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Decay estimates for second order evolution equations with memory

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Abstract

This paper develops a unified method to derive decay estimates for general second order integro-differential evolution equations with semilinear source terms. Depending on the properties of convolution kernels at infinity, we show that the energy of a mild solution decays exponentially or polynomially as $t \to +\infty$. Our approach is based on integral inequalities and multiplier techniques.

These decay results can be applied to various partial differential equations. We discuss three examples: a semilinear viscoelastic wave equation, a linear anisotropic elasticity model, and a Petrovsky type system.

Keywords: Stability; Global existence; Exponential decay; Polynomial decay; Integro-differential equations; Nonlinear wave equations; Linear elasticity; Petrovsky system

1. Introduction

It is well known that viscoelastic materials exhibit natural damping, which is due to the special property of these materials to keep memory of their past history. From the mathematical point of

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view, these damping effects are modeled by integro-differential operators. A simple example is the viscoelastic membrane equation

$$\begin{cases} \partial_t^2 u(t,\xi) - \Delta u(t,\xi) + \displaystyle\int_{-\infty}^{t} \beta(t-s)\Delta u(s,\xi)\,ds = 0, & t \geqslant 0,\ \xi \in \Omega, \\ u(t,\xi) = 0, & t \geqslant 0,\ \xi \in \partial\Omega, \end{cases} \tag{1}$$

in a bounded open domain $\Omega \subset \mathbb{R}^N$, see, e.g. [11,12,14,18,22,25].

A semilinear initial value problem related to Eq. (1) is considered in [8], where the balanced damping effects of friction and viscoelasticity are studied, and in [5], where the equation

$$\begin{cases} \partial_t^2 u(t,\xi) - \Delta u(t,\xi) + \displaystyle\int_{0}^{t} \beta(t-s)\Delta u(s,\xi)\,ds = \left|u(t,\xi)\right|^{\gamma} u(t,\xi), & t \geqslant 0,\ \xi \in \Omega, \\ u(t,\xi) = 0, & t \geqslant 0,\ \xi \in \partial\Omega, \end{cases} \tag{2}$$

is analyzed. Both papers [5] and [8] use a Lyapunov type technique for some perturbed energy, following the method introduced by Komornik and Zuazua [17]. In particular, the modified Lyapunov function introduced in [5] allows to weaken some of the technical assumptions of [8] for convolution kernels.

The above considerations explain, in part, our interest in the abstract integro-differential equation

$$u''(t) + Au(t) - \int_0^t \beta(t-s)Au(s)\,ds = \nabla F\big(u(t)\big), \quad t \in (0,\infty), \tag{3}$$

in a Hilbert space X, where $A : D(A) \subset X \to X$ is an accretive self-adjoint linear operator with dense domain, and ∇F denotes the gradient of a Gâteaux differentiable functional $F : D(\sqrt{A}) \to \mathbb{R}$. In particular, Eq. (1) fits into this framework as well as several other classical equations of mathematical physics such as the linear elasticity system (see Section 4 for details). Related results for the exponential asymptotic stability of solutions to linear integro-differential equations like (3) were obtained in [3].

The main goal of this paper is to obtain decay estimates for the general equation (3) under minimal assumptions on β: we shall assume that $\beta : [0,\infty) \to [0,\infty)$ is a locally absolutely continuous function satisfying, for some $p \in (2,\infty]$,

$$\beta(0) > 0, \quad \int_0^\infty \beta(t)\,dt < 1, \quad \beta' \leqslant -k\beta^{1+\frac{1}{p}}.$$

We will then show that the above conditions ensure that the energy of any mild solution of (3), with sufficiently small initial data, decays at infinity with the same (exponential or polynomial) rate as β, see Theorems 3.5 and 3.6. Observe that the above assumptions allow for exponential ($p = \infty$), polynomial ($2 < p < \infty$), as well as compactly supported convolution kernels. Moreover, for linear problems ($F \equiv 0$), no restriction on the initial data is required.

Our result differs from the one of [5,8] for both set-up and methodology. Indeed, instead of using a Lyapunov type technique for some perturbed energy, we rather concentrate on the original energy, showing that this one satisfies a nonlinear integral inequality which, in turn, yields the final decay estimate.

The advantage of our approach is clear. By the same general theorem we recover, as special cases, the result in [5] for Eq. (2), the result in [21] (in the case of bounded space domains) allowing for more general initial conditions and convolution kernels, and we can also obtain new results for other partial differential operators such as Petrovsky systems.

The outline of this paper is the following. In Section 2 we fix notations and recall some decay estimates. In Section 3 we prove our main stability results. Finally, in Section 4 we give applications to various partial differential operators.

2. Preliminaries

Let X be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

For any $T \in (0,\infty]$, $C^k([0,T];X)$, $k=0,1,2$, stands for the space of continuous functions from $[0,T]$ to X having continuous derivatives up to the order k in $[0,T]$. In particular, we shall use the notation $C([0,T];X)$ for $C^0([0,T];X)$.

Further, for any $T \in (0,\infty]$ we denote by $L^1(0,T;X)$ and $L^\infty(0,T;X)$ the usual spaces of measurable functions $v:(0,T)\to X$ such that one has

$$\|v\|_1 := \int_0^T \|v(t)\|\,dt < \infty,$$

$$\|v\|_\infty := \operatorname{ess}\sup_{0\leqslant t\leqslant T} \|v(t)\| < \infty,$$

respectively.

In the case $X=\mathbb{R}$, we make use of the abbreviations $L^1(0,T)$ and $L^\infty(0,T)$ to denote the spaces $L^1(0,T;\mathbb{R})$ and $L^\infty(0,T;\mathbb{R})$, respectively.

For any $\psi \in L^1(0,T)$ and any $v \in L^1(0,T;X)$ the symbol $\psi * v$ stands for convolution from 0 to t, that is

$$\psi * v(t) = \int_0^t \psi(t-s)v(s)\,ds, \quad t \in [0,T].$$

For reader's convenience we give a proof of the following lemmas. For more general integral inequalities see also [1,2,13,19].

Lemma 2.1. *Let E be a nonnegative decreasing function defined on* $[0,\infty)$. *If*

$$\int_S^\infty E(t)\,dt \leqslant CE(S) \quad \forall S \geqslant S_0, \tag{4}$$

for some constants $S_0, C > 0$*, then*

$$E(t) \leqslant E(0)\exp\left(1 - \frac{t}{S_0 + C}\right) \quad \forall t \geqslant 0.$$

Proof. If $0 \leqslant S \leqslant S_0$, by (4) we have

$$\int_S^\infty E(t)\,dt = \int_S^{S_0} E(t)\,dt + \int_{S_0}^\infty E(t)\,dt$$
$$\leqslant (S_0 - S)E(S) + CE(S_0) \leqslant (S_0 + C)E(S),$$

whence

$$\int_S^\infty E(t)\,dt \leqslant (S_0 + C)E(S) \quad \forall S \geqslant 0.$$

Therefore, by a well-known decay estimate (see, e.g., [16, Theorem 8.1]) the conclusion follows. □

Using the same arguments as in the proof of Lemma 2.1, one gets

Lemma 2.2. *Let* E *be a nonnegative decreasing function defined on* $[0, \infty)$*. Assume there are positive constants* α*,* C *and* S_0 *such that*

$$\int_S^\infty E^{1+\alpha}(t)\,dt \leqslant CE^\alpha(0)E(S) \quad \forall S \geqslant S_0. \tag{5}$$

Then

$$E(t) \leqslant E(0)\left(\frac{(S_0 + C)(1 + \alpha)}{\alpha t + S_0 + C}\right)^{\frac{1}{\alpha}} \quad \forall t \geqslant 0.$$

3. Stability for the abstract problem

In this section, we shall be concerned with the asymptotic behaviour of solutions to the second order integro-differential equation

$$u''(t) + Au(t) - \int_0^t \beta(t - s)Au(s)\,ds = \nabla F\big(u(t)\big), \quad t \in (0, \infty), \tag{6}$$

in a Hilbert space X. To begin with, we assume the following conditions.

Assumptions (H1).

1. A is a self-adjoint linear operator on X with dense domain $D(A)$, satisfying
$$\langle Ax, x\rangle \geqslant M\|x\|^2 \quad \forall x \in D(A), \tag{7}$$
for some $M > 0$.
2. $\beta : [0,\infty) \to [0,\infty)$ is a locally absolutely continuous function such that
$$\int_0^\infty \beta(t)\,dt < 1 \tag{8}$$
$$\beta(0) > 0, \quad \beta'(t) \leqslant 0 \quad \text{for a.e. } t \geqslant 0. \tag{9}$$
3. $F : D(\sqrt{A}) \to \mathbb{R}$ is a functional such that
   (a) F is Gâteaux differentiable at any point $x \in D(\sqrt{A})$;
   (b) for any $x \in D(\sqrt{A})$ there exists a constant $c(x) > 0$ such that
   $$\left|DF(x)(y)\right| \leqslant c(x)\|y\|, \quad \text{for any } y \in D(\sqrt{A}), \tag{10}$$
   where $DF(x)$ denotes the Gâteaux derivative of F in x; consequently, $DF(x)$ can be extended to the whole space X (and we will denote by $\nabla F(x)$ the unique vector representing $DF(x)$ in the Riesz isomorphism, that is, $\langle \nabla F(x), y\rangle = DF(x)(y)$, for any $y \in X$);
   (c) for any $R > 0$ there exists a constant $C_R > 0$ such that
   $$\left\|\nabla F(x) - \nabla F(y)\right\| \leqslant C_R\|\sqrt{A}x - \sqrt{A}y\| \tag{11}$$
   for all $x, y \in D(\sqrt{A})$ satisfying $\|\sqrt{A}x\|, \|\sqrt{A}y\| \leqslant R$.

