Chapter 3

Dynamics of Domain Walls

This chapter consists in two long sections, each of which is partitioned into a number of subsections. The first section, which is taken with only minor changes from [39], deals with flat walls in motion; the second section is where new and unpublished material on the motion of curved walls is found.

Flat walls

In their path-breaking paper of 1935 [30], Landau and Lifshitz, besides determining the internal structure of a static domain wall in the absence of an external field, considered the case when an external magnetic field parallel to the easy axis sets the wall in motion; they derived an approximate solution of the traveling-wave form

$$\mathbf{m} = \mathbf{m}(\xi)\,, \quad \xi := x - vt\,, \tag{3.1}$$

with which they were able to estimate the dynamical magnetic permeability of a ferromagnet. Some two decades later, in his doctoral thesis, Walker [49, 43, 27] furnished an exact solution of the form (3.1) to the classical (Landau-Lishitz and) Gilbert [22] evolution equation for the magnetization as adapted to the flat-wall case. Walker's solution, being explicit, permits us to dispose of the somewhat casual asymptotics used by Landau and Lifhitz to justify their approximations. Remarkably, constants apart, Walker's dynamic solution depends on the current variable ξ just as Landau-Lifshitz' static solution depends on the spatial coordinate x (Section 4); moreover, during a Walker's evolution, as noted in passing on p. 275 of [27], dissipation is exactly compensated by external working.

This fact draws attention to the class of the Walker processes, that is to say, those evolution processes of the magnetization field during which such a compensation condition is fulfilled. One may ask whether, for ferromagnets whose constitutive response is more general than the standard response, it would still be true that, granted the compensation condition, the traveling-wave solutions to the dynamic problem have the same form as the variational solutions to the associated static problem. One may also ask whether there are other Walker processes - in addition to the steady propagation motions of type (3.1) discussed below - that solve the standard Gilbert equation, or generalizations of it.

We introduce a procedure to derive Walker's solution to the classical Gilbert equation. By applying this procedure, we show that, when the Gilbert equation is generalized by the addition of terms due to either higher-order exchange energy or exchange dissipation (or both) [48, 38, 6, 40], the Walker processes retain their form but do not solve the generalized Gilbert equation. We apply our procedure again in the case when a dry-friction dissipation term is added to the standard Gilbert equation, and we show that, if the applied magnetic field has suitable, nonvanishing components in the directions orthogonal to the easy axis, then there are exact solutions of the Gilbert equation picturing $90^{\mathrm{O}}-$walls; these solutions, however, are not Walker processes.

Curved Walls

We model a curved domain wall by a regular, oriented surface $\mathcal{S}$ evolving smoothly in time. We introduce the parameter $\varepsilon = \beta^{-1}$, and, by using suitable space and time coordinates, we cast the Gilbert equation in the form:

$$\varepsilon^2 \left((\mu\gamma)^{-1}\dot{\mathbf{m}} + \mathbf{m} \times \dot{\mathbf{m}}\right) = \mathbf{m} \times \left(\varepsilon^2 \Delta \mathbf{m} + (\mathbf{m} \cdot \mathbf{e})\mathbf{e} + \varepsilon(\mathbf{h}^{\mathrm{s}} + \mathbf{h}^{\mathrm{e}})\right) . \qquad (3.2)$$

We suppose that, for each ε in a non-empty interval $(0, \overline{\varepsilon})$, there is a vector field $\mathbf{m}_\varepsilon$ which solves (3.2) and admits two regular expansions in powers of ε: an *outer expansion* and an *inner expansion*; the outer expansion is valid away from the domain wall, and captures the behavior of the magnetization in the magnetic domains; the inner expansion describes the magnetization in a narrow neighborhood of the domain wall.

Compatibility of the inner expansion with the field equations yields two ODEs. The first ODE, when solved subject to the boundary data which arise from the matching conditions for the inner and outer expansion, yields the magnetization profile across the domain wall. In particular, for flat 180° walls, this solution is similar to that of Walker's travelling wave.

Solvability of the second ODE requires that the domain wall evolves according to:

$$\rho\, v + \sigma\, k = p^e + p^s\,. \qquad (3.3)$$

Here, given a time t, $v = v(\mathbf{s}, t)$ and $k = k(\mathbf{s}, t)$ are, respectively, the *normal velocity* and *twice the mean curvature* of the wall at point $\mathbf{s} \in \mathcal{S}(t)$; the quantities $\sigma = \sigma(\mathtt{s}, t)$ and $\rho = \rho(\mathtt{s}, t)$ depend on the magnetization profile across the wall and are interpreted, respectively, as a *surface tension* and a *viscous-drag coefficient*; the quantities

$$\begin{aligned} p^e(\mathtt{s}, t) &:= \ \mathbf{h}^{\mathrm{e}}(\mathtt{s}, t) \cdot [\![\mathbf{m}]\!](\mathtt{s}, t)\,, \\ p^s(\mathtt{s}, t) &:= \langle\!\langle \mathbf{h}^{\mathrm{s}} \rangle\!\rangle(\mathtt{s}, t) \cdot [\![\mathbf{m}]\!](\mathtt{s}, t)\,, \end{aligned} \tag{3.4}$$

are interpreted as the *pressures* that the external and internal magnetic field produce on the domain wall.[1]

This result confirms[2] that mean–curvature flow emerges in a natural way from the Gilbert equation for high uniaxial anisotropies. Moreover, an analogy with the theory of phase transitions is suggested: since $\mathbf{h}^{\mathrm{e}}$ is continuous across the interface, the pressure exerted by the external field can be written as

$$p^{\mathrm{e}} = [\![\mathbf{h}^{\mathrm{e}} \cdot \mathbf{m}]\!] = [\![\psi^{\mathrm{e}}]\!] \tag{3.5}$$

and interpreted as the jump in "chemical potential" which, together with surface tension, drives the motion of a phase boundary in the standard theory of phase transitions. Following a different approach, equation (3.3) may be interpreted as a balance of configurational forces [32, 23].

The analogy becomes more evident if we note that the anisotropy–energy density $\psi^{\mathrm{a}}(\mathbf{m}) = \frac{1}{2}\beta(\mathbf{m} \cdot \mathbf{e})^2$ is a non–convex function on the unit sphere (Figure 3.1.a).[3] This allows us to regard the Gilbert equation as a vectorial form of the scalar Ginzburg–Landau equation

$$\varepsilon^2 \dot{u} - \varepsilon^2 \Delta u + \partial_u \psi(u) = 0 \tag{3.6}$$

which is a model equation used to describe first-order phase transitions and to study mean-curvature flow. In the Ginzburg–Landau model, the *order parameter* u, or *phase field*, characterizes the phase of the material; $\partial_u \psi$ is the derivative of a non-convex function ψ (Figure 3.1.b) with two local minima u_a and u_b, which correspond to the two phases of the material. It has been

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[1] Here $\langle\!\langle \mathbf{v} \rangle\!\rangle$ and $[\![\mathbf{v}]\!]$ denote, respectively the *arithmetic mean*

$$\langle\!\langle \mathbf{v} \rangle\!\rangle(\mathtt{s}, t) = \frac{1}{2}\left(\lim_{\mathtt{p}\to\mathtt{s}+} \mathbf{v}(\mathtt{p}, t) + \lim_{\mathtt{p}\to\mathtt{s}-} \mathbf{v}(\mathtt{p}, t)\right)$$

and the *difference*

$$[\![\mathbf{v}]\!](\mathtt{s}, t) = \lim_{\mathtt{p}\to\mathtt{s}+} \mathbf{v}(\mathtt{p}, t) - \lim_{\mathtt{p}\to\mathtt{s}-} \mathbf{v}(\mathtt{p}, t)$$

between the values that a vector field $\mathbf{v}$ assumes at each side of the surface at point $\mathtt{s} \in \mathcal{S}(t)$.

[2] See also [51, 52, 53, 54].

[3] Recall, from Section 1.2.2 that β is the anisotropy constant.

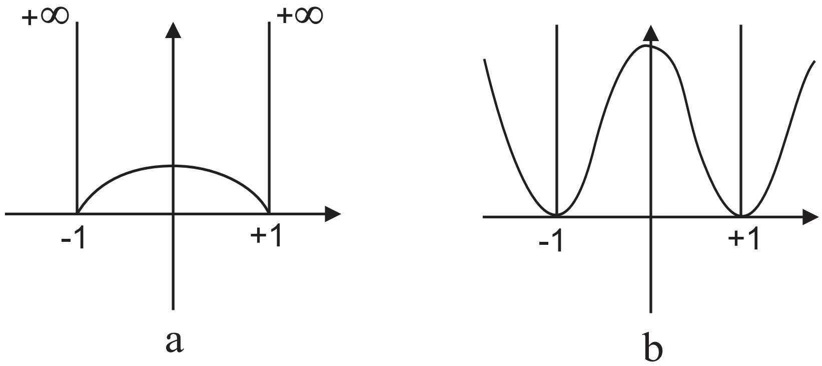

Figure 3.1: (a) Anisotropy energy; (b) Ginzburg-Landau energy.

shown [41, 13, 19, 7] that, for $\varepsilon \to 0$, solutions of (3.6) are patchwise constant, and phase boundaries evolve by mean curvature motion when driving forces are absent:

$$v + k = 0\,. \tag{3.7}$$

By regarding the magnetization vector $\mathbf{m}$ as vectorial order parameter, one is led to interpret magnetic domains as phases and domain walls as phase boundaries. However, recourse to a phase–field theory is appropriate when the two phases have *distinct* physical properties, sharply changing at the phase boundary. In the case of ferromagnets, the situation is different: two neighboring domains are made of the same physical substrate, and the jump in the order parameter at the common wall is more a character of a class of solutions to an initial/boundary–value problem than the result of a localized physical process. Moreover, while a Ginzburg–Landau dynamics is introduced as a somewhat artificial, "generic" tool for a better understanding of phase transitions, the Gilbert equation rests on peculiar and explicit physical grounds. In fact, at variance with, say, a solidification front, and ideal (*i.e.* free from impurities,etc.) domain wall has no definite physical substance; instead, it is merely the site of a spatial discontinuity.

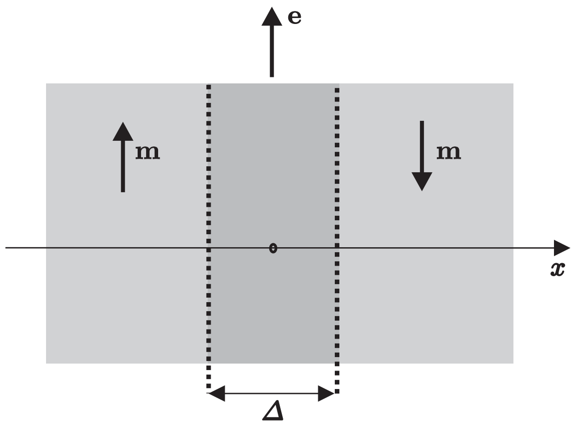

Figure 3.2: A Bloch wall of thickness Δ.

