The Gilbert equation with dry-friction-type damping ⋆,⋆⋆

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Abstract

A modified Gilbert equation for micromagnetics is considered, obtained by augmenting the standard viscous-like dissipation with a rate-independent term. We prove existence of a weak solution both with and without viscous dissipation. By scaling time we show that, if the applied field varies very slowly, then gyromagnetic effects and viscous dissipation become negligible. In the infinitesimally-slow-loading limit, the system thus becomes fully rate-independent.

Key words: Micromagnetics, rate-independent dissipation, variational inequalities, weak solutions, energetic solutions.

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1 Introduction, dynamics of micromagnetism

The evolution of the *magnetization* vector m in rigid ferromagnets is standardly considered as governed by the Gilbert equation [16]:

$$\gamma^{-1}\dot{\mathsf{m}} = \mathsf{m}\times\Big(\mathsf{h}_{\text{eff}} - \mathsf{r}\Big). \tag{1.1}$$

Here $\dot{\mathsf{m}}$ denotes the time derivative of m, and "$\times$" is the vector product in $\mathbb{R}^3$. The constant $\gamma > 0$ is proportional to the gyromagnetic ratio. The *effective field* h_{eff} is the negative (partial) Gâteaux derivative of a possibly nonlocal and time dependent free energy $\mathcal{E}(t, \mathsf{m})$, i.e.:

$$\mathsf{h}_{\text{eff}}(t, \mathsf{m}) := -\frac{\partial}{\partial \mathsf{m}}\mathcal{E}(t, \mathsf{m}). \tag{1.2}$$

The relaxation field r is usually proportional to $\dot{\mathsf{m}}$ through a positive constant α. This kind of viscous-like friction effectively accounts for dissipation mechanisms that dominate at resonance or during relaxation; it is not clear, however, whether it is appropriate to capture the rate-independent response observed during quasistatic evolution, when the system, driven by a slowly-varying applied field, evolves through a series of states of equilibrium, alternated with a series of irregular random bursts, the so-called *Barkhausen jumps*, resulting from the pinning of domain walls by impurities and lattice imperfections.

Baltensperger and Helman suggested in [2] that rate-independent dissipation mechanisms may be phenomenologically accounted for by adding a *dry-friction*-like term to the standard Gilbert damping. Using the notion of subdifferential of a convex function, the prescription for the relaxation field proposed in [2] can be written as:

$$\mathsf{r} \in \partial R_{\alpha,\beta}(\dot{\mathsf{m}}) \qquad \text{where } R_{\alpha,\beta}(\mathsf{a}) := \frac{\alpha}{2}|\mathsf{a}|^2 + \beta|\mathsf{a}| \quad \forall \mathsf{a} \in \mathbb{R}^3. \tag{1.3}$$

Dry-friction dissipation was also proposed by Visintin in [36] as a device to model properly hysteresis in ferromagnets. Visintin [36] modified the Landau-Lifschitz equation [23] by augmenting the effective field h_{eff} with a maximally responsive term (in the sense of [14], i.e. having a positively homogeneous potential describing rate-independent dry-friction-like effects). Although the original Gilbert's and Landau-Lifschitz' equations are equivalent with each other, the resulting augmented equations proposed in [2] and [36] are no longer mutually equivalent. This has been pointed out by Podio-Guidugli in [29], where the conceptual differences between the Gilbert and the Landau-Lifschitz formats have been elucidated, and where several constitutive prescriptions, including (1.3), have been given a precise significance from the standpoint of Continuum Thermodynamics. From this standpoint, the non-negativity of α

and β is a requisite of consistency, in the sense of Coleman and Noll [10], with the Second Law of Thermodynamics.

The standard Gilbert equation with viscous dissipation (that is, with $\alpha > 0$ and $\beta = 0$) has been the object of an impressive amount of mathematical work. Here, we limit ourselves to mentioning a handful of references concerning existence [1,6,17,35], regularity [8,9,17,25] and qualitative behavior of solutions [18,19,38], and we refer to the survey [22] for a more detailed bibliographical account. However, the mathematical literature for micromagnetics with dry-friction-like dissipation appears to be much less developed [21,30,31,33,37].

In this paper we study existence of weak solutions to (1.1) with r given by (1.3), and we identify $\mathcal{E}$ with the following *Gibbs free energy*:

$$\mathcal{E}(t,\mathsf{m}) := \int_\Omega \frac{1}{2}\mu|\nabla\mathsf{m}(x)|^2 + \psi(\mathsf{m}(x)) - \mathsf{h}(t,x)\cdot\mathsf{m}(x)\,\mathrm{d}x, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded open set representing the region occupied by the ferromagnet, $\mu > 0$ is the exchange constant, $\psi : \mathbb{R}^3 \to \mathbb{R}$ is any smooth extension to $\mathbb{R}^3$ of the anisotropy energy (defined on the unit sphere), and h is a time dependent applied field.

In the integral on the right-hand side of (1.4) the first term accounts for exchange effect of quantum-mechanical origin and it penalizes spatial variations of the magnetization; the second term accounts for anisotropy effects which tend to align the magnetization with some favorite directions; the last term accounts for the interaction of the magnetization with the external magnetic field. For simplicity, we neglect the demagnetizing field, whose energetic contribution would not affect the main technical points of our proofs. We point out that the demagnetizing energy is mostly relevant for the explanation and description of magnetic microstructures [11,12].

The reader may consult [5,7] for a detailed explanation of the physical significance of (1.4). With the choice (1.4), the effective field (1.2) becomes

$$\mathsf{h}_{\mathrm{eff}}(t,\cdot) = \mu\Delta\mathsf{m} - \psi'(\mathsf{m}) + \mathsf{h}(t,\cdot), \tag{1.5}$$

where $\psi' : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the derivative of $\psi : \mathbb{R}^3 \to \mathbb{R}$.

The precise strong and weak formulations of the initial-boundary-value problem we study are given in Section 2. Here we point out two important features of solutions to (1.1), which are unaffected by the choice of the free energy or by that of the relaxation field. The first feature is that the norm of m is preserved during evolution. In particular, if $|\mathsf{m}(0,\cdot)| = 1$, then

$$|\mathsf{m}(t,x)| = 1 \quad \forall x \in \Omega, \forall t \geq 0, \tag{1.6}$$

which is the so-called *saturation* or *Heisenberg constraint.* Equation (1.6) can be obtained by taking the scalar product of both sides of (1.1) with m, which gives $\dot{\mathsf{m}} \cdot \mathsf{m} = 0$, whence $\frac{\partial}{\partial t}|\mathsf{m}|^2 = 0$. An even more important feature of (1.1) is its Lyapunov structure [29]. To get an insight, let us take the scalar product of both sides of (1.1) with $\mathsf{m}\times\dot{\mathsf{m}}$, and use the identity

$$(\mathsf{m}\times\mathsf{w}) \cdot (\mathsf{m}\times\mathsf{v}) = |\mathsf{m}|^2(\mathsf{w} \cdot \mathsf{v}) - (\mathsf{m} \cdot \mathsf{v})(\mathsf{m} \cdot \mathsf{w}), \tag{1.7}$$

along with (1.6) to get $(\mathsf{h}_{\rm eff} - \mathsf{r}) \cdot \dot{\mathsf{m}} = 0$. Integrating over Ω, using (1.4) and (1.5), and then applying the divergence theorem, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, \mathsf{m}(t)) + \int_\Omega \mathsf{r} \cdot \dot{\mathsf{m}} + \dot{\mathsf{h}} \cdot \mathsf{m} \,\mathrm{d}x - \int_{\partial\Omega} \frac{\partial\mathsf{m}}{\partial\mathbf{n}} \cdot \dot{\mathsf{m}} \,\mathrm{d}S = 0, \tag{1.8}$$

where $\frac{\partial\mathsf{m}}{\partial\mathbf{n}}$ is the directional derivative of m with respect to the unit outward normal $\mathbf{n}$ at the boundary $\partial\Omega$. Now, observe that, from (1.1), setting $\mathsf{w} = \gamma(\mathsf{h}_{\rm eff} - \mathsf{r})$, we have

$$\frac{\partial\mathsf{m}}{\partial\mathbf{n}} \cdot \dot{\mathsf{m}} = \frac{\partial\mathsf{m}}{\partial\mathbf{n}} \cdot (\mathsf{m}\times\mathsf{w}) = -\Big(\mathsf{m}\times\frac{\partial\mathsf{m}}{\partial\mathbf{n}}\Big) \cdot \mathsf{w} \tag{1.9}$$

(the last equality uses a standard property of the mixed product). Thus, if (1.1) is complemented by the boundary condition $\mathsf{m}\times\frac{\partial\mathsf{m}}{\partial\mathbf{n}} = 0$ on $\partial\Omega$, we have from (1.8)-(1.9), using also (1.3), the following energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\mathcal{E}(t, \mathsf{m}(t))}_{\substack{\text{Gibbs' energy}\\ \text{at time } t}} + \int_\Omega \underbrace{\alpha|\dot{\mathsf{m}}|^2 + \beta|\dot{\mathsf{m}}|}_{\substack{\text{specific dissi-}\\ \text{pation rate}}} \,\mathrm{d}x = -\int_\Omega \underbrace{\dot{\mathsf{h}} \cdot \mathsf{m}}_{\substack{\text{"dual" power}\\ \text{of external forcing}}} \,\mathrm{d}x; \tag{1.10}$$

the adjective "dual" refers to the fact that this power contributes the time variation of Gibbs' energy, as opposed to the standard power which contributes to the time derivative of the Helmholtz' free energy $\mathsf{h} \cdot \dot{\mathsf{m}}$. Infinitesimal variations of m consistent with the condition $|\mathsf{m}| = 1$ have the form $\mathsf{m}\times\mathsf{v}$, with v an arbitrary vector field. These are the natural "test functions" for (1.1). Indeed, testing (1.1) by $\mathsf{m}\times\mathsf{v}$, using (1.5), and employing the identity (1.7) we find

$$-\gamma^{-1}\mathsf{m}\times\dot{\mathsf{m}} = \mu\Delta\mathsf{m} - \psi'(\mathsf{m}) + \mathsf{h} - \mathsf{r} + \lambda\mathsf{m}, \tag{1.11}$$

where λ is the Lagrange multiplier associated to the constraint (1.6). This multiplier is needed because, unlike (1.1), equation (1.11) does not imply (1.6). From the standpoint of Dynamic Micromagnetics, (1.11) is a balance between doublet forces [13]. In particular, the left-hand side of (1.11) is an inertial doublet force that expends null power over actual motions of the system. The fact that the energetic balance (1.10) does not contain γ^{-1} is a consequence of this null expenditure of power.

Due to the non-smoothness of the constitutive prescription (1.3) for the relaxation field, the notions of weak solution provided, for instance, in [1] and [6] cannot be used. Instead, one must formulate (1.1) as a variational inequality; see Section 2, Proposition 2.2 and Definition 2.4. In particular, the case $\alpha = 0$ needs a weak formulation of its own; see Section 3, Theorem 3.2. In fact, if $\alpha = 0$ the best regularity we can expect is that m be a function of bounded variation in time, as suggested by the estimate (1.10). In this case, the dissipation must be expressed in terms of an appropriate notion of variation of m, which we provide in (3.7) below. This having been said, we point out that the strategy we adopt to prove existence of weak solutions in Theorem 3.2 is inspired by [6]: we penalize the non-convex constraint $|\mathsf{m}| = 1$ and, in order to have sufficient compactness to handle the resulting additional nonlinearity, we augment the relaxation field with an exchange-type dissipation $\varepsilon\Delta\dot{\mathsf{m}}$. Then, we pass to the limit as $\varepsilon \to 0$. This regularization itself is physically motivated, namely, as discussed also in [29], it may be interpreted as a physically relevant "dissipative counterpart" of the energy-storing mechanism associated to exchange interactions.

The asymptotic behavior of solutions when $\alpha \to 0$, or $\gamma^{-1} \to 0$, or both, is discussed in Section 3, where we also discuss existence of weak solutions for the corresponding limit cases. In particular, the limit $\alpha \to 0$ and $\gamma^{-1} \to 0$ can be interpreted as infinitesimally slowing the loading rate, and we will show that indeed all rate-dependent effects disappear in the limit. The limit $\alpha = 0$ and $\gamma^{-1} = 0$ itself fits within the theory of rate-independent processes proposed by Mielke and Theil in [28]. In this case it is possible to prove existence of a special class of weak solutions, the so-called energetic solutions, which are particularly suitable for handling nonlinear problems and for performing numerical calculations. Energetic solutions are only a subset of the class of weak solutions. Moreover, explicit examples (in the contest of crack propagation) provided in [20,34] show that the weak solution obtained by the limit of some viscous parameter is different from the energetic solution, and suggest that, also in the present contest, the weak solution obtained by taking the simultaneous limit $\alpha \to 0$ and $\gamma^{-1} \to 0$ may differ from the energetic solution. As uniqueness of the energetic solution still cannot be expected because of the non-convex constraint $|\mathsf{m}| = 1$, also uniqueness of weak solutions cannot be expected.

