-\sigma)\,\sigma f H^{\sigma-2}\left|\nabla_i H\right|_m^2 - 2\,(2p-1)\,H^{\sigma-2}\frac{R}{H^2}\left(\left|\nabla_i A\right|_{Hg}^2 - \left|\nabla_i H\right|_h^2\right) \\ & -(2p-1)\,\sigma f H^{\sigma-1}\left(\left|\nabla A\right|^2 - \left|\nabla H\right|^2\right) \\ = \; & -(2p-1)\left(2\frac{R}{H^2}+\sigma f\right)H^{\sigma-2}\left(\left|\nabla_i A\right|_{Hg}^2 - \left|\nabla_i H\right|_h^2\right) + (2p-\sigma)\,\sigma f H^{\sigma-2}\left|\nabla_i H\right|_m^2 \end{aligned}$$

into the equation for f_σ we have

$$\begin{aligned} \frac{\partial f_\sigma}{\partial t} = \; & 2pR^{p-1}\Big\{\Delta_m f_\sigma - 4\,(1-p)\,RH^{\sigma-5}\left|H\nabla A - A\nabla H\right|^2 \\ & -2\,(1-p)\,\sigma f H^{\sigma-3}\left|H\nabla A - A\nabla H\right|^2 \\ & -\frac{1-p}{2}\sigma f_\sigma \frac{H^3}{R}\left|\nabla f\right|^2 + (1-p)\,\frac{H^{1-\sigma}}{R}\left|\nabla_i f_\sigma\right|_{|A|^2 g - Hh}^2 \\ & -2\,(1-p)\,\sigma f\frac{1}{R}\left\langle \nabla_i f_\sigma, \nabla_i H\right\rangle_{|A|^2 g - Hh} + 2\,(2p-\sigma)\,\frac{1}{H}\left\langle \nabla_i f_\sigma, \nabla_i H\right\rangle_m \\ & +(1-p)\,\sigma^2 f f_\sigma \frac{1}{RH}\left|\nabla_i H\right|_{|A|^2 g - Hh}^2 - (2p-\sigma)\,\sigma f_\sigma \frac{1}{H^2}\left|\nabla_i H\right|_m^2 \\ & -(2p-1)\,H^{\sigma-2}\left(2\frac{R}{H^2}+\sigma f\right)\left(\left|\nabla_i A\right|_{Hg}^2 - \left|\nabla_i H\right|_h^2\right) \\ & -\frac{2p-1}{p}H^{\sigma-3}R\left[\left(HC - |A|^4\right) + \frac{1}{2}\sigma f H^2 |A|^2\right] + \sigma\left(H|A|^2 - C\right)f_\sigma\Big\}. \end{aligned}$$

Since $|\cdot|_{|A|^2 g - Hh}^2 \leq H\,|\cdot|_m^2$ we can estimate

$$\begin{aligned} & (1-p)\,\sigma^2 f f_\sigma \frac{1}{RH}\left|\nabla_i H\right|_{|A|^2 g - Hh}^2 - (2p-\sigma)\,\sigma f_\sigma \frac{1}{H^2}\left|\nabla_i H\right|_m^2 \\ \leq \; & \sigma\left[(1-p)\,\sigma f - (2p-\sigma)\,\frac{R}{H^2}\right]\frac{1}{R}\left|\nabla_i H\right|_m^2 f_\sigma \\ = \; & -\sigma\left[2p\frac{R}{H^2} - \left(f + \frac{R}{H^2} - pf\right)\sigma\right]\frac{1}{R}\left|\nabla_i H\right|_m^2 f_\sigma. \end{aligned}$$

Moreover, by formula (3.9) follows that $f < \frac{1}{n(n-1)}$ implies

$$\lambda_n < \left(\sqrt{\frac{n-1}{n}\gamma} + \frac{1}{n}\right)H = \frac{2}{n}H,$$

then

$$H\,|A|^2 - C = \sum_{i=1}^{n}\lambda_i^2\,(H-\lambda_i) \leq \lambda_n \sum_{i=1}^{n}\lambda_i\,(H-\lambda_i) < \frac{2}{n}HR. \tag{3.17}$$

Now using Proposition 3.9 ii) and the estimate (3.10) we derive for $\frac{1}{2} < p \leq 1$

$$\begin{aligned}\frac{\partial f_\sigma}{\partial t} \leq\ & 2pR^{p-1}\Bigg\{\Delta_m f_\sigma + (1-p)\frac{H^{1-\sigma}}{R}\left|\nabla_i f_\sigma\right|^2_{|A|^2 g - Hh} \\ & -2(1-p)\sigma\frac{1}{R}f\left\langle\nabla_i f_\sigma, \nabla_i H\right\rangle_{|A|^2 g - Hh} + 2(2p-\sigma)\frac{1}{H}\left\langle\nabla_i f_\sigma, \nabla_i H\right\rangle_m \\ & -\sigma\left[2p\frac{R}{H^2} - \left(\frac{n-1}{n} - pf\right)\sigma\right]\frac{1}{R}\left|\nabla_i H\right|^2_m f_\sigma - \left[\frac{2p-1}{p}n\eta^2 - \frac{2}{n}\sigma\right]RHf_\sigma\Bigg\}.\end{aligned}$$

The hypothesis $f < \gamma$ is equivalent to

$$\frac{R}{H^2} \geq c(n,p) = \frac{n-1}{n} - \gamma, \tag{3.18}$$

so there exists $\sigma^* > 0$ such that the last two terms are negative and for any $\sigma < \sigma^*$ and we can conclude applying the maximum principle.

In the case $p > 1$, the equations (3.5) and (3.15) yield

$$\begin{aligned}\frac{\partial f_\sigma}{\partial t} =\ & 2pR^{p-1}\Bigg\{\Delta_m f_\sigma - 2\sigma H^{\sigma-1}\left\langle\nabla_i f, \nabla_i H\right\rangle_m - (p-1)\frac{H^{\sigma+1}}{R}\left|\nabla_i f\right|^2_{|A|^2 g - Hh} \\ & + \frac{p-1}{2}\sigma f\frac{H^{\sigma+3}}{R}\left|\nabla f\right|^2 + 4H^{\sigma-1}\left\langle\nabla_i f, \nabla_i H\right\rangle_m \\ & + 4(p-1)H^{\sigma-2}\left\langle\nabla_i f, \nabla_i H\right\rangle_{|A|^2 g - Hh} - 2(p-1)\sigma f H^\sigma\left\langle\nabla f, \nabla H\right\rangle \\ & - 2RH^{\sigma-4}\left(\left|\nabla_i A\right|^2_{Hg} - \left|\nabla_i H\right|^2_h\right) - \sigma f H^{\sigma-2}\left(\left|\nabla_i A\right|^2_{Hg} - \left|\nabla_i H\right|^2_h\right) \\ & + \sigma f H^{\sigma-2}\left|\nabla_i H\right|^2_m + \sigma(1-\sigma)fH^{\sigma-2}\left|\nabla_i H\right|^2_m \\ & - 4(p-1)RH^{\sigma-5}\left|\nabla_i H\right|^2_{|A|^2 g - Hh} + 2(p-1)\sigma f R H^{\sigma-3}\left|\nabla H\right|^2 \\ & - \frac{2p-1}{p}\frac{R}{H^3}H^\sigma\left(HC - |A|^4\right) - \frac{2p-1}{2p}\sigma f H^{\sigma-1}R|A|^2 + \sigma f\left(H|A|^2 - C\right)H^\sigma\Bigg\}.\end{aligned}$$

Some of the first order terms in this equation can be rearranged in such a way to distinguish the positive and the negative part: first we use identity (3.12)

$$\begin{aligned}
& -2\sigma H^{\sigma-1}\left\langle \nabla_i f, \nabla_i H\right\rangle_m - (p-1)\frac{H^{\sigma+1}}{R}\left|\nabla_i f\right|^2_{|A|^2 g - Hh} \\
& +\frac{p-1}{2}\sigma f\frac{H^{\sigma+3}}{R}\left|\nabla f\right|^2 + 4H^{\sigma-1}\left\langle \nabla_i f, \nabla_i H\right\rangle_m \\
& +4(p-1)H^{\sigma-2}\left\langle \nabla_i f, \nabla_i H\right\rangle_{|A|^2 g - Hh} - 2(p-1)\sigma f H^{\sigma}\left\langle \nabla f, \nabla H\right\rangle \\
= \; & -2\sigma H^{\sigma-1}\left\langle \nabla_i f, \nabla_i H\right\rangle_m - (p-1)\frac{H^{\sigma+2}}{R}\left|\nabla_i f\right|^2_m + (p-1)H^{\sigma+1}\left|\nabla f\right|^2 \\
& +\frac{p-1}{2}\sigma f\frac{H^{\sigma+3}}{R}\left|\nabla f\right|^2 + 4H^{\sigma-1}\left\langle \nabla_i f, \nabla_i H\right\rangle_m \\
& +4(p-1)H^{\sigma-1}\left\langle \nabla_i f, \nabla_i H\right\rangle_m - 4(p-1)H^{\sigma-2}R\left\langle \nabla f, \nabla H\right\rangle \\
& -2(p-1)\sigma f H^{\sigma}\left\langle \nabla f, \nabla H\right\rangle, \\
= \; & -(p-1)\frac{H^{\sigma+2}}{R}\left|\nabla_i f\right|^2_m + (p-1)H^{\sigma+1}\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)\left|\nabla f\right|^2 \\
& +2(2p-\sigma)H^{\sigma-1}\left\langle \nabla_i f, \nabla_i H\right\rangle_m - 4(p-1)\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)RH^{\sigma-2}\left\langle \nabla f, \nabla H\right\rangle
\end{aligned}$$

then we use the equalities of page 50

$$\begin{aligned}
= \; & -(p-1)\frac{H^{\sigma+2}}{R}\left|\nabla_i f\right|^2_m + (p-1)\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)H^{1-\sigma}\left|\nabla f_\sigma\right|^2 \\
& -2(p-1)\sigma f\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)\left\langle \nabla f_\sigma, \nabla H\right\rangle + (p-1)\sigma^2 f^2\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)H^{\sigma-1}\left|\nabla H\right|^2 \\
& +2(2p-\sigma)\frac{1}{H}\left\langle \nabla_i f_\sigma, \nabla_i H\right\rangle_m - 2\sigma(2p-\sigma)H^{\sigma-2}f\left|\nabla_i H\right|^2_m \\
& -4(p-1)\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)\frac{R}{H^2}\left\langle \nabla f_\sigma, \nabla H\right\rangle + 4(p-1)\sigma f\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)RH^{\sigma-3}\left|\nabla H\right|^2 \\
= \; & -(p-1)\frac{H^{\sigma+2}}{R}\left|\nabla_i f\right|^2_m + (p-1)\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)H^{1-\sigma}\left|\nabla f_\sigma\right|^2 \\
& +2(2p-\sigma)\frac{1}{H}\left\langle \nabla_i f_\sigma, \nabla_i H\right\rangle_m - 2\sigma(2p-\sigma)H^{\sigma-2}f\left|\nabla_i H\right|^2_m \\
& -2(p-1)\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)\left(2\frac{R}{H^2}+\sigma f\right)\left\langle \nabla f_\sigma, \nabla H\right\rangle \\
& +(p-1)\sigma f\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)\left(4\frac{R}{H^2}+\sigma f\right)H^{\sigma-1}\left|\nabla H\right|^2.
\end{aligned}$$

Thus we have

$$
\begin{aligned}
\frac{\partial f_\sigma}{\partial t} &= 2pR^{p-1}\Bigg\{\Delta_m f_\sigma - (p-1)\frac{H^{\sigma+2}}{R}\left|\nabla_i f\right|_m^2 + (p-1)\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)H^{1-\sigma}\left|\nabla f_\sigma\right|^2 \\
&\quad +2(2p-\sigma)\frac{1}{H}\left\langle\nabla_i f_\sigma,\nabla_i H\right\rangle_m - 2(p-1)\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)\left(2\frac{R}{H^2}+\sigma f\right)\left\langle\nabla f_\sigma,\nabla H\right\rangle \\
&\quad +(p-1)\sigma f\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)\left(4\frac{R}{H^2}+\sigma f\right)H^{\sigma-1}\left|\nabla H\right|^2 \\
&\quad -2RH^{\sigma-4}\left(\left|\nabla_i A\right|_{Hg}^2-\left|\nabla_i H\right|_h^2\right)-\sigma f H^{\sigma-2}\left(\left|\nabla_i A\right|_{Hg}^2-\left|\nabla_i H\right|_h^2\right) \\
&\quad -2\sigma(2p-\sigma)H^{-2}f_\sigma\left|\nabla_i H\right|_m^2+\sigma(2-\sigma)fH^{\sigma-2}\left|\nabla_i H\right|_m^2 \\
&\quad -4(p-1)RH^{\sigma-5}\left|\nabla_i H\right|_{|A|^2g-Hh}^2+2(p-1)\sigma fRH^{\sigma-3}\left|\nabla H\right|^2 \\
&\quad -\frac{2p-1}{p}\frac{R}{H^3}H^\sigma\left(HC-|A|^4\right)-\frac{2p-1}{2p}\sigma fH^{\sigma-1}R\left|A\right|^2+\sigma f\left(H\left|A\right|^2-C\right)H^\sigma\Bigg\}.
\end{aligned}
$$