3.1 Flat Walls

We consider an infinite ferromagnetic body partitioned into two domains by a flat wall parallel to the easy axis **e**. We suppose that the wall is of the $180^{\rm O}$ type, i.e., that the magnetization field **m**, while having constant direction in each domain, rotates from **e** to $-\,\mathbf{e}$ across the wall thickness Δ (Fig. 3.2). For the external magnetic field driving the wall motion we choose

$$\mathbf{h}^{\rm e} = h\,\mathbf{e}\,, \quad h = \text{a constant}\,. \tag{3.8}$$

Following the procedure of Landau & Lifshitz and Walker, for $\{\mathbf{o}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ an orthonormal cartesian frame with $\mathbf{c}_3 = \mathbf{e}$ and $\mathbf{c}_1$ orthogonal to the domain wall, we consider solutions of the Gilbert equation in the form of a traveling wave:

$$\mathbf{m} = \mathbf{m}(\xi)\,, \quad \xi = x_1 - vt\,. \tag{3.9}$$

For the one-dimensional problem we address, the dependence of the stray field $\mathbf{h}^{\rm s}$ on the magnetization **m** can be made explicit:

$$\mathbf{h}^{\rm s} = -(\mathbf{m}\cdot\mathbf{c}_1 + c)\mathbf{c}_1\,. \tag{3.10}$$

If we set $c = 0$,[4] the stray-field energy density (1.17) takes the form:

$$\psi^{\rm s} = \frac{1}{2}\,(\mathbf{m}\cdot\mathbf{c}_1)^2\,. \tag{3.11}$$

[4]Whenever $c \neq 0$ the physical effect of the part $\mathbf{c}_1$ of the stray field can be cancelled by the addition of a suitable external field.

To prove (3.10), we introduce the usual representation of the stray field in terms of the scalar potential H:

$$\mathbf{h}^{\mathrm{s}} = -\nabla H\,; \tag{3.12}$$

we note that, due to (3.9),[5]

$$\mathrm{div}\,\mathbf{m} = \mathbf{m}' \cdot \mathbf{c}_1 = (\mathbf{m} \cdot \mathbf{c}_1)'\,; \tag{3.13}$$

and, finally, we write equation $(3.101)_2$ as

$$\triangle H = (\mathbf{m} \cdot \mathbf{c}_1)', \tag{3.14}$$

where, for t fixed, the right side is depends at most on x_1. But then the representation formula for the solutions of the Poisson equation implies that $H = H(x_1, t)$, and hence the stray field is parallel to $\mathbf{c}_1$:

$$\mathbf{h}^{\mathrm{s}} = -H'\,\mathbf{c}_1\,, \quad H'(x_1, t) = (\mathbf{m}(x_1, t) \cdot \mathbf{c}_1) + c(t)\,. \tag{3.15}$$

The desired conclusion follows when we dispose of the arbitrary function $c(t)$ by requiring that $\lim_{x_1 \to \pm\infty} \mathbf{h}^{\mathrm{s}}(x_1, t) = \mathbf{0}$.

With (3.11), the density of internal energy can be given the following explicit forms:

$$\psi^{\mathrm{i}} = \frac{1}{2}\,\alpha\,(\mathbf{m}')^2 + \frac{1}{2}\,\mathbf{T}\mathbf{m} \cdot \mathbf{m}\,, \tag{3.16}$$

with

$$\mathbf{T} := -\beta\,\mathbf{e} \otimes \mathbf{e} + \,\mathbf{c}_1 \otimes \mathbf{c}_1\,, \tag{3.17}$$

where use has been made also of (1.9) and (1.11). Substituting (3.16) in (1.18) and using the definition $(1.6)_2$ we find, for the internal magnetic field:

$$\mathbf{h}^{\mathrm{i}} = \alpha \mathbf{m}'' + \beta(\mathbf{e} \cdot \mathbf{m})\mathbf{e} - (\mathbf{c}_1 \cdot \mathbf{m})\mathbf{c}_1\,, \tag{3.18}$$

or rather, with the use of (3.17),

$$\mathbf{h}^{\mathrm{i}} = \alpha \mathbf{m}'' - \mathbf{T}\mathbf{m}\,. \tag{3.19}$$

3.1.1 Preliminaries

Let $\mathbf{e}$, φ and ϑ be, respectively, the polar axis and the parallel and meridional coordinates in a system of spherical coordinates. Moreover, let

$$\mathbf{a} = \mathbf{a}(\varphi) = -\sin\varphi\,\mathbf{c}_1 + \cos\varphi\,\mathbf{c}_2 \tag{3.20}$$

be the unit vector orthogonal to both $\mathbf{e}$ and $\mathbf{m} = \mathbf{m}(\varphi, \vartheta, t)$ and such that $\mathbf{a} \cdot \mathbf{e} \times \mathbf{m} > 0$ (Fig. 3.3). Then, with the use of the orthonormal basis

[5] For f a function of a real variable, f' denotes its derivative.

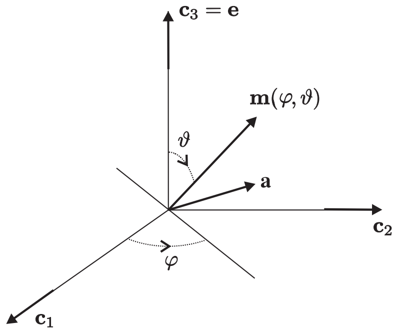

Figure 3.3: Polar coordinates.

$(\mathbf{a}, \mathbf{e}, \mathbf{Ae})$, where $\mathbf{A} = \mathbf{A}(\varphi)$ is the skew tensor uniquely associated to $\mathbf{a}$:

$$\mathbf{Ae} = \mathbf{a} \times \mathbf{e} = \cos\varphi\,\mathbf{c}_1 + \sin\varphi\,\mathbf{c}_2\,, \tag{3.21}$$

we have that

$$\mathbf{m} = \cos\vartheta\,\mathbf{e} + \sin\vartheta\,\mathbf{Ae}\,, \tag{3.22}$$
$$\dot{\mathbf{m}} = \sin\vartheta\,\dot{\varphi}\,\mathbf{a} + \dot{\vartheta}\,\mathbf{Am}\,, \tag{3.23}$$

where

$$\mathbf{Am} = \mathbf{a} \times \mathbf{m} = -\sin\theta\,\mathbf{e} + \cos\theta\,\mathbf{Ae}\,; \tag{3.24}$$

hence,

$$\mathbf{m} \times \dot{\mathbf{m}} = \dot{\vartheta}\,\mathbf{a} - \sin\vartheta\,\dot{\varphi}\,\mathbf{Am}\,, \tag{3.25}$$

so that, in particular,

$$\mathbf{e} \cdot \mathbf{m} \times \dot{\mathbf{m}} = \sin^2\vartheta\,\dot{\varphi}\,. \tag{3.26}$$

3.1.2 Walker processes

We are especially interested in finding circumstances when *the external working balances the dissipation pointwise*:

$$-\,\mathbf{h}^{\mathrm{e}} \cdot \dot{\mathbf{m}} + \mathrm{d} = 0\,.^{6} \tag{3.27}$$

[6]Trivially, this compensation condition implies that, if the external working is null, so is dissipation; but then, as it is not difficult to deduce from (2.5), thermodynamics only allows for static processes. As a matter of fact, the Landau-Lifshitz solution is a *static* Walker process.

Under such circumstances,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi^{\mathrm{e}} + \int_{\Omega} \mathrm{d} = 0\,, \tag{3.28}$$

and hence, due to the Liapounov relation (2.6), *the internal free-energy is globally conserved*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi^{\mathrm{i}} = 0\,. \tag{3.29}$$

More precisely, we are interested in finding solutions, if any, of the generalized Gilbert equation (2.1) by looking into the set of the *Walker processes*, *i.e.*, the solutions of the scalar equation (3.27); we refer to the latter as the *Walker condition*.

In conclusion, the Walker equation (3.27) becomes

$$(-\,h\,\mathbf{e} + \mu\,\dot{\mathbf{m}})\cdot\dot{\mathbf{m}} = 0\,.^{7} \tag{3.30}$$

Moreover, the generalized Gilbert equation (2.1) reduces to the standard form

$$\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m}\times(\mathbf{h}^{\mathrm{i}} + h\,\mathbf{e} -\,\mu\,\dot{\mathbf{m}})\,; \tag{3.31}$$

the equivalent scalar system (2.2) is

$$\begin{array}{rcl} -\,\gamma^{-1}\dot{\mathbf{m}}\cdot\dot{\mathbf{m}} &=& (\mathbf{h}^{\mathrm{i}} + h\,\mathbf{e})\cdot\mathbf{m}\times\dot{\mathbf{m}}\,,\\ 0 &=& (\mathbf{h}^{\mathrm{i}} + h\,\mathbf{e} -\,\mu\,\dot{\mathbf{m}})\cdot\dot{\mathbf{m}}\,. \end{array} \tag{3.32}$$

Thus, for a Walker process (a solution of (3.30)) to be a Gilbert process (a solution of (3.31)), it has to satisfy

$$\begin{array}{rcl} -\,\gamma^{-1}\dot{\mathbf{m}}\cdot\dot{\mathbf{m}} &=& (\alpha\triangle\mathbf{m} - \mathbf{T}\mathbf{m} + h\,\mathbf{e})\cdot\mathbf{m}\times\dot{\mathbf{m}}\,,\\ 0 &=& (\alpha\triangle\mathbf{m} - \mathbf{T}\mathbf{m})\cdot\dot{\mathbf{m}}\,. \end{array} \tag{3.33}$$

We show that one such solution exists in the next section.

3.1.3 Satisfying the Walker condition

With (3.23), the Walker equation (3.30) becomes:

$$h\,\sin\vartheta\,\dot{\vartheta} + \mu\,(\sin^2\vartheta\,\dot{\varphi}^2 + \dot{\vartheta}^2) = 0\,. \tag{3.34}$$

[7] An interesting consequence of (3.30) is that

$$h(\mathbf{m}\cdot\mathbf{e})^{\cdot} \geq 0$$

along all Walker processes.

