On the Dynamic Programming Approach to Incentive Constraint Problems

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Abstract.

In this paper we study a class of infinite horizon optimal control problems with incentive constraints in the discrete time case. More specifically, we establish sufficient conditions under which the value function associated to such problems satisfies the Dynamic Programming Principle.

Key words: incentive compatibility constraints, value function, dynamic programming principle.

1 Introduction

In this paper we study a family of discrete time deterministic dynamic optimization problems with an infinite horizon and an *incentive compatibility constraint*, which, in the sequel, will be called *incentive constrained problems*or ICP for short. These problems (see Section 2 for the detailed formulation) are classical infinite horizon optimal control problems with a constraint on the continuation value of the plan. The continuation value at any current date t must be larger than some prescribed function of the current state and of the current control.

ICP arise in many economic applications [1] (see e.g. [2], [3], [4], [5] for the discrete time case, and [1], [7] for the continuous time case). ICP are not easily manageable due to the specific nature of the incentive constraint. Indeed, such a constraint concerns the future of the strategies, differently than the more commonly used constraints, which appear in standard dynamic optimization problems and concern the states of the system or the range of the admissible controls. For this reason, in general, the Dynamic Programming (DP) approach is not exploitable. Therefore, with standard terminology (see e.g. in [3], [6]), the contract is said to be *not recursive*.

In [3] a method is proposed to deal with incentive constrained problems in discrete time (deterministic or stochastic) when DP method may not be applied. This method is based on the introduction of a new set of variables, the so-called *dual variables*, that still allows to solve the problem in a recursive way. This result seems to be very useful in the general case, though we see two possible restraints:

- strong regularity and boundedness assumptions are needed on the data;
- the new problem to solve is more complex than the original one.

These two points call for a question: although DP approach cannot be exploited in general, is it possible to determine a class of incentive constrained problems such that the DP procedure turns out to be successful? For this class there would be two advantages, related with the two limitations described above:

- weaker assumptions on the data;
- not increasing of the complexity of the original problem.

Towards this direction we are aware of two results in the recent literature (see [4] and [1]) that show the applicability of DP approach to a class of ICP, under the assumption that the incentive running objective is equal to the running objective of the maximizing agent. More precisely, [4] deals with a general discrete time case, and [1] is concerned with the deterministic case in continuous time.

In this paper, we aim to extend the domain of applicability of DP approach to a larger class of ICP, though dealing only with the discrete deterministic case. This is just our preliminary step to tackle the discrete stochastic case and the continuous time cases. Our main result is that the Dynamic Programming Principle (DPP), formally the equation

$$V(x) = \sup_{c \in \mathcal{C}_g(x)} \sum_{t=0}^{T} \beta^t r\left(x_t(x,c), c_t\right) + \beta^T V\left(x_T(x,c)\right), \tag{1}$$

[1]We just recall that in ICP context we are in a *normative perspective*. A social planner seeks an optimal policy among the ones which do not include the termination of the process. The goal is the definition of a social contract, on account that agents can decide to go out of the contract in the future. Such an event is prevented by including the incentive compatibility constraint.

where $\mathcal{C}_g(x)$ is an admissible set of controls associated to the initial point x, holds true under a suitable comonotonicity assumption introduced and explained in Section 2.

The paper has the following structure: in Section 2 we formulate the problem (see 3); in Section 3 we prove DPP (1) for ICP problems via two Lemmas, which are related to some properties of the trajectories: stability under shift (SS) (see Lemma 4) and stability under concatenation (SC) (see Lemma 6).

2 Incentive constrained problems in the discrete-time deterministic case

We first give a general formulation

2.1 The general model

Let $(x_t)_{t\in\mathbb{N}}$ be a controlled discrete-time stationary dynamical system with states in the real Euclidean space $\mathbb{R}^n$. Given a map $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$, the *transition function* of the dynamical system, a nonempty subset $\mathbb{X}_0$ of $\mathbb{R}^n$, the *state constraint* on the dynamical system, and a nonempty-valued correspondence $\Gamma:\mathbb{R}^n\to\mathfrak{P}(\mathbb{R}^m)$, the *technological constraint* on the control, we assume that $(x_t)_{t\in\mathbb{N}}$ is subject to the difference equation

$$\begin{cases} x_{t+1}=f(x_t,c_t), & t\in\mathbb{N},\\ x_0=x\in\mathbb{X}_0, \end{cases} \tag{2}$$

where x is the *initial state* of the dynamical system, and $(c_t)_{t\in\mathbb{N}}$ is a discrete-time control process such that $c_t\in\Gamma(x_t)$ and $f(x_t,c_t)\in\mathbb{X}_0$, for every $t\in\mathbb{N}$. Such a process $(c_t)_{t\in\mathbb{N}}$ is called an *admissible control* for the dynamical system with initial state x. We denote by $\mathcal{C}(x)$ the set of all admissible control processes with initial state x and we write $x_t(x,c)$ for the solution of (2) corresponding to the choice of a specific control process $c\equiv(c_t)_{t\in\mathbb{N}}$ in $\mathcal{C}(x)$.

