Fields Institute Communications
Volume **00**, 0000

Classification of Subsystems, Local Symmetry Generators and Instrinsic Definition of Local Observables

Sebastiano Carpi
Dipartimento di Matematica, Università di Roma "La Sapienza"
Piazzale A. Moro, I-00185 Roma, Italy
carpi@mat.uniroma1.it

Roberto Conti
Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, I-00133 Roma, Italy
conti@mat.uniroma2.it

Dedicated to Sergio Doplicher and John E. Roberts on the occasion of their 60th birthdays.

Abstract. We give a general overview of results about subsystems of local nets of von Neumann algebras in close connection with the problem of characterizing the abstract algebra of observables through the existence of Wightman currents.

1 Introduction

The basic philosophy of algebraic quantum field theory ("local quantum physics" [41]) is that all the information about a physical theory is encoded in the observable net $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$, customarily written just $\mathfrak{A}$ for short. This has been mostly considered as an isotonous (inclusion preserving) correspondence between the set $\mathcal{K}$ of open double cones[1] in 4D Minkowski spacetime $\mathbb{M}_4$ and the family of (C^* or) von Neumann algebras acting on some fixed Hilbert space $\mathcal{H}_0$ satisfying suitable physically meaningful axioms such as

- *locality*: $\mathcal{O}_1 \subset \mathcal{O}_2' \Rightarrow [\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\}$
- *irreducibility*: $(\cup_{\mathcal{O}\in\mathcal{K}}\mathfrak{A}(\mathcal{O}))'' = \mathrm{B}(\mathcal{H}_0)$
- *property B*: if $\overline{\mathcal{O}} \subset \tilde{\mathcal{O}}$ every self-adjoint projection $E \in \mathfrak{A}(\mathcal{O})$ is equivalent to I in $\mathfrak{A}(\tilde{\mathcal{O}})$, namely there is an isometry $W \in \mathfrak{A}(\tilde{\mathcal{O}})$ such that $WW^* = E$

2000 *Mathematics Subject Classification.* 81T05 (primary); 81T40, 81R15, 46Lxx (secondary).

[1] A double cone in $\mathbb{M}_d$ is a region of the form $\{(x+V^+)\cap(y+V^-),\ y-x\in V^+\}$ where V^+ (resp. V^-) is the open forward (resp. backward) light cone.

- *Poincaré covariance*: there is a strongly continuous unitary representation V of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}_0$ such that, for every $L \in \mathcal{P}_+^\uparrow$ and every $\mathcal{O} \in \mathcal{K}$, there holds
$$V(L)\mathfrak{A}(\mathcal{O})V(L)^* = \mathfrak{A}(L\mathcal{O}) \tag{1.1}$$
- *spectrum condition*: the joint spectrum of the generators of the spacetime translations is contained in the closure $\overline{V}_+$ of V_+

among others, although one can consider more general localization regions and spacetimes, and modify the axioms accordingly.

For many purposes it is essential to strenghten locality to

- *Haag duality* : $\mathfrak{A}(\mathcal{O}')' = \mathfrak{A}(\mathcal{O}),\ \mathcal{O} \in \mathcal{K}$

i.e. $\mathfrak{A}^d = \mathfrak{A}$ where $\mathfrak{A}^d$ is the net defined by the *l.h.s.*, or at least to essential duality, i.e. $\mathfrak{A}^d = \mathfrak{A}^{dd}$ (Haag duality for $\mathfrak{A}^d$), which is equivalent to the locality of $\mathfrak{A}^d$. In the sequel when not otherwise stated we always consider observable nets satisfying irreducibility, property B and Haag duality. Once a net $\mathfrak{A}$ has been given, in the next step one has to look for a suitable class of its representations, i.e. the representations of the *quasi-local C^*-algebra* $(\cup_{\mathcal{O}\in\mathcal{K}}\mathfrak{A}(\mathcal{O}))^{-\|\cdot\|}$ still denoted $\mathfrak{A}$.[2] For theories on $\mathbb{M}_4$ describing short-range interactions these are generally chosen according to the DHR *selection criterion* [25]
$$\pi|_{\mathfrak{A}(\mathcal{O}')} \cong \pi_0|_{\mathfrak{A}(\mathcal{O}')} \qquad \forall \mathcal{O} \in \mathcal{K}. \tag{1.2}$$
The collection of all such representations together with their intertwiners form a W*-category $S(\pi_0)$, and the unitary equivalence classes of irreducible elements in $S(\pi_0)$ are the *superselection sectors*. The statistics of a sector is described by the statistical dimension d taking values in $\mathbb{N} \cup \{\infty\}$ and a sign $\pm$ expressing the Bose-Fermi alternative.

Usually in models one starts with Wightman fields (as operator-valued distributions), say $\phi(x), \psi(x), \dots,\ x \in \mathbb{M}_d$; after smearing them out with test functions they generate (through affiliation of closable unbounded operators to von Neumann algebras) a field net $\mathfrak{F}$ of bounded operators which in favourable situations is (graded) local, see e.g. [6]. For a brief survey on the important problem of the relationships between Wightman fields and local nets we refer to [12].

Given a field net $\mathfrak{F}$ acted upon by an internal symmetry group G, the observables $\mathfrak{A}$ are then obtained by a principle of gauge invariance.

It is interesting to know whether one can recover $\mathfrak{F}$ and G from $\mathfrak{A}$. From a conceptual point of view there is a very satisfactory answer in the setting of algebraic QFT provided by the Doplicher-Roberts reconstruction theorem, applicable for any spacetime dimension $D \geq 3$: given $\mathfrak{A}$ satisfying few relevant assumptions one can construct a canonical field net $\mathfrak{F}_\mathfrak{A}$ along with a compact group $G_\mathfrak{A}$ of net automorphisms ($\mathfrak{F}_\mathfrak{A}^{G_\mathfrak{A}} = \mathfrak{A}$) describing all suitably localized charges of the theory. In a sense to be made precise through a new abstract duality theory for compact groups [29, 30] $\mathfrak{F}_\mathfrak{A}$ is obtained as a crossed product construction of $\mathfrak{A}$ by the tensor category of its localized transportable endomorphisms with finite statistics, and $G_\mathfrak{A} = \mathrm{Aut}_\mathfrak{A}(\mathfrak{F}_\mathfrak{A})$. In the case where $\mathfrak{A} = \mathfrak{F}^G$ then it holds $\mathfrak{F} = \mathfrak{F}_\mathfrak{A}^H$ and $G = G_\mathfrak{A}/H$ [31], where H is a closed normal subgroup of $G_\mathfrak{A}$. In some cases, for instance if a Bosonic $\mathfrak{F}$ has no nontrivial sectors, H must be trivial and hence $\mathfrak{F} = \mathfrak{F}_\mathfrak{A}$.

---

[2] Sometimes it is convenient to consider an $\mathfrak{A}$ as *represented* on $\mathcal{H}_0$; then for theories on Minkowski spacetime one refers to the *vacuum* representation, denoted π_0.

However turning back to generic $\mathfrak{F}$ and G it has to be made clear the reason why one should set $\mathfrak{A} := \mathfrak{F}^G$; how is it possible to decide what is the best G? It seems definitely easier to answer only when some generating set of basic fields $\phi(x), \psi(x), ...$ is known together with their transformation properties.

The observable algebra is usually defined in an implicit way through a set of properties of self-consistency, still is absent a criterion telling how to get it in general situations. This also overlaps with the problem of uniqueness of the net of observables: different choices could lead to the same physical consequences such as mass spectrum, scattering theory etc., and it is by no means clear if there is some special criterion dictated by an intrinsic principle.

An important object in a theory is the observable energy-momentum tensor $\Theta^{\mu\nu}(x)$. Given $\Theta^{\mu\nu}(x)$ one might ask what kind of informations are available about $\mathfrak{A}$ or $\mathfrak{F}$. It is a workable idea to consider the "minimal" local net $\mathfrak{A}_0 \subset \mathfrak{A}$ generated by $\Theta^{\mu\nu}(x)$, (a kind of "core" for the observables: if anything had to be detected, these should be through local measurements of energy and momentum), and wonder under what circumstances it happens that $\mathfrak{A}_0 = \mathfrak{A}$. Then $\mathfrak{A}$ would be generated by operators with a clear physical meaning, so that the ambiguities in its definition disappear. Of course a similar situation occurs when $\Theta^{\mu\nu}(x)$ is replaced by other fields with a definite physical interpretation like Noether currents associated with more general symmetries. In this way one obtains nets of subalgebras (*subsystems*) of $\mathfrak{F}$ which are not defined as fixpoint nets under compact group actions.

As an example, in the case of a field net $\mathfrak{F}$ generated by a single massive scalar free field $\varphi(x)$ acting on the symmetric Fock space $\mathcal{H} = e^{L^2(H_m^+, \Omega_m)}$, for which

$$\Theta^{\mu\nu}(x) =: \partial^\mu\varphi(x)\partial^\nu\varphi(x) : -1/2g^{\mu\nu} : \partial^\rho\varphi(x)\partial_\rho\varphi(x) : +1/2m^2g^{\mu\nu} : \varphi(x)^2 :, \quad (1.3)$$

it has been recognized long time ago that $\mathfrak{A}_0$ is generated by the (smeared) Wick polynomial $: \varphi(x)^2 :$, moreover $\mathfrak{A}_0 = \mathfrak{F}^{\mathbb{Z}_2}$ i.e. the fixed point net under the symmetry $\varphi(x) \to -\varphi(x)$ implemented on $\mathcal{H}$ by $e^{i\pi\hat{N}}$ with $\hat{N}$ the number operator [49, 50]. Here one can define the observable net $\mathfrak{A}$ to be either $\mathfrak{F}$ itself or $\mathfrak{F}^{\mathbb{Z}_2}$ corresponding to $G = \{e\}$ and $G = \mathbb{Z}_2$ respectively; in both cases it holds $\mathfrak{F} = \mathfrak{F}_\mathfrak{A}$ since $\mathfrak{F}$ has trivial superselection structure, but only the second choice leads to $\mathfrak{A} = \mathfrak{A}_0$.

This said, it is worthwhile a deeper and abstract analysis of the above situation free from the peculiarities of the model under investigation and without assuming *a priori* the existence of underlying Wightman fields like $\Theta^{\mu\nu}(x)$. Such an analysis requires a substitute for the condition $\mathfrak{A}_0 = \mathfrak{A}$.
This program has been pursued in close connection with a formulation of Quantum Noether Theorem, building upon the nice results made accessible by an extensive use of the *split property*. The split property for $\mathfrak{A}$ (for double cones) states that there is an intermediate type I factor $\mathcal{N}$ lying between $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ for any given $\mathcal{O}_1 \subset\subset \mathcal{O}_2$, where $\subset\subset$ means that $\mathcal{O}_1 + \mathcal{O}_0 \subset \mathcal{O}_2$ for some double cone $\mathcal{O}_0$ centred on the origin. The split property is the algebraic substitute for the existence of a Lagrangian and it is satisfied in all the physically reasonable models (free fields, $P(\varphi)_2$, Y_2, even for free fields on globally hyperbolic curved spacetimes). It is implied by nuclearity, a condition which expresses a good thermodynamical behaviour of the theory. (For a discussion of these facts see [41] and the references therein.) It should be remarked that the validity of the split property for $\mathfrak{F}_\mathfrak{A}$ implies that for $\mathfrak{A}$, while the converse is known only in special situations.

In classical relativistic field theory the existence of conserved currents associated to every one-parameter group of symmetries of the Lagrangian is a consequence of Noether's theorem. For example the existence of the energy-momentum tensor follows from the invariance under spacetime translations. Although the presence of conserved currents related to symmetries is a general feature of models of quantum field theory, the understanding of this relation in this context is less satisfactory than in the classical case. If one starts from general assumptions as the Wightman axioms [58], the existence of such conserved currents is not a consequence of the existence of symmetries.

