Chapter 1

Micromagnetics

1.1 Ferromagnetic Materials

Within the framework of continuum mechanics, magnetizable matter is modelled as the composition of two interacting continua each having its own kinematics: the *lattice continuum* and the *spin continuum* [16, 33]. For a deformable and magnetizable material body with *reference shape* a region Ω of the Euclidean space, at a fixed time, the kinematics of the lattice continuum determines the placement of material points through the *deformation* $\mathsf{f} : \Omega \to \mathcal{E}$; the additional kinematics brought in by the spin continuum consists in the *magnetic moment per unit volume* $\mathbf{m}_{\mathsf{f}} : \Omega_{\mathsf{f}} \to \mathbb{R}^3$, a smooth vector field defined over the *current shape* $\Omega_{\mathsf{f}} = \mathsf{f}(\Omega)$ of the body.

We restrict attention to undeformable, homogeneous ferromagnetic bodies under isothermal conditions, below their *Curie temperature*. Such bodies exhibit *spontaneous magnetization*: denoting with ρ the mass density, the magnitude of the *magnetic moment per unit mass* $\boldsymbol{m}_{\mathsf{f}} = \rho^{-1}\mathbf{m}_{\mathsf{f}}$ is strictly positive, and depends on the temperature only.

When, as we here do, deformational effects are ignored, the current and the reference configuration coincide, and the only unknown is the orientation of $\mathbf{m}_{\mathsf{f}}$, identified with the vector field $\mathbf{m} : \Omega \to S^2$ defined by $\mathbf{m} = \mathrm{vers}(\mathbf{m}_{\mathsf{f}})$, which we will refer to as the *magnetization*.

1.2 Micromagnetics

Micromagnetics [9] is a theory which predicts the equilibrium configurations of the magnetization $\mathbf{m}$. It is based on a variational approach: a functional Ψ is chosen, which associates an energy to each configuration of the magnetization; the system is presumed to attain those states which are local minimizers of Ψ.

1.2.1 Euler-Lagrange equations

The Euler-Lagrange equation associated to the problem

$$\min_{|\mathbf{m}|=1} \Psi\{\mathbf{m}\}\,, \tag{1.1}$$

is

$$\delta_{\mathbf{m}}\Psi + \lambda\mathbf{m} = 0\,, \tag{1.2}$$

where $\delta_{\mathbf{m}}\Psi$ is the variational derivative of Ψ, and where λ is the Lagrange multiplier which enforces the constraint on $\mathbf{m}$. Equation (1.2) can be rewritten in the form

$$\mathbf{m} \times \mathbf{h} = 0\,, \quad \mathbf{h} = -\delta_{\mathbf{m}}\Psi\,, \tag{1.3}$$

where $\mathbf{h}$ is the *effective magnetic field*.

1.2.2 Internal and external energy

We write the energy as

$$\Psi = \Psi^{\mathrm{e}} + \Psi^{\mathrm{i}}\,, \tag{1.4}$$

where Ψ^{e} is the *external energy* and Ψ^{i} is the *internal energy*. The external energy is assumed to be the integral over Ω of an external-energy density:

$$\Psi^{\mathrm{e}}\{\mathbf{m}\} = -\int_\Omega \psi^{\mathrm{e}}(\mathbf{m})\,, \quad \psi^{\mathrm{e}} = -\,\mathbf{h}^{\mathrm{e}} \cdot \mathbf{m}\,, \tag{1.5}$$

where the $\mathbf{h}^{\mathrm{e}}$ is the *external magnetic field*. According to (1.4), the effective field splits as

$$\mathbf{h} = \mathbf{h}^{\mathrm{e}} + \mathbf{h}^{\mathrm{i}}\,, \quad \mathbf{h}^{\mathrm{i}} := -\delta_{\mathbf{m}}\Psi^{\mathrm{i}}\,, \tag{1.6}$$

with $\mathbf{h}^{\mathrm{i}}$ the *internal magnetic field*.

The external field accounts for the interaction between the body and its environment, while the internal field accounts for the interactions of a body part with other parts and with itself. Accordingly, the former is a quantity that, in principle, may be prescribed at will; the latter is determined by the magnetization field $\mathbf{m}$ in the body through constitutive prescriptions which model the response of a particular class of materials (in a variational setting, those prescriptions are specifications of the internal energy); their standard form is illustrated in the next section.