2 The model and its weak solutions.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded Lipschitz domain and let $T > 0$ be a fixed time horizon. We use the following notation:

$$I := (0,T), \qquad \bar{I} := [0,T], \qquad Q := I \times \Omega.$$

Given a Banach space X, we denote by $C_{\rm w}(\bar{I};X)$, $BV(\bar{I};X)$, and $BM(\bar{I};X)$ respectively the space of weakly continuous functions, functions with bounded variation, and the space of bounded measurable functions $\bar{I}\to X$. For $p\geq 1$, we also denote by $L^p(I;X)$ the space of L^p-Bochner integrable functions $I\to X$. We denote by $W^{1,p}(I;X)$ the corresponding Sobolev space. We also denote by m_i, $i=1\dots 3$ the Cartesian components of m.

In this section we consider the following initial-boundary-value problem:

$$\left.\begin{aligned}&\gamma^{-1}\dot{\mathsf{m}}=\mathsf{m}\times\left(\mu\Delta\mathsf{m}-\psi'(\mathsf{m})+\mathsf{h}-\mathsf{r}\right)\\&\mathsf{r}\in\partial R_{\alpha,\beta}(\dot{\mathsf{m}})\end{aligned}\right\}\quad\text{in } Q, \tag{2.1}$$

$$\mathsf{m}\times\frac{\partial\mathsf{m}}{\partial\mathbf{n}}=0 \qquad\text{on } I\times\partial\Omega, \tag{2.2}$$

$$\mathsf{m}(0,\cdot)=\mathsf{m}_0(\cdot) \qquad\text{in } \Omega, \tag{2.3}$$

where the pseudopotential $R_{\alpha,\beta}$ has been defined in (1.3). In this section we make the following assumptions:

$$\alpha>0,\ \beta>0,\ \gamma^{-1}>0,\ \mu>0; \tag{2.4}$$

$$\psi:\mathbb{R}^3\to\mathbb{R}\text{ is convex and of class } C^1; \tag{2.5}$$

$$\exists C_\psi>0\ \forall\mathsf{a}\in\mathbb{R}^3:\quad \psi(\mathsf{a})\geq C_\psi(1+|\mathsf{a}|^6)\ \text{ and }\ |\psi'(\mathsf{a})|\leq C_\psi(1+|\mathsf{a}|^5); \tag{2.6}$$

$$\mathsf{m}_0\in W^{1,2}(\Omega;\mathbb{R}^3)\ \text{ and }\ |\mathsf{m}_0|=1\ \text{ a.e. in }\Omega; \tag{2.7}$$

$$\mathsf{h}\in W^{1,1}(I;L^1(\Omega;\mathbb{R}^3)). \tag{2.8}$$

Remark 2.1. Typical examples of anisotropy energies conventionally used in models of micromagnetics are $\psi(\mathsf{m})=\psi_{\rm u}(\mathsf{m}):=K(m_1^2+m_2^2)$ with $K>0$ for uniaxial anisotropy and $\psi(\mathsf{m})=\psi_{\rm c}(\mathsf{m}):=K(m_1^2m_2^2+m_1^2m_3^2+m_2^2m_3^2)$ for cubic anisotropy with either 3 axis (for $K>0$) or 4 axis (for $K<0$) of easy magnetization [5]. Neither $\psi_{\rm u}$ nor $\psi_{\rm c}$ satisfy the convexity, coercivity and growth assumptions in (2.5) and (2.6). However, due to (1.6), and since $\mathsf{m}\times\psi'(\mathsf{m})$ is perpendicular to m, the right-hand side of (2.1) *depends only on the tangential derivative* of ψ on the unit sphere. Thus, we can replace $\psi_{\rm u}$ and $\psi_{\rm c}$ by $\psi_{\rm u}(\mathsf{m})+|K|(|\mathsf{m}|^2+|\mathsf{m}|^6)$ and $\psi_{\rm c}(\mathsf{m})+|K|(|\mathsf{m}|^2+|\mathsf{m}|^6)$ respectively, which satisfy (2.5)-(2.6).

The notion of a weak solution we are going to introduce stands on the following characterization of smooth solutions to (2.1)-(2.3):

Proposition 2.2. *Assume* $\mathsf{m}\in C^2(\overline{Q};\mathbb{R}^3)$ *satisfies the initial condition* (2.3) *with* $|\mathsf{m}_0|=1$. *Then* m *satisfies* (2.1) *in* Q *and the boundary condition* (2.2) *if and only if* $|\mathsf{m}|=1$ *in* Q *and*

$$\begin{aligned}&\iint_Q R_{\alpha,\beta}(\mathsf{m}\times\mathsf{v})-\mu(\mathsf{m}\divideontimes\nabla\mathsf{m}):\nabla\mathsf{v}-\Big(\gamma^{-1}\dot{\mathsf{m}}+\mathsf{m}\times\psi'(\mathsf{m})-\mathsf{m}\times\mathsf{h}\Big)\cdot\mathsf{v}\,\mathrm{d}x\,\mathrm{d}t\\&\geq\iint_Q R_{\alpha,\beta}(\dot{\mathsf{m}})\,\mathrm{d}x\,\mathrm{d}t+\mathcal{E}(T,\mathsf{m}(T,\cdot))-\mathcal{E}(0,\mathsf{m}(0,\cdot))+\iint_Q\dot{\mathsf{h}}\cdot\mathsf{m}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{2.9}$$

for every test $\mathsf{v} \in C^1(\overline{Q};\mathbb{R}^3)$*, with* $\mathcal{E}(t,\mathsf{m})$ *as in* (1.4) *and with the algebraic operator "*$\divideontimes$*" meaning that, for a matrix* $\mathsf{A} \in \mathbb{R}^{3\times3}$*, the expression* $\mathsf{m}\divideontimes\mathsf{A} \in \mathbb{R}^{3\times3}$ *is defined by*

$$(\mathsf{m}\divideontimes\mathsf{A})\,\mathsf{a} := \mathsf{m}\times(\mathsf{A}\mathsf{a}) \qquad \forall \mathsf{a} \in \mathbb{R}^3. \tag{2.10}$$

Remark 2.3. We will use $\mathsf{m}\divideontimes\mathsf{A}$ from (2.10) always only for $A = \nabla\mathsf{m}$ as in (2.9). We draw the reader's attention to the following interesting calculus needed in what follows:

$$\nabla(\mathsf{m}\times\mathsf{v}) = \mathsf{m}\divideontimes\nabla\mathsf{v} - \mathsf{v}\divideontimes\nabla\mathsf{m}, \tag{2.11}$$

$$\nabla\mathsf{m} : (\mathsf{m}\divideontimes\nabla\mathsf{v}) = -(\mathsf{m}\divideontimes\nabla\mathsf{m}) : \nabla\mathsf{v}, \tag{2.12}$$

as well as to the integration-by-parts formula

$$\int_\Omega (\mathsf{m}\times\Delta\mathsf{m})\cdot\mathsf{v}\,\mathrm{d}x = -\int_\Omega (\mathsf{m}\divideontimes\nabla\mathsf{m}) : \nabla\mathsf{v}\,\mathrm{d}x + \int_{\partial\Omega} \mathsf{m}\times\frac{\partial\mathsf{m}}{\partial\mathbf{n}}\cdot\mathsf{v}\,\mathrm{d}x. \tag{2.13}$$

The latter follows from the identity $(\mathsf{m}\times\Delta\mathsf{m})\cdot\mathsf{v} = -\Delta\mathsf{m}\cdot(\mathsf{m}\times\mathsf{v})$, along with (2.11), (2.12), and the divergence theorem.

Proof of Proposition 2.2.
(i) The "only if" implication. Assume that $\mathsf{m} \in C^2(\overline{Q};\mathbb{R}^3)$ is a classical solution of (2.1)-(2.3). From (2.1), we have $\mathsf{m}\cdot\dot{\mathsf{m}} = 0$ hence $|\mathsf{m}| = 1$ in Q because $|\mathsf{m}_0| = 1$. Let $\mathsf{v} \in C^1(\overline{Q};\mathbb{R}^3)$ and define

$$\mathsf{w} := \gamma\Big(\mu\Delta\mathsf{m} - \psi'(\mathsf{m}) + \mathsf{h} - \mathsf{r}\Big), \tag{2.14}$$

so that, by (2.1),

$$\dot{\mathsf{m}} = \mathsf{m}\times\mathsf{w}. \tag{2.15}$$

Taking the vector product of both sides of (2.14) with m, we obtain, rearranging terms,

$$\mathsf{m}\times(\gamma^{-1}\mathsf{w} - \mu\Delta\mathsf{m} + \psi'(\mathsf{m}) - \mathsf{h}) = -\mathsf{m}\times\mathsf{r}. \tag{2.16}$$

Taking the scalar product of both sides of (2.16) with $\mathsf{v}-\mathsf{w}$, and making use of the identity $(\mathsf{a}\times\mathsf{b})\cdot\mathsf{c} = -\mathsf{b}\cdot(\mathsf{a}\times\mathsf{c})$, we obtain

$$-(\gamma^{-1}\mathsf{w} - \mu\Delta\mathsf{m} + \psi'(\mathsf{m}) - \mathsf{h})\cdot(\mathsf{m}\times\mathsf{v} - \mathsf{m}\times\mathsf{w}) = \mathsf{r}\cdot(\mathsf{m}\times\mathsf{v} - \mathsf{m}\times\mathsf{w}). \tag{2.17}$$

By (2.1) and (2.15) $\mathsf{r} \in \partial R_{\alpha,\beta}(\mathsf{m}\times\mathsf{w})$ and from (2.17), using the definition of subdifferential, we have

$$R_{\alpha,\beta}(\mathsf{m}\times\mathsf{v}) - R_{\alpha,\beta}(\mathsf{m}\times\mathsf{w}) \geq -(\gamma^{-1}\mathsf{w} - \mu\Delta\mathsf{m} + \psi'(\mathsf{m}) - \mathsf{h})\cdot(\mathsf{m}\times\mathsf{v} - \mathsf{m}\times\mathsf{w}). \tag{2.18}$$

We integrate (2.18) over Q to obtain:

$$\iint_Q -\mathsf{m}\times(\gamma^{-1}\mathsf{w} - \mu\Delta\mathsf{m} + \psi'(\mathsf{m}) - \mathsf{h}) \cdot \mathsf{v} + R_{\alpha,\beta}(\mathsf{m}\times\mathsf{v})\,\mathrm{d}x\,\mathrm{d}t$$
$$\geq \iint_Q (-\mu\Delta\mathsf{m} + \psi'(\mathsf{m}) - \mathsf{h}) \cdot (\mathsf{m}\times\mathsf{w}) + R_{\alpha,\beta}(\mathsf{m}\times\mathsf{w})\,\mathrm{d}x\,\mathrm{d}t. \tag{2.19}$$

By (2.15), we replace $\mathsf{m}\times\mathsf{w}$ with $\dot{\mathsf{m}}$ in both sides of the previous inequality. Using then the identity (2.13) with the boundary condition (2.2), we see that the left-hand side of (2.19) coincides with the left-hand side of (2.9); while the definition of $\mathcal{E}$ (1.4) provides the coincidence of the right-hand side of (2.19) with the right-hand side of (2.9).

(ii) The "if" implication. It follows from the assumption $|\mathsf{m}| = 1$ that there exists $\tilde{\mathsf{w}} \in C^2(\overline{Q}; \mathbb{R}^3)$ such that (2.15) holds with w replaced by $\tilde{\mathsf{w}}$. Let

$$\tilde{\mathsf{r}} := -P_{\mathsf{m}}(\gamma^{-1}\tilde{\mathsf{w}} - \mu\Delta\mathsf{m} + \psi'(\mathsf{m}) - \mathsf{h}), \tag{2.20}$$

where $P_{\mathsf{a}}(\mathsf{v}) := -\mathsf{a}\times(\mathsf{a}\times\mathsf{v})$ is the orthogonal projector on the 2-dimensional linear subspace perpendicular to $\mathsf{a} \in \mathbb{R}^3$; note that P_{m} in (2.20) depends on (t, x) since $\mathsf{m} = \mathsf{m}(t, x)$. Since $\forall \mathsf{v} \in \mathbb{R}^3$, $\mathsf{m}\times\mathsf{v}$ and $\mathsf{m}\times\tilde{\mathsf{w}}$ are orthogonal to m, it is immediate from (2.20) that

$$-(\gamma^{-1}\tilde{\mathsf{w}} - \mu\Delta\mathsf{m} + \psi'(\mathsf{m}) - \mathsf{h}) \cdot (\mathsf{m}\times\mathsf{v} - \mathsf{m}\times\tilde{\mathsf{w}}) = \tilde{\mathsf{r}} \cdot (\mathsf{m}\times\mathsf{v} - \mathsf{m}\times\tilde{\mathsf{w}}), \tag{2.21}$$

hence by the arbitrariness of v one verifies that $\mathsf{m}\times(\gamma^{-1}\tilde{\mathsf{w}} - \mu\Delta\mathsf{m} + \psi'(\mathsf{m}) - \mathsf{h}) = -\mathsf{m}\times\tilde{\mathsf{r}}$, and therefore, by $\dot{\mathsf{m}} = \mathsf{m}\times\tilde{\mathsf{w}}$,

$$\gamma^{-1}\dot{\mathsf{m}} = \mathsf{m}\times\Big(\mu\Delta\mathsf{m} - \psi'(\mathsf{m}) + \mathsf{h} - \tilde{\mathsf{r}}\Big). \tag{2.22}$$

Comparing (2.22) and (2.1), we see that the proof is concluded if we show that

$$\tilde{\mathsf{r}} \in \partial R_{\alpha,\beta}(\dot{\mathsf{m}}). \tag{2.23}$$