Since $f<\gamma$, then there exists an $\varepsilon>0$ such that $f\le\frac{4(n-1)}{n(n+2)^2}\left(\frac{1-\varepsilon}{2p-1}\right)^2$, hence we can apply Proposition 3.9 iii) and estimate the last part of the previous equation in the following way

$$
\begin{aligned}
& (p-1)\sigma f\left(4\frac{R}{H^2}+3\sigma f+\frac{1}{2}\sigma^2f^2\frac{H^2}{R}\right)H^{\sigma-1}\left|\nabla H\right|^2 \\
& -2RH^{\sigma-4}\left(\left|\nabla_i A\right|_{Hg}^2-\left|\nabla_i H\right|_h^2\right)-\sigma fH^{\sigma-2}\left(\left|\nabla_i A\right|_{Hg}^2-\left|\nabla_i H\right|_h^2\right) \\
& -2(2p-1)\sigma fH^{\sigma-2}\left|\nabla_i H\right|_m^2+\sigma^2fH^{\sigma-2}\left|\nabla_i H\right|_m^2 \\
& -4(p-1)RH^{\sigma-5}\left|\nabla_i H\right|_{|A|^2g-Hh}^2+2(p-1)\sigma fRH^{\sigma-3}\left|\nabla H\right|^2 \\
& -\left[\frac{2p-1}{p}\frac{R}{H^3}\left(HC-|A|^4\right)+\frac{2p-1}{2p}\sigma f\frac{R}{H}\left|A\right|^2-\sigma f\left(H\left|A\right|^2-C\right)\right]H^\sigma \\
\le\; & (p-1)\sigma f\left(6\frac{R}{H^2}+3\sigma f+\frac{1}{2}\sigma^2f^2\frac{H^2}{R}\right)H^{\sigma-1}\left|\nabla H\right|^2 \\
& -2\left(\frac{2(n-1)}{n(n+2)}\frac{2(p-1)+\varepsilon}{2p-1}\right)RH^{\sigma-3}\left|\nabla H\right|^2-\sigma f\left(\frac{2(n-1)}{n(n+2)}\frac{2(p-1)+\varepsilon}{2p-1}\right)H^{\sigma-1}\left|\nabla H\right|^2 \\
& +\sigma^2fH^{\sigma-1}\left|\nabla H\right|^2+4(p-1)\frac{2(n-1)}{n(n+2)}\left(\frac{1-\varepsilon}{2p-1}\right)RH^{\sigma-3}\left|\nabla H\right|^2 \\
& -\left[\frac{2p-1}{p}n\eta^2-\sigma\frac{2}{n}\right]RHf_\sigma
\end{aligned}
$$

where we also used formulas (3.14), (3.10) and (3.17). Finally we write the same expression isolating terms independent from σ

$$
\begin{aligned}
= \quad & -\frac{2(n-1)}{n(n+2)}\left[\frac{4(p-1)+2\varepsilon}{2p-1}-4(p-1)\frac{1-\varepsilon}{2p-1}\right]RH^{\sigma-3}|\nabla H|^2 \\
& -\sigma f\left(\frac{2(n-1)}{n(n+2)}\frac{2(p-1)+\varepsilon}{2p-1}\right)H^{\sigma-1}|\nabla H|^2+\sigma^2 fH^{\sigma-1}|\nabla H|^2 \\
& +\sigma f(p-1)\left(6\frac{R}{H^2}+3\sigma f+\frac{1}{2}\sigma^2 f^2\frac{H^2}{R}\right)H^{\sigma-1}|\nabla H|^2 \\
& -\left[\frac{2p-1}{p}n\eta^2-\sigma\frac{2}{n}\right]RHf_\sigma
\end{aligned}
$$

and obtain

$$
\begin{aligned}
\frac{\partial f_\sigma}{\partial t} \quad < \quad & 2pR^{p-1}\Big\{\Delta_m f_\sigma+(p-1)\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)H^{1-\sigma}|\nabla f_\sigma|^2 \\
& +2(2p-\sigma)\frac{1}{H}\langle\nabla_i f_\sigma,\nabla_i H\rangle_m-2(p-1)\left(1+\frac{1}{2}\sigma f\frac{H^2}{R}\right)\left(2\frac{R}{H^2}+\sigma f\right)\langle\nabla f_\sigma,\nabla H\rangle \\
& -\frac{4(n-1)\varepsilon}{n(n+2)}RH^{\sigma-3}|\nabla H|^2-\sigma f\varepsilon\frac{2(n-1)}{n(n+2)}H^{\sigma-1}|\nabla H|^2+\sigma^2 fH^{\sigma-1}|\nabla H|^2 \\
& +\sigma f(p-1)\left(6\frac{R}{H^2}+3\sigma f+\frac{1}{2}\sigma^2 f^2\frac{H^2}{R}\right)H^{\sigma-1}|\nabla H|^2 \\
& -\left[\frac{2p-1}{p}n\eta^2-\sigma\frac{2}{n}\right]RHf_\sigma.
\end{aligned}
$$

Now we can choose a constant $\sigma^*=\sigma^*(n,\varepsilon,\eta)>0$ sufficiently small in such a way that the expression in the last three lines is negative. Then an application of the maximum principle assures that f_σ, with $\sigma<\sigma^*$, is nonincreasing on all the existence time interval of the solution.

We have shown that in both cases $\frac{1}{2}<p\leq 1$ and $p>1$

$$
f_\sigma(t)\leq c_1=\sup_{M_0} fH^\sigma
$$

hence $f\leq c_1 H^{-\sigma}$. ■

3.3 Convergence to a point

In this section we prove the first part of Theorem 3.1. We begin proving that the solutions to the problem (3.1) cannot exist for all times because the minimum of the scalar curvature diverges in a finite time T. Then we show that, as long as there exists a sphere of positive radius enclosed in M_t, the principal curvatures stay bounded, hence the regularity theorem (Corollary 1.5) gives the existence of the flow for a longer time interval. As a consequence it follows that the inner radius of the M_t's tends to zero as $t \to T$. Furthermore, using the pinching of principal curvatures, one can estimate the outer radius by the inner radius from above and deduce the desired convergence to a point.

Proposition 3.11 *The maximum time interval of existence of the solution to (3.1) is finite.*

Proof. Let us consider the evolution equation for R given in formula (3.3)

$$\frac{\partial R}{\partial t} = 2pR^{p-1}\Delta_m R + 2p\,(p-1)\,R^{p-2}\,|\nabla_i R|_m^2 + 2\left(H\,|A|^2 - C\right)R^p.$$

In order to prove that the scalar curvature of the evolving surfaces blows up in finite time, it is necessary to estimate $H\,|A|^2 - C$ from below by a power of R. From Lemma 2.2 in [15] we have $C \leq |A|^3$, hence, using the inequality $|A|^2 \geq \frac{1}{n}H^2$, we obtain

$$H\,|A|^2 - C \geq |A|^2\,(H - |A|) = \frac{|A|^2\,R}{H + |A|} \geq \frac{|A|\,R}{\sqrt{n}+1} \geq \frac{1}{\sqrt{n-1}\,(\sqrt{n}+1)}R^{\frac{3}{2}} \tag{3.19}$$

where the last estimate follows by $R \leq (n-1)\,|A|^2$.

Now we define the function $\varphi\,(t) := \min_{M_t} R$. The evolution equation for φ is an ordinary differential equation such that the estimate

$$\frac{d\varphi}{dt} \geq \frac{1}{\sqrt{n-1}\,(\sqrt{n}+1)}\varphi^{p+\frac{3}{2}}$$

holds in a weak sense for all $t \geq 0$. The statement is proved because

$$\begin{aligned} t\,(\varphi) &\leq \sqrt{n-1}\left(\sqrt{n}+1\right)\int_{\varphi_0}^{\infty}\varphi^{-p-\frac{3}{2}}d\varphi \\ &\leq \sqrt{n-1}\left(\sqrt{n}+1\right)\int_{\varphi_0}^{\infty}\varphi^{-2}d\varphi \end{aligned}$$

where it is known that the last integral converges. ■

Definition 3.12 *We define the inner and outer radius respectively as:*

$$\rho_- := \sup\left\{r \in [0,\infty)\ \ s.\ t.\ B_r(\mathbf{y})\ \text{ is enclosed in } \mathbf{F}_t(M)\ \text{ for some } \mathbf{y} \in \mathbb{R}^{n+1}\right\}$$

$$\rho_+ := \inf\left\{r \in [0,\infty)\ \ s.\ t.\ B_r(\mathbf{y})\ \text{ encloses } \mathbf{F}_t(M)\ \text{ for some } \mathbf{y} \in \mathbb{R}^{n+1}\right\}$$

where $B_r(\mathbf{y})$ is the ball of radius r centred in $\mathbf{y}$.

In what follows we prove that the principal curvatures of the evolving surfaces stay bounded as long as the inner radius is positive. The technique used, involving a lower bound on the support function, first appeared in [25] and then was applied also in [1] and [22].

Lemma 3.13 *Let us consider the flow (3.1). If the initial surface satisfies the pinching condition $f < \frac{1}{n(n-1)}$, then the following estimate holds*

$$\rho_+(t) \le c_2 \rho_-(t)\ \ \forall t \in [0,T). \tag{3.20}$$

Proof. We denote by $\gamma^* = \sup\limits_{M_0} f$, with $\gamma^* < \frac{1}{n(n-1)}$ as in the hypothesis of Theorem 3.1. Then by the inequalities (3.8) and (3.9) we have on M_0

$$\lambda_1 \ge \left(\frac{1}{n} - \sqrt{\frac{n-1}{n}\gamma^*}\right) H \ge \frac{\frac{1}{n} - \sqrt{\frac{n-1}{n}\gamma^*}}{\sqrt{\frac{n-1}{n}\gamma^* + \frac{1}{n}}}\lambda_n.$$

The same estimate holds for all times $t \in [0,T)$ thank to Corollary 3.10. This allows us to apply Theorem 5.1 and Theorem 5.4 of [1] and deduce the statement. ■

Proposition 3.14 *If there exists a ball of radius $r > 0$ enclosed in M_t for all $t \in [0,T')$ then*

$$\max_{M_t} R \le c_5 \left(1 + \frac{1}{r^2}\right)\ \ \forall t \in \left[0,T'\right)$$

with c_5 depending only on n, p and M_0.

Proof. Let us consider a time $t^* \in (0,T')$ such that $\max\limits_{M\times[0,t^*]} R = R(y^*,t^*)$ and set $r = \rho_-(t^*)$. The monotonicity of $\rho_-(t)$ implies $r = \rho_-(t^*) \le \rho_-(t)\,\forall t \in [0,t^*]$, hence, there exists $\mathbf{y}_0 \in$

$\mathbb{R}^{n+1}$ such that the sphere $B_r(\mathbf{y}_0)$ is enclosed in M_t for any $t \in [0, t^*]$. Now take $\mathbf{Y} := \mathbf{y} - \mathbf{y}_0$ the position vector field with origin $\mathbf{y}_0$ and note that

$$2\langle \mathbf{Y}, \boldsymbol{\nu} \rangle - r \geq r > 0, \tag{3.21}$$

thus the quantity $v := R^p / (2\langle \mathbf{Y}, \boldsymbol{\nu} \rangle - r)$ is well defined on $M \times [0, t^*]$. The evolution equation for v has been found in Lemma 1.11 for a generic flow, in this case, from formula (1.10) we have

$$\frac{\partial v}{\partial t} = 2pR^{p-1}\left[\Delta_m v + \frac{4\langle \nabla_i \langle \mathbf{Y}, \boldsymbol{\nu} \rangle, \nabla_i v \rangle_m}{2\langle \mathbf{F}, \boldsymbol{\nu} \rangle - r} - \frac{r\left(H|A|^2 - C\right)v}{2\langle \mathbf{F}, \boldsymbol{\nu} \rangle - r}\right] + 2(2p+1)v^2.$$

In order to apply the maximum principle we use formula (3.19) to estimate

$$\frac{\partial v}{\partial t} \leq 2pR^{p-1}\left(\Delta_m v + \frac{4\langle \nabla_i \langle \mathbf{Y}, \boldsymbol{\nu} \rangle, \nabla_i v \rangle_m}{2\langle \mathbf{Y}, \boldsymbol{\nu} \rangle - r}\right) + 2\left[(2p+1) - pr\frac{R^{\frac{1}{2}}}{\sqrt{n-1}\left(\sqrt{n}+1\right)}\right]v^2.$$

Assume $v(\mathbf{y}, t)$ to attain a local maximum $v(\mathbf{y}_0, t_0) = c_3$ with $t_0 > 0$, then at $(\mathbf{y}_0, t_0)$ must be $\Delta_m v \leq 0$, $\nabla v = 0$ and $\frac{\partial v}{\partial t} \geq 0$, so we have

$$R^{\frac{1}{2}}(\mathbf{y}_0, t_0) \leq \frac{2p+1}{p}\sqrt{n-1}\left(\sqrt{n}+1\right)\frac{1}{r}.$$

From condition (3.21) it follows $R^p(\mathbf{y}_0, t_0) = v(\mathbf{y}_0, t_0)(2\langle \mathbf{Y}, \boldsymbol{\nu} \rangle - r) \geq c_3 r$, hence we can conclude that

$$c_3 \leq \max\left\{\max v(\mathbf{y}, 0), \frac{1}{r}\left(\frac{2p+1}{p}\sqrt{n-1}\left(\sqrt{n}+1\right)\frac{1}{r}\right)^{2p}\right\}.$$

If $\frac{1}{r}\left(\frac{2p+1}{p}\sqrt{n-1}\left(\sqrt{n}+1\right)\frac{1}{r}\right)^{2p} \geq \max v(\mathbf{y}, 0)$, then recalling the definition of r and using the estimate (3.20) we have

$$\begin{aligned} R^p(\mathbf{y}^*, t^*) &= v(\mathbf{y}^*, t^*)(2\langle \mathbf{y}^* - \mathbf{y}_0, \boldsymbol{\nu} \rangle - r) \\ &\leq c_4 \frac{1}{r^{2p+1}}\left(2\rho_+(t^*) - r\right) \leq c_4(2c_2 - 1)\frac{1}{r^{2p}}, \end{aligned}$$

otherwise we have $R^p(\mathbf{y}^*, t^*) \leq c_3 (2c_2 - 1) \rho_- (0)$. Hence

$$\max_{M_t} R \leq c_5 \left(1 + \frac{1}{r^2}\right) \ \forall t \in [0, t^*].$$

The statement follows for all $t \in [0, T')$ because t^* is arbitrarily chosen. ■

Corollary 3.15 *Let the surfaces $M_t = \mathbf{F}_t(M)$ satisfy the hypotheses of Theorem 3.1. Then all the principal curvatures are bounded as long as there exists a ball of positive radius enclosed in all M_t's.*

Proof. The previous proposition guarantees that if $\rho_-(t) > 0$ for all $t \in [0, T')$, then the scalar curvature R is bounded in the same time interval. Recalling that, as proved in Corollary 3.10, the condition required on the initial surface $\frac{R}{H^2} > c = \frac{n-1}{n} - \gamma > 0$ is preserved for all times, we deduce that H^2 is bounded for all $t \in [0, T')$. Moreover, from the definition of $R = H^2 - |A|^2$, it is clear that also $|A|^2$ is bounded and this concludes the proof because the boundedness of $|A|^2 = \sum_{i=1}^n \lambda_i^2$ is equivalent to the boundedness of all principal curvatures. ■

Now we can apply the regularity result (Corollary 1.5) to deduce the smoothness of the limiting surface.