1.2.3 Standard constitutive assumptions

The standard choice for the internal energy Ψ^{i}, is

$$\Psi^{\mathrm{i}} = \Psi^{\mathrm{xc}} + \Psi^{\mathrm{a}} + \Psi^{\mathrm{s}}\,, \tag{1.7}$$

where the terms on the right-hand side are, in the order, the *exchange* energy, the *anisotropy* energy and the *stray-field* energy.

Exchange Energy
In ferromagnetic materials, exchange interactions penalize misalignment of neighboring spins. At the macroscopic level, this effect is accounted for by introducing an *exchange-energy density* $\psi^{\mathrm{xc}}(\nabla\mathbf{m})$, and writing:

$$\Psi^{\mathrm{xc}}\{\mathbf{m}\} = \int_\Omega \psi^{\mathrm{xc}}(\nabla\mathbf{m})\,. \tag{1.8}$$

A common assumption is that ψ^{xc} is isotropic and homogeneous of degree two:

$$\psi^{\mathrm{xc}}(\nabla\mathbf{m}) = \frac{1}{2}\,\alpha|\nabla\mathbf{m}|^2\,, \quad \alpha > 0\,; \tag{1.9}$$

α is called *exchange constant.*

Anisotropy energy
The energy of the system depends also on the orientation of the magnetization with respect to the body's lattice. Anisotropy effects are modeled by introducing a "coarse-grain" energy with density $\psi^{\mathrm{a}}(\mathbf{m})$, whose value measures the alignment of $\boldsymbol{m}$ with certain "easy" ($\equiv$ preferred) directions or planes. Accordingly, we write:

$$\Psi^{\mathrm{a}}\{\mathbf{m}\} = \int_\Omega \psi^{\mathrm{a}}(\mathbf{m})\,. \tag{1.10}$$

For uniaxial materials, a common choice for the anisotropy-energy density is

$$\psi^{\mathrm{a}}(\mathbf{m}) = -\frac{1}{2}\,\beta(\mathbf{m}\cdot\mathbf{e})^2\,; \tag{1.11}$$

the sign of the *anisotropy constant* β indicates whether the material has an *easy axis* of magnetization ($\beta > 0$) or an *easy plane* ($\beta < 0$); we restrict attention to the former case.

Stray-field energy
The *stray field* $\mathbf{h}^{\mathrm{s}}_\Omega\{\mathbf{m}\}$ generated by a magnetized body with magnetization

$\mathbf{m}$ occupying the region Ω is the unique square-integrable solution of the quasistatic Maxwell's equations

$$\begin{aligned} \operatorname{curl} \mathbf{h}^{\mathrm{s}} &= 0; \\ \operatorname{div} \mathbf{h}^{\mathrm{s}} &= -\operatorname{div}\left(\chi_\Omega \mathbf{m}\right), \end{aligned} \tag{1.12}$$

where χ_Ω is the characteristic function of Ω. Equation $(1.12)_1$ implies that there is scalar potential H such that

$$\mathbf{h}^{\mathrm{s}} = -\nabla H. \tag{1.13}$$

By taking the potential as the unknown, the remaining equation $(1.12)_2$ becomes the Poisson equation

$$\Delta H = \operatorname{div}\left(\chi_\Omega \mathbf{m}\right), \tag{1.14}$$

and a standard representation formula applies:

$$H(\mathtt{p}) = -\frac{1}{4\pi}\int_\Omega \frac{\operatorname{div}\mathbf{m}}{|\mathtt{p}-\mathtt{y}|}\,\mathrm{dV}_{\mathtt{y}} + \frac{1}{4\pi}\int_{\partial\Omega}\frac{\mathbf{m}\cdot\mathbf{n}}{|\mathtt{p}-\mathtt{y}|}\,\mathrm{dS}_{\mathtt{y}}. \tag{1.15}$$

The stray-field energy is the energy needed to assemble the magnetic dipoles so as to constitute the magnetic body under examination; it is given by

$$\Psi^{\mathrm{s}}\{\mathbf{m}\} = \frac{1}{2}\int_{\mathcal{E}} |\mathbf{h}^{\mathrm{s}}_\Omega\{\mathbf{m}\}|^2\,, \tag{1.16}$$

and, by virtue of a standard reciprocity theorem, it can also be written as the integral over Ω of a suitable energy density [10]:

$$\Psi^{\mathrm{s}} = \int_\Omega \psi^{\mathrm{s}}\,, \quad \psi^{\mathrm{s}} = -\frac{1}{2}\,\mathbf{m}\cdot\mathbf{h}^{\mathrm{s}}_\Omega\{\mathbf{m}\}\,. \tag{1.17}$$

