Renormalization Group, hidden symmetries and approximate Ward identities in the XYZ model.

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Abstract 1. *Using renormalization group methods, we study the Heisenberg-Ising XYZ chain in an external magnetic field directed as the z axis, in the case of small coupling J_3 in the z direction. In particular, we focus our attention on the asymptotic behaviour of the spin correlation function in the direction of the magnetic field and the singularities of its Fourier transform.*

An expansion for the ground state energy and the effective potential is derived, which is convergent if the running coupling constants are small enough. Moreover, by using hidden symmetries of the model, we show that this condition is indeed verified, if J_3 is small enough, and we derive an expansion for the spin correlation function. We also prove, by means of an approximate Ward identity, that a critical index, related with the asymptotic behaviour of the correlation function, is exactly vanishing, together with other properties, so obtaining a rather detailed description of the XYZ correlation function.

1. Introduction

1.1 If $(S^1_x, S^2_x, S^3_x) = \frac{1}{2}(\sigma^1_x, \sigma^2_x, \sigma^3_x)$, for $i = 1, 2, ..., L$, σ^α_i, $\alpha = 1, 2, 3$, being the Pauli matrices, the Hamiltonian of the *Heisenberg-Ising XYZ chain* is given by

$$H = -\sum_{x=1}^{L-1} [J_1 S^1_x S^1_{x+1} + J_2 S^2_x S^2_{x+1} + J_3 S^3_x S^3_{x+1} + h S^3_x] - h S^3_L + U^1_L \ , \tag{1.1}$$

where the last term, to be fixed later, depends on the boundary conditions. The space-time *spin correlation function* at temperature β^{-1} is given by

$$\Omega^\alpha_{L,\beta}(\mathbf{x}) = < S^\alpha_{\mathbf{x}} S^\alpha_{\mathbf{0}} >_{L,\beta} - < S^\alpha_{\mathbf{x}} >_{L,\beta} < S^\alpha_{\mathbf{0}} >_{L,\beta} \ , \tag{1.2}$$

where $\mathbf{x} = (x, x_0)$, $S^\alpha_{\mathbf{x}} = e^{Hx_0} S^\alpha_x e^{-Hx_0}$ and $< . >_{L,\beta} = Tr[e^{-\beta H} .]/Tr[e^{-\beta H}]$ denotes the expectation in the grand canonical ensemble. We shall use also the notation $\Omega^\alpha(\mathbf{x}) \equiv \lim_{L,\beta\to\infty} \Omega^\alpha_{L,\beta}(\mathbf{x})$.

The Hamiltonian (1.1) can be written [LSM] as a *fermionic interacting spinless Hamiltonian.* In fact, it is easy to check that the operators

$$a^\pm_x \equiv \left[\prod_{y=1}^{x-1} (-\sigma^3_y)\right] \sigma^\pm_x \tag{1.3}$$

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are a set of anticommuting operators and that, if $\sigma_x^\pm = (\sigma_x^1 \pm i\sigma_x^2)/2$, we can write

$$\sigma_x^- = e^{-i\pi \sum_{y=1}^{x-1} a_y^+ a_y^-} a_x^- \ , \quad \sigma_x^+ = a_x^+ e^{i\pi \sum_{y=1}^{x-1} a_y^+ a_y^-} \ , \quad \sigma_x^3 = 2a_x^+ a_x^- - 1 \ . \tag{1.4}$$

Hence, if we fix the units so that $J_1 + J_2 = 2$ and we introduce the *anisotropy* $u = (J_1 - J_2)/(J_1 + J_2)$, we get

$$\begin{aligned} H = \sum_{x=1}^{L-1} \Big\{ &-\frac{1}{2}[a_x^+ a_{x+1}^- + a_{x+1}^+ a_x^-] - \frac{u}{2}[a_x^+ a_{x+1}^+ + a_{x+1}^- a_x^-] - \\ &-J_3(a_x^+ a_x^- - \frac{1}{2})(a_{x+1}^+ a_{x+1}^- - \frac{1}{2}) \Big\} - h \sum_{x=1}^{L} (a_x^+ a_x^- - \frac{1}{2}) + U_L^2 \ , \end{aligned} \tag{1.5}$$

where U_L^2 is the boundary term in the new variables. We choose it so that the fermionic Hamiltonian (1.5) coincides with the Hamiltonian of a fermion system on the lattice with periodic boundary conditions, that is we put U_L^2 equal to the term in the first sum in the r.h.s. of (1.5) with $x = L$ and $a_{L+1}^\pm = a_1^\pm$ (in [LMS] this choice for the XY chain is called "c-cyclic"). It is easy to see that this choice corresponds to fix the boundary conditions for the spin variables so that

$$U_L^1 = -\frac{1}{2}[\sigma_L^+ e^{i\pi\mathcal{N}} \sigma_1^- + \sigma_L^- e^{i\pi\mathcal{N}} \sigma_1^+] - \frac{u}{2}[\sigma_L^+ e^{i\pi\mathcal{N}} \sigma_1^+ + \sigma_L^- e^{i\pi\mathcal{N}} \sigma_1^-] - \frac{J_3}{4}\sigma_L^3 \sigma_1^3 \ , \tag{1.6}$$

where $\mathcal{N} = \sum_{x=1}^{L} a_x^+ a_x$. Strictly speaking, with this choice U_L^1 does not look really like a boundary term, because $\mathcal{N}$ depends on all the spins of the chain. However $[(-1)^\mathcal{N}, H] = 0$; hence the Hilbert space splits up in two subspaces on which $(-1)^\mathcal{N}$ is equal to 1 or to -1 and on each of these subspaces U_L^1 really depends only on the boundary spins. One expects that, in the $L \to \infty$ limit, the correlation functions are independent on the boundary term, but we shall not face here this problem.

1.2 The Heisenberg XYZ chain has been the subject of a very active research over many years with a variety of methods.

A first class of results is based on the *exact solutions.* If one of the three parameters is vanishing (**e.g.** $J_3 = 0$), the model is called XY *chain.* Its solution is based on the fact that the hamiltonian, in the fermionic form (1.5), is quadratic in the fermionic fields, so that it can be diagonalized (see [LSM], [LSM1]) by a Bogoliubov transformation. If $u = 0$, we get the free Fermi gas with Fermi momentum $p_F = \arccos(-h)$; if $|u| > 0$, it turns out that the energy spectrum has a gap at p_F.

The equal time correlation functions $\Omega^\alpha(x, 0)$ were explicitly calculated in [Mc] (even at finite L and β), in the case $h = 0$, that is $p_F = \pi/2$. Note that, while $\Omega^3(\mathbf{x})$ coincides with the correlation function of the density in the fermionic representation of the model, $\Omega^1(\mathbf{x})$ and $\Omega^2(\mathbf{x})$ are given by quite complicated expressions. It turns out, for example, that, if $|u| < 1$, $\Omega^3(x, 0)$ is of the following form:

$$\Omega^3(x, 0) = -\frac{\alpha^{|x|}}{\pi^2 x^2} \sin^2\left(\frac{\pi x}{2}\right) F(-|x| \log\alpha, |x|) \ , \quad \alpha = (1 - |u|)/(1 + |u|) \ , \tag{1.7}$$

where $F(\gamma,n)$ is a bounded function, such that, if $\gamma \le 1$, $F(\gamma,n) = 1 + O(\gamma\log\gamma) + O(1/n)$, while, if $\gamma \ge 1$ and $n \ge 2\gamma$, $F(\gamma,n) = \pi/2 + O(1/\gamma)$.

For $|h| > 0$, it is not possible to get a so explicit expression for $\Omega^3(x,0)$. However, it is not difficult to prove that, if $|u| < \sin p_F$, $|\Omega^3(x,0)| \le \alpha^{|x|}$ and, if $x \ne 0$ and $|ux| \le 1$

$$\Omega^3(x,0) = -\frac{1}{\pi^2 x^2}\sin^2(p_F x)[1 + O(|ux|\log|ux|) + O(1/|x|)] \ . \tag{1.8}$$

Note that, if $u = 0$, a very easy calculation shows that $\Omega^3(x,0) = -(\pi^2 x^2)^{-2}\sin^2(p_F x)$.

We want to stress that the only case in which the correlation functions and their asymptotic behaviour can be computed explicitly in a rigorous way is just the $J_3 = 0$ case.

If two parameters are equal (e.g. $J_1 = J_2$), but $J_3 \ne 0$, the model is called XXZ model. In the case $h = 0$, it was solved in [YY] via the *Bethe-ansatz*, in the sense that the Hamiltonian was diagonalized. However, it was not possible till now to obtain the correlation functions from the exact solution. Such solution is a particular case of the general solution of the XYZ model by Baxter [B], but again *only in the case of zero magnetic field.* The ground state energy has been computed and it has been proved that there is a gap in the spectrum, which, if $J_1 - J_2$ and J_3 are not too large, is given approximately by (see [LP])

$$\Delta = 8\pi\frac{\sin\mu}{\mu}|J_1|\left(\frac{|J_1^2 - J_2^2|}{16(J_1^2 - J_3^2)}\right)^{\frac{\pi}{2\mu}} \tag{1.9}$$

with $\cos\mu = -J_3/J_1$.

The solution is based on the fact that the XYZ chain with periodic boundary conditions is equivalent to the *eight vertex* model, in the sense that H is proportional to the logarithmic derivative with respect to a parameter of the eight vertex transfer matrix, if a suitable identification of the parameters is done, see [S], [B]. The eight vertex model is obtained by putting arrows in a suitable way on a two-dimensional lattice with M rows, L columns and periodic boundary conditions. There are eight allowed vertices, and with each of them an energy is associated in a suitable way (there are four different values of the energy). With the above choice of the parameters and $T - T_c < 0$ and small, $u = O(|T - T_c|)$, so that the critical temperature of the eight vertex model corresponds to no anisotropy in the XYZ chain. Moreover, see [JKM], the correlation function C_x between two vertical arrows in a row, separated by x vertices, is given, in the limit $M \to \infty$, by $C_x = < S_0^3 S_x^3 >$. However, an explicit expression for the correlation functions cannot be derived for the XYZ or the eight vertex model. In [JKM] the correlation length of C_x was computed heuristically under some physical assumptions (an exact computation is difficult because it does not depend only on the largest and the next to the largest eigenvalues). The result is $\xi^{-1} = (T - T_c)^{\frac{\pi}{2\mu}}$, if ξ is the correlation length. One sees that the critical index of the correlation length is *non universal.*

Another interesting observation is that the XYZ model is equivalent to two interpenetrating two-dimensional Ising lattices with nearest-neighbor coupling, interacting via a four spins coupling (which is proportional to J_3). The *four spin correlation function* is identical to C_x. In the decoupling limit $J_3 = 0$ the two Ising lattices are independent and one can see that the Ising model solution can be reduced to the diagonalization, via a Bogoliubov transformation, of a quadratic Fermi Hamiltonian, see [LSM1].

Recent new results using the properties of the transfer matrix can be found in [EFIK], in which an integro-difference equation for the correlation function of the XXZ chain is obtained. It is however not clear how to deduce the physical properties of the correlation function from this equation.

1.3 Since it is very difficult to extract detailed information on the behaviour of the correlation functions from the above exact solutions, the XYZ model has been studied by quantum field theory methods, see [LP]. The idea is to approximate the fermionic hamiltonian (1.5) by the hamiltonian of the *massive Thirring model*, describing a massive relativistic spinning particle on the continuum $d = 1$ space interacting with a local current-current potential (for a heuristic justification of this approximation, see [A]).

As a relativistic field theory, the massive Thirring model is plagued by ultraviolet divergences, which were absent in the original model, defined on a lattice; one can heuristically remove this problem by introducing "by hand" an ultraviolet cut-off. A way to introduce it could be to consider a short-ranged instead of a local potential; if $J_1 = J_2$, this means that we have approximated the XXZ-chain with the *Luttinger model*, whose correlation functions can be explicitly computed, see [ML], [BGM].

The Luttinger model is defined in terms of two fields $\psi_{\mathbf{x},\omega}$, $\omega = \pm 1$, and one expects that, if $|h| < 1$ and J_3 is small enough, the large distance asymptotic behaviour of $\Omega^3(\mathbf{x})$ is qualitatively similar to that of the truncated correlation of the operator $\rho_{\mathbf{x}} = \psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}$, where $\psi^\sigma_{\mathbf{x}} = \sum_\omega \exp(i\sigma\omega p_F x)\psi^\sigma_{\mathbf{x},\omega}$, if some "reasonable" relationship between the parameters of the two models is assumed. One can make for instance the substitutions $\lambda \to -J_3$ and $p_0^{-1} \to a = 1$, if λ is the coupling in the Luttinger model, a is the chain step and p_0^{-1} is the potential range. Moreover, one expects that it is possible to choose a constant ν of order J_3, so that $h = h_0 + \nu$ and $p_F = \arccos(J_3 - h_0)$, see §1.4 below.