Let $u_0, u_1 \in X$ and consider the Cauchy problem
$$\begin{cases} u''(t) + Au(t) - \displaystyle\int_0^t \beta(t-s)Au(s)\,ds = \nabla F\big(u(t)\big), \quad t \in (0,\infty), \\ u(0) = u_0, \\ u'(0) = u_1. \end{cases} \tag{12}$$

We recall that a *mild solution* of (12) in $[0,T]$, $T > 0$, is a function $u \in C([0,T]; D(\sqrt{A}))$ that satisfies the integral equation
$$u(t) = S(t)u_0 + \int_0^t S(\tau)u_1\,d\tau + \int_0^t 1 * S(t-\tau)\nabla F\big(u(\tau)\big)\,d\tau, \quad \forall t \in [0,T], \tag{13}$$

where $\{S(t)\}$ is the resolvent for the linear equation

$$u''(t) + Au(t) - \int_0^t \beta(t-s)Au(s)\,ds = 0, \quad t \in (0,\infty),$$

see [6,7,24].

Another useful notion of generalized solution of (6) is the so-called *weak solution*, that is a function $u \in C^1([0,T];X) \cap C([0,T];D(\sqrt{A}))$ such that, for any $v \in D(\sqrt{A})$, $\langle u'(t), v\rangle \in C^1([0,T])$ and

$$\frac{d}{dt}\langle u'(t), v\rangle + \langle \sqrt{A}u(t), \sqrt{A}v\rangle - \left\langle \int_0^t \beta(t-s)\sqrt{A}u(s)\,ds, \sqrt{A}v\right\rangle = \langle \nabla F(u(t)), v\rangle, \quad \forall t \in [0,T]. \tag{14}$$

Adapting a classical argument due to Ball [4], one can show that any mild solution of (6) is also a weak solution, and the two notions of solution are equivalent when $F \equiv 0$.

For more regular data, one should expect more regular solutions. In particular, we will call a *strong solution* of Eq. (6) any function $u \in C^2([0,T];X) \cap C([0,T];D(A))$ verifying the equation for all $t \in [0,T]$.

Local existence, uniqueness, and regularity for (12) is guaranteed by the following result.

Proposition 3.1. *For any $u_0 \in D(\sqrt{A})$ and $u_1 \in X$, a number $T > 0$ exists so that* (12) *admits a unique mild solution u on* $[0,T]$. *Moreover, u belongs to* $C^1([0,T];X)$.

Furthermore, if $u_0 \in D(A)$ and $u_1 \in D(\sqrt{A})$, then the mild solution of (12) *is a strong solution and belongs to* $C^1([0,T];D(\sqrt{A}))$.

The proof of the above theorem is a variant of a well-known linear result that can be found in [24], see [6] for details.

3.1. Dissipation

Extending to the present abstract set-up an idea due to [22], we define the energy of a mild solution u of (12) on a given interval $[0,T]$ as

$$E_u(t) := \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\left(1 - \int_0^t \beta(s)\,ds\right)\|\sqrt{A}u(t)\|^2 - F(u(t)) + \frac{1}{2}\int_0^t \beta(t-s)\|\sqrt{A}u(s) - \sqrt{A}u(t)\|^2\,ds, \quad t \in [0,T]. \tag{15}$$

Then, we obtain the following preliminary properties.

Theorem 3.2. *Under Assumptions* (H1)*, the energy of any mild solution u of* (12) *is a decreasing function.*

Proof. Suppose, first, that u is a strong solution of (6) on $[0,T]$. Differentiating formula (15), we obtain

$$\begin{aligned}
\frac{d}{dt}E_u(t) &= \langle u''(t), u'(t)\rangle + \langle \sqrt{A}u(t), \sqrt{A}u'(t)\rangle - \frac{1}{2}\beta(t)\|\sqrt{A}u(t)\|^2 \\
&\quad - \frac{1}{2}\left(\int_0^t \beta(s)\,ds\right)\frac{d}{dt}\|\sqrt{A}u(t)\|^2 \\
&\quad + \frac{1}{2}\int_0^t \beta'(t-s)\|\sqrt{A}u(t) - \sqrt{A}u(s)\|^2\,ds \\
&\quad - \int_0^t \beta(t-s)\langle \sqrt{A}u(s) - \sqrt{A}u(t), \sqrt{A}u'(t)\rangle\,ds \\
&\quad - \langle \nabla F(u(t)), u'(t)\rangle \\
&= \langle u''(t), u'(t)\rangle + \langle \sqrt{A}u(t), \sqrt{A}u'(t)\rangle - \frac{1}{2}\beta(t)\|\sqrt{A}u(t)\|^2 \\
&\quad - \frac{1}{2}\left(\int_0^t \beta(s)\,ds\right)\frac{d}{dt}\|\sqrt{A}u(t)\|^2 \\
&\quad + \frac{1}{2}\int_0^t \beta'(t-s)\|\sqrt{A}u(t) - \sqrt{A}u(s)\|^2\,ds - \langle \beta * \sqrt{A}u(t), \sqrt{A}u'(t)\rangle \\
&\quad + \left(\frac{1}{2}\int_0^t \beta(s)\,ds\right)\frac{d}{dt}\|\sqrt{A}u(t)\|^2 - \langle \nabla F(u(t)), u'(t)\rangle \\
&= \langle u''(t), u'(t)\rangle + \langle Au(t), u'(t)\rangle - \langle \beta * Au(t), u'(t)\rangle - \langle \nabla F(u(t)), u'(t)\rangle \\
&\quad - \frac{1}{2}\beta(t)\|\sqrt{A}u(t)\|^2 + \frac{1}{2}\int_0^t \beta'(t-s)\|\sqrt{A}u(t) - \sqrt{A}u(s)\|^2\,ds \\
&= -\frac{1}{2}\beta(t)\|\sqrt{A}u(t)\|^2 + \frac{1}{2}\int_0^t \beta'(t-s)\|\sqrt{A}u(t) - \sqrt{A}u(s)\|^2\,ds
\end{aligned} \tag{16}$$

where the last identity holds because u is a solution of (6). Since $\beta \geqslant 0$ and $\beta' \leqslant 0$, the right-hand side above is negative. We have thus shown that the energy $E_u(t)$ of any strong solution is decreasing. An approximation argument suffices to extend such a conclusion to mild solutions. □

Applying Theorem 3.2 we can prove a global existence result for problem (12) with sufficiently small initial conditions.

Theorem 3.3. *Let Assumptions* (H1) *be satisfied. Suppose, in addition, that there exists a strictly increasing continuous function* $\psi : [0, \infty) \to [0, \infty)$ *such that*

$$\left|F(x)\right| \leqslant \psi\left(\|\sqrt{A}x\|\right)\|\sqrt{A}x\|^2 \quad \forall x \in D(\sqrt{A}). \tag{17}$$

Then a number $\rho_0 > 0$ *exists such that, for any* $u_0 \in D(\sqrt{A})$ *and* $u_1 \in X$ *satisfying*

$$\|\sqrt{A}u_0\| + \|u_1\| < \rho_0, \tag{18}$$

problem (12) *admits a unique mild solution* u *on* $[0, \infty)$. *Moreover,* $E_u(t)$ *is positive and*

$$E_u(0) \leqslant \rho_0^2, \tag{19}$$

$$E_u(t) \geqslant \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{4}\left(1 - \int_0^\infty \beta(s)\, ds\right)\left\|\sqrt{A}u(t)\right\|^2, \tag{20}$$

$$\psi\left(\left\|\sqrt{A}u(t)\right\|\right) \leqslant \frac{1}{4}\left(1 - \int_0^\infty \beta(s)\, ds\right) \tag{21}$$

for every $t \geqslant 0$. *Furthermore, if* $u_0 \in D(A)$ *and* $u_1 \in D(\sqrt{A})$, *then* u *is a strong solution of* (6).

Proof. Let $[0, T)$ be the maximal domain of definition of the mild solution u of (6) and let

$$\nu := \frac{1}{2}\left(1 - \int_0^\infty \beta(s)\, ds\right).$$

Observe that $E_u(0) \geqslant 0$ if $\psi(\|\sqrt{A}u_0\|) < \nu/2$, for

$$E_u(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|\sqrt{A}u_0\|^2 - F(u_0) \geqslant \frac{1}{2}\|u_1\|^2 + \frac{1-\nu}{2}\|\sqrt{A}u_0\|^2.$$

We claim that, if

$$\psi\left(\|\sqrt{A}u_0\|\right) < \frac{\nu}{2} \quad \text{and} \quad \psi\left(\left(\frac{2E_u(0)}{\nu}\right)^{\frac{1}{2}}\right) < \frac{\nu}{2}, \tag{22}$$

then

$$E_u(t) \geqslant \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{\nu}{2}\left\|\sqrt{A}u(t)\right\|^2 \quad \forall t \in [0, T). \tag{23}$$

Indeed, let τ be the supremum of all $s \in [0, T)$ such that (23) holds true for any $t \in [0, s]$. Suppose $\tau < T$. By continuity,

$$E_u(\tau) \geqslant \frac{1}{2}\left\|u'(\tau)\right\|^2 + \frac{\nu}{2}\left\|\sqrt{A}u(\tau)\right\|^2 \geqslant 0.$$

Hence, by Theorem 3.2 we have that

$$\psi\left(\left\|\sqrt{A}u(\tau)\right\|\right) \leqslant \psi\left(\left(\frac{2E_u(\tau)}{\nu}\right)^{\frac{1}{2}}\right) \leqslant \psi\left(\left(\frac{2E_u(0)}{\nu}\right)^{\frac{1}{2}}\right) < \frac{\nu}{2}.$$

Moreover, assumption (17) and the above inequality yield

$$\begin{aligned} E_u(\tau) &\geqslant \frac{1}{2}\left\|u'(\tau)\right\|^2 + \nu\left\|\sqrt{A}u(\tau)\right\|^2 - F\big(u(\tau)\big) \\ &> \frac{1}{2}\left\|u'(\tau)\right\|^2 + \left(\nu - \frac{\nu}{2}\right)\left\|\sqrt{A}u(\tau)\right\|^2 = \frac{1}{2}\left\|u'(\tau)\right\|^2 + \frac{\nu}{2}\left\|\sqrt{A}u(\tau)\right\|^2. \end{aligned}$$

This contradicts the maximality of τ.

