PBW THEOREMS AND FROBENIUS STRUCTURES FOR QUANTUM MATRICES

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Abstract. Let $G \in \{ \text{Mat}_n(\mathbb{C}), \text{GL}_n(\mathbb{C}), \text{SL}_n(\mathbb{C}) \}$, let $O_q(G)$ be the quantum function algebra — over $\mathbb{Z}[q,q^{-1}]$ — associated to $G$, and let $O_\varepsilon(G)$ be the specialisation of the latter at a root of unity $\varepsilon$, whose order $\ell$ is odd. There is a quantum Frobenius morphism that embeds $O(G)$, the function algebra of $G$, in $O_\varepsilon(G)$ as a central Hopf subalgebra, so that $O_\varepsilon(G)$ is a module over $O(G)$. When $G = \text{SL}_n(\mathbb{C})$, it is known by [BG], [BGStaf] that (the complexification of) such a module is free, with rank $\ell \dim(G)$. In this note we prove a PBW-like theorem for $O_q(G)$, and we show that — when $G$ is $\text{Mat}_n$ or $\text{GL}_n$ — it yields explicit bases of $O_\varepsilon(G)$ over $O(G)$. As a direct application, we prove that $O_\varepsilon(GL_n)$ and $O_\varepsilon(M_n)$ are free Frobenius extensions over $O(GL_n)$ and $O(M_n)$, thus extending some results of [BGStro].

§ 1 The general setup

Let $G$ be a complex semisimple, connected, simply connected affine algebraic group. One can introduce a quantum function algebra $O_q(G)$, a Hopf algebra over the ground ring $\mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate, as in [DL]. If $\varepsilon$ is any root of 1, one can specialize $O_q(G)$ at $q = \varepsilon$, which means taking the Hopf algebra $O_\varepsilon(G) := O_q(G)/(q - \varepsilon)O_q(G)$. In particular, for $\varepsilon = 1$ one has $O_1(G) \cong O(G)$, the classical (commutative) function algebra over $G$. Moreover, if the order $\ell$ of $\varepsilon$ is odd, then there exists a Hopf algebra monomorphism $\Phi: O(G) \cong O_1(G) \longrightarrow O_\varepsilon(G)$, called quantum Frobenius morphism for $G$, which embeds $O(G)$ into $O_\varepsilon(G)$ as a central Hopf subalgebra. Therefore, $O_\varepsilon(G)$ is naturally a module over $O(G)$. It is proved in [BGStaf] and in [BG] that such a module is free, with rank $\ell \dim(G)$. In the special case of $G = \text{SL}_2$, a stronger result was given in [DRZ], where an explicit basis was found. We shall give similar results when $G$ is $\text{GL}_n$ or $M_n := \text{Mat}_n$; namely we provide explicit bases of $O_\varepsilon(G)$ as a free module over $O(G)$, where in addition everything is defined replacing $\mathbb{C}$ with $\mathbb{Z}$. The proof is via some (more or less known) PBW theorems for $O_q(M_n)$ and $O_q(GL_n)$ — and $O_q(\text{SL}_n)$ as well — as modules over $\mathbb{Z}[q,q^{-1}]$.

Let $M_n := \text{Mat}_n(\mathbb{C})$. The algebra $O(M_n)$ of regular functions on $M_n$ is the unital associative commutative $\mathbb{C}$-algebra with generators $\bar{t}_{i,j}$ $(i,j = 1, \ldots, n)$. The semigroup structure on $M_n$ yields on $O(M_n)$ the natural bialgebra structure given by matrix product — see [CP],

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Ch. 7. We can also consider the semigroup-scheme \((M_n)_{\Z}\) associated to \(M_n\), for which a like analysis applies: in particular, its function algebra \(\mathcal{O}^Z(M_n)\) is a \(\Z\)-bialgebra, with the same presentation as \(\mathcal{O}(M_n)\) but over the ring \(\Z\).

Now we define quantum function algebras. Let \(R\) be any commutative ring with unity, and let \(q \in R\) be invertible. We define \(\mathcal{O}_q^R(M_n)\) as the unital associative \(R\)-algebra with generators \(t_{i,j} \ (i, j = 1, \ldots, n)\) and relations

\[
t_{i,j} t_{i,k} = q t_{i,k} t_{i,j}, \quad t_{i,k} t_{h,k} = q t_{h,k} t_{i,k} \quad \forall \ j < k, i < h,
\]

\[
t_{i,l} t_{j,k} = t_{j,k} t_{i,l}, \quad t_{i,k} t_{j,l} - t_{j,l} t_{i,k} = (q - q^{-1}) t_{i,l} t_{j,k} \quad \forall \ i < j, k < l.
\]

It is known that \(\mathcal{O}_q^R(M_n)\) is a bialgebra, but we do not need this extra structure in the present work (see [CP] for further details — cf. also [AKP] and [PW]).

As to specialisations, set \(\Z_q := \Z[q, q^{-1}]\), let \(\ell \in \N_+\) be odd, let \(\phi_\ell(q)\) be the \(\ell\)-th cyclotomic polynomial in \(q\), and let \(\varepsilon := \bar{\bar{q}} \in \Z_\varepsilon := \Z_q / (\phi_\ell(q))\), so that \(\varepsilon\) is a (formal) primitive \(\ell\)-th root of 1 in \(\Z_\varepsilon\). Then

\[
\mathcal{O}_{\varepsilon^\mathbb{Z}}^q(M_n) = \mathcal{O}_q^q(M_n) / (\phi_\ell(q)) \mathcal{O}_q^q(M_n) \cong \Z_{\varepsilon} \otimes_{\Z} \mathcal{O}_q^q(M_n).
\]