A new approach towards a quantum Noether's theorem has been proposed by Doplicher in [22] and developed by Doplicher, Longo and Buchholz in [26] and [8]. In these works it has been proved that, in a theory where the field net satisfies the split property, the global symmetries, including discrete symmetries, spacetime symmetries and supersymmetries, can be locally implemented by local unitary operators which are canonically constructed from the theory in question. If a part of the symmetries considered forms a connected Lie group, then the generators of the corresponding local implementations can be considered as the analogue of the zero component of Wightman conserved currents, smeared with appropriate test functions with support in the region of localization, thereby establishing the existence of a rigorous analogue of the current algebra known from particle physics.

It has been suggested by Doplicher in [22] that the canonical local generators constructed using the split property could be used to construct Wightman currents by an appropriate scaling limit in which the region of localization shrinks to a point. The success of this program would give us a complete quantum Noether's theorem and a general prescription to construct Wightman fields with a definite physical meaning, directly from the algebra of observables. See [36, 42] for related issues.

We now consider a Poincaré covariant field net, and let T denote the group of unitaries implementing spacetime translations. If Ψ_Λ is the *universal localizing map* attached to the standard and split inclusion of von Neumann algebras $\Lambda = (\mathfrak{F}_{\mathfrak{A}}(\mathcal{O}_\Lambda), \mathfrak{F}_{\mathfrak{A}}(\tilde{\mathcal{O}}_\Lambda), \Omega)$ [27, 8] associated to the pair of double cones $\mathcal{O}_\Lambda \subset\subset \tilde{\mathcal{O}}_\Lambda$ and to the vacuum vector Ω, then $T_\Lambda(x) := \Psi_\Lambda(T(x)) = e^{iP^\mu_\Lambda x_\mu}$ defines the canonical local implementations of spacetime translations (whose generators are the abstract analogue for the local energy-momentum tensor), namely a group of unitaries in $\mathfrak{F}_{\mathfrak{A}}(\tilde{\mathcal{O}}_\Lambda)$ implementing "small" spacetime translations on the algebra $\mathfrak{F}_{\mathfrak{A}}(\mathcal{O}_\Lambda)$. We introduce a new net $\mathcal{A}$ by setting

$$\mathcal{A}(\mathcal{O}) := \{T_\Lambda(x) \mid \tilde{\mathcal{O}}_\Lambda \subset \mathcal{O},\ x \in \mathbb{M}_4\}''. \tag{1.4}$$

Then by the properties of the maps Ψ_Λ one can easily check that $\mathcal{A}$ is a Poincaré covariant subsystem of $\mathfrak{F}_{\mathfrak{A}}$ (in fact of $\mathfrak{A}$).[3] Similarly given a compact group G of unbroken internal symmetries of $\mathfrak{F}_{\mathfrak{A}}$ one could define a local net $\mathcal{A}_G$ generated by $\mathcal{A}$ and the local operators $\Psi_\Lambda(Z)$, $Z \in G' \cap G''$ as a local shadow of the global superselection structure [18], but for the time being we will restrict our attention only to $\mathcal{A}$. Actually a universal localizing map can be defined from every net with the split property, e.g. $\mathfrak{A}$ itself or more generally $\mathfrak{F}_{\mathfrak{A}}^H$ with H a nontrivial compact subgroup of $G_{\mathfrak{A}}$. For these examples one obtains local operators implementing

[3] As a matter of fact the localization of the whole Poincaré group essentially does not produce anything different from $\mathcal{A}$ (up to the passage to the dual nets), see Sect. 3.

symmetry transformations on $\mathfrak{A}$ but not necessarily on $\mathfrak{F}_{\mathfrak{A}}$. For this reason we disregard this possibility in the definition of $\mathcal{A}$, cf. [8].

Proceeding further, there are two possible directions:

(i) investigate the main properties of the subsystem $\mathcal{A} \subset \mathfrak{A}$ (cf. [48] for a more abstract approach)

(ii) try to define in a suitable limit of operators in $\mathcal{A}$ a pointlike localized energy-momentum tensor $\Theta^{\mu\nu}(x)$, generating a local net $\mathcal{A}_W \subset \mathcal{A}$.

There are natural related problems: find a characterization of the cases $\mathcal{A} = \mathfrak{A}$ and $\mathcal{A}_W = \mathfrak{A}$ (whenever the solution to point (*ii*) above has been found) respectively.

Actually the equality of nets $\mathcal{A} = \mathfrak{A}$ is *a priori* less interesting, mainly because the physical meaning of T_Λ is prejudged by the possibility of perturbing it with operators in $\mathfrak{F}_{\mathfrak{A}}(\mathcal{O})' \cap \mathfrak{F}_{\mathfrak{A}}(\tilde{\mathcal{O}})$ (cf. [26, 34]) so that the second equality would be desiderable. However it always holds

$$\mathcal{A}_W \subset \mathcal{A} \subset \mathfrak{A}$$

therefore $\mathcal{A}_W = \mathfrak{A}$ implies $\mathcal{A} = \mathfrak{A}$.

A general solution to the problem (*ii*) above is not yet known. There are some indications that in some cases it is not possible to get $\mathcal{A}_W$ (cf. [16]), still its "algebraic counterpart" $\mathcal{A}$ is always present and thus provides a useful substitute. In fact, as we shall see, under reasonable hypotheses we have $\mathcal{A} = \mathcal{A}_W$ provided the latter is defined and both nets satisfy Haag duality. Thus in any case the analysis of the net $\mathcal{A}$ is *a posteriori* justified.

In [18] it is shown that $\mathcal{A}$ is irreducible in the field net, namely $\mathcal{A}' \cap \mathfrak{F}_{\mathfrak{A}} = \mathbb{C}$; it also holds $\mathcal{A}(\mathcal{O}')' \cap \mathfrak{F}_{\mathfrak{A}} = \mathfrak{F}_{\mathfrak{A}}(\mathcal{O})$. Furthermore, as a consequence of the covariance properties of the universal localizing maps [8] and the fact that, due to the split property, the group $G_{\max}$ of all the (unbroken) internal symmetries of $\mathfrak{F}_{\mathfrak{A}}$ is compact in the strong operator topology and commutes with the Poincaré transformations [27], it is possible to see that $\mathcal{A} \subset \mathfrak{F}_{\mathfrak{A}}^{G_{\max}}$, so that $\mathcal{A} = \mathfrak{A}$ implies $G = G_{\max}$. Actually in presence of broken symmetries it is clear that $\mathcal{A} \subset \mathfrak{F}_{\mathfrak{A}^d}^{G_{\max}} \subset \mathfrak{A}^d$ but under reasonable hypotheses it can be shown that $\mathcal{A} \not\subset \mathfrak{A}$. The trials for a better understanding of the potential equality

$$\mathcal{A} = \mathfrak{F}_{\mathfrak{A}}^{G_{\max}} \tag{1.5}$$

which, together with the necessary condition $G = G_{\max}$, implies the equality $\mathcal{A} = \mathfrak{A}$ motivated the general classification results for subsystems expounded in Sect. 3. We'll see that the equality 1.5 is verified under quite general conditions providing a satisfactory understanding of the 4D situation which comprises a computation of the dual net of $\mathcal{A}$ in concrete models.

Similar problems can be addressed in 2D CFT. Here the net $\mathcal{A}$ can still be introduced by a similar procedure (although there is no analog of the canonical field net in this context) at the price of some arbitrariness in its definition. Classification results allowing to identify $\mathcal{A}$ are made available for some specific models but no general result is known. On the positive side, the reconstruction of $\mathcal{A}_W$ from $\mathcal{A}$ is possible for a larger class of models. We report on this matter in Sect. 5.

2 Subsystems and Superselection Structure

In order to continue there is a supply of results of general nature about subsystems. Since at this stage this causes no further complications we can work them out in rather wide generality, and provide a thorough analysis of the situation where we are given an arbitrary pair system-subsystem (mostly in 4D).

Along the way we will provide answers to some puzzling questions like what happens by iterating the DR construction, namely what is the field net of (the Bose part of) the field net, and, more generally, what are the superselection sectors of certain subsystems $\mathfrak{B} \subset \mathfrak{F}_{\mathfrak{A}}$ (e.g. intermediate nets $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{F}_{\mathfrak{A}}$).

Although this should be intuitively clear from the previous discussion we start by formally defining what is meant by a general *subsystem* (also called *subtheory*, *subnet* in the literature). In one form or another they already appear e.g. in [49, 50, 24, 63, 60, 2, 5, 21, 51, 18, 20, 17, 14, 15, 52, 13, 37, 54, 64, 65, 66, 4, 46, 55].

Let us consider an isotonous net $\mathfrak{F}$ of von Neumann algebras over a suitable family $\mathcal{Q}$ of (open) subsets of some "spacetime" manifold $\mathcal{M}$ acting on a Hilbert space $\mathcal{H} = \mathcal{H}_{\mathfrak{F}}$. Usually one requires $\mathcal{Q}$ to be a base for the topology on $\mathcal{M}$. Note that however $\mathcal{Q}$ in general might not be directed under inclusion, thus we use the word "net" in a wider sense than usual precosheaf being a more precise expression, see e.g. [7]. For instance $\mathcal{M}$ and $\mathcal{Q}$ could be the Minkowski spacetime and the directed set $\mathcal{K}$ of open double cones, a (globally hyperbolic) 4D spacetime and the set of regular diamonds [40], $\mathbb{R}$ and the set of all bounded open intervals, S^1 and the set of all nondense open intervals, etc. One could also consider more general situations where $\mathcal{M}$ is not given and $\mathcal{Q}$ is some abstract partially ordered index set, including the extreme case of $\{1, 2\}$ with the natural order giving rise to an inclusion of von Neumann algebras (e.g. a subfactor), see [3, 51, 57]. We will not further consider this possibility. Given a net $\mathfrak{F}$ over $\mathcal{Q}$ and an open set $\mathcal{S} \subset \mathcal{M}$ one can define the von Neumann algebra $\mathfrak{F}(\mathcal{S})$ by additivity, namely $\mathfrak{F}(\mathcal{S}) = \bigvee_{\mathcal{O}\in\mathcal{Q},\mathcal{O}\subset\mathcal{S}} \mathfrak{F}(\mathcal{O})$, and thus extend the net to all open sets.

Definition 2.1 A *subsystem* $\mathfrak{B}$ of $\mathfrak{F}$ consists in an isotonous net of von Neumann algebras over $\mathcal{Q}$, acting on $\mathcal{H}$, such that

$$\mathfrak{B}(\mathcal{O}) \subset \mathfrak{F}(\mathcal{O}), \qquad \forall \mathcal{O} \in \mathcal{Q}. \tag{2.1}$$

It readily follows that $\mathfrak{B}(\mathcal{S}) \subset \mathfrak{F}(\mathcal{S})$ for all open sets $\mathcal{S}$. We will often use the notation $\mathfrak{B} \subset \mathfrak{F}$ to denote the fact $\mathfrak{B}$ is a subsystem of $\mathfrak{F}$. If all the algebras $\mathfrak{B}(\mathcal{O})$ and $\mathfrak{F}(\mathcal{O})$ are (infinite) factors the pair $\mathfrak{B} \subset \mathfrak{F}$ is also called a *net of subfactors* [51]. If $\mathcal{M}$ has a causal structure one can assume the net $\mathfrak{F}$ to be local. In this case each susystem would be automatically local.

For simplicity in the sequel of this section we specialize to the case $\mathcal{M} = \mathbb{M}_4$ although some of the results below hold in more general situations.