Of course such identification is completely arbitrary, but one can hope that for large distances the function $\Omega^3(\mathbf{x})$ has something to do with the truncated correlation of $\rho_{\mathbf{x}}$, which can be obtained by the general formula (2.5) of [BGM], based on the exact solution of [ML]. There is apparently a problem, since the expectation of $\rho_{\mathbf{x}}$ is infinite; however, it is possible to see that there exists the limit, as $\varepsilon_1, \varepsilon_2 \to 0^+$, of $[<\rho_{\mathbf{x},\varepsilon_1}\rho_{\mathbf{y},\varepsilon_2}> - <\rho_{\mathbf{x},\varepsilon_1}><\rho_{\mathbf{y},\varepsilon_2}>]$, where $\rho_{\mathbf{x},\varepsilon} = \psi^+_{(x,x_0+\varepsilon)}\psi^-_{(x,x_0)}$, and it is natural to take this quantity, let us call it $G(\mathbf{x}-\mathbf{y})$, as the truncated correlation of $\rho_{\mathbf{x}}$.

Let us define $v_0 = \sin p_F$; from (2.5) of [BGM] (by inserting a missing $(-\varepsilon_i\varepsilon_j)$ in the last sum), it follows that, for $|\mathbf{x}| \to \infty$

$$G(\mathbf{x}) \simeq [1+\lambda a_1(\lambda)] \frac{\cos(2p_F x)}{2\pi^2[(v_0^* x_0)^2 + x^2]^{1+\lambda a_3(\lambda)}} + \frac{(v_0 x_0)^2 - x^2}{2\pi^2[(v_0 x_0)^2 + x^2]^2} , \tag{1.10}$$

where $v_0^* = v_0[1+\lambda a_2(\lambda)]$ and $a_i(\lambda)$, $i = 1,2,3$, are bounded functions. Note that, in the second term in the r.h.s. of (1.10), the bare Fermi velocity v_0 appears, instead of the renormalized one, v_0^*, as one could maybe expect.

In the physical literature, it is more usual the introduction of other ultraviolet cutoffs, such that the resulting model is not exactly soluble, even if $J_1 = J_2$; however, it can be studied heuristically, see [LP], and the resulting density-density correlation function is more or less of the form (1.10).

If $J_1 \neq J_2$, there is no soluble model suitable for a similar analysis of the large distance behaviour of $\Omega^3(\mathbf{x})$. However, one can guess that the asymptotic behaviour is still of the form (1.10), if $1 << |\mathbf{x}| << 1/|u|^\alpha$, for some α. We shall prove that this is indeed true, with $\alpha = 1 + O(J_3)$.

1.4 In this paper we develop a rigorous renormalization group analysis for the XYZ Hamiltonian in its fermionic form (some "not optimal" bounds for the correlation function $\Omega^3(\mathbf{x})$ were already found in [M2]). As we said before, $\Omega^3(\mathbf{x})$ can be obtained from the exact solution only in the case $J_3 = 0$, when the fermionic theory is a non interacting one. In particular, if $\mathbf{x} = (x,0)$ and $|ux| << 1$, (1.8) and a more detailed analysis of the "small" terms in the r.h.s. (in order to prove that their derivatives of order n decay as $|x|^{-n}$), show that $\Omega^3(x,0)$ is a sum of "oscillating" functions with frequencies $(np_F)/\pi \bmod 1$, $n = 0, \pm 1$, where $p_F = \arccos(-h)$; this means that its Fourier transform has to be a smooth function, even for $u = 0$, in the neighborhood of any momentum $k \neq 0, \pm 2p_F$. These frequencies are proportional to p_F, so they depend only on the external magnetic field h.

If $J_3 \neq 0$, a similar property is satisfied for the leading terms in the asymptotic behaviour, as we shall prove, but the value of p_F depends in general also on u and J_3. For example, if $u = 0$, the Hamiltonian (1.5) is equal, up to a constant, to the Hamiltonian of a free fermion gas with Fermi momentum $p_F = \arccos(J_3 - h)$ plus an interaction term proportional to J_3. As it is well known, the interaction modifies the Fermi momentum of the system by terms of order J_3 and it is convenient (see [BG], for example), in order to study the interacting model, to fix the Fermi momentum to an interaction independent value, by adding a counterterm to the hamiltonian. We proceed here in a similar way, that is we fix p_F and h_0 so that

$$h = h_0 - \nu , \qquad \cos p_F = J_3 - h_0 , \tag{1.11}$$

and we look for a value of ν, depending on u, J_3, h_0, such that, as in the $J_3 = 0$ case, the leading terms in the asymptotic behaviour of $\Omega^3_{L,\beta}(\mathbf{x})$ can be represented as a sum of oscillating functions with frequencies $(np_F)/\pi \bmod 1$, $n = 0, \pm 1$.

As we shall see, we can realize this program only if J_3 is small enough and it turns out that ν is of order J_3. It follows that we can only consider magnetic fields such that $|h| < 1$. Moreover, it is clear that the equation $h = h_0 - \nu(u, J_3, h_0)$ can be inverted, once the function $\nu(u, J_3, h_0)$ has been determined, so that p_F is indeed a function of the parameters appearing in the original model.

If $J_1 = J_2$, it is conjectured, on the base of heuristic calculations, that to fix p_F is equivalent to the impose the condition that, in the limit $L, \beta \to \infty$, the density is fixed ("Luttinger Theorem") to the free model value $\rho = p_F/\pi$. Remembering that $\rho - \frac{1}{2}$ is the magnetization in the 3-direction for the original spin variables, this would mean that to fix p_F is equivalent to fix the magnetization in the 3 direction, by suitably choosing the magnetic field.

If $J_1 \neq J_2$, there is in any case no simple relation between p_F and the mean magnetization, as one can see directly in the case $J_3 = 0$, where one can do explicit calculations. The only exception is the case $p_F = \pi/2$, where one can see that, in the limit $L \to \infty$, $\nu = J_3$ (so that $h = 0$ by (1.11)) and $< S^3_x >= 0$. This last property easily follows from the observation that, if one choose $h = 0$ in the original Hamiltonian (1.1), then the expectation of S^3_x has to be equal to zero, by symmetry reasons, up to terms which go to 0 for $L \to \infty$.

Our main achievement is an expansion of $\Omega^3_{L,\beta}(\mathbf{x})$, which provides a very detailed and explicit description of it. We state in the following theorem some of its properties, but we stress that many other interesting properties of $\Omega^3_{L,\beta}(\mathbf{x})$ can be extracted from the expansion.

1.5 THEOREM. *Suppose that the equations (1.11) are satisfied and that $v_0 = \sin p_F \geq \bar{v}_0 > 0$, for some value of $\bar{v}_0$ fixed once for all, and let us define $a_0 = \min\{p_F/2, (\pi - p_F)/2\}$; then the following is true.*

a) There exists a constant ε, such that, if $(u, J_3) \in \mathcal{A}$, with

$$\mathcal{A} = \{(u, J_3) : |u| \leq \frac{a_0}{8(1+\sqrt{2})}, |J_3| \leq \varepsilon\} , \tag{1.12}$$

it is possible to choose ν, so that $|\nu| \leq c|J_3|$, for some constant c independent of L, β, u, J_3, p_F, and the spin correlation function $\Omega^3_{L,\beta}(\mathbf{x})$ is a bounded (uniformly in L, β, p_F and $(u, J_3) \in \mathcal{A}$) function of $\mathbf{x} = (x, x_0)$, $x = 1, \ldots, L$, $x_0 \in [0, \beta]$, periodic in x and x_0 of period L and β respectively, continuous as a function of x_0.

b) We can write

$$\Omega^3_{L,\beta}(\mathbf{x}) = \cos(2p_F x)\Omega^{3,a}_{L,\beta}(\mathbf{x}) + \Omega^{3,b}_{L,\beta}(\mathbf{x}) + \Omega^{3,c}_{L,\beta}(\mathbf{x}) , \tag{1.13}$$

with $\Omega^{3,i}_{L,\beta}(\mathbf{x})$, $i = a, b, c$, continuous bounded functions, which are infinitely times differentiable as functions of x_0, if $i = a, b$. Moreover, there exist two constants η_1 and η_2 of the form

$$\eta_1 = a_1 J_3 + O(J_3^2), \qquad \eta_2 = -a_2 J_3 + O(J_3^2) , \tag{1.14}$$

a_1 and a_2 being positive constants, uniformly bounded in L, β, p_F and $(u, J_3) \in \mathcal{A}$, such that the following is true.

Let us define

$$\mathbf{d}(\mathbf{x}) = (\frac{L}{\pi}\sin(\frac{\pi x}{L}), \frac{\beta}{\pi}\sin(\frac{\pi x_0}{\beta})) \tag{1.15}$$

and suppose that $|\mathbf{d}(\mathbf{x})| \geq 1$. Then, given any positive integers n and N, there exist positive constants $\vartheta < 1$ and $C_{n,N}$, independent of L, β, p_F and $(u, J_3) \in \mathcal{A}$, so that, for any integers $n_0, n_1 \geq 0$ and putting $n = n_0 + n_1$,

$$|\partial_{x_0}^{n_0} \bar{\partial}_x^{n_1} \Omega^{3,a}_{L,\beta}(\mathbf{x})| \leq \frac{1}{|\mathbf{d}(\mathbf{x})|^{2+2\eta_1+n}} \frac{C_{n,N}}{1 + [\Delta|\mathbf{d}(\mathbf{x})|]^N} , \tag{1.16}$$

$$|\partial_{x_0}^{n_0} \bar{\partial}_x^{n_1} \Omega^{3,b}_{L,\beta}(\mathbf{x})| \leq \frac{1}{|\mathbf{d}(\mathbf{x})|^{2+n}} \frac{C_{n,N}}{1 + [\Delta|\mathbf{d}(\mathbf{x})|]^N} , \tag{1.17}$$

$$|\Omega^{3,c}_{L,\beta}(\mathbf{x})| \leq \frac{1}{|\mathbf{d}(\mathbf{x})|^2} \left[\frac{1}{|\mathbf{d}(\mathbf{x})|^\vartheta} + \frac{(\Delta|\mathbf{d}(\mathbf{x})|)^\vartheta}{|\mathbf{d}(\mathbf{x})|^{\min\{0,2\eta_1\}}} \right] \frac{C_{0,N}}{1 + [\Delta|\mathbf{d}(\mathbf{x})|]^N} , \tag{1.18}$$

where $\bar{\partial}_x$ denotes the discrete derivative and

$$\Delta = \max\{|u|^{1+\eta_2}, \sqrt{(v_0\beta)^{-2} + L^{-2}}\} . \tag{1.19}$$

c) There exist the limits $\Omega^{3,i}(\mathbf{x}) = \lim_{L,\beta\to\infty} \Omega^{3,i}_{L,\beta}(\mathbf{x})$, $\mathbf{x} \in \mathbb{Z} \times \mathbb{R}$; they satisfy the bounds (1.16), with $|\mathbf{x}|$ in place of $|\mathbf{d}(\mathbf{x})|$. Moreover, $\Omega^{3,a}(\mathbf{x})$ and $\Omega^{3,b}(\mathbf{x})$ are even functions of $\mathbf{x}$ and there exists a constant δ^*, of order J_3, such that, if $1 \leq |\mathbf{x}| \leq \Delta^{-1}$ and $v_0^* = v_0(1+\delta^*)$, given any $N > 0$

$$\begin{aligned} \Omega^{3,a}(\mathbf{x}) &= \frac{1 + A_1(\mathbf{x})}{2\pi^2[x^2 + (v_0^* x_0)^2]^{1+\eta_1}} , \\ \Omega^{3,b}(\mathbf{x}) &= \frac{1}{2\pi^2[x^2 + (v_0^* x_0)^2]} \left\{ \frac{x_0^2 - (x/v_0^*)^2}{x^2 + (v_0^* x_0)^2} + A_2(\mathbf{x}) \right\} , \end{aligned} \tag{1.20}$$

$$|A_i(\mathbf{x})| \leq C_N \left\{ \frac{1}{1 + |\mathbf{x}|^N} + |J_3| + (\Delta|\mathbf{x}|)^{1/2} \right\} , \tag{1.21}$$

for some constant C_N.