Starting from (2.9), and reversing the argument that leads from (2.18) to (2.9), we obtain (2.18) with w replaced by $\tilde{\mathsf{w}}$, namely:

$$R_{\alpha,\beta}(\mathsf{m}\times\mathsf{v}) - R_{\alpha,\beta}(\mathsf{m}\times\tilde{\mathsf{w}}) \geq -(\gamma^{-1}\tilde{\mathsf{w}} - \mu\Delta\mathsf{m} + \psi'(\mathsf{m}) - \mathsf{h}) \cdot (\mathsf{m}\times\mathsf{v} - \mathsf{m}\times\tilde{\mathsf{w}}). \tag{2.24}$$

Combining (2.24) with (2.21), and using the fact that $\tilde{\mathsf{r}} \cdot \lambda\mathsf{m} = 0$, we find that, for all $\lambda \in \mathbb{R}$,

$$R_{\alpha,\beta}(\mathsf{m}\times\mathsf{v}) - R_{\alpha,\beta}(\mathsf{m}\times\tilde{\mathsf{w}}) \geq \tilde{\mathsf{r}} \cdot (\mathsf{m}\times\mathsf{v} + \lambda\mathsf{m}) - \tilde{\mathsf{r}} \cdot (\mathsf{m}\times\tilde{\mathsf{w}}). \tag{2.25}$$

From the mutual orthogonality in $\mathbb{R}^3$ of $\mathsf{m}\times\mathsf{v}$ and $\lambda\mathsf{m}$ it follows that $|\mathsf{m}\times\mathsf{v} + \lambda\mathsf{m}| \geq |\mathsf{m}\times\mathsf{v}|$; moreover, by (1.3), $R_{\alpha,\beta}(\mathsf{z})$ is monotone increasing with respect to $|\mathsf{z}|$; hence

$$R_{\alpha,\beta}(\mathsf{m}\times\mathsf{v} + \lambda\mathsf{m}) - R_{\alpha,\beta}(\mathsf{m}\times\mathsf{v}) \geq 0. \tag{2.26}$$

As λ varies in $\mathbb{R}$ and v varies in $\mathbb{R}^3$, the vectors $\mathsf{m}\times\mathsf{v}+\lambda\mathsf{m}$ span all $\mathbb{R}^3$, hence from (2.25), (2.26), and the identity $\dot{\mathsf{m}}=\mathsf{m}\times\tilde{\mathsf{w}}$ we obtain the inclusion (2.23), as claimed. □

Definition 2.4 (Weak solutions with $\alpha>0$). *Assume that* (2.4)-(2.8) *hold. We say that* $\mathsf{m}\in C_{\mathrm{w}}(\bar{I};W^{1,2}(\Omega;\mathbb{R}^3))\cap W^{1,2}(\bar{I};L^2(\Omega;\mathbb{R}^3))$ *is a weak solution to* (2.1) *with boundary conditions* (2.2) *and initial conditions* (2.3) *if:*

(i) m *satisfies* (2.9) *for all* $\mathsf{v}\in C^1(\overline{Q};\mathbb{R}^3)$*;*

(ii) $|\mathsf{m}|=1$ *a.e. in* Q*;*

(iii) $\mathsf{m}(0,\cdot)=\mathsf{m}_0$.

For all $\mathsf{w}\in L^2(\Omega;\mathbb{R}^3)$ we define

$$\mathcal{R}_{\alpha,\beta}(\mathsf{w}):=\int_\Omega R_{\alpha,\beta}(\mathsf{w})\,\mathrm{d}x=\int_\Omega\frac{\alpha}{2}|\mathsf{w}|^2+\beta|\mathsf{w}|\,\mathrm{d}x. \tag{2.27}$$

Proposition 2.5. *Under the assumptions in Definition 2.4, problem* (2.1)-(2.3) *has a weak solution* $\mathsf{m}\in C_{\mathrm{w}}(\bar{I};W^{1,2}(\Omega;\mathbb{R}^3))\cap W^{1,2}(\bar{I};L^2(\Omega;\mathbb{R}^3))$*. Moreover, for every* $s\in\bar{I}$*,*

$$\begin{aligned}&\int_0^s\mathcal{R}_{\alpha,\beta}(\mathsf{m}\times\mathsf{v})\,\mathrm{d}t-\int_0^s\!\!\int_\Omega\mu(\mathsf{m}\divideontimes\nabla\mathsf{m}){:}\nabla\mathsf{v}+\Big(\gamma^{-1}\dot{\mathsf{m}}+\mathsf{m}\times\psi'(\mathsf{m})-\mathsf{m}\times\mathsf{h}\Big)\cdot\mathsf{v}\,\mathrm{d}x\,\mathrm{d}t\\&\geq\int_0^s\mathcal{R}_{\alpha,\beta}(\dot{\mathsf{m}})\,\mathrm{d}t+\mathcal{E}(s,\mathsf{m}(s,\cdot))-\mathcal{E}(0,\mathsf{m}(0,\cdot))+\int_0^s\!\!\int_\Omega\dot{\mathsf{h}}\cdot\mathsf{m}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{2.28}$$

for all $\mathsf{v}\in C^1(\overline{Q};\mathbb{R}^3)$*, and there exists a constant* $C_0>0$*, which does not depend on* α *and* γ*, such that*

$$\|\mathsf{m}\|_{L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3))}\leq C_0, \tag{2.29}$$

$$\|\dot{\mathsf{m}}\|_{L^1(Q;\mathbb{R}^3)}\leq C_0, \tag{2.30}$$

$$\|\dot{\mathsf{m}}\|_{L^2(Q;\mathbb{R}^3)}\leq\frac{C_0}{\sqrt{\alpha}}\,. \tag{2.31}$$

Before giving the proof, which for the reader's convenience will be divided in seven steps, let us first outline the strategy we follow. In the first three steps, we prove the existence of some m_ε solving the following "penalized" and "regularized" version of (1.11):

$$\left.\begin{aligned}&\gamma^{-1}\mathsf{m}_\varepsilon\times\dot{\mathsf{m}}_\varepsilon+\mu\Delta\mathsf{m}_\varepsilon-\psi'(\mathsf{m}_\varepsilon)-\tfrac{1}{\varepsilon}\xi'(\mathsf{m}_\varepsilon)+\mathsf{h}=\mathsf{r}_\varepsilon\\&\mathsf{r}_\varepsilon\in\partial R_{\alpha,\beta}(\dot{\mathsf{m}}_\varepsilon)-2\varepsilon\Delta\dot{\mathsf{m}}_\varepsilon\end{aligned}\right\}\text{ in }Q, \tag{2.32}$$

with the boundary condition

$$\frac{\partial\mathsf{m}_\varepsilon}{\partial\mathbf{n}}=0\quad\text{on }I\times\partial\Omega, \tag{2.33}$$

where

$$\xi(\mathsf{a}) := (1 - |\mathsf{a}|^2)^2 \qquad \forall \mathsf{a} \in \mathbb{R}^3, \tag{2.34}$$

and $0 < \varepsilon \ll 1$. We emphasize that (2.32) is a classical formulation and later will be treated only in a weak form. In the fourth step, we perform a test of (2.32) by $\dot{\mathsf{m}}_\varepsilon$ to obtain energy estimates which yield the uniform bounds (2.67)-(2.70) displayed below. In the fifth step, integrating by parts on $(0, s)$, and using the strict positivity of the regularizing term $\varepsilon|\nabla\dot{\mathsf{m}}_\varepsilon|^2$ and of the penalization term $\frac{1}{\varepsilon}\xi(\mathsf{m}_\varepsilon)$, we obtain from (2.32) that the weak form of (2.32)–(2.33), i.e. inequality (2.71) below, holds true. Then we select $\mathsf{z} := \mathsf{m}_\varepsilon \times \mathsf{v}$, a choice that allows us to get rid of the term $\frac{1}{\varepsilon}\xi'(\mathsf{m}_\varepsilon)$ and, in the sixth step, we let ε tend to 0 and we show that m_ε converges to a limit m that satisfies (2.28). All the previous steps will be done by considering a $W^{1,2}(I; L^2(\Omega; \mathbb{R}^3))$-regularization of $\mathsf{h} \in W^{1,1}(I; L^1(\Omega; \mathbb{R}^3))$ from (2.8). In the seventh and final step, we will make a limit passage for such h to get rid of this regularization.

Proof of Proposition 2.5.

Step 1: time-discrete problems. Let $\mathsf{h}_n \in W^{1,2}(I; L^2(\Omega; \mathbb{R}^3))$ with $\mathsf{h}_n \to \mathsf{h}$ in $W^{1,1}(I; L^1(\Omega; \mathbb{R}^3))$. To obtain a solution to (2.32), we use the Rothe method. We fix $\varepsilon > 0$ and $n \in \mathbb{N}$, and introduce a uniform discretization of the time interval $\bar{I} = [0, T]$ with a time step $\tau = T/N$, with $N \in \mathbb{N}$. For every $\mathsf{w} \in W^{1,2}(\Omega; \mathbb{R}^3)$ we define

$$\mathcal{R}^\varepsilon_{\alpha,\beta}(\mathsf{w}) := \int_\Omega R_{\alpha,\beta}(\mathsf{w}) + \varepsilon|\nabla\mathsf{w}|^2 \,\mathrm{d}x = \mathcal{R}_{\alpha,\beta}(\mathsf{w}) + \int_\Omega \varepsilon|\nabla\mathsf{w}|^2 \,\mathrm{d}x. \tag{2.35}$$

We look for an approximating solution of (2.32). To construct such solution, we let $\mathsf{m}^0_\tau = \mathsf{m}_0$ and by recursion we look for a solution $\mathsf{m}^k_\tau \in W^{1,2}(\Omega; \mathbb{R}^3)$ of the following variational inequality:

$$\begin{aligned}\mathcal{R}^\varepsilon_{\alpha,\beta}(\mathsf{w}) - \mathcal{R}^\varepsilon_{\alpha,\beta}\Big(\frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau}\Big) \geq - \int_\Omega \mu\nabla\mathsf{m}^k_\tau : \nabla\Big(\mathsf{w} - \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau}\Big) \,\mathrm{d}x \\ + \int_\Omega \Big(\gamma^{-1}\mathsf{m}^{k-1}_\tau \times \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau} + \mathsf{h}^k_\tau - \psi'(\mathsf{m}^k_\tau) - \frac{1}{\varepsilon}\xi'(\mathsf{m}^k_\tau)\Big) \cdot \\ \cdot \Big(\mathsf{w} - \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau}\Big) \,\mathrm{d}x,\end{aligned} \tag{2.36}$$

for all $\mathsf{w} \in C^1(\overline{\Omega}; \mathbb{R}^3)$. Here the time samples $\mathsf{h}^k_\tau := \mathsf{h}_n(k\tau)$ are well-defined thanks to the qualification $\mathsf{h}_n \in W^{1,2}(I; L^2(\Omega; \mathbb{R}^3))$. Existence of at least one solution $\mathsf{m}^k_\tau \in W^{1,2}(\Omega; \mathbb{R}^3)$ follows standardly by monotonicity arguments, compactness of lower order terms, and coercivity.

Step 2: a priori estimates. Let us define:

$$\mathcal{E}_\tau^k(\mathsf{w}) := \int_\Omega \frac{1}{2}\mu|\nabla \mathsf{w}|^2 + \psi(\mathsf{w}) - \mathsf{h}_\tau^k \cdot \mathsf{w}\,\mathrm{d}x \qquad \forall \mathsf{w} \in W^{1,2}(\Omega;\mathbb{R}^3). \tag{2.37}$$

For the remaining part of this step it is convenient to split the function ξ, given by (2.34), by

$$\xi = \xi_1 + \xi_2 \qquad \text{with } \xi_1 := \mathrm{co}(\xi) \text{ and } \xi_2 := \xi - \xi_1 \tag{2.38}$$

where $\mathrm{co}(\xi)$ denotes the convexification of ξ. In case of (2.34), we have simply $\xi_1(\mathsf{a}) = \xi(\mathsf{a})$ for $|\mathsf{a}| > 1$, while $\xi_1(\mathsf{a}) = 0$ for $|\mathsf{a}| \le 1$. We are going to use the facts that ξ_1 is convex and ξ_2' is bounded.