We here restrict attention to traveling-wave solutions to (3.34) being of type (3.1) and such that

$$\varphi(\xi) = \varphi_o = \text{a constant}, \tag{3.35}$$

so that, in particular,

$$\dot{\vartheta} = -\, v\, \vartheta' \,. \tag{3.36}$$

Under the provisional assumptions that both the propagation velocity v and ϑ' be not null and that the signs of v and the datum h be the same, we write equation (3.34) in the simple form

$$\vartheta'(\xi) = c\, \sin \vartheta(\xi), \quad \xi \in (-\infty, +\infty), \quad c = \frac{h}{\mu v}\,. \tag{3.37}$$

Equation (3.37) is directly reminiscent of the equation derived by Landau and Lifshitz in their classical paper:

$$\vartheta'^2 = \frac{\beta}{\alpha}\, \sin^2 \vartheta\,, \tag{3.38}$$

that is, equation (8) of [30]. The solution of (3.37) can be read off equation (9) of [30], and is

$$\vartheta(\xi) = \arccos \frac{1 - \exp(2\, c\, \xi)}{1 + \exp(2\, c\, \xi)}\,. \tag{3.39}$$

Remark 1. For a flat domain wall parallel to the easy axis $\mathbf{e}$ and perpendicular to $\mathbf{c}_1$, centered at $\xi = 0$, and of thickness $2\xi_0$, we expect the conditions

$$\mathbf{m}(\mp\xi_0) = \pm\, \mathbf{e}\,. \tag{3.40}$$

to be satisfied at the boundary. However, Dirichlet-type conditions such as (3.40) do not seem physically realizable in micromagnetics: instead, Neumann conditions, such as

$$\partial_{\mathbf{c}_1} \mathbf{m}(\mp\xi_0) = \mathbf{0}\,, \tag{3.41}$$

have a physical sense that is not questioned.[8] Walker's solution effectively concentrates about $\xi = 0$ most of the rotation of $\mathbf{m}$ from $\mathbf{e}$ to $-\,\mathbf{e}$, although it spreads that rotation over the whole real line. Moreover, for any traveling-wave process of the type we are considering,

$$\nabla \mathbf{m} = \mathbf{m}' \otimes \mathbf{c}_1, \quad \mathbf{m}' = \vartheta' \mathbf{A} \mathbf{m}\,. \tag{3.42}$$

Thus, the boundary condition (3.41) takes for Walker's solution the limit form

$$\vartheta'(\pm\, \infty) = 0\,, \tag{3.43}$$

[8] In the present case, the boundary normal is $\mathbf{n} = \pm\, \mathbf{c}_1$ for $\xi = \pm\, \xi_0$; $\partial_{\mathbf{c}_1} \mathbf{m} = (\nabla \mathbf{m})\mathbf{c}_1$.

or rather, with the use of $(3.37)_1$,

$$\sin\vartheta(\pm\infty) = 0\,; \tag{3.44}$$

this last condition, with (3.22), yields

$$\mathbf{m}(\vartheta(\mp\infty)) = \pm\,\mathbf{e}\,, \tag{3.45}$$

in agreement with (3.40). ♢

3.1.4 Solving the Gilbert equation

Consider now the vectorial equation (3.31) and, with (3.23)-(3.26), replace it by the following system of two scalar evolution equations:

$$\begin{aligned} -\gamma^{-1}\,\dot\vartheta &= (\mathbf{h}^{\rm i}+\mathbf{d})\cdot\mathbf{a}\,,\\ \gamma^{-1}\sin\vartheta\,\dot\varphi &= (\mathbf{h}^{\rm i}+\mathbf{d})\cdot\mathbf{Am} - h\sin\vartheta\,, \end{aligned} \tag{3.46}$$

where of course $\mathbf{h}^{\rm i}$ and $\mathbf{d}$ are given by (3.19) and (2.11), respectively. This system is equivalent to system (3.32) and, as we proceed to show, more convenient to arrive to a quick and complete derivation of the traveling-wave solutions to equation (3.31); the derivations one finds in the literature move instead from (3.32).

For a steadily propogating magnetization process consistent with (3.35), that is to say, for $\mathbf{m}(\vartheta(x-vt))$, we have that

$$\dot{\mathbf{m}} = -v\,\mathbf{m}', \quad \triangle\mathbf{m} = \mathbf{m}''\,, \quad \mathbf{m}'' = \vartheta''\,\mathbf{Am} - \vartheta'^2\,\mathbf{m}\,, \tag{3.47}$$

(*cf.* (3.17) and (3.19)) so that

$$\begin{gathered} \mathbf{d}\cdot\mathbf{a} = 0\,, \quad \mathbf{d}\cdot\mathbf{Am} = \mu v\,\vartheta'\,,\\ \mathbf{h}^{\rm i} = \alpha(\vartheta''\mathbf{Am} - \vartheta'^2\mathbf{m}) - \mathbf{Tm}\,, \quad \mathbf{Tm} = -\beta(\mathbf{m}\cdot\mathbf{e})\mathbf{e} + (\mathbf{m}\cdot\mathbf{c}_1)\mathbf{c}_1\,, \end{gathered} \tag{3.48}$$

moreover,

$$\begin{aligned} \mathbf{Tm}\cdot\mathbf{a} &= -\sin\varphi_o\cos\varphi_o\sin\vartheta\,,\\ \mathbf{Tm}\cdot\mathbf{Am} &= (\beta+\cos^2\varphi_o)\sin\vartheta\cos\vartheta\,. \end{aligned} \tag{3.49}$$

Thus, the system (3.46) reduces to

$$\begin{aligned} v\,\vartheta' &= \gamma\sin\varphi_o\cos\varphi_o\sin\vartheta\,,\\ 0 &= \alpha\,\vartheta'' - (\beta+\cos^2\varphi_o)\sin\vartheta\cos\vartheta + \mu v\,\vartheta' - h\sin\vartheta\,. \end{aligned} \tag{3.50}$$

For the Walker solution (3.39) of equation (3.37) to be a solution of this system as well, the second equation must further reduce to

$$\alpha\,\vartheta'' = (\beta+\cos^2\varphi_o)\sin\vartheta\cos\vartheta\,.^9 \tag{3.51}$$

[9] Note that the system (3.33) reduces precisely to the system of this equation and the first of (3.50).

In addition, the so-far indeterminate constants v and φ_o must satisfy the two consistency conditions with the datum h resulting from substitution of (3.39) and its derivative into, respectively, $(3.50)_1$ and (3.51). These conditions are:

$$h = \mu\gamma \sin\varphi_o \cos\varphi_o\,, \tag{3.52}$$

and

$$\left(\frac{h}{\mu v}\right)^2 = \frac{\beta + \cos^2\varphi_o}{\alpha}\,. \tag{3.53}$$

Just as the Walker condition (3.37), the first equation of the Gilbert system (3.50) requires that ϑ' and $\sin\vartheta$ be proportional; for it to be consistent with the second, both (3.52) and (3.53) must hold. There is no need to determine the actual shape (3.39) of the Walker solution to deduce directly from (3.52)-(3.53) that, for whatever external field satisfying

$$h \leq \frac{1}{2}\mu\gamma\,, \tag{3.54}$$

the steady propagation of a plane magnetization wave $\mathbf{m}(\vartheta(x - vt))$ is possible, with the one or the other of the two velocities:

$$v = \frac{h}{\mu}\sqrt{\frac{\alpha}{\beta + \cos^2\varphi_o}}\,, \tag{3.55}$$

$$\cos^2\varphi_o = \frac{1}{2}\left(1 \pm \sqrt{1 - \left(\frac{2\,h}{\mu\gamma}\right)^2}\,\right). \tag{3.56}$$

Combination of the first of these relations with the last of (3.37) yields for the constant c the value Δ^{-1}, with

$$\Delta := \sqrt{\frac{\alpha}{\beta + \cos^2\varphi_o}}\;; \tag{3.57}$$

we can take Δ, the material parameter that would drive a conceivable sharp-interface asymptotics, as a measure of the *wall thickness*.

Remark 2. For h, and hence v, equal to zero, (3.52) gives $\varphi_o = \pi/2$: the wall is a Bloch wall (no stray field), of thickness $\Delta = (\alpha/\beta)^{1/2}$, as Landau and Lifshitz [30] found by solving the extremum problem

$$\int_{-\infty}^{+\infty}\left(\frac{1}{2}\alpha\vartheta'^2 - \frac{1}{2}\beta\cos^2\vartheta\right)dx = \min\,, \tag{3.58}$$

whose Euler-Lagrange equation is

$$\alpha\,\vartheta'' - \beta\sin\vartheta\cos\vartheta = 0 \tag{3.59}$$

(*cf.* (3.51)). ♢

3.1.5 High-order exchange energy and exchange dissipation

We now investigate whether the method we propose to generate the Walker solution continues to work for ferromagnets of more general constitutive response than the standard response. We take the expressions for the internal energy density and for the dissipation potential to be

$$\psi^{\mathrm{i}} = \frac{1}{2}\,\alpha\,|\nabla\mathbf{m}|^2 + \frac{1}{2}\,\mathbf{Tm}\cdot\mathbf{m} + \frac{1}{2}\,\lambda\,|\triangle\mathbf{m}|^2\,, \quad \lambda > 0\,, \tag{3.60}$$

and

$$\chi = \frac{1}{2}\,\mu\,|\dot{\mathbf{m}}|^2 + \frac{1}{2}\,\tau\,|\nabla\dot{\mathbf{m}}|^2\,, \quad \tau > 0 \tag{3.61}$$

(cf., respectively, (3.16) and (2.10)). Then, the Walker equation becomes

$$(-\,h\,\mathbf{e} + \mu\,\dot{\mathbf{m}})\cdot\dot{\mathbf{m}} + \tau\,\nabla\dot{\mathbf{m}}\cdot\nabla\dot{\mathbf{m}} = 0\,. \tag{3.62}$$

Moreover, the internal magnetic field and the dissipation field become

$$\mathbf{h}^{\mathrm{i}} = \alpha\,\triangle\mathbf{m} - \mathbf{Tm} - \lambda\,\triangle\triangle\mathbf{m} \tag{3.63}$$

and

$$\mathbf{d} = -\,\mu\,\dot{\mathbf{m}} + \tau\,\triangle\dot{\mathbf{m}}\,,^{10} \tag{3.64}$$

so that the corresponding generalized Gilbert equation is

$$\begin{aligned}\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m}\times(\alpha\triangle\mathbf{m}\, - \mathbf{Tm}\, - \lambda\triangle\triangle\mathbf{m} \\ +h\,\mathbf{e}\, - \,\mu\,\dot{\mathbf{m}} + \tau\triangle\dot{\mathbf{m}})\,.\end{aligned} \tag{3.65}$$