It is also known that there is a bialgebra isomorphism

\[
\mathcal{O}_{\varepsilon}^q(M_n) \cong \mathcal{O}_{\bar{\bar{q}}}^q(M_n) / (q-1) \mathcal{O}_{\bar{\bar{q}}}^q(M_n) \hookrightarrow \mathcal{O}_{\varepsilon}^q(M_n), \quad t_{i,j} \mod (q-1) \mathcal{O}_{\bar{\bar{q}}}^q(M_n) \mapsto \bar{t}_{i,j}
\]

and a bialgebra monomorphism, called quantum Frobenius morphism \((\varepsilon\text{ and } \ell\text{ as above})\),

\[
\mathfrak{F}\varepsilon^q : \mathcal{O}_{\varepsilon}^q(M_n) \cong \mathcal{O}_{\varepsilon}^q(M_n) \hookrightarrow \mathcal{O}_{\varepsilon}^q(M_n), \quad \bar{t}_{i,j} \mapsto t_{i,j}^\ell |_{q=\varepsilon}
\]

whose image is central in \(\mathcal{O}_{\varepsilon}^q(M_n)\). Thus \(\mathcal{O}_{\varepsilon}^q(M_n) := \Z_{\varepsilon} \otimes_{\Z} \mathcal{O}_{\varepsilon}^q(M_n)\) becomes identified — via \(\mathfrak{F}\varepsilon\varepsilon\), which clearly extends to \(\mathcal{O}_{\varepsilon}^q(M_n)\) by scalar extension — with a central subbialgebra of \(\mathcal{O}_{\varepsilon}^q(M_n)\), so the latter can be seen as an \(\mathcal{O}_{\varepsilon}^q(M_n)\)-module. By the result in [BGStaf] and [BG] mentioned above, we can expect this module to be free, with rank \(\ell^2n^2\).

All the previous framework also extends to \(GL_n\) and to \(SL_n\) instead of \(M_n\). Indeed, consider the quantum determinant \(D_q := \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)}t_{1,\sigma(1)}t_{2,\sigma(2)} \cdots t_{n,\sigma(n)} \in \mathcal{O}_q^q(M_n)\), where \(\ell(\sigma)\) denotes the length of any permutation \(\sigma\) in the symmetric group \(S_n\). Then \(D_q\) belongs to the centre of \(\mathcal{O}_q^q(M_n)\), hence one can extend \(\mathcal{O}_q^R(M_n)\) by a formal inverse to \(D_q\), i.e. defining the algebra \(\mathcal{O}_q^R(GL_n) := \mathcal{O}_q^R(M_n)[D_q^{-1}]\). Similarly, we can define also \(\mathcal{O}_q^R(SL_n) := \mathcal{O}_q^R(M_n) / (D_q - 1)\). Now \(\mathcal{O}_q^R(GL_n)\) and \(\mathcal{O}_q^R(SL_n)\) are Hopf \(R\)-algebras, and the maps \(\mathcal{O}_q^R(M_n) \hookrightarrow \mathcal{O}_q^R(GL_n), \quad \mathcal{O}_q^R(GL_n) \longrightarrow \mathcal{O}_q^R(SL_n), \quad \mathcal{O}_q^R(M_n) \longrightarrow \mathcal{O}_q^R(SL_n)\) (the third one being the composition of the first two) given by \(t_{i,j} \mapsto t_{i,j}\) are epimorphisms of \(R\)-bialgebras, and even of Hopf \(R\)-algebras in the second case. The specialisations

\[
\mathcal{O}_{\varepsilon}^Z(GL_n) = \mathcal{O}_q^q(GL_n) / (\phi_\ell(q)) \mathcal{O}_q^q(GL_n) \cong \Z_{\varepsilon} \otimes_{\Z} \mathcal{O}_q^q(GL_n)
\]

\[
\mathcal{O}_{\varepsilon}^Z(SL_n) = \mathcal{O}_q^q(SL_n) / (\phi_\ell(q)) \mathcal{O}_q^q(SL_n) \cong \Z_{\varepsilon} \otimes_{\Z} \mathcal{O}_q^q(SL_n)
\]

enjoy the same properties as above, namely there exist isomorphisms \(\mathcal{O}_{\varepsilon}^Z(GL_n) \cong \mathcal{O}_Z(GL_n)\) and \(\mathcal{O}_{\varepsilon}^Z(SL_n) \cong \mathcal{O}_Z(SL_n)\) and there are quantum Frobenius morphisms

\[
\mathfrak{F}\varepsilon : \mathcal{O}_Z(GL_n) \cong \mathcal{O}_{\varepsilon}^Z(GL_n) \hookrightarrow \mathcal{O}_{\varepsilon}^Z(GL_n), \quad \mathfrak{F}\varepsilon : \mathcal{O}_Z(SL_n) \cong \mathcal{O}_{\varepsilon}^Z(SL_n) \hookrightarrow \mathcal{O}_{\varepsilon}^Z(SL_n)
\]
described by the same formula as for $M_n$. Moreover, $D_q^\pm 1 \mod (q - 1) \mapsto D_q^\pm 1$ in the isomorphisms and $D_q^\pm 1 \cong D_q^\pm 1 \mod (q - 1) \mapsto D_q^{\pm \ell} \mod (q - \varepsilon)$ in the quantum Frobenius morphisms for $GL_n$ (which extend those of $M_n$). In addition, all these isomorphisms and quantum Frobenius morphisms are compatible (in the obvious sense) with the natural maps which link $O_q^{\mathbb{Z}_n}(M_n)$, $O_q^{\mathbb{Z}_n}(GL_n)$ and $O_q^{\mathbb{Z}_n}(SL_n)$, and their specialisations, to each other.