Let $\mathfrak{F}$ be a net over the double cones of $\mathbb{M}_4$. Assume that the Hilbert space $\mathcal{H}_{\mathfrak{F}}$ contains a preferred unit vector Ω, the *vacuum* vector, which is cyclic for $\mathfrak{F}(\mathbb{M}_4)$. Given a subsystem $\mathfrak{B} \subset \mathfrak{F}$ we set $\mathcal{H}_{\mathfrak{B}}$ (the vacuum Hilbert space of $\mathfrak{B}$) to be the closure of $\mathfrak{B}(\mathbb{M}_4)\Omega$ and call $E_{\mathfrak{B}}$ the corresponding orthogonal projection. We consider the restriction $\hat{\mathfrak{B}}$ of $\mathfrak{B}$ to the subspace $\mathcal{H}_{\mathfrak{B}}$, and for a given $B \in \mathfrak{B}(\mathcal{O})$ we write $\hat{B} \in \mathrm{B}(\mathcal{H}_{\mathfrak{B}})$ for $E_{\mathfrak{B}} B E_{\mathfrak{B}}|_{\mathcal{H}_{\mathfrak{B}}}$. Then $B \to \hat{B}$ gives the "vacuum representation" for $\mathfrak{B}$. Usually we consider $\mathfrak{F}$ to be irreducible on $\mathcal{H}_{\mathfrak{F}}$, then in some interesting cases $\hat{\mathfrak{B}}$ will be irreducible as well. Sometimes we write $\mathfrak{B}$ instead of $\hat{\mathfrak{B}}$ since typically these two nets are isomorphic and use $\hat{\mathfrak{B}}$ only when spatial properties of the vacuum representation are involved. We also remark that if $\mathfrak{C} \subset \mathfrak{F}$ is another subsystem such that in addition $\mathfrak{C}(\mathcal{O}) \subset \mathfrak{B}(\mathcal{O})$ for all $\mathcal{O} \in \mathcal{K}$ then in obvious way (the restriction of) $\mathfrak{C}$ can be considered as a subsystem of $\hat{\mathfrak{B}}$, still we simply write $\mathfrak{C} \subset \mathfrak{B}$ when no confusion arises.

Definition 2.2 $\mathfrak{B} \subset \mathfrak{F}$ is called a *Haag-dual* subsystem if the net $\hat{\mathfrak{B}}$ satisfies Haag duality on $\mathcal{H}_{\mathfrak{B}}$.

Besides its usefulness this property may prevent the occurrence of certain pathologies like $\mathfrak{B}(\mathcal{O}) = \mathbb{C}I$ for some $\mathcal{O}$.

The global isometry group of Minkowski spacetime is the Poincaré group, let $\mathcal{P}_+^\uparrow$ be its connected component of the identity and $\widetilde{\mathcal{P}_+^\uparrow}$ the universal covering of $\mathcal{P}_+^\uparrow$.

Assume now that $\mathfrak{F}$ is Poincaré covariant, i.e. that there is a strongly continuous representation V of $\widetilde{\mathcal{P}_+^\uparrow}$ on $\mathcal{H}_{\mathfrak{F}}$, leaving Ω invariant, such that

$$V(L)\mathfrak{F}(\mathcal{O})V(L)^* = \mathfrak{F}(L\mathcal{O}), \qquad \forall L \in \widetilde{\mathcal{P}_+^\uparrow}. \tag{2.2}$$

We denote α_L the induced action of L on $\mathfrak{F}$. Note that if $\mathfrak{F}$ is Bosonic (local) and the usual spin-statistics connection holds [58, 25, 39, 47] then the representation V factorize through a representation of $\mathcal{P}_+^\uparrow$ via the natural projection map, thus in this case we can replace $\widetilde{\mathcal{P}_+^\uparrow}$ with $\mathcal{P}_+^\uparrow$.

Later on the following definition plays a central role.

Definition 2.3 A Poincaré covariant subsystem of $\mathfrak{F}$ is a subsystem $\mathfrak{B} \subset \mathfrak{F}$ for which it holds

$$V(L)\mathfrak{B}(\mathcal{O})V(L)^* = \mathfrak{B}(L\mathcal{O}), \qquad \forall L \in \widetilde{\mathcal{P}_+^\uparrow}. \tag{2.3}$$

Then of course the action of (the universal covering of) the Poincaré group restricts to an action on $\mathfrak{B}$, still denoted α. When there is no danger of confusion we just say that $\mathfrak{B}$ is a covariant subsystem. Similarly one can define conformal covariant subsystems in the case where $\mathfrak{F}$ is a conformally covariant net (taking some care in the choice of the set $\mathcal{Q}$, cf. [7]).

There are many examples of subsystems of quite different type, we list some of them for theories on $\mathbb{M}_4$ without any claim to be exhaustive: for instance an observable net $\mathfrak{A}$ (not necessarily satisfying Haag duality) is a subsystem of its outer regularized net $\mathfrak{A}^r$ defined by $\mathfrak{A}^r(\mathcal{O}) := \cap_{\mathcal{K}\ni\mathcal{O}_1\supset\supset\mathcal{O}}\mathfrak{A}(\mathcal{O}_1)$, moreover $\mathfrak{A}$, being local, is a subsystem of $\mathfrak{A}^d$ as well and also of its canonical field net $\mathfrak{F}_{\mathfrak{A}^d}$ whenever $\mathfrak{A}^d$ is local. Actually if $\mathcal{G} \supset G_{\mathfrak{A}^d}$ is the group of all (broken or unbroken) internal symmetries one has

$$\mathfrak{A} \subset \mathfrak{F}_{\mathfrak{A}^d}^{\mathcal{G}} \subset \mathfrak{A}^d = \mathfrak{F}_{\mathfrak{A}^d}^{G_{\mathfrak{A}^d}} \subset \mathfrak{F}_{\mathfrak{A}^d},$$

see [9], and furthermore we have already mentioned the Noether subsystem $\mathcal{A} \subset \mathfrak{A}^d$, sitting in generic position with respect to $\mathfrak{A}$. One can also consider subsystems of the form $\mathfrak{F}_{\mathfrak{A}}^M \subset \mathfrak{F}_{\mathfrak{A}}^N$ where N is a (not necessarily normal) subgroup of $M \subset G_{\max}$, and in the next section we'll meet relative commutant (coset) subsystems. If an irreducible net is generated by a set of Wightman fields then each (sufficiently regular) field in the corresponding Borchers class, like a Wick polynomial in the case of free field theories or a linear combination of the originally given fields, gives rise to a subsystem. Given a single selfadjoint local operator $X \in \mathfrak{A}(\mathcal{O}_0)$ one can naively define a subsystem $\mathfrak{A}_X \subset \mathfrak{A}$ by $\mathfrak{A}_X(\mathcal{O}) = \{V(L)XV(L)^*;\ L \in \mathcal{P}_+^\uparrow, L\mathcal{O}_0 \subset \mathcal{O}\}''$, likewise one defines $\mathfrak{A}_\mathcal{X} \subset \mathfrak{A}$ for any generic family $\mathcal{X} = \{X_1, X_2, \ldots\}$ of selfadjoint local operators, cf. [63]. In a similar vein if $\mathcal{R}$ is a von Neumann algebra globally invariant under the adjoint action of V which contains some local operator (e.g. $\mathcal{R}_X = \{V(L)XV(L)^*,\ L \in \mathcal{P}_+^\uparrow\}''$), one can define $\tilde{\mathfrak{A}}_\mathcal{R}$ via $\tilde{\mathfrak{A}}_\mathcal{R}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}) \cap \mathcal{R}$ (of course $\mathfrak{A}_X \subset \tilde{\mathfrak{A}}_{\mathcal{R}_X}$). In some situation it is important to consider subsystems that are only covariant under spacetime translations [11].

In order to illustrate some basic ideas and techniques in the sequel we briefly report on the recent paper [20] motivated by the task of studying some functorial properties of the embedding of systems, e.g. the subsystem $\mathfrak{A} \subset \mathfrak{B}$ clearly induces an embedding $\mathfrak{A}^r \subset \mathfrak{B}^r$ of the outer regularized nets, but there are much more interesting cases dealing with the embedding of the dual nets or of the canonical field nets. Hereafter a subsystem $\mathfrak{A} \subset \mathfrak{B}$ is intended as compatible normal inclusions $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{B}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ with $\mathfrak{A}, \mathfrak{B}$ (possibly local, or at least relatively local) nets, each acting (faithfully and) irreducibly on its vacuum Hilbert space. The ultimate goal of this analysis is to clarify the relationships between the superselection structures of a system and that of a subsystem, both living on the Minkowski spacetime.[4] As far as we are concerned with Haag dual nets it is convenient to work with endomorphisms instead of representations, in fact if π is a DHR representation of $\mathfrak{A}$ then $\pi \cong \pi_0 \circ \rho$, where ρ is an endomorphism of $\mathfrak{A}$ which is *localized* in some $\mathcal{O}$, namely $\rho(A) = A$ for $A \in \cup_{\mathcal{O}_1 \subset \mathcal{O}'} \mathfrak{A}(\mathcal{O}_1)$, and *transportable*, i.e. inner equivalent to endomorphisms localized in any double cone. The family of all such endomorphisms and their intertwiners gives raise to a category equivalent to $S(\pi_0)$ with the bonus of having an obvious monoidal structure.

Theorem 2.4 *(*[19, 20]*) Given an inclusion of local nets $\mathfrak{A} \subset \mathfrak{B}$, both satisfying duality on their own vacuum Hilbert space, there are monoidal embedding functors $\mathcal{T}(\mathfrak{A}) \to \mathcal{T}(\mathfrak{B})$ and, by restriction, $\mathcal{T}_f(\mathfrak{A}) \to \mathcal{T}_f(\mathfrak{B})$ acting identically on the arrows. More precisely there is a localization-preserving map $\rho \in \Delta(\mathfrak{A}) \mapsto \hat{\rho} \in \Delta(\mathfrak{B})$ such that $\hat{\rho}$ is an extension of ρ and furthermore*

i) $\hat{\iota}_\mathfrak{A} = \iota_\mathfrak{B}$,
ii) $\widehat{\rho_1\rho_2} = \hat{\rho}_1\hat{\rho}_2$,
iii) $(\oplus_i\rho_i)\hat{} = \oplus_i\hat{\rho}_i$,
iv) $\hat{\bar{\rho}} = \bar{\hat{\rho}}$, $\quad \rho \in \Delta_f(\mathfrak{A})$,
v) $\epsilon(\hat{\rho}, \hat{\sigma}) = \epsilon(\rho, \sigma)$,
vi) $d(\hat{\rho}) = d(\rho)$,
vii) $T \in (\rho_1, \rho_2) \Rightarrow T \in (\hat{\rho}_1, \hat{\rho}_2)$,
viii) $\phi_{\hat{\rho}}|_\mathfrak{A} = \phi_\rho$
ix) *If $\mathfrak{B}$ is Poincaré covariant and $\mathfrak{A}$ is a covariant subsystem of $\mathfrak{B}$, then*

$$\widehat{\rho_L} = (\hat{\rho})_L, \quad L \in \widetilde{\mathcal{P}}_+^\uparrow\ .$$

Here $\Delta(\mathfrak{A})$ (resp. $\Delta_f(\mathfrak{A})$) is the semigroup of all localized transportable morphisms (resp. with finite statistics) of $\mathfrak{A}$, $\mathcal{T}(\mathfrak{A})$ (resp. $\mathcal{T}_f(\mathfrak{A})$) is the monoidal category with objects the elements of $\Delta(\mathfrak{A})$ (resp. of $\Delta_f(\mathfrak{A})$) and arrows their intertwiners in $\mathfrak{A}$,

[4] A closely related issue is the classification of subsystems; a possible strategy is to classify subsystems satisfying duality on their vacuum Hilbert space first (as treated in the next section), and then try to relax duality afterwards.

$\iota_{\mathfrak{A}}$ *denotes the monoidal unit of* $\mathcal{T}(\mathfrak{A})$*,* $\epsilon(\rho,\sigma)$ *are the statistical operators,* $d(\cdot)$ *is the dimension function,* ϕ_ρ *is the standard left inverse of* ρ *and* $\rho_L = \alpha_L \circ \rho \circ \alpha_L^{-1}$ *(*$\Delta(\mathfrak{B})$ *etc. are analogously defined).*

Using the previous properties it is not difficult to prove the following uniqueness result for the extensions.