From (1.15) we see that the stray-field energy vanishes if $\operatorname{div}\mathbf{m} = 0$ in the interior of the body and $\mathbf{m}\cdot\mathbf{n} = 0$ on the boundary. This means that, from a qualitative point of view, the stray-field energy is minimized whenever the magnetization vector follows closed paths inside the body; this statement is also known as *principle of pole avoidance* (see [4], §7.1).

Internal field and effective field

With the standard constitutive choices, from (1.7), (1.8), (1.10) and $(1.17)_1$, the internal energy can be represented as the integral over Ω of an *internal energy density*:

$$\Psi^{\mathrm{i}} = \int_\Omega \psi^{\mathrm{i}}\,, \quad \psi^{\mathrm{i}} := \psi^{\mathrm{xc}} + \psi^{\mathrm{a}} + \psi^{\mathrm{s}}\,; \tag{1.18}$$

with the use of $(1.6)_2$, (1.9), (1.11), $(1.17)_2$ we find, for the internal field:

$$\mathbf{h}^{\mathrm{i}} = \alpha\Delta\mathbf{m} + \beta(\mathbf{m}\cdot\mathbf{e})\mathbf{e} + \mathbf{h}^{\mathrm{s}}\,, \tag{1.19}$$

where $\mathbf{h}^{\mathrm{s}}$ is a shorthand for $\mathbf{h}^{\mathrm{s}}_\Omega\{\mathbf{m}\}$. The explicit form of the energy is

$$\Psi\{\mathbf{m}\} = \int_\Omega \left(\frac{1}{2}\alpha|\nabla\mathbf{m}|^2 - \frac{1}{2}\beta\,(\mathbf{e}\cdot\mathbf{m})^2 - (\frac{1}{2}\mathbf{h}^{\mathrm{s}} + \mathbf{h}^{\mathrm{e}})\cdot\mathbf{m}\right)\,, \tag{1.20}$$

and, from $(1.6)_1$ and (1.19), we have that

$$\mathbf{h} = \alpha\Delta\mathbf{m} + \beta(\mathbf{m}\cdot\mathbf{e})\mathbf{e} + \mathbf{h}^{\mathrm{s}} + \mathbf{h}^{\mathrm{e}}\,. \tag{1.21}$$

Finally, the Euler-Lagrange equation (1.3) takes the form:

$$\mathbf{m}\times(\alpha\Delta\mathbf{m} + \beta(\mathbf{m}\cdot\mathbf{e})\mathbf{e} + \mathbf{h}^{\mathrm{s}} + \mathbf{h}^{\mathrm{e}}) = 0\,. \tag{1.22}$$

1.3 Magnetic Domains and Domain Walls

Measurements performed at the surface of a ferromagnetic specimen may show a patchwise-constant magnetization, indicating that the interior of the body is partitioned into regions in each of which the orientation of $\mathbf{m}$ is spatially constant; these regions are called *magnetic domains.* The rotation of the magnetization from one domain to the other does not take place with an abrupt jump, but with a continuous transition in a narrow layer, called *domain wall*; typically, the thickness of domain walls is 10^4 times smaller that the size of magnetic domains and, at the length scale of observation of magnetic domains, a domain wall appears as a sharp interface.

Magnetic domains and domain walls are not the rule, and the magnetization observed in some specimens is not patchwise-constant. One may ask why magnetic domains develop in some bodies and not in others. The answer, in principle, might be drawn from micromagnetics through energy minimization. However, this turns out to be a difficult task to accomplish, because the functional Ψ is both nonlinear and nonlocal.

A detailed discussion of the solutions of the variational problem of micromagnetics is beyond the scope of this thesis. The rest of this section is a brief account of the features of a ferromagnetic body that are believed to be the most relevant to the formation of domains and walls.