Remark 3. The scalar system equivalent to (3.65) can still be written in the form (3.46), of course with $\mathbf{h}^{\mathrm{i}}$ and $\mathbf{d}$ now given by (3.63) and (3.64). The mathematical effects of the high-order exchange terms in (3.65) have been studied in [6] and [40] (see also [38], [24]). From the physical point of view, the relative importance of the these terms is measured by two additional material parameters, both having the dimensions of a length: these are $\Delta_e := (\lambda/\alpha)^{1/2}$ and $\Delta_d := (\tau/\mu)^{1/2}$. ♢

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[10] Note that, in the place of relation (2.12), we now have

$$\mathbf{d}\cdot\dot{\mathbf{m}} + \mathrm{d} = \tau(\triangle\dot{\mathbf{m}} + \nabla\dot{\mathbf{m}}\cdot\nabla\dot{\mathbf{m}}) = \tau\,\mathrm{div}\,((\nabla\dot{\mathbf{m}})^{\mathrm{T}}\dot{\mathbf{m}})\,.$$

Under the present circumstances, it is convenient to supplement (3.42) and (3.47) with the additional relations

$$\begin{aligned} \nabla\dot{\mathbf{m}} = -\,v\,\mathbf{m}'' \otimes \mathbf{c}_1\,, \quad \triangle\dot{\mathbf{m}} = -\,v\,\mathbf{m}''', \quad \triangle\triangle\mathbf{m} = \mathbf{m}''''\,, \\ \mathbf{m}''' = (\vartheta''' - \vartheta'^3)\,\mathbf{Am} - 3\vartheta'\vartheta''\,\mathbf{m}\,, \\ \mathbf{m}'''' = (\vartheta'''' - 6\,\vartheta'^2\vartheta'')\mathbf{Am} - (4\vartheta'\vartheta''' - \vartheta'^4 + 3\,\vartheta''^2)\mathbf{m}\,. \end{aligned} \tag{3.66}$$

With the help of these formulae, equation (3.62) can be written in the following form, when restricted to processes of the type $\mathbf{m}(\vartheta(x - vt))$:

$$-\,h\sin\vartheta\,\vartheta' + \mu v\,\vartheta'^2 + \tau v\,(\vartheta''^2 + \vartheta'^4) = 0\,. \tag{3.67}$$

It is not difficult to check that this equation admits solutions of the form (3.37). In fact, the admissible value of the constant $C := c^2$ is the only real and positive solution $C_0 = C_0(\mu, \tau, v, h)$ of the following algebraic system:

$$\frac{\tau}{\mu}C^3 + C - \frac{h}{\mu v} = 0. \tag{3.68}$$

(note that (3.53) is recovered from (3.68) for τ equal to zero).

It remains for us to check whether the solution we found for (3.67) also solves the generalized Gilbert equation (3.65). It is easy to predict a negative outcome. In fact, with the generalized energy density (3.60), the Landau-Lifshitz functional (3.58) becomes

$$\int_{-\infty}^{+\infty} (\frac{1}{2}\alpha\vartheta'^2 - \frac{1}{2}\beta\cos^2\vartheta + \frac{1}{2}\lambda\vartheta''^2)\,dx = \min\,, \tag{3.69}$$

and the associated Euler-Lagrange equation,

$$\alpha\,\vartheta'' - \beta\sin\vartheta\cos\vartheta - \lambda\,\vartheta'''' = 0\,, \tag{3.70}$$

has no solution of type (3.39). However, to perform a thorough, conclusive check, we observe that, when $\mathbf{m} = \mathbf{m}(\vartheta(x - vt))$, the generalized Gilbert equation reads

$$\begin{aligned} -v\gamma^{-1}\mathbf{m}' = \mathbf{m} \times (\alpha\,\mathbf{m}'' - \mathbf{Tm} - \lambda\,\mathbf{m}'''' \\ +\,h\,\mathbf{e} + \mu v\,\mathbf{m}' - \tau v\,\mathbf{m}''')\,, \end{aligned} \tag{3.71}$$

and is equivalent to a system consisting of the first equation of (3.50) (because both higher-order terms have a null orthogonal projection in the direction of **a**) and the following modification of the second

$$\begin{aligned} 0 = \alpha\,\vartheta'' - (\beta + \cos^2\varphi_o)\sin\vartheta\cos\vartheta + \mu v\,\vartheta' - h\sin\vartheta \\ -\lambda(\vartheta'''' - 6\,\vartheta'^2\vartheta'') - \tau v(\vartheta''' - \vartheta'^3)\,. \end{aligned} \tag{3.72}$$

For these equations to be compatible, once again there must be such a constant C that

$$\vartheta' = C \sin\vartheta\,, \tag{3.73}$$

with

$$v\,C = \gamma \sin\varphi_o \cos\varphi_o\,, \tag{3.74}$$

and with the following algebraic condition satisfied whatever the angle ϑ in $(0, \pi)$:

$$\begin{aligned} 0 = (\alpha\, C^2 - (\beta + \cos^2\varphi_o))\cos\vartheta + \mu v\, C - h_{ext} \\ -\lambda\, C^4(12\cos^2\vartheta - 11)\cos\vartheta - \tau v\, C^3(3\cos^2\vartheta - 2), \end{aligned} \tag{3.75}$$

However, this last requirement is impossible to satisfy exactly, unless of course both λ and τ are equal to zero.[11]

3.1.6 Dry-friction dissipation

We now propose a generalization of Walker's solution to the case when a *dry-friction* term is included in the dissipation vector. Precisely, we take the dissipation potential to be

$$\chi = \frac{1}{2}\,\mu\,|\dot{\mathbf{m}}|^2 + \eta\,|\dot{\mathbf{m}}|\,, \tag{3.78}$$

with μ and η positive constants, so that the dissipation vector $\mathbf{d}$ becomes

$$\mathbf{d} = -\,\mu\,\dot{\mathbf{m}} + \eta\,\mathbf{f}(\dot{\mathbf{m}}) \tag{3.79}$$

where

$$\begin{aligned} -\,\mathbf{f}(\dot{\mathbf{m}}) &= |\dot{\mathbf{m}}|^{-1}\dot{\mathbf{m}} \quad \text{for } \dot{\mathbf{m}} \neq \mathbf{0}\,, \\ -\,\mathbf{f}(\mathbf{0}) &\in \{\mathbf{v}\,|\,|\mathbf{v}| \leq 1\}\,, \end{aligned} \tag{3.80}$$

[11] Relation (3.73) has the following differential consequences:

$$\begin{aligned} \vartheta'' &= C^2 \sin\vartheta\cos\vartheta, \\ \vartheta''' &= C^3(\cos^2\vartheta - \sin^2\vartheta)\sin\vartheta, \\ \vartheta'''' &= C^4(\cos^2\vartheta - 5\,\sin^2\vartheta)\sin\vartheta\cos\vartheta, \end{aligned} \tag{3.76}$$

whence

$$\begin{aligned} \vartheta'''' - 6\vartheta'^2\vartheta'' &= C^4(\cos^2\vartheta - 11\sin^2\vartheta)\sin\vartheta\cos\vartheta \\ \vartheta''' - \vartheta'^3 &= C^3(\cos^2\vartheta - 2\sin^2\vartheta)\sin\vartheta\,. \end{aligned} \tag{3.77}$$

Condition (3.75) obtains when we substitute (3.77) into (3.72).

is the *dry-friction mapping.* Next, we replace the prescription (3.8) for the applied magnetic field by the following more general prescription:

$$\mathbf{h}^{\mathrm{e}} = h\,\mathbf{e} + h_a\mathbf{a} + h_{Ae}\mathbf{Ae}, \tag{3.81}$$

where all three components of $\mathbf{h}^{\mathrm{e}}$, not only h, are control parameters we can assign the constant values we wish. With (3.79)-(3.81), the Walker condition (3.27) takes the general form

$$\begin{array}{c} -h_a \sin\vartheta\,\dot\varphi + (h\sin\theta - h_{Ae}\cos\theta)\dot\vartheta \\ +\mu(\sin^2\vartheta\,\dot\varphi^2 + \dot\vartheta^2) + \eta\,|(\sin^2\vartheta\,\dot\varphi^2 + \dot\vartheta^2)^{1/2}| = 0\,, \end{array} \tag{3.82}$$

and the simpler form

$$-\,h\sin\theta + h_{Ae}\cos\theta + \mu v\,\vartheta' + \eta\,\mathrm{sign}\,(v\vartheta') = 0 \tag{3.83}$$

for processes of the type $\mathbf{m}(\vartheta(x - vt))$. Likewise, with (3.81) we can write the Gilbert system as

$$\begin{array}{rcl} -\gamma^{-1}\,\dot\vartheta &=& (\mathbf{h}^{\mathrm{i}} + \mathbf{d})\cdot\mathbf{a} + h_a\,, \\ \gamma^{-1}\sin\vartheta\,\dot\varphi &=& (\mathbf{h}^{\mathrm{i}} + \mathbf{d})\cdot\mathbf{am} - h\sin\vartheta + h_{Ae}\cos\vartheta\,; \end{array} \tag{3.84}$$

with (3.79) and (3.80), and for processes of the type $\mathbf{m}(\vartheta(x-vt))$, this system becomes

$$\begin{array}{rcl} v\,\vartheta' &=& \gamma\,\sin\varphi_o\cos\varphi_o\sin\vartheta - \gamma h_a\,, \\ 0 &=& \alpha\,\vartheta'' - (\beta + \cos^2\varphi_o)\sin\vartheta\cos\vartheta + \mu v\,\vartheta' - h\sin\vartheta \\ && +\,h_{Ae}\cos\vartheta + \eta\,\mathrm{sign}\,(v\vartheta')\,. \end{array} \tag{3.85}$$

As a glance to (3.83) and the first of (3.85) makes evident, no solution of the former equation can also solve the latter, unless perhaps $h_{Ae} = 0$. Now, mutual consistency of the equations (3.85) is guaranteed provided that the constants h, v and φ_o satisfy the conditions (3.52)-(3.53) and that, in addition, the remaining components of the control field be such that

$$h_a = \frac{\eta}{\mu\gamma}\mathrm{sign}\,(v\vartheta')\,,\quad h_{Ae} = \frac{\alpha\eta}{(\mu v)^2}\,h\,\mathrm{sign}\,(v\vartheta')\,. \tag{3.86}$$