Lemma 3.16 *If $\max_{M_t} |A|^2 \leq c_6(M_0, r)$ for all $t \in [0, T')$ then the surfaces M_t defined by the initial value problem (3.1) converge to a smooth limiting surface $M_{T'}$.*

Proof. Note that the boundedness of $|A|^2$, and hence of all principal curvatures, allows us to describe locally the evolving surfaces as the graph of a function u

$$\mathbf{F}(\mathbf{p}) = (x_1, ..., x_n, u(x_1, ..., x_n)) = (\mathbf{x}, u(\mathbf{x})),$$

like in the general setting, and the evolution equation for u (1.5) in this case becomes

$$\frac{\partial u}{\partial t} = -\sqrt{1 + |Du(\mathbf{x})|^2} \left\langle \frac{\partial}{\partial t}\mathbf{F}, \boldsymbol{\nu} \right\rangle = R^p \sqrt{1 + |Du(\mathbf{x})|^2}. \tag{3.22}$$

Recalling formula (1.3), we can write the scalar curvature as a function of u

$$\begin{aligned} R &= H^2 - |A|^2 = g^{ij} h_{ij} g^{kl} h_{kl} - h_{ij} g^{jk} h_{kl} g^{li} \\ &= \left(g^{ij} g^{kl} - g^{jk} g^{li}\right) \frac{D_{ij}^2 u(\mathbf{x}) D_{kl}^2 u(\mathbf{x})}{1 + |Du(\mathbf{x})|^2}, \end{aligned}$$

hence we have

$$\frac{\partial u}{\partial t} = G\left(D^2 u, Du\right) = \left(\sqrt{1 + |Du|^2}\right)^{1-2p} \left[\left(g^{ij} g^{lk} - g^{jk} g^{li}\right) D_{ij}^2 u(\mathbf{x}) D_{kl}^2 u(\mathbf{x})\right]^p.$$

If we define $a(Du) = \left(\sqrt{1 + |Du|^2}\right)^{1-2p} > 0$ and $L\left(D^2 u, Du\right) = \left[\left(g^{ij} g^{lk} - g^{il} g^{kj}\right) D_{lk}^2 u D_{ij}^2 u\right]^{\frac{1}{2}}$, then

$$G\left(D^2 u, Du\right) = a(Du)\left[L\left(D^2 u, Du\right)\right]^{2p}$$

and the function $\phi(y) = y^{2p}$ is an increasing function of its argument.

Since $l = R^{\frac{1}{2}}$ is a concave function of the λ_j's, and the estimate (1.6) holds:

$$\frac{\dot{l}^k - \dot{l}^j}{\lambda_k - \lambda_j} = \frac{1}{2} R^{-\frac{1}{2}} \frac{(H - \lambda_k) - (H - \lambda_j)}{\lambda_k - \lambda_j} = -\frac{1}{2} R^{-\frac{1}{2}} \leq 0 \quad \forall k \neq j$$

Remark 1.6 allows us to apply Corollary 1.5. ■

We can finally infer that the evolving surfaces stay smooth until they degenerate to a point.

Theorem 3.17 *The solution to the problem (3.1) exists in a finite time interval* $[0, T)$ *and the manifolds* M_t *converge uniformly to a point as* $t \to T$*.*

Proof. The previous two statements show that as long as there exists a ball of positive radius enclosed in $M_{T'}$, then the latter fulfills the hypotheses of the short time existence theorem, and the flow can start again. From Proposition 3.11 follows that the inner radius has to tend to zero as $t \to T$. Furthermore the inequality (3.20) implies that also the outer radius tends to zero, hence the evolving surfaces shrink to a point. ■

3.4 Convergence to a sphere

In order to prove the second part of Theorem 3.1 we normalize the solutions of (3.1) requiring the volume of rescaled surfaces to be bounded both from above and from below during the evolution.

Let us consider for any $t \in [0, T)$ the maps $\tilde{\mathbf{F}}(\mathbf{x}, t) = \psi(t)\mathbf{F}(\mathbf{x}, t)$ where ψ is a positive scalar function. We choose the rescaling factor ψ in such a way that spheres are constant solutions for the rescaled flow.

If M_0 is a sphere of radius r_0 centred in $\mathbf{0} \in \mathbb{R}^{n+1}$, then the position vector $\mathbf{F}$ is parallel to the normal direction $\boldsymbol{\nu}$ and its scalar curvature is given by $R = \frac{n(n-1)}{r^2(t)}$. It follows that the evolving surfaces M_t are all spheres and their radius $r(t)$ satisfies the following ordinary differential equation

$$\begin{aligned} \frac{dr}{dt} &= -n^p (n-1)^p r^{-2p} \\ \frac{d}{dt} r^{2p+1} &= -n^p (n-1)^p (2p+1), \end{aligned}$$

whose solution is

$$r(t) = \left[r_0^{2p+1} - n^p (n-1)^p (2p+1)\, t\right]^{\frac{1}{2p+1}}. \tag{3.23}$$

Hence we set $c' = n^p (n-1)^p (2p+1)$ and define $\psi(t) := [c'(T-t)]^{-\frac{1}{2p+1}}$, obtaining

$$\tilde{\mathbf{F}}(\mathbf{x}, t) = [n^p (n-1)^p (2p+1)(T-t)]^{-\frac{1}{2p+1}} \mathbf{F}(\mathbf{x}, t).$$

The time parameter τ is defined as

$$\tau(t) := \int_0^t \frac{1}{c'(T-t')} dt' = -\frac{1}{c'} \ln\left(1 - \frac{t}{T}\right),$$

in such a way that $\tau \to +\infty$ as $t \to T$.

The following lemma is useful to deduce the evolution equation of geometric quantities evolving by the rescaled flow from their evolution by the non rescaled one.

Lemma 3.18 *Let P and Q be expressions formed by the elements of the second fundamental* *form and the metric g such that*

$$\frac{\partial P}{\partial t} = 2pR^{p-1}\left(\Delta_m P + Q\right),$$

and let $\tilde{P} = \psi^{\alpha} P$. Then the equation

$$\frac{\partial \tilde{P}}{\partial \tau} = 2p\tilde{R}^{p-1}\left(\Delta_{\tilde{m}}\tilde{P} + \tilde{Q}\right) + \alpha n^p (n-1)^p \tilde{P}$$

holds with $\tilde{Q} = \psi^{\alpha-3} Q$.

Proof. From the definitions given at page 5 it is easy to check that if $\tilde{\mathbf{F}} = \psi \mathbf{F}$ then $\tilde{g}_{ij} = \psi^2 g_{ij}$, $\tilde{g}^{ij} = \psi^{-2} g_{ij}$ and $\tilde{h}_{ij} = \psi h_{ij}$. Moreover we compute

$$\frac{d\tau}{dt} = \frac{1}{c'(T-t)} = \psi^{2p+1}$$

and

$$\frac{d\psi}{dt} = \frac{d}{dt}\left[c'(T-t)\right]^{-\frac{1}{2p+1}} = \frac{c'}{2p+1}\left[c'(T-t)\right]^{-\frac{2p+2}{2p+1}} = n^p (n-1)^p \psi^{2p+2}.$$

Now we calculate the evolution of $\tilde{P}$

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial \tau} &= \frac{dt}{d\tau}\frac{\partial}{\partial t}\left(\psi^{\alpha} P\right) = \psi^{-2p-1}\left(\alpha\psi^{\alpha-1}\frac{d\psi}{dt}P + \psi^{\alpha}\frac{\partial P}{\partial t}\right) = \\ &= \psi^{-2p-1}\psi^{\alpha}\left[\alpha n^p (n-1)^p \psi^{2p+1} P + 2pR^{p-1}\left(\Delta_m P + Q\right)\right]. \end{aligned}$$

Since for the second order operator the following equality holds

$$\Delta_{\tilde{m}} P = \tilde{m}^{ij}\nabla_i\nabla_j P = \psi^{-3} m^{ij}\nabla_i\nabla_j P = \psi^{-3}\Delta_m P,$$

we have

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial \tau} &= 2p\tilde{R}^{p-1}\psi^{\alpha-3}\left(\Delta_m P + Q\right) + \alpha n^p (n-1)^p \tilde{P} \\ &= 2p\tilde{R}^{p-1}\left(\Delta_{\tilde{m}}\tilde{P} + \psi^{\alpha-3} Q\right) + \alpha n^p (n-1)^p \tilde{P}. \end{aligned}$$

■

As a consequence of the previous lemma we can define the rescaled R^p-flow by the equation

$$\frac{\partial}{\partial t}\tilde{\mathbf{F}} = -\tilde{R}^p\tilde{\boldsymbol{\nu}} + n^p\left(n-1\right)^p\tilde{\mathbf{F}}. \tag{3.24}$$

Following the proof of Theorem 7.1 in [1], just with a different rescaling factor, it is possible to prove the estimate

$$\frac{1}{c_1'} \leq \tilde{\rho}_- \leq 1 \leq \tilde{\rho}_+ \leq c_1' \tag{3.25}$$

and thus check the assumption that the volume of $\tilde{M}_\tau := \tilde{\mathbf{F}}\left(\cdot,\tau\right)$ is bounded from above and from below.

Since $\psi \to \infty$ as $\tau \to \infty$ we have an estimate from above also for the rescaled scalar curvature due to Proposition 3.14:

$$\max_{\tilde{M}_\tau}\tilde{R} = \psi^{-2}\max_{M_t}R \leq \psi^{-2}c_5\left[1+\frac{1}{\rho_-\left(t\right)^2}\right] \leq c_5'\frac{1}{\tilde{\rho}_-\left(\tau\right)^2}, \tag{3.26}$$

this is necessary to bound the principal curvatures and prove the long time existence. But is important to observe that the rescaled scalar curvature is not uniformly bounded away from zero: though $\tilde{R} > 0$ holds on all $\tilde{M}_\tau \times [0,+\infty)$ we have not a positive lower bound on $\tilde{R}$. This means that the rescaled flow is degenerate, hence we cannot use the standard theory for uniformly parabolic equations to prove the convergence to a sphere. The proof then uses a different argument (shown in paragraph 3 of [23]): we write the equation for the rescaled scalar curvature as a porous medium equation and deduce the necessary interior Hölder estimates applying Theorem 1.2 in [9].

Corollary 3.19 *The rescaled scalar curvature $\tilde{R}$ satisfies the equation*

$$\frac{\partial}{\partial \tau}\tilde{R} = 2p\tilde{R}^{p-1}\left\{\Delta_{\tilde{m}}\tilde{R} + \left(p-1\right)\frac{1}{\tilde{R}}\left|\nabla_i\tilde{R}\right|^2_{\tilde{m}} + \frac{1}{p}\left(\tilde{H}\left|\tilde{A}\right|^2 - \tilde{C}\right)\tilde{R}\right\} - 2n^p\left(n-1\right)^p\tilde{R}$$

or, equivalently, the equation

$$\frac{\partial}{\partial t}\tilde{R} = 2\Delta_{\tilde{m}}\tilde{R}^p + 2\left(\tilde{H}\left|\tilde{A}\right|^2 - \tilde{C}\right)\tilde{R}^p - 2n^p\left(n-1\right)^p\tilde{R}. \tag{3.27}$$

Proof. Using the identity

$$\Delta_m R^p = m^{ij}\nabla_i\nabla_j R^p = m^{ij}\nabla_i\left(pR^{p-1}\nabla_j R\right) = p\left(p-1\right)R^{p-2}\left|\nabla_i R\right|_m^2 + pR^{p-1}\Delta_m R$$

the equation (3.3) for R can be written as a porous medium equation:

$$\frac{\partial}{\partial t}R = 2\Delta_m R^p + 2\left(H\left|A\right|^2 - C\right)R^p.$$

The claim is a consequence of previous lemma. ■

The next lemma gives an estimate on the gradient of $\tilde{R}$ both in space and in time that is essential for the proof of the long time existence of the flow.

Lemma 3.20 *The rescaled scalar curvature $\tilde{R}$ of the evolving surfaces $\tilde{M}_\tau$ satisfies the estimate*

$$\int_{\tau_1}^{\tau_2}\int\left|\nabla\tilde{R}^p\right|_{\tilde{m}}^2 d\tilde{\mu}d\tau \le c_3'\left(1+\tau_2-\tau_1\right).$$

Proof. First we need to prove the following identity based on the integration by parts: for any two functions $a, b \in C^2$ we have

$$\begin{aligned}
\int a\Delta_m b &= \int a\left(Hg^{ij}-h^{ij}\right)\nabla_i\nabla_j b \\
&= -\int Hg^{ij}\nabla_i a\nabla_j b - \int ag^{ij}\nabla_i H\nabla_j b + \int h^{ij}\nabla_i a\nabla_j b + \int a\nabla_i h^{ij}\nabla_j b \\
&= -\int\left(Hg^{ij}-h^{ij}\right)\nabla_i a\nabla_j b + \int a\left(g^{ij}g^{lk}\nabla_i h_{lk} - g^{il}g^{kj}\nabla_i h_{lk}\right)\nabla_j b \\
&= -\int\left\langle\nabla a,\nabla b\right\rangle_m + \int a\left(m^{ij}\nabla_i h_{ij}\right)\nabla_j b = -\int\left\langle\nabla a,\nabla b\right\rangle_m
\end{aligned}$$

where the last equality is due to Codazzi equation.

Then we compute the evolution equation of the rescaled area element $\tilde{\mu} = \psi^n\mu$: applying Lemma 3.18 to equation (3.2) it follows

$$\frac{\partial\tilde{\mu}}{\partial t} = -\tilde{R}^p\tilde{H}\tilde{\mu} + n^{p+1}\left(n-1\right)^p\tilde{\mu}.$$

Hence, recalling also formula (3.24), we have

$$\begin{aligned}\frac{\partial}{\partial t}\int \tilde{R}^{p+1}d\tilde{\mu} &= (p+1)\int \tilde{R}^{p}\frac{\partial}{\partial t}\tilde{R}d\tilde{\mu}+\int \tilde{R}^{p+1}\frac{\partial}{\partial t}d\tilde{\mu}\\ &= 2(p+1)\int \tilde{R}^{p}\Delta_{\tilde{m}}\tilde{R}^{p}d\tilde{\mu}+2(p+1)\int\left(\tilde{H}\left|\tilde{A}\right|^{2}-\tilde{C}\right)\tilde{R}^{2p}d\tilde{\mu}\\ &\quad -2(p+1)\,n^{p}(n-1)^{p}\int \tilde{R}^{p+1}d\tilde{\mu}\\ &\quad -\int \tilde{R}^{2p+1}\tilde{H}d\tilde{\mu}+n^{p+1}(n-1)^{p}\int \tilde{R}^{p+1}d\tilde{\mu}\\ &= -2(p+1)\int\left|\nabla\tilde{R}^{p}\right|_{\tilde{m}}^{2}d\tilde{\mu}+2(p+1)\int\left(\tilde{H}\left|\tilde{A}\right|^{2}-\tilde{C}\right)\tilde{R}^{2p}d\tilde{\mu}\\ &\quad -\int \tilde{R}^{2p+1}\tilde{H}d\tilde{\mu}+n^{p}(n-1)^{p}(n-2p-2)\int \tilde{R}^{p+1}d\tilde{\mu}.\end{aligned}$$

Now we can estimate $\int\left|\nabla\tilde{R}^{p}\right|_{\tilde{m}}^{2}d\tilde{\mu}$ by the inequality (3.17):

$$\begin{aligned}\int\left|\nabla\tilde{R}^{p}\right|_{\tilde{m}}^{2}d\tilde{\mu} &= -\frac{1}{2(p+1)}\frac{\partial}{\partial t}\int \tilde{R}^{p+1}d\tilde{\mu}+\int\left(\tilde{H}\left|\tilde{A}\right|^{2}-\tilde{C}\right)\tilde{R}^{2p}d\tilde{\mu}\\ &\quad -\frac{1}{2(p+1)}\int \tilde{R}^{2p+1}\tilde{H}d\tilde{\mu}+n^{p}(n-1)^{p}\frac{n-2p-2}{2(p+1)}\int \tilde{R}^{p+1}d\tilde{\mu}\\ &\leq -\frac{1}{2(p+1)}\frac{\partial}{\partial t}\int \tilde{R}^{p+1}d\tilde{\mu}+\left(\frac{2}{n}-\frac{1}{2(p+1)}\right)\int \tilde{R}^{2p+1}\tilde{H}d\tilde{\mu}\\ &\quad +n^{p}(n-1)^{p}\frac{n-2p-2}{2(p+1)}\int \tilde{R}^{p+1}d\tilde{\mu}.\end{aligned}$$

Let us observe that convexity of $\tilde{M}_\tau$ implies $\int_{\tilde{M}_\tau} d\tilde{\mu}\leq c_2'\tilde{\rho}_+^n$, while $\tilde{R}$ is bounded for all times by inequality (3.26). Then the last two terms are bounded by formula (3.25) and the statement follows integrating in time. ■

Theorem 3.21 *The rescaled hypersurfaces converge to a smooth one in C^∞ norm as $\tau\to\infty$.*

Proof. For every $(\mathbf{p},\tau)\in\tilde{M}_\tau\times(\varepsilon,\infty)$ with $\varepsilon>0$, we have to bound the α- Hölder norm in space-time of $\tilde{R}$ on $B_\varepsilon(\mathbf{p})\times(\tau-\varepsilon,\tau+\varepsilon)$.