Like for $M_n$, the image of the quantum Frobenius morphisms are central in $O_q^{\mathbb{Z}_n}(GL_n)$ and in $O_q^{\mathbb{Z}_n}(SL_n)$. Thus $O_q^{\mathbb{Z}_n}(GL_n) := \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} O_q^{\mathbb{Z}_n}(GL_n)$ identifies to a central Hopf subalgebra of $O_q^{\mathbb{Z}_n}(GL_n)$, and $O_q^{\mathbb{Z}_n}(SL_n) := \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} O_q^{\mathbb{Z}_n}(SL_n)$ identifies to a central Hopf subalgebra of $O_q^{\mathbb{Z}_n}(SL_n)$; so $O_q^{\mathbb{Z}_n}(GL_n)$ is an $O_q^{\mathbb{Z}_n}(GL_n)$–module and $O_q^{\mathbb{Z}_n}(SL_n)$ is an $O_q^{\mathbb{Z}_n}(SL_n)$–module.

In §2, we shall prove (Theorem 2.1) a PBW-like theorem providing several different bases for $O_q^R(M_n)$, $O_q^R(GL_n)$ and $O_q^R(SL_n)$ as $R$–modules. As an application, we find (Theorem 2.2) explicit bases of $O_q^{\mathbb{Z}_n}(M_n)$ as an $O_q^{\mathbb{Z}_n}(M_n)$–module, which then in particular is free of rank $\ell^{\dim(M_n)}$. The same bases are also $O_q^{\mathbb{Z}_n}(GL_n)$–bases for $O_q^{\mathbb{Z}_n}(GL_n)$, which then is free of rank $\ell^{\dim(GL_n)}$. Both results can be seen as extensions of some results in [BGStf].

Finally, in §3 we use the above mentioned bases to prove that $O_q^{\mathbb{Z}_n}(M_n)$ is a free Frobenius extension of its central subalgebra $O_q^{\mathbb{Z}_n}(M_n)$, and to explicitly compute the associated Nakayama automorphism. The same we do for $O_q^{\mathbb{Z}_n}(GL_n)$ as well. Everything follows from the ideas and methods in [BGStro], now applied to the explicit bases given by Theorem 2.2.

§2 PBW–like theorems

Theorem 2.1. (PBW theorem for $O_q^R(M_n)$, $O_q^R(GL_n)$ and $O_q^R(SL_n)$ as $R$–modules)

Assume $(q - 1)$ is not invertible in $R_q := (q, q^{-1})$, the subring of $R$ generated by $q$ and $q^{-1}$.

(a) Let any total order be fixed in $\{1, \ldots, n\}^2$. Then the following sets of ordered monomials are $R$–bases of $O_q^R(M_n)$, resp. $O_q^R(GL_n)$, resp. $O_q^R(SL_n)$, as modules over $R$:

\[
B_M := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \mid N_{i,j} \in \mathbb{N} \forall i, j \right\}
\]

\[
B^\land_{GL} := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} D_q^{-N} \mid N_{i,j} \in \mathbb{N} \forall i, j ; \min \{\{N_{i,i}\}_{1 \leq i \leq n} \cup \{N\}\} = 0 \right\}
\]

\[
B^\lor_{GL} := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} D_q^2 \mid Z \in \mathbb{Z}, N_{i,j} \in \mathbb{N} \forall i, j ; \min \{\{N_{i,i}\}_{1 \leq i \leq n}\} = 0 \right\}
\]

\[
B_{SL} := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \mid N_{i,j} \in \mathbb{N} \forall i, j ; \min \{\{N_{i,i}\}_{1 \leq i \leq n} = 0 \right\}
\]

(b) Let $\preceq$ be any total order fixed in $\{1, \ldots, n\}^2$ such that $(i, j) \preceq (h, k) \preceq (l, m)$ whenever $j > n+1-i$, $k = n+1-h$, $m < n+1-l$. Then the following sets of ordered monomials are $R$–bases of $O_q^R(GL_n)$, resp. $O_q^R(SL_n)$, as modules over $R$:

\[
B^\land_{GL} := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} D_q^{-N} \mid N, N_{i,j} \in \mathbb{N} \forall i, j ; \min \{\{N_{i,n+1-i}\}_{1 \leq i \leq n} \cup \{N\}\} = 0 \right\}
\]

\[
B^\lor_{GL} := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} D_q^2 \mid Z \in \mathbb{Z}, N, N_{i,j} \in \mathbb{N} \forall i, j ; \min \{\{N_{i,n+1-i}\}_{1 \leq i \leq n} = 0 \right\}
\]

\[
B_{SL} := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \mid N, N_{i,j} \in \mathbb{N} \forall i, j ; \min \{\{N_{i,n+1-i}\}_{1 \leq i \leq n} = 0 \right\}
\]
Proof. Roughly speaking, our method is a (partial) application of the diamond lemma (see [Be]): however, we do not follow it in all details, as we use a specialisation trick as a shortcut.

If we prove our results for the algebras defined over $R_q$ instead of $R$, then the same results will hold as well by scalar extension. Thus we can assume $R = R_q$, and then we note that, by our assumption, the specialised ring $\overline{R} := R/(q-1)R \neq \{0\}$ is non-trivial.

Proof of (a): (see also [Ko], Theorem 3.1, and [PW], Theorem 3.5.1)

We begin with $O_q^R(M_n)$. It is clearly spanned over $R$ by the set of all (possibly unordered) monomials in the $t_{ij}$'s: so we must only prove that any such monomial belongs to the $R$–span of the ordered monomials. In fact, the latter are linearly independent, since such are their images via specialisation $O_q^R(M_n) \longrightarrow O_q^R(M_n)/(q-1)O_q^R(M_n) \cong O_1^\overline{R}(M_n)$.