Thus, for h_{Ae} to be null, h should be null as well, a circumstance when, as is easily seen, the system (3.85) has no solution for $\eta \neq 0$. We must then conclude that, in the presence of dry friction, no Walker process solves the Gilbert equation. This notwithstanding, an explicit solution to the Gilbert equation can be found. Assuming that all consistency conditions hold, we can write $(3.85)_1$ in the form

$$\vartheta'(\xi) = \frac{1}{\Delta}(\sin\vartheta(\xi) + r)\,,\quad r = -\frac{\eta}{|h|}\,\mathrm{sign}\vartheta'\,, \tag{3.87}$$

with Δ given by (3.57). An easy continuity argument shows that the sign of ϑ' must be constant for class-C^1 solutions of (3.87).[12] Hence, we treat r in (3.87) as a constant parameter. Granted this, solutions of (3.87) exist only for $|r| < 1$, and have the form

$$\vartheta(\xi) = -\mathrm{sign}\,(r) \arccos \frac{1 - f^2(\xi)}{1 + f^2(\xi)}, \tag{3.88}$$

(cf. (3.39)), where

$$f(\xi) = \frac{f_2\, F \exp{(\xi/\Delta_r)} - f_1}{F \exp{(x/\Delta_r)} - 1}\,, \tag{3.89}$$

$$f_1 = \frac{-1 + \sqrt{1 - r^2}}{r}\,, \quad f_2 = \frac{-1 - \sqrt{1 - r^2}}{r}\,, \tag{3.90}$$

$$F = \frac{1 - |r| - \sqrt{1 - r^2}}{1 - |r| + \sqrt{1 - r^2}}\,; \tag{3.91}$$

moreover,

$$\Delta_r = \frac{1}{\sqrt{1 - r^2}}\, \Delta \tag{3.92}$$

is the thickness of the transition layer when dry-friction is accounted for. Note that, for r small, $f_1 \approx -r/2$, $f_2 \approx -2/r$ and $F \approx r/2$; therefore, when the dry-friction coefficient η is small, $f(\xi) \approx -\exp(\xi/\Delta)$ and Walker's solution is recovered. Note also that the wall thickness becomes larger when the dry-friction coefficient increases (Figure 3.4). Finally, as to the limit values of the magnetization, one finds that

$$\lim_{\xi \to \pm\infty} \mathbf{m}(\xi) = \pm\sqrt{1 - r^2}\, \mathbf{e} + r\, \mathbf{e} \times \mathbf{a}\,. \tag{3.93}$$

3.2 Curved Walls

3.2.1 Dimensionless equations

We begin by introducing the dimensionless variables

$$x = \frac{\bar{x}}{L}, \quad t = \frac{\bar{t}}{T}, \tag{3.94}$$

where $\bar{x}$ and $\bar{t}$ are the original space and time variables, and L and T are suitable length and time scales to be selected at a later stage. In dimensionless

[12] Interestingly, it also follows from (3.87) that, for class-C^0 solutions, the jump in ϑ' must equal $2\,|r|$.

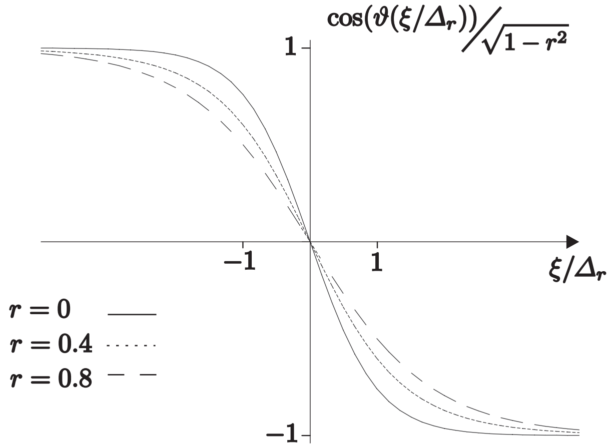

Figure 3.4: The effect of dry friction.

variables, the Gilbert equation (2.13) reads

$$\frac{1}{\beta}\frac{\mu}{T}\left((\mu\gamma)^{-1}\dot{\mathbf{m}} + \mathbf{m}\times\dot{\mathbf{m}}\right) = \mathbf{m}\times\left(\frac{1}{\beta}\frac{\alpha}{L^2}\Delta\mathbf{m} + (\mathbf{m}\cdot\mathbf{e})\mathbf{e} + \frac{1}{\beta}(\mathbf{h}^{\mathrm{s}} + \mathbf{h}^{\mathrm{e}})\right), \tag{3.95}$$

while the Maxwell's equations are unchanged.

Next, with a view toward a formal asymptotic analysis of (3.95), we introduce the smallness parameter

$$\varepsilon = \beta^{-1}. \tag{3.96}$$

From a physical point of view, we are considering a class of *hard ferromagnetic materials* with increasing anisotropy energy, with a view toward constructing a corresponding class of solutions. The aim is to fetch as much information as we can about the behavior of these solutions when the anisotropy energy goes to infinity.

The analysis of flat walls yields the following estimates for the dimensionless thickness Δ and velocity c of a flat domain wall:

$$\Delta \approx \sqrt{\frac{\alpha}{\beta}}\frac{1}{L}; \tag{3.97}$$

$$c \approx \frac{1}{\mu}\sqrt{\frac{\alpha}{\beta}}\ \frac{T}{L}\ |\mathbf{h}^{\mathrm{e}}|. \tag{3.98}$$

If we are to regard a domain wall as a sharp interface, it is natural to ask that Δ be a $O(\varepsilon)$ quantity. Further, it is reasonable to select a time scale T

such that c is $O(1)$. This motivates us to select

$$L = \varepsilon^{-1}\sqrt{\frac{\alpha}{\beta}}\,, \quad T = \varepsilon^{-1}\mu\,. \tag{3.99}$$

The Gilbert equation now reads

$$\varepsilon^2\left((\mu\gamma)^{-1}\dot{\mathbf{m}} + \mathbf{m}\times\dot{\mathbf{m}}\right) = \mathbf{m}\times\left(\varepsilon^2\Delta\mathbf{m} + (\mathbf{m}\cdot\mathbf{e})\mathbf{e} + \varepsilon(\mathbf{h}^{\mathrm{s}} + \mathbf{h}^{\mathrm{e}})\right)\,, \tag{3.100}$$

where, we recall, $\mathbf{h}^{\mathrm{s}}$ is the stray field, and $\mathbf{h}^{\mathrm{e}}$ is the external field.

The Maxwell equations retain their form:

$$\begin{aligned} \operatorname{div}\,(\mathbf{h}^{\mathrm{s}} + \chi_{\Omega_\varepsilon}\mathbf{m}) = 0\,, \\ \operatorname{curl}\,\mathbf{h}^{\mathrm{s}} = \mathbf{0}\,, \end{aligned} \tag{3.101}$$

where Ω_ε is the image of Ω under the scaling $(3.99)_1$.

3.2.2 Normal coordinates with respect to an evolving surface

Consider a smooth oriented surface $\mathcal{S}(t)$ evolving smoothly in an Euclidean space $\mathcal{E}$, and let $\mathbf{n} = \check{\mathbf{n}}(\mathtt{s}, t)$ be the positively-oriented unit vector orthogonal to $\mathcal{S}(t)$ at point $\mathtt{s} \in \mathcal{S}(t)$. We denote with $\mathcal{W}_{\varepsilon(t)}$ the *tubular neighborhood* of thickness $2\varepsilon(t)$ of $\mathcal{S}(t)$, that is, the set formed by points $\mathtt{p} \in \mathcal{E}$ such that

$$\mathtt{p} = \mathtt{s} + q\,\check{\mathbf{n}}(\mathtt{s}, t)\,, \quad \mathtt{s} \in \mathcal{S}(t)\,,\ q \in (-\varepsilon, +\varepsilon)\,. \tag{3.102}$$

Moreover, we let $\varepsilon(t)$ be the largest ε such that for each $\mathtt{p} \in \mathcal{W}_\varepsilon(t)$ there is exactly one point $\mathtt{s} \in \mathcal{S}(t)$ which solves the minimum problem

$$\min_{\mathtt{y}\in\mathcal{S}(t)} |\mathtt{p} - \mathtt{y}|\,. \tag{3.103}$$

Now, fix a time $t_1 \in \mathbb{R}$, and an interval $T = (t_0, t_2)$ containing t_1. If T is small enough, the set defined by

$$\mathcal{W}_\varepsilon = \bigcap_{t\in T}\mathcal{W}_{\varepsilon(t)}\,, \quad \varepsilon = \min_{t\in T}\varepsilon(t) \tag{3.104}$$

(see Fig. 3.5) is not empty, and for each $(\mathtt{p}, t) \in \mathcal{W}_\varepsilon \times T$, the minimum problem (3.103) has exactly one solution $\mathtt{s} \in \mathcal{S}(t)$. We denote by $\hat{\mathtt{s}}$ the mapping that associates such minimal distance point to $\mathtt{p}$:

$$\mathtt{s} = \hat{\mathtt{s}}(\mathtt{p}, t)\,; \tag{3.105}$$

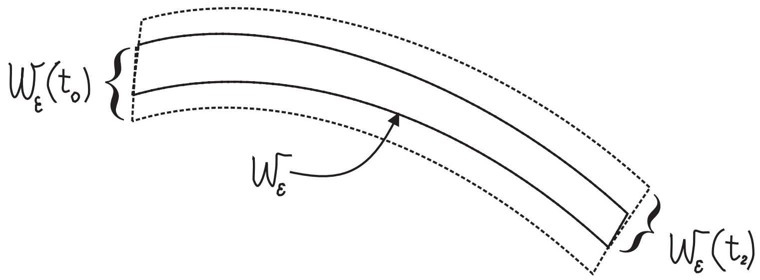

Figure 3.5: The set $\mathcal{W}_\varepsilon$.

and we denote by r the *scaled* and *signed* distance map

$$r = \hat{r}(\mathtt{p}, t) := \varepsilon^{-1}\Big((\mathtt{p} - \hat{\mathtt{s}}(\mathtt{p}, t)) \cdot \check{\mathbf{n}}(\hat{\mathtt{s}}(\mathtt{p}, t), t)\Big)\,. \tag{3.106}$$

We observe that: (i) for each $t \in T$, the image of $\mathcal{W}_\varepsilon$ under the mapping $\hat{\mathtt{s}}(\cdot, t)$ is the surface $\mathcal{S}(t)$; (ii) εr is the signed distance of the point $\mathtt{p}$ from $\mathcal{S}(t)$, and $r \in (-1, +1)$; (iii) the relation

$$\mathtt{p} = \hat{\mathtt{s}}(\mathtt{p}, t) + \varepsilon\, \hat{r}(\mathtt{p}, t)\hat{\mathbf{n}}(\mathtt{p}, t)\,, \quad \hat{\mathbf{n}}(\mathtt{p}, t) := \check{\mathbf{n}}(\hat{\mathtt{s}}(\mathtt{p}, t), t)\,, \tag{3.107}$$

is identically verified in $\mathcal{W}_\varepsilon \times T$.