Proposition 2.5 *Given* $\mathfrak{A} \subset \mathfrak{B}$ *as above, if* $\tilde{\rho}$ *is a localized endomorphism of* $\mathfrak{B}$ *transportable in* $\mathfrak{A}$ *then* $\rho := \tilde{\rho}|_{\mathfrak{A}}$ *is a localized transportable endomorphism of* $\mathfrak{A}$ *such that* $\hat{\rho} = \tilde{\rho}$*.*

Remark 2.6 (i) In general the irreducibility is not preserved by the extension.
(ii) The extension procedure for endomorphisms neither need the finiteness of $\mathrm{Ind}(\mathfrak{A} \subset \mathfrak{B})$ nor the local algebras to be factors. Actually Haag duality can be relaxed to some extent, relative duality for both $\mathfrak{A}$ and $\mathfrak{B}$ being sufficient: the functor above corresponds to the embedding $Z^1(\mathfrak{A}) \subset Z^1(\mathfrak{B})$ (1-cocycles in net cohomology).
(iii) For nets of subfactors on the real line similar results hold true (homomorphism properties for α-induction [4]); however due to braid group symmetry there are two possible extensions, a fact which is relevant for the search of modular invariants, and the understanding of the occurrence of soliton sectors.

Some application of further results concerning conditional expectations implemented by projections gives some useful criteria for the existence of conditional expectations onto subsystems, and the embedding of the corresponding dual nets. As a sample, we quote a particular case; recall that a wedge region $\mathcal{W}$ is the image of $\{x \in \mathbb{M}_4 \ : \ x^1 > |x^0|\}$ under a Poincaré transformation.

Proposition 2.7 *(*[20]*) Let* $\mathfrak{B}$ *be a net satisfying Haag duality and the Reeh-Schlieder property with respect to the vacuum vector* $\Omega \in \mathcal{H}_{\mathfrak{B}}$*, and consider a subsystem* $\mathfrak{A} \subset \mathfrak{B}$*, with* $\mathfrak{A}$ *satisfying, on its vacuum Hilbert space, wedge duality (*$\hat{\mathfrak{A}}(\mathcal{W})' = \hat{\mathfrak{A}}(\mathcal{W}')$ *for every wedge region* $\mathcal{W}$*) but not Haag duality. Then there is an embedding* $\nu : \mathfrak{A}^d \to \mathfrak{B}$ *such that* $\nu(A)\Omega = A\Omega,\ A \in \mathfrak{A}^d$ *and a (unique) conditional expectation* $m : \mathfrak{B} \to \nu(\mathfrak{A}^d)$ *such that* $m(B)E = EBE,\ B \in \mathfrak{B}$*, where* E *is the orthogonal projection from* $\mathcal{H}_{\mathfrak{B}}$ *onto* $\mathcal{H}_{\mathfrak{A}}$*. Moreover it holds* $\nu(\mathfrak{A}^d)(\mathcal{O}) = \mathfrak{B}(\mathcal{O}) \cap \{E\}'$ *for every* $\mathcal{O} \in \mathcal{K}$*.*

It is convenient to recall in more detail some basic notions and results in [31].

Definition 2.8 A *field system with gauge symmetry* for $\mathfrak{A}$ is a triple $\{\tilde{\pi}, G, \mathfrak{F}\}$ where:

- $\tilde{\pi}$ is a representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H} \supset \mathcal{H}_0$ containing π_0 as a subrepresentation
- G is a strongly compact group of unitaries on $\mathcal{H}$ leaving $\mathcal{H}_0$ pointwise fixed
- $\mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$ is a net of von Neumann algebras on $\mathcal{H}$ such that
  $\alpha)$ $g \in G$ induces automorphisms of every $\mathfrak{F}(\mathcal{O})$, and $\mathfrak{F}(\mathcal{O})^G = \tilde{\pi}(\mathfrak{A}(\mathcal{O}))$
  $\beta)$ $\mathfrak{F}$ is irreducible
  $\gamma)$ $\mathcal{H}_0$ is cyclic for $\mathfrak{F}(\mathcal{O})$
  $\delta)$ the fields are local relative to the observables.

A triple as above is *normal* if there is $k \in \mathcal{Z}(G)$ with $k^2 = e$ such that $\mathfrak{F}$ obeys graded local commutativity for the $\mathbb{Z}_2$-grading defined by k, and *complete* if each equivalence class of irreducible DHR representations of $\mathfrak{A}$ [5] with finite statistics is realized in $\tilde{\pi}$. The following theorem is of fundamental importance.

Theorem 2.9 *([31]) Assume irreducibility, property B and Haag duality for $\mathfrak{A}$, then there exists a (unique up to isomorphism, thus canonical) complete normal field system with gauge symmetry $(\tilde{\pi}_{\mathfrak{A}}, \mathfrak{F}_{\mathfrak{A}}, G_{\mathfrak{A}})$.*[6]

When $\mathfrak{A}$ is also Poincaré covariant with spectrum condition there is a corresponding existence result for a canonical Poincaré covariant field net $\mathfrak{F}_{\mathfrak{A},c}$ which is complete w.r.t. the covariant DHR representations of $\mathfrak{A}$ with finite statistics.

Theorem 2.10 *Let $\{\tilde{\pi}, G, \mathfrak{F}\}$ be a field system with gauge symmetry, then*

- $\tilde{\pi}(\mathfrak{A})' \cap \mathfrak{F} = \mathbb{C}$
- $\gamma \in \mathrm{Aut}(\mathfrak{F})$ *is of the form* $\mathrm{Ad}g$ *for some* $g \in G$ *iff* $\gamma|_{\tilde{\pi}(\mathfrak{A})} = \mathrm{id}$*, i.e.* $G = \mathrm{Aut}_{\tilde{\pi}(\mathfrak{A})}(\mathfrak{F})$
- $\tilde{\pi}(\mathfrak{A})' = G''$, $\tilde{\pi} = \oplus_{\xi \in \hat{G}} d(\xi)\pi_\xi$ *and accordingly* $\mathcal{H} \cong \oplus_{\xi \in \hat{G}} \mathcal{H}_\xi \otimes \mathbb{C}^{d(\xi)}$*, where* π_ξ *are inequivalent irreducible DHR representations of* $\mathfrak{A}$ *with parastatistics of finite order* $d(\xi)$ *equal to the dimension of the corresponding irreducible representation of* G
- *(if the system is normal)* $\mathfrak{F}$ *satisfies twisted duality, i.e. setting* $\mathfrak{F}^t(\mathcal{O}) = V\mathfrak{F}(\mathcal{O})V^*$, $V = (I + ik)/\sqrt{2}$ *it holds* $\mathfrak{F}^t(\mathcal{O}) = \mathfrak{F}(\mathcal{O}')'$.

Note that the Bosonic net $\mathfrak{F}^b_{\mathfrak{A}}$ of even elements under the grading is a local net, thus it is meaningful to speak about its superselection sectors; moreover if $\mathfrak{A}$ has no Fermionic sector then the grading is trivial and hence $\mathfrak{F}_{\mathfrak{A}}$ is local.

Next important result shows that an inclusion of local nets induces an inclusion of the corresponding canonical field nets [20].

Theorem 2.11 *Let $\mathfrak{A} \subset \mathfrak{B}$ be an inclusion of nets both satisfying property B and Haag duality, then there is a canonical embedding $\mathfrak{F}_{\mathfrak{A}} \subset \mathfrak{F}_{\mathfrak{B}}$. Also, if $\mathfrak{B}$ is Poincaré covariant with spectrum condition and $\mathfrak{A}$ is a covariant subsystem of $\mathfrak{B}$, the same conclusion holds true for the corresponding canonical covariant field nets, namely $\mathfrak{F}_{\mathfrak{A},c} \subset \mathfrak{F}_{\mathfrak{B},c}$, and furthermore $\mathfrak{F}_{\mathfrak{A},c}$ is a covariant subsystem of $\mathfrak{F}_{\mathfrak{B},c}$. Finally if $\mathfrak{F}_{\mathfrak{B}} = \mathfrak{F}_{\mathfrak{B},c}$ then $\mathfrak{F}_{\mathfrak{A}} = \mathfrak{F}_{\mathfrak{A},c}$.*

Proof (sketch) The copy of $\mathfrak{F}_{\mathfrak{A}}(\mathcal{O})$ inside $\mathfrak{F}_{\mathfrak{B}}(\mathcal{O})$ is generated by the Hilbert spaces $H_{\hat{\rho}} = \{\psi \in \mathfrak{F}_{\mathfrak{B}} \mid \psi B = \hat{\rho}(B)\psi,\ B \in \mathfrak{B}\}$ in $\mathfrak{F}_{\mathfrak{B}}$ implementing the canonical extensions to $\mathfrak{B}$ of the transportable localized morphisms ρ of $\mathfrak{A}$ with finite statistics which are localized in $\mathcal{O}$, see [20].
The second statement basically follows from Theorem 2.4, properties $vii), ix)$: if $\rho \in \Delta(\mathfrak{A})$ is covariant then $\hat{\rho} \in \Delta(\mathfrak{B})$ is also covariant, and for a given $\psi \in H_{\hat{\rho}}$ it holds $V(L)\psi V(L)^* \in H_{(\hat{\rho})_L} \subset \mathfrak{F}_{\mathfrak{A},c}(L\mathcal{O}),\ L \in \widetilde{\mathcal{P}}^\uparrow_+$ by a straightforward calculation. Similarly one shows that if $\mathfrak{F}_{\mathfrak{B}} = \mathfrak{F}_{\mathfrak{B},c}$ then it contains $\mathfrak{F}_{\mathfrak{A}}$ as a covariant subsystem. But then, arguing as in [27, Theorem 10.4], one deduces that the actions of $G_{\mathfrak{A}}$ and of $\widetilde{\mathcal{P}}^\uparrow_+$ on $\mathfrak{F}_{\mathfrak{A}}$ commute. From this it is easy to show that all the sectors of $\mathfrak{A}$ with finite statistics are in fact covariant, cf. [24], and thus the conclusion follows. □

[5] Here the charges are localizable in double cones of Minkowski spacetime. There is also a version for topological charges, i.e. those localizable in space-like cones.

[6] To make the notation easier sometimes we drop $\tilde{\pi}$ and $\tilde{\pi}_{\mathfrak{A}}$ leaving the reader the task of deciding from the context the right Hilbert space on which $\mathfrak{A}$ acts.

Notice that the canonical representation $V = V_{\mathfrak{B}}$ of the universal covering of the Poincaré group making $\mathfrak{F}_{\mathfrak{B},c}$ covariant restricts to the corresponding representation for $\mathfrak{F}_{\mathfrak{A},c}$.

Actually the functor $\hat{\cdot}$ (in restriction to $\mathcal{T}_f(\mathfrak{A})$) is induced by a homomorphism $h : G_{\mathfrak{B}} \to G_{\mathfrak{A}}$, cf. [30, Theorem 6.10]. Let us define $N := \mathrm{Ker}(h)$, $M = h(G_{\mathfrak{B}})$.

Corollary 2.12 *(*[20]*) With the above notation $\mathfrak{F}_{\mathfrak{A}}$ is $G_{\mathfrak{B}}$-stable, h is given by restriction, and $\mathfrak{F}_{\mathfrak{A}} \vee \mathfrak{B} = \mathfrak{F}_{\mathfrak{B}}^N$, $\mathfrak{F}_{\mathfrak{A}} \cap \mathfrak{B} = \mathfrak{F}_{\mathfrak{A}}^M$.*

In particular if $\mathfrak{F}_{\mathfrak{A}} = \mathfrak{F}_{\mathfrak{B}}$ then h is just the inclusion $G_{\mathfrak{B}} \subset G_{\mathfrak{A}}$ and the extension procedure for endomorphisms with finite statistics corresponds (via the bijection between superselection sectors with finite statistics and classes of irreducible representations of the gauge group) to the restriction of group representations (while the restriction of DR-representations corresponds to the induction of group representations).

By appealing to Theorem 2.4 it is easy to show that the field net of $\mathfrak{F}_{\mathfrak{A}}^K$ is $\mathfrak{F}_{\mathfrak{A}}$, for every compact $K \supset G_{\mathfrak{A}}$ [20]. Next proposition includes a classification result for intermediate nets.