The function $\Omega^{3,a}(\mathbf{x})$ is the restriction to $\mathbb{Z}\times\mathbb{R}$ of a function on $\mathbb{R}^2$, satisfying the symmetry relation

$$\Omega^{3,a}(x, x_0) = \Omega^{3,a}\left(x_0 v_0^*, \frac{x}{v_0^*}\right) . \tag{1.22}$$

d) Let $\hat{\Omega}^3(\mathbf{k})$, $\mathbf{k} = (k, k_0) \in [-\pi, \pi] \times \mathbb{R}^1$, the Fourier transform of $\Omega^3(\mathbf{x})$. For any fixed $\mathbf{k}$ with $\mathbf{k} \neq (0,0), (\pm 2p_F, 0)$, $\hat{\Omega}^3(\mathbf{k})$ is uniformly bounded as $u \to 0$; moreover, for some constant c_2,

$$\begin{aligned} |\hat{\Omega}^3(0,0)| &\leq c_2 \left[1 + |J_3| \log \frac{1}{\Delta} \right] , \\ |\hat{\Omega}^3(\pm 2p_F, 0)| &\leq c_2 \frac{1 - \Delta^{2\eta_1}}{2\eta_1} . \end{aligned} \tag{1.23}$$

Finally, if $u = 0$, $|\hat{\Omega}^3(\mathbf{k})| \leq c_2[1 + |J_3| \log |\mathbf{k}|^{-1}]$ near $\mathbf{k} = (0,0)$, and, at $\mathbf{k} = (\pm 2p_F, 0)$, it is singular only if $J_3 < 0$; in this case it diverges as $|\mathbf{k} - (\pm 2p_F, 0)|^{2\eta_1}/|\eta_1|$.

e) Let $G(x) = \Omega^3(x,0)$ and $\hat{G}(k)$ its Fourier transform. For any fixed $k \neq 0, \pm 2p_F$, $\hat{G}(k)$ is uniformly bounded as $u \to 0$, together with its first derivative; moreover

$$
\begin{aligned}
|\partial_k \hat{G}(0)| &\le c_2 \ , \\
|\partial_k \hat{G}(\pm 2p_F)| &\le c_2(1+\Delta^{2\eta_1}) \ .
\end{aligned}
\tag{1.24}
$$

Finally, if $u = 0$, $\partial_k \hat{G}(k)$ has a first order discontinuity at $k = 0$, with a jump equal to $1 + O(J_3)$, and, at $k = \pm 2p_F$, it is singular only if $J_3 < 0$; in this case it diverges as $|k - (\pm 2p_F)|^{2\eta_1}$.

1.6 REMARKS.

a)The above theorem holds for any magnetic field h such that $\sin p_F > 0$; remember that the exact solution given in [B] is valid only for $h = 0$. Moreover u has not to be very small, but we only need a bound of order 1 on its value, see (1.12); the only perturbative parameter is J_3. However the interesting (and more difficult) case is when also u is small.

b)A naive estimate of ε is $\varepsilon = c(\sin p_F)^\alpha$, with c, α positive numbers; in other words we must take smaller and smaller J_3 for p_F closer and closer to 0 or π, *i.e.* for magnetic fields of size close to 1. It is unclear at the moment if this is only a technical problem or a property of the model.

c)If $J_1 \neq J_2$ and $J_3 \neq 0$, one can distinguish, like in the $J_3 = 0$ case (1.7), two different regimes in the asymptotic behaviour of the correlation function $\Omega^3(\mathbf{x})$, discriminated by an intrinsic length ξ, which is approximately given by the inverse of spectral gap, whose size, is of order $|u|^{1+\eta_2}$, see (1.19), in agreement with (1.9), found by the exact solution.

If $1 << |\mathbf{x}| << \xi$, the bounds for the correlation function are the same as in the gapless $J_1 = J_2$ case; if $\xi << |\mathbf{x}|$, there is a faster than any power decay with rate of order ξ^{-1}. In the first region we can obtain the exact large distance asymptotic behaviour of $\Omega^3(\mathbf{x})$, see (1.20),(1.21); in the second region only an upper bound is obtained. Note that, even in the $J_3 = 0$ case, it is not so easy to obtain a more precise result, if $h \neq 0$, see §1.2.

The spin interaction in the z direction has the effect that the gap becomes anomalous, in the sense that it acquires a *critical index* η_2; the ratio between the "renormalized" and the "bare" gap is very small or very large, if u is small, depending on the sign of J_3.

d)It is useful to compare the expression for the large distance behaviour of $\Omega^3(\mathbf{x})$ in the case $u = 0$ with its analogous for the Luttinger model, see §1.3. A first difference is that, while in the Luttinger model the Fermi momentum is independent of the interaction, in the XYZ model in general *it is changed non trivially* by the interaction, unless the magnetic external field is zero, *i.e.* $p_F = \frac{\pi}{2}$. The reason is that the Luttinger model has special parity properties which are not satisfied by the XYZ chain (except if the magnetic field is vanishing).

e)Another peculiar property of the Luttinger model correlation function is that it depends on p_F only through the factor $\cos(2p_F x)$; this is true not only for the asymptotic behaviour

(1.10), but also for the complete expression given in [BGM], and is due to a special symmetry of the Luttinger model (the Fermi momentum disappears from the Hamiltonian if a suitable redefinition of the fermionic fields is done, see [BGM]). This property is of course not true in the XYZ model and in fact the dependence on p_F of $\Omega^3(\mathbf{x})$ is very complicated. However we prove that $\Omega^3(\mathbf{x})$ can be written as sum of three terms, see (1.13), and the first two terms are very similar to the two terms in the r.h.s. of (1.10). In particular, the functions $\Omega^{3,a}(\mathbf{x})$ and $\Omega^{3,b}(\mathbf{x})$ have the same power decay as the analogous functions in the Luttinger model and are "free of oscillations", in the sense that each derivative increases the decay power of one unit, see (1.16),(1.17).

This is not true for the third term $\Omega^{3,c}(\mathbf{x})$, which does not satisfy a similar bound, because of the presence of oscillating contributions. However we can prove that such term, if $u = 0$, is negligible for large distances, see (1.18) (note that ϑ is J_3 and u independent, unlike η_1). Of course this is true only for small J_3 and it could be that $\Omega^{3,c}(\mathbf{x})$ plays an important role for larger J_3.

If we compare, in the case $u = 0$, the functions $\Omega^{3,a}(\mathbf{x})$ and $\Omega^{3,b}(\mathbf{x})$, see (1.20), with the corresponding ones in the Luttinger model, see (1.10), we see that they differ essentially for the non oscillating functions $A_i(\mathbf{x})$, containing terms of higher order in our expansion. However, this difference is not important in the case of $\Omega^{3,a}(\mathbf{x})$, which also satisfies the same symmetry property (1.22) as the analogue in the Luttinger model, of course with different values of v_0^*; note that the validity of (1.22) allows to interpret v_0^* as the *renormalized Fermi velocity*. Guided by the analogy with the Luttinger model, one would like to prove a similar property for $\Omega^{3b}(\mathbf{x})$ with v_0 replacing v_0^*; such property holds in fact for the Luttinger model, see (1.10). However we were not able to prove a similar properties for $A_2(\mathbf{x})$, and this has some influence on our results, see below.

f)Another important property of the Luttinger model correlation function is the fact that the "not oscillating term", that is the term corresponding to $\Omega^{3,b}(\mathbf{x})$, *does not acquire a critical index*, contrary to what happens for the term corresponding to $\cos(2p_F x)\Omega^{3,a}(\mathbf{x})$. Hence one is naturally led to the conjecture that the critical index of $\Omega^{3,b}_{L,\beta}(\mathbf{x})$ is still vanishing, see for instance [Sp]. In our expansion, the critical index of $\Omega^{3,b}(\mathbf{x})$ is represented as a convergent series, but, even if an explicit computation of the first order term gives a vanishing result, it is not obvious that this is true at any order. However, due to some hidden symmetries of the model (*i.e.* symmetries approximately enjoyed by the relevant part of the effective interaction), we can prove a suitable *approximate Ward identity*, implying that all the coefficients of the series are indeed vanishing.

g) The above properties can be used to study the Fourier transform $\hat{G}(k)$ of the equal time correlation function $G(x) = \Omega^3(x,0)$. If $J_3 = 0$, $\hat{G}(k)$ is bounded together with its first order derivative up to $u = 0$; in fact, the possible logarithmic divergence at $k = \pm 2p_F$ and $k = 0$ (if $u = 0$) of $\partial\hat{G}(k)$ is changed by the parity properties of $G(x)$ in a first order discontinuity.

If $J_3 \neq 0$, $\partial\hat{G}(k)$ behaves near $k = \pm 2p_F$ in a completely different way. In fact it is bounded and continuous if $J_3 > 0$, while it has a power like singularity, if $u = 0$ and $J_3 < 0$, see item e) of Theorem (1.5). This is due to the fact that the critical index η_1, characterizing the asymptotic behaviour of $\Omega^{3,a}(\mathbf{x})$, has the same sign of J_3 (note that η_1 has nothing to do with the critical index η related with the two point fermionic Schwinger function, which is $O((J_3)^2)$).

On the other hand, the behaviour of $\partial\hat{G}(k)$ near $k = 0$ is the same for the Luttinger model, the XYZ model and the free fermionic gas ($J_1 = J_2$, $J_3 = 0$) (see also [Sp] for a heuristic explanation). This is due to the vanishing of the critical index related with $\Omega^{3,b}(\mathbf{x})$ and to the parity properties of the leading terms, which change, as in the $J_3 = 0$ case, the apparent dimensional logarithmic divergence in a first order discontinuity.

h) If $u = 0$, the (two dimensional) Fourier transform can be singular only at $\mathbf{k} = (0, 0)$ and $\mathbf{k} = (\pm 2p_F, 0)$. If $J_3 = 0$, the singularity is logarithmic at $\mathbf{k} = (\pm 2p_F, 0)$; if $J_3 \neq 0$, the singularity is removed if $J_3 > 0$, while it is enhanced to a power like singularity if $J_3 < 0$, see item d) in the Theorem (1.5). Hence, the singularity at $\mathbf{k} = (\pm 2p_F, 0)$ is of the same type as in the Luttinger model, see (1.10).

However, we can not conclude that the same is true for the Fourier transform at $\mathbf{k} = 0$, which is bounded in the Luttinger model, while we can not exclude a logarithmic divergence. In order to get such a stronger result, it would be sufficient to prove that the function $\Omega^{3,b}(\mathbf{x})$ is odd in the exchange of (x, x_0) with $(x_0 v, x/v)$, for some v; this property is true for the leading term corresponding to $\Omega^{3,b}(\mathbf{x})$ in (1.10), with $v = v_0$, but seems impossible to prove on the base of our expansion. We can only see this symmetry for the leading term, with $v = v_0^*$ (or any other value v differing for terms of order J_3, since the substitution of v_0^* with v would not affect the bound (1.21)), but this is only sufficient to prove that the singularity has to be of order J_3, at least.

i) If $u = 0$, the critical indices and ν can be computed with any prefixed precision; we write explicitly in the theorem only the first order for simplicity. However, if $u \neq 0$, they are not fixed uniquely; for what concerns ν, this means that, in the gapped case, the system is insensitive to variations of the magnetic field much smaller than the gap size.

l) There is no reason to restrict the analysis to a nearest-neighbor Hamiltonian like (1.1); it will be clear in the following that our results still holds for non nearest-neighbor spin hamiltonians; see also [Spe].

m) The same techniques could perhaps be used to study $\Omega^1_{L,\beta}(\mathbf{x})$ and $\Omega^2_{L,\beta}(\mathbf{x})$, however this problem is more difficult, as one has to study the average of the exponential of the sum of fermionic density operators, see(1.4). In the $J_3 = 0$ case the evaluation of $\Omega^1_{L,\beta}(\mathbf{x})$ and

$\Omega^2_{L,\beta}(\mathbf{x})$ was done in [Mc].

1.7 In order to prove Theorem 1.5, we use the well known representation of $\Omega^3_{\mathbf{x}-\mathbf{y}}$ in terms of a *Grassmanian integral*, that is

$$\Omega^3_{\mathbf{x}-\mathbf{y}} = \frac{\int \mathcal{D}\psi e^{A(\psi)} \rho_{\mathbf{x}} \rho_{\mathbf{y}}}{\int \mathcal{D}\psi e^{A(\psi)}} - \frac{\int \mathcal{D}\psi e^{A(\psi)} \rho_{\mathbf{x}}}{\int \mathcal{D}\psi e^{A(\psi)}} \cdot \frac{\int \mathcal{D}\psi e^{A(\psi)} \rho_{\mathbf{y}}}{\int \mathcal{D}\psi e^{A(\psi)}} \ , \tag{1.25}$$

where $\psi^{\pm}_{\mathbf{x}}$ are elements of a *Grassmanian algebra*, $\mathcal{D}\psi$ is the usual *Lebesgue measure* on the algebra, $A(\psi)$ is the action corresponding to (1.5) and $\rho_{\mathbf{x}} = \psi^{+}_{\mathbf{x}} \psi^{-}_{\mathbf{x}}$. Of course, in order to give a meaning to (1.25), we have to regularize the model so that the Grassmanian algebra is finite dimensional, hence we introduce an ultraviolet cutoff also in the time variable (in the space variable such cut-off is provided by the lattice, while the finite volume and temperature provide natural infrared cutoffs); see §2.1 for the precise definitions. This procedure allows to write expansions for the physical quantities, which satisfy uniform bounds in the various cutoffs and admit a well defined limit as the ultraviolet cutoff is removed and as L, β go to infinity.