Differentiating the identity (3.107) with respect to time, we obtain

$$\mathbf{0} = \dot{\mathtt{s}} + \varepsilon(\dot{r}\mathbf{n} + r\dot{\mathbf{n}})\,, \tag{3.108}$$

whence the following expression for $V := \dot{\mathtt{s}} \cdot \mathbf{n}$, the *normal velocity* of the surface $\mathcal{S}(t)$:

$$V = -\varepsilon\dot{r}\,; \tag{3.109}$$

the field $V = \hat{V}(\mathtt{p}, t)$ delivers the normal velocity of the point $\mathtt{s} = \hat{\mathtt{s}}(\mathtt{p}, t)$ of $\mathcal{S}(t)$. Furthermore, taking the gradient of (3.107) we obtain

$$\mathbf{1} = \nabla\mathbf{s} + \varepsilon(r\nabla\mathbf{n} + \mathbf{n} \otimes \nabla r)\,. \tag{3.110}$$

Since the normal field is unitary and since $(\nabla\mathbf{s}^{\mathrm{T}})\mathbf{n}$, it follows from this relation that

$$\nabla r = \varepsilon^{-1}\mathbf{n}\,; \tag{3.111}$$

consequently, (3.110) yields

$$\nabla\mathbf{s} = \mathbf{P} - \varepsilon r\nabla\mathbf{n}\,, \quad \mathbf{P} := \mathbf{1} - \mathbf{n} \otimes \mathbf{n}\,, \tag{3.112}$$

where $\mathbf{P}$ is orthogonal projector on the tangent plane to $\mathcal{S}(t)$.

3.2.3 Matched asymptotic expansions

We assume that the domain wall has the form of the regular surface $\mathcal{S}(t)$ devised in the previous section.

Let $\mathbf{m}_\varepsilon$ and $\mathbf{h}^{\mathrm{s}}_\varepsilon$ be solutions of the scaled Gilbert equation (3.100) and the Maxwell equations (3.101) for ε in an open interval $(0, \overline{\varepsilon})$, and assume that there exist two regular expansions in powers of ε:
the *outer expansion*

$$\begin{aligned} \mathbf{m}_\varepsilon(\mathtt{p}, t) &= \hat{\mathbf{m}}_0(\mathtt{p}, t) + \varepsilon \hat{\mathbf{m}}_1(\mathtt{p}, t) + o(\varepsilon)\,, \\ \mathbf{h}^{\mathrm{s}}_\varepsilon(\mathtt{p}, t) &= \hat{\mathbf{h}}^{\mathrm{s}}{}_0(\mathtt{p}, t) + \varepsilon \hat{\mathbf{h}}^{\mathrm{s}}{}_1(\mathtt{p}, t) + o(\varepsilon)\,; \end{aligned} \tag{3.113}$$

and the *inner expansion*

$$\begin{aligned} \mathbf{m}_\varepsilon(\mathtt{p}, t) &= \check{\mathbf{m}}_0 \hat{r}(\mathtt{p}, t), \hat{\mathtt{s}}(\mathtt{p}, t), t) + \varepsilon \check{\mathbf{m}}_1(\hat{r}(\mathtt{p}, t), \hat{\mathtt{s}}(\mathtt{p}, t), t) + o(\varepsilon)\,, \\ \mathbf{h}^{\mathrm{s}}_\varepsilon(\mathtt{p}, t) &= \check{\mathbf{h}}^{\mathrm{s}}{}_0(\hat{r}(\mathtt{p}, t), \hat{\mathtt{s}}(\mathtt{p}, t), t) + \varepsilon \check{\mathbf{h}}^{\mathrm{s}}{}_1(\hat{r}(\mathtt{p}, t), \hat{\mathtt{s}}(\mathtt{p}, t), t) + o(\varepsilon)\,. \end{aligned} \tag{3.114}$$

At time t, the fields which compose the outer expansion (3.113) are assumed to be smooth in $\mathcal{E} \backslash \mathcal{S}(t)$, and are presumed to represent the solution $\mathbf{m}_\varepsilon$ in $\mathcal{E} \backslash \mathcal{W}_\varepsilon(t)$, that is, away from the domain wall; the inner expansion (3.114) is supposed to hold in a tubular neighborhood $\mathcal{W}_{h(\varepsilon)}(t)$ of $\mathcal{S}(t)$, with $h(\varepsilon)$ such that

$$\lim_{\varepsilon \to 0} h(\varepsilon) = 0\,, \quad \lim_{\varepsilon \to 0} h(\varepsilon)/\varepsilon = \infty\,. \tag{3.115}$$

Since the domains of validity of the two expansions overlap, we stipulate that the *matching conditions*

$$\lim_{r \to \pm\infty} \check{\mathbf{m}}_0(r, \mathtt{s}, t) = \hat{\mathbf{m}}_0(\mathtt{s}\pm, t) \tag{3.116}$$

hold for all $\mathtt{s} \in \mathcal{S}(t)$ at time t (a justification is given below in small writings). Here $\hat{\mathbf{m}}_0(\mathtt{s}+, t)$ and $\hat{\mathbf{m}}_0(\mathtt{s}-, t)$ denote the limits of $\hat{\mathbf{m}}_0(\mathtt{p}, t)$ obtained by letting $\mathtt{p}$ approach $\mathtt{s} \in \mathcal{S}(t)$, respectively, from the positive and negative side of the oriented surface $\mathcal{S}(t)$. Analogous relations hold for $\mathbf{h}^{\mathrm{s}}$.

Both the inner and the outer expansion are valid for $1 < r < h(\varepsilon)/\varepsilon$, therefore we can write (omitting time dependence for brevity)

$$\sum_{n=0}^{N} \varepsilon^n \left(\check{\mathbf{m}}_n(r, \mathtt{s}) - \hat{\mathbf{m}}_n(\mathtt{s} + \varepsilon r \mathbf{n}) \right) + o(\varepsilon^N) = 0\,, \quad \forall\, \varepsilon, r \text{ s.t. } 1 < r < \frac{h(\varepsilon)}{\varepsilon}\,.$$

By expanding the right-hand side in powers of ε we obtain

$$\sum_{n=0}^{N} \varepsilon^n \left(\hat{\mathbf{m}}_n(r, \mathtt{s}) - P_n(r, \mathtt{s}) \right) + o(\varepsilon^N) = 0\,,$$

where

$$P_n(r, \mathtt{s}) = \sum_{k=0}^{n} \frac{r^{n-k}}{(n-k)!} \partial_{\mathbf{n}}^{(n-k)} \hat{\mathbf{m}}_k(\mathtt{s})$$

is a polynomial in the variable r, of order at most n. Therefore, we have

$$\check{\mathbf{m}}_n(r, s) \to P_n(r, \mathtt{s}) \quad \text{for} \quad r \to \infty\,.$$

The matching condition (3.116) obtains for $n = 0$.

3.2.4 Magnetic domains

For the first expansion to be compatible with the constraint $|\mathbf{m}| = 1$ we stipulate that:

$$|\hat{\mathbf{m}}_0| = 1\,, \quad \hat{\mathbf{m}}_0 \cdot \hat{\mathbf{m}}_1 = 0. \tag{3.117}$$

Substituting the expansion (3.113) in the Maxwell equations (3.101), we have

$$\begin{aligned} \operatorname{div} \hat{\mathbf{h}^{\mathrm{s}}}_i &= -\operatorname{div} \hat{\mathbf{m}}_i\,, \\ \operatorname{curl} \hat{\mathbf{h}^{\mathrm{s}}}_i &= \mathbf{0}\,, \end{aligned} \tag{3.118}$$

for all i. Substituting (3.113) in the the scaled form (3.100) of the Gilbert equation, and retaining the $O(1)$ and $O(\varepsilon)$ terms, we obtain, respectively,

$$\hat{\mathbf{m}}_0 \times (\mathbf{e} \cdot \hat{\mathbf{m}}_0)\mathbf{e} = \mathbf{0} \tag{3.119}$$

and

$$\hat{\mathbf{m}}_1 \times (\mathbf{e} \cdot \hat{\mathbf{m}}_0)\mathbf{e} - \hat{\mathbf{m}}_0 \times (\mathbf{e} \otimes \mathbf{e})\hat{\mathbf{m}}_1 - \hat{\mathbf{m}}_0 \times (\hat{\mathbf{h}^{\mathrm{s}}}_0 + \mathbf{h}^{\mathrm{e}}) = \mathbf{0}. \tag{3.120}$$

Equation (3.119) implies that, in the limit $\varepsilon \to 0$, $\hat{\mathbf{m}}_0$ must be either parallel or orthogonal to $\mathbf{e}$.

3.2.5 Estimates

We now compute some estimates of the derivatives of the fields $\mathbf{m}$ and $\mathbf{h}^{\mathrm{s}}$ in terms of the variables r, $\mathtt{s}$ and t. As pointed out in [21], some care is needed because $(r, \mathtt{s})$ is a time-dependent coordinate system.

To fix ideas, let $\phi = \hat{\phi}(\mathtt{p}, t)$ be a scalar field, and let

$$\check{\phi}_t : \mathbb{R} \times \mathcal{S}(t) \to \mathbb{R}\,, \quad t \in \mathbb{R}\,, \tag{3.121}$$

be a family of mappings such that

$$\hat{\phi}(\mathtt{p}, t) = \check{\phi}_t(r(\mathtt{p}, t), \mathtt{s}(\mathtt{p}, t))\,. \tag{3.122}$$

In order to compute $\phi^{\boldsymbol{\cdot}}$ in terms of the function $\check{\phi}_t$, we may be tempted to apply the chain rule to (3.122) to obtain:

$$\phi^{\boldsymbol{\cdot}} = \partial_{\mathtt{s}}\check{\phi}_t \cdot \dot{\mathtt{s}} + \partial_r\check{\phi}_t\,\dot{r} + \partial_t\check{\phi}_t\,. \tag{3.123}$$

However, the partial derivative of $\check{\phi}_t$ with respect to t holding $\mathtt{s}$ fixed is not defined because $\mathtt{s}$ belongs to the time-dependent surface $\mathcal{S}(t)$. We overcome this difficulty by introducing the smooth mapping $\check{\phi} : \mathbb{R}\times\mathcal{E}\times\mathbb{R} \to \mathbb{R}$ defined by

$$\check{\phi}(r, \mathtt{p}, t) := \check{\phi}_t(r, \hat{\mathtt{s}}(\mathtt{p}, t), t)\,. \tag{3.124}$$

Note that the requirement that $\mathcal{S}(t)$ evolves smoothly in time is crucial to ensure that the mapping $\check{\phi}$ be smooth with respect to all its arguments.