Thus, take any (possibly unordered) monomial in the $t_{ij}$'s, say $t := t_{i_1,j_1} t_{i_2,j_2} \cdots t_{i_h,j_h}$, where $k$ is the degree of $t$: we associate to it its weight, defined as

$$w(t) := (k, d_1, 1, d_2, 2, \ldots, d_i, n, d_{i+1}, 1, \ldots, d_{n-1}, n, d_{n+1}, 2, \ldots, d_{n,n})$$

where $d_{i,j} := \{|s \in \{1, \ldots, k\} | (i_s, j_s) = (i, j)\}|$ is number of occurrences of $t_{i,j}$ in $t$. Then $w(t) \in \mathbb{N}^{n^2+1}$, and we consider $\mathbb{N}^{n^2+1}$ as a totally ordered set with respect to the (total) lexicographic order $\leq_{lex}$. By a quick look at the defining relations of $O_q^R(M_n)$, namely

$$t_{i,j} t_{k,l} = t_{k,j} t_{i,l} \quad t_{i,j} t_{h,k} = q t_{h,k} t_{i,j} \quad t_{i,l} t_{j,k} = t_{j,k} t_{i,l} \quad t_{i,k} t_{h,l} - t_{j,l} t_{i,k} = (q-q^{-1}) t_{i,l} t_{j,k}$$

one easily sees that the weight defines an algebra filtration on $O_q^R(M_n)$.

Now, using these same relations, one can re-order the $t_{ij}$'s in any monomial according to the fixed total order. During this process, only two non-trivial things may occur, namely:

-1) some powers of $q$ show up as coefficients (when a relation in first line is employed);
-2) a new summand is added (when the bottom-right relation is used);

If only steps of type 1) occur, then the process eventually stops with an ordered monomial in the $t_{ij}$'s multiplied by a power of $q$. Whenever instead a step of type 2) occurs, the newly added term is just a coefficient $(q-q^{-1})$ times a (possibly unordered) monomial in the $t_{ij}$'s, call it $t'$: however, by construction $w(t') \leq_{lex} w(t)$. Then, by induction on the weight, we can assume that $t'$ lies in the $R$–span of the ordered monomials, so we can ignore the new summand. The process stops in finitely many steps, and we are done with $O_q^R(M_n)$.

Second, we look at $O_q^R(GL_n)$. Let us consider $f \in O_q^R(GL_n)$. By definition, there exists $N \in \mathbb{N}$ such that $f D_q^N \in O_q^R(M_n)$; therefore, by the result for $O_q^R(M_n)$ just proved, we can expand $f D_q^N$ as an $R$–linear combination of ordered monomials, call them $t = \prod_{i,j=1}^n t_{i,j}^{N_{i,j}}$. Thus, $f$ itself is an $R$–linear combination of monomials $t D_q^{-N}$, so the latter span $O_q^R(GL_n)$.

Now consider an ordered monomial $t = \prod_{i,j=1}^n t_{i,j}^{N_{i,j}}$ in which $N_{i,i} > 0$ for all $i$. Then we can re-arrange the $t_{i,i}$'s in $t$ so to single out a factor $t_{1,1} t_{2,2} \cdots t_{n-1,n-1} t_{n,n}$, up to “paying the cost” (perhaps) of producing some new summands of lower weight: the outcome reads

$$t = q^s t_0 t_{1,1} t_{2,2} \cdots t_{n-1,n-1} t_{n,n} + l.t.'s$$  \hspace{1cm} (2.1)

for some $s \in \mathbb{Z}$, with $t_0 \equiv \prod_{i,j=1}^n t_{i,j}^{N_{i,j}-\delta_{i,j}}$ having lower weight than $t$, and the expression $l.t.'s$ standing for an $R$–linear combination of some monomials $\bar{t}$ such that $w(\bar{t}) \leq_{lex} w(t)$.
Then we re-write the monomial \( t_{1,1} t_{2,2} \cdots t_{n-1,n-1} t_{n,n} \) using the identity

\[
t_{1,1} t_{2,2} \cdots t_{n-1,n-1} t_{n,n} = D_q - \sum_{\sigma \in S_n \setminus \{id\}} (-q)^{l(\sigma)} t_{1,\sigma(1)} t_{2,\sigma(2)} \cdots t_{n,\sigma(n)} = D_q + \text{l.t.'s} \tag{2.2}
\]

and we replace the right-hand side of (2.2) inside (2.1). We get \( \xi = q^{s} t_{0} D_q + \text{l.t.'s} \) (for \( D_q \) is central), where now \( t_{0} \) and all monomials within l.t.'s have strictly lower weight than \( t_{0} \).

If we look now at \( t \cdot D_q^z \) (for some \( z \in \mathbb{Z} \)), we can re-write \( t \) as above, thus getting

\[
t \cdot D_q^z = q^{s} t_{0} D_q D_q^z + \text{l.t.'s} = q^{s} t_{0} D_q^{z+1} + \text{l.t.'s} \tag{2.3}
\]

where l.t.'s is an \( R \)–linear combination of monomials \( t \cdot D_q^{z+1} \) such that \( w(t) \leq_{\text{lex}} w(t_0) \).

By repeated use of (2.3) as reduction argument, we can easily show — by induction on the weight — that any monomial of type \( t \cdot D_q^N \) \((N \in \mathbb{N})\) can be expanded as an \( R \)–linear combination of elements of \( B_{GL}^\wedge \) or elements of \( B_{GL}^\vee \). Thus, both these sets do span \( \mathcal{O}^{\mathcal{R}}(GL_n) \).

To finish with, both \( B_{GL}^\wedge \) and \( B_{GL}^\vee \) are \( R \)–linearly independent, as their image through the specialisation epimorphism \( \mathcal{O}_q^R(GL_n) \rightarrow \mathcal{O}_q^R(GL_n) \cong \mathcal{O}_q^R(GL_n) \) are \( \mathcal{R} \)–bases of \( \mathcal{O}_q^R(GL_n) \).