We can write the evolving surfaces as the graph of a function u, as we did for the non rescaled flow, and transform the second order operator of the evolution equation for $\tilde{R}$ in divergence form:

$$\tilde{\Delta}_{\tilde{m}}\tilde{R}^p = \tilde{m}^{ij}\left(D_iD_j\tilde{R}^p - \Gamma^l_{ij}D_l\tilde{R}^p\right) = D_i\left(\tilde{m}^{ij}D_j\tilde{R}^p\right) - D_i\tilde{m}^{ij}D_j\tilde{R}^p - \tilde{m}^{ij}\Gamma^l_{ij}D_l\tilde{R}^p.$$

Let us observe that this operator is not necessarily elliptic because the functions $\tilde{m}^{ij}$ with $i, j = 1, ..., n$ tend to zero if $\tilde{R} \to 0$. Then we have to transform again the second order operator as

$$\tilde{\Delta}_{\tilde{m}}\tilde{R}^p = \frac{2p}{2p+1}D_i\left(\tilde{R}^{-\frac{1}{2}}\tilde{m}^{ij}D_j\tilde{R}^{p+\frac{1}{2}}\right) - \frac{2p}{2p+1}\tilde{R}^{-\frac{1}{2}}D_i\tilde{m}^{ij}D_j\tilde{R}^{p+\frac{1}{2}} - \tilde{m}^{ij}\Gamma^l_{ij}D_l\tilde{R}^p$$

and prove that the coefficients $\tilde{R}^{-\frac{1}{2}}\tilde{m}^{ij}$ are bounded. Using the estimate (3.11) and $H^2 \geq R$ we have

$$R^{-\frac{1}{2}}m^{ij} \geq (H - \lambda_n)\, HR^{-\frac{1}{2}} > \left(\frac{n-1}{n} - \sqrt{\frac{n-1}{n}\gamma}\right),$$

moreover, from the inequality (3.18) follows

$$R^{-\frac{1}{2}}m^{ij} \leq R^{-\frac{1}{2}}H < \sqrt{\frac{n-1}{n} - \gamma}. \tag{3.28}$$

Hence the equation (3.27) can be written as a porous medium equation

$$\frac{\partial}{\partial t}\tilde{R} - \frac{4p}{2p+1}D_i\left(\tilde{R}^{-\frac{1}{2}}\tilde{m}^{ij}D_j\tilde{R}^{p+\frac{1}{2}}\right) = -\frac{4p}{2p+1}\tilde{R}^{-\frac{1}{2}}D_i\tilde{m}^{ij}D_j\tilde{R}^{p+\frac{1}{2}} + k\left(\tilde{R}, \nabla\tilde{R}\right) \tag{3.29}$$

with $\tilde{R}^{-\frac{1}{2}}\tilde{m}^{ij}\xi_i\xi_j \geq k_1' |\xi|$ and $\left|\tilde{R}^{-\frac{1}{2}}\tilde{m}^{ij}\right| \leq k_2'$ as required by Theorem 1.2 in [9].

Furthermore, by a property of symmetric functions, we have $\nabla_i\tilde{m}^{ij} = D_i\tilde{m}^{ij} + \Gamma^{ij}_i\tilde{m}^{ij} = 0$, hence we can estimate $D_i\tilde{m}^{ij}$ by $\tilde{m}^{ij}$, and then obtain an estimate on the right hand side of (3.29) in terms of $\nabla\tilde{R}^{p+\frac{1}{2}}$. Now formula (3.28) and Lemma 3.20 imply

$$\int_{\tau-2\varepsilon}^{\tau+2\varepsilon}\int_{B_{2\varepsilon}(\mathbf{p}_0)\cap T_{\mathbf{p}_0}\tilde{M}_{\tau_0}}\left|\nabla\tilde{R}^{p+\frac{1}{2}}\right|^2_{\tilde{g}} d\mathbf{x} < c_4'\int_{\tau-2\varepsilon}^{\tau+2\varepsilon}\int_{B_{2\varepsilon}(\mathbf{p}_0)\cap T_{\mathbf{p}_0}\tilde{M}_{\tau_0}}\left|\nabla\tilde{R}^p\right|^2_{\tilde{m}}\tilde{R}^{\frac{1}{2}}d\mathbf{x} < \infty$$

and Theorem 1.2 of [9] gives the desired $C^{2,\alpha}$ interior estimates.

Finally C^∞ estimates, long time existence of the solution and convergence to a sphere follows by standard arguments as in [8] or [23]. ■

The following theorem proves the second part of the statement of Theorem 3.1.

Theorem 3.22 *Under the hypotheses of Theorem 3.1 the immersions* $\tilde{\mathbf{F}}_\tau$ *converge to a limit immersion* $\tilde{\mathbf{F}}_\infty$*, whose image is a sphere, with exponential speed.*

Proof. Let us consider the function $\tilde{f} = \frac{|\tilde{A}|^2}{\tilde{H}^2} - \frac{1}{n}$. Since its homogeneity degree is zero, we have $\tilde{f} = f$, hence the evolution equation for $\tilde{f}$ is given by formulas (3.4) and (3.5). Then applying Proposition 3.9 and using formula (3.10) we can estimate for $\frac{1}{2} < p \leq 1$

$$\frac{\partial \tilde{f}}{\partial t} \leq 2p\tilde{R}^{p-1}\left\{\Delta_{\tilde{m}}\tilde{f} + (1-p)\frac{\tilde{H}}{\tilde{R}}\left|\nabla_i \tilde{f}\right|^2_{|\tilde{A}|^2\tilde{g}-\tilde{H}\tilde{h}} + 4p\frac{1}{\tilde{H}}\left\langle \nabla_i \tilde{f}, \nabla_i \tilde{H}\right\rangle_{\tilde{m}} - \frac{2p-1}{2p}n\eta^2\tilde{R}\tilde{H}\tilde{f}\right\}$$

and for $p > 1$

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial t} \leq\ & 2p\tilde{R}^{p-1}\left\{\Delta_{\tilde{m}}\tilde{f} - (p-1)\frac{\tilde{H}}{\tilde{R}}\left|\nabla_i \tilde{f}\right|^2_{|\tilde{A}|^2\tilde{g}-\tilde{H}\tilde{h}} + 4\frac{1}{\tilde{H}}\left\langle \nabla_i \tilde{f}, \nabla_i \tilde{H}\right\rangle_{\tilde{m}}\right.\\ & \left. +4(p-1)\frac{1}{\tilde{H}^2}\left\langle \nabla_i \tilde{f}, \nabla_i \tilde{H}\right\rangle_{|\tilde{A}|^2\tilde{g}-\tilde{H}\tilde{h}} - \frac{2p-1}{p}n\eta^2\tilde{R}\tilde{H}\tilde{f}\right\}.\end{aligned}$$

Then $\tilde{f}(\tau) \leq c_4' e^{-\delta\tau}$ for a suitable small $\delta > 0$. The rest of the proof is similar to that of Theorem 3.5 in [23]. ■

3.5 Singularity formation

This last section provides a counterexample to Theorem 3.1 in the case of a non convex initial surface with positive scalar curvature. We construct a surface that has the shape of dumbbell and prove that, as in the case of mean curvature flow, it forms a neckpinch singularity in finite time evolving by equation (3.1).

Proposition 3.23 *Let* $\mathbf{F}_0 : M \longrightarrow \mathbb{R}^{n+1}$ *be a smooth immersion in* $\mathbb{R}^{n+1}$ *with* $n \geq 2$ *and assume that* $H > 0$ *and* $R > 0$ *hold at any point of* $M_0 = \mathbf{F}_0(M)$*. Then the function* s *associated to* $\mathcal{S} = R^p$ *satisfies the condition*

$$\frac{\partial}{\partial \lambda_i} s(\lambda_1(\mathbf{y}), ..., \lambda_n(\mathbf{y})) > 0 \qquad \forall i = 1, ..., n.$$

It follows that there exists a unique solution of the initial value problem (3.1) and the properties $H > 0$*,* $R > 0$ *hold as long as the solution of the flow exists.*

Proof. Lemma 2.4 in [16] assures that, if the hypothesis $H > 0$, $R > 0$ on M_0 is satisfied, then $H - \lambda_i > 0$ for any $i = 1, ..., n$ on all the surface M_0. This guarantees the short time existence and uniqueness of the solution of the flow as in Theorem 3.2. Furthermore, since

$$H |A|^2 - C = \sum_{i=1}^{n} \lambda_i^2 (H - \lambda_i) > 0,$$

we can apply the maximum principle to the evolution equation of R (3.3), and deduce that the estimate $R \geq \min_{M_0} R > 0$ holds on all the time interval $[0, T)$ of existence of the flow. Finally, if there were an instant $t \in [0, T)$ in which the mean curvature vanishes at a point, we would find a contradiction to the inequality $H^2 > R > 0$, hence the positivity of H is preserved too. ■

The counterexample we are going to describe is a compact, rotationally symmetric surface. The following lemma shows an important property of this kind of surfaces that will be used in the proof of the singularity formation.

Lemma 3.24 *Let M be a surface with $H > 0$ and $R > 0$. If M is rotationally symmetric with $\lambda_1 \leq \lambda_2 = ... = \lambda_n$ then at any point we have $\lambda_n < \frac{2}{n} H$ and the norm associated to the metric $m_{ij} = H g_{ij} - h_{ij}$ can be estimated by the standard norm $|\cdot|_g$:*

$$\frac{n-2}{n} H |\cdot|_g < |\cdot|_m < \frac{2(n-1)}{n} H |\cdot|_g .$$

The same estimate holds on any surface at those points where $\lambda_n < \frac{2}{n} H$.

Proof. Given a point on the surface M the eigenvalues of the matrix m^i_j are $H - \lambda_i$ for $i = 1, ..., n$. First we prove that if the n-tuple $(\lambda_1, ..., \lambda_n)$ with $\lambda_1 \leq \lambda_2 = ... = \lambda_n$ satisfy

$$H = \sum_{i=1}^{n} \lambda_i > 0 \qquad \text{and} \qquad R = \sum_{i=1}^{n} \lambda_i (H - \lambda_i) > 0,$$

then we have the estimate

$$\frac{n-2}{n} H < H - \lambda_i < \frac{2(n-1)}{n} H.$$

Since $H > 0$ and $R > 0$, Lemma 2.4 in [16] implies, as before, that $H - \lambda_n > 0$. Then we can set $H - \lambda_n = \varepsilon H$, it is equivalent to $\lambda_n = (1 - \varepsilon) H$ with $0 < \varepsilon < 1$. Hence λ_1 can be expressed as $\lambda_1 = H - (n-1)\lambda_n = \varepsilon H - (n-2)\lambda_n$ and the function R becomes

$$\begin{aligned} R &= \lambda_1 (H - \lambda_1) + (n-1)\lambda_n (H - \lambda_n) \\ &= [\varepsilon H - (n-2)\lambda_n](n-1)\lambda_n + (n-1)\lambda_n \varepsilon H \\ &= (n-1)(1-\varepsilon)[2\varepsilon - (n-2)(1-\varepsilon)] H^2 \\ &= (n-1)(1-\varepsilon)[n\varepsilon - (n-2)] H^2 \end{aligned}$$

The last factor is positive if and only if $\frac{n-2}{n} < \varepsilon < 1$, hence

$$H - \lambda_n = \varepsilon H > \frac{n-2}{n} H,$$

$\lambda_n < \frac{2}{n} H$ and

$$H - \lambda_1 = (n-1)\lambda_n H = (n-1)(1-\varepsilon) H < \frac{2(n-1)}{n} H.$$

This estimates prove the desired property of the n-tuple $(\lambda_1, ..., \lambda_n)$.

Now take a general surface M with $H > 0$, $R > 0$ and take a point where $\lambda_n < \frac{2}{n} H$. We define the n-tuple $\left(\tilde{\lambda}_1, ..., \tilde{\lambda}_n\right)$ setting $\tilde{\lambda}_2 = ... = \tilde{\lambda}_n = \lambda_n$ and $\tilde{\lambda}_1 = H - (n-1)\lambda_n$; note that $H \leq n\lambda_n$ implies

$$\tilde{\lambda}_1 = H - (n-1)\lambda_n \leq \lambda_n = \tilde{\lambda}_2 = ... = \tilde{\lambda}_n.$$

Then we check that the n-tuple $\left(\tilde{\lambda}_1, ..., \tilde{\lambda}_n\right)$ fulfills the hypothesis of the first part:

$$\tilde{H} = \tilde{\lambda}_1 + ... + \tilde{\lambda}_n = \tilde{\lambda}_1 + (n-1)\tilde{\lambda}_n = H > 0$$

and function $\tilde{R}$ satisfies

$$\begin{aligned} \tilde{R} &= \tilde{\lambda}_1 \left(\tilde{H} - \tilde{\lambda}_1\right) + (n-1)\tilde{\lambda}_n \left(H - \tilde{\lambda}_n\right) \\ &= [H - (n-1)\lambda_n](n-1)\lambda_n + (n-1)\lambda_n (H - \lambda_n) \\ &= (n-1)\lambda_n (2H - n\lambda_n) \end{aligned}$$

where the last factor is positive thank to the hypothesis $\lambda_n < \frac{2}{n}H$. This allows us to apply the previous part and deduce the lower bound

$$\frac{H-\lambda_n}{H} = \frac{\tilde{H}-\tilde{\lambda}_n}{\tilde{H}} > \frac{n-2}{n},$$

hence $|\cdot|_m \geq (H-\lambda_n)\,|\cdot|_g > \frac{n-2}{n}H\,|\cdot|_g$. On the other hand we have

$$\frac{H-\lambda_1}{H} \leq \frac{(n-1)\,\lambda_n}{H} = \frac{(n-1)\,\tilde{\lambda}_n}{\tilde{H}} = \frac{\tilde{H}-\tilde{\lambda}_1}{\tilde{H}} < \frac{2\,(n-1)}{n},$$

thus $|\cdot|_m \leq (H-\lambda_1)\,|\cdot|_g < \frac{2(n-1)}{n}H\,|\cdot|_g$. ∎

We use a comparison principle to describe the evolution of the surface we want to define. It encloses two spheres in a symmetric position with respect to the origin and it is enclosed in an hyperboloid whose axis of symmetry links the centres of the spheres.