Since in this paper we are not concerned in dealing with the minimal hypotheses that assure the existence of solutions of (2), we assume straightforwardly that for every $x\in\mathbb{X}_0$ the set $\mathcal{C}(x)$ is nonempty.

Given a function $r:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, the *running objective*, and $\beta\in(0,1)$, the *discount factor*, for any $x\in\mathbb{X}_0$ and any $c\in\mathcal{C}(x)$, we introduce the *objective functional* $J_r:\mathbb{X}_0\times\mathcal{C}(x)\to\mathbb{R}$, given by

$$J_r\left(x;c\right)\overset{\text{def}}{=}\sum_{t=0}^{+\infty}\beta^t r\left(x_t\left(x,c\right),c_t\right).$$

Then, the standard optimization problem for the functional $J_r:\mathbb{X}_0\times\mathcal{C}(x)\to\mathbb{R}$ is to compute the value function $V_0:\mathbb{X}_0\to\tilde{\mathbb{R}}$, where $\tilde{\mathbb{R}}\equiv\mathbb{R}\cup\{-\infty,+\infty\}$ stands for the extended Euclidean real line, given by

$$V_0(x)\overset{\text{def}}{=}\sup_{c\in\mathcal{C}(x)}J_r\left(x;c\right),$$

and, possibly, to determine an optimal control $c^*\in\mathcal{C}(x)$ such that

$$V_0(x)=J_r\left(x;c^*\right),$$

for every $x\in\mathbb{X}_0$.

Now, given the functions $g_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, the *initial incentive constraint*, and $g_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, the *running incentive constraint*, and given $N \in \mathbb{N} \cup \{+\infty\}$, for any $x \in \mathbb{X}_0$, we introduce the set $\mathcal{C}_g(x)$ of all controls $c \in \mathcal{C}(x)$ fulfilling the incentive constraint

$$g_1\left(x_t\left(x,c\right),c_t\right) + \sum_{n=1}^{N} \beta^n g_2\left(x_{t+n}\left(x,c\right),c_{t+n}\right) \geq 0, \qquad \forall t \in \mathbb{N}. \tag{3}$$

We assume that the set $\mathbb{X} \equiv \{x \in \mathbb{X}_0 \mid \mathcal{C}_g(x) \neq \emptyset\}$ is nonempty [2]. The incentive constraint problem for the functional $J_r : \mathbb{X} \times \mathcal{C}_g(x) \to \mathbb{R}$ is to compute the value function $V : \mathbb{X} \to \tilde{\mathbb{R}}$ given by

$$V(x) \stackrel{\text{def}}{=} \sup_{c \in \mathcal{C}_g(x)} J_r\left(x;c\right), \tag{4}$$

and, possibly, to determine an optimal control $c^* \in \mathcal{C}_g(x)$ such that

$$V(x) = J_r\left(x;c^*\right),$$

for every $x \in \mathbb{X}$.

In the sequel, for simplicity, we set

$$\sum_{t=0}^{+\infty} \beta^t g_2\left(x_t\left(x,c\right),c_t\right) \equiv J_{g_2}\left(x;c\right),$$

and

$$\sum_{t=0}^{N-h} \beta^t g_2\left(x_t\left(x,c\right),c_t\right) \equiv J_{g_2}^h\left(x;c\right),$$

for every $N \in \mathbb{N}$, and $h = 1, ..., N$.

2.2 Main Assumptions and statement of DPP

To tackle the constrained optimization problem (4) via DP approach, we begin by assuming that:

Assumption 1 *For every $x \in \mathbb{X}$ we have $V(x) \in \mathbb{R}$.*

This is clearly a restriction and sometimes one would like to consider problems where V may be infinite. On the other hand, the latter case is not considered in most of the quoted papers treating ICP. We formulate this assumption for sake of simplicity. Indeed, sometimes the finiteness of V arises from compactness, some other times by boundedness, other times again by suitable state-control constraints. In this general part, the treatment of all these different topics would result in useless technical complications.