Now, the partial derivatives of $\check{\phi}$ are well defined, and we can apply the chain rule to obtain

$$\nabla\hat{\phi}(\mathtt{p}, t) = \varepsilon^{-1}\partial_r\check{\phi}(r, \mathtt{s}, t)\nabla\hat{r}(\mathtt{p}, t) + \nabla\hat{\mathtt{s}}(\mathtt{p}, t)^{\mathrm{T}}\nabla_{\mathtt{s}}\check{\phi}(r, \mathtt{s}, t)\,, \tag{3.125}$$

$$\hat{\phi}^{\boldsymbol{\cdot}}(\mathtt{p}, t) = \partial_t\check{\phi}(r, \mathtt{s}, t) + \partial_r\check{\phi}(r, \mathtt{s}, t)\hat{r}^{\boldsymbol{\cdot}}(\mathtt{p}, t) + \nabla_{\mathtt{s}}\check{\phi}(r, \mathtt{s}, t)\cdot\hat{\mathtt{s}}^{\boldsymbol{\cdot}}(\mathtt{p}, t)\,, \tag{3.126}$$

where we have set

$$\nabla_{\mathtt{s}}\check{\phi}(r, \mathtt{s}, t) := \nabla_{\mathtt{y}}\check{\phi}(r, \mathtt{y}, t)|_{\mathtt{y}=\mathtt{s}}\,. \tag{3.127}$$

Note that

$$\nabla_{\mathtt{s}}\check{\phi}(r, \mathtt{s}, t)\cdot\hat{\mathbf{n}}(\mathtt{s}, t) = 0. \tag{3.128}$$

To give (3.125) and (3.126) a more usable format, some further computations are needed. To make our notation lighter, we will use ϕ as a shorthand both for $\hat{\phi}$ and for $\check{\phi}$, being understood that we are referring to the former when we write $\nabla\phi$ and $\phi^{\boldsymbol{\cdot}}$, and to the latter when we write $\partial_r\phi$, $\nabla_{\mathtt{s}}\phi$ and $\partial_t\phi$. In addition, we will omit the dependence on the variables $(\mathtt{p}, t)$ or $(r, \mathtt{s}, t)$ when there is little risk of confusion.

By virtue of (3.109)-(3.112) and (3.128), equations (3.125) and (3.126) become, respectively,

$$\nabla\phi = \varepsilon^{-1}\partial_r\phi\,\mathbf{n} + \nabla_{\mathtt{s}}\phi - \varepsilon r\nabla\mathbf{n}^{\mathrm{T}}\nabla_{\mathtt{s}}\phi \tag{3.129}$$

and

$$\phi^{\boldsymbol{\cdot}} = -\varepsilon^{-1}V\,\partial_r\phi + \partial_t\phi - \varepsilon r\nabla_{\mathtt{s}}\phi\cdot\dot{\mathbf{n}}\,. \tag{3.130}$$

By applying the above considerations to a vector field $\mathbf{v}$, we find, in a similar fashion:

$$\nabla\mathbf{v} = \varepsilon^{-1}\partial_r\mathbf{v}\otimes\mathbf{n} + \nabla_{\mathtt{s}}\mathbf{v} - \varepsilon r\nabla_{\mathtt{s}}\mathbf{v}\nabla\mathbf{n}\,; \tag{3.131}$$

$$\dot{\mathbf{v}} = -\varepsilon^{-1}V\,\partial_r\mathbf{v} + \partial_t\mathbf{v} - \varepsilon r\nabla_{\mathtt{s}}\mathbf{v}\cdot\dot{\mathbf{n}}\,. \tag{3.132}$$

Finally, by making repeated use of the identitities (3.129) and (3.131), we can compute $\Delta\mathbf{v}$:

$$\Delta\mathbf{v} = \varepsilon^{-2}\partial_{rr}\mathbf{v} + \varepsilon^{-1}(\mathrm{div}\,\mathbf{n})\partial_r\mathbf{v} + \mathrm{div}\,\nabla_{\mathsf{s}}\mathbf{v} - \varepsilon r\mathrm{div}\,(\nabla_{\mathsf{s}}\mathbf{v}\nabla\mathbf{n}). \tag{3.133}$$

By applying (3.131) to $\mathbf{n}$ and taking into account that $\partial_r\mathbf{n} = \mathbf{0}$, we have $\nabla\mathbf{n} = \nabla_{\mathsf{s}}\mathbf{n}(\mathbf{P} - \varepsilon r\nabla\mathbf{n})$. Moreover, since the surface gradient of $\mathbf{n}$ does not depend on r, [13] we can write

$$\nabla_{\mathsf{s}}\mathbf{n} = -\mathbf{L}\,, \tag{3.134}$$

where

$$\mathbf{L}(\mathbf{s},t) \equiv -\nabla_{\mathsf{s}}\mathbf{n}(0,\mathbf{s},t)$$

is the *Weingarten Tensor*, and hence

$$\mathrm{div}\,\mathbf{n} = -\mathrm{tr}\,(\mathbf{L} + \varepsilon r\nabla\mathbf{n}) = -K - \varepsilon r\,\nabla_{\mathsf{s}}\mathbf{n}\cdot\nabla\mathbf{n}\,, \tag{3.135}$$

where K is twice the the *mean curvature* of $\mathcal{S}(t)$ at the point $\mathbf{s} = \hat{\mathbf{s}}(\mathrm{p}, t)$, and we conclude that:

$$\Delta\mathbf{v} = \varepsilon^{-2}\partial_{rr}\mathbf{v} - \varepsilon^{-1}K\partial_r\mathbf{v} + \mathrm{div}\,\nabla_{\mathsf{s}}\mathbf{v} - r(\nabla_{\mathsf{s}}\mathbf{n}\cdot\nabla\mathbf{n})\partial_r\mathbf{v} - \varepsilon r\mathrm{div}\,(\nabla_{\mathsf{s}}\mathbf{v}\nabla\mathbf{n}). \tag{3.136}$$

By applying (3.131), (3.132) and (3.136) to the inner expansions (3.114), we obtain the following estimates:

$$\begin{aligned} \dot{\mathbf{m}} &= -\varepsilon^{-1}V\partial_r\mathbf{m}_0 + \partial_t\mathbf{m}_0 - V\partial_r\mathbf{m}_1 + O(\varepsilon); \\ \mathrm{div}\,\mathbf{m} = \mathrm{tr}\,\nabla\mathbf{m} &= \varepsilon^{-1}\partial_r\mathbf{m}_0\cdot\mathbf{n} + \mathrm{tr}\,\nabla_{\mathsf{s}}\mathbf{m}_1 + \partial_r\mathbf{m}_1\cdot\mathbf{n} + O(\varepsilon); \\ \Delta\mathbf{m} &= \varepsilon^{-2}\partial_{rr}\mathbf{m}_0 - \varepsilon^{-1}(K\partial_r\mathbf{m}_0 - \partial_{rr}\mathbf{m}_1) + O(1). \end{aligned} \tag{3.137}$$

Similar estimates hold for the stray field $\mathbf{h}^{\mathrm{s}}$; in particular, we need the following:

$$\mathrm{curl}\,\mathbf{h}^{\mathrm{s}} = \varepsilon^{-1}\mathbf{n}\times\partial_r\mathbf{h}^{\mathrm{s}}_0 + (\mathrm{curl}_{\mathsf{s}}\mathbf{v})\mathbf{n} + \mathbf{n}\times\partial_r\mathbf{h}^{\mathrm{s}}_1 + O(\varepsilon)\,, \tag{3.138}$$

where $\mathrm{curl}_{\mathsf{s}}\mathbf{v} = \mathrm{tr}\,\nabla_{\mathsf{s}}(\mathbf{n}\times\mathbf{v})$.

[13] By definition

$$\nabla_{\mathsf{s}}\mathbf{n}(r,\mathbf{s},t) = \nabla_{\mathrm{y}}\mathbf{n}(r,\mathrm{y},t)|_{\mathrm{y}=\mathbf{s}},$$

and hence $\partial_r(\nabla_{\mathbf{s}}\mathbf{n}) = \partial_r(\nabla_{\mathrm{y}}\mathbf{n}) = \nabla_{\mathrm{y}}(\partial_r\mathbf{n}) = \mathbf{0}$.

3.2.6 Motion by curvature of domain walls

Substituting the estimates (3.137) in the scaled Gilbert equation (3.100), and retaining the $O(1)$ term, we find that $\check{\mathbf{m}}_0(\cdot, \mathbf{s}, t)$ must solve the following ODE

$$0 = \check{\mathbf{m}}_0(r, \mathbf{s}, t) \times \left(\partial_{rr}\check{\mathbf{m}}_0(r, \mathbf{s}, t) + (\check{\mathbf{m}}_0(r, \mathbf{s}, t) \cdot \mathbf{e})\mathbf{e}\right), \tag{3.139}$$

with boundary data at $r = \pm\infty$ determined by the outer expansion through the matching conditions (3.116).

Remark 4. For a 180° wall, with $\hat{\mathbf{m}}_0(\mathbf{s}-, t) = \mathbf{e}$ and $\hat{\mathbf{m}}_0(\mathbf{s}+, t) = -\mathbf{e}$ (Fig. 3.6a), the ODE (3.139) is solved if

$$\check{\mathbf{m}}_0(r, \mathbf{s}, t) \cdot \mathbf{e} = -\tanh(r + r_0), \quad r_0 = \text{a constant}, \tag{3.140}$$

and $\check{\mathbf{m}}_0(r, \mathbf{s}, t) \times \mathbf{e}$ has constant direction. In this case, a Walker-like magnetization profile is recovered (Fig. 3.6b). ♢

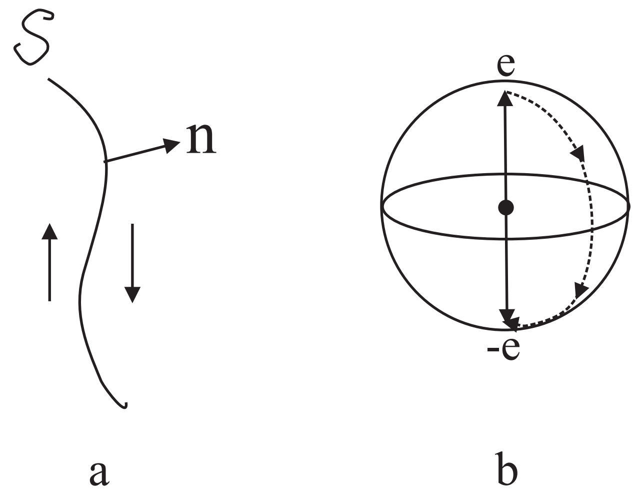

Figure 3.6: (a) A curved 180°-wall; (b) the magnetization goes from the $\mathbf{e}$ to $-\mathbf{e}$ along a meridian of the unit sphere.