Observe that

$$\nabla \mathsf{m}_\tau^k : (\nabla \mathsf{m}_\tau^k - \nabla \mathsf{m}_\tau^{k-1}) \ge \frac{1}{2}|\nabla \mathsf{m}_\tau^k|^2 - \frac{1}{2}|\nabla \mathsf{m}_\tau^{k-1}|^2, \tag{2.39}$$

$$(\psi'(\mathsf{m}_\tau^k) + \xi_1'(\mathsf{m}_\tau^k)) \cdot (\mathsf{m}_\tau^k - \mathsf{m}_\tau^{k-1}) \ge \psi(\mathsf{m}_\tau^k) + \xi_1(\mathsf{m}_\tau^k) - \psi(\mathsf{m}_\tau^{k-1}) - \xi_1(\mathsf{m}_\tau^{k-1}), \tag{2.40}$$

$$\mathsf{h}_\tau^k \cdot (\mathsf{m}_\tau^k - \mathsf{m}_\tau^{k-1}) = \mathsf{h}_\tau^k \cdot \mathsf{m}_\tau^k - \mathsf{h}_\tau^{k-1} \cdot \mathsf{m}_\tau^{k-1} - (\mathsf{h}_\tau^k - \mathsf{h}_\tau^{k-1}) \cdot \mathsf{m}_\tau^{k-1}, \tag{2.41}$$

where (2.40) follows from the convexity of ψ, which we assume in (2.5), and of ξ_1 (cf. (2.38)). Also, note that

$$\Big(\mathsf{m}_\tau^{k-1} \times \frac{\mathsf{m}_\tau^k - \mathsf{m}_\tau^{k-1}}{\tau}\Big) \cdot \frac{\mathsf{m}_\tau^k - \mathsf{m}_\tau^{k-1}}{\tau} = 0. \tag{2.42}$$

Using (2.39)–(2.42), and recalling (2.37)–(2.38), we obtain the following inequality:

$$\begin{aligned}
&\int_\Omega \mu \nabla \mathsf{m}_\tau^k : \nabla \frac{\mathsf{m}_\tau^k - \mathsf{m}_\tau^{k-1}}{\tau} - \gamma^{-1} \mathsf{m}_\tau^{k-1} \times \frac{\mathsf{m}_\tau^k - \mathsf{m}_\tau^{k-1}}{\tau} \cdot \frac{\mathsf{m}_\tau^k - \mathsf{m}_\tau^{k-1}}{\tau}\,\mathrm{d}x \\
&\quad - \int_\Omega \Big(\mathsf{h}_\tau^k - \psi'(\mathsf{m}_\tau^k) - \frac{1}{\varepsilon}\xi'(\mathsf{m}_\tau^k)\Big) \cdot \frac{\mathsf{m}_\tau^k - \mathsf{m}_\tau^{k-1}}{\tau}\,\mathrm{d}x \\
&\ge \frac{\mathcal{E}_\tau^k(\mathsf{m}_\tau^k) - \mathcal{E}_\tau^{k-1}(\mathsf{m}_\tau^{k-1})}{\tau} + \int_\Omega \frac{\mathsf{h}_\tau^k - \mathsf{h}_\tau^{k-1}}{\tau} \cdot \mathsf{m}_\tau^{k-1}\,\mathrm{d}x \\
&\quad + \frac{1}{\varepsilon} \int_\Omega \frac{\xi_1(\mathsf{m}_\tau^k) - \xi_1(\mathsf{m}_\tau^{k-1})}{\tau} + \xi_2'(\mathsf{m}_\tau^k) \cdot \frac{\mathsf{m}_\tau^k - \mathsf{m}_\tau^{k-1}}{\tau}\,\mathrm{d}x.
\end{aligned} \tag{2.43}$$

Combining (2.36) and (2.43) we arrive at:

$$\begin{aligned}
&\mathcal{R}^{\varepsilon}_{\alpha,\beta}(\mathsf{w}) + \int_{\Omega} \mu \nabla \mathsf{m}^k_\tau : \nabla \mathsf{w}\,\mathrm{d}x \\
&\quad - \int_{\Omega} \left(\gamma^{-1} \mathsf{m}^{k-1}_\tau \times \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau} + \mathsf{h}^k_\tau - \psi'(\mathsf{m}^k_\tau) - \frac{1}{\varepsilon}\xi'(\mathsf{m}^k_\tau) \right) \cdot \mathsf{w}\,\mathrm{d}x \\
&\quad \geq \mathcal{R}^{\varepsilon}_{\alpha,\beta}\left(\frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau} \right) + \frac{\mathcal{E}^k_\tau(\mathsf{m}^k_\tau) - \mathcal{E}^{k-1}_\tau(\mathsf{m}^{k-1}_\tau)}{\tau} + \int_{\Omega} \frac{\mathsf{h}^k_\tau - \mathsf{h}^{k-1}_\tau}{\tau} \cdot \mathsf{m}^{k-1}_\tau \,\mathrm{d}x \\
&\quad + \frac{1}{\varepsilon} \int_{\Omega} \frac{\xi_1(\mathsf{m}^k_\tau) - \xi_1(\mathsf{m}^{k-1}_\tau)}{\tau} + \xi_2'(\mathsf{m}^k_\tau) \cdot \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau}\,\mathrm{d}x, \qquad (2.44)
\end{aligned}$$

for all $\mathsf{w} \in C^1(\overline{\Omega};\mathbb{R}^3)$. By Young's inequality, $|\int_{\Omega} \frac{1}{\varepsilon}\xi_2'(\mathsf{m}^k_\tau) \cdot \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau}\,\mathrm{d}x| \leq \frac{\widehat{C}}{\alpha\varepsilon^2} + \frac{\alpha}{4}\int_{\Omega} |\frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau}|^2\,\mathrm{d}x$ (where $\widehat{C} > 0$ is such that $\int_{\Omega} |\xi_2'(\mathsf{m}^k_\tau)|^2\,\mathrm{d}x \leq \widehat{C}$); also, by (1.3) and (2.35) we have $\mathcal{R}^{\varepsilon}_{\alpha,\beta}(\mathsf{w}) \geq \int_{\Omega} \frac{\alpha}{2}|\mathsf{w}|^2 + \varepsilon|\nabla \mathsf{w}|^2\,\mathrm{d}x$; therefore, choosing $\mathsf{w} = 0$ in (2.44) and using Hölder's inequality we obtain:

$$\begin{aligned}
0 \geq &\int_{\Omega} \frac{\alpha}{4} \left| \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau} \right|^2 + \varepsilon \left| \nabla \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau} \right|^2 \mathrm{d}x + \frac{\mathcal{E}^k_\tau(\mathsf{m}^k_\tau) - \mathcal{E}^{k-1}_\tau(\mathsf{m}^{k-1}_\tau)}{\tau} \\
&+ \int_{\Omega} \frac{\xi_1(\mathsf{m}^k_\tau) - \xi_1(\mathsf{m}^{k-1}_\tau)}{\varepsilon\tau} - \frac{1}{2}\left| \frac{\mathsf{h}^k_\tau - \mathsf{h}^{k-1}_\tau}{\tau} \right|^2 - \frac{1}{2}|\mathsf{m}^{k-1}_\tau|^2\,\mathrm{d}x - \frac{\widehat{C}}{\alpha\varepsilon^2}. \qquad (2.45)
\end{aligned}$$

Henceforth C denotes a positive constant which may change from line to line. By (2.37) and (2.6),

$$\int_{\Omega} |\mathsf{m}^{k-1}_\tau|^2\,\mathrm{d}x \leq C \left(\mathcal{E}^{k-1}_\tau(\mathsf{m}^{k-1}_\tau) + \int_{\Omega} |\mathsf{h}^{k-1}_\tau|^2\,\mathrm{d}x \right). \qquad (2.46)$$

Multiplying both sides of (2.45) by τ, and using (2.46), we obtain:

$$\begin{aligned}
&\mathcal{E}^k_\tau(\mathsf{m}^k_\tau) - \mathcal{E}^{k-1}_\tau(\mathsf{m}^{k-1}_\tau) \\
&\quad + \int_{\Omega} \frac{\xi_1(\mathsf{m}^k_\tau) - \xi_1(\mathsf{m}^{k-1}_\tau)}{\varepsilon} + \tau\frac{\alpha}{4}\left| \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau} \right|^2 + \tau\varepsilon \left| \nabla \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau} \right|^2 \mathrm{d}x \\
&\quad \leq \tau C \left(\mathcal{E}^{k-1}_\tau(\mathsf{m}^{k-1}_\tau) + \int_{\Omega} |\mathsf{h}^{k-1}_\tau|^2\,\mathrm{d}x \right) + \frac{\tau}{2}\int_{\Omega} \left| \frac{\mathsf{h}^k_\tau - \mathsf{h}^{k-1}_\tau}{\tau} \right|^2 \mathrm{d}x + \frac{\widehat{C}}{\alpha\varepsilon^2}\tau. \qquad (2.47)
\end{aligned}$$

Given any $1 \leq \ell \leq N$, summing (2.47) for $k = 1 \ldots \ell$ gives:

$$\begin{aligned}
\mathcal{E}^\ell_\tau(\mathsf{m}^\ell_\tau) + \int_{\Omega} \frac{\xi_1(\mathsf{m}^\ell_\tau)}{\varepsilon} + \tau \sum_{k=1}^{\ell} \left(\frac{\alpha}{4}\left| \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau} \right|^2 + \varepsilon \left| \nabla \frac{\mathsf{m}^k_\tau - \mathsf{m}^{k-1}_\tau}{\tau} \right|^2 \right) \mathrm{d}x \\
\leq C_{\varepsilon,\tau} + \tau C \sum_{k=0}^{\ell-1} \mathcal{E}^k_\tau(\mathsf{m}^k_\tau), \qquad (2.48)
\end{aligned}$$

where

$$C_{\varepsilon,\tau} = \mathcal{E}(0,\mathsf{m}_0) + \ell\tau\frac{\widehat{C}}{\alpha\varepsilon^2} + \tau\sum_{k=0}^{\ell-1}\Big(\int_\Omega \frac{1}{2}\left|\frac{\mathsf{h}_\tau^k - \mathsf{h}_\tau^{k-1}}{\tau}\right|^2 + C|\mathsf{h}_\tau^{k-1}|^2\,\mathrm{d}x\Big). \quad (2.49)$$

Using a discrete version of the Gronwall's lemma we get, from (2.48)-(2.49),

$$\mathcal{E}_\tau^\ell(\mathsf{m}_\tau^\ell) \le C_{\varepsilon,\tau}\exp(C\tau\ell) \le C_\varepsilon, \quad (2.50)$$

where C_ε is some positive constant independent of τ. The existence of such C_ε is ensured by the assumption $\mathsf{h}_n \in W^{1,2}(I;L^2(\Omega;\mathbb{R}^3))$, which gives a bound (uniform in τ) to $C_{\varepsilon,\tau}$.

Now, we introduce interpolants $\overline{\mathsf{m}}_\tau$, $\underline{\mathsf{m}}_\tau$, and m_τ defined by:

$$\left.\begin{aligned} &\overline{\mathsf{m}}_\tau(t) := \mathsf{m}_\tau^k \\ &\underline{\mathsf{m}}_\tau(t) := \mathsf{m}_\tau^{k-1} \\ &\mathsf{m}_\tau(t) := \tfrac{t-(k-1)\tau}{\tau}\mathsf{m}_\tau^k + \tfrac{k\tau - t}{\tau}\mathsf{m}_\tau^{k-1} \end{aligned}\right\} \text{ for } t \in ((k-1)\tau, k\tau]. \quad (2.51)$$

Using (2.51) and recalling (2.37), the bound (2.50) and formula (2.48) imply that there exists a positive constant C_ε (independent of τ) such that

$$\|\mathsf{m}_\tau\|_{L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3))} \le C_\varepsilon, \quad (2.52)$$
$$\|\dot{\mathsf{m}}_\tau\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^3))} \le C_\varepsilon. \quad (2.53)$$

Step 3: limit passage as $\tau \to 0$. By (2.52)-(2.53) there exists a sequence $\{N_k\}_{k\in\mathbb{N}}$ such that

$$\mathsf{m}_\tau \overset{*}{\rightharpoonup} \mathsf{m} \text{ in } L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3)), \quad (2.54)$$
$$\dot{\mathsf{m}}_\tau \rightharpoonup \dot{\mathsf{m}} \text{ in } L^2(I;W^{1,2}(\Omega;\mathbb{R}^3)). \quad (2.55)$$

Here and henceforth we write τ as a shorthand for T/N_k. We also have

$$\mathsf{m} \in C_{\mathrm{w}}(\bar{I};W^{1,2}(\Omega;\mathbb{R}^3)), \quad \mathsf{m}(0) = \mathsf{m}_0, \quad (2.56)$$
$$\mathsf{m}_\tau(t) \rightharpoonup \mathsf{m}(t) \text{ in } W^{1,2}(\Omega;\mathbb{R}^3) \quad \forall t \in \bar{I}, \quad (2.57)$$
$$\mathsf{m}_\tau \to \mathsf{m} \text{ in } L^\eta(I;L^{6-1/\eta}(\Omega;\mathbb{R}^3)) \quad \forall 1 \le \eta < +\infty. \quad (2.58)$$

Indeed, from the identity $\mathsf{m}_\tau(t) = \mathsf{m}_0 + \int_0^t \dot{\mathsf{m}}_\tau(s)\,\mathrm{d}s$, using (2.54) and (2.55) gives (2.56) and (2.57), while (2.58) is a consequence of the Aubin-Lions theorem.