Having assumed the finiteness of the value function, we still need to introduce the following hypothesis on the structure of the family of all admissible controls, which will turn out to be crucial to exploit a DP approach to ICP.

Assumption 2 *For any $c \in \mathcal{C}_g(x)$, any $T > 0$, and any $\varepsilon > 0$ there exists a ε-optimal control at $x_T(x,c)$, denoted by c^ε, such that either*

1.

$$J_{g_2}(x_T(x,c);c^\varepsilon) \geq J_{g_2}(x_T(x,c);c^T),$$

in the case $N = +\infty$; or

[2] We recall that in [1] the set $\mathbb{X}$ is one of the unknown to be found.

2.

$$J^h_{g_2}(x_T(x,c);c^\varepsilon) \geq J^h_{g_2}(x_T(x,c);c^T), \qquad h=1,...,N,$$

in the case $N < +\infty$;

where c^T is the control at $x_T(x,c)$ given by

$$c^T_t \stackrel{def}{=} c_{T+t}, \quad t \in \mathbb{N}.$$

Assumption 2 calls for some comment.
First, we remember that c^ε is an ε-*optimal control* at $x_T(x,c)$ when

$$V(x_T(x,c)) < J_r(x_T(x,c);c^\varepsilon) + \varepsilon.$$

The possibility of finding an ε-optimal control is assured by the finiteness of the value function. The relevance of Assumption 2 is then the requirement that there exists at least one of such controls fulfilling also either 1 or 2.
Second, observe that, as required by 1 and 2, the control c^T actually belongs to $\mathcal{C}_g(x_T(x,c))$. This will be shown in subsequent Lemma 4.
Third, Assumption 2 may be seen as an hypothesis of *comonotonicity* of the functionals $J_{g_2}(x;\cdot)$ (or $J^h_{g_2}(x;\cdot)$) with respect to $J_r(x;\cdot)$. [3] Actually, if this is the case, 1 and 2 clearly hold true. This has a plain economic explanation. Our "comonotonicity" means that the running objectives r and g_2 are compatible. This compatibility means that the social planner and the agent maximizing $J_{g_2}(x;\cdot)$ (or $J^h_{g_2}(x;\cdot)$) and $J_r(x;\cdot)$ have the same structure of preferences along optimal (or almost optimal) strategies. In this case, the incentive does not affect the recursivity.

Finally, we note that Assumption 2 is satisfied in [4] and [1], where $g_2 = r$.

We are now in a position to state DP approach to our ICP.

Theorem 3 *For any $x \in \mathbb{X}$, any $c \in \mathcal{C}_g(x)$, and any $T \in \mathbb{N}$, write*

$$J_{r,T}(x;c) \equiv \sum_{t=0}^{T} \beta^t r(x_t(x,c), c_t) + \beta^T V(x_T(x,c)),$$

Then, under Assumptions 1 and 2, we have

$$V(x) = \sup_{c \in \mathcal{C}_g(x)} J_{r,T}(x;c).$$

The proof is obtained by considering the two inequalities shown in Propositions 5 and 7.

3 Dynamic Programming: statement and proof

The main idea beyond our result is the following (contained also "in nuce" in [1, Section 3]).

The classical proof of the Dynamic Programming Principle (1) depends on two key properties of the family of the sets of admissible strategies $\{\mathcal{C}_g(x)\}_{x\in\mathbb{X}}$

[3] The functional $J_{g_2}(x;\cdot)$ is comonotone with respect to $J_r(x;\cdot)$ when for all controls $c_1, c_2 \in \mathcal{C}_g(x)$ the condition

$$J_r(x;c_1) \leq J_r(x;c_2)$$

implies

$$J_{g_2}(x;c_1) \leq J_{g_2}(x;c_2).$$

Clearly, the same holds for $J^h_{g_2}(x;\cdot)$

The first one is the “stability under shift” (SS), while the second is the “stability under concatenation” (SC).

Property (SS) states that $c \in \mathcal{C}_g(x)$ implies $c^T \in \mathcal{C}_g(x_T(x,c))$. Starting with an admissible control c at $t = 0$ from $x_0 = x$, shifting it at T then c^T is admissible starting at $t = 0$ from $x_0 = x_T(x,c)$. Even in cases more general than ours, (SS) is true as it is shown in Lemma 4 and it yields the inequality $\leq$ in (1). This is proved in Subsection 3.1.