As to \( \mathcal{O}_q^R(SL_n) \), we can repeat the argument for \( \mathcal{O}_q^R(GL_n) \). First, \( B_{SL} \) is linearly independent, for its image through specialisation \( \mathcal{O}_q^R(SL_n) \rightarrow \mathcal{O}_q^R(SL_n) \cong \mathcal{O}_q^R(SL_n) \) is an \( \mathcal{R} \)–basis of \( \mathcal{O}_q^R(SL_n) \). Second, the epimorphism \( \mathcal{O}_q^R(M_n) \rightarrow \mathcal{O}_q^R(SL_n) \) \((t_{i,j} \mapsto t_{i,j})\), and the result for \( \mathcal{O}_q^R(M_n) \), imply that the \( R \)–span of \( S_{SL} := \{ \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} \mid N_{i,j} \in \mathbb{N} \forall i, j \} \) is \( \mathcal{O}_q^R(SL_n) \). Thus one is only left to prove that each monomial \( \xi = \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} \in S_{SL} \) belongs to the \( R \)–span of \( B_{SL} \): as before, this can be done by induction on the weight, using the reduction formula \( \xi = q^{s} t_{0} D_q + \text{l.t.'s} \) (see above), and plugging it in the relation \( D_q = 1 \).

Alternatively, we remind there is an isomorphism \( \mathcal{O}_q^R(SL_n) \otimes_R R[x, x^{-1}] \cong \mathcal{O}_q^R(GL_n) \) (of \( R \)–algebras) by \( t_{i,j} \otimes x^\delta \mapsto D_q^\delta t_{i,j} \cdot D_q^{-\delta} \) (cf. [LS]). This along with the result about \( B_{GL}^\vee \) clearly implies that also \( B_{SL} \) is an \( R \)–basis for \( \mathcal{O}_q^R(SL_n) \), as claimed.

**Proof of (b):** First look at \( \mathcal{O}_q^R(GL_n) \). If \( f \in \mathcal{O}_q^R(GL_n) \), like in the proof of (a) we expand \( f \cdot D_q^N \) as an \( R \)–linear combination of ordered (according to \( \leq \)) monomials of type \( t = t^- t^+ \), with \( t^- := \prod_{j>n+1-i} t_{i,j}^{N_{i,j}} \), \( t^+ := \prod_{j<n+1-i} t_{i,j}^{N_{i,j}} \) and \( t^0 := \prod_{j<n+1-i} t_{i,j}^{N_{i,j}} \). So \( f \) is an \( R \)–linear combination of monomials \( t^- t^+ D_q^- \), hence the latter span \( \mathcal{O}_q^R(GL_n) \).

We show that each (ordered) monomial \( t^- t^+ D_q^- \in \mathcal{O}_q^R(GL_n) \) belongs both to the \( R \)–span of \( B_{GL}^\wedge \) and of \( B_{GL}^\vee \), by induction on the (total) degree of the monomial \( t^- \). The basis of induction is \( \text{deg}(t^-) = 0 \), so that \( t^- = 1 \) and \( t^- t^+ D_q^- = t^- t^+ D_q^- \in B_{GL}^\wedge \cap B_{GL}^\vee \).

As a matter of notation, let \( \mathcal{N}^- \), resp. \( \mathcal{N}^+ \), be the \( R \)–subalgebra of \( \mathcal{O}_q^R(M_n) \) generated by the \( t_{i,j} \)’s with \( j > n+1-i \), resp. \( j > n+1-i \), resp. \( j < n+1-i \). Note that \( \mathcal{H} \) is Abelian, and \( t^- \in \mathcal{N}^- \), \( t^+ \in \mathcal{H} \), \( t^0 \in \mathcal{N}^+ \).

Now assume that all the exponents \( N_{i,n+1-i} \)’s in the factor \( t^- \) are strictly positive. As \( \mathcal{H} \) is Abelian, we can draw out of \( t^- \) (even out of \( t^- t^+ t^0 \) ) a factor \( t_{n,1} t_{n-1,2} \cdots t_{2,n-1} t_{1,n} \).

Now recall that \( D_q \) can be expanded as \( D_q = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{n,\sigma(n)} t_{n-1,\sigma(n-1)} \cdots t_{2,\sigma(2)} t_{1,\sigma(1)} \) (see, e.g., [PW] or [Ko]). Then we can re-write the monomial \( t_{n,1} t_{n-1,2} \cdots t_{2,n-1} t_{1,n} \) as

\[
t_{n,1} t_{n-1,2} \cdots t_{1,n} = (-q)^{-l(\sigma_0)} D_q - \sum_{\sigma \in S_n \setminus \{\sigma_0\}} (-q)^{l(\sigma)-l(\sigma_0)} t_{n,\sigma(n)} t_{n-1,\sigma(n-1)} \cdots t_{1,\sigma(1)} \tag{2.4}
\]
where $\sigma_0 \in S_n$ is the permutation $i \mapsto (n+1-i)$. Note also that we can reorder the factors in the summands of (2.4) so that all factors $t_{i,j}$ from $\mathcal{N}^-$ are on the left of those from $\mathcal{N}^+$.  