For pedagogical reasons we begin our analysis not directly from (1.25) but from the normalization $\int \mathcal{D}\psi e^{A(\psi)}$, which is much easier to study; the expansion for $\Omega^3_{\mathbf{x}}$ will be clearer once the expansion for $\int \mathcal{D}\psi e^{A(\psi)}$ is understood. The simplest way to evaluate such Grassmanian integral is to write

$$\frac{1}{\mathcal{N}} \int \mathcal{D}\psi e^{A(\psi)} = \int P(d\psi) e^{-\mathcal{V}(\psi)} = \sum_{n=0}^{\infty} \int P(d\psi) \frac{[-\mathcal{V}(\psi)]^n}{n!} \ , \tag{1.26}$$

where $\mathcal{N}$ is a suitable constant, $P(d\psi)$ is the Grassmanian measure generated by the quadratic terms of (1.5) for $u = 0$, to be called the *free measure*, while $\mathcal{V}(\psi)$ contains the other terms together with the counterterm for the chemical potential, in agreement with the discussion of §1.4. In other words we are considering as reference model the *isotropic* XY model with hamiltonian (1.5) with $J_3 = u = 0$ and $h = -\cos p_F$.

If $Q(\psi)$ is a monomial in the Grassmanian variables, it is easy to see that $\int P(d\psi) Q(\psi)$ is given by the anticommutative *Wick rule*; the corresponding propagator has two singularities in momentum space, as $L, \beta \to \infty$, at $\mathbf{k} = (\pm p_F, 0)$. As a consequence, one can see that the r.h.s. of (1.26) is, when the ultraviolet cutoff is removed, a series convergent for $|u|, |J_3| \leq \varepsilon_{L,\beta}$, with $\varepsilon_{L,\beta} \to_{L,\beta\to\infty} 0$, so that we cannot control the zero temperature infinite volume limit by the trivial perturbative expansion.

If $u \neq 0$, the failure of the above expansion is quite clear, since we are expanding around the isotropic XY model, so considering the anisotropy, which is a sort of *mass*, as a perturbation. We could instead include the anisotropy in the free measure by writing

$$\frac{1}{\mathcal{N}} \int \mathcal{D}\psi e^{A(\psi)} = \int \tilde{P}(d\psi) e^{-\tilde{\mathcal{V}}(\psi)} = \sum_{n=0}^{\infty} \int \tilde{P}(d\psi) \frac{[-\tilde{\mathcal{V}}(\psi)]^n}{n!} \ , \tag{1.27}$$

where $\tilde{P}(d\psi)$ is the Grassmanian measure generated by all the quadratic terms of (1.5). In this case the propagator in momentum space corresponding to $\tilde{P}(d\psi)$ has no singularities, for $L, \beta \to \infty$; in fact, it is easy to see, by a *Bogoliubov transformation*, that there is an $O(u)$ mass term which plays the role of an infrared cutoff, so that one can indeed prove the convergence of the r.h.s. of (1.27) in the $L, \beta \to \infty$ limit. However convergence holds only for $|J_3| \le \varepsilon_u$, with $\varepsilon_u \to_{u \to 0} 0$, *i.e.* it is not uniform in the anisotropy. Then also this expansion fails in providing results in the critical region of parameters we are interested in. The point is that the evaluation of the correlation functions in the critical region by simple power series in J_3 cannot work (even if one introduces a counterterm in order to fix the singularities of the correlation functions at the same point when $J_3 = 0$ or $J_3 \neq 0$), as the $J_3 = 0$ theory in not *analytically close* to the $J_3 \neq 0$ theory; this is clear if one looks, for instance, to the gap (1.9), which cannot be expanded in a power series of J_3 for u small enough.

We have then to set-up a much more complicated procedure to evaluate the correlation function (1.25) and the partition function (1.26). This procedure is based on (Wilsonian) Renormalization group as implemented in [BG1]. The idea is to take as a reference model the *isotropic* XY model like in (1.26), by considering u and J_3 as perturbations. However, we do not simply expand in power series of u and J_3 as in (1.26). The first step (see §(2)) is to decompose the measure $P(d\psi)$ as a product of independent measures $P(d\psi) = \prod_{h=-\infty}^{1} P(d\psi^{(h)})$, where the momentum space propagator corresponding to $P(d\psi^{(h)})$ is not singular, but $O(\gamma^{-h})$, for $L, \beta \to \infty$, γ being a fixed *scaling parameter* greater than 1. This decomposition is realized by slicing in a smooth way the momentum space, so that $\psi^{(h)}$, if $h \le 0$, depends only on the momenta between γ^{h-1} and γ^{h+1}. Then we integrate each field iteratively, starting from $\psi^{(1)}$, so obtaining a sequence of *effective potentials* $\mathcal{V}^{(h)}$ in the following way. We write, if $\psi^{(\le j)} = \sum_{h=-\infty}^{j} \psi^{(h)}$,

$$\int P(d\psi) e^{-\mathcal{V}(\psi)} = \int \prod_{h=-\infty}^{1} P(d\psi^{(h)}) e^{-\mathcal{V}(\psi^{(\le +1)})} = \int P(d\psi^{(\le 0)}) e^{-\mathcal{V}^{(0)}(\psi^{(\le 0)})} \qquad (1.28)$$

where $\mathcal{V}^{(0)}(\psi)$ can be written as a sum (finite, since we work with a finite algebra) of monomials in the Grassmanian variables, with coefficients which are perturbative finite expansions converging, uniformly in L and β, to well defined power series, as the ultraviolet cutoff is removed.

According to Renormalization Group, one has to identify in the effective potential the *relevant*, *marginal* and *irrelevant* terms. We write then $\mathcal{V}^{(0)} = \mathcal{L}\mathcal{V}^{(0)} + \mathcal{R}\mathcal{V}^{(0)}$, where $\mathcal{R} = 1 - \mathcal{L}$ and $\mathcal{L}$ is a linear operator, called *localization operator*, whose role is to extract from $\mathcal{V}^{(0)}$ some local relevant or marginal terms, so that the remainder is irrelevant. It turns out that all the monomials with six Grassmanian variables or more are irrelevant, while the terms quartic in ψ and similar to the original interaction (but with a different coupling) are marginal. The relevant terms are all quadratic in the Grassmanian variables; more exactly there are terms like $\psi^{+(\le 0)}\psi^{-(\le 0)}$, representing the shift in the chemical potential, and terms like $\psi^{+(\le 0)}\psi^{+(\le 0)}$ or $\psi^{-(\le 0)}\psi^{-(\le 0)}$, which behave as mass terms. Note that the definition of relevant terms is not simply based on power counting arguments but also on momentum conservation considerations, *i.e.* the power counting must be improved with respect to the trivial one, as we shall see in §2. The relevant terms corresponding to the shift of the chemical potential are controlled by choosing in a suitable way the counterterm ν so that they are smaller and smaller at each Renormalization Group iteration. On the contrary, the relevant mass terms must be included in the reference free measure, which then acquires a mass. Among the marginal terms there are also quadratic terms of the form $\psi^{+(\le 0)}\partial\psi^{-(\le 0)}$, related with the wave function renormalization, which must be also included in the reference measure; then we write (1.28) as

$$\int P(d\psi^{(\le 0)}) e^{-\mathcal{L}\mathcal{V}^0(\psi^{\le 0})-\mathcal{R}\mathcal{V}^0(\psi^{\le 0})} = \frac{1}{\mathcal{N}}\int \bar{P}(d\psi^{(\le 0)}) e^{-\mathcal{L}\bar{\mathcal{V}}^0(\psi^{\le 0})-\mathcal{R}\mathcal{V}^0(\psi^{\le 0})} \tag{1.29}$$

where $\bar{P}(d\psi^{(h)})$ is the new reference measure, obtained by absorbing in the old one the terms in $\mathcal{L}\mathcal{V}^{(0)}$ which are quadratic in the Grassmanian variables and are related with the mass and the wave function renormalization. We then integrate the field $\psi^{(0)}$ and the procedure is iterated, so that at each step we get new contributions to the mass and wave function renormalization and the field $\psi^{(h)}$ is integrated by a measure with mass σ_h and wave function renormalization Z_h; the iteration stops as soon as σ_h, which at the beginning is of size $|u| \le 1$, becomes of order γ^h, that is the same order of the momenta contributing to $\psi^{(h)}$. If we call h^* the corresponding value of h, the integration of the field $\psi^{(\le h^*)}$ can be performed in a single step, since σ_{h^*} acts as an infrared cutoff on the momentum scale γ^{h^*}.

Of course $h^* \to -\infty$ as $u \to 0$, but the dependence of the *effective mass* σ_{h^*} on u and J_3 is highly non trivial. In fact σ_{h^*}/u tends to 0 or ∞, as $u \to 0$, depending on the sign of J_3; this result can be expressed in terms of a critical index, see (1.19). Note that the inclusion of the mass term in the reference measure means essentially that we have to perform a different Bogoliubov transformation at each integration step (up to $h = h^*$), instead of a single one as in (1.28), in order to take into account the *anomalous* dependence of the effective mass on u.

Also the dependence of the wave function renormalization Z_{h^*} on u and J_3 is non trivial; it turns out that $Z_{h^*} \simeq \gamma^{cJ_3^2|h^*|}$, with $c > 0$, so it diverges as $u \to 0$. This result is strictly related with the fact that, for $u = 0$, $Z_h \simeq \gamma^{cJ_3^2|h|}$, implying an anomalous asymptotic behaviour of the field correlation function.

This iterative procedure allows to write the effective potential on any scale as a sum of monomials in the Grassmanian variables, with coefficients which are perturbative expansions (well defined as the ultraviolet cutoff is removed, uniformly in L and β) in terms of a few *running coupling constants* and *renormalization constants*. The running coupling constants are λ_h, which is the effective coupling of the interaction between fermions, ν_h, related with the chemical potential renormalization, and δ_h, related with the shift of the Fermi velocity. The renormalization constants are σ_h and Z_h. The running coupling constants and the renormalization constants verify a recursive equation called *Beta function*.

In §3 we prove that such expansions can be controlled, uniformly in L and β (even if the renormalization constants are diverging or go to 0 as $h \to -\infty$), if the running coupling constants are small enough, see Theorem 3.12. §3 is the more technical section of the paper; the expansion is written as sum over *trees* and we use determinant bounds for the fermionic expectations. The proof of the convergence requires some care as the power counting has to be improved. Moreover we pay attention to perform all the estimates taking finite L, β; this requires some care, as the preceding analysis of similar problems were not so careful about this point.

In §5 we build an expansion for the correlation function in the direction of the magnetic field $\Omega^3_{\mathbf{x}}$, which is very similar to the previous one. The idea is to note that $\Omega^3_{\mathbf{x}-\mathbf{y}} = [\partial^2 \mathcal{S}(\phi)/\partial\phi_{\mathbf{x}}\partial\phi_{\mathbf{y}}]|_{\phi=0}$, where $\phi_{\mathbf{x}}$ is a bosonic external field and

$$e^{\mathcal{S}(\phi)} = \int P(d\psi) e^{-\mathcal{V}(\psi)+\int d\mathbf{x}\phi_{\mathbf{x}}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}} \ . \tag{1.30}$$

Hence, the previous analysis can be applied, by adding a new term to the interaction. As we shall see, this implies that we have to introduce two new renormalization constants, $Z^{(1)}_h$ and $Z^{(2)}_h$, related respectively with the oscillating and non oscillating part of the correlation function (*i.e.* the first two addenda in (1.13)). We prove the convergence also for this expansion and careful estimates on the Fourier transform are obtained, always under the hypothesis that the running coupling constants are small enough, see Theorem 5.8.

Once the convergence problems of the renormalized expansions are solved, one has still to face two main problems: the first one is to show that the running coupling constants indeed remain small if J_3 is small enough; the second one is to prove that the ratio between Z_h and $Z^{(2)}_h$, both diverging as $h \to -\infty$ (which is an important property for u small) is close to one. This last property is not essential to prove the convergence of the series, but it is crucial to obtain the correct asymptotic behaviour of the correlation function as it is related to the vanishing of a critical index appearing in the non oscillating part of the correlation function.

Both problems are solved in §4 and §7 by a careful analysis of the cancellations arising in our expansions as a consequence of some symmetry properties. We write the beta function governing the flow of λ_h, δ_h and of $Z^{(2)}_h/Z_h$ as a sum of several terms, and we show that only one term is really crucial, while the other ones have a little effect on the flow in absence of the first one, if the finite counterterm ν is chosen in a proper way. On the other hand, one recognizes that such crucial contribution to the Beta function of the XYZ model is coinciding with the Beta function obtained by applying the same Renormalization group analysis to the Luttinger model. For such model many properties are true, like local gauge invariance and exact solubility (thanks to the possibility of representing its Hamiltonian as a quadratic bosonic one, [ML]); these properties are not enjoyed by the XYZ hamiltonian but the model is close, in a Renormalization group sense, to a model enjoying them.