Let

$$\rho_0 := \sqrt{\frac{\nu}{2}\psi^{-1}\left(\frac{\nu}{2}\right)} > 0.$$

Then $\psi(\|\sqrt{A}u_0\|) < \nu/2$ for any $u_0 \in D(\sqrt{A})$ and any $u_1 \in X$ such that $\|\sqrt{A}u_0\| + \|u_1\| < \rho_0$. In addition, (19) is satisfied since

$$E_u(0) \leqslant \frac{1}{2}\|u_1\|^2 + \|\sqrt{A}u_0\|^2 \leqslant \big(\|u_1\| + \|\sqrt{A}u_0\|\big)^2 \leqslant \rho_0^2 = \frac{\nu}{2}\psi^{-1}\left(\frac{\nu}{2}\right)^2.$$

Moreover,

$$\psi\left(\left(\frac{2E_u(0)}{\nu}\right)^{\frac{1}{2}}\right) < \frac{\nu}{2}.$$

So, we have shown that, for $\|\sqrt{A}u_0\| + \|u_1\| < \rho_0$, conditions (22) are satisfied, and hence (23) holds true. Thus, the energy of u is nonnegative on $[0, T)$: E_u is bounded and u is global.

Finally, estimate (23) yields (20), which in turn implies (21) since

$$\psi\left(\left\|\sqrt{A}u(t)\right\|\right) \leqslant \psi\left(\left(\frac{2E_u(t)}{\nu}\right)^{\frac{1}{2}}\right) \leqslant \psi\left(\left(\frac{2E_u(0)}{\nu}\right)^{\frac{1}{2}}\right) < \frac{\nu}{2} \quad \forall t \geqslant 0. \qquad \square$$

Remark 3.4. In the linear case $F \equiv 0$ restriction (18) is unnecessary: the energy of any mild solution is nonnegative and for any $u_0 \in D(\sqrt{A})$ and $u_1 \in X$ one gets a unique global solution, given by

$$u(t) = S(t)u_0 + \int_0^t S(\tau)u_1\,d\tau, \quad \forall t \geqslant 0.$$

3.2. Main results

In this section, we will study the asymptotic behaviour at ∞ of a mild solution of (6) providing conditions to ensure that the energy of the solution decays at ∞ at exponential or polynomial rate. Such conditions are obtained by strengthening Assumptions (H1) as follows.

Assumptions (H2).

1. There exist $p \in (2, \infty]$ and $k > 0$ such that
$$\beta'(t) \leqslant -k\beta^{1+\frac{1}{p}}(t) \quad \text{for a.e. } t \geqslant 0 \tag{24}$$
(here we have set $\frac{1}{p} = 0$ for $p = \infty$).
2. $F(0) = 0$, $\nabla F(0) = 0$, and there is a strictly increasing continuous function $\psi : [0, \infty) \to [0, \infty)$ such that
$$\left|\left\langle \nabla F(x), x\right\rangle\right| \leqslant \psi\left(\|\sqrt{A}x\|\right)\|\sqrt{A}x\|^2 \quad \forall x \in D(\sqrt{A}). \tag{25}$$

Our main results are the following.

Theorem 3.5. *Assume* (H1) *and* (H2) *with* $p = \infty$. *Then, there exist positive numbers* ρ_0[1] *and* C *such that, for all initial data* $(u_0, u_1) \in D(\sqrt{A}) \times X$ *satisfying*
$$\|\sqrt{A}u_0\| + \|u_1\| < \rho_0,$$
the energy $E_u(t)$ *of the mild solution* u *of* (12) *decays as*
$$E_u(t) \leqslant E_u(0)\exp(1 - Ct) \quad \forall t \geqslant 0.$$
Moreover, one can take $\rho_0 = \infty$ *if* $F \equiv 0$.

Theorem 3.6. *Assume* (H1) *and* (H2) *with* $p \in (2, \infty)$. *Then, there exist positive numbers* ρ_0 *and* C *such that, for all initial data* $(u_0, u_1) \in D(\sqrt{A}) \times X$ *satisfying*
$$\|\sqrt{A}u_0\| + \|u_1\| < \rho_0,$$
the energy $E_u(t)$ *of the mild solution* u *of* (12) *decays as*
$$E_u(t) \leqslant E_u(0)\left(\frac{C(1+p)}{t+pC}\right)^p \quad \forall t \geqslant 0.$$
Moreover, one can take $\rho_0 = \infty$ *if* $F \equiv 0$.

The above theorems are an immediate consequence of Lemmas 2.1 and 2.2, respectively, and of the following technical result.

[1] ρ_0 is given by Theorem 3.3.

Theorem 3.7. *Let Assumptions* (H1) *and* (H2) *be satisfied and let* $S_0 > 0$. *Then, there exist positive numbers* ρ_0 *and* C *such that, for any* $u_0 \in D(\sqrt{A})$ *and* $u_1 \in X$ *with* $\|\sqrt{A}u_0\| + \|u_1\| < \rho_0$, *the energy* $E_u(t)$ *of the mild solution* u *of* (12) *satisfies*

$$\int_S^\infty E_u^{1+\frac{1}{p}}(t)\,dt \leqslant C E_u^{\frac{1}{p}}(0) E_u(S) \quad \forall S \geqslant S_0. \tag{26}$$

Moreover, one can take $\rho_0 = \infty$ *if* $F \equiv 0$.

Remark 3.8.

1. As we shall see in the sequel, the above set-up can be used to treat kernels with either polynomial or exponential decay. For instance, for any $p \in (2, \infty)$, $\beta(t) := (1+t)^{-p}$ satisfies Assumptions (H2)-1 and is a typical kernel with polynomial decay, whereas $\beta(t) := e^{-\alpha t}$, $\alpha > 0$, satisfies the same condition with $p = \infty$.
2. We note that for $p \in (2, \infty)$ it follows from (24)

$$\beta(t) \leqslant \frac{K}{(1+t)^p}, \quad \forall t \geqslant 0,$$

for some constant $K > 0$. Therefore, we have

$$\beta^\theta \in L^1(0, \infty), \quad \text{for any } \theta > 1/p. \tag{27}$$

3. Observe that assumption (H2)-2 ensures that hypothesis (17) is satisfied. Indeed, for $x \in D(\sqrt{A})$

$$\bigl|F(x)\bigr| \leqslant \int_0^1 \bigl|\bigl\langle \nabla F(tx), x\bigr\rangle\bigr|\,dt \leqslant \|\sqrt{A}x\|^2 \int_0^1 \psi\bigl(t\|\sqrt{A}x\|\bigr)t\,dt \leqslant \frac{1}{2}\psi\bigl(\|\sqrt{A}x\|\bigr)\|\sqrt{A}x\|^2.$$

Proof of Theorem 3.7. Let $u_0 \in D(\sqrt{A})$ and $u_1 \in X$ be fixed so as to satisfy $\|\sqrt{A}u_0\| + \|u_1\| < \rho_0$ and let u be the mild solution of (12). Then, owing to Theorem 3.3, the solution is global. Moreover, we shall suppose, first, that u is a strong solution, which is indeed the case if $u_0 \in D(A)$ and $u_1 \in D(\sqrt{A})$. Such an extra assumption will be removed later by a standard approximation argument.

To prove (26), we rewrite the left-hand side of (26) using the definition (15) of the energy. Having fixed $0 \leqslant S \leqslant T$, we obtain

$$\int_S^T E_u^{\frac{1}{p}}(t) E_u(t)\,dt$$
$$= \frac{1}{2}\int_S^T E_u^{\frac{1}{p}}(t)\left(1 - \int_0^t \beta(s)\,ds\right)\bigl\|\sqrt{A}u(t)\bigr\|^2\,dt + \frac{1}{2}\int_S^T E_u^{\frac{1}{p}}(t)\bigl\|u'(t)\bigr\|^2\,dt$$

$$-\int_S^T E_u^{\frac{1}{p}}(t)F\big(u(t)\big)\,dt+\frac{1}{2}\int_S^T E_u^{\frac{1}{p}}(t)\int_0^t \beta(t-s)\big\|\sqrt{A}u(s)-\sqrt{A}u(t)\big\|^2\,ds\,dt. \tag{28}$$

We will now use multipliers techniques to bound all the right-hand side terms of (28).

Lemma 1. *If $\phi:[0,\infty)\to[0,\infty)$ is a function with negative derivative, then we have for any $S_0>0$ and for any $T\geqslant S\geqslant S_0$*

$$\begin{aligned}&\frac{1}{2}\int_S^T \phi(t)\left(1-\int_0^t\beta(s)\,ds\right)\big\|\sqrt{A}u(t)\big\|^2\,dt+\frac{1}{2}\int_S^T\phi(t)\big\|u'(t)\big\|^2\,dt\\&\quad-\int_S^T\phi(t)F\big(u(t)\big)\,dt\leqslant C\phi(0)E_u(S)\end{aligned} \tag{29}$$

for some constant $C>0$.

To prove the above inequality, we need the following three lemmas.

Lemma 2. *For any $T\geqslant S\geqslant 0$ we have*

$$\int_S^T \phi(t)\left(1-\int_0^t\beta(s)\,ds\right)\big\|\sqrt{A}u(t)\big\|^2\,dt\leqslant C\int_S^T\phi(t)\big\|u'(t)\big\|^2\,dt+C\phi(0)E_u(S) \tag{30}$$

for some constant $C>0$.

Proof of Lemma 2. We split the reasoning into four steps.