Proposition 2.13 *(*[20]*) Consider the situation $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{F}_{\mathfrak{A}}$ where $\mathfrak{B}$ is an intermediate net satisfying Haag duality, then*

(a) *$\mathfrak{B} = \mathfrak{F}_{\mathfrak{A}}^H$ for some closed subgroup $H \subset G$ ("Galois correspondence", cf. below and* [21, 43, 33]*),*

(b) *H is normal in G if and only if $\mathfrak{B}$ is G-stable if and only if $\mathfrak{B}$ is generated by Hilbert spaces inducing elements of $\Delta_f(\mathfrak{A})$,*

(c) *if a local $\mathfrak{F}_{\mathfrak{A}}$ has no sectors, then $\mathfrak{F}_{\mathfrak{B}} = \mathfrak{F}_{\mathfrak{A}}$.*

Proof We only show (c): $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{F}_{\mathfrak{A}} \subset \mathfrak{F}_{\mathfrak{B}} \subset \mathfrak{F}_{\mathfrak{F}_{\mathfrak{A}}}$ □

There are two known sets of structural hypotheses [32, 56], allowing one to conclude that a Bosonic net in 4D has no sectors (cf. [53, 46] for 2D theories). For instance, the argument in [56] ("quasi–triviality of the 1–cohomology of a $\mathfrak{F}$") requires (a weak form of) the split property for a field algebra, but not its completeness (which follows at once). However it needs another assumption which seems much harder to prove, namely that

$$\bigcap_{\partial p \,=\, \partial b} \mathfrak{F}(|p|) \;=\; \mathfrak{F}(|\partial_0 b|) \vee \mathfrak{F}(|\partial_1 b|),$$

for each $b \in \Sigma_1(\mathcal{K})$, where we use the symbol p to denote a generic *path* in Roberts' cohomology.

Using the above results it is possible to show that a (Bosonic) canonical field net is "DR self-complete", namely there is no way it can be embedded in a bigger system by performing the DR procedure simply because it has no sectors. We say that $\mathfrak{A}$ is *rational* if it has only finitely many superselection sectors with finite statistics (still it could have infinitely many sector with infinite statistics!).

Theorem 2.14 *(*[19, 52]*) If $\mathfrak{A}$ is rational then $\mathfrak{F}_{\mathfrak{A}}^b$ has no non-trivial bosonic superselection sectors with finite statistics. In other words $\mathfrak{F}_{\mathfrak{F}_{\mathfrak{A}}^b} = \mathfrak{F}_{\mathfrak{A}}$, and the gauge group of $\mathfrak{F}_{\mathfrak{A}}^b$ is generated by the element $k \in \mathcal{Z}(G_{\mathfrak{A}})$ giving the grading.*

There are 3 proofs available by now that we sketch below (for the Bosonic case):

a) uses an extension procedure for automorphisms [10], and it is interesting for a problem about topologies: since $\mathfrak{A} \overset{G_{\mathfrak{A}}}{\subset} \mathfrak{F}_{\mathfrak{A}} \overset{H}{\subset} \mathfrak{F}_{\mathfrak{F}_{\mathfrak{A}}}$, setting $\tilde{G} :=$ the group of extensions of $G_{\mathfrak{A}}$ (so that $H \subset \tilde{G}$ is the fiber over the unit of $G_{\mathfrak{A}}$), then there is a short exact sequence $1 \to H \overset{i}{\to} \tilde{G} \overset{r}{\to} G_{\mathfrak{A}} \to 1$ but $\tilde{G}$ has to be compact (for a net $\mathfrak{A}$ with the split property this would be automatic if the implication "split for $\mathfrak{A} \Rightarrow$ split for $\mathfrak{F}_{\mathfrak{A}}$" were true), therefore need the quotient topology on $G_{\mathfrak{A}} = \tilde{G}/H$ to be weaker (thus equal) to the strong topology of $\mathrm{Aut}(\mathfrak{F}_{\mathfrak{A}})$, i.e. the restriction r (=the quotient map) to be open; in general this is a serious obstruction but in the finite case it is obviously overcome;

b), in the context of (standard) nets of subfactors, makes use of the analysis in [51] together with a useful "extension-restriction" argument $\sigma \prec (\sigma^{rest})\tilde{}$ [19] (cf. [4, 46]) applied to localized endomorphisms with finite statistics of $\mathfrak{F}$ (the general case including Fermi statistics then follows e.g. by some kind of Frobenius duality argument);

c) see the paragraph following the next theorem.

Here follows the most updated version concerning the absence of sectors for a canonical field net.

Theorem 2.15 *(*[20]*) Let $\mathfrak{A}$ be an observable net on a separable Hilbert space and suppose that (1) the dual net satisfies Property B and (2) every representation of $\mathfrak{A}$ satisfying the selection criterion is a direct sum of (possibly uncountably many) irreducible representations with finite statistics, then every representation of an intermediate Bosonic net $\mathfrak{B} = \mathfrak{F}_{\mathfrak{A}}^H$ satisfying the selection criterion is a direct sum of sectors with finite statistics and these are labelled by the equivalence classes of irreducible representations of H.*

In particular under the above assumptions, which rule out the occurrence of sectors of $\mathfrak{A}$ with infinite statistics, if each sector of $\mathfrak{A}$ is Bose then the local net $\mathfrak{F}_{\mathfrak{A}}$ has no nontrivial DHR representations (thus it has no sectors with finite or infinite statistics), whilst when $\mathfrak{F}_{\mathfrak{A}}$ contains Fermi elements then the Bose part of $\mathfrak{F}_{\mathfrak{A}}$ has precisely two simple sectors. In the case where $\mathfrak{A}$ is rational i.e. $|G_{\mathfrak{A}}| < \infty$ (equivalently $\mathrm{Ind}(\mathfrak{A} \subset \mathfrak{F}_{\mathfrak{A}}) < \infty$ when $\mathfrak{A} \subset \mathfrak{F}$ is a net of subfactors), condition (2) is automatically satisfied for the "restrictions to $\mathfrak{A}$" of the DHR representations of (the Bose part of) $\mathfrak{F}_{\mathfrak{A}}$ with finite statistical dimension. This is sufficient to recover Theorem 2.14 arguing as in [20].

Corollary 2.16 *Under the hypothesis of the previous theorem, the field nets of $\mathfrak{A}$ and $\mathfrak{B}$ coincide, $\mathfrak{F}_{\mathfrak{A}} = \mathfrak{F}_{\mathfrak{B}}$.*

In the light of the previous theorem (cf. also Prop. 3.1) there is a close connection between the two conditions: the Bose part of the canonical field net has at most one sector; the observable net has no sectors with infinite statistical dimension. It would be very interesting to find conditions on general physical grounds to rule out the occurrence of DHR sectors with infinite dimension. Although there are 2D conformal field theory models where such sectors show up [35], a fact that seems to be quite natural in that context [54], there are no examples in 4D.

Remark 2.17 In a sense a Bosonic $\mathfrak{F}_{\mathfrak{A}}$ is a local analog of the compact operators (no matter what its isomorphism class is): it has only one relevant irreducible representation.

Most of the above arguments should work for theories living on a high dimensional globally hyperbolic curved spacetime, cf. [40].

3 Classification Results in 4D

In this section we discuss a classification result for subsystems of a local net $\mathfrak{F}$ on the 4D Minkowski spacetime satisfying standard assumptions plus the absence of nontrivial sectors with *any* statistics. As a consequence of the previous Theorem 2.15 this condition is satisfied by the canonical field net $\mathfrak{F}_{\mathfrak{A}}$ of a observable net $\mathfrak{A}$ with the split property and countably many DHR sectors, all Bosonic and with finite statistical dimension [17]. On the model side, the validity of the above condition for Bosonic free fields is well-known. Note that we don't exclude the possibility for $\mathfrak{F}$ to have representations with weaker localization properties like those corresponding to topological charges.

Our result will enable us to deduce that any irreducible covariant subsystem $\mathfrak{B}$ of $\mathfrak{F}$, satisfying Haag duality, arises as a fixpoint net under a compact group action, and thus to prove the equality 1.5 (up to the substitution of $\mathcal{A}$ with its dual net) on fairly general grounds.

To be more precise throughout this section we assume the local net $\mathfrak{F}$, given on the Hilbert space $\mathcal{H}$, to satisfy Poincaré covariance, spectrum condition, existence and uniqueness of the vacuum vector Ω, Reeh-Schlieder property, Haag duality, geometric modular action (for the algebras associated with wedges) and the split property, in addition to the above condition about the superselection structure. All these properties hold for $\mathfrak{F}_{\mathfrak{A}}$ under reasonable conditions for $\mathfrak{A}$, see [17] for more details.

Let $\mathfrak{B}$ (acting on $\mathcal{H}$) be a covariant subsystem of $\mathfrak{F}$. Then $\mathfrak{B}$ automatically satisfies wedge duality on its vacuum Hilbert space. The dual net of $\mathfrak{B}$ is embedded as a covariant subsystem of $\mathfrak{F}$ through the relation $\mathfrak{B}^d(\mathcal{O}) = \cap_{\mathcal{W}\supset\mathcal{O}}\mathfrak{B}(\mathcal{W})$ where $\mathcal{W}$ runs over the set of all wedges containing $\mathcal{O} \in \mathcal{K}$. Therefore $\mathfrak{B}$ is *Haag-dual* if $\mathfrak{B}(\mathcal{O}) = \cap_{\mathcal{W}\supset\mathcal{O}}\mathfrak{B}(\mathcal{W})$ for all $\mathcal{O} \in \mathcal{K}$. Actually as a consequence of covariance one also has $\cap_{\mathcal{W}\supset\mathcal{O}}\mathfrak{B}(\mathcal{W}) = \cap_{\mathcal{W}\supset\overline{\mathcal{O}}}\mathfrak{B}(\mathcal{W})$. It follows that Haag-dual covariant subsystems of $\mathfrak{F}$ are outer regular, cf. [60].

One can also consider the net $\mathfrak{B}^c$ defined by

$$\mathfrak{B}^c(\mathcal{O}) = \mathfrak{B}(\mathbb{R}^4)' \cap \mathfrak{F}(\mathcal{O}), \tag{3.1}$$

cf. [21, 5, 66]. If $\mathfrak{B}^c$ is trivial, then we say that $\mathfrak{B}$ is *full* (in $\mathfrak{F}$). An irreducible subsystem (i.e. a subsystem for which the quasi-local C^*-algebra satisfies $\mathfrak{B}' \cap \mathfrak{F} = \mathbb{C}$) is clearly full. If $\mathfrak{B}^c$ is nontrivial, then it is easy to check that it is a Haag-dual covariant subsystem of $\mathfrak{F}$ (*the coset subsystem*). It follows from the definition that $\mathfrak{B} \subset \mathfrak{B}^{cc}$, and $\mathfrak{B}^c = \mathfrak{B}^{ccc}$.

The following basic result provides a kind of converse to Theorem 2.15.

Proposition 3.1 *Let $\mathfrak{F}$ be a local net as above and let $\mathfrak{B} \subset \mathfrak{F}$ be a Haag-dual covariant subsystem, then every irreducible DHR representation σ of $\hat{\mathfrak{B}}$ is equivalent to a subrepresentation of the representation induced by the embedding of $\mathfrak{B}$ in $\mathfrak{F}$.*

Moreover σ is covariant with positive energy and it has finite statistical dimension.

Now the Theorem 2.11 is crucial since it shows that $\mathfrak{F}_{\mathfrak{B}}$ can be considered as a covariant subsystem of $\mathfrak{F}$, since the latter coincides with its own canonical field net. Next result describes the relative position of $\mathfrak{F}_{\mathfrak{B}}$ in $\mathfrak{F}$.