Note that, despite the fact that the Luttinger model Hamiltonian is formally gauge invariant, the ultraviolet and infrared cutoffs introduced to perform our Renormalization group analysis have the effect that gauge invariance is lost even in that model. Nevertheless in §7 we can derive an approximate Ward identity (approximate as the gauge invariance is only approximately true), which tells us that in the Luttinger model

$$\frac{Z_h^{(2)}}{Z_h} = 1 + O(\lambda) \ . \tag{1.31}$$

Note that the formal Ward identity obtained in absence of cutoffs would give exactly one in the r.h.s. of (1.31). This result is obtained by considering a tree expansion also for the corrections to the formal Ward identity (the bounds for the corrections are only sketched and more details will be published elsewhere). In §5 we show how to use (1.31) to prove that the critical index related with the asymptotic behaviour of the leading non oscillating part of the XYZ model correlation function (the second term in the r.h.s. of (1.13)) is *exactly* vanishing.

By using another important property of the Luttinger model, *i.e.* its exact solubility, it was proved in [GS], [BGPS], [BM1] that the beta function of the Luttinger model for the running coupling constants is vanishing; this means that the crucial contribution to the XYZ beta function is vanishing. This result is used in §4 to prove that the running coupling constants are small for any h.

Finally in §(7) we complete the proof of the main theorem, deriving the correlation function properties listed in the main theorem. In particular we prove, for J_3 small enough

1) upper bounds for the asymptotic behaviour, see (1.16), (1.17), (1.18);
2) a rather explicit expression for the asymptotic behaviour for distances smaller than the inverse of the gap, see (1.20), (1.21);
3) bounds for the Fourier transform of the correlation function and its derivatives in the limit $L = \beta = \infty$.

2. Multiscale decomposition and anomalous integration

2.1 The Hamiltonian (1.5) can be written, if U_L^2 is chosen as explained in §1.1 and the definitions (1.11) are used, in the following way (by neglecting a constant term):

$$H = \sum_{x\in\Lambda} \Big\{ (\cos p_F + \nu) a_x^+ a_x^- \; - \frac{1}{2}[a_x^+ a_{x+1}^- + a_{x+1}^+ a_x^-] - \\ - \frac{u}{2}[a_x^+ a_{x+1}^+ + a_{x+1}^- a_x^-] + \lambda (a_x^+ a_x^-)(a_{x+1}^+ a_{x+1}^-) \Big\} \; , \tag{2.1}$$

where Λ is an interval of L points on the one-dimensional lattice of step one, which will chosen equal to $(-[L/2], [(L-1)/2])$, the fermionic field $a_x^\pm$ satisfies periodic boundary conditions and

$$\lambda = -J_3 \; . \tag{2.2}$$

The Hamiltonian (2.1) will be considered as a perturbation of the Hamiltonian H_0 of a system of free fermions in Λ with unit mass and chemical potential $\mu = 1 - \cos p_F$ ($u = J_3 = \nu = 0$); p_F is the *Fermi momentum*. This system will have, at zero temperature, density $\rho = p_F/\pi$, corresponding to magnetization $\rho - 1/2$ in the 3-direction for the original spin system. Since p_F is not uniquely defined at finite volume, we choose it so that

$$p_F = \frac{2\pi}{L}(n_F + \frac{1}{2}) \; , \quad n_F \in \mathbb{N} \; , \quad \lim_{L\to\infty} p_F = \pi\rho \tag{2.3}$$

This means, in particular, that p_F is not an allowed momentum of the fermions.

We consider also the operators $a_{\mathbf{x}}^\pm = e^{x_0 H} a_x^\pm e^{-Hx_0}$, with

$$\mathbf{x} = (x, x_0) \; , \quad -\beta/2 \le x_0 \le \beta/2 \; , \tag{2.4}$$

for some $\beta > 0$; on x_0, which we shall call the time variable, antiperiodic boundary conditions are imposed.

Many interesting physical properties of the fermionic system at inverse temperature β can be expressed in terms of the *Schwinger functions*, that is the truncated expectations in the Grand Canonical Ensemble of the time order product of the field $a_{\mathbf{x}}^\pm$ at different space-time points. There is of course a relation between these functions and the expectations of some suitable observables in the spin system. However, by looking at (1.4), one sees that this relation is simple enough only in the case of the truncated expectations of the time order product of the fermionic *density operator* $\rho_{\mathbf{x}} = a_{\mathbf{x}}^+ a_{\mathbf{x}}^-$ at different space-time points, which we shall call the *density Schwinger functions*; they coincide with the truncated expectations of the time order product of the operator $S_{\mathbf{x}}^3 = e^{x_0 H} S_x^3 e^{-Hx_0}$ at different space-time points.

As it is well known, the Schwinger functions can be written as power series in λ and u, convergent for $|\lambda|, |u| \le \varepsilon_\beta$, for some constant ε_β (the only trivial bound of ε_β goes to zero, as $\beta \to \infty$). This power expansion is constructed in the usual way in terms of Feynman graphs, by using as *free propagator* the function

$$g^{L,\beta}(\mathbf{x}-\mathbf{y}) = \frac{\mathrm{Tr}\left[e^{-\beta H_0}\mathbf{T}(a^-_{\mathbf{x}}a^+_{\mathbf{y}})\right]}{\mathrm{Tr}[e^{-\beta H_0}]} = \\ = \frac{1}{L}\sum_{k\in\mathcal{D}_L} e^{-ik(x-y)}\left\{\frac{e^{-\tau e(k)}}{1+e^{-\beta e(k)}}\mathbf{1}(\tau>0) - \frac{e^{-(\beta+\tau)e(k)}}{1+e^{-\beta e(k)}}\mathbf{1}(\tau\le 0)\right\}\ , \tag{2.5}$$

where $\mathbf{T}$ is the time order product, $N=\sum_{x\in\Lambda} a^+_x a^+_x$, $\tau = x_0 - y_0$, $\mathbf{1}(E)$ denotes the indicator function ($\mathbf{1}(E)=1$, if E is true, $\mathbf{1}(E)=0$ otherwise),

$$e(k) = \cos p_F - \cos k\ , \tag{2.6}$$

and $\mathcal{D}_L \equiv \{k = 2\pi n/L, n\in\mathbb{Z}, -[L/2]\le n\le [(L-1)/2]\}$.

It is also well known that, if $x_0\neq y_0$, $g^{L,\beta}(\mathbf{x}-\mathbf{y}) = \lim_{M\to\infty} g^{L,\beta,M}(\mathbf{x}-\mathbf{y})$, where

$$g^{L,\beta,M}(\mathbf{x}-\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{-ik_0+\cos p_F-\cos k}\ , \tag{2.7}$$

$\mathbf{k}=(k,k_0)$, $\mathbf{k}\cdot\mathbf{x} = k_0x_0+kx$, $\mathcal{D}_{L,\beta}\equiv\mathcal{D}_L\times\mathcal{D}_\beta$, $\mathcal{D}_\beta\equiv\{k_0=2(n+1/2)\pi/\beta, n\in Z, -M\le n\le M-1\}$. Note that $g^{L,\beta,M}(\mathbf{x}-\mathbf{y})$ is real, $\forall M$.

Hence, if we introduce a finite set of Grassmanian variables $\{\hat a^\pm_{\mathbf{k}}\}$, one for each $\mathbf{k}\in\mathcal{D}_{L,\beta}$, and a linear functional $P(da)$ on the generated Grassmanian algebra, such that

$$\int P(da)\hat a^-_{\mathbf{k}_1}\hat a^+_{\mathbf{k}_2} = L\beta\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat g(\mathbf{k}_1)\ ,\quad \hat g(\mathbf{k}) = \frac{1}{-ik_0+\cos p_F-\cos k}\ , \tag{2.8}$$

we have

$$\lim_{M\to\infty}\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\hat g(\mathbf{k}) = \lim_{M\to\infty}\int P(da)a^-_{\mathbf{x}}a^+_{\mathbf{y}} \equiv g^{L,\beta}(\mathbf{x};\mathbf{y})\ , \tag{2.9}$$

where the *Grassmanian field* $a_{\mathbf{x}}$ is defined by

$$a^\pm_{\mathbf{x}} = \frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\hat a^\pm_{\mathbf{k}}e^{\pm i\mathbf{k}\cdot\mathbf{x}}\ . \tag{2.10}$$

The "Gaussian measure" $P(da)$ has a simple representation in terms of the "Lebesgue Grassmanian measure" $\prod_{\mathbf{k}\in\mathcal{D}_{L,\beta}} da^+_{\mathbf{k}}da^-_{\mathbf{k}}$, defined as the linear functional on the Grassmanian algebra, such that, given a monomial $Q(a^-,a^+)$ in the variables $a^-_{\mathbf{k}}, a^+_{\mathbf{k}}$, $\mathbf{k}\in\mathcal{D}_{L,\beta}$, its value is 0, except in the case $Q(a^-,a^+)=\prod_{\mathbf{k}}\hat a^-_{\mathbf{k}}\hat a^+_{\mathbf{k}}$, up to a permutation of the variables. In this case the value of the functional is determined, by using the anticommuting properties of the variables, by the condition

$$\int\left[\prod_{\mathbf{k}\in\mathcal{D}_{L,\beta}} da^+_{\mathbf{k}}da^-_{\mathbf{k}}\right]\prod_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\hat a^-_{\mathbf{k}}\hat a^+_{\mathbf{k}} = 1\ . \tag{2.11}$$

We have

$$P(da) = \Big\{\prod_{\mathbf{k}}(L\beta\hat g_{\mathbf{k}})\hat a^+_{\mathbf{k}}\hat a^-_{\mathbf{k}}\Big\}\exp\Big\{-\sum_{\mathbf{k}}(L\beta\hat g_{\mathbf{k}})^{-1}\hat a^+_{\mathbf{k}}\hat a^-_{\mathbf{k}}\Big\}\ . \tag{2.12}$$

Note that, since $(\hat a^-_{\mathbf k})^2 = (\hat a^+_{\mathbf k})^2 = 0$, $e^{-z\hat a^+_{\mathbf k}\hat a_{\mathbf k}} = 1 - z\hat a^+_{\mathbf k}\hat a_{\mathbf k}$, for any complex z.

Remark. The *ultraviolet cutoff* M on the k_0 variable was introduced so that the Grassmanian algebra is finite; this implies that the Grassmanian integration is indeed a simple algebraic operation and all quantities that appear in the calculations are finite sums. However, M does not play any essential role in this paper, since all bounds will be uniform with respect to M and they easily imply the existence of the limit. The only problem is that, if $x_1 = y_1$, the propagator (2.5) has a first order discontinuity at $x_0 - y_0 = 0$, where it has to be defined as the limit from the left, while $\lim_{M\to\infty} g^{L,\beta,M}(0,0) = [g^{L,\beta}(0,0^+) + g^{L,\beta}(0,0^-)]/2$. One could take care of this problem, by adding to the r.h.s. of (2.10) a factor $\exp(i\delta_M k_0)$, where δ_M is a suitable positive constant proportional to $\beta/\sqrt{M}$, and by leaving unchanged (2.8); then the r.h.s. of (2.7) is multiplied by $\exp(2i\delta_M k_0)$ and it is easy to see that the new propagator has the right value in $\mathbf{x} = 0$ for $M \to \infty$. In order to simplify the notation, we shall neglect this minor problem in the following and we shall not stress the dependence on M of the various quantities we shall study.