Property (SC) states that picking two admissible controls $c \in \mathcal{C}_g(x)$ and $\hat{c} \in \mathcal{C}_g(x_T(x,c))$ then the control $\tilde{c}$ given by

$$\tilde{c}_t \stackrel{def}{=} \begin{cases} c_t, & \text{if } t < T \\ \hat{c}_{t-T} & \text{if } t \geq T \end{cases}, \tag{5}$$

belongs to $\mathcal{C}_g(x)$, hence the concatenated control $\tilde{c}$ is admissible starting at $t = 0$ from $x_0 = x$. Unfortunately, even in simple cases (SC) is not true. Of course, the failure of (SC) does not mean that DP principle is false since (SC) is a sufficient condition. In Subsection 3.2 we show (see Lemma 6) that (SC) holds if $\hat{c}$ is “better” than c along the functional J_{g_2}. This is done by analyzing the family $\{\mathcal{C}_g(x)\}_{x\in\mathbb{X}}$. Given this fact, the inequality $\geq$ is an easy consequence of Assumption 2.

In next two subsection we consider the two inequalities separately

$$V(x) \leq \sup_{c\in\mathcal{C}_g(x)} \sum_{t=0}^{T} \beta^t r\left(x_t(x,c), c_t\right) + \beta^T V\left(x_T(x,c)\right), \tag{6}$$

$$V(x) \geq \sup_{c\in\mathcal{C}_g(x)} \sum_{t=0}^{T} \beta^t r\left(x_t(x,c), c_t\right) + \beta^T V\left(x_T(x,c)\right). \tag{7}$$

3.1 The easy inequality

We begin by proving the above mentioned Property (SS).

Lemma 4 *For any $x \in \mathbb{X}$, any $c \in \mathcal{C}_g(x)$, and any $T > 0$, the control c^T given by*

$$c_t^T \stackrel{def}{=} c_{T+t}, \quad t \in \mathbb{N}$$

belongs to $\mathcal{C}_g\left(x_T(x,c)\right)$.

Proof. Since the transition function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ of the dynamical system is stationary, for any $x \in \mathbb{X}$, any $c \in \mathcal{C}(x)$, and any $T > 0$, we clearly have

$$x_{T+t}(x,c) = x_t(x_T(x,c), c^T) \tag{8}$$

for every $t \in \mathbb{N}$. Hence, $c_T \in \mathcal{C}(x_T(x,c))$. Assuming in addition $c \in \mathcal{C}_g(x)$, we have also

$$g_1(x_s(x,c), c_s) + \sum_{n=1}^{N} \beta^n g_2(x_{s+n}(x,c), c_{s+n}) \geq 0,$$

for every $s \in \mathbb{N}$. Thus, setting $s \equiv t + T$, it follows

$$g_1(x_{T+t}(x,c), c_{T+t}) + \sum_{n=1}^{N} \beta^n g_2(x_{T+t+n}(x,c), c_{T+t+n}) \geq 0$$

for every $t \in \mathbb{N}$. Therefore, on account of (8) and of the definition of c^T, we obtain

$$g_1(x_t(x_T(x,c),c^T),c_t^T) + \sum_{n=1}^{N} \beta^n g_2(x_{t+n}(x_T(x,c),c^T),c_{t+n}^T) \geq 0,$$

which completes the proof. $\square$

¿From (SS) it easily follows

Proposition 5 *For any $x \in \mathbb{X}$, any $c \in \mathcal{C}_g(x)$, and any $T \in \mathbb{N}$ we have*

$$V(x) \leq \sup_{c \in \mathcal{C}_g(x)} J_{r,T}(x;c). \tag{9}$$

Proof. To prove (9), it is sufficient to observe that, for every $\varepsilon > 0$, there exists $c^\varepsilon \in \mathcal{C}_g(x)$ such that

$$J_r(x;c^\varepsilon) + \varepsilon > V(x). \tag{10}$$

On the other hand, on account of Lemma 4, for any $T \in \mathbb{N}$, we can write

$$\begin{aligned} J_r(x;c^\varepsilon) &= \sum_{t=0}^{T} \beta^t r(x_t(x;c^\varepsilon),c_t^\varepsilon) + \beta^T J(x_T(x;c^\varepsilon);c^{\varepsilon,T}) \\ &\leq \sum_{t=0}^{T} \beta^t r(x_t(x;c^\varepsilon),c_t^\varepsilon) + \beta^T V(x_T(x;c^\varepsilon)) \\ &= J_{r,T}(x;c^\varepsilon). \end{aligned} \tag{11}$$