Theorem 3.2 *There exists a unitary isomorphism of $\mathfrak{F}$ with $\widehat{\mathfrak{F}_{\mathfrak{B}}} \otimes \widehat{\mathfrak{B}^c}$ which maps FB into $\hat{F} \otimes \hat{B}$ for every $\mathcal{O} \in \mathcal{K}$, $F \in \mathfrak{F}_{\mathfrak{B}}(\mathcal{O})$ and $B \in \mathfrak{B}^c(\mathcal{O})$. In particular $\mathfrak{F}_{\mathfrak{B}} = \mathfrak{B}^{cc}$, and if $\mathfrak{B}$ is full in $\mathfrak{F}$ then $\mathfrak{F}_{\mathfrak{B}} = \mathfrak{F}$.*

It follows immediately that every Haag-dual subsystem $\mathfrak{B}$ of $\mathfrak{F}$ as above is of the form $\mathfrak{F}_1^H \otimes I$ for some tensor product decomposition $\mathfrak{F} = \mathfrak{F}_1 \otimes \mathfrak{F}_2$ and some compact group H of unbroken internal symmetries of $\mathfrak{F}_1$.

Corollary 3.3 *If $\mathfrak{B}$ is a full Haag-dual covariant subsystem of $\mathfrak{F}$ then there exists a compact group H of unbroken internal symmetries of $\mathfrak{F}$ such that $\mathfrak{B} = \mathfrak{F}^H$, and in fact H coincides with the canonical gauge group $G_{\mathfrak{B}}$.*

We may sum up the results to date in this section about $\mathfrak{A} = \mathfrak{F}_{\mathfrak{A}}^{G_{\mathfrak{A}}}$ and $\mathcal{A}$:

Theorem 3.4 *Keeping the same assumptions on $\mathfrak{F} = \mathfrak{F}_{\mathfrak{A}}$ as above, we have*

$$\mathcal{A}^d = \mathfrak{F}_{\mathfrak{A}}^{G_{\max}} .$$

Moreover, the following conditions are equivalent:

(1) $\mathfrak{A} = \mathcal{A}^d$
(2) $G_{\mathfrak{A}} = G_{\max}$ *(i.e.* $\mathfrak{A} = \mathfrak{A}_{\min}$*)*
(3) $\mathfrak{A}$ *has no proper full subsystem*
(4) *any Haag dual subsystem $\mathfrak{B} \subset \mathfrak{A}$ is of the form $\mathfrak{B}_1 \otimes I$ for some tensor product decomposition $\mathfrak{A} = \mathfrak{B}_1 \otimes \mathfrak{B}_2$*

Proof (sketch) (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are already shown in [17].
As to (4), $\Leftarrow$ is clear. "$\Rightarrow$": let $\mathfrak{B}$ be a subsystem of $\mathfrak{A}$. We know that $\mathfrak{B}$ is full in $\mathfrak{A}$ if and only if $\mathfrak{B}$ is full in $\mathfrak{F}_{\mathfrak{A}}$. Hence if $\mathfrak{B}$ is not full in $\mathfrak{A}$ then $\mathfrak{F}_{\mathfrak{A}} = \mathfrak{F}_{\mathfrak{B}} \vee \mathfrak{B}^c$ is unitarily equivalent to $\widehat{\mathfrak{F}_{\mathfrak{B}}} \otimes \widehat{\mathfrak{B}^c}$. From $\mathfrak{A} = \mathfrak{F}_{\mathfrak{A}}^{G_{\max}}$ it follows that

$$\begin{aligned} \mathfrak{A} \subset \mathfrak{F}_{\mathfrak{A}}^{G_{\mathfrak{B}} \times e} &= \mathfrak{B} \vee \mathfrak{B}^c \\ &\cong \widehat{\mathfrak{F}_{\mathfrak{B}}}^{G_{\mathfrak{B}}} \otimes \widehat{\mathfrak{B}^c} . \end{aligned}$$

Therefore the chain of inclusions $\mathfrak{B} \subset \mathfrak{A} \subset \mathfrak{B} \vee \mathfrak{B}^c$ is spatially isomorphic to a chain of the type $\mathfrak{B}_1 \otimes I \subset \tilde{\mathfrak{A}} \subset \mathfrak{B}_1 \otimes \widehat{\mathfrak{B}^c}$ and, using the results in [38], one can show that $\tilde{\mathfrak{A}} = \mathfrak{B}_1 \otimes \mathfrak{B}_2$, i.e. $\mathfrak{A} \cong \mathfrak{B}_1 \otimes \mathfrak{B}_2$ for a certain $\mathfrak{B}_2 \subset \widehat{\mathfrak{B}^c}$, cf. [17, Section 3]. □

Remark 3.5 In any case $\mathcal{A}^d$ will not have full subsystems; this can be read off from the previous result applied to $\mathcal{A}^d$ in place of $\mathfrak{A}$, recalling that $\mathfrak{F}_{\mathfrak{A}} = \mathfrak{F}_{\mathcal{A}^d}$.

Remark 3.6 Theorem 3.4 also holds true if $\mathcal{A}$ is replaced by the net $\tilde{\mathcal{A}}$ obtained by localizing the whole Poincaré group, in particular $\mathcal{A}^d = \tilde{\mathcal{A}}^d$.

There are some interesting consequences. In the presence of suitable interactions the possibility that $\mathfrak{A} \simeq \mathfrak{B}_1 \otimes \mathfrak{B}_2$ has to be excluded (the corresponding S-matrix would factorize in the form $S = S_1 \otimes S_2$). In this case it turns out that $\mathfrak{A} = \mathcal{A}^d$ is equivalent to the statement that "$\mathfrak{A}$ is *minimal*", i.e. it has no proper subsystem.

A little more effort playing with free theories reveals examples for which $\mathfrak{A} = \mathcal{A}^d$ is minimal and also other ones for which $\mathfrak{A} = \mathcal{A}^d = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ is not. For instance if $\mathfrak{F} = \mathfrak{F}_{m_1} \otimes \mathfrak{F}_{m_2}$ is generated by the fields φ_1, φ_2 with masses $m_1 \neq m_2$ it holds $G_{\max} = \mathbb{Z}_2 \times \mathbb{Z}_2$, therefore

$$\mathcal{A}^d = \mathfrak{F}_{m_1}^{\mathbb{Z}_2} \otimes \mathfrak{F}_{m_2}^{\mathbb{Z}_2} = \mathcal{A}_1^d \otimes \mathcal{A}_2^d$$

and hence $\mathcal{A}^d$ is not minimal. However, if $m_1 = m_2 = m$ we have $G_{\max} = O(2)$ so that $\mathcal{A}^d = (\mathfrak{F}_m \otimes \mathfrak{F}_m)^{O(2)} = \mathfrak{F}^{O(2)}$. On the other side $\mathfrak{F}_m \otimes \mathfrak{F}_m$ is the unique tensor product decomposition of $\mathfrak{F}$ (up to inner automorphisms) [17, Appendix]; if we had $\mathcal{A}^d = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ it should hold

$$\mathfrak{F} = \mathfrak{F}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2} = \mathfrak{F}_{\mathfrak{A}_1} \otimes \mathfrak{F}_{\mathfrak{A}_2} \cong \mathfrak{F}_m \otimes \mathfrak{F}_m \ ,$$

and thus $\mathcal{A}^d$ would be isomorphic to

$$\mathfrak{F}_m^{G_{\mathfrak{A}_1}} \otimes \mathfrak{F}_m^{G_{\mathfrak{A}_2}} \supset \mathfrak{F}_m^{\mathbb{Z}_2} \otimes \mathfrak{F}_m^{\mathbb{Z}_2} \not\supset (\mathfrak{F}_m \otimes \mathfrak{F}_m)^{O(2)} = \mathcal{A}^d \ .$$

The above results lends support to the conjecture that $\mathcal{A}^d = \mathfrak{F}^{G_{\max}}$ even under less restrictive assumptions, but exhibiting a complete answer seems really tough.

One may also state variants of the above results for the local net $\mathcal{A}_G$ generated by $\mathcal{A}$ and the local operators measuring the charge content of the theory with gauge group G. Here are some related problems: (i) study the dependence of $\mathcal{A}_G$ on G; even more, one could see what happens replacing the reference vector Ω in the definition of the universal localizing maps with other suitably chosen (families of) vectors, e.g. those analytic for the energy (cf. [48]). (ii) it has been shown that every (metrizable) compact group may appear as a G [28] but as far as we know there are only few results about the possible $G_{\max}$.

4 Back to Wightman Currents

As discussed in the introduction the net $\mathcal{A}$ is generated by the canonical implementations of spacetime translations $T_\Lambda(x) := \Psi_\Lambda(T(x)) = e^{iP_\Lambda^\mu x_\mu}$ which can be defined when the field net associated with the observable net $\mathfrak{A}$ has the split property. If $\mathcal{O}_\Lambda \subset\subset \tilde{\mathcal{O}}_\Lambda$ are the double cones defining the triple Λ then there hold

$$T_\Lambda(x) \in \mathfrak{A}(\tilde{\mathcal{O}}_\Lambda) \tag{4.1}$$

$$T_\Lambda(x) F T_\Lambda(-x) = T(x) F T(-x) \qquad \text{if} \quad F, T(x)FT(-x) \in \mathfrak{F}_{\mathfrak{A}}(\mathcal{O}_\Lambda) \tag{4.2}$$

Now if there is an (observable) energy-momentum tensor $\Theta^{\mu\nu}(x)$ one has the (formal) equality

$$P^\mu = \int_{\mathbb{R}^3} \Theta^{0\mu}(x) \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3. \tag{4.3}$$

It turns out that one can choose a test function f_Λ with support in $\tilde{\mathcal{O}}_\Lambda$ such that the previous relations for $T_\Lambda(x)$ also hold for $\mathrm{e}^{i\Theta^{0\nu}(f_\Lambda)}$. In this sense the local generators P_Λ^μ provide an abstract algebraic analogue for $\Theta^{0\nu}(f_\Lambda)$. A similar situation occurs for internal symmetries and the corresponding conserved Wightmann currents. Let us now choose a suitable family Λ_R for $R > 0$ where $\tilde{\mathcal{O}}_{\Lambda_R}$ is the double cone centered at the origin with base of radius R. The previous analogy suggests that $T(x) P_{\Lambda_R}^\mu T(x)^* / R^3$ should converge to (a multiple of) $\Theta^{0\mu}(x)$ for $R \to 0$ in the sense of distributions, on a suitable domain. This prescription would also allow to define an energy-momentum tensor from purely algebraic concepts when the latter is not *a priori* given. To avoid the difficulties with the domains of the unbounded operators

$P^{\mu}_{\Lambda_R}$ one can try to replace them with suitably chosen bounded substitutes, e.g. $e^{iP^{\mu}_{\Lambda_R}} - (\Omega, e^{iP^{\mu}_{\Lambda_R}}\Omega)$.

A preliminary investigation in this direction has been done by Aita [1]. First results have been given by one of the authors of this report in [16] where it is shown that in an interesting class of models of chiral 2D conformal field theory the energy-momentum tensor can be recovered from the canonical local implementation of translations, with a procedure of the type described above (see also the next section). On the negative side in [16] an example is given in which the scaling limit for the U(1) internal symmetry of the model vanishes and the corresponding Wightman current does not exist. Finally partial results have been given by Tomassini for the 4D massless charged free field [61].

In absence of results of general nature we consider here some consequences of the assumption that an energy-momentum tensor $\Theta^{\mu\nu}(x)$ can be defined using scaling limits of local operators in $\mathcal{A}$. It should be clear from the above discussion that if this limits exists in a sufficiently strong sense then $\Theta^{\mu\nu}(x)$ generates a covariant subsystem $\mathcal{A}_W$ of $\mathcal{A}$, cf. [14]. It turns out that despite the intrinsic interest of the existence the pointwise limit, the associated net essentially coincides with its abtract version.