Now we pass to the limit in (2.36). Let $\mathsf{z} \in C^1(\overline{Q};\mathbb{R}^3)$ and $\mathsf{z}_\tau^k(\cdot) := \mathsf{z}(\tau k,\cdot)$. We define the interpolants $\bar{\mathsf{z}}_\tau$ and $\overline{\mathsf{h}}_\tau$ in terms of z_τ^k and of h_τ^k, respectively, as in (2.51). For each $t \in \bar{I}$, we substitute $\mathsf{w} = \mathsf{z}_\tau(t,\cdot)$ in (2.36), and we integrate with respect to t over I to obtain

$$\begin{aligned}\int_0 \mathcal{R}^\varepsilon_{\alpha,\beta}(\bar{\mathsf{z}}_\tau)\,\mathrm{d}t + \iint_Q \mu\nabla\overline{\mathsf{m}}_\tau : \nabla\bar{\mathsf{z}}_\tau - (\gamma^{-1}\underline{\mathsf{m}}_\tau \times \dot{\mathsf{m}}_\tau + \bar{\mathsf{h}}_\tau - \psi'(\overline{\mathsf{m}}_\tau) - \frac{1}{\varepsilon}\xi'(\overline{\mathsf{m}}_\tau))\cdot\bar{\mathsf{z}}_\tau\,\mathrm{d}x\,\mathrm{d}t \\ \geq \int_0 \mathcal{R}^\varepsilon_{\alpha,\beta}(\dot{\mathsf{m}}_\tau)\,\mathrm{d}t + \iint_Q \mu\nabla\overline{\mathsf{m}}_\tau : \nabla\dot{\mathsf{m}}_\tau - (\bar{\mathsf{h}}_\tau - \psi'(\overline{\mathsf{m}}_\tau) - \frac{1}{\varepsilon}\xi'(\overline{\mathsf{m}}_\tau))\cdot\dot{\mathsf{m}}_\tau\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.59}$$

Note that $\iint_Q(\nabla\overline{\mathsf{m}}_\tau - \nabla\mathsf{m}_\tau) : \nabla\dot{\mathsf{m}}_\tau\,\mathrm{d}x\mathrm{d}t = \mathcal{O}(\tau)$ because $\|\nabla\overline{\mathsf{m}}_\tau - \nabla\mathsf{m}_\tau\|_{L^2(Q;\mathbb{R}^3)} = \frac{\tau}{\sqrt{3}}\|\nabla\dot{\mathsf{m}}_\tau\|_{L^2(Q;\mathbb{R}^3)}$, hence

$$\begin{aligned}\liminf_{\tau\to 0}\iint_Q \nabla\overline{\mathsf{m}}_\tau : \nabla\dot{\mathsf{m}}_\tau\,\mathrm{d}x\,\mathrm{d}t &= \liminf_{\tau\to 0}\iint_Q \nabla\mathsf{m}_\tau : \nabla\dot{\mathsf{m}}_\tau\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \lim_{\tau\to 0}\iint_Q (\nabla\overline{\mathsf{m}}_\tau - \nabla\mathsf{m}_\tau) : \nabla\dot{\mathsf{m}}_\tau\,\mathrm{d}x\,\mathrm{d}t \\ &= \liminf_{\tau\to 0}\int_\Omega \frac{1}{2}|\nabla\mathsf{m}_\tau(T)|^2\,\mathrm{d}x - \int_\Omega \frac{1}{2}|\nabla\mathsf{m}_0|^2\,\mathrm{d}x \\ &\geq \int_\Omega \frac{1}{2}|\nabla\mathsf{m}(T)|^2\,\mathrm{d}x - \int_\Omega \frac{1}{2}|\nabla\mathsf{m}_0|^2\,\mathrm{d}x \\ &= \iint_Q \nabla\mathsf{m} : \nabla\dot{\mathsf{m}}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{2.60}$$

where we have used (2.57). Using (2.54)–(2.58) and (2.60) we can pass to the limit as $\tau \to 0$ in (2.59) to obtain

$$\begin{aligned}\int_0 \mathcal{R}^\varepsilon_{\alpha,\beta}(\mathsf{z})\,\mathrm{d}t + \iint_Q \mu\nabla\mathsf{m} : \nabla\mathsf{z} - (\gamma^{-1}\mathsf{m}\times\dot{\mathsf{m}} + \mathsf{h}_n - \psi'(\mathsf{m}) - \frac{1}{\varepsilon}\xi'(\mathsf{m}))\cdot\mathsf{z}\,\mathrm{d}x\,\mathrm{d}t \\ \geq \int_0 \mathcal{R}^\varepsilon_{\alpha,\beta}(\dot{\mathsf{m}})\,\mathrm{d}t + \iint_Q \mu\nabla\mathsf{m} : \nabla\dot{\mathsf{m}} - (\mathsf{h}_n - \psi'(\mathsf{m}) - \frac{1}{\varepsilon}\xi'(\mathsf{m}))\cdot\dot{\mathsf{m}}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.61}$$

The inequality (2.61) provides the weak formulation of problem (2.32) with the initial-boundary conditions (2.3) and (2.33). Note that, by density, we can assume $\mathsf{z} \in L^2(I; W^{1,2}(\Omega;\mathbb{R}^3))$ in (2.61).

Step 4: estimates independent of ε. To stress the dependence on ε, we now denote by m_ε the function m obtained in the previous step. In (2.61) we can choose z such that $\mathsf{z} = \dot{\mathsf{m}}_\varepsilon$ on (s,T). Carrying out the integration with respect to time in the second term on the right-hand side of (2.61), and then using

(2.56) together with $\xi(\mathsf{m}_0) = 0$ which follows from assumption (2.7), we obtain:

$$\begin{aligned}
&\int_0^s \mathcal{R}^\varepsilon_{\alpha,\beta}(\mathsf{z})\,\mathrm{d}t + \int_0^s\!\!\int_\Omega \mu\nabla\mathsf{m}_\varepsilon : \nabla\mathsf{z} - (\gamma^{-1}\mathsf{m}_\varepsilon\times\dot{\mathsf{m}}_\varepsilon \\
&\quad + \mathsf{h}_n - \psi'(\mathsf{m}_\varepsilon))\cdot\mathsf{z} + \frac{1}{\varepsilon}\xi'(\mathsf{m}_\varepsilon)\cdot\mathsf{z}\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_0^s \mathcal{R}^\varepsilon_{\alpha,\beta}(\dot{\mathsf{m}}_\varepsilon)\,\mathrm{d}t + \mathcal{E}(s,\mathsf{m}_\varepsilon(s)) - \mathcal{E}(0,\mathsf{m}_0) \\
&\quad + \int_\Omega \frac{1}{\varepsilon}\xi(\mathsf{m}_\varepsilon(s))\,\mathrm{d}x + \int_0^s\!\!\int_\Omega \dot{\mathsf{h}}_n\cdot\mathsf{m}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{2.62}$$

for all $s \in \bar{I}$ and for all $\mathsf{z} \in L^2(I; W^{1,2}(\Omega;\mathbb{R}^3))$. Choosing $\mathsf{z} = 0$ in (2.62) and recalling that $\mathcal{R}^\varepsilon_{\alpha,\beta}(\mathsf{w}) \geq \mathcal{R}_{\alpha,\beta}(\mathsf{w})$ we obtain, for all $s \in \bar{I}$,

$$\mathcal{E}(s,\mathsf{m}_\varepsilon(s)) + \int_\Omega \frac{1}{\varepsilon}\xi(\mathsf{m}_\varepsilon(s))\,\mathrm{d}x + \int_0^s \mathcal{R}_{\alpha,\beta}(\dot{\mathsf{m}}_\varepsilon)\,\mathrm{d}t \leq \mathcal{E}(0,\mathsf{m}_0) - \int_0^s\!\!\int_\Omega \dot{\mathsf{h}}_n\cdot\mathsf{m}_\varepsilon\,\mathrm{d}x\mathrm{d}t\,. \tag{2.63}$$

By the non-negativity of ξ and $\mathcal{R}_{\alpha,\beta}$, it follows from (2.63) that

$$\begin{aligned}
\mathcal{E}(s,\mathsf{m}_\varepsilon(s)) &\leq \mathcal{E}(0,\mathsf{m}_0) + \int_0^s \|\dot{\mathsf{h}}_n\|_{L^2(\Omega;\mathbb{R}^3)}\|\mathsf{m}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)}\mathrm{d}t \\
&\leq \mathcal{E}(0,\mathsf{m}_0) + \int_0^s \frac{1}{2}\|\dot{\mathsf{h}}_n\|_{L^2(\Omega;\mathbb{R}^3)}(1 + \|\mathsf{m}_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^3)})\mathrm{d}t.
\end{aligned} \tag{2.64}$$

Note that

$$\|\mathsf{m}_\varepsilon(s)\|^2_{L^2(\Omega;\mathbb{R}^3)} \leq C_1\left(\mathcal{E}(s,\mathsf{m}_\varepsilon(s)) + \|\mathsf{h}_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \leq C_1\Big(\mathcal{E}(s,\mathsf{m}_\varepsilon(s)) + C_2\Big) \tag{2.65}$$

for some $C_1 > 0$, $C_2 > 0$ (the last inequality follows from the assumption $\mathsf{h}_n \in W^{1,2}(I; L^2(\Omega;\mathbb{R}^3))$). By using (2.65) we can apply the integral version of Gronwall's lemma to (2.64) to obtain

$$\max_{s\in\bar{I}} \mathcal{E}(s,\mathsf{m}_\varepsilon(s)) \leq C, \tag{2.66}$$

where C is a positive constant. Using (2.66) and (2.63) we obtain

$$\|\mathsf{m}_\varepsilon\|_{L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3))} \leq C_0, \tag{2.67}$$

$$\|\dot{\mathsf{m}}_\varepsilon\|_{L^1(Q;\mathbb{R}^3)} \leq C_0, \tag{2.68}$$

$$\sqrt{\alpha}\|\dot{\mathsf{m}}_\varepsilon\|_{L^2(Q;\mathbb{R}^3)} \leq C_0, \tag{2.69}$$

$$\max_{s\in\bar{I}} \int_\Omega \xi(\mathsf{m}_\varepsilon(s,x))\,\mathrm{d}x \leq \varepsilon C_0, \tag{2.70}$$

where the constant $C_0 > 0$ does not depend on α and γ.

Step 5: selection of test functions. Our argument is based on an appropriate choice of the test functions, see (2.72) below. By the non-negativity of ξ and by $\mathcal{R}^\varepsilon_{\alpha,\beta}(\mathsf{w}) \geq \mathcal{R}_{\alpha,\beta}(\mathsf{w})$, from (2.62) we have

$$\begin{aligned}\int_0^s \mathcal{R}^\varepsilon_{\alpha,\beta}(\mathsf{z})\,\mathrm{d}t + \int_0^s\!\!\int_\Omega \mu\nabla\mathsf{m}_\varepsilon : \nabla\mathsf{z} \\ - (\gamma^{-1}\mathsf{m}_\varepsilon\times\dot{\mathsf{m}}_\varepsilon + \mathsf{h}_n - \psi'(\mathsf{m}_\varepsilon))\cdot\mathsf{z} + \frac{1}{\varepsilon}\xi'(\mathsf{m}_\varepsilon)\cdot\mathsf{z}\,\mathrm{d}x\,\mathrm{d}t \\ \geq \int_0^s \mathcal{R}_{\alpha,\beta}(\dot{\mathsf{m}}_\varepsilon)\,\mathrm{d}t + \mathcal{E}(s,\mathsf{m}_\varepsilon(s)) - \mathcal{E}(0,\mathsf{m}_0) + \int_0^s\!\!\int_\Omega \dot{\mathsf{h}}_n\cdot\mathsf{m}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t. \quad (2.71)\end{aligned}$$

We do not pass to the limit in (2.71) as it is, because we do not have control over the term $\frac{1}{\varepsilon}\xi'(\mathsf{m}_\varepsilon)\cdot\mathsf{z}$. However, by (2.34), $\xi'(\mathsf{z})$ and z are parallel vectors, hence the term that we do not control vanishes for all tests z of the form

$$\mathsf{z} = \mathsf{m}_\varepsilon\times\mathsf{v} \quad \text{with } \mathsf{v}\in C^1(\overline{Q};\mathbb{R}^3). \quad (2.72)$$

Note also that, by (2.11)-(2.12) we have

$$\int_\Omega \nabla\mathsf{m}_\varepsilon : \nabla(\mathsf{m}_\varepsilon\times\mathsf{v})\,\mathrm{d}x = -\int_\Omega (\mathsf{m}_\varepsilon \divideontimes \nabla\mathsf{m}_\varepsilon) : \nabla\mathsf{v}\,\mathrm{d}x. \quad (2.73)$$

Thus, substituting (2.72) in (2.71), and using (2.73), along with the identity:

$$(\mathsf{m}_\varepsilon\times\dot{\mathsf{m}}_\varepsilon)\cdot(\mathsf{m}_\varepsilon\times\mathsf{v}) = |\mathsf{m}_\varepsilon|^2(\dot{\mathsf{m}}_\varepsilon\cdot\mathsf{v}) - (\dot{\mathsf{m}}_\varepsilon\cdot\mathsf{m}_\varepsilon)(\mathsf{m}_\varepsilon\cdot\mathsf{v}),$$

we obtain

$$\begin{aligned}\int_0^s \mathcal{R}^\varepsilon_{\alpha,\beta}(\mathsf{m}_\varepsilon\times\mathsf{v})\,\mathrm{d}t + \int_0^s\!\!\int_\Omega \gamma^{-1}(\dot{\mathsf{m}}_\varepsilon\cdot\mathsf{m}_\varepsilon)(\mathsf{m}_\varepsilon\cdot\mathsf{v})\,\mathrm{d}x\,\mathrm{d}t \\ - \int_0^s\!\!\int_\Omega \mu(\mathsf{m}_\varepsilon\divideontimes\nabla\mathsf{m}_\varepsilon) : \nabla\mathsf{v} + \Big(\gamma^{-1}|\mathsf{m}_\varepsilon|^2\dot{\mathsf{m}}_\varepsilon - \mathsf{m}_\varepsilon\times(\mathsf{h}_n - \psi'(\mathsf{m}_\varepsilon))\Big)\cdot\mathsf{v}\,\mathrm{d}x\,\mathrm{d}t \\ \geq \int_0^s \mathcal{R}_{\alpha,\beta}(\dot{\mathsf{m}}_\varepsilon)\,\mathrm{d}t + \mathcal{E}(s,\mathsf{m}_\varepsilon(s)) - \mathcal{E}(0,\mathsf{m}_0) + \int_0^s\!\!\int_\Omega \dot{\mathsf{h}}_n\cdot\mathsf{m}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t. \quad (2.74)\end{aligned}$$