Combining (10) and (11), it then follows

$$J_{r,T}(x;c^\varepsilon) + \varepsilon > V(x),$$

and the latter clearly implies the desired (9). $\square$

3.2 The hard inequality

Now, we prove Property (SC)

Lemma 6 *For any $x \in \mathbb{X}$, any $c \in \mathcal{C}_g(x)$ and any $T > 0$, let $\hat{c} \in \mathcal{C}_g(x_T(x,c))$ satisfying either the condition*

$$J_{g_2}(x_T(x,c);\hat{c}) \geq J_{g_2}\left(x_T(x,c);c^T\right) \tag{12}$$

in the case $N = +\infty$, or the condition

$$J_{g_2}^h(x_T(x,c);\hat{c}) \geq J_{g_2}^h\left(x_T(x,c);c^T\right), \qquad \forall h = 1,...,N, \tag{13}$$

in the case $N < +\infty$. Then the control $\tilde{c}$ given by

$$\tilde{c}_t \overset{def}{=} \begin{cases} c_t, & \text{if } t < T \\ \hat{c}_{t-T} & \text{if } t \geq T \end{cases}, \tag{14}$$

belongs to $\mathcal{C}_g(x)$.

Proof. For any $x \in \mathbb{X}$, any $c \in \mathcal{C}_g(x)$ and any $T > 0$, by (14), we have

$$x_t(x, \tilde{c}) = \begin{cases} x_t(x, c), & \text{if } t < T \\ x_{t-T}(x_T(x, c), \hat{c}), & \text{if } t \geq T \end{cases} . \tag{15}$$

Hence, $\tilde{c} \in \mathcal{C}(x)$. Thus, to prove that $\tilde{c} \in \mathcal{C}_g(x)$, we are left with the task of showing that the incentive constraint (3) is satisfied for every $t \geq 0$. To this task, we start by dealing with the case $N = +\infty$.

Actually, for $0 \leq t < T$, we can write

$$\begin{aligned} & g_1(x_t(x,\tilde{c}), \tilde{c}_t) + \sum_{n=1}^{+\infty} \beta^n g_2(x_{t+n}(x,\tilde{c}), \tilde{c}_{t+n}) \\ &= g_1(x_t(x,\tilde{c}), \tilde{c}_t) + \sum_{n=1}^{T-t-1} \beta^n g_2(x_{t+n}(x,\tilde{c}), \tilde{c}_{t+n}) \\ &\quad + \sum_{n=T-t}^{+\infty} \beta^n g_2(x_{t+n}(x,\tilde{c}), \tilde{c}_{t+n}) \\ &= g_1(x_t(x,c), c_t) + \sum_{n=1}^{T-t-1} \beta^n g_2(x_{t+n}(x,c), c_{t+n}) \\ &\quad + \sum_{n=T-t}^{+\infty} \beta^n g_2(x_{t-T+n}(x_T(x,c),\hat{c}), \hat{c}_{t-T+n}). \end{aligned}$$

Hence, adding and subtracting the term

$$\sum_{n=T-t}^{+\infty} \beta^n g_2(x_{t+n}(x,c), c_{t+n}) = \sum_{n=T-t}^{+\infty} \beta^n g_2(x_{t-T+n}(x_T(x,c), c^T), c^T_{t-T+n}),$$

we obtain

$$\begin{aligned} & g_1(x_t(x,\tilde{c}), \tilde{c}_t) + \sum_{n=1}^{+\infty} \beta^n g_2(x_{t+n}(x,\tilde{c}), \tilde{c}_{t+n}) \\ &= g_1(x_t(x,c), c_t) + \sum_{n=1}^{+\infty} \beta^n g_2(x_{t+n}(x,c), c_{t+n}) \\ &\quad + \sum_{n=T-t}^{+\infty} \beta^n g_2(x_{t-T+n}(x_T(x,c),\hat{c}), \hat{c}_{t-T+n}) \\ &\quad - \sum_{n=T-t}^{+\infty} \beta^n g_2(x_{t-T+n}(x_T(x,c), c^T), c^T_{t-T+n}). \end{aligned}$$