Proposition 4.1 *Let $\mathfrak{A}$ be a local net of observables whose canonical field net satisfies the properties of the previous sections. Assume also that the (re)construction procedure for Wightman energy-momentum tensor can be succesfully afforded, and let $\mathcal{A}_W$ be the local net generated by the (smeared) energy-momentum tensor. Then*

$$(\mathcal{A}_W)^d = \mathfrak{F}_{\mathfrak{A}}^{G_{\max}} = \mathcal{A}^d \ .$$

Proof $\mathcal{A}_W \subset \mathcal{A}$ is full in $\mathfrak{F}_{\mathfrak{A}}$, whence we can apply Corollary 3.3. □

The following proposition shows that in favourable situations $\mathcal{A}$ is Haag-dual and hence can be completely computed.

Proposition 4.2 *Assume that $\mathcal{A}_W \subset \mathcal{A}$ can be constructed, then it holds $\mathcal{A}^d = \mathcal{A}^r$ and furthermore $\mathcal{A}^d = \mathcal{A}$ whenever $\mathcal{O} \mapsto \mathfrak{F}_{\mathfrak{A}}(\mathcal{O})$ is continuous from inside.*

Proof $\mathcal{A}_W$, being generated by Wightman fields, is weakly additive, therefore for every $\mathcal{O} \in \mathcal{K}$ it holds

$$\mathcal{A}^d(\mathbb{M}_4) = \mathcal{A}_W(\mathbb{M}_4) = \bigvee_{x\in\mathbb{M}_4} \mathcal{A}_W(\mathcal{O}+x) \subset \bigvee_{x\in\mathbb{M}_4} \mathcal{A}(\mathcal{O}+x) \subset \mathcal{A}(\mathbb{M}_4) = \mathcal{A}^d(\mathbb{M}_4) \ ,$$

so that $\mathcal{A}$ is weakly additive as well. Then, from

$$\mathcal{A}^d(\mathcal{O}_\Lambda) \subset \Psi_\Lambda(\mathcal{A}^d(\mathbb{M}_4)) = \Psi_\Lambda(\mathcal{A}(\mathbb{M}_4)) = \Psi_\Lambda(\bigvee_{x\in\mathbb{M}_4} T(x)\mathcal{A}(\mathcal{O}_\Lambda)T(x)^*) \subset \mathcal{A}(\tilde{\mathcal{O}}_\Lambda) \tag{4.4}$$

(cf. [18]), $\mathcal{A}^d$ being outer regular, it follows that $\mathcal{A}^d = \mathcal{A}^r$. The continuity of $\mathcal{O} \mapsto \mathfrak{F}_{\mathfrak{A}}(\mathcal{O})$ implies that of $\mathcal{O} \mapsto \mathcal{A}^d(\mathcal{O}) = \mathfrak{F}_{\mathfrak{A}}(\mathcal{O})^{G_{\max}}$ and we are done. □

5 2D Chiral Field Theories

We denote $\mathcal{I}_0$ the set of all (non-empty) open bounded intervals on $\mathbb{R}$. A local chiral net $\mathfrak{F}$ is an isotonous net of von Neumann algebras over $\mathcal{I}_0$ acting on a Hilbert space $\mathcal{H}$, satisfying

- *Locality*: $[\mathfrak{F}(I_1), \mathfrak{F}(I_2)] = \{0\}$ whenever $I_1, I_2 \in \mathcal{I}_0$ and $I_1 \cap I_2 = \emptyset$
- *Conformal covariance*: there is a strongly continuous unitary representation V of $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{H}$ such that
$$V(\alpha)\mathfrak{F}(I)V(\alpha)^* = \mathfrak{F}(\alpha I)$$
whenever $I, \alpha I \in \mathcal{I}_0$, where $\mathrm{SL}(2,\mathbb{R})$ acts on "the circle" $\mathbb{R}\cup\{\infty\}$ by Moebius transformations
- *Positivity*: The one-parameter subgroup of the rotations has positive generator
- *Existence, uniqueness and ciclycity of the vacuum*: There is a unique (up to a phase) V-invariant unit vector $\Omega \in \mathcal{H}$, which is also cyclic for $\mathfrak{F}(\mathbb{R})$.

By conformal covariance such a net can always be extended to a net on the family $\mathcal{I}$ of all (nonempty) open nondense intervals of S^1.

Interesting examples are constructed starting from the Virasoro or the Kac Moody algebras, hereafter called $\mathfrak{A}_{\mathrm{Vir}(c)}$ and $\mathfrak{F}_{G_k}$ respectively. $\mathfrak{A}_{\mathrm{Vir}(c)}$ is the Haag-Kastler net on the real line, in its vacuum representation, generated by the energy-momentum tensor $\Theta(x)$ of a chiral CFT, for which it holds

$$[\Theta(x), \Theta(y)] = i\delta'(x-y)(\Theta(x)+\Theta(y)) - i\frac{c}{24\pi}\delta'''(x-y),\ x, y \in \mathbb{R},$$

where the central charge c satisfies $c \geq 1$ or belongs to the discrete series $c = 1 - 6/(m+2)(m+3),\ m \in \mathbb{N}$. Analogously for a simply connected compact simple Lie group G and a level $k \in \mathbb{N}$, $\mathfrak{F}_{G_k}$ is generated by currents $j^X(x)$ such that

$$[j^X(x), j^Y(y)] = ij^{[X,Y]}(x)\delta(x-y) - i\frac{k}{4h\pi}\delta'(x-y)\mathrm{Tr}(\mathrm{Ad}X \cdot \mathrm{Ad}Y),\ x, y \in \mathbb{R}$$

where h is the dual Coxeter number of $\mathrm{Lie}(G)$. In a different way $\mathfrak{F}_{G_k}$ can be defined through a loop group construction, cf. [62]. For any such net there is a representation of G in the unbroken internal symmetry group of $\mathfrak{F}_{G_k}$.

Appealing to our general discussion in Sect. 2 it is now clear how to define the $\mathrm{SL}(2,\mathbb{R})$-covariant subsystems of a given chiral net, which we simply call *conformal subsystems*. For instance $\mathfrak{A}_{\mathrm{Vir}(c)} \subset \mathfrak{F}^G_{G_k}$ for $c = k \cdot \dim(G)/k + h \geq 1$ by the Sugawara formula. Other examples are obtained considering subsystems of $\mathfrak{F}_{G_k}$ generated by currents $j^X(x)$ with X in a Lie subalgebra of $\mathrm{Lie}(G)$.

It seems difficult for several reasons to adapt the methods discused in the Sections 2 and 3 to obtain complete classification results for subsystems in the context of chiral nets. There is no analog of the canonical field net, and the extended endomorphisms of localized endomorphisms might not be localized better than in half-lines ("soliton sectors"). However for some of the aforementioned models, the following classification results are known:

Theorem 5.1 *([14]) $\mathfrak{A}_{\mathrm{Vir(c)}}$ has no proper $SL(2,\mathbb{R})$-covariant subsystem for all the allowed values of the central charge.*

Theorem 5.2 *([15]) All covariant subsystems of $\mathfrak{F}_{\mathrm{SU}(2)_1}$ are obtained as fixed points under a closed subgroup of the internal symmetry group $SO(3)$.*

Using the fact that the net $\mathfrak{F}_{\mathrm{U(1)}}$ generated by a single $U(1)$-current $j(x)$, for which

$$[j(x), j(y)] = i\delta'(x-y),$$

can be embedded in $\mathfrak{F}_{\mathrm{SU(2)}_1}$ (in fact in every $\mathfrak{F}_{G_k}$) it can be shown that the former has only two distinct proper conformal subsystems, the one generated by the energy-momentum tensor (with $c = 1$) which is proportional to $: j(x)^2 :$ (with infinite index), and the $\mathbb{Z}_2$ fixed point net [15]. Although the $U(1)$ chiral current algebra is considered as the chiral analogue of the Hermitian scalar field, in contrast with the 4D case these two nets do not coincide.

Some remarks are in order. First of all for different G's or k's there are subsystems of $\mathfrak{F}_{G_k}$ which are not fixpoint nets under a compact action, for instance $\mathfrak{A}_{\mathrm{Vir(c)}}$ generated by the Sugawara energy-momentum tensor [54] and $\mathfrak{F}_{\mathrm{U(1)}}$ generated by a single current $j^X(x)$ for a fixed non-zero $X \in \mathrm{Lie}(G)$ [15]. For such nets no complete classification results are known. Also the techniques needed to prove the foregoing theorems are quite different from those used in the previous section. In particular, differently from the higher dimensional case, these results heavily rely on the existence of Wightman fields, especially the energy-momentum tensor. An important ingredient for their proofs (as in the case of the achievements in [16]) is the paper of Fredenhagen and Jörß [36] where it is shown that chiral nets are always generated by underlying pointlike localized fields constructed from these nets by a scaling limit procedure, cf. also [44].

Mimicking the 4D situation given a chiral net $\mathfrak{F}$ with the split property one can define a new chiral net by localizing the whole Moebius group, namely

$$\mathcal{A}(I) := \{\Psi_{(\mathfrak{F}(I_1),\mathfrak{F}(\tilde{I}_1),\Omega)}(V(\mathrm{SL}(2,\mathbb{R}))) \ ; \ I_1, \tilde{I}_1 \in \mathcal{I}_0, I_1 \subset\subset \tilde{I}_1 \subset I\}'' \ , \qquad (5.1)$$

thus producing a conformal subsystem $\mathcal{A} = \mathcal{A}_{\mathfrak{F}} \subset \mathfrak{F}$. In the light of the above results we trivially have that $\mathcal{A} = \mathfrak{F}$ for $\mathfrak{F} = \mathfrak{A}_{\mathrm{Vir(c)}}$, while $\mathcal{A} = \mathfrak{F}^{G_{\max}} = \mathfrak{A}_{\mathrm{Vir(1)}}$ for $\mathfrak{F} = \mathfrak{F}_{\mathrm{SU(2)}_1}$, where $G_{\max} = \mathrm{SO(3)}$ is the full symmetry group of $\mathfrak{F}$. Note that for $\mathfrak{F}_{\mathrm{U(1)}} = \mathfrak{F}^{\mathrm{U(1)}}_{\mathrm{SU(2)}_1}$ one has

$$\mathcal{A}_{\mathfrak{F}_{\mathrm{SU(2)}_1}} \subset \mathcal{A}_{\mathfrak{F}_{\mathrm{U(1)}}} \subset \mathfrak{F}^{\mathbb{Z}_2}_{\mathrm{U(1)}}$$

one of the two inclusions being necessarily an equality. Besides, for $\mathfrak{F} = \mathfrak{F}_{G_k}$ one can always show that

$$\mathfrak{A}_{\mathrm{Vir(c)}} \subset \mathcal{A}_{\mathfrak{F}} \subset \mathfrak{F}^{G_{\max}}, \qquad (5.2)$$

with $c = k \cdot \dim(G)/k + h$, due to the fact that in this context the reconstruction procedure for the (chiral) energy-momentum tensor as scaling limit of local operators in $\mathcal{A}_{\mathfrak{F}}$ has a positive end [16], cf. also [14] ($\mathfrak{A}_{\mathrm{Vir(c)}}$ playing the role of the net $\mathcal{A}_W$ considered before).

Finally we point out an ambiguity which does not arise in 4D. Let $\mathfrak{A}$ be a chiral "observable net" and assume that $\mathfrak{A} = \mathfrak{F}^G$ for some larger net $\mathfrak{F}$ and compact group G. One can define two conformal subsystems of $\mathfrak{A}$: $\mathcal{A}_{\mathfrak{A}}$ and $\mathcal{A}_{\mathfrak{F}}$. In analogy with the 4D case the latter should be preferable, but given $\mathfrak{A}$ a canonical choice of such a $\mathfrak{F}$, which is maximal in some sense (like the field net in 4D), does not seem to exist in general. For example in the case of the U(1)-model there are countably many choices for maximal $\mathfrak{F}$'s, see [13]. $\mathcal{A}_{\mathfrak{F}}$ could depend on this choice.

6 Final Comments

As we have already mentioned several important problems remain open, and we pointed out some topics for further research.

For the sake of completeness we collect few more questions which have not been touched in the main text.

In Sect. 3 we stated classification results only for subsystems of Bosonic field nets. Although we believe that Fermi fields can be settled in a similar way without changing too much the overall picture, such a generalization would be desiderable.