Step 6: limit passage as $\varepsilon\to 0$. By (2.67)-(2.69) there exists a subsequence (not relabeled) such that

$$\mathsf{m}_\varepsilon \overset{*}{\rightharpoonup} \mathsf{m} \text{ in } L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3)), \quad (2.75)$$
$$\dot{\mathsf{m}}_\varepsilon \rightharpoonup \dot{\mathsf{m}} \text{ in } L^2(Q;\mathbb{R}^3), \quad (2.76)$$

as $\varepsilon\to 0$. Moreover, by the same argument used in Step 3, (2.75)-(2.76) imply

$$\mathsf{m}\in C_{\mathrm{w}}(\bar{I};W^{1,2}(\Omega;\mathbb{R}^3)), \quad \mathsf{m}(0)=\mathsf{m}_0, \quad (2.77)$$
$$\mathsf{m}_\varepsilon(t)\rightharpoonup\mathsf{m}(t) \text{ in } W^{1,2}(\Omega;\mathbb{R}^3) \quad \forall t\in\bar{I}. \quad (2.78)$$

By the Aubin-Lions theorem we have, for all $5\leq\eta<+\infty$,

$$\mathsf{m}_\varepsilon \to \mathsf{m} \text{ in } L^\eta(I;L^{6-1/\eta}(\Omega;\mathbb{R}^3)), \quad (2.79)$$
$$|\mathsf{m}_\varepsilon|^2\dot{\mathsf{m}}_\varepsilon \rightharpoonup |\mathsf{m}|^2\dot{\mathsf{m}} \text{ in } L^{2-1/\eta}(I;L^{6/5-1/\eta}(\Omega;\mathbb{R}^3)), \quad (2.80)$$
$$\mathsf{m}_\varepsilon\cdot\dot{\mathsf{m}}_\varepsilon \rightharpoonup 0 \text{ in } L^{2-1/\eta}(I;L^{3/2-1/\eta}(\Omega;\mathbb{R}^3)). \quad (2.81)$$

By (2.70) and (2.79), and recalling (2.34), we also have

$$|\mathsf{m}| = 1 \text{ a.e. in } Q. \tag{2.82}$$

By (2.76) and the convexity of $\mathcal{R}_{\alpha,\beta}$, and by (2.78), we have, for all $s \in \bar{I}$,

$$\liminf_{\varepsilon\to 0} \int_0^s \mathcal{R}_{\alpha,\beta}(\dot{\mathsf{m}}_\varepsilon)\,\mathrm{d}t \geq \int_0^s \mathcal{R}_{\alpha,\beta}(\dot{\mathsf{m}})\,\mathrm{d}t, \tag{2.83}$$

$$\liminf_{\varepsilon\to 0} \mathcal{E}(s,\mathsf{m}_\varepsilon(s)) \geq \mathcal{E}(s,\mathsf{m}(s)). \tag{2.84}$$

Moreover, by (2.11) we have $|\nabla(\mathsf{m}_\varepsilon \times \mathsf{v})|^2 \leq 2|\mathsf{m}_\varepsilon \circledast \nabla\mathsf{v}|^2 + 2|\mathsf{v} \circledast \nabla\mathsf{m}_\varepsilon|^2$, hence by (2.67) we have

$$\lim_{\varepsilon\to 0} \int_0^s\!\!\int_\Omega \varepsilon|\nabla(\mathsf{m}_\varepsilon \times \mathsf{v})|^2\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2.85}$$

By (2.85) and (2.78)-(2.82), the left-hand side of (2.74) converges to the left-hand side of (2.28), with h replaced by h_n, as ε tends to 0. Thus, taking the liminf as ε tends to 0 of the right-hand side of (2.74), using (2.83)-(2.84) and (2.57) with the compact embedding $W^{1,2}(\Omega;\mathbb{R}^3) \subset L^2(\Omega;\mathbb{R}^3)$, we obtain (2.28) with h and $\dot{\mathsf{h}}$ replaced by h_n and $\dot{\mathsf{h}}_n$, respectively.

Step 7: limit passage as $n \to \infty$. Let us denote by m_n the function obtained in the previous step. This function satisfies, for all $\mathsf{v} \in C^1(\overline{Q};\mathbb{R}^3)$,

$$\begin{aligned}\int_0^s \mathcal{R}_{\alpha,\beta}(\mathsf{m}_n \times \mathsf{v})\,\mathrm{d}t - \int_0^s\!\!\int_\Omega &\mu(\mathsf{m}_n \circledast \nabla\mathsf{m}_n) : \nabla\mathsf{v} \\ &+ \Big(\gamma^{-1}\dot{\mathsf{m}}_n + \mathsf{m}_n \times \psi'(\mathsf{m}_n) - \mathsf{m}_n \times \mathsf{h}_n\Big) \cdot \mathsf{v}\,\mathrm{d}x\,\mathrm{d}t \\ \geq \int_0^s \mathcal{R}_{\alpha,\beta}(\dot{\mathsf{m}}_n)\,\mathrm{d}t + \mathcal{E}(s,\mathsf{m}_n(s)) &- \mathcal{E}(0,\mathsf{m}_0) + \int_0^s\!\!\int_\Omega \dot{\mathsf{h}}_n \cdot \mathsf{m}_n\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.86}$$

Having $|\mathsf{m}_n| = 1$ a.e. on Q, we can now improve the estimate (2.64). In fact, from (2.86) we deduce $\mathcal{E}(s,\mathsf{m}_n(s)) \leq \mathcal{E}(0,\mathsf{m}_0) + \int_0^s \|\dot{\mathsf{h}}_n\|_{L^1(\Omega;\mathbb{R}^3)}\mathrm{d}t$ for all $s \in \bar{I}$. Arguing as in Step 4 of this proof, we conclude that m_n satisfies the bounds (2.67)–(2.69). Since these bounds are uniform with respect to n, there exists a subsequence of $\{\mathsf{m}_n\}_{n\in\mathbb{N}}$ (not relabeled) and a limit function $\mathsf{m} \in C_{\mathrm{w}}(\bar{I};W^{1,2}(\Omega;\mathbb{R}^3))$ such that $\mathsf{m}_n \overset{*}{\rightharpoonup} \mathsf{m}$ in $L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3))$, $\dot{\mathsf{m}}_n \rightharpoonup \dot{\mathsf{m}}$ in $L^2(Q;\mathbb{R}^3)$, $\mathsf{m}(0) = \mathsf{m}_0$, and $\mathsf{m}_n(t) \rightharpoonup \mathsf{m}(t)$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ for all $t \in \bar{I}$, as $n \to \infty$. Using the strong convergence $\mathsf{h}_n \to \mathsf{h}$ in $W^{1,1}(I;L^1(\Omega;\mathbb{R}^3))$, we can pass to the limit in the integrals $\int_0^s\!\int_\Omega(\mathsf{h}_n \times \mathsf{m}_n)\cdot v\,\mathrm{d}x\,\mathrm{d}t$ and $\int_0^s\!\int_\Omega \dot{\mathsf{h}}_n \cdot \mathsf{m}_n\,\mathrm{d}x\,\mathrm{d}t$ in (2.86). Arguing as in the previous step for the other terms, it follows in conclusion that the limit function m satisfies (2.28) as well as the estimates (2.29)–(2.31). □

Remark 2.6. The time discretization used in Proposition 2.5 may be exploited to construct approximate solutions to (2.1)-(2.3). However, for numerical purposes a more appropriate approach should be based on the finite-element scheme proposed in [4] for the standard Landau-Lifschitz/Gilbert equation. Denoting by $h > 0$ and $\tau > 0$ respectively the mesh size in the spatial domain and the time step, and by m_h^0 the discretized initial datum, the scheme proposed in [4] gives approximating magnetization fields $\mathsf{m}_{h,\tau}^k$, with $k \in \{0 \dots T/\tau\}$, satisfying $\mathsf{m}_{h,\tau}^k(\hat{x}) = \mathsf{m}_h^0(\hat{x})$ for every nodal point $\hat{x}$ of the mesh. If the initial datum satisfies the saturation constraint at nodal points, then the approximate solution $\mathsf{m}_{h,\tau} : Q \to \mathbb{R}^3$ obtained from $\mathsf{m}_{h,\tau}^k$ by piecewise-affine interpolation (with respect to time) converges, as $h, \tau \to 0$, to a limit m satisfying $|\mathsf{m}| = 1$ a.e. in Q. Thus the saturation constraint is recovered in the limit without introducing any penalization. An interesting question is how to adapt the scheme proposed in [4] to account for the additional rate-independent term we consider in this paper.

3 The regimes $\alpha \to 0$ and/or $\gamma^{-1} \to 0$.

In the Introduction we pointed out that the limit $\alpha \to 0$ and $\gamma^{-1} \to 0$ is expected to describe the behavior of the system for slowly-varying applied fields (see also [32, Section 6]), that is: taking the limit $\alpha \to 0$ and $\gamma^{-1} \to 0$ corresponds to taking the limit of the "frequency" $\omega \to 0$ in a system where $\alpha, \gamma^{-1} > 0$ are fixed, while the external field h is scaled in time by ωt. More precisely, let us fix $\mathsf{h} \in W^{1,1}(I; L^1(\Omega; \mathbb{R}^3))$, and let us consider the family of loadings

$$\tilde{\mathsf{h}}_\omega(t, \cdot) := \mathsf{h}(\omega t, \cdot), \tag{3.1}$$

with $\omega > 0$. When the "frequency" ω tends to 0, the applied field $\tilde{\mathsf{h}}_\omega$ exhibits a slower and slower rate of variation. We denote by $\tilde{\mathsf{m}}_\omega$ the solution to

$$\left.\begin{aligned} &\gamma^{-1}\dot{\tilde{\mathsf{m}}}_\omega = \tilde{\mathsf{m}}_\omega \times \Big(\mu\Delta\tilde{\mathsf{m}}_\omega - \psi'(\tilde{\mathsf{m}}_\omega) + \tilde{\mathsf{h}}_\omega - \mathsf{r}\Big) \\ &\mathsf{r} \in \partial R_{\alpha,\beta}(\dot{\tilde{\mathsf{m}}}_\omega) \end{aligned}\right\} \text{ in } Q; \tag{3.2}$$

see (2.1)-(2.3) with applied field $\tilde{\mathsf{h}}_\omega = \tilde{\mathsf{h}}_\omega(t)$.

To show the equivalence between this system, where $\alpha > 0$ and $\gamma^{-1} > 0$ are fixed while the applied field $\tilde{\mathsf{h}}_\omega$ varies as $\omega \to 0$, and the system where h is fixed, and α and γ^{-1} tend to 0, we need to perform some scaling in time. For any fixed $\omega > 0$ let us define, for all times t,

$$\mathsf{m}_\omega(\omega t) := \tilde{\mathsf{m}}_\omega(t). \tag{3.3}$$

By using (3.1), (3.2) and (3.3), and then defining $s := \omega t$, it is easy to see that

m_ω solves

$$\left.\begin{aligned} &(\omega\gamma^{-1})\dot{\mathsf{m}}_\omega = \mathsf{m}_\omega \times \Big(\mu\Delta\mathsf{m}_\omega - \psi'(\mathsf{m}_\omega) + \mathsf{h} - \mathsf{r}\Big) \\ &\mathsf{r} \in \partial R_{(\omega\alpha),\beta}(\dot{\mathsf{m}}_\omega) \end{aligned}\right\} \text{ in } Q, \tag{3.4}$$

since $\partial R_{\alpha,\beta}(\omega\dot{\mathsf{m}}_\omega) = \partial R_{\omega\alpha,\beta}(\dot{\mathsf{m}}_\omega)$. Hence, taking the limit $\omega \to 0$ in (3.2) is equivalent, thanks to (3.4), to taking the limit $\alpha \to 0$, $\gamma^{-1} \to 0$ in (2.1).

Since the constant C_0 in (2.29)-(2.30) does not depend on α neither on γ^{-1}, we expect that weak solutions obtained in Proposition 2.5 will converge weakly to some limit m, which we would like to identify with a weak solution of

$$\left.\begin{aligned} &0 = \mathsf{m} \times \Big(\mu\Delta\mathsf{m} - \psi'(\mathsf{m}) + \mathsf{h} - \mathsf{r}\Big) \\ &\mathsf{r} \in \partial R_{0,\beta}(\dot{\mathsf{m}}) \\ &|\mathsf{m}| = 1 \end{aligned}\right\} \text{ in } Q. \tag{3.5}$$

Note that, at variance with (3.4), the first equation in (3.5) does not imply (1.6), even if the initial condition satisfies (2.7). This motivates the additional condition $|\mathsf{m}| = 1$ in (3.5). For the sake of completeness, we consider three cases: $\alpha \to 0$ with $\gamma^{-1} > 0$ fixed; $\alpha > 0$ fixed with $\gamma^{-1} \to 0$; and eventually the announced "slow-loading limit" $\alpha \to 0$ and $\gamma^{-1} \to 0$. In the third case, since the system is rate-independent, an alternative way to prove existence of weak solutions is to show that there exist special weak solutions (called "energetic" in the sense of [28]) by limiting directly the time-discrete problems without any vanishing-viscosity approach, which we briefly touch at the end of this section, too.