Now we replace the right-hand side of (2.4) in the factor $t^{-\ell}$ within $t^{-\ell} t^{+\ell} t^{-\ell}$, thus

$$t^{-\ell} t^{+\ell} = (-q)^{- \ell(\sigma_0)} t^{-\ell} t_0^{-\ell} D_q^+ + \text{l.t.'s} = (-q)^{\ell(\sigma_0)} t^{-\ell} t_0^{+\ell} D_q + \text{l.t.'s}$$

Here $t_0^{-\ell} := t^{-\ell} (t_{n,1} t_{n-1,2} \cdots t_{2,n-1} t_{1,n})^{-1}$ has lower (total) degree than $t^{-\ell}$, and the expression l.t.'s stands for an $R$-linear combination of some other monomials $t^{-\ell} t^{\ell} t^{+\ell}$ (like $t^{-\ell} t^{+\ell}$ above) in which again the degree of $t^{-\ell}$ is lower than the degree of $t^{+\ell}$. In fact, this holds because when any factor $t_{i,\sigma(i)} \in \mathcal{N}^-$ is pulled from the right to the left of any monomial in $t^{-\ell} \in \mathcal{H}$, the degree of $t^{-\ell}$ is not increased. By induction on this degree, we can easily conclude that every ordered monomial $t^{-\ell} t^{+\ell} D_q^z$ (with $z \in \mathbb{Z}$) belongs to both the $R$-span of $B^\Lambda_{GL}$ and the $R$-span of $B^\vee_{GL}$, which is span $O^R_q(GL_n)$. Eventually, both $B^\Lambda_{SL}$ and $B^\vee_{SL}$ are linearly independent, as their image through the specialisation epimorphism $O^R_q(GL_n) \rightarrow \mathcal{H}^R(GL_n) \cong \mathcal{H}^R(GL_n)$ are $R$-bases of $\mathcal{H}^R(GL_n)$.

Second, we look at $O^R_q(SL_n)$. Like for claim (a), we can repeat again — mutatis mutandis — the argument for $O^R_q(GL_n)$, which does work again — one only has to plug in the additional relation $D_q = 1$ too. Otherwise, as an alternative proof, we can note that the isomorphism $O^R_q(SL_n) \otimes_R R[x,x^{-1}] \cong O^R_q(GL_n)$ together with the result about $B^\vee_{GL}$ easily implies that $B^\Lambda_{SL}$ too is an $R$-basis for $O^R_q(SL_n)$, q.e.d.

**Remarks 2.2:** (1) Claim (a) of Theorem 2.1 for $M_n$ only was independently proved in [PW] and in [Ko], but taking a field as ground ring. In [Ko], claim (b) for $GL_n$ only was proved as well. Similarly, the analogue of claim (b) for $SL_n$ only was proved in [Ga], §7, but taking as ground ring the field $k(q)$ — for any field $k$ of zero characteristic. Our proof then provide an alternative, unifying approach, which yields stronger results over $R$.

(2) We would better point out a special aspect of the basic assumption of Theorem 2.1 about $q$ and $R$. Namely, if the subring (1) of $R$ generated by 1 has prime characteristic (hence it is a finite field) then the condition on $(q-1)$ is equivalent to $q$ being trascendental over $R_q$ or $q = 1$. But if instead the characteristic of (1) is zero or positive non-prime, then $(q-1)$ might be non-invertible in $R_q$ even though $q$ is algebraic (or even integral) over (1).

The end of the story is that Theorem 2.1 holds true in the “standard” case of trascendental values of $q$, but also in more general situations.

(3) The argument used in the proof of Theorem 2.1 to get the result for $O^R_q(SL_n)$ from those for $O^R_q(GL_n)$, via the isomorphism $O^R_q(SL_n) \otimes_R R[x,x^{-1}] \cong O^R_q/GL_n$, actually work both ways. Therefore, one can also prove the results directly for $O^R_q(SL_n)$ — as we sketched above — and from them deduce those for $O^R_q/GL_n$. Even more, as we have proved independently the results for $O^R_q/GL_n$ — i.e. $B^\Lambda_{GL}$ and $B^\vee_{GL}$ are $R$-bases — and for $O^R_q(SL_n)$ — i.e., $B_{SL}$ and $B^\vee_{SL}$ are $R$-bases — we can use them to prove that the algebra morphism $O^R_q(SL_n) \otimes_R R[x,x^{-1}] \rightarrow O^R_q(GL_n)$ is in fact bijective.

(4) The orders considered in claim (b) of Theorem 2.1 refer to a triangular decomposition of $O^R_q/GL_n$ and $O^R_q(SL_n)$ which is opposite to the standard one. This opposite decomposition was introduced — and its importance was especially pointed out — in [Ko].

We are now ready to state and proof the main result of this paper:
Theorem 2.3 (PBW theorem for $O_{\varepsilon}^{Z_e}(G)$ as an $O_{\varepsilon}^{Z_e}(G)$–module, for $G \in \{M_n, GL_n\}$).

Let any total order be fixed in $\{1, \ldots, n\}^{\times^2}$. Then the set of ordered monomials

$$B_{GL}^M := \left\{ \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} \mid 0 \leq N_{i,j} \leq \ell - 1, \forall i, j \right\}$$

thought of as a subset of $O_{\varepsilon}^{Z_e}(M_n) \subset O_{\varepsilon}^{Z_e}(GL_n)$, is a basis of $O_{\varepsilon}^{Z_e}(M_n)$ as a module over $O_{\varepsilon}^{Z_e}(M_n)$, and a basis of $O_{\varepsilon}^{Z_e}(GL_n)$ as a module over $O_{\varepsilon}^{Z_e}(GL_n)$.

In particular, both modules are free of rank $\ell^{\dim(G)}$, with $G \in \{M_n, GL_n\}$.