The next proposition shows that if the initial surface encloses a sphere, then the evolving surfaces enclose its evolution as long as the flow exists.

Proposition 3.25 *If the initial surface M_0 encloses a sphere B_{r_0} of radius r_0 then the surface M_t encloses the sphere of radius*

$$r\,(t) = \left[r_0^{2p+1} - (2p+1)\,n^p\,(n-1)^p\,t\right]^{\frac{1}{2p+1}}$$

whose maximal time of existence is

$$T'^{*} = \frac{r_0^{2p+1}}{(2p+1)\,n^p\,(n-1)^p}.$$

Proof. We assume M_0 to be tangent to B_{r_0} from outside. Let $\bar{\mathbf{y}}$ be the centre of the sphere B_{r_0} and consider the function $d\,(\mathbf{y},t) = |\mathbf{y}-\bar{\mathbf{y}}|^2$on M_t.

We compute $\Delta_m d$:

$$m^{ij}\nabla_i\nabla_j\,|\mathbf{y}-\bar{\mathbf{y}}|^2 = 2m^{ij}\nabla_i\,(y_j-\bar{y}_j) = 2\left(Hg^{ij}-h^{ij}\right)g_{ik}\delta_j^k = 2\,(n-1)\,H,$$

then the evolution equation for d is

$$\left(\frac{\partial}{\partial t} - R^{p-1}\Delta_m\right) d = -R^{p-1}\Delta_m d = -2(n-1) H R^{p-1}.$$

If we define $r(t) = \sqrt{\min_{M_t} d(\mathbf{y},t)}$, then the surface M_t is tangent from outside to $B_{r(t)}(\bar{\mathbf{y}})$ and in points of contact the curvatures are bounded by the curvatures of the sphere:

$$H \leq \frac{n}{r(t)} \qquad \text{and} \qquad R \leq \frac{n(n-1)}{r^2(t)}.$$

As a consequence, the evolution equation for d implies

$$\begin{aligned} \frac{d}{dt} r^2 &\geq -2(n-1) H R^{p-1} \geq -2n^p (n-1)^p r^{-2p+1} \\ \frac{d}{dt} r &\geq -n^p (n-1)^p r^{-2p}, \end{aligned}$$

hence the minimum of the distance function $|\mathbf{y} - \bar{\mathbf{y}}|$ is bounded from below by the radius of the evolution of B_{r_0} derived in (3.23). ■

In analogy with the previous proposition, the following one shows that if the initial surface is enclosed in an hyperboloid then the evolving surfaces are enclosed in its evolution until it degenerates.

Proposition 3.26 *Let us consider a compact hypersurface M_0 such that $|y_{n+1}| \leq N$. Moreover assume M_0 to be enclosed in the hyperboloid B_0 of equation*

$$2|\mathbf{y}|^2 - (n-\beta) y_{n+1}^2 - 2\eta^2 = 0,$$

where $0 < \beta < n-2$. Then the surface M_t is enclosed in the hyperboloid B_t of equation

$$2|\mathbf{y}|^2 - (n-\beta) y_{n+1}^2 + \frac{4(n-1)}{n}\beta c_0 t - 2\eta^2 = 0,$$

that develops a singularity at time $T^* = \frac{n\eta^2}{2(n-1)c_0\beta}$ where $c_0 = c_0(\beta, N)$.

Proof. If one defines $\hat{\mathbf{y}}$ such that $\mathbf{y} = (\hat{\mathbf{y}}, y_{n+1})$, the equation of the hyperboloid B_0 can be written as

$$2\left|\hat{\mathbf{y}}\right|^2 = (n-2-\beta)\, y_{n+1}^2 + 2\eta^2$$

and it can be considered as the rotation around the y_{n+1} axis of the function

$$s_H\left(y_{n+1}\right) := \sqrt{\frac{n-2-\beta}{2} y_{n+1}^2 + \eta^2}.$$

The principal curvatures of B_0 can the be computed using the derivatives of s_H (see e. g. [17] paragraph 3C):

$$\lambda_1 = \frac{-s_H''}{\left[1+\left(s_H'\right)^2\right]^{\frac{3}{2}}} \qquad \text{and} \qquad \lambda_k = \frac{1}{s_H\left[1+\left(s_H'\right)^2\right]^{\frac{1}{2}}} \quad \forall k = 2, ..., n. \tag{3.30}$$

We have $s_H'\left(y_{n+1}\right) = \frac{n-2-\beta}{2}\frac{y_{n+1}}{s_H}$, that implies

$$1+\left(s_H'\right)^2 = \frac{s_H^2 + \left(\frac{n-2-\beta}{2}\right)^2 y_{n+1}^2}{s_H^2} = \frac{(n-2-\beta)\,(n-\beta)\, y_{n+1}^2 + 4\eta^2}{4s_H^2},$$

and

$$\begin{aligned} s_H''\left(y_{n+1}\right) &= \frac{n-2-\beta}{2}\left(\frac{1}{s_H} - \frac{y_{n+1}^2}{s_H^3}\frac{n-2-\beta}{2}\right) \\ &= \frac{n-2-\beta}{2}\frac{s_H^2 - \left(\frac{n-2-\beta}{2}\right) y_{n+1}^2}{s_H^3} = \frac{n-2-\beta}{2}\frac{\eta^2}{s_H^3}. \end{aligned} \tag{3.31}$$

Hence the principal curvatures are

$$\lambda_1 = -\frac{(n-2-\beta)\, 4\eta^2}{\left[(n-2-\beta)\,(n-\beta)\, y_{n+1}^2 + 4\eta^2\right]^{\frac{3}{2}}}$$

and

$$\lambda_n = \frac{2}{\left[(n-2-\beta)\,(n-\beta)\, y_{n+1}^2 + 4\eta^2\right]^{\frac{1}{2}}}.$$

Let us check that on B_0 we have $H > 0$ and $R > 0$.

The mean curvature is positive if and only if

$$\begin{aligned} & -(n-2-\beta)\,4\eta^2 + 2\,(n-1)\left[(n-2-\beta)\,(n-\beta)\,y_{n+1}^2 + 4\eta^2\right] \\ = \;& 2\,(n-1)\,(n-2-\beta)\,(n-\beta)\,y_{n+1}^2 + 4\eta^2\,(n+\beta) > 0 \end{aligned}$$

and it is always true. For any rotationally symmetric surface with $\lambda_1 \leq \lambda_2 = ... = \lambda_n$ the scalar curvature is given by

$$\begin{aligned} R &= \lambda_1\,(n-1)\,\lambda_n + (n-1)\,\lambda_n\,(H-\lambda_n) \\ &= (n-1)\,\lambda_n\,[2\lambda_1 + (n-2)\,\lambda_n]\,. \end{aligned} \tag{3.32}$$

Note that

$$\frac{\lambda_1}{\lambda_n} = -\frac{(n-2-\beta)\,2\eta^2}{(n-2-\beta)\,(n-\beta)\,y_{n+1}^2 + 4\eta^2} \geq -\frac{n-2-\beta}{2},$$

then we have the following estimate on B_0:

$$R \geq (n-1)\,\beta\lambda_n^2 = \frac{4\,(n-1)\,\beta}{(n-2-\beta)\,(n-\beta)\,y_{n+1}^2 + 4\eta^2} > 0.$$

Moreover on $B_0 \cap [-N, N]$ we have

$$R > \frac{4\,(n-1)\,\beta}{(n-2-\beta)\,(n-\beta)\,N^2 + 4\eta^2} = c_0\,(\eta, \beta, N)\,.$$

Let us observe that fixing a positive constant $c_0\,(\eta, \beta, N)$, the same constant is valid for any $\eta' < \eta$ and that the estimate found on B_0 holds as long as the flow exists because R is increasing along the flow.

Now consider the function

$$\begin{aligned} h &= 2\,|\mathbf{y}|^2 - (n-\beta)\,y_{n+1}^2 + \frac{4\,(n-1)}{n}\beta c_0 t - 2\eta^2 \\ &= 2\,|\hat{\mathbf{y}}|^2 - (n-2-\beta)\,y_{n+1}^2 + \frac{4\,(n-1)}{n}\beta c_0 t - 2\eta^2 \end{aligned}$$

vanishing on the hyperboloid B_t and negative inside. The evolution equation for h is

$$\left(\frac{\partial}{\partial t} - R^{p-1}\Delta_m\right) h = \frac{4(n-1)}{n}\beta c_0 - 4(n-1) HR^{p-1} + 2(n-\beta) R^{p-1} |\nabla_i y_{n+1}|^2_m ,$$

it can be estimated using Lemma 3.24 and the inequality $|\nabla y_{n+1}|^2 \leq 1$:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - R^{p-1}\Delta_m\right) h &< \frac{4(n-1)}{n}\beta c_0 - \frac{4(n-1)}{n}\left[n - (n-\beta)|\nabla y_{n+1}|^2\right] HR^{p-1} \\ &\leq \frac{4(n-1)}{n}\beta c_0 - \frac{4(n-1)}{n}\beta HR^{p-1}; \end{aligned}$$

finally the last expression is negative because $HR^{p-1} \geq R^{p-\frac{1}{2}} > c_0^{p-\frac{1}{2}}(\beta, N)$.

By the maximum principle we infer that $\max h < 0$ for all $t \in [0, T^*)$, hence the evolving surfaces are enclosed in the hyperboloids B_t until they degenerate at time T^*. ■

Now we give an explicit example of a surface with positive scalar curvature that does not converge to a round point.

Example 3.27 *There exist a surface M_0 with $H > 0$ and $R > 0$ that develops a neckpinch singularity evolving by the flow (3.1).*

Proof. Let us consider the hyperboloid B_0 of the previous lemma that is the rotation around the y_{n+1} axis of the function

$$s_H(y_{n+1}) := \sqrt{\frac{n-2-\beta}{2} y_{n+1}^2 + \eta^2}.$$

We define the function

$$s_E(y_{n+1}) := \sqrt{B^2 - A^2(|y_{n+1}| - L_1)^2}$$

in the interval $L_1 - \frac{B}{A} \leq |y_{n+1}| \leq L_1 + \frac{B}{A}$, and choose the constant L_1 in such a way that the ellipses obtained rotating s_E are tangent to B_0.

If we define $\alpha := \frac{n-2-\beta}{2} + A^2$, the condition $s_H(y_{n+1}) = s_E(y_{n+1})$ is equivalent to

$$s_H^2 - s_E^2 = \alpha y_{n+1}^2 - 2A^2 L_1 |y_{n+1}| + A^2 L_1^2 - B^2 + \eta^2 = 0,$$

and the tangential condition becomes

$$A^4 L_1^2 - \alpha \left(A^2 L_1^2 - B^2 + \eta^2 \right) = 0.$$

It follows that the ellipses are tangent to the hyperboloid if and only if they are centred in $(\mathbf{0}, \pm L_1)$ with

$$L_1 = \sqrt{\frac{\alpha \left(B^2 - \eta^2 \right)}{A^2 \left(\alpha - A^2 \right)}};$$

in this case

$$s_H^2 - s_E^2 = \alpha \left(|y_{n+1}| - \frac{A^2}{\alpha} L_1 \right)^2 \tag{3.33}$$

and the $(n+1)$-th coordinate of the contact points are $\pm L_0$ with $L_0 := \frac{A^2}{\alpha} L_1$.

Now we want to define a function s connecting s_H with s_E such that the surface M_0, obtained rotating s around the y_{n+1} axis is a twice differentiable surface with positive scalar curvature. The surface M_0 will be C^2 rather then C^∞ as we usually assume; however, by the smoothening properties of parabolic equalities, our calculations remain valid also in this case. We can also consider a C^∞ initial surface $\tilde{M}_0$ which is close enough to M_0 in C^2-norm to satisfy the same conditions on the curvatures which we impose on M_0.

We consider the function φ

$$\varphi \left(y_{n+1} \right) := 1 - \frac{\left(L_1 - |y_{n+1}| \right)^3}{\left(L_1 - L_0 \right)^3},$$

such that $\varphi \left(L_0 \right) = 0$ and $\varphi \left(L_1 \right) = 1$, note that its derivatives

$$\varphi' \left(y_{n+1} \right) = 3 \frac{\left(L_1 - |y_{n+1}| \right)^2}{\left(L_1 - L_0 \right)^3} \frac{y_{n+1}}{|y_{n+1}|} \qquad \text{and} \qquad \varphi'' \left(y_{n+1} \right) = -6 \frac{L_1 - |y_{n+1}|}{\left(L_1 - L_0 \right)^3}$$

satisfy $\varphi' \left(L_1 \right) = \varphi'' \left(L_1 \right) = 0$. Then we define the function

$$s \left(y_{n+1} \right) := \begin{cases} s_H \left(y_{n+1} \right) & 0 \leq |y_{n+1}| \leq L_0 \\ s_R \left(y_{n+1} \right) & L_0 \leq |y_{n+1}| \leq L_1 \\ s_E \left(y_{n+1} \right) & L_1 \leq |y_{n+1}| \leq L_1 + \frac{B}{A}, \end{cases}$$

where s_R is the convex combination of s_E and s_H by φ

$$\begin{aligned} s_R(y_{n+1}) \quad : & = \varphi(y_{n+1})\, s_E(y_{n+1}) + [1-\varphi(y_{n+1})]\, s_H(y_{n+1}) \\ & = s_H(y_{n+1}) + \varphi(y_{n+1})\, [s_E(y_{n+1}) - s_H(y_{n+1})] \,. \end{aligned}$$

Now we can prove that M_0 is C^2 differentiable. Note that the rotation of s_H at $y_{n+1}=0$ and the rotation of s_E at $|y_{n+1}| = L_1 + \frac{B}{A}$ are smooth, and all the functions involved are smooth inside their domain. Hence we only need to check continuity and differentiability of s on the connection points L_0 and L_1: once computed the derivatives of s_R

$$\begin{aligned} s_R' & = s_H' + \varphi\left(s_E' - s_H'\right) + \varphi'\left(s_E - s_H\right) \\ s_R'' & = s_H'' + \varphi\left(s_E'' - s_H''\right) + 2\varphi'\left(s_E' - s_H'\right) + \varphi''\left(s_E - s_H\right) \end{aligned}$$

they are easy consequences of the properties of φ and s_E.