2.1: Let us take the inner product of both sides of Eq. (6) with $\phi(t)u$ and integrate over $[S,T]$. We obtain

$$\int_S^T\phi(t)\big\langle u''(t)+Au(t)-\beta*Au(t),u(t)\big\rangle\,dt=\int_S^T\phi(t)\big\langle\nabla F\big(u(t)\big),u(t)\big\rangle\,dt \tag{31}$$

whence, integrating by parts,

$$\begin{aligned}&-\int_S^T\phi(t)\big\|u'(t)\big\|^2\,dt-\int_S^T\phi'(t)\big\langle u'(t),u(t)\big\rangle\,dt+\int_S^T\phi(t)\big\|\sqrt{A}u(t)\big\|^2\,dt\\&\quad-\int_S^T\phi(t)\big\langle\beta*\sqrt{A}u(t),\sqrt{A}u(t)\big\rangle\,dt+\big[\phi(t)\big\langle u'(t),u(t)\big\rangle\big]_S^T\end{aligned}$$

$$= \int_S^T \phi(t)\big\langle \nabla F\big(u(t)\big), u(t)\big\rangle\, dt. \tag{32}$$

Therefore,

$$\begin{aligned}
&\int_S^T \phi(t)\left(1 - \int_0^t \beta(s)\, ds\right)\big\|\sqrt{A}u(t)\big\|^2\, dt \\
&\quad = \int_S^T \phi(t)\big\|u'(t)\big\|^2\, dt + \int_S^T \phi(t)\left\langle \int_0^t \beta(t-s)\big(\sqrt{A}u(s) - \sqrt{A}u(t)\big)\, ds, \sqrt{A}u(t)\right\rangle dt \\
&\qquad + \int_S^T \phi'(t)\big\langle u'(t), u(t)\big\rangle\, dt + \int_S^T \phi(t)\big\langle \nabla F\big(u(t)\big), u(t)\big\rangle\, dt - \big[\phi(t)\big\langle u'(t), u(t)\big\rangle\big]_S^T.
\end{aligned} \tag{33}$$

2.2: In order to bound the term

$$\int_S^T \phi(t)\left|\left\langle \int_0^t \beta(t-s)\big(\sqrt{A}u(s) - \sqrt{A}u(t)\big)\, ds, \sqrt{A}u(t)\right\rangle\right| dt$$

we observe that, for any $\varepsilon > 0$,

$$\begin{aligned}
&\int_S^T \phi(t)\big\|\sqrt{A}u(t)\big\| \int_0^t \beta(t-s)\big\|\sqrt{A}u(s) - \sqrt{A}u(t)\big\|\, ds\, dt \\
&\quad \leqslant \frac{\varepsilon}{2}\int_S^T \phi(t)\big\|\sqrt{A}u(t)\big\|^2\, dt \\
&\qquad + \frac{1}{2\varepsilon}\int_S^T \phi(t)\left(\int_0^t \beta(t-s)\big\|\sqrt{A}u(s) - \sqrt{A}u(t)\big\|\, ds\right)^2 dt.
\end{aligned} \tag{34}$$

Recalling (27) and $\beta'(t) \leqslant -k\beta^{1+\frac{1}{p}}(t)$, we get

$$\begin{aligned}
&\int_S^T \phi(t)\left(\int_0^t \beta(t-s)\big\|\sqrt{A}u(s) - \sqrt{A}u(t)\big\|\, ds\right)^2 dt \\
&\quad \leqslant \int_S^T \phi(t)\left(\int_0^t \beta^{1-\frac{1}{p}}(s)\, ds\right)\left(\int_0^t \beta^{1+\frac{1}{p}}(t-s)\big\|\sqrt{A}u(s) - \sqrt{A}u(t)\big\|^2\, ds\right) dt
\end{aligned}$$

$$\leqslant \int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds \int_S^T \phi(t) \int_0^t \beta^{1+\frac{1}{p}}(t-s)\big\|\sqrt{A}u(s)-\sqrt{A}u(t)\big\|^2\,ds\,dt$$
$$\leqslant -\frac{1}{k}\int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds \int_S^T \phi(t) \int_0^t \beta'(t-s)\big\|\sqrt{A}u(s)-\sqrt{A}u(t)\big\|^2\,ds\,dt.$$

So, thanks also to formula (16) for $E_u'(t)$,

$$\int_S^T \phi(t)\left(\int_0^t \beta(t-s)\big\|\sqrt{A}u(s)-\sqrt{A}u(t)\big\|\,ds\right)^2 dt$$
$$\leqslant -\frac{2}{k}\int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds \int_S^T \phi(t)E_u'(t)\,dt \leqslant \frac{2}{k}\int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds\,\phi(0)E_u(S). \tag{35}$$

Therefore, (34) and (35) yield

$$\int_S^T \phi(t)\left|\left\langle\int_0^t \beta(t-s)\big(\sqrt{A}u(s)-\sqrt{A}u(t)\big)\,ds, \sqrt{A}u(t)\right\rangle\right| dt$$
$$\leqslant \frac{\varepsilon}{2}\int_S^T \phi(t)\big\|\sqrt{A}u(t)\big\|^2\,dt + \frac{1}{\varepsilon k}\int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds\,\phi(0)E_u(S). \tag{36}$$

2.3: First, we note that from (20) it follows

$$\frac{1}{2}\big\|\sqrt{A}u(t)\big\|^2 \leqslant \frac{2}{1-\int_0^\infty \beta(s)\,ds}E_u(t), \tag{37}$$

and hence, using also (7), we get

$$\frac{1}{2}\big\|u(t)\big\|^2 \leqslant \frac{1}{2M}\big\|\sqrt{A}u(t)\big\|^2 \leqslant \frac{2}{M(1-\int_0^\infty \beta(s)\,ds)}E_u(t). \tag{38}$$

Applying the above inequality we have

$$\big|\big\langle u'(t), u(t)\big\rangle\big| \leqslant \frac{1}{2}\big\|u'(t)\big\|^2 + \frac{1}{2}\big\|u(t)\big\|^2$$
$$\leqslant \frac{1}{2}\big\|u'(t)\big\|^2 + \frac{2}{M(1-\int_0^\infty \beta(s)\,ds)}E_u(t)$$
$$\leqslant \left(1+\frac{2}{M(1-\int_0^\infty \beta(s)\,ds)}\right)E_u(t). \tag{39}$$

Therefore,

$$\begin{aligned}
\int_S^T \phi'(t)\langle u'(t),u(t)\rangle\,dt &\leqslant -\int_S^T \phi'(t)\bigl|\langle u'(t),u(t)\rangle\bigr|\,dt\\
&\leqslant -\left(1+\frac{2}{M(1-\int_0^\infty \beta(s)\,ds)}\right)\int_S^T \phi'(t)E_u(t)\,dt\\
&\leqslant -\left(1+\frac{2}{M(1-\int_0^\infty \beta(s)\,ds)}\right)\int_S^T \phi'(t)\,dt\,E_u(S)\\
&\leqslant \left(1+\frac{2}{M(1-\int_0^\infty \beta(s)\,ds)}\right)\phi(0)E_u(S). \qquad (40)
\end{aligned}$$

Now, using assumption (25) we obtain

$$\int_S^T \phi(t)\bigl\langle \nabla F(u(t)),u(t)\bigr\rangle\,dt \leqslant \int_S^T \phi(t)\psi\bigl(\bigl\|\sqrt{A}u(t)\bigr\|\bigr)\bigl\|\sqrt{A}u(t)\bigr\|^2\,dt.$$

Now, recalling Remark 3.8-3 we can invoke Theorem 3.3 to obtain (21). Therefore,

$$\int_S^T \phi(t)\bigl\langle \nabla F\bigl(u(t)\bigr),u(t)\bigr\rangle\,dt \leqslant \frac{1}{4}\int_S^T \phi(t)\left(1-\int_0^t \beta(s)\,ds\right)\bigl\|\sqrt{A}u(t)\bigr\|^2\,dt. \qquad (41)$$

Next, again by (39) and since $\phi(t)$ and $E(t)$ are decreasing, we have

$$-\bigl[\phi(t)\bigl\langle u'(t),u(t)\bigr\rangle\bigr]_S^T \leqslant 2\left(1+\frac{2}{M(1-\int_0^\infty \beta(s)\,ds)}\right)\phi(0)E_u(S). \qquad (42)$$

2.4: Using estimates (36), (40), (41) and (42) to bound the terms in the right-hand side of (33), we conclude that, for any $\varepsilon>0$,

$$\begin{aligned}
&\int_S^T \phi(t)\left(1-\int_0^t \beta(s)\,ds\right)\bigl\|\sqrt{A}u(t)\bigr\|^2\,dt\\
&\quad\leqslant \int_S^T \phi(t)\bigl\|u'(t)\bigr\|^2\,dt+\frac{\varepsilon}{2}\int_S^T \phi(t)\bigl\|\sqrt{A}u(t)\bigr\|^2\,dt+\frac{1}{\varepsilon k}\int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds\,\phi(0)E_u(S)\\
&\qquad+3\left(1+\frac{2}{M(1-\int_0^\infty \beta(s)\,ds)}\right)\phi(0)E_u(S)
\end{aligned}$$

$$+\frac{1}{4}\int_S^T \phi(t)\left(1-\int_0^t \beta(s)\,ds\right)\left\|\sqrt{A}u(t)\right\|^2 dt.$$