1.3.1 Small and large bodies

Following a scaling argument taken from [14], let $|\Omega| = \lambda^3$ be the volume of Ω and $\Sigma = g(\Omega)$ be the image of Ω under the scaling

$$g : \mathtt{p} \mapsto \lambda^{-1}(\mathtt{p} - \mathtt{o}) + \mathtt{o}\,, \quad \mathtt{o}\in\mathcal{E} \text{ fixed}\,. \tag{1.23}$$

We say that λ is the *size* of the body, and Σ is its *shape*.[1] Consider the vector field $\mathbf{m}_\lambda : \Sigma \to S^2$ and the functional Ψ_λ defined, respectively, as

$$\mathbf{m}_\lambda := \mathbf{m} \circ \mathbf{g}^{-1}\,, \quad \Psi_\lambda\{\mathbf{m}_\lambda\} := \Psi\{\mathbf{m}\}/|\Omega|\,. \tag{1.24}$$

It is easily checked that the solution $\mathbf{h}^{\rm s}_\lambda$ of the Maxwell's equations (1.12) with $\mathbf{m}_\lambda$ and Σ in the place of, respectively, $\mathbf{m}$ and Ω, is given by $\mathbf{h}^{\rm s}_\lambda = \mathbf{h}^{\rm s} \circ \mathbf{g}^{-1}$. If we assume $\mathbf{h}^{\rm e} = \mathbf{0}$, the problem of minimizing the functional Ψ is equivalent to the minimization of the *scaled energy*

$$\Psi_\lambda\{\mathbf{m}\} = \int_\Sigma \left(\left(\frac{\lambda_{ex}}{\lambda}\right)^2 |\nabla \mathbf{m}_\lambda|^2 - \frac{1}{2}\beta\,(\mathbf{e}\cdot\mathbf{m}_\lambda)^2 - \frac{1}{2}\mathbf{h}^{\rm s}_\lambda \cdot \mathbf{m}_\lambda \right)\,, \tag{1.25}$$

where

$$\lambda_{ex} := \sqrt{\frac{\alpha}{2}} \tag{1.26}$$

is the *exchange length*.

The formulation (1.25) shows that *anisotropy and stray-field energy do not scale with the size of the body, while exchange energy does.* This suggests that, for a fixed shape, the relative importance of these energetic contributions is determined not only by the material parameters α and β, but also by the size λ, whose natural unit appears to be λ_{ex}; accordingly, we say that Ω is *large* if $\lambda >> \lambda_{ex}$, *small* if $\lambda << \lambda_{ex}$.

1.3.2 Exchange vs. stray-field energy: size effects

Exchange interactions induce states in which the orientation of the magnetization is spatially uniform; on the other hand, according to the principle of pole avoidance, stray-field energy is minimized whenever the magnetization follows closed paths inside the body. These energy contributions are in competition the one with the other in the sense that they both want to be as small as possible, although they cannot achieve this at the same time (Fig. 1.1). We expect that exchange prevails in small bodies, and that stray-field effect are important in large bodies.

It can be shown[2] that the limiting behavior for large bodies, *i.e.*, for $\lambda_{ex}/\lambda \to 0$, is captured by the functional

$$\Psi_\infty\{\mathbf{m}\} = -\frac{1}{2}\int_\Sigma \left(\beta(\mathbf{e}\cdot\mathbf{m})^2 + \mathbf{h}^{\rm s}_\Sigma\{\mathbf{m}\}\cdot\mathbf{m}\right)\,, \tag{1.27}$$

[1] Note that Σ has unit volume.

[2] See [14] and the references quoted therein.

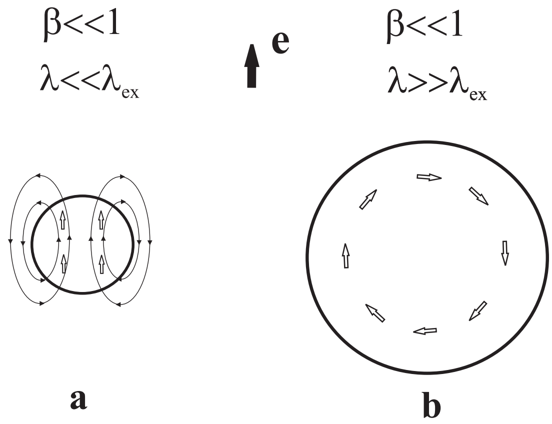

Figure 1.1: For a body of small size (a), a uniform magnetization is favored, although such a magnetization generates a stray field. As the size λ increases, divergence-free magnetizations become energetically convenient. If anisotropy is negligible ($\beta << 1$), a "vortex" configuration (b) may develop in a sphere-shaped body; this configuration does not generate a stray field.