We also have to prove that the scalar curvature of M_0 is positive. From equation (3.32) and formulas (3.30) it follows that a rotationally symmetric surface with $\lambda_1 \leq \lambda_2 = ... = \lambda_n$ have positive scalar curvature if and only if

$$\begin{aligned} R & = (n-1)\,\lambda_n\,[2\lambda_1 + (n-2)\,\lambda_n] \\ & = \frac{n-1}{s^2\left[1+(s')^2\right]^2}\left[-2ss'' + (n-2)\left(1+(s')^2\right)\right] > 0. \end{aligned}$$

Let us observe that, as shown in (3.31) the function s_H generating the hyperboloid satisfies the stronger estimate

$$s_H s_H'' = \frac{n-2-\beta}{2}\frac{\eta^2}{s_H^2} < \frac{n-2}{2},$$

hence it suffices to show that for $L_0 \leq |y_{n+1}| \leq L_1$ we have

$$s_R(y_{n+1}) \leq s_H(y_{n+1}) \qquad \text{and} \qquad s_H''(y_{n+1}) \leq s_H''(y_{n+1})$$

so that

$$s_R s_R'' \leq s_H s_H'' < \frac{n-2}{2} < \frac{n-2}{2}\left[1+\left(s_R^{'}\right)^2\right]$$

and the positivity of the scalar curvature is guaranteed on all the surface M_0. From equation (3.33) follows

$$s_H - s_E = \frac{\alpha\left(|y_{n+1}| - L_0\right)^2}{s_H + s_E} > 0,$$

hence $s_R = s_H - \varphi\left(s_H - s_E\right) < s_H$. Computing

$$s'_H\left(y_{n+1}\right) = \frac{n-2-\beta}{2}\frac{y_{n+1}}{s_H} \qquad \text{and} \qquad s'_E\left(y_{n+1}\right) = -\frac{A^2\left(|y_{n+1}| - L_1\right)}{s_E}\frac{y_{n+1}}{|y_{n+1}|}$$

we have

$$\begin{aligned}
s'_H - s'_E &= \frac{1}{s_H s_E}\left[\left(\alpha - A^2\right) y_{n+1} s_E + A^2 \frac{y_{n+1}}{|y_{n+1}|}\left(|y_{n+1}| - L_1\right) s_H\right] \\
&= \frac{1}{s_H s_E}\left[\left(\alpha - A^2\right)|y_{n+1}| s_E + \left(A^2|y_{n+1}| - \alpha L_0\right) s_H\right]\frac{y_{n+1}}{|y_{n+1}|} \\
&= \frac{1}{s_H s_E}\left[\left(\alpha - A^2\right)\left(s_E - s_H\right)|y_{n+1}| + \alpha\left(|y_{n+1}| - L_0\right) s_H\right]\frac{y_{n+1}}{|y_{n+1}|} \\
&= \frac{\alpha\left(|y_{n+1}| - L_0\right)}{\left(s_H + s_E\right) s_H s_E}\left[-\left(\alpha - A^2\right)\left(|y_{n+1}| - L_0\right)|y_{n+1}| + s_H^2 + s_H s_E\right]\frac{y_{n+1}}{|y_{n+1}|} \\
&= \frac{\alpha\left(|y_{n+1}| - L_0\right)}{s_H + s_E}\left[\frac{\alpha - A^2}{s_H s_E} L_0 |y_{n+1}| + \frac{\eta^2}{s_H s_E} + 1\right]\frac{y_{n+1}}{|y_{n+1}|}.
\end{aligned}$$

The second derivatives of s_H and s_E are

$$\begin{aligned}
s''_H\left(y_{n+1}\right) &= \frac{n-2-\beta}{2}\left(\frac{1}{s_H} - \frac{y_{n+1}^2}{s_H^3}\frac{n-2-\beta}{2}\right) \\
s''_E\left(y_{n+1}\right) &= -A^2\left(\frac{1}{s_E} + \frac{A^2}{s_E^3}\left(|y_{n+1}| - L_1\right)^2\right),
\end{aligned}$$

thus

$$s''_H - s''_E = \frac{n-2-\beta}{2}\frac{\eta^2}{s_H^3} + \frac{A^2 B^2}{s_E^3} > \frac{A^2 B^2}{s_E^3}.$$

It follows the estimate

$$
\begin{aligned}
s''_H - s''_R &= \varphi\left(s''_H - s''_E\right) + 2\varphi'\left(s'_H - s'_E\right) + \varphi''\left(s_H - s_E\right) \\
&> \frac{\left(L_1 - L_0\right)^3 - \left(L_1 - |y_{n+1}|\right)^3}{\left(L_1 - L_0\right)^3} \frac{A^2 B^2}{s_E^3} + \\
&\quad +6 \frac{\left(L_1 - |y_{n+1}|\right)^2}{\left(L_1 - L_0\right)^3} \frac{\alpha\left(|y_{n+1}| - L_0\right)}{s_H + s_E} \left(\frac{\alpha - A^2}{s_H s_E} L_0 |y_{n+1}| + \frac{\eta^2}{s_H s_E} + 1\right) \\
&\quad -6 \frac{L_1 - |y_{n+1}|}{\left(L_1 - L_0\right)^3} \frac{\alpha\left(|y_{n+1}| - L_0\right)^2}{s_H + s_E},
\end{aligned}
$$

moreover for $L_0 \leq |y_{n+1}| \leq L_1$ we have $\frac{A^2 B^2}{s_E^3} > \frac{A^2}{s_E}$ and

$$
\begin{aligned}
s''_H - s''_R &> \frac{|y_{n+1}| - L_0}{\left(L_1 - L_0\right)^3} \left[\left(L_1 - L_0\right)^2 + \left(L_1 - L_0\right)\left(L_1 - |y_{n+1}|\right) + \left(L_1 - |y_{n+1}|\right)^2\right] \frac{A^2}{s_E} \\
&\quad +6\alpha \frac{L_1 - |y_{n+1}|}{\left(L_1 - L_0\right)^3} \frac{|y_{n+1}| - L_0}{s_H + s_E} \left[\left(L_1 - |y_{n+1}|\right) - \left(|y_{n+1}| - L_0\right)\right].
\end{aligned}
$$

Looking at the last line above it is clear that the inequality $s''_H - s''_R > 0$ holds for $|y_{n+1}| \leq \frac{L_1 + L_0}{2}$.

Otherwise, since $s_H > s_E$ implies $-\frac{1}{s_H + s_E} > -\frac{1}{2 s_E}$, for $\frac{L_1 + L_0}{2} < |y_{n+1}| < L_1$ we have

$$
s''_H - s''_R > \frac{|y_{n+1}| - L_0}{s_E \left(L_1 - L_0\right)^3} \left[A^2 \left(L_1 - L_0\right)^2 - 3\alpha \left(L_1 - |y_{n+1}|\right)\left(|y_{n+1}| - L_0\right)\right].
$$

The negative term assume its minimum at $|y_{n+1}| = \frac{L_1 + L_0}{2}$, hence $s''_H - s''_R > 0$ if

$$
\frac{|y_{n+1}| - L_0}{s_E \left(L_1 - L_0\right)} \left(A^2 - \frac{3}{4}\alpha\right) \geq 0
$$

and the difference is positive provided that $A^2 - \frac{3}{4}\alpha = \frac{1}{4}A^2 - \frac{n-2-\beta}{2} \geq 0$.

Now we can choose $A^2 = 2(n - 2 - \beta) + 1$ and $B = A r_0$ so that the surface M_0 encloses two spheres of radius r_0 as in Proposition 3.25. As a consequence M_0 cannot vanish before time $T'^* = \frac{r_0^{2p+1}}{(2p+1) n^p (n-1)^p}$.

Furthermore let us observe that

$$
M_0 \subset \left\{|y_{n+1}| \leq L_1 + \frac{B}{A}\right\} = \left\{|y_{n+1}| \leq L_1 + r_0\right\}
$$

and that

$$L_1 + r_0 = \sqrt{\frac{2\alpha\left(A^2 r_0^2 - \eta^2\right)}{A^2\left(n-2-\beta\right)}} + r_0 < \left(\sqrt{\frac{2\alpha}{n-2-\beta}} + 1\right) r_0,$$

hence M_0 the hypotheses of Proposition 3.26 with $N = N\left(n, \beta, r_0\right)$. Since M_0 is symmetric with respect to the origin, it has to shrink in the middle as the hyperboloid at most at time $T^* = \frac{n\eta^2}{2(n-1)c_0\beta}$.

Finally η can be chosen such that $T^* \leq T^{'*}$:

$$\eta^2 \leq \frac{2c_0\left(\beta\right)\beta r_0^{2p+1}}{\left(2p+1\right) n^{p+1}\left(n-1\right)^{p-1}},$$

then M_0 develops a neckpinch singularity because it shrinks in the middle when it still contains a sphere of positive radius in both sides of the y_{n+1} axis. ■

Chapter 4

Evolution of entire graphs

In this chapter we study the geometric flow (1.4)

$$\begin{cases} \frac{\partial \mathbf{F}}{\partial t}(\mathbf{p},t) = & -\mathcal{S}(\mathcal{W})\,\boldsymbol{\nu}(\mathbf{F}(\mathbf{p},t)) \\ \mathbf{F}(\mathbf{p},0) = & \mathbf{F}_0(\mathbf{p}). \end{cases}$$

with the hypothesis that the initial surface is the graph of a smooth function u_0 over $\mathbb{R}^n \times \{0\}$:

$$\mathbf{F}_0(\mathbf{p}) = (x_1, ..., x_n, u_0(x_1, ..., x_n)) = (\mathbf{x}, u_0(\mathbf{x})).$$

First of all we state the short time existence theorem.

Theorem 4.1 (Short time existence) *Let us consider the initial value problem (1.4) and assume that for any $\mathbf{y} \in M_0$ the function s associated to $\mathcal{S}$ satisfies*

$$\frac{\partial}{\partial \lambda_i} s(\lambda_1(\mathbf{y}), ..., \lambda_n(\mathbf{y})) > 0 \qquad \forall i = 1, ..., n.$$

If M_0 is the graph of a smooth function u_0 such that $\|\nabla u_0\|_{C^{1,\alpha}} < \infty$, then for a short time interval there exists a unique smooth solution $u(\mathbf{x},t) : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ with $u(\cdot,0) = u_0$ and $\|u - u_0\|_{C^{2,\alpha}} < \infty$.

Proof. The statement can be proved by a standard fixed point argument in a similar way to Theorem 8.5.4 of [19]. Note that, instead of $\|u\|_{C^{2,\alpha}} < \infty$, the hypothesis above is sufficient because the equation of the flow involves only terms in ∇u and $\nabla^2 u$. ■

From now on we assume that on M_0 the hypotheses of the Theorem 4.1 are verified. Let us observe that the boundedness of the gradient function ∇u_0 forces M_0 to have at most a linear growth at infinity. As a consequence of the previous theorem we have that $|u(\mathbf{x},t) - u_0(\mathbf{x})|$, $\|\nabla u\|$ and $\|\nabla^2 u\|$ are all bounded quantities for all $(\mathbf{x},t) \in \mathbb{R}^n \times [0,T)$.

Since M_t's are non compact surfaces we cannot use the statement of the maximum principle given in chapter 1, then we prove here a specific statement of the maximum principle over graphs.

Theorem 4.2 (Maximum principle over graphs) *Let $M_t = (\mathbf{x}, u(\mathbf{x},t))$ for $t \in [0,T)$ be the surfaces defined by the flow (1.4) with the assumptions of Theorem 4.1 and let f be a smooth function defined on $M_t \times [0,T)$. Assume that for the evolution equation of f the following inequality holds*

$$\begin{cases} \frac{\partial f}{\partial t} \leq \dot{\mathcal{S}} g^* (Hess_\nabla f) + \dot{\mathcal{S}} g^* (\mathbf{a}, \nabla f) \\ f(M_0) = f_0 \end{cases}$$

where the vector field $\mathbf{a}$ *and the function* f_0 *satisfy* $\sup\limits_{M \times [0,T)} |\mathbf{a}| \leq c_1 < \infty$ *and* $\|f_0\|_{C^2} < \infty$.

Then if $\sup\limits_{M_t \times [0,T)} |f|$ *is bounded we have* $\sup\limits_{M_t \times [0,T)} f \leq \sup f_0$.

Proof. Since $|f(M_t)|$ is bounded, say

$$\sup_{M_t \times [0,T)} |f| = N,$$

then for any $\varepsilon > 0$ there exists a point and a time $(\bar{\mathbf{y}}, \bar{t})$ such that $f(\bar{\mathbf{y}}, \bar{t}) > N - \varepsilon$. We define for any $\eta > 0$ a new function f_η as

$$f_\eta(\mathbf{y},t) = f(\mathbf{y},t) - \frac{\eta^2}{2} |\mathbf{y} - \bar{\mathbf{y}}|^2 - K(\eta)\, t$$

where

$$K(\eta) > \eta \left[\eta + 2\sqrt{N} c_1\right] \cdot \sup_{M_t \times [0,T)} Tr\left(\dot{\mathcal{S}}\right) > 0; \tag{4.1}$$

note that $\sup\limits_{M_t \times [0,T)} Tr\left(\dot{\mathcal{S}}\right) < \infty$ because it depends on the principal curvatures that are bounded by $\|\nabla^2 u\|$.

We have

$$\nabla f_\eta = \nabla f - \eta^2 (\mathbf{y} - \bar{\mathbf{y}}) \qquad \text{and} \qquad Hess_\nabla f_\eta = Hess_\nabla f - \eta^2 Id,$$

hence the function f_η satisfies the evolution equation given by

$$\begin{aligned} \frac{\partial f_\eta}{\partial t} &= \frac{\partial f}{\partial t} - K(\eta) \le \dot{\mathcal{S}}g^*(Hess_\nabla f) + \dot{\mathcal{S}}g^*(\mathbf{a},\nabla f) - K(\eta) \\ &= \dot{\mathcal{S}}g^*\left(Hess_\nabla f_\eta + \eta^2 I\right) + \dot{\mathcal{S}}g^*\left(\mathbf{a},\nabla f_\eta + \eta^2(\mathbf{y} - \bar{\mathbf{y}})\right) - K(\eta) \\ &= \dot{\mathcal{S}}g^*(Hess_\nabla f_\eta) + \dot{\mathcal{S}}g^*(\mathbf{a},\nabla f_\eta) + \eta^2 \dot{\mathcal{S}}g^*(I) + \eta^2 \dot{\mathcal{S}}g^*(\mathbf{a},(\mathbf{y} - \bar{\mathbf{y}})) - K(\eta). \quad (4.2) \end{aligned}$$

In what follows we estimate the last terms of this equation.