Then, choosing $\varepsilon = 1-\int_0^\infty \beta(s)\,ds > 0$, we achieve that

$$\int_S^T \phi(t)\left(1-\int_0^t \beta(s)\,ds\right)\left\|\sqrt{A}u(t)\right\|^2 dt$$
$$\leqslant 4\int_S^T \phi(t)\left\|u'(t)\right\|^2 dt + 4\left(\frac{1}{\varepsilon k}\int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds + 3 + \frac{6}{M(1-\int_0^\infty \beta(s)\,ds)}\right)\phi(0)E_u(S),$$

and so it follows (30). This completes the proof of Lemma 2. $\square$

Lemma 3. *The following identity holds true for any $T \geqslant S \geqslant 0$:*

$$\int_S^T \phi(t)\left(\int_0^t \beta(s)\,ds\right)\left\|u'(t)\right\|^2 dt$$
$$= -\left[\phi(t)\left\langle u'(t), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle\right]_S^T$$
$$+\int_S^T \phi'(t)\left\langle u'(t), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt$$
$$+\int_S^T \phi(t)\left\langle u'(t), \int_0^t \beta'(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt$$
$$+\int_S^T \phi(t)\left(\int_0^t \beta(s)\,ds - 1\right)\left\langle \sqrt{A}u(t), \int_0^t \beta(t-s)\big(\sqrt{A}u(s)-\sqrt{A}u(t)\big)\,ds\right\rangle dt$$
$$+\int_S^T \phi(t)\left\|\int_0^t \beta(t-s)\big(\sqrt{A}u(s)-\sqrt{A}u(t)\big)\,ds\right\|^2 dt$$
$$+\int_S^T \phi(t)\left\langle \nabla F\big(u(t)\big), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt. \tag{43}$$

Proof of Lemma 3. Multiplying both sides of Eq. (6) by $\phi(t)\int_0^t \beta(t-s)(u(s)-u(t))\,ds$ and integrating over $[S,T]$ gives

$$\int_S^T \phi(t)\left\langle u''(t)+Au(t)-\beta * Au(t)-\nabla F\big(u(t)\big), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt = 0. \quad (44)$$

Integrating by parts, we obtain

$$\begin{aligned}
&\int_S^T \phi(t)\left\langle u''(t), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt \\
&\quad = \left[\phi(t)\left\langle u'(t), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle\right]_S^T \\
&\qquad - \int_S^T \phi'(t)\left\langle u'(t), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt \\
&\qquad - \int_S^T \phi(t)\left\langle u'(t), \int_0^t \beta'(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt + \int_S^T \phi(t)\left(\int_0^t \beta(s)\,ds\right)\|u'(t)\|^2\,dt.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\int_S^T \phi(t)\left\langle Au(t)-\beta * Au(t), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt \\
&\quad = \int_S^T \phi(t)\left\langle \sqrt{A}u(t), \int_0^t \beta(t-s)\big(\sqrt{A}u(s)-\sqrt{A}u(t)\big)\,ds\right\rangle dt \\
&\qquad - \int_S^T \phi(t)\left\langle \int_0^t \beta(t-s)\sqrt{A}u(s)\,ds, \int_0^t \beta(t-s)\big(\sqrt{A}u(s)-\sqrt{A}u(t)\big)\,ds\right\rangle dt \\
&\quad = \int_S^T \phi(t)\left(1-\int_0^t \beta(s)\,ds\right)\left\langle \sqrt{A}u(t), \int_0^t \beta(t-s)\big(\sqrt{A}u(s)-\sqrt{A}u(t)\big)\,ds\right\rangle dt \\
&\qquad - \int_S^T \phi(t)\left\|\int_0^t \beta(t-s)\big(\sqrt{A}u(s)-\sqrt{A}u(t)\big)\,ds\right\|^2 dt.
\end{aligned}$$

Therefore, plugging the above two identities into (44) we get (43). $\square$

Lemma 4. *For any $S_0 > 0$, for any $T \geqslant S \geqslant S_0$ and for any $\varepsilon > 0$ we have*

$$\int_S^T \phi(t)\|u'(t)\|^2\,dt \leqslant \varepsilon \int_S^T \phi(t)\|\sqrt{A}u(t)\|^2\,dt + C\phi(0)E_u(S) \tag{45}$$

where $C = C(S_0, \varepsilon)$ is a positive constant.

Proof of Lemma 4. We split the proof into two steps: first, we evaluate all the terms in the right-hand side of Eq. (43), then we obtain the conclusion.

4.1: We note that

$$\begin{aligned}
&\left|\left\langle u'(t), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle\right| \\
&\quad\leqslant \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\left(\int_0^t \beta(t-s)\|u(s)-u(t)\|\,ds\right)^2 \\
&\quad\leqslant E_u(t) + \frac{1}{2M}\left(\int_0^t \beta(t-s)\|\sqrt{A}u(s)-\sqrt{A}u(t)\|\,ds\right)^2 \\
&\quad\leqslant E_u(t) + \frac{1}{2M}\int_0^t \beta(s)\,ds \int_0^t \beta(t-s)\|\sqrt{A}u(s)-\sqrt{A}u(t)\|^2\,ds \\
&\quad\leqslant \left(1+\frac{1}{M}\right)E_u(t).
\end{aligned} \tag{46}$$

So,

$$-\left[\phi(t)\left\langle u'(t), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle\right]_S^T \leqslant 2\left(1+\frac{1}{M}\right)\phi(0)E_u(S). \tag{47}$$

Now, to estimate the second term in the right-hand side of Eq. (43), we use again (46) to get

$$\begin{aligned}
&\int_S^T \phi'(t)\left\langle u'(t), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt \\
&\quad\leqslant -\left(1+\frac{1}{M}\right)\int_S^T \phi'(t)E_u(t)\,dt \leqslant \left(1+\frac{1}{M}\right)\phi(0)E_u(S).
\end{aligned} \tag{48}$$

In addition, for any $\delta > 0$ we have

$$\int_S^T \phi(t)\left\langle u'(t), \int_0^t \beta'(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt$$
$$\leqslant \frac{\delta}{2}\int_S^T \phi(t)\left\|u'(t)\right\|^2 dt + \frac{1}{2\delta}\int_S^T \phi(t)\left(\int_0^t \left|\beta'(t-s)\right|\left\|u(s)-u(t)\right\| ds\right)^2 dt.$$

Since $\beta'(t) \leqslant 0$ and thanks to (16), we have

$$\left(\int_0^t \left|\beta'(t-s)\right|\left\|u(s)-u(t)\right\| ds\right)^2$$
$$\leqslant -\int_0^t \beta'(s)\,ds \int_0^t \left|\beta'(t-s)\right|\left\|u(s)-u(t)\right\|^2 ds$$
$$\leqslant -\beta(0)\int_0^t \beta'(t-s)\left\|u(s)-u(t)\right\|^2 ds$$
$$\leqslant -\frac{\beta(0)}{M}\int_0^t \beta'(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds$$
$$\leqslant -\frac{2\beta(0)}{M}E_u'(t).$$

Therefore,

$$\int_S^T \phi(t)\left\langle u'(t), \int_0^t \beta'(t-s)\big(u(s)-u(t)\big)\,ds\right\rangle dt$$
$$\leqslant \frac{\delta}{2}\int_S^T \phi(t)\left\|u'(t)\right\|^2 dt + \frac{\beta(0)}{\delta M}\phi(0)E_u(S). \tag{49}$$

Now, since $\beta(t) \geqslant 0$ and $\int_0^\infty \beta(s)\,ds < 1$, (36) yields

$$\int_S^T \phi(t)\left(\int_0^t \beta(s)\,ds - 1\right)\left\langle \sqrt{A}u(t), \int_0^t \beta(t-s)\big(\sqrt{A}u(s)-\sqrt{A}u(t)\big)\,ds\right\rangle dt$$
$$\leqslant \int_S^T \phi(t)\left|\left\langle \sqrt{A}u(t), \int_0^t \beta(t-s)\big(\sqrt{A}u(s)-\sqrt{A}u(t)\big)\,ds\right\rangle\right| dt$$

$$\leqslant \frac{\varepsilon}{2}\int_S^T \phi(t)\big\|\sqrt{A}u(t)\big\|^2\,dt + \frac{1}{\varepsilon k}\int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds\,\phi(0)E_u(S). \tag{50}$$

We have already bounded the fifth term in the right-hand side of (43), see (35). So, we proceed to estimate the rightmost term in (43). To this end, we need to prove that

$$\big\|\nabla F\big(u(t)\big)\big\| \leqslant C\big\|\sqrt{A}u(t)\big\| \quad \forall t \geqslant 0. \tag{51}$$

To derive (51) we can apply Theorem 3.3 thanks to Remark 3.8-3: by (20) and (19) it follows that, for any $t \geqslant 0$,

$$\big\|\sqrt{A}u(t)\big\| \leqslant \frac{2E_u^{1/2}(t)}{(1-\int_0^\infty \beta(s)\,ds)^{1/2}} \leqslant \frac{2E_u^{1/2}(0)}{(1-\int_0^\infty \beta(s)\,ds)^{1/2}} \leqslant \frac{2\rho_0}{(1-\int_0^\infty \beta(s)\,ds)^{1/2}}.$$

Therefore, assumption (H1)-3(c), the fact that $\nabla F(0) = 0$ and the above inequality yield (51).