by dropping some regularity assumptions on $\mathbf{m}$ and by extending in a suitable manner the notion of minimizer of Ψ_λ. It can also be shown that the limiting behavior for small bodies, *i.e.*, for $\lambda_{ex}/\lambda \to \infty$, is recovered by the functional (1.27), provided that the following additional constraint is set on $\mathbf{m}$:

$$\int_\Sigma |\nabla \mathbf{m}|^2 = 0\,. \tag{1.28}$$

This result shows that, in order for a nonuniform magnetization to develop, it is necessary that the body be large in the sense just explained. However, for the formation of magnetic domains and domain walls this condition is, in general, insufficient.

1.3.3 Shape anisotropy

Anisotropy and stray-field energy scale with λ according to the same law, as pointed out in §1.3.1. However, they do not behave in the same way with respect to a change in the shape Σ. We illustrate this fact by considering the case when Σ is an ellipsoid and λ is so small that $\mathbf{m}$ is spatially constant.[3]

[3]It can be shown that there exists a critical size λ_Σ such that spatially constant minimizers of Ψ_∞ are also local minimizers of Ψ_λ for $\lambda < \lambda_\Sigma$, that is to say, the magnetization is constant if λ is small enough. An explicit computation of λ_Σ for a body of spherical shape may be found in [8].

With these assumptions, the stray field inside the body has the simple form[4]

$$\mathbf{h}^{\mathrm{s}} = -\mathbf{B}\mathbf{m}\,, \tag{1.29}$$

with $\mathbf{B} = \sum_{i=1}^{3} \beta_i\, \mathbf{e}_i \otimes \mathbf{e}_i$, where the unit vectors $\mathbf{e}_i$ are parallel to the principal axes of the ellipsoid and the *demagnetizing factors* $\{\beta_i, i = 1 \ldots 3\}$ are positive quantities which depend on the ratios between the lengths $\{\rho_i, i = 1 \ldots 3\}$ of the principal axes, and satisfy $\beta_1 + \beta_2 + \beta_3 = 1$. For an ellipsoid of revolution having elongated shape ($\rho_3 > \rho_1 = \rho_2$), it is found that $\beta_3 < \beta_1 = \beta_2$ and hence, while the scaled anisotropy energy

$$\lambda^{-3}\Psi^{\mathrm{a}} = -\frac{1}{2}\,\beta(\mathbf{e}_3 \cdot \mathbf{m})^2 \tag{1.30}$$

is not affected by Σ, the scaled stray-field energy does, and, *modulo* an additive constant, we may write

$$\lambda^{-3}\Psi^{\mathrm{s}} = -\frac{1}{2}\,\beta_m(\mathbf{e}_3 \cdot \mathbf{m})^2\,, \tag{1.31}$$

where $\beta_m = \beta_1 - \beta_3 > 0$. In this particular case, the effect of the stray field is equivalent to that of a material anisotropy, and is often referred to as *shape anisotropy*, since its easy axis and anisotropy constant depend on the shape of the body (Figure 1.2). Shape anisotropy is responsible for the fact that the size up to which a specimen admits a stable state of uniform magnetization is larger for an elongated shape than for a spherical shape.

[4]See [9] or [4].

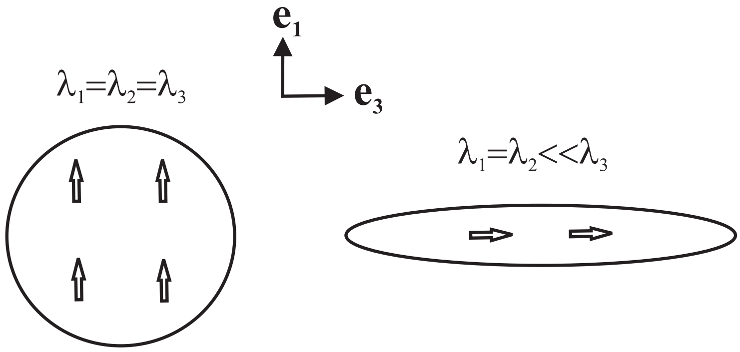

Figure 1.2: In an ellipsoid of revolution, shape anisotropy alignes the magnetization with the longest axis.