Proof. When specialising, Theorem 2.1(a) implies that $O_{\varepsilon}^{Z_e}(M_n)$ is a free $Z_e$–module with $B_{M}|_{q=\varepsilon} = \left\{ \prod_{i,j=1}^{n} t_{ij}^{N_{ij}} \mid N_{ij} \in \mathbb{N} \forall i, j \right\}$ as basis — where, by abuse of notation, we write again $t_{ij}$ for $t_{ij}|_{q=\varepsilon}$ now. Whenever the exponent $N_{ij}$ is a multiple of $\ell$, the power $t_{ij}^{N_{ij}}$ belongs to the isomorphic image $\mathfrak{fr}_Z(O_{\varepsilon}^{Z_e}(M_n))$ of $O_{\varepsilon}^{Z_e}(M_n)$ inside $O_{\varepsilon}^{Z_e}(M_n)$, hence it is a scalar for the $O_{\varepsilon}^{Z_e}(M_n)$–module structure of $O_{\varepsilon}^{Z_e}(M_n)$. Therefore, reducing all exponents modulo $\ell$ we find that $B_{GL}^M$ is a spanning set for the $O_{\varepsilon}^{Z_e}(M_n)$–module $O_{\varepsilon}^{Z_e}(M_n)$. In addition, $O_{\varepsilon}^{Z_e}(M_n)$ clearly admits as $Z$–basis the set $B_M = \left\{ \prod_{i,j=1}^{n} t_{ij}^{N_{ij}} \mid N_{ij} \in \mathbb{N} \forall i, j \right\}$. It follows that $B_M$ is also a $Z_e$–basis of $O_{\varepsilon}^{Z_e}(M_n)$, so $\mathfrak{fr}_Z(B_M) = \left\{ \prod_{i,j=1}^{n} t_{ij}^{N_{ij}} \mid N_{ij} \in \mathbb{N} \forall i, j \right\}$ is a $Z_e$–basis of $\mathfrak{fr}_Z(O_{\varepsilon}^{Z_e}(M_n))$. This last fact easily implies that $B_{GL}^M$ is also $O_{\varepsilon}^{Z_e}(M_n)$–linearly independent, hence it is a basis of $O_{\varepsilon}^{Z_e}(M_n)$ over $O_{\varepsilon}^{Z_e}(M_n)$ as claimed.

As to $O_{\varepsilon}^{Z_e}(GL_n)$, from definitions and the analysis in §1 we get (with $D_{\varepsilon} := D_{q}|_{\varepsilon}$)

$$O_{\varepsilon}^{Z_e}(GL_n) = O_{\varepsilon}^{Z_e}(M_n)[D_{\varepsilon}^{-1}] = O_{\varepsilon}^{Z_e}(M_n)[D_{\varepsilon}^{-\ell}] = O_{\varepsilon}^{Z_e}(M_n)[D^{-1}] \bigotimes_{O_{\varepsilon}^{Z_e}(M_n)} O_{\varepsilon}^{Z_e}(M_n) = O_{\varepsilon}^{Z_e}(GL_n) \bigotimes_{O_{\varepsilon}^{Z_e}(M_n)} O_{\varepsilon}^{Z_e}(M_n)$$

thus the result for $O_{\varepsilon}^{Z_e}(GL_n)$ follows at once from that for $O_{\varepsilon}^{Z_e}(M_n)$.

\[\Box\]

§ 3 Frobenius structures

3.1 Frobenius extensions and Nakayama automorphisms. Following [BGStro], we say that a ring $R$ is a free Frobenius extension over a subring $S$, if $R$ is a free $S$–module of finite rank, and there is an isomorphism $F : R \rightarrow \text{Hom}_S(R, S)$ of $R$–$S$–bi-modules. Then $F$ provides a non-degenerate associative $S$–bilinear form $\mathbb{B} : R \times R \rightarrow S$, via $\mathbb{B}(r, t) = F(t)(r)$. Conversely, one can characterise Frobenius extensions using such forms. When $S = Z$ is contained in the centre of $R$, there is a $Z$–algebra automorphism $\nu : R \rightarrow R$, given by $r F(1) = F(1) \nu(r)$ (for all $r \in R$), and such $\mathbb{B}(x, y) = \mathbb{B}(\nu(y), x)$. This is called the Nakayama automorphism, and it is uniquely determined by the pair $Z \subseteq R$, up to $\text{Int}(R)$.

Proposition 3.2. (cf. [BGStro], §2)

Let $R$ be a ring, $Z$ an affine central subalgebra of $R$. Assume that $R$ is free of finite rank as a $Z$–module, with a $Z$–basis $B$ that satisfies the following condition: there exists a $Z$–linear functional $\Phi : R \rightarrow Z$ such that for any non-zero $a = \sum_{b \in B} z_b b \in R$ there exists $x \in R$ for which $\Phi(xa) = uz_b$ for some unit $u \in Z$ and some non-zero $z_b \in Z$.

Then $R$ is a free Frobenius extension of $Z$. Moreover, for any maximal ideal $m$ of $Z$, the finite dimensional quotient $R/mR$ is a finite dimensional Frobenius algebra.
This result is used in [BGStro] to show that many families of algebras — in particular, some related to $O_\varepsilon(G)$, where $G$ is a (complex, connected, simply-connected) semisimple affine algebraic group — are indeed free Frobenius extensions. But the authors could not prove the same for $O_\varepsilon(G)$, as they did not know an explicit $O(G)$–basis of $O_\varepsilon(G)$. Now, following their strategy and using Theorem 2.3, I shall now prove that $O^Z_\varepsilon(G)$ is free Frobenius over $O^Z_\varepsilon(G)$ when $G = M_n$ or $GL_n$.

**Theorem 3.3.** Let $G$ be $M_n$ or $GL_n$. Then $O^Z_\varepsilon(G)$ is a free Frobenius extension of $O^Z_\varepsilon(G)$, with Nakayama automorphism $\nu$ given by $\nu(t_{i,j}) = \varepsilon^{2(i+j-n-1)} t_{i,j}$ $(i, j = 1, \ldots, n)$.

**Proof.** We prove that there exists a suitable $O^Z_\varepsilon(G)$–linear functional $\Phi : O^Z_\varepsilon(G) \rightarrow O^Z_\varepsilon(G)$ as required in Proposition 3.2, so that that result applies to $R := O^Z_\varepsilon(G)$ and $\mathcal{Z} := O^Z_\varepsilon(G)$.