Now, by (51), (34) and (35) we conclude that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\int_S^T \phi(t)\Big\langle \nabla F\big(u(t)\big), \int_0^t \beta(t-s)\big(u(s)-u(t)\big)\,ds\Big\rangle\,dt \\
&\quad\leqslant \frac{C}{\sqrt{M}}\int_S^T \phi(t)\big\|\sqrt{A}u(t)\big\|\int_0^t \beta(t-s)\big\|\sqrt{A}u(s)-\sqrt{A}u(t)\big\|\,ds\,dt \\
&\quad\leqslant \frac{\varepsilon}{2}\int_S^T \phi(t)\big\|\sqrt{A}u(t)\big\|^2\,dt + \frac{C^2}{\varepsilon M k}\int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds\,\phi(0)E_u(S).
\end{aligned} \tag{52}$$

4.2: Combining formulas (47)–(50), (35) and (52) with (43), we obtain

$$\begin{aligned}
&\int_S^T \phi(t)\left(\int_0^t \beta(s)\,ds - \frac{\delta}{2}\right)\big\|u'(t)\big\|^2\,dt \\
&\quad\leqslant \varepsilon\int_S^T \phi(t)\big\|\sqrt{A}u(t)\big\|^2\,dt + \frac{\beta(0)}{\delta M}\phi(0)E_u(S) + 3\left(1+\frac{1}{2M}\right)\phi(0)E_u(S) \\
&\qquad + \frac{1}{k}\left(\frac{1}{\varepsilon}+2+\frac{C^2}{\varepsilon M}\right)\int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds\,\phi(0)E_u(S).
\end{aligned} \tag{53}$$

Since β is continuous and $\beta(0) > 0$, for any $S_0 > 0$ we have

$$\int_0^S \beta(s)\,ds \geqslant \int_0^{S_0} \beta(s)\,ds > 0, \quad \forall S \geqslant S_0.$$

Now, if we choose $\delta > 0$ so small that

$$\delta < \int_0^{S_0} \beta(s)\,ds,$$

then, by (53), for any $T > S \geqslant S_0$ we have

$$\begin{aligned}
&\frac{1}{2}\int_0^{S_0} \beta(s)\,ds \int_S^T \phi(t)\big\|u'(t)\big\|^2\,dt \\
&\quad\leqslant \varepsilon \int_S^T \phi(t)\big\|\sqrt{A}u(t)\big\|^2\,dt + \frac{\beta(0)}{\delta M}\phi(0)E_u(S) + 3\left(1+\frac{1}{2M}\right)\phi(0)E_u(S) \\
&\qquad + \frac{1}{k}\left(\frac{1}{\varepsilon}+2+\frac{C^2}{\varepsilon M}\right)\int_0^\infty \beta^{1-\frac{1}{p}}(s)\,ds\,\phi(0)E_u(S).
\end{aligned}$$

Thus (45) follows. □

Proof of Lemma 1. Plugging estimate (45) into (30), we obtain

$$\int_S^T \phi(t)\left(1-\int_0^t \beta(s)\,ds\right)\big\|\sqrt{A}u(t)\big\|^2\,dt \leqslant \varepsilon C\int_S^T \phi(t)\big\|\sqrt{A}u(t)\big\|^2\,dt + C\phi(0)E_u(S),$$

whence, for $\varepsilon > 0$ small enough,

$$\int_S^T \phi(t)\left(1-\int_0^t \beta(s)\,ds\right)\big\|\sqrt{A}u(t)\big\|^2\,dt \leqslant C\phi(0)E_u(S), \tag{54}$$

or

$$\int_S^T \phi(t)\big\|u'(t)\big\|^2\,dt \leqslant C\phi(0)E_u(S) \tag{55}$$

for some constant $C > 0$. Now, recalling Remark 3.8-3 and using (21), (54), we have

$$-\int_S^T \phi(t) F\big(u(t)\big)\,dt \leqslant \int_S^T \phi(t)\psi\big(\|\sqrt{A}u(t)\|\big)\|\sqrt{A}u(t)\|^2\,dt$$
$$\leqslant \frac{1}{4}\int_S^T \phi(t)\left(1-\int_0^t \beta(s)\,ds\right)\|\sqrt{A}u(t)\|^2\,dt \leqslant C\phi(0)E_u(S). \quad (56)$$

So, combining (54)–(56) we obtain (29). □

Proof of Theorem 3.7 *(continued).* Case $p=\infty$: taking $\phi(t)=1$ in (29), we obtain

$$\frac{1}{2}\int_S^T \left(1-\int_0^t \beta(s)\,ds\right)\|\sqrt{A}u(t)\|^2\,dt + \frac{1}{2}\int_S^T \|u'(t)\|^2\,dt - \int_S^T F\big(u(t)\big)\,dt \leqslant C E_u(S). \quad (57)$$

So, by (28) it remains to bound the last term of the energy. To this end, we use condition $\beta'(t) \leqslant -k\beta(t)$ as follows:

$$\frac{1}{2}\int_S^T\int_0^t \beta(t-s)\|\sqrt{A}u(s)-\sqrt{A}u(t)\|^2\,ds\,dt$$
$$\leqslant -\frac{1}{2k}\int_S^T\int_0^t \beta'(t-s)\|\sqrt{A}u(s)-\sqrt{A}u(t)\|^2\,ds\,dt$$
$$\leqslant -\frac{1}{k}\int_S^T E_u'(t)\,dt \leqslant \frac{1}{k}E_u(S). \quad (58)$$

The proof is completed combining (57) and (58) with (28) and taking the limit as $T\to\infty$.

Case $p\in(2,\infty)$: we need some auxiliary lemmas to bound the last term of the energy.

Lemma 5. *Define, for any $m \geqslant 1$,*

$$\varphi_m(t) := \int_0^t \beta^{1-\frac{1}{m}}(t-s)\|\sqrt{A}u(s)-\sqrt{A}u(t)\|^2\,ds, \quad t \geqslant 0. \quad (59)$$

Then, we have for any $T \geqslant S \geqslant 0$

$$\int_S^T E_u^{\frac{m}{p}}(t)\int_0^t \beta(t-s)\|\sqrt{A}u(s)-\sqrt{A}u(t)\|^2\,ds\,dt$$
$$\leqslant C E_u^{\frac{p}{p+m}}(S)\left(\int_S^T E_u^{1+\frac{m}{p}}(t)\varphi_m(t)\,dt\right)^{\frac{m}{p+m}} \quad (60)$$

for some constant $C>0$.

Proof of Lemma 5. First, applying Hölder's inequality with conjugate exponents given by $\frac{p+m}{m}$ and $\frac{p+m}{p}$, we get

$$\int_0^t \beta(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds$$

$$= \int_0^t \beta^{\frac{m-1}{p+m}}(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^{\frac{2m}{p+m}} \beta^{1-\frac{m-1}{p+m}}(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^{\frac{2p}{p+m}} ds$$

$$\leqslant \varphi_m^{\frac{m}{p+m}}(t)\left(\int_0^t \beta^{1+\frac{1}{p}}(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds\right)^{\frac{p}{p+m}}.$$

Therefore,

$$\int_S^T E_u^{\frac{m}{p}}(t)\int_0^t \beta(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds\,dt$$

$$\leqslant \int_S^T E_u^{\frac{m}{p}}(t)\varphi_m^{\frac{m}{p+m}}(t)\left(\int_0^t \beta^{1+\frac{1}{p}}(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds\right)^{\frac{p}{p+m}} dt.$$

Applying again Hölder's inequality with conjugate exponents given by $\frac{p+m}{m}$ and $\frac{p+m}{p}$, we have

$$\int_S^T E_u^{\frac{m}{p}}(t)\int_0^t \beta(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds\,dt$$

$$\leqslant \left(\int_S^T E_u^{1+\frac{m}{p}}(t)\varphi_m(t)\,dt\right)^{\frac{m}{p+m}}\left(\int_S^T\int_0^t \beta^{1+\frac{1}{p}}(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds\,dt\right)^{\frac{p}{p+m}}. \tag{61}$$

Using condition $\beta'(t)\leqslant -k\beta^{1+\frac{1}{p}}(t)$, we obtain

$$\int_S^T E_u^{\frac{m}{p}}(t)\int_0^t \beta(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds\,dt$$

$$\leqslant \frac{1}{k^{\frac{p}{p+m}}}\left(\int_S^T E_u^{1+\frac{m}{p}}(t)\varphi_m(t)\,dt\right)^{\frac{m}{p+m}}\left(-\int_S^T\int_0^t \beta'(t-s)\left\|\sqrt{A}u(s)-\sqrt{A}u(t)\right\|^2 ds\,dt\right)^{\frac{p}{p+m}}$$

$$\leqslant \left(\frac{2}{k}\right)^{\frac{p}{p+m}} \left(\int_S^T E_u^{1+\frac{m}{p}}(t)\varphi_m(t)\,dt\right)^{\frac{m}{p+m}} \left(-\int_S^T E_u'(t)\,dt\right)^{\frac{p}{p+m}}$$

$$\leqslant \left(\frac{2}{k}\right)^{\frac{p}{p+m}} \left(\int_S^T E_u^{1+\frac{m}{p}}(t)\varphi_m(t)\,dt\right)^{\frac{m}{p+m}} E_u^{\frac{p}{p+m}}(S).$$

The proof is thus complete. □

Lemma 6. *Suppose that, for some $m \geqslant 1$, the function φ_m defined in* (59) *is bounded. Then, for any $S_0 > 0$ there is a positive constant C such that*

$$\int_S^\infty E_u^{1+\frac{m}{p}}(t)\,dt \leqslant C\big(E_u^{\frac{m}{p}}(0) + \|\varphi_m\|_\infty^{\frac{m}{p}}\big)E_u(S) \quad \forall S \geqslant S_0. \tag{62}$$

Proof of Lemma 6. Let us rewrite the left-hand side of (62) using the definition of the energy in (15). For all $T \geqslant S \geqslant S_0$, we have

$$\begin{aligned}\int_S^T E_u^{1+\frac{m}{p}}(t)\,dt &= \frac{1}{2}\int_S^T E_u^{\frac{m}{p}}(t)\left(1 - \int_0^t \beta(s)\,ds\right)\big\|\sqrt{A}u(t)\big\|^2\,dt + \frac{1}{2}\int_S^T E_u^{\frac{m}{p}}(t)\big\|u'(t)\big\|^2\,dt \\ &\quad - \int_S^T E_u^{\frac{m}{p}}(t)F\big(u(t)\big)\,dt \\ &\quad + \frac{1}{2}\int_S^T E_u^{\frac{m}{p}}(t)\int_0^t \beta(t-s)\big\|\sqrt{A}u(s) - \sqrt{A}u(t)\big\|^2\,ds\,dt.\end{aligned}$$