By using standard arguments (see, for example, [NO], where a different regularization of the propagator is used), one can show that the partition function and the Schwinger functions can be calculated as expectations of suitable functions of the Grassmanian field with respect to the "Gaussian measure" $P(da)$. In particular the partition function $\mathrm{Tr}[e^{-\beta H}]$ is equal to $\mathcal{Z}_{L,\beta}\mathrm{Tr}[e^{-\beta H_0}]$, with

$$\mathcal{Z}_{L,\beta} = \int P(da) e^{-\mathcal{V}(a)} , \tag{2.13}$$

where

$$\begin{aligned}
&\mathcal{V}(a) = uV_u(a) + \lambda V_\lambda(a) + \nu N(a) ,\\
&V_\lambda(a) = \sum_{x,y\in\Lambda} \int_{-\beta/2}^{\beta/2} dx_0 \int_{-\beta/2}^{\beta/2} dy_0 v_\lambda(\mathbf{x}-\mathbf{y}) a^+_{\mathbf x} a^+_{\mathbf y} a^-_{\mathbf y} a^-_{\mathbf x} , \qquad N(a) = \sum_{x\in\Lambda} \int_{-\beta/2}^{\beta/2} dx_0 a^+_{\mathbf x} a^-_{\mathbf x} ,\\
&V_u(a) = \sum_{x,y\in\Lambda} \int_{-\beta/2}^{\beta/2} dx_0 \int_{-\beta/2}^{\beta/2} dy_0 v_u(\mathbf{x}-\mathbf{y}) \left[a^+_{\mathbf x} a^+_{\mathbf y} - a^-_{\mathbf x} a^-_{\mathbf y}\right]
\end{aligned} \tag{2.14}$$

where

$$v_\lambda(\mathbf{x}-\mathbf{y}) = \frac{1}{2}\delta_{1,|x-y|}\delta(x_0-y_0) , \quad v_u(\mathbf{x}-\mathbf{y}) = \frac{1}{2}\delta_{x,y+1}\delta(x_0-y_0) . \tag{2.15}$$

Note that the parameter ν has been introduced in order to fix the singularities of the interacting propagator to the values of the free model, that is $\mathbf{k} = (0, \pm p_F)$. Hence ν is a function of λ, u, p_F, which has to be fixed so that the perturbation expansion is convergent (uniformly in L, β). This choice of ν has also the effect of fixing the singularities of the spin correlation function Fourier transform, as we explained in the introduction, see §1.4.

Note that, if $p_F = \pi/2$, one can prove that $\nu = -\lambda$, by using simple symmetry properties of our expansion; this implies, by using (1.11), that $h = 0$.

If $u = 0$, it is conjectured, on the base of heuristic calculations, that this condition is equivalent to the condition that, in the limit $L, \beta \to \infty$, the density is fixed ("Luttinger Theorem") to the free model value $\rho = p_F/\pi$. If $u \neq 0$, there is no simple relation between the value of p_F and the density, as one can see directly in the case $\lambda = 0$, where one can do explicit calculations.

2.2 We shall begin our analysis by rewriting the *potential* $\mathcal{V}(a)$ as

$$\mathcal{V}(a) = \mathcal{V}^{(1)}(a) + uV_u(a) + \delta^* V_\delta(a) , \tag{2.16}$$

where

$$\mathcal{V}^{(1)}(a) = \lambda V_\lambda(a) + \nu N(a) - \delta^* V_\delta(a) , \tag{2.17}$$

and

$$V_\delta(a) = \frac{1}{L\beta} \sum_{\mathbf{k}} e(k) \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}} . \tag{2.18}$$

δ^* is an arbitrary parameter, to be fixed later, of modulus smaller than $1/2$; its introduction is not really necessary, but allows to simplify the discussion of the spin correlation function asymptotic behaviour. In terms of the Fermionic system, it will describe the modification of the Fermi velocity due to the interaction.

Afterwards we "move" the terms $uV_u(a)$ and $\delta^* V_\delta(a)$ from the interaction to the Gaussian measure. In order to describe the properties of the new Gaussian measure, it is convenient to introduce a new set of Grassmanian variables $\hat{b}^\sigma_{\mathbf{k},\omega}$, $\omega = \pm 1$, $\mathbf{k} \in \mathcal{D}^+_{L,\beta}$, by defining

$$\mathcal{D}^\omega_{L,\beta} = \{\mathbf{k} \in \mathcal{D}_{L,\beta} : \omega k > 0\} \cup \{\mathbf{k} \in \mathcal{D}_{L,\beta} : k = 0, \omega k_0 > 0\} , \tag{2.19}$$

$$\hat{b}^\sigma_{\mathbf{k},\omega} = \hat{a}^{\sigma\omega}_{\omega\mathbf{k}} , \tag{2.20}$$

so that, by using (2.10)

$$a^\sigma_{\mathbf{x}} = \frac{1}{L\beta} \sum_{\mathbf{k} \in \mathcal{D}^+_{L,\beta},\ \omega = \pm 1} \hat{b}^{\sigma\omega}_{\mathbf{k},\omega} e^{i\sigma\omega\mathbf{k}\cdot\mathbf{x}} . \tag{2.21}$$

It is easy to see that

$$\mathcal{Z}_{L,\beta} = e^{-L\beta t_1} \int P(db) e^{-\tilde{\mathcal{V}}^{(1)}(b)} , \tag{2.22}$$

with $\tilde{\mathcal{V}}^{(1)}(b) = \mathcal{V}^{(1)}(a)$, where a has to be interpreted as the r.h.s. of (2.21),

$$\begin{aligned} P(db) = \Big\{ \prod_{\mathbf{k} \in \mathcal{D}^+_{L,\beta}} \frac{(L\beta)^2}{-k_0^2 - (1+\delta^*)^2 e(k)^2 - u^2 \sin^2 k} \prod_{\omega=\pm 1} \hat{b}^+_{\mathbf{k},\omega} \hat{b}^-_{\mathbf{k},\omega} \Big\} \cdot \\ \cdot \exp\Big\{ -\frac{1}{L\beta} \sum_{\mathbf{k} \in \mathcal{D}^+_{L,\beta}} \sum_{\omega,\omega'} \hat{b}^+_{\mathbf{k},\omega} T_{\omega,\omega'}(\mathbf{k}) \hat{b}^-_{\mathbf{k},\omega} \Big\} , \end{aligned} \tag{2.23}$$

$$T(\mathbf{k}) = \begin{pmatrix} -ik_0 + (1+\delta^*)e(k) & iu\sin k \\ -iu\sin k & -ik_0 - (1+\delta^*)e(k) \end{pmatrix} , \tag{2.24}$$

$$t_1 = -\frac{1}{L\beta} \sum_{\mathbf{k} \in \mathcal{D}^+_{L,\beta}} \log \frac{k_0^2 + (1+\delta^*)e(k)^2 + u^2 \sin^2 k}{k_0^2 + e(k)^2} . \tag{2.25}$$

Note that t_1 is uniformly bounded as $L,\beta\to\infty$, if $|\delta^*|\le 1/2$, as we are supposing. For $\lambda=\nu=\delta^*=0$, it represents the free energy for lattice site of $H-H_0$.

2.3 For $\lambda=\nu=0$, all the properties of the model can be analyzed in terms of the Grassmanian measure (2.23). In particular, we have

$$\int P(db) a_{\mathbf{x}}^{\sigma_1} a_{\mathbf{y}}^{\sigma_2} = \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}^+} \left[e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} T^{-1}(\mathbf{k})_{-\sigma_1,\sigma_2} - e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} T^{-1}(\mathbf{k})_{-\sigma_2,\sigma_1} \right] , \tag{2.26}$$

where $T^{-1}(\mathbf{k})$ denotes the inverse of the matrix $T(\mathbf{k})$. This matrix is defined for any $\mathbf{k}\in\mathcal{D}_{L,\beta}$ and satisfies the symmetry relation

$$T^{-1}(\mathbf{k})_{-\sigma_2,\sigma_1} = -T^{-1}(-\mathbf{k})_{-\sigma_1,\sigma_2} , \tag{2.27}$$

so that we can write (2.26) also in the form

$$\int P(db) a_{\mathbf{x}}^{\sigma_1} a_{\mathbf{y}}^{\sigma_2} = \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} T^{-1}(\mathbf{k})_{-\sigma_1,\sigma_2} . \tag{2.28}$$

If $\lambda\ne 0$, we shall study the model, for λ small, in terms of a perturbative expansion, based on a multiscale decomposition of the measure (2.23), by using the methods introduced in [BG] and extended in various other papers ([BGPS], [BM1], [M1]). In order to discuss the structure of the expansion, it is convenient to explain first how it works in the case of the free energy for site of $H-H_0$

$$E_{L,\beta} = -\frac{1}{L\beta}\log \mathcal{Z}_{L,\beta} . \tag{2.29}$$

Let T^1 be the one dimensional torus, $||k-k'||_{T^1}$ the usual distance between k and k' in T^1 and $||k||=||k-0||$. We introduce a *scaling parameter* $\gamma>1$ and a positive function $\chi(\mathbf{k}')\in C^\infty(T^1\times R)$, $\mathbf{k}'=(k',k_0)$, such that

$$\chi(\mathbf{k}') = \chi(-\mathbf{k}') = \begin{cases} 1 & \text{if } |\mathbf{k}'| < t_0 \equiv a_0 v_0^*/\gamma , \\ 0 & \text{if } |\mathbf{k}'| > a_0 v_0^* , \end{cases} \tag{2.30}$$

where

$$|\mathbf{k}'| = \sqrt{k_0^2 + (v_0^* ||k'||_{T^1})^2} , \tag{2.31}$$

$$a_0 = \min\{p_F/2, (\pi-p_F)/2\} , \tag{2.32}$$

$$v_0^* = v_0(1+\delta^*) , \qquad v_0 = \sin p_F . \tag{2.33}$$

In order to give a well defined meaning to the definition (2.30), $v_0^*>0$ has to be positive. Hence we shall suppose that

$$v_0 \ge \bar v_0 > 0 , \quad |\delta^*| \le \frac{1}{2} , \tag{2.34}$$

where $\bar v_0$ is fixed once for all. All our results will be uniform in v_0, under the conditions (2.34), but we shall not stress this fact anymore in the following.

The definition (2.30) is such that the supports of $\chi(k-p_F,k_0)$ and $\chi(k+p_F,k_0)$ are disjoint and the C^∞ function on $T^1\times R$

$$\hat f_1(\mathbf{k})\equiv 1-\chi(k-p_F,k_0)-\chi(k+p_F,k_0) \tag{2.35}$$

is equal to 0, if $[v_0^*||(|k|-p_F)||_{T^1}]^2+k_0^2<t_0^2$.

We define also, for any integer $h\le 0$,

$$f_h(\mathbf{k}')=\chi(\gamma^{-h}\mathbf{k}')-\chi(\gamma^{-h+1}\mathbf{k}')\ ; \tag{2.36}$$

we have, for any $\bar h<0$,

$$\chi(\mathbf{k}')=\sum_{h=\bar h+1}^{0} f_h(\mathbf{k}')+\chi(\gamma^{-\bar h}\mathbf{k}')\ . \tag{2.37}$$

Note that, if $h\le 0$, $f_h(\mathbf{k}')=0$ for $|\mathbf{k}'|<t_0\gamma^{h-1}$ or $|\mathbf{k}'|>t_0\gamma^{h+1}$, and $f_h(\mathbf{k}')=1$, if $|\mathbf{k}'|=t_0\gamma^h$, so that

$$f_{h_1}(\mathbf{k}')f_{h_2}(\mathbf{k}')=0\ ,\quad \text{if}\ \ |h_1-h_2|>1\ . \tag{2.38}$$

We finally define, for any $h\le 0$:

$$\hat f_h(\mathbf{k})=f_h(k-p_F,k_0)+f_h(k+p_F,k_0)\ ; \tag{2.39}$$

This definition implies that, if $h\le 0$, the support of $\hat f_h(\mathbf{k})$ is the union of two disjoint sets, A_h^+ and A_h^-. In A_h^+, k is strictly positive and $||k-p_F||_{T^1}\le a_0\gamma^h\le a_0$, while, in A_h^-, k is strictly negative and $||k+p_F||_{T^1}\le a_0\gamma^h$.