One can look for more general versions of the results in Sect. 2 in the case where the charges are localized in space-like cones.

Although the classification results in Sect. 3 are potentially not confined to free theories, it would be interesting to state results directly applicable to the few known interacting models like φ_3^4 and $P(\varphi)_2$ which for different reasons are not covered by the present analysis.

Other issues are concerned with the geometry of the space-time.

One of them consists in extending the results (even just for the free fields) to the case of (globally hyperbolic) curved spacetimes. Here we have no translations, therefore we have to replace the usual spectrum condition. Perhaps it seems feasible to consider the μSC in terms of wavefront sets.

Hopefully some of the techniques used in the 4D context, like the functorial properties of the extension procedure, could play an important role in the classification program for subsystems of low dimensional theories. The latter is related to the problem of the possible values for the index of a net of subfactors. In the 4D situation the index of a subsystem is clearly always infinite, or an integer. Moreover any integer value is in fact realized.[7] However in a broader context (e.g. inclusions of chiral nets) the computation of these values seems an interesting problem.

Acknowledgements. The topics considered in this article have been strongly influenced by the works and the ideas of John E. Roberts and Sergio Doplicher. They also obtained some of the exposed results in a joint collaboration with one of the authors. We would like to express our gratitude to them for many valuable discussions and hints.

References

[1] Aita, M. *Cariche e correnti in teoria quantistica dei campi. Tesi di Laurea*, Università di Roma *La Sapienza* (1989).

[2] Araki, H. *Symmetries in the theory of local observables and the choice of the net of local algebras.* Rev. Math. Phys. **Special Issue** (1992), 1–14.

[3] Baumgärtel H., Wollenberg M. *Causal Nets of Operator Algebras. Mathematical Aspects of Algebraic Quantum Field Theory.* Akademie Verlag, Berlin 1992.

[4] Böckenhauer J., Evans D. *Modular Invariants, Graphs and α–Induction for Nets of Subfactors I.* Comm. Math. Phys. **197** (1998), 361–386.

[5] Borchers, H.J. *On the revolutionizing quantum field theory with Tomita's modular theory.* J. Math. Phys. **41** (2000), 3604–3673.

[6] Borchers, H.J., Yngvason J. *From quantum fields to local von Neumann algebras.* Rev. Math. Phys. **Special Issue** (1992), 15–47.

[7] Brunetti R., Guido D., Longo R. *Modular structure and duality in conformal quantum field theory.* Commun. Math. Phys. **156** (1993), 201–219.

[7] To see this, consider the fixpoint net of the complex scalar free field under the subgroup $\mathbb{Z}_n$ of the gauge group S^1.

[8] Buchholz D., Doplicher S., Longo R. *On Noether's theorem in quantum field theory.* Ann. Phys. **170** (1986), 1–17.

[9] Buchholz D., Doplicher S., Longo R., Roberts J. E. *A new look at Goldstone theorem.* Rev. Math. Phys. **Special Issue** (1992), 47–82.

[10] Buchholz D., Doplicher S., Longo R., Roberts J. E. *Extension of automorphisms and gauge symmetries.* Comm. Math. Phys. **155** (1993), 123–134.

[11] Buchholz D., Doplicher S., Morchio G., Roberts J. E., Strocchi F. *A model for charges of electromagnetic type.*, in *Operator algebras and quantum field theory*, editors S. Doplicher, R. Longo, J.E. Roberts, L. Zsido, International Press, Cambridge U.S. I997, pp. 647–660.

[12] Buchholz D., Haag R. *The quest for understanding in relativistic quantum physics.* J. Math. Phys. **41** (2000), 3674–3697.

[13] Buchholz D., Mack G., Todorov I. *The current algebra on the circle as a germ for local field theories.* in "Conformal field theories and related topics", eds. P. Binetruy et al., *Nucl. Phys. B* (Proc. Suppl.) **5B**, 20, (1988).

[14] Carpi, S. *Absence of subsystems for the Haag-Kastler net generated by the energy-momentum tensor in two dimensional conformal field theory.* Lett. Math. Phys. **45** (1998), 259–267.

[15] Carpi, S. *Classification of subsystems for the Haag-Kastler nets generated by $c = 1$ chiral current algebras.* Lett. Math. Phys. **47** (1999), 353–364.

[16] Carpi, S. *Quantum Noether's theorem and conformal field theory: a study of some models.* Rev. Math. Phys. **11** (1999), 519–532.

[17] Carpi S., Conti R. *Classification of subsystems for local nets with trivial superselection structure.* Comm. Math. Phys., to appear.

[18] Conti, R. *On the intrinsic definition of local observables.* Lett. Math. Phys. **35** (1995), 237–250.

[19] Conti, R. *Inclusioni di algebre di von Neumann e teoria algebrica dei campi. Ph.D. Thesis*, Università di Roma *Tor Vergata* (1996).

[20] Conti R., Doplicher S., Roberts J. E. *Superselection theory for subsystems.* Preprint, Università di Roma La Sapienza (2000).

[21] Davidson, D. R. *Classification of subsystems of local algebras. Ph.D. Thesis*, University of California at Berkeley (1993).

[22] Doplicher S. *Local aspects of superselection rules.* Comm. Math. Phys. **85** (1982), 73–85.

[23] Doplicher, S. *Progress and problems in algebraic quantum field theory.* S. Albeverio et al. (eds.), *Ideas and Methods in Quantum and Statistical Physics* Vol. 2. Cambridge Univ. Press, Cambridge 1992, pp. 390-404.

[24] Doplicher S., Haag, R., Roberts J.E. *Fields, observables and gauge transformations I, II.* Comm. Math. Phys. **13** (1969), 1–23; Comm. Math. Phys. **15** (1969), 173–200.

[25] Doplicher S., Haag, R., Roberts J.E. *Local observables and particle statistics I, II.* Comm. Math. Phys. **23** (1971), 199–230; Comm. Math. Phys. **35** (1974), 49–85.

[26] Doplicher S., Longo R. *Local aspects of superselection rules II.* Comm. Math. Phys. **88** (1983), 399–409.

[27] Doplicher S., Longo R. *Standard and split inclusions of von Neumann algebras.* Invent. Math. **75** (1984), 493–536.

[28] Doplicher S., Piacitelli G. In preparation.

[29] Doplicher S., Roberts J. E. *Endomorphisms of C^*–algebras, cross products and duality for compact groups.* Ann. Math. **130** (1989), 75–119.

[30] Doplicher S., Roberts J. E. *A new duality theory for compact groups.* Invent. Math. **98** (1989), 157–218.

[31] Doplicher S., Roberts J.E. *Why there is a field algebra with a compact gauge group describing the superselection structure structure in particle physics.* Comm. Math. Phys. **131** (1990), 51–107.

[32] Driessler, W. *Duality and absence of locally generated sectors for CCR-type algebras.* Comm. Math. Phys. **70** (1979), 213–220.

[33] Enock, M. *Sous-facteurs intermédiaires et groupes quantiques measurés* J. Operator Theory **42** (1999), 305–330.

[34] Fidaleo, F. *On the local implementation of gauge symmetries in local quantum theory.* Comm. Math. Phys. **107** (1986), 233–240.

[35] Fredenhagen, K. *Superselection sectors with infinite statistical dimension.* in "Subfactors" (H. Araki et al. eds.), World Scientific, Singapore, 1995, pp. 242–258.

[36] Fredenhagen K., Jörß M. *Conformal Haag-Kastler nets, pointlike localized fields and the existence of operator product expansion.* Comm. Math. Phys. **176** (1996), 541.

[37] Fredenhagen K., Rehren K.-H., Schroer, B. *Superselection sectors with braid group statistics and exchange algebras II: geometric aspects and conformal covariance.* Rev. Math. Phys. **Special Issue** (1992), 113–157.

[38] Ge L., Kadison R. V. *On tensor products of von Neumann algebras.* Invent. Math. **123** (1996), 453–466.

[39] Guido D., Longo R. *An algebraic spin and statistics theorem.* Comm. Math. Phys. **172** (1995), 517–533.

[40] Guido, D., Longo, R., Roberts, J. E., Verch, R. *Charged sectors, spin and statistics in quantum field theory on curved spacetimes.* Rev. Math. Phys., to appear.

[41] Haag, R. *Local Quantum Physics.* 2nd ed. Springer-Verlag, New York Berlin Heidelberg 1996.

[42] Haag, R., Ojima, I. *On the problem of defining a specific theory within the frame of local quantum physics.* Ann. Inst. H. Poincaré **64** (1996), 385–393.

[43] Izumi, M., Longo, R., Popa, S. *A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras.* J. Funct. Anal. **155** (1998), 25–63.

[44] Jörß M. *The construction of pointlike localized charged fields from conformal Haag–Kastler nets* Lett. Math. Phys. **38** (1996), 257–274.

[45] Kastler, D. ed. *The algebraic theory of superselection sectors.* World Scientific, Singapore, 1990.

[46] Kawahigashi Y., Longo R., Müger M. *Multi-interval subfactors and modularity of representations in conformal field theory.* Preprint, Università di Roma Tor Vergata (1999).

[47] Kuckert B. *A new approach to Spin and Statistics.* Lett. Math. Phys. **35** (1995), 319–331.

[48] Kunhardt, W. *Über die universelle algebra der lokalen translationsoperatoren in der algebraischen quantenfeldtheorie. Diplomarbeit* Hamburg University (1994).

[49] Langerholc J., Schroer B. *On the structure of the von Neumann algebras generated by local functions of the free Bose field.* Comm. Math. Phys. **1** (1965), 215–239.

[50] Langerholc J., Schroer B. *Can current operators determine a complete theory?* Comm. Math. Phys. **4** (1967), 123–136.

[51] Longo R., Rehren K.-H. *Nets of subfactors.* Rev. Math. Phys. **7** (1995), 567–597.

[52] Müger, M. *On charged fields with group symmetry and degeneracies of Verlinde's matrix S.* Ann. Inst. H. Poincaré **71** (1999), 359–394.

[53] Müger, M. *Superselection structure of massive quantum field theories in 1+1 dimensions.* Rev. Math. Phys. **10** (1998), 1147–1170.

[54] Rehren, K.H. *A new view of the Virasoro algebra.* Lett. Math. Phys. **30** (1994), 125–130.

[55] Rehren, K.H. *Chiral observables and modular invariants.* Comm. Math. Phys. **208** (2000), 689–712.

[56] Roberts, J.E. *Lectures on algebraic quantum field theory.* In [45].

[57] Roberts, J. E. Lectures given at the CIME school on Non-commutative geometry, Martina Franca 2000.

[58] Streater R. F., Wightman A. S. *PCT, spin, statistics and all that.* Benjamin, 1964.

[59] Summers, S. J. *Normal product states for fermions and twisted duality for CCR– and CAR–type algebras with applications to the Yukawa$_2$ quantum field model.* Comm. Math. Phys. **86** (1982), 111–141.

[60] Thomas L.J. III, Wichmann E. H. *Standard forms of local nets in quantum field theory.* J. Math. Phys. **39** (1998), 2643–2681.

[61] Tomassini L. *Sul teorema di Noether quantistico: studio del campo libero di massa zero a quattro dimensioni. Tesi di Laurea*, Università di Roma *La Sapienza* (1999).

[62] Wassermann A. *Operator algebras and conformal field theory III: Fusion of positive energy representations of $SU(N)$ using bounded operators.* Invent. Math. **133** (1998), 467–538.

[63] Wichmann, E. H. *On systems of local operators and the duality condition.* J. Math. Phys. **24** (1983), 1633–1644.

[64] Xu, F. *New braided endomorphisms from conformal inclusions.* Comm. Math. Phys. **192** (1998), 349–403.

[65] Xu, F. *Jones–Wassermann subfactors for disconnected intervals.* Preprint *q-alg/9704003.*

[66] Xu, F. *Algebraic coset conformal field theories.* Comm. Math. Phys. **211** (2000), 1–43.