Since f is bounded, then $f_\eta \to -\infty$ as $|\mathbf{y}| \to \infty$, hence the function f_η assumes its maximum value in $M_t \times [0,T)$, say $\max\limits_{M_t \times [0,T)} f_\eta = f_\eta(\tilde{\mathbf{y}}, \tilde{t})$. From

$$f(\bar{\mathbf{y}},0) = f_\eta(\bar{\mathbf{y}},0) \le \max_{M_t \times [0,T)} f_\eta = f_\eta(\tilde{\mathbf{y}},\tilde{t}) = f(\tilde{\mathbf{y}},\tilde{t}) - \frac{\eta^2}{2} |\tilde{\mathbf{y}} - \bar{\mathbf{y}}|^2 - K(\eta)\tilde{t},$$

it follows

$$\frac{\eta^2}{2} |\tilde{\mathbf{y}} - \bar{\mathbf{y}}|^2 \le f(\tilde{\mathbf{y}},\tilde{t}) - f(\bar{\mathbf{y}},0) \le 2 \sup_{M_t \times [0,T)} |f| = 2N$$

and $\eta |\tilde{\mathbf{y}} - \bar{\mathbf{y}}| \le 2\sqrt{N}$.

Recalling that $\dot{\mathcal{S}}g^*(I)$ is the trace of the matrix $\dot{\mathcal{S}}$ we have

$$\eta^2 \dot{\mathcal{S}}g^*(\mathbf{a},(\tilde{\mathbf{y}} - \bar{\mathbf{y}})) \le \eta^2 |\mathbf{a}| |\tilde{\mathbf{y}} - \bar{\mathbf{y}}| Tr\left(\dot{\mathcal{S}}\right) \le 2\eta\sqrt{N}c_1 \cdot \sup_{M_t \times [0,T)} Tr\left(\dot{\mathcal{S}}\right)$$

and the last terms of equation (4.2) are estimated by the assumption (4.1):

$$\eta^2 Tr\left(\dot{\mathcal{S}}\right) + \eta^2 \dot{\mathcal{S}}g^*(\mathbf{a},(\tilde{\mathbf{y}} - \bar{\mathbf{y}})) - K(\eta) \le \eta\left[\eta + 2\sqrt{N}c_1\right] \cdot \sup_{M_t \times [0,T)} Tr\left(\dot{\mathcal{S}}\right) - K(\eta) < 0.$$

If $\tilde{t} > 0$ then

$$\dot{\mathcal{S}}g^*(Hess_\nabla f_\eta)(\tilde{\mathbf{y}},\tilde{t}) \le 0, \qquad \dot{\mathcal{S}}g^*(\mathbf{a},\nabla f_\eta)(\tilde{\mathbf{y}},\tilde{t}) = 0 \qquad \text{and} \qquad \frac{\partial f_\eta}{\partial t}(\tilde{\mathbf{y}},\tilde{t}) \ge 0,$$

but this would lead to the contradiction to the evolution equation for f_η (4.2) estimated by the previous inequality

$$\frac{\partial f_\eta}{\partial t}\left(\tilde{\mathbf{y}},\tilde{t}\right)\leq\eta^2\dot{\mathcal{S}}g^*\left(I\right)+\eta^2\dot{\mathcal{S}}g^*\left(\mathbf{a},\left(\tilde{\mathbf{y}}-\bar{\mathbf{y}}\right)\right)-K\left(\eta\right)<0.$$

We deduce that f_η assumes its maximum value at time $\tilde{t}=0$, then we have

$$f_\eta\left(\tilde{\mathbf{y}},\tilde{t}\right)=f_0\left(\tilde{\mathbf{y}}\right)-\frac{\eta^2}{2}\left|\tilde{\mathbf{y}}-\bar{\mathbf{y}}\right|^2\leq\sup f_0-\frac{\eta^2}{2}\left|\tilde{\mathbf{y}}-\bar{\mathbf{y}}\right|^2$$

and

$$f_\eta\left(\tilde{\mathbf{y}},\tilde{t}\right)=\max_{M_t\times[0,T)}f_\eta\geq f_\eta\left(\bar{\mathbf{y}},\bar{t}\right)=f\left(\bar{\mathbf{y}},\bar{t}\right)-K\left(\eta\right)\bar{t}>N-\varepsilon-K\left(\eta\right)\bar{t}.$$

Comparing the previous two inequalities, it follows

$$N<\sup f_0-\frac{\eta^2}{2}\left|\tilde{\mathbf{y}}-\bar{\mathbf{y}}\right|^2+\varepsilon+K\left(\eta\right)\bar{t}.$$

Since ε and η are arbitrary chosen and $K\left(\eta\right)\to 0$ as $\eta\to 0$, we obtain $N=\sup_{M_t\times[0,T)}f\leq\sup f_0$. ■

Adapting the argument shown in [10] for the mean curvature flow, now we show that the evolving surfaces $M_t=\mathbf{F}\left(\cdot,t\right)$ are all graphs as long as the flow exists.

Since M_0 is a graph there exists a unit vector $\boldsymbol{\omega}\in\mathbb{R}^{n+1}$, orthogonal to the hyperplane where u_0 is defined, satisfying the condition $\left\langle\boldsymbol{\nu}\left(\mathbf{x}\right),\boldsymbol{\omega}\right\rangle>0$ for all $\mathbf{x}\in\mathbb{R}^n$. In order to prove that this characteristic condition of a graph over $\mathbb{R}^n\times\{0\}$ is preserved along the flow we introduce on $M_t\times[0,T)$ the function $v:=\left\langle\boldsymbol{\nu},\boldsymbol{\omega}\right\rangle^{-1}$ and verify that it is nonincreasing in time. In fact if v stays bounded, it means that the normal vector $\boldsymbol{\nu}$ is never orthogonal to the fixed vector $\boldsymbol{\omega}$, hence the evolving surface can be still written as a graph over the same hyperplane of u_0.

Lemma 4.3 *The quantity $v=\left\langle\boldsymbol{\nu},\boldsymbol{\omega}\right\rangle^{-1}$ satisfies the following evolution equation*

$$\frac{\partial v}{\partial t}=\dot{\mathcal{S}}g^*\left(Hess_{\nabla}v\right)-2v^{-1}\dot{\mathcal{S}}g^*\left(\nabla v,\nabla v\right)-v\dot{\mathcal{S}}g^*\left(\mathcal{II}^2\right).$$

Then the evolution of a graph stays a graph as long as the flow exists.

Proof. We compute the time derivative of v

$$\frac{\partial v}{\partial t} = -v^2 \left\langle \frac{\partial}{\partial t} \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle = -v^2 \left\langle \nabla \mathcal{S}, \boldsymbol{\omega} \right\rangle,$$

and the gradient of v^{-1}

$$\nabla_j \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle = \frac{\partial}{\partial x_j} \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle = \left\langle \frac{\partial \boldsymbol{\nu}}{\partial x_j}, \boldsymbol{\omega} \right\rangle = h_j^k \left\langle \frac{\partial \mathbf{F}}{\partial x_k}, \boldsymbol{\omega} \right\rangle.$$

Moreover the equalities (1.8) imply for the Hessian of v^{-1}

$$\begin{aligned}
\nabla_i \left(\nabla_j \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle\right) &= \frac{\partial}{\partial x_i} \left(\nabla_j v^{-1}\right) - \nabla_l v^{-1} \Gamma_{ij}^l \\
&= \frac{\partial}{\partial x_i} \left(h_j^k \left\langle \frac{\partial \mathbf{F}}{\partial x_k}, \boldsymbol{\omega} \right\rangle \right) - \Gamma_{ij}^l h_l^k \left\langle \frac{\partial \mathbf{F}}{\partial x_k}, \boldsymbol{\omega} \right\rangle \\
&= \frac{\partial h_j^k}{\partial x_i} \left\langle \frac{\partial \mathbf{F}}{\partial x_k}, \boldsymbol{\omega} \right\rangle + h_j^k \left\langle \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_k}, \boldsymbol{\omega} \right\rangle - \Gamma_{ij}^l h_l^k \left\langle \frac{\partial \mathbf{F}}{\partial x_k}, \boldsymbol{\omega} \right\rangle \\
&= \frac{\partial h_j^k}{\partial x_i} \left\langle \frac{\partial \mathbf{F}}{\partial x_k}, \boldsymbol{\omega} \right\rangle + h_j^k \Gamma_{ik}^l \left\langle \frac{\partial \mathbf{F}}{\partial x_l}, \boldsymbol{\omega} \right\rangle - h_{ik} h_j^k \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle - \Gamma_{ij}^l h_l^k \left\langle \frac{\partial \mathbf{F}}{\partial x_k}, \boldsymbol{\omega} \right\rangle \\
&= \left(\frac{\partial h_j^k}{\partial x_i} + h_j^l \Gamma_{il}^k - \Gamma_{ij}^l h_l^k \right) \left\langle \frac{\partial \mathbf{F}}{\partial x_k}, \boldsymbol{\omega} \right\rangle - h_{ik} h_j^k \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle \\
&= \nabla_i h_j^k \left\langle \frac{\partial \mathbf{F}}{\partial x_k}, \boldsymbol{\omega} \right\rangle - h_{ik} h_j^k \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle = \nabla^k h_{ij} \left\langle \frac{\partial \mathbf{F}}{\partial x_k}, \boldsymbol{\omega} \right\rangle - h_{ik} h_j^k \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle \\
&= \left\langle \nabla h_{ij}, \boldsymbol{\omega} \right\rangle - h_{ik} h_j^k \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle.
\end{aligned}$$

Then we have

$$\begin{aligned}
\nabla_i \nabla_j v &= \nabla_i \left(-v^2 \nabla_j \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle\right) = -v^2 \nabla_i \nabla_j \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle + 2v^3 \nabla_i \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle \nabla_j \left\langle \boldsymbol{\nu}, \boldsymbol{\omega} \right\rangle \\
&= -v^2 \left\langle \nabla h_{ij}, \boldsymbol{\omega} \right\rangle - v h_{ik} h_j^k + 2v^{-1} \nabla_i v \nabla_j v
\end{aligned}$$

and the evolution equation for v in the statement is a consequence of the following equality

$$\begin{aligned}
\dot{\mathcal{S}} g^* \left(Hess_{\nabla} v\right) &= -v^2 \left\langle \dot{\mathcal{S}} g^* \left(\nabla \mathcal{II}\right), \boldsymbol{\omega} \right\rangle + v \dot{\mathcal{S}} g^* \left(\mathcal{II}^2\right) + 2v^{-1} \dot{\mathcal{S}} g^* \left(\nabla v, \nabla v\right) \\
&= -v^2 \left\langle \nabla \mathcal{S}, \boldsymbol{\omega} \right\rangle + 2v^{-1} \dot{\mathcal{S}} g^* \left(\nabla v, \nabla v\right) + v \dot{\mathcal{S}} g^* \left(\mathcal{II}^2\right).
\end{aligned}$$

Finally the application of maximum principle (Theorem 4.2) ensures that v is nonincreasing, hence $\langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle$ is bounded from below as required. ■

Now we prove, using the evolution equations for the product $\mathcal{S}v$, that the function used as the speed of the flow is bounded for all times.

Proposition 4.4 *Under the assumptions of Theorem 4.1 the speed $\mathcal{S}$ of the flow (1.4) stay bounded for all times.*

Proof. From the evolution equation for $\mathcal{S}$ (1.7) and the evolution of v in the previous lemma we can compute the evolution of the quantity $\mathcal{S}v$:

$$\begin{aligned}
\frac{\partial}{\partial t}(\mathcal{S}v) &= v\dot{\mathcal{S}}g^*\left(Hess_\nabla \mathcal{S}\right) + \mathcal{S}\dot{\mathcal{S}}g^*\left(Hess_\nabla v\right) - 2v^{-1}\mathcal{S}\dot{\mathcal{S}}g^*\left(\nabla v, \nabla v\right) \\
&= \dot{\mathcal{S}}g^*\left(Hess_\nabla \mathcal{S}v\right) - 2\dot{\mathcal{S}}g^*\left(\nabla \mathcal{S}, \nabla v\right) - 2v^{-1}\mathcal{S}\dot{\mathcal{S}}g^*\left(\nabla v, \nabla v\right) \\
&= \dot{\mathcal{S}}g^*\left(Hess_\nabla \mathcal{S}v\right) - 2v^{-1}\dot{\mathcal{S}}g^*\left(\nabla\left(\mathcal{S}v\right), \nabla v\right).
\end{aligned}$$

Applying the maximum principle in Theorem 4.2 to this equation it follows immediately that $\mathcal{S}v$ is nonincreasing in time. Since $\langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle \leq 1$ the definition of v implies $v \geq 1$, hence by the assumption of linear growth we have $\max_{M_t}\mathcal{S} \leq \max_{M_t}\mathcal{S}v \leq \sup_{M_0}\mathcal{S}v$. ■

4.1 Evolution by powers of scalar curvatures

In this section we study the evolution of a graph by the speed $\mathcal{S} = R^p$ with $p \geq \frac{1}{2}$. As in the previous chapter we consider the function $f = \frac{|A|^2}{H^2} - \frac{1}{n}$ and assume a pinching condition on the principal curvatures bounding f from above with the constant γ defined in Theorem 3.7. In contrast to the previous chapter we assume that the inequality $f \leq \gamma$ holds in the weak sense: it means that M_0 is convex, but not necessarily compact, so that it is allowed to be an unbounded graph.

Proposition 4.5 *Let us consider the problem*

$$\begin{cases}
\frac{\partial}{\partial t}\mathbf{F}(\mathbf{p},t) = -R^p(\mathbf{p},t)\,\boldsymbol{\nu}(\mathbf{p},t), & t \geq 0 \\
\mathbf{F}(\cdot,0) = \mathbf{F}_0(M)
\end{cases}$$

with $p \geq \frac{1}{2}$. Assume the initial surface M_0 to be the graph of a smooth function u_0 such that $\|\nabla u_0\|_{C^{1,\alpha}} < \infty$. If on M_0 we have $H > 0$, $R > 0$ and the estimate $f \leq \gamma$ holds, where γ is defined as in Theorem 3.7, then the flow exists for all times.

Proof. The short time existence of the flow is assured by Theorem 4.1 and the first part of Proposition 3.23. Assume by contradiction that $[0, T)$ with $T < \infty$ is the maximum time interval of existence of the flow.

We know that if $f \leq \gamma$ on M_0 then $f \leq \gamma$ on M_t. In fact for $p = \frac{1}{2}$ it is an immediate consequence of Corollary 4.4 in [8], while for all $p > \frac{1}{2}$ it is stated in Corollary 3.10. From the previous proposition we deduce that $\max_{M_t} R^p$ is bounded, hence R is bounded for all $t \in [0, T)$. Moreover the inequality (3.18) implies that there exists a constant c_2 such that $H^2 \leq c_2 R$. Since $R = H^2 - |A|^2 > 0$ we have $|A|^2 \leq H^2 \leq c_2 R < \infty$, hence all the principal curvatures are bounded for all $t \in [0, T)$.