On the other hand, observe that $c \in \mathcal{C}_g(x)$ implies that

$$g_1(x_t(x,c), c_t) + \sum_{n=1}^{+\infty} \beta^n g_2(x_{t+n}(x,c), c_{t+n}) \geq 0.$$

Moreover, setting $k \equiv n - (T - t)$, we have

$$\sum_{n=T-t}^{+\infty} \beta^n g_2(x_{t-T+n}(x_T(x,c),\hat{c}), \hat{c}_{t-T+n}) \tag{16}$$

$$\begin{aligned} &= \beta^{T-t} \sum_{k=0}^{+\infty} \beta^k g_2(x_k(x_T(x,c),\hat{c}), \hat{c}_k) \\ &= \beta^{T-t} J_{g_2}(x_T(x,c); \hat{c}), \end{aligned} \tag{17}$$

and

$$\sum_{n=T-t}^{+\infty} \beta^n g_2(x_{t-T+n}(x_T(x,c),c^T),c^T_{t-T+n}) \tag{18}$$

$$= \beta^{T-t}\sum_{k=0}^{+\infty} \beta^k g_2(x_k(x_T(x,c),c^T),c^T_k)$$

$$= \beta^{T-t} J_{g_2}(x_T(x,c);c^T). \tag{19}$$

Therefore, combining (17) and (19) with (12), thanks to (14), it follows that, for every $0 \leq t < T$,

$$g_1(x_t(x,\tilde{c}),\tilde{c}_t) + \sum_{n=1}^{+\infty} \beta^n g_2(x_{t+n}(x,\tilde{c}),\tilde{c}_{t+n}) \geq 0.$$

Finally, to complete the proof in the case $N = +\infty$, it is sufficient to observe that, on account of (14) and (15), for every $t \geq T$ we have

$$g_1(x_t(x,\tilde{c}),\tilde{c}_t) + \sum_{n=1}^{+\infty} \beta^n g_2(x_{t+n}(x,\tilde{c}),\tilde{c}_{t+n})$$

$$= g_1(x_{t-T}(x_T(x,c),\hat{c}),\hat{c}_{t-T}) + \sum_{n=1}^{+\infty} \beta^n g_2(x_{t-T+n}(x_T(x,c),\hat{c}),\hat{c}_{t-T+n})$$

and that

$$g_1(x_{t-T}(x_T(x,c),\hat{c}),\hat{c}_{t-T}) + \sum_{n=1}^{+\infty} \beta^n g_2(x_{t-T+n}(x_T(x,c),\hat{c}),\hat{c}_{t-T+n}) \geq 0$$

for $\hat{c} \in \mathcal{C}_g(x_T(x,c))$.

Now, with regard to the case $N \in \mathbb{N}$, let us observe first that for every $0 \leq t < T - N$ we clearly have

$$g_1(x_t(x,\tilde{c}),\tilde{c}_t) + \sum_{n=1}^{N} \beta^n g_2(x_{t+n}(x,\tilde{c}),\tilde{c}_{t+n})$$

$$= g_1(x_t(x,c),c_t) + \sum_{n=1}^{N} \beta^n g_2(x_{t+n}(x,c),c_{t+n}) \geq 0.$$

Second, for every $T - N \leq t < T$, namely $T - t \leq N$,

$$g_1(x_t(x,\tilde{c}),\tilde{c}_t) + \sum_{n=1}^{N} \beta^n g_2(x_{t+n}(x,\tilde{c}),\tilde{c}_{t+n})$$

$$= g_1(x_t(x,\tilde{c}),\tilde{c}_t) + \sum_{n=1}^{T-t-1} \beta^n g_2(x_{t+n}(x,\tilde{c}^T),\tilde{c}^T_{t+n})$$

$$+ \sum_{n=T-t}^{N} \beta^n g_2(x_{t+n}(x,\tilde{c}^T),\tilde{c}^T_{t+n})$$

$$= g_1(x_t(x,c),c_t) + \sum_{n=1}^{T-t-1} \beta^n g_2(x_{t+n}(x,c),c_{t+n})$$

$$+ \sum_{n=T-t}^{N} \beta^n g_2(x_{t-T+n}(x_T(x,c),\hat{c}),\hat{c}_{t-T+n}),$$

Hence, adding and subtracting the term

$$\sum_{n=T-t}^{N} \beta^n g_2(x_{t+n}(x,c), c_{t+n}) = \sum_{n=T-t}^{N} \beta^n g_2(x_{t-T+n}(x_T(x,c), c^T), c^T_{t-T+n})$$