Substituting the inner expansions into the Maxwell equations and retaining the $O(1)$ terms, we have:

$$\partial_r(\check{\mathbf{h}}^{\mathrm{s}}{}_0 + \check{\mathbf{m}}_0) \cdot \mathbf{n} = 0, \quad \partial_r\check{\mathbf{h}}^{\mathrm{s}}{}_0 \times \mathbf{n} = 0. \tag{3.141}$$

Using again (3.116) for $\hat{\mathbf{m}}$, and the analogous relation

$$\lim_{r\to\pm\infty} \check{\mathbf{h}}^{\mathrm{s}}{}_0(r, \mathbf{s}, t) = \hat{\mathbf{h}}^{\mathrm{s}}{}_0(\mathbf{s}\pm, t), \tag{3.142}$$

for $\hat{\mathbf{h}}^s$, equations (3.141) yields the *jump conditions* at the surface:

$$[\![\hat{\mathbf{h}}^s{}_0 + \hat{\mathbf{m}}_0]\!] \cdot \mathbf{n} = 0\,; \qquad [\![\hat{\mathbf{h}}^s{}_0]\!] \times \mathbf{n} = \mathbf{0}\,. \tag{3.143}$$

Collecting all the $O(\varepsilon)$-terms in the Gilbert equation, we obtain

$$\begin{aligned} 0 \;=\;& \check{\mathbf{m}}_0 \times (\partial_{rr}\check{\mathbf{m}}_1 - K\partial_r\check{\mathbf{m}}_0 + (\mathbf{e}\otimes\mathbf{e})\check{\mathbf{m}}_1 + \mathbf{h}^e + \check{\mathbf{h}}^s{}_0) \\ &+ \check{\mathbf{m}}_1 \times (\partial_{rr}\check{\mathbf{m}}_0 + (\check{\mathbf{m}}_0\cdot\mathbf{e})\mathbf{e}) \\ &+ V((\gamma\mu)^{-1}\partial\check{\mathbf{m}}_0 + \check{\mathbf{m}}_0 \times \partial_r\check{\mathbf{m}}_0)\,. \end{aligned} \tag{3.144}$$

Taking the cross product of both sides of (3.144) with $\check{\mathbf{m}}_0$ we obtain the ODE

$$\mathbf{A}_0[\check{\mathbf{m}}_1(\cdot, r, t)] = \mathbf{b}_0(\cdot, \mathbf{s}, t)\,, \tag{3.145}$$

where $\mathbf{A}_0$ is the differential operator defined by

$$\mathbf{A}_0[\mathbf{v}] = -\mathbf{P}_0(\partial_{rr}\mathbf{v} + (\mathbf{e}\otimes\mathbf{e})\mathbf{v}) + \left((\mathbf{m}\cdot\mathbf{e})^2 - |\partial_r\check{\mathbf{m}}_0|^2\right)\mathbf{v}\,, \tag{3.146}$$

with $\mathbf{P}_0 = \mathbf{I} - \check{\mathbf{m}}_0 \otimes \check{\mathbf{m}}_0$, and[14]

$$\mathbf{b}_0 = V\left(\partial_r\check{\mathbf{m}}_0 - (\mu\gamma)^{-1}\check{\mathbf{m}}_0 \times \partial_r\check{\mathbf{m}}_0\right) - K\partial_r\check{\mathbf{m}}_0 + \mathbf{P}_0(\mathbf{h}^e_0 + \check{\mathbf{h}}^s{}_0)\,. \tag{3.147}$$

Differentiating (3.139) with respect to r, and taking the cross-product with $\check{\mathbf{m}}_0$, one finds [15]

$$\mathbf{A}_0\partial_r\check{\mathbf{m}}_0 = \mathbf{0}; \tag{3.148}$$

furthermore, a standard integration-by-parts argument yields:

$$\int_{-\infty}^{+\infty} \mathbf{A}\,\check{\mathbf{m}}_1 \cdot \partial_r\check{\mathbf{m}}_0 \mathrm{d}r = -\int_{-\infty}^{+\infty} \mathbf{A}\,\partial_r\check{\mathbf{m}}_0 \cdot \check{\mathbf{m}}_1 \mathrm{d}r\,. \tag{3.149}$$

By taking the scalar product of both sides of (3.145) with $\partial_r\check{\mathbf{m}}_0$, integrating with respect to r, and using (3.148) and (3.149), we find

$$\int_{-\infty}^{+\infty} \mathbf{b}_0 \cdot \partial_r\check{\mathbf{m}}_0\,\mathrm{d}r = 0\,. \tag{3.150}$$

This is a necessary condition that $\mathbf{b}_0$ must fulfill in order for (3.145) to admit a solution, and it can be written as

$$(V - K)g - p^e - p^s = 0, \tag{3.151}$$

[14]We recall that V is the normal velocity of the surface and K is twice its mean curvature.

[15]This reflects the fact that equation (3.139) is translation invariant with respect to r.

where[16]

$$g(\mathbf{s},t) = \int_{-\infty}^{+\infty} |\partial_r \check{\mathbf{m}}_0(r,\mathbf{s},t)|^2 dr, \tag{3.152}$$

and

$$\begin{aligned} p^e(\mathbf{s},t) &= -\int_{-\infty}^{+\infty} \mathbf{h}^e(\mathbf{s},t)\cdot\partial_r\check{\mathbf{m}}_0(r,\mathbf{s},t)\,dr \\ p^s(\mathbf{s},t) &= -\int_{-\infty}^{+\infty} \check{\mathbf{h}}^s{}_0(r,\mathbf{s},t)\cdot\partial_r\check{\mathbf{m}}_0(r,\mathbf{s},t)dr. \end{aligned} \tag{3.153}$$

As proven below in small writings, equations (3.153) may be rewritten as

$$p^e = -\mathbf{h}^e\cdot[\![\hat{\mathbf{m}}_0]\!]\,,\quad p^s = -\langle\!\langle\hat{\mathbf{h}}^s{}_0\rangle\!\rangle\cdot[\![\hat{\mathbf{m}}_0]\!]\,, \tag{3.154}$$

where

$$\langle\!\langle\hat{\mathbf{h}}^s{}_0\rangle\!\rangle = \frac{1}{2}\Big(\hat{\mathbf{h}}^s{}_0(\mathbf{s}+) + \hat{\mathbf{h}}^s{}_0(\mathbf{s}-)\Big) \tag{3.155}$$

is the mean between the values assumed by $\hat{\mathbf{h}}^s{}_0$ at the two sides of the surface.

The first of (3.154) is a straightforward consequence of the matching conditions between $\check{\mathbf{m}}_0$ and $\hat{\mathbf{m}}_0$, repeated here for the reader's sake:

$$\lim_{r\to\pm\infty}\check{\mathbf{m}}_0(r,\mathbf{s},t) = \hat{\mathbf{m}}_0(\mathbf{s}\pm,t)\,. \tag{3.156}$$

To prove $(3.154)_2$, note that, as a consequence of (3.141), the vector

$$\check{\mathbf{h}}^s{}_0(r,\mathbf{s},t) + (\check{\mathbf{m}}_0\cdot\mathbf{n})(r,\mathbf{s},t)\,\mathbf{n}(\mathbf{s},t) \tag{3.157}$$

is constant with r. Then, use the matching conditions (3.156) and the analogous conditions for $\check{\mathbf{h}}^s{}_0$ and $\hat{\mathbf{h}}^s{}_0$:

$$\lim_{r\to\pm\infty}\check{\mathbf{h}}^s{}_0(r,\mathbf{s},t) = \hat{\mathbf{h}}^s{}_0(\mathbf{s}\pm,t) \tag{3.158}$$

to obtain

$$\check{\mathbf{h}}^s{}_0(r,\mathbf{s},t) = \langle\!\langle\hat{\mathbf{h}}^s{}_0\rangle\!\rangle(\mathbf{s},t) + \Big(\langle\!\langle\hat{\mathbf{m}}_0\cdot\mathbf{n}\rangle\!\rangle(\mathbf{s},t) - (\check{\mathbf{m}}_0\cdot\mathbf{n})(r,\mathbf{s},t)\Big)\,\mathbf{n}(\mathbf{s},t)\,. \tag{3.159}$$

Finally, substitute the above equation in $(3.153)_2$ and integrate to find

$$-p^s = \langle\!\langle\hat{\mathbf{h}}^s{}_0\rangle\!\rangle\cdot[\![\hat{\mathbf{m}}_0]\!] + \langle\!\langle\hat{\mathbf{m}}_0\cdot\mathbf{n}\rangle\!\rangle[\![\hat{\mathbf{m}}_0\cdot\mathbf{n}]\!] - \frac{1}{2}[\![(\hat{\mathbf{m}}_0\cdot\mathbf{n})^2]\!]\,, \tag{3.160}$$

then use the identity

$$[\![a^2]\!] = 2[\![a]\!]\langle\!\langle a\rangle\!\rangle \tag{3.161}$$

[16] For a 180°-wall (cf. Remark 4), we have $g = 1$.

to obtain the desired conclusion. ◇

Going back to the original space and time variables, the normal velocity v and the curvature k are given by

$$v = T^{-1}L\,V\,, \quad k = L^{-1}\,K\,, \tag{3.162}$$

and equation (3.151) reads

$$\rho\,v - \sigma\,k = p^e + p^s\,, \tag{3.163}$$

where,

$$\rho := g\mu\sqrt{\frac{\beta}{\alpha}} \tag{3.164}$$

is the *viscous–drag coefficient* and

$$\sigma := g\sqrt{\alpha\beta} \tag{3.165}$$

is the *surface tension* of the domain wall; p^e and p^s are the *driving forces* produced, respectively, by the external and the stray field.

Remark 5. If the effect of curvature and stray field is negligible ($k = 0$, $\mathbf{h}^{\mathrm{s}} = \mathbf{0}$), the normal velocity is proportional to $\mathbf{h}^{\mathrm{e}}$:

$$v = (g\mu)^{-1}\sqrt{\frac{\alpha}{\beta}}\,[\![\hat{\mathbf{m}}_0]\!]\cdot\mathbf{h}^{\mathrm{e}}\,. \tag{3.166}$$

Moreover, for 180°-domain walls (see Remark 4), we have $g = 2$ and $[\![\hat{\mathbf{m}}_0]\!] = 2\mathbf{e}$, and the above equation becomes:

$$v = \frac{h}{\mu}\sqrt{\frac{\alpha}{\beta}}\,, \quad h = \mathbf{e}\cdot\mathbf{h}^{\mathrm{e}}\,. \tag{3.167}$$

Note that the same result obtains by differentiating with respect to h the velocity of Walker's travelling solution as given by (3.55) and (3.56). ◇