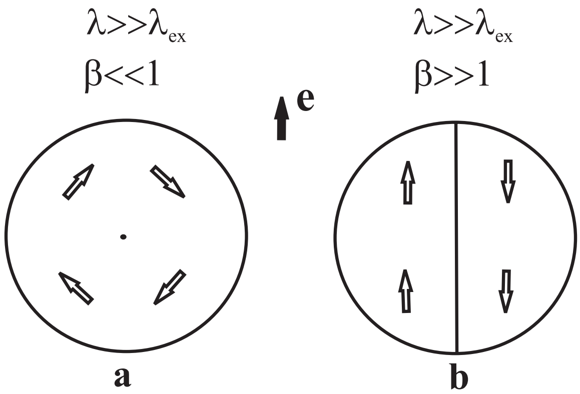

Figure 1.3: (a) Vortex configuration in a soft material; (b) domain configuration in a hard material.

1.3.4 Hard and soft materials

A ferromagnetic material is said to be *soft* if $\beta << 1$, *hard* if β is of the order of the unity or larger. Magnetic domains are more likely to develop in a body made of a hard material.

To understand the role of material anisotropy, consider the functional (1.25), with $\beta << 1$, so that the effect of material anisotropy is negligible, and with $\lambda >> \lambda_{ex}$, so that a divergence-free magnetization field develops which minimizes the stray-field energy (Fig. 1.3a). By letting β gradually increase, regions where the magnetization points in an easy direction grow bigger and become magnetic domains, regions where the magnetization points in difficult directions shrink (Fig. 1.3b) and become domain walls. Although this argument does not indicate for what value of β domains develop, it shows that the formation of magnetic domains and domain walls

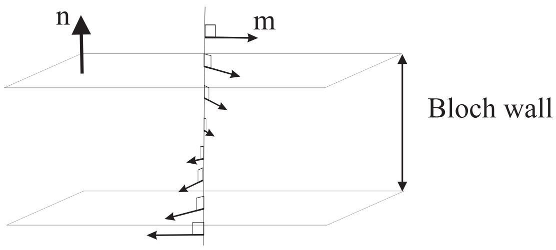

Figure 1.4: A Bloch wall.

may be driven by anisotropy.[5]

1.3.5 Anisotropy vs. exchange: the internal structure of a domain wall

Domain walls are commonly classified according to the relative directions that the magnetization has in the two magnetic domains that the wall separates; 180° *wall* separate magnetic domains in which the magnetization points in opposite directions.

Denoting with $\mathbf{n}$ the unit-vector orthogonal to a flat wall, a *Bloch wall* is defined as a 180° wall in which the magnetization is perpendicular to $\mathbf{n}$ (Fig. 1.4). The problem of estimating the thickness of a Bloch wall was tackled by Landau and Lifshitz [30] with a reasoning that runs as follows. They consider a uniaxial, homogeneous body Ω with easy axis $\mathbf{e}$, and they conjecture that: 1) the body is partitioned in magnetic domains in which the magnetization is constant; 2) in the bulk, *i.e.*, away from the boundary $\partial\Omega$ of the body, those domains are arranged in parallel layers of thickness D, with alternating magnetization $\mathbf{m} = \pm\mathbf{e}$; 3) the rotation that $\mathbf{m}$ undergoes in passing from one such magnetic domain to the other is localized in a narrow layer in between (Fig. 1.5), with thickness much smaller than D. To estimate the thickness of this layer, Landau and Lifshitz consider the ideal situation in which an infinite ferromagnetic body with easy axis $\mathbf{e}$ is partitioned in two magnetic domains by a flat wall. For $\{\mathbf{o}; \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ an orthonormal frame with $\mathbf{c}_1$ perpendicular to the wall plane and $\mathbf{c}_3 = \mathbf{e}$ (Fig. 6), they look for a magnetization $\mathbf{m}$ which

- depends on the coordinate x_1 only;
- agrees with the direction of the magnetization in each of the two domains:

[5]Magnetic domains may also be observed in soft ferromagnets. For example, magnetic domains develop in thin film elements with straight edges (see [27], §3.3.3).