we obtain

$$\begin{aligned} & g_1(x_t(x,\tilde{c}), \tilde{c}_t) + \sum_{n=1}^{N} \beta^n g_2(x_{t+n}(x,\tilde{c}), \tilde{c}_{t+n}) \\ &= g_1(x_t(x,c), c_t) + \sum_{n=1}^{N} \beta^n g_2(x_{t+n}(x,c), c_{t+n}) \\ &\quad \sum_{n=T-t}^{N} \beta^n g_2(x_{t-T+n}(x_T(x,c), \hat{c}), \hat{c}_{t-T+n}) \\ &\quad - \sum_{n=T-t}^{N} \beta^n g_2(x_{t-T+n}(x_T(x,c), c^T), c^T_{t-T+n}). \end{aligned}$$

On the other hand, observe that $c \in \mathcal{C}_g(x)$ implies that

$$g_1(x_t(x,c), c_t) + \sum_{n=1}^{N} \beta^n g_2(x_{t+n}(x,c), c_{t+n}) \geq 0.$$

Moreover, setting $k = n - T - t$, we have

$$\begin{aligned} & \sum_{n=T-t}^{N} \beta^n g_2(x_{t-T+n}(x_T(x,c), \hat{c}), \hat{c}_{t-T+n}) \\ &= \beta^{T-t} \sum_{k=0}^{N-(T-t)} \beta^k g_2(x_k(x_T(x,c), \hat{c}), \hat{c}_k) \\ &= \beta^{T-t} J^{T-t}_{g_2}(x_T(x,c); \hat{c}) \end{aligned} \tag{20}$$

and

$$\begin{aligned} & \sum_{n=T-t}^{N} \beta^n g_2(x_{t-T+n}(x_T(x,c), c^T), c^T_{t-T+n}) \\ &= \beta^{T-t} \sum_{k=0}^{N-(T-t)} \beta^k g_2(x_k(x_T(x,c), c^T), c^T_k) \\ &= \beta^{T-t} J^{T-t}_{g_2}(x_T(x,c); c^T) \end{aligned} \tag{21}$$

Therefore, combining (20) and (21) with (13), and observing that $T-N \leq t < T$ means $T - t = 1, ..., N$, thanks to (14), we obtain again

$$g_1(x_t(x,\tilde{c}^T), \tilde{c}^T_t) + \sum_{n=1}^{N} \beta^n g_2(x_{t+n}(x,\tilde{c}^T), \tilde{c}^T_{t+n}) \geq 0.$$

Finally, for $t \geq T$ we can argue exactly as in the case $N = +\infty$. This completes the proof. $\square$

We are now in a position to prove

Proposition 7 *Under Assumption 2, for any $x \in \mathbb{X}$ and any $T \in \mathbb{N}$, we have*

$$V(x) \geq \sup_{c \in \mathcal{C}_g(x)} J_{r,T}(x;c). \tag{22}$$

Proof. Fix any control $c \in \mathcal{C}_g(x)$ and consider

$$J_{r,T}(x;c) = \sum_{t=0}^{T} \beta^t r(x_t(x,c), c_t) + \beta^T V(x_T(x,c)).$$

Now, consider the corresponding control $c^T \in \mathcal{C}_g(x_T(x,c))$ (see Lemma 4). If c^T turns out to be optimal at $x_T(x,c)$, we have

$$\begin{aligned} V(x_T(x,c)) = & \sup_{\hat{c} \in \mathcal{C}_g(x_T(x,c))} \sum_{t=0}^{+\infty} \beta^t r(x_t(x_T(x,c), \hat{c}), \hat{c}_t) \\ = & \sum_{t=0}^{+\infty} \beta^t r(x_t(x_T(x,c), c^T), c_t^T). \end{aligned}$$

Thus,

$$\begin{aligned} \beta^T V(x_T(x,c)) = & \sum_{t=0}^{+\infty} \beta^{T+t} r(x_t(x_T(x,c), c^T), c_t^T) \\ = & \sum_{t=T}^{+\infty} \beta^t r(x_{t-T}(x_T(x,c), c^T), c_{t-T}^T), \end{aligned}$$

and

$$J_{r,T}(x;c) = \sum_{t=0}^{T} \beta^t r(x_t(x,c), c_t) + \sum_{t=T}^{+\infty} \beta^t r(x_{t-T}(x_T(x,c), c^T), c_{t-T}^T).$$