Now, apply Lemma 1 with $\phi(t) = E_u^{\frac{m}{p}}(t)$ to obtain

$$\begin{aligned}&\frac{1}{2}\int_S^T E_u^{\frac{m}{p}}(t)\left(1 - \int_0^t \beta(s)\,ds\right)\big\|\sqrt{A}u(t)\big\|^2\,dt + \frac{1}{2}\int_S^T E_u^{\frac{m}{p}}(t)\big\|u'(t)\big\|^2\,dt \\ &\quad - \int_S^T E_u^{\frac{m}{p}}(t)F\big(u(t)\big)\,dt \leqslant C E_u^{\frac{m}{p}}(0)E_u(S).\end{aligned} \tag{63}$$

Moreover, since φ_m is bounded, by (60) and Young's inequality we have that, for any $\varepsilon > 0$,

$$\frac{1}{2}\int_S^T E_u^{\frac{m}{p}}(t)\int_0^t \beta(t-s)\big\|\sqrt{A}u(s) - \sqrt{A}u(t)\big\|^2\,ds\,dt$$

$$\leqslant C\left(\int_S^T E_u^{1+\frac{m}{p}}(t)\,dt\right)^{\frac{m}{p+m}} \|\varphi_m\|_\infty^{\frac{m}{p+m}} E_u^{\frac{p}{p+m}}(S)$$

$$\leqslant \varepsilon \int_S^T E_u^{1+\frac{m}{p}}(t)\,dt + C(\varepsilon)\|\varphi_m\|_\infty^{\frac{m}{p}} E_u(S), \tag{64}$$

for some constant $C(\varepsilon) > 0$. Combining (63) and (64), we conclude that

$$\int_S^T E_u^{1+\frac{m}{p}}(t)\,dt \leqslant \varepsilon \int_S^T E_u^{1+\frac{m}{p}}(t)\,dt + C(\varepsilon)\big(E_u^{\frac{m}{p}}(0) + \|\varphi_m\|_\infty^{\frac{m}{p}}\big) E_u(S).$$

Thus, for $\varepsilon > 0$ small enough,

$$\int_S^T E_u^{1+\frac{m}{p}}(t)\,dt \leqslant C\big(E_u^{\frac{m}{p}}(0) + \|\varphi_m\|_\infty^{\frac{m}{p}}\big) E_u(S).$$

Taking the limit as $T \to \infty$, we get (62). $\square$

Proof of Theorem 3.7 *(continued).* Case $p \in (2, \infty)$: first, we claim that φ_2 (defined in (59)) is bounded. Indeed, thanks to Remark 3.8-2, we know that $\beta^{1/2} \in L^1(0, \infty)$ since $p > 2$. Hence, recalling (20),

$$\big|\varphi_2(t)\big| \leqslant C \int_0^t \beta^{1/2}(t-s)\big(E_u(s) + E_u(t)\big)\,ds \leqslant 2C \int_0^\infty \beta^{1/2}(s)\,ds\, E_u(0), \quad \forall t \geqslant 0.$$

Thus, $\|\varphi_2\|_\infty \leqslant C E_u(0)$ as claimed.

Next, thanks to Lemma 6 we have

$$\int_S^\infty E_u^{1+\frac{2}{p}}(t)\,dt \leqslant C E_u^{\frac{2}{p}}(0) E_u(S) \quad \forall S \geqslant S_0. \tag{65}$$

Hence, applying Lemma 2.2 to E_u with $\alpha = 2/p$, we get

$$E_u(t) \leqslant E_u(0)\left(\frac{(S_0 + C)(2 + p)}{2t + p(S_0 + C)}\right)^{\frac{p}{2}} \quad \forall t \geqslant 0. \tag{66}$$

Recalling (20) and the fact that $p > 2$, by (66) we obtain

$$\big|\varphi_1(t)\big| \leqslant C\left(\int_0^t E_u(s)\,ds + tE_u(t)\right) \leqslant C\left(\int_0^\infty E_u(s)\,ds + \sup_{t \geqslant 0}\big(tE_u(t)\big)\right) \leqslant C E_u(0), \quad \forall t \geqslant 0,$$

whence $\|\varphi_1\|_\infty \leqslant C E_u(0)$. So, applying Lemma 6 with $m = 1$, we obtain the conclusion of the theorem:

$$\int_S^\infty E_u^{1+\frac{1}{p}}(t)\,dt \leqslant C E_u^{\frac{1}{p}}(0) E_u(S) \quad \forall S \geqslant S_0. \qquad \square$$

4. Applications

We shall now give applications of our stability result to concrete models for various partial differential operators. In this section, Ω will denote a bounded open domain in $\mathbb{R}^N$, $N \geqslant 3$, with sufficiently smooth boundary $\partial\Omega$. The degree of smoothness of $\partial\Omega$ will depend on the specific example under consideration. Points in Ω will be denoted by the Greek letter ξ. Moreover, the lower dimensional cases $N = 1, 2$ can be treated by the same method in an even easier way.

In the following examples, as for the convolution kernel β, we shall assume that $\beta : [0, \infty) \to [0, \infty)$ is a locally absolutely continuous function satisfying (8), (9) and (24) as in Section 3.

Our first example concerns the semilinear wave equation with memory that was analyzed in [5]. As we shall see, our abstract theorem subsumes the result of [5].

Example 4.1. Let us consider the semilinear problem

$$\begin{cases} \partial_t^2 u(t,\xi) - \Delta u(t,\xi) + \int_0^t \beta(t-s)\Delta u(s,\xi)\,ds = \left|u(t,\xi)\right|^\gamma u(t,\xi), & t \geqslant 0,\ \xi \in \Omega, \\ u(t,\xi) = 0, \quad t \geqslant 0,\ \xi \in \partial\Omega, \\ u(0,\xi) = u_0(\xi), \quad \xi \in \Omega, \\ \partial_t u(0,\xi) = u_1(\xi), \quad \xi \in \Omega. \end{cases} \tag{67}$$

Here, $u(t,\xi)$ is real-valued, and we have denoted by $\partial_t u$ the time derivative of u and by Δu the Laplacian of u with respect to space variable ξ. Also, $\gamma > 0$ satisfies a suitable restriction to be specified later. We can rewrite (67) as an abstract problem of the type (12). Indeed, let $X = L^2(\Omega)$ be endowed with the usual inner product and norm

$$\|x\| := \left(\int_\Omega \left|x(\xi)\right|^2 d\xi\right)^{1/2}, \quad x \in L^2(\Omega).$$

We consider the operator $A : D(A) \subset X \to X$ defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$
$$Ax(\xi) = -\Delta x(\xi), \quad x \in D(A),\ \xi \in \Omega \text{ a.e.}$$

It is well known that A verifies Assumptions (H1)-1. Moreover, the fractional power $\sqrt{A}$ of A is well defined and $D(\sqrt{A}) = H_0^1(\Omega)$. Next, consider the functional

$$F(x) := \frac{1}{\gamma+2}\int_\Omega \left|x(\xi)\right|^{\gamma+2} d\xi, \quad x \in H_0^1(\Omega),$$

which, if $0 < \gamma \leqslant 4/(N-2)$, is well defined in view of Sobolev's embedding theorem. Observe that F is Gâteaux differentiable at any $x \in H_0^1(\Omega)$ and the Gâteaux derivative of F is given by

$$DF(x)(y) = \int_\Omega |x(\xi)|^\gamma x(\xi) y(\xi)\, d\xi, \quad y \in H_0^1(\Omega).$$

Assume now the more restrictive condition

$$0 < \gamma \leqslant \frac{2}{N-2}. \tag{68}$$

Then, $2(\gamma+1) \leqslant 2^* = 2N/(N-2)$. So, again by Sobolev's theorem,

$$\begin{aligned} |DF(x)(y)| &\leqslant \int_\Omega |x(\xi)|^{\gamma+1} |y(\xi)|\, d\xi \leqslant \left(\int_\Omega |x(\xi)|^{2(\gamma+1)}\, d\xi \right)^{1/2} \|y\| \\ &\leqslant C \|\nabla x\|^{\gamma+1} \|y\|, \quad y \in H_0^1(\Omega). \end{aligned}$$

Therefore, F satisfies Assumptions (H1)-3(a) and (b). Consequently, for any $x \in H_0^1(\Omega)$, the linear operator $DF(x)$ has a unique extension to $L^2(\Omega)$ represented (up to Riesz isomorphism) by

$$\nabla F(x)(\xi) = |x(\xi)|^\gamma x(\xi), \quad x \in H_0^1(\Omega),\ \xi \in \Omega \text{ a.e.} \tag{69}$$

Let us check that Assumption (H1)-3(c) holds. Since

$$\big| |a|^\gamma a - |b|^\gamma b \big| \leqslant (\gamma+1)\big(|a| + |b|\big)^\gamma |a-b| \quad \forall a, b \in \mathbb{R},$$

we have

$$\begin{aligned} &\int_\Omega \big| |x(\xi)|^\gamma x(\xi) - |y(\xi)|^\gamma y(\xi) \big|^2 d\xi \\ &\quad \leqslant (\gamma+1)^2 \int_\Omega \big(|x(\xi)| + |y(\xi)|\big)^{2\gamma} |x(\xi) - y(\xi)|^2\, d\xi \\ &\quad \leqslant (\gamma+1)^2 \left(\int_\Omega \big(|x(\xi)| + |y(\xi)|\big)^{2(\gamma+1)} d\xi \right)^{\frac{\gamma}{\gamma+1}} \left(\int_\Omega |x(\xi) - y(\xi)|^{2(\gamma+1)} d\xi \right)^{\frac{1}{\gamma+1}} \\ &\quad \leqslant C \left(\int_\Omega \big(|\nabla x(\xi)|^2 + |\nabla y(\xi)|^2 \big)\, d\xi \right)^\gamma \int_\Omega |\nabla x(\xi) - \nabla y(\xi)|^2\, d\xi. \end{aligned}$$