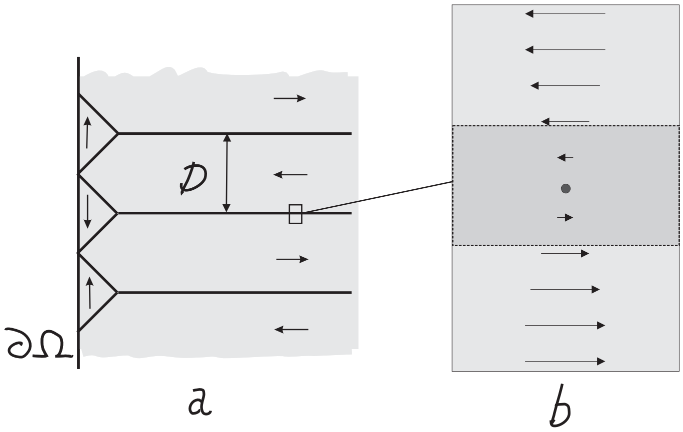

Figure 1.5: (a) Magnetic domains; (b) domain wall.

$$\mathbf{m}(\pm\infty) = \pm\mathbf{e}\,; \tag{1.32}$$

- minimizes the *energy per unit wall area*, that is to say,[6]

$$\sigma = \frac{1}{2}\int_{-\infty}^{+\infty} \alpha\,|\mathbf{m}'(x_1)|^2 - \beta\,(\mathbf{e}\cdot\mathbf{m}(x_1))^2 + (\mathbf{c}_1\cdot\mathbf{m}(x_1))^2\;dx_1\,, \tag{1.33}$$

where $\mathbf{m}' \equiv d\mathbf{m}/dx_1$.[7]
Landau and Lifshitz note that σ is minimized when $\mathbf{m}\cdot\mathbf{c}_1 = 0$,[8] hence by introducing a scalar function ϑ such that $\cos\vartheta(x_1) = \mathbf{m}(x_1)\cdot\mathbf{e}$, the magnetization can be represented as

$$\mathbf{m}(x_1) = \cos\vartheta(x_1)\mathbf{c}_3 + \sin\vartheta(x_1)\mathbf{c}_2\,, \tag{1.34}$$

[6]The dependence of the stray field $\mathbf{h}^s$ on the magnetization $\mathbf{m}$ can be made explicit (see Chapter 3):

$$\mathbf{h}^s = -(\mathbf{m}\cdot\mathbf{c}_1)\mathbf{c}_1\,,$$

and hence the stray-field energy density (1.17) takes the form:

$$\psi^s = \frac{1}{2}\,(\mathbf{m}\cdot\mathbf{c}_1)^2\,,$$

which is the last term in the integrand of (1.33).

[7]Here and henceforth, for functions of a real variable, a superscript prime denotes differentiation with respect to the argument.

[8]By expressing the magnetization as $\mathbf{m}(x_1) = \cos\vartheta(x_1)\,\mathbf{e} + \sin\vartheta(x_1)\,\mathbf{a}(x_1)$, with $\mathbf{a}\cdot\mathbf{e} = 0$, we may write

$$2\sigma\{\mathbf{m}\} = \int \alpha\,(\vartheta'^2 + (\sin\vartheta)^2|\mathbf{a}'|^2) - \beta(\cos\vartheta)^2 + (\sin\vartheta)^2(\mathbf{a}\cdot\mathbf{c}_1)^2\,.$$

Then, by defining $\widehat{\mathbf{m}}(x_1) = \cos\vartheta(x_1)\mathbf{c}_3 + \sin\vartheta(x_1)\mathbf{c}_2$, we easily check that $\sigma\{\mathbf{m}\} \geq \sigma\{\widehat{\mathbf{m}}\}$.

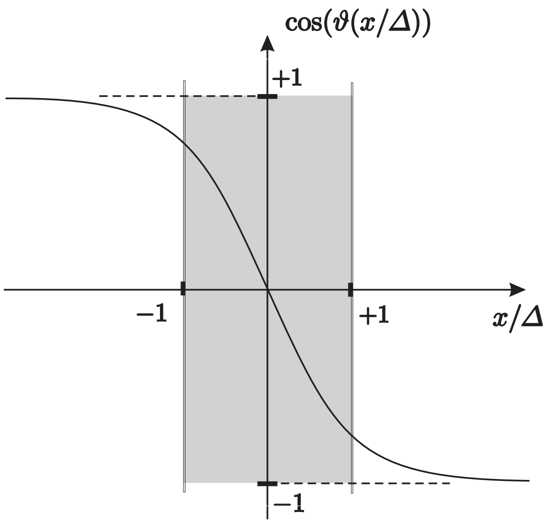

Figure 1.6: Magnetization profile in a Bloch wall.