Therefore, by (8), we can write

$$J_{r,T}(x;c) = \sum_{t=0}^{+\infty} \beta^t r(x_t(x,c), c_t) = J_r(x;c),$$

and the latter clearly implies

$$J_{r,T}(x;c) \leq V(x),$$

On the other hand, if c^T is not optimal at $x_T(x,c)$, then for any $\varepsilon > 0$ we can find an ε-optimal control $c^\varepsilon \in \mathcal{C}_g(x_T(x,c))$ satisfying Assumption 2. Then, we have

$$V(x_T(x,c)) < J_r(x_T(x,c); c^\varepsilon) + \varepsilon. \tag{23}$$

$$J_r(x_T(x,c); c^T) \leq J_r(x_T(x,c); c^\varepsilon). \tag{24}$$

This implies that

$$
\begin{aligned}
& J_{r,T}(x;c) \\
= \quad & \sum_{t=0}^{T} \beta^t r(x_t(x,c),c_t) + \beta^T V(x_T(x,c)) \\
\leq \quad & \sum_{t=0}^{T} \beta^t r(x_t(x,c),c_t) + \beta^T J_r(x_T(x,c);c^\varepsilon) + \beta^T \varepsilon \\
= \quad & \sum_{t=0}^{T} \beta^t r(x_t(x,c),c_t) + \sum_{t=0}^{+\infty} \beta^{T+t} r(x_t(x_T(x,c),c^\varepsilon),c_t^\varepsilon) + \beta^T \varepsilon \\
= \quad & \sum_{t=0}^{T} \beta^t r(x_t(x,c),c_t) + \sum_{t=T}^{+\infty} \beta^{t} r(x_{t-T}(x_T(x,c),c^\varepsilon),c_{t-T}^\varepsilon) + \beta^T \varepsilon \qquad (25)
\end{aligned}
$$

Now, thanks to Lemma 6, the concatenation of c and c^ε, i.e. the control

$$
\tilde{c}_t^\varepsilon \stackrel{def}{=} \begin{cases} c_t, & \text{if } t < T \\ c_{t-T}^\varepsilon & \text{if } t \geq T \end{cases},
$$

belongs to $\mathcal{C}_g(x)$. Hence, by (8),

$$
\sum_{t=0}^{T} \beta^t r(x_t(x,c),c_t) + \sum_{t=T}^{+\infty} \beta^{t} r(x_{t-T}(x_T(x,c),c^\varepsilon),c_{t-T}^\varepsilon) = J_r\left(x;\tilde{c}^\varepsilon\right),
$$

Thus, substituting the latter into (25), we obtain

$$
J_{r,T}(x;c) \leq J\left(x;\tilde{c}^\varepsilon\right) + \beta^T\varepsilon \leq V(x) + \beta^T\varepsilon,
$$

and for the arbitrariness of ε, it follows

$$
J_T(x;c) \leq V(x).
$$

Finally, the arbitrariness of $c \in \mathcal{C}_g(x)$ yields the desired (22). $\square$

4 Conclusions

We have provided sufficient conditions to apply a DP approach to ICP, which are conditions (12) and (13) such that property (SC) holds true. They are basically comonotonicity conditions relating the objective functionals to the running functional. As a future development of this paper we are studying how to apply the direct approach presented here to stochastic optimal control problems with incentive constraint, both in the discrete and in the continuous case.

References

[1] Barucci, E., Gozzi, F.and Swiech, A: Incentive compatibility constraints and dynamic programming in continuous time, J.M.E., n.34, p.471–508, 2000.

[2] Dilip, A., Pearce, D. and Stacchetti, E.: Towards a theory of discounted repeated games with imperfect monitoring, Econometrica vol.58 n.5, p.1041–1064, 1990.

[3] Marcet, A. and Marimon: R., Recursive contracts, EUI, Florence, 1998.

[4] Rustichini, A.: Dynamic programming solutions of incentive constrained problems, J.E.T., n.78 (2), p.329–354, 1998.

[5] Rustichini, A.: Lagrange multipliers in incentive–constrained problems, J.M.E., n.29 (4), p.365–380, 1998.

[6] Stokey, N.L., Lucas, R.E. and Prescott, E.C.: Recursive methods in economic dynamics, Harvard University Press, Cambridge, Ma., 1989.

[7] Von Thadden, E.L.: An incentive problem in the dynamic theory of banking, J.M.E., n.38 (1-2), p.271–292, 2002.