Define $\Phi$ on the elements of the $O^Z_\varepsilon(G)$–basis $B^M_{GL}$ of $O^Z_\varepsilon(G)$ (see Theorem 2.3) by

$$
\Phi \left( \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \right) := \prod_{i,j=1}^n \delta_{N_{i,j}, \ell-1} = \begin{cases} 1, & \text{if } N_{i,j} = \ell - 1 \ \forall i, j \\ 0, & \text{if not} \end{cases}
$$

(3.1)

(for all $0 \leq N_{i,j} \leq \ell - 1$), and extend to all of $O^Z_\varepsilon(G)$ by $O^Z_\varepsilon(G)$–linearity. In other words, $\Phi$ is the unique $O^Z_\varepsilon(G)$–valued linear functional on $O^Z_\varepsilon(G)$ whose value is 1 on the basis element $t_{\ell-1} := \prod_{i,j=1}^n t_{i,j}^{\ell-1}$ and is zero on all other elements of the $O^Z_\varepsilon(G)$–basis $B^M_{GL}$.

We claim that $\Phi$ satisfies the assumptions of Proposition 3.2, so the latter applies and proves our statement. Indeed, let us consider any non-zero $a = \sum_{\ell \in B^M_{GL}} z_{\ell} t_{\ell} \in O^Z_\varepsilon(G)$, and let $t_0 = \prod_{i,j=1}^n t_{i,j}^{N_{i,j}}$ in $B^M_{GL}$ be such that $z_{t_0} \neq 0$ and $w(t_0)$ is maximal (w.r.t. $\leq_{\text{lex}}$). Then define $t_0' := \prod_{i,j=1}^n t_{i,j}^{N_{i,j}'}$ $(\in B^M_{GL})$ with $N_{i,j}' := \ell - 1 - N_{i,j}$ for all $i, j = 1, \ldots, n$. Quoting from the proof of Theorem 2.1(a), we know that $t_0' z_{t_0} = \varepsilon^s t_{\ell-1} + \ell t$ for $s \in \mathbb{Z}$ and the expression $t t$'s now stands for an $O^Z_\varepsilon(G)$–linear combination of monomials $\ell \in B^M_{GL}$ such that $w(\ell) \leq_{\text{lex}} w(t_{\ell-1})$; in particular, $\Phi(\ell) = 0$ for all these $\ell$, hence eventually $\Phi(t_0' z_{t_0}) = \varepsilon^s \Phi(t_{\ell-1}) = \varepsilon^s$. Similarly, if $t' \in B^M_{GL}$ is such that $w(t') <_{\text{lex}} w(t_{\ell})$, then $t_0' t'$ is an $O^Z_\varepsilon(G)$–linear combination of PBW monomials whose weight is at most $w(t_0' t')$, hence $\Phi(t_0' t') = 0$. As we chose $t_0$ so that $w(t_0)$ is maximal, we eventually find

$$
\Phi(t_0' a) = \sum_{\ell \in B^M_{GL}} z_{\ell} \Phi(\ell) = z_{t_0} \Phi(t_0) = \varepsilon^s z_{t_0}
$$

where $\varepsilon^s$ is a unit in $O^Z_\varepsilon(G)$. So $\Phi$ satisfies the assumptions of Proposition 3.2, as claimed.

As to the Nakayama automorphism $\nu : O^Z_\varepsilon(G) \rightarrow O^Z_\varepsilon(G)$, it is characterized (see §3.1) by the property that $B(x, y) = B(\nu(y), x)$ for all $x, y \in R$. Here $B$ is a $\mathcal{Z}$–bilinear form as in §3.1, which now is related to $\Phi$ by the formula $B(x, y) = \Phi(xy)$ for all $x, y \in R$.

As $\Phi$ is an automorphism, and $O^Z_\varepsilon(G)$ is generated — over $O^Z_\varepsilon(G)$ — by the $t_{i,j}$'s, the claim about $\nu$ is proved if we show that

$$
\Phi \left( \prod_{r,s=1}^n t_{r,s}^{e_{r,s}} \cdot t_{i,j} \right) = \varepsilon^{2(i+j-n-1)} t_{i,j} \cdot \prod_{r,s=1}^n t_{r,s}^{e_{r,s}}
$$

(3.2)

Now, our usual argument shows that the expansions of the product of a generator $t_{i,j}$ and a PBW monomial $\prod_{r,s=1}^n t_{r,s}^{e_{r,s}}$ (in either order of the factors) as an $O^Z_\varepsilon(G)$–linear combination of elements of the $O^Z_\varepsilon(G)$–basis $B^M_{GL}$ are of the form
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\begin{equation}
\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{i,j} = \varepsilon^{i+j-2n} \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}} + \text{l.t.'s}
\end{equation}

This along with (3.1) gives

\begin{equation}
\Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{i,j}\right) = \varepsilon^{i+j-2n} \Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}}\right) = \varepsilon^{i+j-2n} \text{ if } e_{r,s} = \ell - 1 - \delta_{r,i}\delta_{j,s}
\end{equation}

\begin{equation}
\Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{i,j}\right) = \varepsilon^{i+j-2n} \Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}}\right) = 0 \text{ if not}
\end{equation}

and similarly

\begin{equation}
\Phi\left(t_{i,j} \cdot \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}}\right) = \varepsilon^{2-i-j} \Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}}\right) = \varepsilon^{2-i-j} \text{ if } e_{r,s} = \ell - 1 - \delta_{r,i}\delta_{j,s}
\end{equation}

\begin{equation}
\Phi\left(t_{i,j} \cdot \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}}\right) = \varepsilon^{2-i-j} \Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}}\right) = 0 \text{ if not}
\end{equation}

Direct comparison now shows that (3.2) holds, q.e.d. □

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