and the functional (1.33) becomes

$$\sigma = \frac{1}{2}\int_{-\infty}^{+\infty} \alpha\,\vartheta'^2 - \beta\cos^2\vartheta \; dx_1 \,. \tag{1.35}$$

Then, under the assumption that $\lim_{x_1\to\infty} \vartheta'(x_1) = 0$, which is the natural boundary condition for (1.33), the Euler–Lagrange equation

$$\alpha\,\vartheta'' + \beta\sin\vartheta = 0 \tag{1.36}$$

obtains which, with another integration, gives

$$\alpha\,\vartheta'^2 = \beta\,\cos^2\vartheta \,. \tag{1.37}$$

Solutions of (1.37) have the form

$$\vartheta(x_1) = \nu\arctan\exp\left(\frac{x_1 - c}{\Delta}\right)\,, \tag{1.38}$$

where $\nu = \pm 1$, c is a constant, and $\Delta = \sqrt{\alpha/\beta}$; Landau & Lifshitz' choice is $\nu = -1$, $c = 0$ (Fig. 5). Noting that most of the the rotation is concentrated in the interval $(-\frac{\Delta}{2}, +\frac{\Delta}{2})$, Landau and Lifhitz estimate the wall thickness with the parameter Δ, and the wall energy per unit area with

$$\sigma = 2\sqrt{\alpha\beta}\,. \tag{1.39}$$

Equation (1.37) states the important fact that inside an ideal Bloch wall the density of exchange energy is equal to the density of anisotropy energy. This suggests the interpretation of the thickness of a domain wall as the optimal tradeoff between making Δ small to reduce the volume of the regions where $\mathbf{m}$ is unfavorably oriented, and making Δ large to avoid rapid spatial variations of $\mathbf{m}$.

1.4 Domain Theory as a Sharp-Interface Theory

Domain theory is a simplification of micromagnetics which presumes that the energy is minimized by a patchwise-constant magnetization, and regards domain walls as sharp interfaces endowed with surface energy. An additional simplification may be introduced by appealing to the principle of pole avoidance: if the magnetization is assumed to be divergence free,[9] the stray-field energy vanishes, and the total energy may be computed as the sum of the surface energy contained in the walls, plus the anisotropy energy contained in the magnetic domains. Usually, a solution is sought by choosing a domain configuration which is believed to be a good candidate for minimization; for this configuration, the geometry of the domains is made dependent on a set of scalar parameters, and these parameters are adjusted to minimize the energy; this procedure may be repeated for several configurations, and the configuration which attains the lowest energy is retained.

The basic ideas behind domain theory are contained in the paper of Landau and Lifshitz; further developments are due to Néel [35], Kittel [29], and many others. As anticipated in the previous section, Landau and Lifshitz approached the problem in two steps involving different scales. The first step consists in the computation of the magnetization profile in an ideal Bloch wall, which allows for an estimate of the wall thickness Δ and the total energy per unit wall area σ for a specific choice of the material parameters. With this result at hand, Landau and Lifshitz considered a homogeneous body with size much larger that Δ, and assumed it to be partitioned in

[9] For a patchwise-constant magnetization $\mathbf{m}$ this requirement corresponds to asking that the difference between the magnetizations in two neighboring magnetic domains be parallel to the domain wall in between. In other words, if $[\![\mathbf{m}]\!]$ is the difference between the two magnetizations, and $\mathbf{n}$ is a vector orthogonal to the wall, it is required that

$$[\![\mathbf{m}]\!] \cdot \mathbf{n} = 0\,.$$

domains. They postulated a specific geometry for the magnetic domains, following the principle of pole avoidance, and computed the total energy of the system as the sum between a "bulk" term, obtained by performing a volume-integration of the anisotropy energy density in the domains, and a "surface" term, given by the total area of the domain walls times their surface energy, assumed to be σ. By letting the geometry of that domain structure to depend on a restricted set of scalar parameters, they reduced the original problem to an easier minimization over a finite dimensional space, which allowed them to estimate the size D of the magnetic domains obtaining $D >> \Delta$, a result consistent with the starting assumptions that domain walls have a thickness which is much smaller than the typical size of magnetic domains.