The long time existence is then obtained as a consequence of Lemma 3.16: the regularity of M_T allows us to define a new flow with initial surface M_T, but this contradicts the assumption on $[0, T)$ and proves the statement. ■

To conclude we give some examples of graphs evolving by powers of scalar curvature.

We consider in particular rotationally symmetric graphs

$$\mathbf{F}(\mathbf{p}) = (x_1, ..., x_n, u(r)) \qquad \text{where} \qquad r(\mathbf{x}) := |\mathbf{x}| = \sqrt{x_1^2 + ... + x_n^2},$$

for which it is possible to describe the profile more explicitly. If we denote by u_r and u_{rr} the first and second derivatives of u with respect of r, then the principal curvatures of the graph of u are

$$\lambda_1 = \frac{u_{rr}}{(1 + u_r^2)^{\frac{3}{2}}} \qquad \text{and} \qquad \lambda_k = \frac{u_r}{r(1 + u_r^2)^{\frac{1}{2}}} \quad \forall k = 2, ..., n$$

(see e.g. [17] paragraph 3C). For the scalar curvature the following formula holds

$$\begin{aligned} R &= 2(n-1)\lambda_1\lambda_n + (n-1)(n-2)\lambda_n^2 \\ &= (n-1)\lambda_n(2\lambda_1 + (n-2)\lambda_n) \\ &= \frac{n-1}{r^2(1+u_r^2)^2}\left[2ru_r u_{rr} + (n-2)u_r^2\left(1+u_r^2\right)\right], \end{aligned}$$

hence the evolution equation for u (3.22) becomes

$$\frac{\partial u}{\partial t} = \left(1+u_r^2\right)^{\frac{1}{2}-2p} \left\{ \frac{n-1}{r^2} \left[2ru_r u_{rr} + (n-2)\, u_r^2 \left(1+u_r^2\right)\right] \right\}^p . \tag{4.3}$$

Example 4.6 *Let us consider the flow (3.1), with $n \geq 4$ and $\frac{1}{2} \leq p \leq \frac{3n}{2(n+2)}$. If the initial surface M_0 is the graph of the function*

$$u_0(\mathbf{x}) = \sqrt{1+|\mathbf{x}|^2}$$

then it satisfies all the hypotheses of the previous proposition, hence there exists a unique smooth solution of the flow for all times.

Proof. First we check that u_0 satisfies the assumption $\nabla u_0 = \frac{\mathbf{x}}{\sqrt{1+|\mathbf{x}|^2}}$ bounded in $C^{1,\alpha}$ norm. If we set $r = |\mathbf{x}|$ and $u_0(\mathbf{x}) = w(r) = \sqrt{1+r^2}$, then

$$\nabla u_0 = w_r \nabla |\mathbf{x}| = w_r \frac{\mathbf{x}}{|\mathbf{x}|}.$$

Hence we can estimate $|\nabla u_0|$ with the derivatives of w with respect to r. We compute

$$w_r = \frac{r}{\sqrt{1+r^2}}, \qquad w_{rr} = \frac{1}{\sqrt{1+r^2}} - \frac{r^2}{\left(\sqrt{1+r^2}\right)^3} = \frac{1}{\left(\sqrt{1+r^2}\right)^3}$$

and $w_{rrr} = -\frac{3r}{\left(\sqrt{1+r^2}\right)^5}$, the boundedness of these functions for all $r \in [0, +\infty)$ implies the estimate on ∇u_0.

Moreover in order to apply the previous proposition we need to compute H, R and the function f.

Since $1 + w_r^2 = 1 + \frac{r^2}{1+r^2} = \frac{2r^2+1}{r^2+1}$, then the principal curvature of the surface M_0 are

$$\lambda_1 = \frac{1}{\left(\sqrt{2r^2+1}\right)^3} \qquad \text{and} \qquad \lambda_k = \frac{1}{\sqrt{2r^2+1}} \quad \forall k = 2, ..., n.$$

Thus we can compute the mean curvature

$$H = \lambda_1 + (n-1)\,\lambda_k = \frac{1}{\left(\sqrt{2r^2+1}\right)^3} + \frac{n-1}{\sqrt{2r^2+1}} = \frac{1+(n-1)\left(2r^2+1\right)}{\left(\sqrt{2r^2+1}\right)^3}$$

and the scalar curvature

$$R = (n-1)\,\lambda_n\,(2\lambda_1 + (n-2)\,\lambda_n) = \frac{n-1}{(2r^2+1)^2}\left[2+(n-2)\left(2r^2+1\right)\right]$$

are both positive. From equation (3.13) it follows that the function f on M_0 is given by

$$\begin{aligned} f &= \frac{n-1}{n} - \frac{R}{H^2} \\ &= \frac{n-1}{n} - \frac{(n-1)\left(2r^2+1\right)\left[2+(n-2)\left(2r^2+1\right)\right]}{\left[1+(n-1)\,(2r^2+1)\right]^2} \\ &= (n-1)\,\frac{1-2\left(2r^2+1\right)+\left(2r^2+1\right)^2}{n\left[1+(n-1)\,(2r^2+1)\right]^2} = \frac{4\,(n-1)\,r^4}{n\left[1+(n-1)\,(2r^2+1)\right]^2}. \end{aligned}$$

Note that for our choice of p and n the constants γ_1 and γ_2 are such that $\gamma_2 \geq \gamma_1$, thus the assumption $f \leq \gamma$ becomes $f \leq \frac{1}{n(n-1)}$. This condition is satisfied because

$$n\,(n-1)\,f = \frac{4\,(n-1)^2\,r^4}{\left[1+(n-1)\,(2r^2+1)\right]^2} \leq 1$$

is equivalent to $\frac{2(n-1)r^2}{1+(n-1)(2r^2+1)} \leq 1$. ■

The next example, instead concerns the 2-dimensional translating solutions of the flow (3.1).

Example 4.7 *The translating solutions of a flow are characterized by the condition* $\frac{\partial u}{\partial t} = 1$*. Then from the equation (4.3) we find*

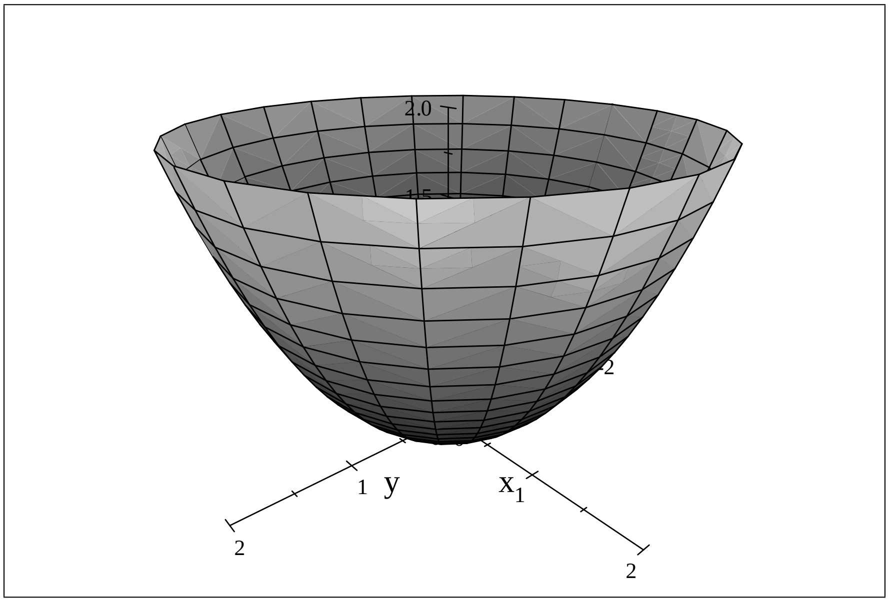
$$\left(1+u_r^2\right)^{\frac{1-4p}{2p}}\left[2ru_ru_{rr} + (n-2)\,u_r^2\left(1+u_r^2\right)\right] = \frac{r^2}{n-1}.$$

In dimension $n = 2$ we have an explicit solution:

$$\left(1+u_r^2\right)^{\frac{1-4p}{2p}}\left(1+u_r^2\right)_r = r.$$

For $p = \frac{1}{2}$ *this leads to* $\frac{\left(1+u_r^2\right)_r}{1+u_r^2} = r$, *then* $\log\left(1+u_r^2\right) = \frac{r^2}{2}$. *It follows that* u_r *satisfies* $u_r = \sqrt{\exp\left(\frac{r^2}{2}\right) - 1}$ *and the solution is*

$$u = \int \sqrt{\exp\left(\frac{r^2}{2}\right) - 1}dr.$$



For $p > \frac{1}{2}$ *we have*

$$\frac{2p}{1-2p}\left(1+u_r^2\right)^{\frac{1-2p}{2p}} - \frac{2p}{1-2p} = \frac{r^2}{2}$$

hence

$$1 + u_r^2 = \left(1 + \frac{1-2p}{4p}r^2\right)^{-\frac{2p}{2p-1}}$$

where the right hand side is well definite only if $r^2 < \frac{4p}{2p-1}$.

Then inside the disc of radius $r = \sqrt{\frac{4p}{2p-1}}$ *the function* u_r *is*

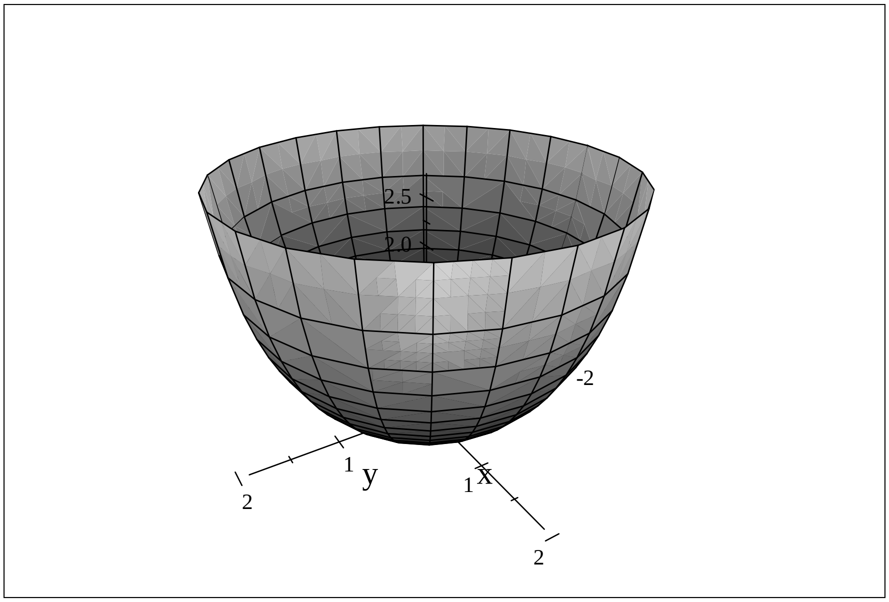
$$u_r = \sqrt{\left(1 - \frac{2p-1}{4p}r^2\right)^{\frac{2p}{1-2p}} - 1}.$$

Note that $\lim_{r \to \sqrt{\frac{4p}{2p-1}}} u_r$ *has the same behavior of* $\lim_{y \to 0} y^{-\frac{p}{2p-1}}$, *then* u_r *is an integrable function if and only if* $\frac{p}{2p-1} < 1$ *i.e.* $p > 1$.

This means that for any $p > \frac{1}{2}$ *the graph of* u *is contained in the cylinder of radius* $\sqrt{\frac{4p}{2p-1}}$*. In the case* $p > 1$ *the graph is bounded, whereas for* $\frac{1}{2} < p \leq 1$ *it is unbounded. In particular if* $p = 1$ *then*

$$u = \int \sqrt{\left(1 - \frac{1}{4}r^2\right)^{-2} - 1}dr$$

appear as follows.



Bibliography

[1] B. Andrews, *Contraction of convex hypersurfaces in Euclidean space*, Calc. Var. **2** (1994) 151-171.

[2] B. Andrews, *Gauss curvature flow: the fate of rolling stones*, Invent. Math. **138** (1999) 151-161.

[3] B. Andrews, *Fully nonlinear parabolic equations in two space variables*, arXiv:math.AP/0402235 vl (2004)

[4] B. Andrews, *Moving surfaces by non-concave curvature functions*, arXiv:math.DG/0402273 vl (2004).

[5] B. Andrews, *Pinching estimates and motion of hypersurfaces by curvature functions*, arXiv:math.DG/0402311 vl (2004).

[6] L. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations*, Ann. Math 130 (1989) 189-213.

[7] B. Chow, *Deforming convex hypersurfaces by the nth root of the Gaussian curvature*, J. Diff. Geom. **22** (1985) 117-138.

[8] B. Chow, *Deforming convex hypersurfaces by the square root of the scalar curvature*, Invent. Math. **87** (1987) 63-82.

[9] E. Di Benedetto, A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems*, J. reine angew. Math. **357** (1985) 1-22.

[10] K. Ecker, G. Huisken, *Mean curvature evolution of entire graphs*, Ann. Math. **130** (1989) 453-471.

[11] K. Ecker, G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991) 547-569.

[12] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982) 255-306.

[13] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Diff. Geom. **20** (1984) 237-266.

[14] G. Huisken, A. Polden, *Geometric evolution equation for hypersurfaces*, Calc. Var. and Geom. Evo. Probl. CIME, Lecture notes of Cetraro, Springer (1996).

[15] G. Huisken, C. Sinestrari, *Mean curvature flow singularities for mean convex surfaces*, Acta Math. **183** (1999) 45-70.

[16] G. Huisken, C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Calc. Var. **8** (1999) 1-14.

[17] W. Kühnel, *Differential geometry-Curves, surfaces, manifolds*, translated by B. Hunt, AMS Vol.16, 2000.

[18] N. V. Krylov, *Nonlinear elliptic and parabolic differential equations of second order*, D. Reidel, 1978.

[19] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser **16**, 1995.

[20] M. Protter, H. F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

[21] O. C. Schnürer, *Surfaces contracting with speed* $|A|^2$, J. Diff. Geom. **71** (2005) 347-363.

[22] F. Schulze, *Evolution of convex hypersurfaces by powers of the mean curvature*, Math. Zeit. **251** (2005) 721-733.

[23] F. Schulze, *Convexity estimates for flows by powers of the mean curvature*, Ann. Scuola Norm. Sup. Pisa Cl. Sci (5) Vol V (2006) 261-277.

[24] M. Spivak, *A comprehensive introduction to Differential Geometry*, Vol I-V, Wilmington, Del., 1979.

[25] K. Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature*, Comm. Pure App. Math **38** (1985) 867-882.