The label h is called the *scale* or *frequency* label. Note that, if $\mathbf{k}\in\mathcal{D}_{L,\beta}$, then $|\mathbf{k}\pm(p_F,0)|\ge\sqrt{(\pi\beta^{-1})^2+(v_0^*\pi L^{-1})^2}$, by (2.3) and the definition of $\mathcal{D}_{L,\beta}$. Therefore

$$\hat f_h(\mathbf{k})=0\quad\forall h<h_{L,\beta}=\min\{h:t_0\gamma^{h+1}>\sqrt{(\pi\beta^{-1})^2+(v_0^*\pi L^{-1})^2}\}\ , \tag{2.40}$$

and, if $\mathbf{k}\in\mathcal{D}_{L,\beta}$, the definitions (2.35) and (2.39), together with the identity (2.37), imply that

$$1=\sum_{h=h_{L,\beta}}^{1}\hat f_h(\mathbf{k})\ . \tag{2.41}$$

We now introduce, for each scale label h, such that $h_{L,\beta}\le h\le 1$, a set of Grassmanian variables $b_{\mathbf{k},\omega}^{(h)\sigma}$ and a corresponding Gaussian measure $P(db^{(h)})$, such that, if $h=1$, then $\mathbf{k}\in\mathcal{D}_{L,\beta}$ and

$$\int P(db^{(1)})b_{\mathbf{k}_1,\omega_1}^{(1)-\sigma_1}b_{\mathbf{k}_2,\omega_2}^{(1)\sigma_2}=L\beta\sigma_1\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}\frac{1}{2}T^{-1}(\mathbf{k}_1)_{\omega_1,\omega_2}\hat f_1(\mathbf{k}_1)\ , \tag{2.42}$$

while, if $h\le 0$, then $\mathbf{k}\in\mathcal{D}_{L,\beta}^+$ and

$$\int P(db^{(h)})b_{\mathbf{k}_1,\omega_1}^{(h)-\sigma_1}b_{\mathbf{k}_2,\omega_2}^{(h)\sigma_2}=L\beta\sigma_1\delta_{\sigma_1,\sigma_2}\delta_{\mathbf{k}_1,\mathbf{k}_2}T^{-1}(\mathbf{k}_1)_{\omega_1,\omega_2}f_h(k_1-p_F,k_0)\ . \tag{2.43}$$

The support properties of the r.h.s. of (2.42) and (2.43) allow to impose the condition

$$b^{(h)\sigma}_{\mathbf{k},\omega} = 0, \quad \text{if } \hat{f}_h(\mathbf{k}) = 0 \; . \tag{2.44}$$

By using (2.26) and (2.27), it is easy to see that

$$\int P(db) a^{\sigma_1}_{\mathbf{x}} a^{\sigma_2}_{\mathbf{y}} = \sum_{h=h_{L,\beta}}^{1} \sum_{\omega_1,\omega_2} \int P(db^{(h)}) b^{(h)\sigma_1\omega_1}_{\mathbf{x},\omega_1} b^{(h)\sigma_2\omega_2}_{\mathbf{y},\omega_2} \; , \tag{2.45}$$

where, if $h \le 0$,

$$b^{(h)\sigma}_{\mathbf{x},\omega} = \frac{1}{L\beta} \sum_{\mathbf{k}\in\mathcal{D}^+_{L,\beta}} \hat{b}^{(h)\sigma}_{\mathbf{k},\omega} e^{i\sigma\mathbf{k}\cdot\mathbf{x}} \; , \tag{2.46}$$

while, if $h = 1$, a similar definition is used, with $\mathcal{D}_{L,\beta}$ in place of $\mathcal{D}^+_{L,\beta}$. Note that this different definition, which is at the origin of the factor 1/2 in the r.h.s. of (2.42), is not really necessary, but implies that $\int P(db^{(1)}) b^{(1)-}_{\mathbf{x},\omega_1} b^{(h)+}_{\mathbf{y},\omega_2}$ is bounded for $M \to \infty$, a property which should otherwise be true only for $\sum_{\omega_1,\omega_2} \int P(db^{(1)}) b^{(1)\sigma_1\omega_1}_{\mathbf{x},\omega_1} b^{(h)\sigma_2\omega_2}_{\mathbf{y},\omega_2}$. In the following, we shall use this property in order to simplify the discussion in some minor points.

The identity (2.45), as it is well known, implies that, if $F(a)$ is any function of the variables $a^\sigma_{\mathbf{x}}$, then

$$\int P(da) F(a) = \int \prod_{h=h_{L,\beta}}^{1} P(db^{(h)}) F\Big(\sum_{h=h_{L,\beta}}^{1} a^{(h)} \Big) \; , \tag{2.47}$$

where

$$a^{(h)\sigma}_{\mathbf{x}} = \sum_{\omega=\pm1} b^{(h)\sigma\omega}_{\mathbf{x},\omega} \; . \tag{2.48}$$

It is now convenient to introduce a variable which measures the distance of the momentum from the Fermi surface, by putting $k = k' + p_F$, with $k' \in \mathcal{D}'_L = \{k' = 2(n+1/2)\pi/L, n \in \mathbb{Z}, -[L/2] \le n \le [(L-1)/2]\}$. Moreover, we rename the Grassmanian variables, by defining

$$\hat{\psi}^{(h)\sigma}_{\mathbf{k}',\omega} = \hat{b}^{(h)\sigma}_{\mathbf{k}'+\mathbf{p}_F,\omega} \; , \quad \psi^{(h)\sigma}_{\mathbf{x},\omega} = \frac{1}{L\beta} \sum_{\mathbf{k}'\in\mathcal{D}'_{L,\beta}} e^{i\sigma\mathbf{k}'\mathbf{x}} \hat{\psi}^{(h)\sigma}_{\mathbf{k}',\omega} \; , \tag{2.49}$$

where $\mathcal{D}'_{L,\beta} = \mathcal{D}'_L \times \mathcal{D}_\beta$, $\mathbf{k}' = (k', k_0)$ and $\mathbf{p}_F = (p_F, 0)$. Note that, by (2.44),

$$\hat{\psi}^{(h)\sigma}_{\mathbf{k}',\omega} = 0 \;\; \text{if} \quad \hat{f}_h(\mathbf{k}' + \mathbf{p}_F) = 0 \; . \tag{2.50}$$

The definition (2.49) allows to write (2.48) in the form

$$a^{(h)\sigma}_{\mathbf{x}} = \sum_{\omega} e^{i\sigma\omega\mathbf{p}_F\mathbf{x}} \psi^{(h)\sigma\omega}_{\mathbf{x},\omega} \; . \tag{2.51}$$

The measure $P(db^{(h)})$ can be thought in a natural way as a measure on the variables $\psi^{(h)\sigma}_{\mathbf{x},\omega}$, that we shall denote $P(d\psi^{(h)})$. Then, (2.43) and (2.49) imply that, if $h \le 1$,

$$\int P(d\psi^{(h)}) \, \hat{\psi}^{(h)-\sigma_1}_{\mathbf{k}'_1,\omega_1} \hat{\psi}^{(h)\sigma_2}_{\mathbf{k}'_2,\omega_2} = (1 - \frac{1}{2}\delta_{h,1}) L\beta \sigma_1 \delta_{\sigma_1,\sigma_2} \delta_{\mathbf{k}'_1,\mathbf{k}'_2} \tilde{g}^{(h)}_{\omega_1,\omega_2}(\mathbf{k}'_1) \; , \tag{2.52}$$

where, if $f_1(\mathbf{k}') \equiv \hat{f}_1(\mathbf{k}' + \mathbf{p}_F)$,

$$\tilde{g}^{(h)}(\mathbf{k}') = \frac{f_h(\mathbf{k}')}{-k_0^2 - E(k')^2 - u^2 \sin^2(k' + p_F)} \begin{pmatrix} -ik_0 - E(k') & -iu\sin(k' + p_F) \\ iu\sin(k' + p_F) & -ik_0 + E(k') \end{pmatrix} , \quad (2.53)$$

$$E(k') = v_0^* \sin k' + (1 + \delta^*)(1 - \cos k') \cos p_F \; . \quad (2.54)$$

In the following we shall use also the notation

$$\psi^{(\leq h)\sigma}_{\mathbf{x},\omega} = \sum_{h'=h_{L,\beta}}^{h} \psi^{(h')\sigma}_{\mathbf{x},\omega} \quad , \quad P(d\psi^{(\leq h)}) = \prod_{h'=h_{L,\beta}}^{h} P(d\psi^{(h')}) \; , \quad (2.55)$$

which allows to write the identity (2.47) as

$$\int P(da) F(a) = \int P(d\psi^{(\leq 1)}) \tilde{F}(\psi^{(\leq 1)}) \; , \quad (2.56)$$

where $\tilde{F}(\psi^{(\leq 1)})$ is obtained from $F(\sum_h a^{(h)})$, by using (2.51).

Remark. Note that the sum over k_0 in (2.49) can be thought as a finite sum for any M, if $h \leq 0$, because of the support properties of $\hat{\psi}^{(h)\sigma}_{\mathbf{k}',\omega}$. Hence, all quantities that we shall calculate will depend on M only trough the propagator $\tilde{g}^{(1)}(\mathbf{k}')$, if M is large enough.

2.4 If we apply (2.56) to $\mathcal{Z}_{L,\beta}$ and we use (2.29) and (2.22), we get

$$e^{-L\beta E_{L,\beta}} = e^{-L\beta t_1} \int P(d\psi^{(\leq 1)}) e^{-\mathcal{V}^{(1)}(\psi^{(\leq 1)})} \; , \quad (2.57)$$

where

$$\mathcal{V}^{(1)}(\psi^{(\leq 1)}) = \lambda V_\lambda (\sum_{h=h_{L,\beta}}^{1} a^{(h)}) + \nu N (\sum_{h=h_{L,\beta}}^{1} a^{(h)}) - \delta^* V_\delta (\sum_{h=h_{L,\beta}}^{1} a^{(h)}) \; . \quad (2.58)$$

Let us now perform the integration over $\psi^{(1)}$; we get

$$e^{-L\beta E_{L,\beta}} = e^{-L\beta(\tilde{E}_1 + t_1)} \int P(d\psi^{(\leq 0)}) \, e^{-\bar{\mathcal{V}}^{(0)}(\psi^{(\leq 0)})} \quad , \quad \bar{\mathcal{V}}^{(0)}(0) = 0 \; , \quad (2.59)$$

$$e^{-\bar{\mathcal{V}}^{(0)}(\psi^{(\leq 0)}) - L\beta\tilde{E}_1} = \int P(d\psi^{(+1)}) e^{-\mathcal{V}^{(1)}(\psi^{(\leq 0)} + \psi^{(+1)})} \; . \quad (2.60)$$

It is easy to see that $\bar{\mathcal{V}}^{(0)}(\psi^{(\leq 0)})$ can be written in the form

$$\begin{aligned} \bar{\mathcal{V}}^{(0)}(\psi^{(\leq 0)}) = \sum_{n=1}^{\infty} \frac{1}{(L\beta)^{2n}} \sum_{\underline{\sigma},\underline{\omega}} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n}} \prod_{i=1}^{2n} \hat{\psi}^{(\leq 0)\sigma_i}_{\mathbf{k}'_i,\omega_i} \cdot \\ \cdot \; \hat{W}^{(0)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{k}'_1, ..., \mathbf{k}'_{2n-1}) \; \delta(\sum_{i=1}^{2n} \sigma_i(\mathbf{k}'_i + \mathbf{p}_F)) \; , \end{aligned} \quad (2.61)$$

where $\underline{\sigma} = (\sigma_1, \ldots, \sigma_{2n})$, $\underline{\omega} = (\omega_1, \ldots, \omega_{2n})$ and we used the notation

$$\delta(\mathbf{k}) = \delta(k)\delta(k_0) \; , \quad \delta(k) = L \sum_{n\in\mathbb{Z}} \delta_{k,2\pi n} \; , \quad \delta(k_0) = \beta\delta_{k_0,0} \; . \quad (2.62)$$

As we shall prove in §3, the kernels $\hat W^{(0)}_{2n,\underline\sigma,\underline\omega}({\bf k}'_1,\ldots,{\bf k}'_{2n-1})$, as well as $\tilde E_1$, are expressed as power series of λ,ν, convergent for $\varepsilon\equiv Max(|\lambda|,|\nu|)\le\varepsilon_0$, for ε_0 small enough. Moreover there exists a constant C, such that, uniformly in L,β, $|\tilde E_1|\le C\varepsilon$ and $|\hat W^{(0)}_{2n,\underline\sigma,\underline\omega}|\le C^n\varepsilon^{\max(1,n-1)}$.

Remark - The conservation of momentum and the support property (2.50) of $\hat\psi^{(\le 0)\sigma}_{{\bf k}',\omega}$ imply that, if $n=1$, only the terms with $\sigma_1+\sigma_2=0$ contribute to the sum in (2.61).