Motivated by this scaling, $\beta > 0$ will be kept fixed thorough the whole section.

The limit $\alpha \to 0$.

For purpose of limiting α, let us denote a weak solution to (2.1)-(2.3) by m_α and investigate the collection $\{\mathsf{m}_\alpha\}_{\alpha>0}$. Using the uniform estimate (2.29), we obtain

$$\mathsf{m}_\alpha \overset{*}{\rightharpoonup} \mathsf{m} \text{ in } L^\infty(I; W^{1,2}(\Omega; \mathbb{R}^3)) \tag{3.6}$$

as $\alpha \to 0$, in the sense of subsequences. Of course, we cannot identify the limit m with a weak solution in the sense of Definition 2.4 with $\alpha = 0$. In fact, the upper bound in (2.31) blows up as $\alpha \to 0$, and the L^1-type estimate (2.30) gives only a weak* convergence of $\dot{\mathsf{m}}_\alpha$ in the space of $L^1(\Omega; \mathbb{R}^3)$-valued measures with support on $\bar{I}$. Consequently, $\iint_Q \gamma^{-1}\dot{\mathsf{m}} \cdot \mathsf{v}\, \mathrm{d}x\, \mathrm{d}t$ and $\iint_Q R_{0,\beta}(\dot{\mathsf{m}})\, \mathrm{d}x\, \mathrm{d}t$ may lose sense as Lebesgue's integrals. Yet, the limit m can be interpreted as a weak solution to (2.1), provided that, in Definition 2.4, the regularity assumptions on m are weakened, and (2.9) is modified suitably, inspired by the definition of weak solution introduced in [31]. To this goal, given $\mathsf{m} : \bar{I} \to \{\mathsf{z} \in L^\infty(\Omega;\mathbb{R}^3)\,;\, \mathsf{z}(\cdot) \in S(2) \text{ a.e. on } \Omega\}$, where $S(2)$ denotes the unit sphere on $\mathbb{R}^3$, we define the *geodesic variation* of m over the time interval $[\underline{t},\bar{t}]$ by

$$\mathrm{Var}_{S(2)}(\mathsf{m};\underline{t},\bar{t}) := \sup_{\substack{\underline{t}=t_0<t_1<\cdots<t_k=\bar{t}\\ k\in\mathbb{N}}} \sum_{i=1}^{k} \int_\Omega \mathrm{dist}_{S(2)}\Big(\mathsf{m}(t_i,x),\mathsf{m}(t_{i-1},x)\Big)\,\mathrm{d}x \quad \text{where} \tag{3.7}$$

$$\mathrm{dist}_{S(2)}(\mathsf{m}_1,\mathsf{m}_2) := \begin{cases} \arccos(\mathsf{m}_1\cdot\mathsf{m}_2) \text{ if } |\mathsf{m}_1| = 1 = |\mathsf{m}_2|, \\ +\infty \qquad\qquad\qquad \text{elsewhere.} \end{cases} \tag{3.8}$$

The function $\mathrm{dist}_{S(2)} : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^+ \cup \{+\infty\}$ is called a *geodesic distance*; a notion similar to (3.8) can be found in [26, Section 7.4] and [27, Section 5.6]. Note that $\mathrm{Var}_{S(2)}(\mathsf{m};\underline{t},\bar{t}) < +\infty$ implies in particular that $\mathsf{m}(t,\cdot) \in S(2)$ a.e. on Ω for all $t \in [\underline{t},\bar{t}]$.

A generalization of Helly's principle to separable, reflexive Banach spaces, see [3, Chap. 1, Theorem 3.5], states that if a sequence of $f_n : \bar{I} \to X$ is bounded in $BV(\bar{I};X)$, then there exists $f \in BV(\bar{I};X)$ and a subsequence (not relabeled) such that $f_n(t) \rightharpoonup f(t)$ for all $t \in \bar{I}$. It can be shown that the same result applies to (not linear) sequentially compact topological Hausdorff space instead of the Banach space X, see [24, Theorem 3.2] or, under additional continuity of f_n, also [**?**, Theorem 7.1]. Here the topological space we consider is

$$\mathcal{M} = \{\mathsf{z} \in W^{1,2}(\Omega;\mathbb{R}^3) : \ \mathsf{z} \in S(2) \text{ a.e. in } \Omega\} \tag{3.9}$$

equipped with the weak topology of $W^{1,2}(\Omega;\mathbb{R}^3)$. Although $\mathcal{M}$ itself, being unbounded in $W^{1,2}(\Omega;\mathbb{R}^3)$, is not sequentially compact, its intersections with the balls $\{\mathsf{z} \in W^{1,2}(\Omega;\mathbb{R}^3) : \ \|\mathsf{z}\|_{W^{1,2}(\Omega;\mathbb{R}^3)} \le C\}$ enjoys this property for any $C < +\infty$.

Lemma 3.1. (Mainik & Mielke [24], special case) *Let* $\{\mathsf{m}_\alpha : \bar{I} \to \mathcal{M}\}_{\alpha\in\mathbb{N}} \subset C_{\mathrm{w}}(I;W^{1,2}(\Omega;\mathbb{R}^3))$ *with*

$$\|\mathsf{m}_\alpha\|_{L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3))} + \mathrm{Var}_{S(2)}(\mathsf{m}_\alpha;0,T) \le C, \tag{3.10}$$

for some positive constant C (independent of α). Then there exists a subsequence $\{\mathsf{m}_{\alpha_j}\}_{j\in\mathbb{N}}$ such that

$$\mathsf{m}_{\alpha_j}(t) \rightharpoonup \mathsf{m}(t) \ \textit{in } \mathcal{M} \textit{ for all } t \in \bar{I}. \tag{3.11}$$

Proposition 3.2. *For each $\alpha > 0$, let m_α be a weak solution to* (2.1)-(2.3) *in the sense of Definition 2.4. There exists a sequence $\alpha_k \to 0$ such that m_{α_k} converges, in the sense of* (3.6)*, to $\mathsf{m} \in BM(\bar{I};W^{1,2}(\Omega;\mathbb{R}^3))$ satisfying*

$\mathrm{Var}_{S(2)}(\mathsf{m}; 0, T) < +\infty$. *Moreover,*

$$|\mathsf{m}(t,\cdot)| = 1 \qquad \textit{a.e. on } \Omega \textit{ for all } t \in [0,T], \tag{3.12}$$

$\mathsf{m}(0,\cdot) = \mathsf{m}_0$, *and*

$$\begin{aligned} &\int_0^s \mathcal{R}_{0,\beta}(\mathsf{m}\times\mathsf{v})\,\mathrm{d}t - \int_0^s\!\!\int_\Omega \mu(\mathsf{m}\divideontimes\nabla\mathsf{m}) : \nabla\mathsf{v} + \Big(\mathsf{m}\times\big(\psi'(\mathsf{m}) - \mathsf{h}\big)\Big)\cdot\mathsf{v}\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_0^s\!\!\int_\Omega \gamma^{-1}\mathsf{m}\cdot\dot{\mathsf{v}}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \gamma^{-1}(\mathsf{m}(s,x)\cdot\mathsf{v}(s,x) - \mathsf{m}_0\cdot\mathsf{v}(0,x))\,\mathrm{d}x \\ &\qquad \geq \beta\,\mathrm{Var}_{S(2)}(\mathsf{m}; 0, s) + \mathcal{E}(s, \mathsf{m}(s,\cdot)) - \mathcal{E}(0, \mathsf{m}_0) + \int_0^s\!\!\int_\Omega \dot{\mathsf{h}}\cdot\mathsf{m}\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.13}$$

for all $s \in \bar{I}$ *and for all* $\mathsf{v} \in C^1(\overline{Q}; \mathbb{R}^3)$.

Proof. By assumption, each m_α satisfies (2.28). Integrating by parts in time the term $\gamma^{-1}\dot{\mathsf{m}}_\alpha\cdot\mathsf{v}$, and using $R_{\alpha,\beta}(\mathsf{a}) \geq \beta|\mathsf{a}|$, we obtain

$$\begin{aligned} I_{1,\alpha}(s) := &\int_0^s \mathcal{R}_{\alpha,\beta}(\mathsf{m}_\alpha\times\mathsf{v})\,\mathrm{d}t \\ &- \int_0^s\!\!\int_\Omega \mu(\mathsf{m}\divideontimes\nabla\mathsf{m}_\alpha) : \nabla\mathsf{v} + \Big(\mathsf{m}_\alpha\times\psi'(\mathsf{m}_\alpha) - \mathsf{m}_\alpha\times\mathsf{h}\Big)\cdot\mathsf{v}\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^s\!\!\int_\Omega \gamma^{-1}\mathsf{m}_\alpha\cdot\dot{\mathsf{v}}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \gamma^{-1}(\mathsf{m}_\alpha(s,x)\cdot\mathsf{v}(s,x) - \mathsf{m}_0\cdot\mathsf{v}(0,x))\,\mathrm{d}x \\ \geq\ &\int_0^s\!\!\int_\Omega \beta|\dot{\mathsf{m}}_\alpha|\,\mathrm{d}x\,\mathrm{d}t + \mathcal{E}(s, \mathsf{m}_\alpha(s,\cdot)) \\ &- \mathcal{E}(0, \mathsf{m}(0,\cdot)) + \int_0^s\!\!\int_\Omega \dot{\mathsf{h}}\cdot\mathsf{m}_\alpha\,\mathrm{d}x\,\mathrm{d}t =: I_{2,\alpha}(s) \end{aligned} \tag{3.14}$$

for all $s \in \bar{I}$ and for all $\mathsf{v} \in C^1(\overline{Q}; \mathbb{R}^3)$. By (2.29) and weak* compactness, there is a sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ such that $\alpha_j \to 0$ and (3.6) holds. By the estimate (2.30), and by the Aubin-Lions theorem, we have

$$\mathsf{m}_{\alpha_j} \to \mathsf{m} \text{ in } L^\eta(I; L^{6-1/\eta}(\Omega;\mathbb{R}^3)) \quad \forall 1 \leq \eta < +\infty. \tag{3.15}$$

By (3.15) and (3.6) we have, passing to the limit as $\alpha_j \to 0$,

$$\begin{aligned} I_{1,\alpha_j}(s) \to &\int_0^s\!\!\int_\Omega \beta|\mathsf{m}\times\mathsf{v}| - \mu(\mathsf{m}\divideontimes\nabla\mathsf{m}) : \nabla\mathsf{v} - \Big(\mathsf{m}\times\psi'(\mathsf{m}) - \mathsf{m}\times\mathsf{h}\Big)\cdot\mathsf{v}\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^s\!\!\int_\Omega \gamma^{-1}\mathsf{m}\cdot\dot{\mathsf{v}}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \gamma^{-1}(\mathsf{m}(s,x)\cdot\mathsf{v}(s,x) - \mathsf{m}_0\cdot\mathsf{v}(0,x))\,\mathrm{d}x. \end{aligned} \tag{3.16}$$

For every fixed partition of $0 = t_0 < t_1 < \ldots t_k = s$ of the interval $[0, s]$, we have

$$\begin{aligned} \int_0^s\!\!\int_\Omega |\dot{\mathsf{m}}_{\alpha_j}|\,\mathrm{d}x\,\mathrm{d}t &= \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}}\!\!\int_\Omega |\dot{\mathsf{m}}_{\alpha_j}|\,\mathrm{d}x\,\mathrm{d}t = \sum_{i=0}^{k-1} \int_\Omega \int_{t_i}^{t_{i+1}} |\dot{\mathsf{m}}_{\alpha_j}|\,\mathrm{d}t\,\mathrm{d}x \\ &\geq \sum_{i=0}^{k-1} \int_\Omega \mathrm{dist}_{S(2)}\Big(\mathsf{m}_{\alpha_j}(t_i), \mathsf{m}_{\alpha_j}(t_{i+1})\Big)\,\mathrm{d}x, \end{aligned} \tag{3.17}$$

because $|\mathsf{m}_{\alpha_j}(t,\cdot)| = 1$ a.e. on Ω for all $t \in [0,s]$. Fixing j, and taking the supremum over all partitions, and using (2.30), we obtain

$$\mathrm{Var}_{S(2)}(\mathsf{m}_{\alpha_j};0,s) \le C_0 \quad \forall j \in \mathbb{N} \text{ and } \forall s \in [0,T]. \tag{3.18}$$

Using (2.29) and (3.18), assumption (3.10) is satisfied and we can apply Lemma 3.1 to obtain (3.11). An immediate consequence of (3.11) is that, due to the convexity of $\mathcal{E}(s,\cdot)$,

$$\liminf_{\alpha_j\to 0} \mathcal{E}(s,\mathsf{m}_{\alpha_j}(s,\cdot)) \ge \mathcal{E}(s,\mathsf{m}(s,\cdot)) \quad \forall s \in \bar{I}. \tag{3.19}$$

Now, we fix the partition in (3.17) and we let $j \to \infty$. By compact embedding, (3.11) implies that $\mathsf{m}_{\alpha_j}(t) \to \mathsf{m}(t)$ in $L^{6-1/\eta}(\Omega;\mathbb{R}^3)$, η as in (3.15), and for every chosen $s \in \bar{I}$, hence passing to the limit in (3.17) we find, by the Fatou lemma,

$$\liminf_{\alpha_j\to 0} \int_0^s\int_\Omega |\dot{\mathsf{m}}_{\alpha_j}|\,\mathrm{d}x\,\mathrm{d}t \ge \sum_{i=0}^{k-1}\int_\Omega \mathrm{dist}_{S(2)}\Big(\mathsf{m}(t_i),\mathsf{m}(t_{i+1})\Big)\,\mathrm{d}x. \tag{3.20}$$

Taking the supremum over all possible partitions of the interval $[0,s]$ in (3.20), we obtain

$$\liminf_{\alpha_j\to 0} \int_0^s\int_\Omega |\dot{\mathsf{m}}_{\alpha_j}|\,\mathrm{d}x\,\mathrm{d}t \ge \mathrm{Var}_{S(2)}(\mathsf{m};0,s). \tag{3.21}$$

By (3.19) and (3.21) we have

$$\liminf_{\alpha_j\to 0} I_{2,\alpha_j}(s) \ge \beta\,\mathrm{Var}_{S(2)}(\mathsf{m};0,s) + \mathcal{E}(s,\mathsf{m}(s,\cdot)) - \mathcal{E}(0,\mathsf{m}_0) + \int_0^s\int_\Omega \dot{\mathsf{h}}\cdot\mathsf{m}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.22}$$

Combining (3.14), (3.16), and (3.22) we obtain (3.13). Note that the limit satisfies (3.12) because the left-hand side of (3.21) with $s = T$ is finite (indeed, as already observed, $\mathrm{Var}_{S(2)}(\mathsf{m};0,T) < +\infty$ implies that $\mathsf{m}(t,\cdot) \in S(2)$ a.e. on Ω for all $t \in [0,T]$). □

Remark 3.3. A discussion of the format (3.13) in the context of weak solutions for rate-independent systems may be found in [32]. A similar notion may be found in [37].