This yields (11). Finally, we observe that Assumption (H2)-2 is also satisfied with $\psi(s) = Cs^\gamma$, where

$$\langle \nabla F(x), x \rangle = \int_\Omega |x(\xi)|^{\gamma+2} d\xi \leqslant C \left(\int_\Omega |\nabla x(\xi)|^2\, d\xi \right)^{\frac{\gamma}{2}} \int_\Omega |\nabla x(\xi)|^2\, d\xi, \quad x \in H_0^1(\Omega).$$

Therefore, by Theorem 3.3 we conclude that, if the initial conditions $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ are sufficiently small, that is,

$$\int_\Omega \left(|\nabla u_0|^2 + |u_1|^2\right) d\xi < \rho$$

for some $\rho > 0$, then problem (67) admits a unique mild solution u on $[0, \infty)$. Moreover, as recalled at the beginning of Section 3, u is a weak solution of the equation in (67), that is,

$$u \in C^1\big([0,\infty); L^2(\Omega)\big) \cap C\big([0,\infty); H_0^1(\Omega)\big)$$

and, for all $v \in H_0^1(\Omega)$ such that $t \mapsto \int_\Omega \partial_t u(t,\xi) v(\xi)\, d\xi$ is of class C^1, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \partial_t u(t,\xi) v(\xi)\, d\xi + \int_\Omega \nabla u(t,\xi)\cdot \nabla v(\xi)\, d\xi - \int_\Omega d\xi \int_0^t \beta(t-s) \nabla u(s,\xi)\cdot\nabla v(\xi)\, ds \\
&\quad = \int_\Omega \big|u(t,\xi)\big|^\gamma u(t,\xi) v(\xi)\, d\xi, \qquad \forall t \geqslant 0.
\end{aligned}$$

Defining the energy of u by

$$\begin{aligned}
E_u(t) := {}& \frac{1}{2}\int_\Omega \big|\partial_t u(t,\xi)\big|^2 d\xi + \frac{1}{2}\left(1 - \int_0^t \beta(s)\, ds\right)\int_\Omega \big|\nabla u(t,\xi)\big|^2 d\xi \\
& - \frac{1}{\gamma+2}\int_\Omega \big|u(t,\xi)\big|^{\gamma+2} d\xi \\
& + \frac{1}{2}\int_0^t \beta(t-s)\left(\int_\Omega \big|\nabla u(s,\xi) - \nabla u(t,\xi)\big|^2 d\xi\right) ds,
\end{aligned}$$

we can invoke Theorems 3.5 and 3.6 to obtain the following decay estimates (that depend on the value of p in assumption (24)): there is a constant $C > 0$ such that, for all $t \geqslant 0$,

$$E_u(t) \leqslant \begin{cases} E_u(0)\exp(1 - Ct) & \text{if } p = \infty, \\ E_u(0)\big(\frac{C(1+p)}{t+pC}\big)^p & \text{if } p \in (2,\infty). \end{cases} \tag{70}$$

Next, we shall study the asymptotic stability problem for the linear elasticity system with memory. Such a problem was studied in [20] for exponential decay, and in [21] for polynomial decay. As we show below, applying our abstract theorem we recover the decay results of both papers—for more general initial conditions and less regular convolution kernels. It has to be noted, however, that the kernel considered in [21] has a more general structure than the one of our model.

Example 4.2. Let us consider the linear anisotropic elasticity model

$$\begin{cases} \partial_t^2 u_i(t,\xi) - \displaystyle\sum_{j,k,l=1}^{N} \partial_j\big(a_{ijkl}(\xi)e_{kl}\big(u(t,\xi)\big)\big) \\ \quad + \displaystyle\int_0^t \beta(t-s) \sum_{j,k,l=1}^{N} \partial_j\big(a_{ijkl}(\xi)e_{kl}\big(u(s,\xi)\big)\big)\,ds = 0, \quad t \geqslant 0,\ \xi \in \Omega,\ i=1,\dots,N, \\ u(t,\xi) = 0, \quad t \geqslant 0,\ \xi \in \partial\Omega, \\ u(0,\xi) = v_0(\xi), \quad \xi \in \Omega, \\ \partial_t u(0,\xi) = v_1(\xi), \quad \xi \in \Omega. \end{cases} \tag{71}$$

Here, $u(t,\xi) = (u_1(t,\xi),\dots,u_N(t,\xi))$ and

$$e_{kl}\big(u(t,\xi)\big) := \frac{1}{2}\big(\partial_k u_l(t,\xi) + \partial_l u_k(t,\xi)\big),$$

where we denote by $\partial_k u_l$ the spatial derivative of u_l respect to the variable ξ_k.

The elasticity tensor (a_{ijkl}), with $a_{ijkl} : \overline{\Omega} \to \mathbb{R}$ and $a_{ijkl} \in L^2(\Omega)$, is symmetric and coercive, that is

$$a_{ijkl}(\xi) = a_{jikl}(\xi) = a_{klij}(\xi), \quad \text{for any } \xi \in \Omega, \tag{72}$$

and there exists a constant $\alpha > 0$ such that, for any symmetric matrix (f_{kl}), we have

$$\sum_{i,j,k,l=1}^{N} a_{ijkl}(\xi) f_{kl} f_{ij} \geqslant \alpha \sum_{i,j=1}^{N} f_{ij}^2, \quad \text{for any } \xi \in \Omega. \tag{73}$$

We can rewrite (71) as an abstract problem of the type (12). Indeed, let $X = L^2(\Omega;\mathbb{R}^N)$ be endowed with the usual inner product and norm

$$\|x\| := \left(\int_\Omega \big|x(\xi)\big|^2\,d\xi\right)^{1/2}, \quad x \in L^2\big(\Omega;\mathbb{R}^N\big),$$

and let $A : D(A) \subset X \to X$ be the operator defined by

$$D(A) = H^2\big(\Omega;\mathbb{R}^N\big) \cap H_0^1\big(\Omega;\mathbb{R}^N\big),$$

$$(Ax)_i(\xi) = -\sum_{j,k,l=1}^{N} \partial_j\big(a_{ijkl}(\xi)e_{kl}\big(x(\xi)\big)\big), \quad x \in D(A),\ \xi \in \Omega \text{ a.e.}$$

It is well known that A verifies Assumptions (H1)-1 and $D(\sqrt{A}) = H_0^1(\Omega;\mathbb{R}^N)$ (see, e.g., [15]). Therefore, by Theorem 3.3 and Remark 3.4 we conclude that, for all initial conditions

$$(v_0, v_1) \in H_0^1\big(\Omega;\mathbb{R}^N\big) \times L^2\big(\Omega;\mathbb{R}^N\big),$$

problem (71) admits a unique mild solution u on $[0,\infty)$. Moreover, as explained in the above example, u is a weak solution of the elasticity system in the sense of (14). Defining energy by

$$\begin{aligned} E_u(t) &:= \frac{1}{2}\int_\Omega \left|\partial_t u(t,\xi)\right|^2 d\xi \\ &\quad + \frac{1}{2}\left(1-\int_0^t \beta(s)\,ds\right)\sum_{i,j,k,l=1}^N \int_\Omega a_{ijkl}(\xi)e_{kl}\big(u(t,\xi)\big)e_{ij}\big(u(t,\xi)\big)\,d\xi \\ &\quad + \frac{1}{2}\int_0^t \beta(t-s)\left(\sum_{i,j,k,l=1}^N \int_\Omega a_{ijkl}(\xi)e_{kl}\big(u(s,\xi)-u(t,\xi)\big)\right. \\ &\quad \left. \times e_{ij}\big(u(s,\xi)-u(t,\xi)\big)\,d\xi\right)ds, \end{aligned}$$

we can invoke Theorems 3.5 and 3.6 to deduce that $E_u(t)$ decays as in (70) for some constant $C>0$.

Our last example consists of a Petrovsky system related to a plate model with nonlocal dissipation (see also [23] and [9]).

Example 4.3. Let us consider the Petrovsky type system

$$\begin{cases} \partial_t^2 u(t,\xi) + \Delta^2 u(t,\xi) - \int_0^t \beta(t-s)\Delta^2 u(s,\xi)\,ds = 0, & t \geqslant 0,\ \xi \in \Omega, \\ u(t,\xi) = \partial_\nu u(t,\xi) = 0, & t \geqslant 0,\ \xi \in \partial\Omega, \\ u(0,\xi) = u_0(\xi), & \xi \in \Omega, \\ \partial_t u(0,\xi) = u_1(\xi), & \xi \in \Omega. \end{cases} \tag{74}$$

Here, we denote by $\partial_\nu u$ the normal derivative of u.

We can rewrite (74) as an abstract problem of the type (12). Indeed, let $X = L^2(\Omega)$ be endowed with the usual inner product and norm. We consider the operator $A : D(A) \subset X \to X$ defined by

$$\begin{aligned} &D(A) = H^4(\Omega) \cap H_0^2(\Omega), \\ &Ax(\xi) = \Delta^2 x(\xi), \quad x \in D(A),\ \xi \in \Omega \text{ a.e.} \end{aligned}$$

It is well known that A verifies Assumptions (H1)-1 and $D(\sqrt{A}) = H_0^2(\Omega)$ (see, e.g., [10, pp. 28–29]). Then, from Theorems 3.5 and 3.6 it follows that, for all $u_0 \in H_0^2(\Omega)$ and $u_1 \in L^2(\Omega)$, $E_u(t)$ decays as in (70) for some constant $C>0$.

Remark 4.4. The method described in Example 4.3 can be used to study the Petrovsky system in (74) with the boundary condition

$$u(t,\xi) = \Delta u(t,\xi) = 0, \quad t \geqslant 0,\ \xi \in \partial\Omega.$$

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