THE LIMIT $\gamma^{-1} \to 0$.

The case $\alpha > 0$ fixed and $\gamma^{-1} \to 0$ is much easier to handle, because the estimates (2.29)-(2.31) do not involve γ. Moreover, the only term containing γ in the weak formulation (2.9) depends linearly on $\dot{\mathsf{m}}$. Indeed, let $\alpha > 0$ and $\beta > 0$ be fixed. For each $\gamma > 0$, let m_γ be a solution to (2.1) with boundary condition (2.2) and initial condition (2.3). Due to the bounds (2.29) and (2.31), one can select a sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ such that $\gamma_k^{-1} \to 0$, $\mathsf{m}_{\gamma_k} \overset{*}{\rightharpoonup} \mathsf{m}$ in $L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3))$, and $\dot{\mathsf{m}}_{\gamma_k} \rightharpoonup \dot{\mathsf{m}}$ in $L^2(Q;\mathbb{R}^3)$ as $k \to \infty$. Then $\iint_Q \gamma_k^{-1}\dot{\mathsf{m}}_{\gamma_k}\cdot \mathsf{v}\,\mathrm{d}x\,\mathrm{d}t \to 0$ for all $\mathsf{v} \in C^1(\overline{Q};\mathbb{R}^3)$ and, by an application of the compactness arguments used in Steps 5–6 of the proof of Proposition 2.5, one can show that m satisfies (2.9) with $\gamma^{-1} = 0$.

As a matter of fact, the strict positivity of γ^{-1} is not essential, and we can extend the notion of weak solution to the case $\gamma^{-1} = 0$. However, for $\gamma^{-1} = 0$, (2.1) alone does not guarantee $|\mathsf{m}| = 1$ and the strong formulation corresponding to (2.9) is

$$\left.\begin{aligned} &0 = \mathsf{m}\times\Big(\mu\Delta\mathsf{m} - \psi'(\mathsf{m}) + \mathsf{h} - \mathsf{r}\Big) \\ &\mathsf{r} \in \partial R_{\alpha,\beta}(\dot{\mathsf{m}}) \\ &|\mathsf{m}| = 1 \end{aligned}\right\} \text{ in } Q. \tag{3.23}$$

THE LIMIT $\gamma^{-1} \to 0$ AND $\alpha \to 0$.

We consider two sequences $\alpha_k \to 0$ and $\gamma_k^{-1} \to 0$, and for each k we let m_k be a weak solution in the sense of Definition 2.4 with $\alpha = \alpha_k$ and $\gamma^{-1} = \gamma_k^{-1}$.

Proposition 3.4. *There exist subsequences (not relabeled) $\alpha_k \to 0$ and $\gamma_k^{-1} \to 0$ such that, with m_k being as specified above,*

$$\mathsf{m}_k \overset{*}{\rightharpoonup} \mathsf{m} \text{ in } L^\infty(I; W^{1,2}(\Omega;\mathbb{R}^3)) \tag{3.24}$$

with $\mathsf{m} \in BM(\bar{I}; W^{1,2}(\Omega;\mathbb{R}^3))$ and $\mathrm{Var}_{S(2)}(\mathsf{m}; 0, T) < +\infty$. Moreover, every m obtained by this way satisfies (3.12), $\mathsf{m}(0,\cdot) = \mathsf{m}_0$, and (3.13) with $\gamma^{-1} = 0$, i.e. m is a weak solution to (3.23).

Sketch of the proof. The estimate (2.29) (which does not depend on γ) gives $\iint_Q \gamma_k^{-1}\mathsf{m}_k \cdot \dot{\mathsf{v}}\,\mathrm{d}x\,\mathrm{d}t \to 0$ as $\gamma_k^{-1} \to 0$. The proof follows now the same arguments as in the proof of Proposition 3.2. □

The considerations made so far lead to the following existence result.

Corollary 3.5. *Let $\alpha = 0$ and $\gamma^{-1} \geq 0$. Assume that (2.5)–(2.8) hold. There exists $\mathsf{m} \in BM(\bar{I}; W^{1,2}(\Omega;\mathbb{R}^3))$, satisfying $\mathrm{Var}_{S(2)}(\mathsf{m}; 0, T) < +\infty$, that solves (2.1) (or (3.23) if $\gamma^{-1} = 0$) with boundary/initial conditions (2.2)-(2.3) in the weak sense, i.e. m satisfies (3.12)–(3.13) and $\mathsf{m}(0,\cdot) = \mathsf{m}_0$.*

ENERGETIC SOLUTIONS.

The notion of energetic solutions [24,28] associated with an energy functional $\mathcal{E}$ and a dissipation distance $\mathcal{D}$ is based on two ingredients: a Stability Condition for a configuration on $\mathcal{M}$ at current times and an Energy-Balance Condition along a trajectory $t \mapsto \mathsf{m}(t) \in \mathcal{M}$. In the present context, the energy functional $\mathcal{E} : \bar{I}\times\mathcal{M} \to \mathbb{R}$ is the one defined in (1.4), while the dissipation distance

$\mathcal{D} : \mathcal{M}\times\mathcal{M} \to [0,\infty)$ is given by

$$\mathcal{D}(\mathsf{m}_1,\mathsf{m}_2) := \beta \int_\Omega \mathrm{dist}_{S(2)}(\mathsf{m}_1,\mathsf{m}_2)\,\mathrm{d}x \tag{3.25}$$

with $\mathrm{dist}_{S(2)}$ from (3.8). A state $\widehat{\mathsf{m}} \in \mathcal{M}$ satisfies the *Stability Condition* at a given time t if

$$\mathcal{E}(t,\widehat{\mathsf{m}}) \le \mathcal{E}(t,\widetilde{\mathsf{m}}) + \mathcal{D}(\widehat{\mathsf{m}},\widetilde{\mathsf{m}}) \quad \forall \widetilde{\mathsf{m}} \in \mathcal{M}. \tag{3.26}$$

A trajectory $t \mapsto \mathsf{m}(t)$ satisfies the *Energy Balance Condition* on $\bar{I}$ if

$$\mathcal{E}(t,\mathsf{m}(t)) + \beta\,\mathrm{Var}_{S(2)}(\mathsf{m};0,t) = \mathcal{E}(0,\mathsf{m}(0)) - \int_0^t\!\!\int_\Omega \dot{\mathsf{h}}\cdot\mathsf{m}\,\mathrm{d}x\mathrm{d}s \quad \forall t \in \bar{I}. \tag{3.27}$$

Definition 3.6 (Energetic solutions)**.** *We say that* $\mathsf{m} : \bar{I} \mapsto \mathcal{M}$ *is an energetic solution to* (3.5) *with boundary conditions* (2.2) *and initial conditions* (2.3) *if:*

(i) the function $t \mapsto \frac{\partial}{\partial t}\mathcal{E}(t,\mathsf{m}(t))$ *belongs to* $L^1(I)$*;*

(ii) $\mathsf{m}(t)$ *satisfies the Stability Condition* (3.26) *for all* $t \in \bar{I}$*;*

(iii) the trajectory $t \mapsto \mathsf{m}(t)$ *satisfies the Energy Balance Condition* (3.27)*.*

Energetic solutions are a subclass of weak solutions [32, Proposition 5.2].

Using the existence results of Mainik and Mielke [24, Theorem 4.5], or Francfort and Mielke [15, Theorem 3.4], it is relatively easy to prove existence of energetic solutions under the additional assumptions:

$$\mathsf{m}_0 \text{ is stable at } t = 0 \text{ in the sense of (3.26);} \tag{3.28}$$

$$\mathsf{h} \in W^{1,\infty}(I; L^1(\Omega;\mathbb{R}^3)). \tag{3.29}$$

Proposition 3.7. *In addition to the assumptions made in Corollary 3.5, suppose that* (3.28)-(3.29) *hold. Then there exists an energetic solution* $t \mapsto \mathsf{m}(t)$ *to* (3.5) *with boundary conditions* (2.2) *and initial conditions* (2.3)*.*

Outline of the proof. We endow the manifold $\mathcal{M}$ defined in (3.9) with the weak topology of $W^{1,2}(\Omega;\mathbb{R}^3)$. To apply Theorem 4.5 of [24] it suffices to verify that conditions (A1)-(A9) listed in [24] hold true. It is easy to verify that the dissipation distance $\mathcal{D}$ defined in (3.25) satisfies the triangle inequality, i.e. Condition (A1) in [24]. Also, due to (3.29), the map $\frac{\partial}{\partial t}\mathcal{E}(\cdot,\mathsf{m})$ is Lipschitz continuous, i.e. Condition (A2) in [24] holds. Since $\mathrm{dist}_{S(2)}(\cdot,\cdot)$ is continuous on $S(2) \times S(2)$, we have that $\mathcal{D}$ is continuous on $\mathcal{M} \times \mathcal{M}$. Moreover, the energy functional $\mathcal{E}$, by its definition and assumption (2.5), is lower semicontinuous. Therefore Condition (A3) (lower semicontinuity of $\mathcal{D}$), Condition (A9) ($\mathcal{E}$ is lower semicontinuous), and Condition (A6) (lower semicontinuity of $\mathcal{E}(t,\cdot)+\mathcal{D}(\widetilde{\mathsf{m}},\cdot)$ for all $t \in [0,T]$ and all $\widetilde{\mathsf{m}} \in \mathcal{M}$) are satisfied. Since $\frac{\partial}{\partial t}\mathcal{E}(t,\cdot)$ is continuous for a.e. $t \in [0,T]$, also Condition (A8) holds true.

Further conditions involve the *stability* and the *reachable* sets

$$\mathcal{S}(t) := \Big\{\mathsf{m} \in \mathcal{M} : \ \mathsf{m} \text{ is stable at time } t \text{ in the sense of (3.26)}\Big\},$$
$$\mathcal{R}(t) := \Big\{\mathsf{m} \in \mathcal{M} : \ \mathcal{E}(t,\mathsf{m}) + \mathcal{D}(\mathsf{m},\mathsf{m}_0) \leq \mathcal{E}(t,\mathsf{m}_0) + Lt + 1\Big\},$$

where $L = \operatorname{ess\,sup}_{t\in I} \|\dot{\mathsf{h}}\|_{L^1(\Omega;\mathbb{R}^3)}$. Also, we have conditions involving the sets

$$\mathcal{S}_{[0,T]} := \bigcup_{t\in[0,T]} \{t\}\times\mathcal{S}(t), \quad \mathcal{R}_{[0,T]} := \bigcup_{t\in[0,T]} \{t\}\times\mathcal{R}(t), \quad \mathcal{V}_{[0,T]} := \mathcal{S}_{[0,T]} \cap \mathcal{R}_{[0,T]}.$$

Condition (A4) in [24] reads as

$$\left.\begin{array}{l} (t_k,\mathsf{m}_k) \in \mathcal{V}_{[0,T]} \\ t_k \to t \\ \mathcal{D}(\mathsf{m}_k,\mathsf{m}) \to 0 \end{array}\right\} \quad \Rightarrow \quad \mathsf{m}_k \rightharpoonup \mathsf{m} \text{ in } W^{1,2}(\Omega;\mathbb{R}^3),$$

and follows easily thanks to the definition of $\mathcal{V}_{[0,T]}$ and the definition of $\mathcal{D}$, by using also the fact that $\mathsf{m}_k \in \mathcal{R}(t_k) \implies \|\mathsf{m}_k\|_{W^{1,2}(\Omega;\mathbb{R}^3)}$ is bounded uniformly with respect to k. From this property and the lower semicontinuity of $\mathcal{E}(T,\cdot)$, we also deduce that $\mathcal{R}(T)$ is compact, which corresponds to Condition (A5). By using in addition the continuity of $\mathcal{D}$ on $\mathcal{M}\times\mathcal{M}$, the remaining Condition (A7), i.e. the compactness of $\mathcal{V}_{[0,T]}$, is satisfied. □

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