Let us now define ${\bf k}^*=(k,-k_0)$. It is possible to show, by using the symmetries of the interaction and of the covariance $\tilde g^{(1)}({\bf k}')$, that

$$\begin{aligned}\hat W^{(0)}_{n,\underline\sigma,\underline\omega}({\bf k}^*_1,\ldots,{\bf k}^*_{n-1}) &= (-1)^{\frac12\sum_{i=1}^n\sigma_i\omega_1}[\hat W^{(0)}_{n,\underline\sigma,\underline\omega}({\bf k}_1,\ldots,{\bf k}_{n-1})]^* =\\ &= (-1)^{\frac12\sum_{i=1}^n\sigma_i\omega_i}\hat W^{(0)}_{n,-\underline\sigma,-\underline\omega}({\bf k}_1,\ldots,{\bf k}_{n-1})\;.\end{aligned}\tag{2.63}$$

2.5 The integration of the fields of scale $h\le 0$ is performed iteratively.We define a sequence of positive constants Z_h, $h=h_{L,\beta},\ldots,0$, a sequence of *effective potentials* $\mathcal V^{(h)}(\psi)$, a sequence of constants E_h and a sequence of functions $\sigma_h({\bf k}')$, such that

$$Z_0=1,\quad E_0=\tilde E_1+t_1,\quad \sigma_0({\bf k}')=u\sin(k'+p_F)\;,\tag{2.64}$$

and

$$e^{-L\beta E_{L,\beta}}=\int P_{Z_h,\sigma_h,C_h}(d\psi^{(\le h)})\,e^{-\mathcal V^{(h)}(\sqrt{Z_h}\psi^{(\le h)})-L\beta E_h}\;,\quad \mathcal V^{(h)}(0)=0\;,\tag{2.65}$$

where

$$\begin{aligned}P_{Z_h,\sigma_h,C_h}(d\psi^{(\le h)}) &= \prod_{{\bf k}':C_h^{-1}({\bf k}')>0}\prod_{\omega=\pm1}\frac{d\hat\psi^{(\le h)+}_{{\bf k}',\omega}d\hat\psi^{(\le h)-}_{{\bf k}',\omega}}{\mathcal N({\bf k}')}\cdot\\ &\exp\left\{-\frac{1}{L\beta}\sum_{{\bf k}':C_h^{-1}({\bf k}')>0}C_h({\bf k}')Z_h\sum_{\omega,\omega'=\pm1}\hat\psi^{(\le h)+}_{{\bf k}',\omega}T^{(h+1)}_{\omega,\omega'}\hat\psi^{(\le h)-}_{{\bf k}',\omega'}\right\}\;,\end{aligned}\tag{2.66}$$

$$\mathcal N({\bf k}')=\frac{C_h({\bf k}')Z_h}{L\beta}[k_0^2+E(k')^2+\sigma_h({\bf k}')^2]^{1/2}\;,\tag{2.67}$$

$$C_h({\bf k}')^{-1}=\sum_{j=h_{L,\beta}}^{h}f_j({\bf k}')\;,\tag{2.68}$$

and the 2×2 matrix $T_h({\bf k}')$ is given by

$$T_h({\bf k}')=\begin{pmatrix}-ik_0+E(k') & i\sigma_{h-1}({\bf k}')\\ -i\sigma_{h-1}({\bf k}') & -ik_0-E(k')\end{pmatrix}\;.\tag{2.69}$$

We shall also prove that the $\mathcal V^{(h)}$ can be represented as

$$\begin{aligned}\mathcal V^{(h)}(\psi^{(\le h)}) &= \sum_{n=1}^\infty\frac{1}{(L\beta)^{2n}}\sum_{{\bf k}'_1,\ldots,{\bf k}'_{2n},\,\underline\sigma,\underline\omega}\prod_{i=1}^{2n}\hat\psi^{(\le h)\sigma_i}_{{\bf k}'_i,\omega_i}\;\cdot\\ &\cdot\;\hat W^{(h)}_{2n,\underline\sigma,\underline\omega}({\bf k}'_1,\ldots,{\bf k}'_{2n-1})\delta(\sum_{i=1}^{2n}\sigma_i({\bf k}'_i+{\bf p}_F))\;,\end{aligned}\tag{2.70}$$

with the kernels $\hat W^{(h)}_{2n,\underline\sigma,\underline\omega}$ verifying the symmetry relations

$$\begin{aligned}\hat W^{(h)}_{n,\underline\sigma,\underline\omega}(\mathbf{k}_1^*,\ldots,\mathbf{k}_{n-1}^*) &= (-1)^{\frac12\sum_{i=1}^n \sigma_i\omega_i}[\hat W^{(h)}_{n,\underline\sigma,\underline\omega}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1})]^* = \\ &= (-1)^{\frac12\sum_{i=1}^n \sigma_i\omega_i}\hat W^{(h)}_{n,-\underline\sigma,-\underline\omega}(\mathbf{k}_1,\ldots,\mathbf{k}_{n-1})\ .\end{aligned} \tag{2.71}$$

The previous claims are true for $h=0$, by (2.59), (2.61), (2.64) and (2.53). In order to prove them for any $h \ge h_{L,\beta}$, we must explain how $\mathcal{V}^{(h-1)}(\psi)$ is calculated, given $\mathcal{V}^{(h)}(\psi)$. It is convenient, for reasons which will be clear below, to split $\mathcal{V}^{(h)}$ as $\mathcal{L}\mathcal{V}^{(h)}+\mathcal{R}\mathcal{V}^{(h)}$, where $\mathcal{R}=1-\mathcal{L}$ and $\mathcal{L}$, the *localization operator*, is a linear operator on functions of the form (2.70), defined in the following way by its action on the kernels $\hat W^{(h)}_{2n,\underline\sigma,\underline\omega}$.

1) If $2n=4$, then

$$\mathcal{L}\hat W^{(h)}_{4,\underline\sigma,\underline\omega}(\mathbf{k}'_1,\mathbf{k}'_2,\mathbf{k}'_3) = \hat W^{(h)}_{4,\underline\sigma,\underline\omega}(\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++})\ , \tag{2.72}$$

where

$$\bar{\mathbf{k}}_{\eta\eta'} = \left(\eta\frac{\pi}{L},\eta'\frac{\pi}{\beta}\right)\ . \tag{2.73}$$

Note that this definition depends on the the field variables order in the r.h.s. of (2.70), if $\sum_{i=1}^4\sigma_i\neq 0$. In fact, since $\sigma_4\mathbf{k}'_4 = -\sum_{i=1}^3\sigma_i\mathbf{k}'_i - \mathbf{p}_F\sum_{i=1}^4\sigma_i$ (modulo $(2\pi,0)$), if $\mathbf{k}'_i=\bar{\mathbf{k}}_{++}$ for $i=1,2,3$, $\mathbf{k}'_4=\bar{\mathbf{k}}_{++}$ only if $\sum_{i=1}^4\sigma_i=0$. This is apparently a problem, because the representation (2.70) is not uniquely defined (the terms which differ by a common permutation of the $\underline\sigma$ and $\underline\omega$ indices are equivalent). However, it is easy to see, by using the anticommuting property of the field variables, that the contribution to $\mathcal{L}\mathcal{V}^{(h)}$ of the terms with $2n=4$ is equal to 0, unless, after a suitable permutation of the fields, $\underline\sigma=(+,-,+,-)$, $\underline\omega=(+1,-1,-1,+1)$.

The previous discussion implies that we are free to change the order of the field variables as we like, before applying the definition (2.72); this freedom will be useful in the construction of the main expansion in §3.

2) If $2n=2$ and, possibly after a suitable permutation of the fields, $\underline\sigma=(+,-)$ ($\sigma_1+\sigma_2=0$, by the remark following (2.62)), then

$$\begin{aligned}\mathcal{L}\hat W^{(h)}_{2,\underline\sigma,\underline\omega}(\mathbf{k}') = \frac14\sum_{\eta,\eta'=\pm1} &\hat W^{(h)}_{2,\underline\sigma,\underline\omega}(\bar{\mathbf{k}}_{\eta\eta'})\,\cdot \\ &\cdot\left\{1+\delta_{\omega_1,\omega_2}\left[\eta\frac{L}{\pi}\left(b_L+a_L\frac{E(k')}{v_0^*}\right)+\eta'\frac{\beta}{\pi}k_0\right]\right\}\ ,\end{aligned} \tag{2.74}$$

where

$$a_L\frac{L}{\pi}\sin\frac{\pi}{L}=1\ ,\qquad \frac{\cos p_F}{v_0}(1-\cos\frac{\pi}{L})+b_L\frac{L}{\pi}\sin\frac{\pi}{L}=0\ . \tag{2.75}$$

In order to better understand this definition, note that, if $L=\beta=\infty$,

$$\mathcal{L}\hat W^{(h)}_{2,\underline\sigma,\underline\omega}(\mathbf{k}') = \hat W^{(h)}_{2,\underline\sigma,\underline\omega}(0)+\delta_{\omega_1,\omega_2}\left[\frac{E(k')}{v_0^*}\frac{\partial\hat W^{(h)}_{2,\underline\sigma,\underline\omega}}{\partial k'}(0)+k_0\frac{\partial\hat W^{(h)}_{2,\underline\sigma,\underline\omega}}{\partial k_0}(0)\right]\ . \tag{2.76}$$

Hence, $\mathcal{L}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{k}')$ has to be understood as a discrete version of the Taylor expansion up to order 1. Since $a_L = 1 + O(L^{-2})$ and $b_L = O(L^{-2})$, this property would be true also if $a_L = 1$ and $b_L = 0$; however the choice (2.75) has the advantage to share with (2.76) another important property, that is $\mathcal{L}^2\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{k}') = \mathcal{L}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{k}')$.

3) In all the other cases

$$\mathcal{L}\hat{W}^{h}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1}) = 0\;. \tag{2.77}$$

By (2.72) and the remark following (2.76), the operator $\mathcal{L}$ satisfies the relation

$$\mathcal{R}\mathcal{L} = 0\;. \tag{2.78}$$

By using the anticommuting properties of the Grassmanian variables (see discussion in item 1) above) and the symmetry relations (2.71), we can write $\mathcal{L}\mathcal{V}^{(h)}$ in the following way:

$$\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)}) = \gamma^h n_h F^{(\le h)}_\nu + s_h F^{(\le h)}_\sigma + z_h F^{(\le h)}_\zeta + a_h F^{(\le h)}_\alpha + l_h F^{(\le h)}_\lambda\;, \tag{2.79}$$

where n_h, s_h, z_h, a_h and l_h are real numbers and

$$\begin{aligned}
F^{(\le h)}_\nu &= \sum_{\omega=\pm1}\frac{\omega}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}'_{L,\beta}}\hat\psi^{(\le h)+}_{\mathbf{k}',\omega}\hat\psi^{(\le h)-}_{\mathbf{k}',\omega}\;,\\
F^{(\le h)}_\sigma &= \sum_{\omega=\pm1}\frac{i\omega}{(L\beta)}\sum_{\mathbf{k}'\in\mathcal{D}'_{L,\beta}}\hat\psi^{(\le h)+}_{\mathbf{k}',\omega}\hat\psi^{(\le h)-}_{\mathbf{k}',-\omega}\;,\\
F^{(\le h)}_\alpha &= \sum_{\omega=\pm1}\frac{\omega}{(L\beta)}\sum_{\mathbf{k}'\in\mathcal{D}'_{L,\beta}}\frac{E(k')}{v_0^*}\hat\psi^{(\le h)+}_{\mathbf{k}',\omega}\hat\psi^{(\le h)-}_{\mathbf{k}',\omega}\;,\\
F^{(\le h)}_\zeta &= \sum_{\omega=\pm1}\frac{1}{(L\beta)}\sum_{\mathbf{k}'\in\mathcal{D}'_{L,\beta}}(-ik_0)\hat\psi^{(\le h)+}_{\mathbf{k}',\omega}\hat\psi^{(\le h)-}_{\mathbf{k}',\omega}\;,\\
F^{(\le h)}_\lambda &= \frac{1}{(L\beta)^4}\sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_4\in\mathcal{D}'_{L,\beta}}\hat\psi^{(\le h)+}_{\mathbf{k}'_1,+1}\hat\psi^{(\le h)-}_{\mathbf{k}'_2,-1}\hat\psi^{(\le h)+}_{\mathbf{k}'_3,-1}\hat\psi^{(\le h)-}_{\mathbf{k}'_4,+1}\delta(\mathbf{k}'_1-\mathbf{k}'_2+\mathbf{k}'_3-\mathbf{k}'_4)\;.
\end{aligned} \tag{2.80}$$

By using (2.72) and (2.74), it is easy to see that, if $\varepsilon\equiv\max\{|\lambda|,|\nu|\}$,

$$\begin{aligned}
l_0 &= 4\lambda\sin^2(p_F+\pi/L)+O(\varepsilon^2)\;,\quad a_0=-\delta^* v_0 + c_0^\delta\lambda_1 + O(\varepsilon^2)\;,\\
s_0 &= O(u\varepsilon)\;,\quad z_0 = O(\varepsilon^2)\;,\quad n_0 = \nu + O(\varepsilon)\;,
\end{aligned} \tag{2.81}$$

where c_0^δ is a constant, bounded uniformly in L,β.

We now *renormalize* the free measure $P_{Z_h,\sigma_h,C_h}(d\psi^{(\le h)})$, by adding to it part of the r.h.s. of (2.79). We get

$$\begin{aligned}
&\int P_{Z_h,\sigma_h,C_h}(d\psi^{(\le h)})\,e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} =\\
&\qquad = e^{-L\beta t_h}\int P_{\tilde Z_{h-1},\sigma_{h-1},C_h}(d\psi^{(\le h)})\,e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}\;,
\end{aligned} \tag{2.82}$$

where $P_{\tilde Z_{h-1},\sigma_{h-1},C_h}(d\psi^{(\le h)})$ is obtained from $P_{Z_h,\sigma_h,C_h}(d\psi^{(\le h)})$ by substituting Z_h with

$$\tilde Z_{h-1}(\mathbf{k}') = Z_h[1 + C_h^{-1}(\mathbf{k}')z_h] \tag{2.83}$$

and $\sigma_h(\mathbf{k}')$ with

$$\sigma_{h-1}(\mathbf{k}') = \frac{Z_h}{\tilde{Z}_{h-1}(\mathbf{k}')}[\sigma_h(\mathbf{k}') + C_h^{-1}(\mathbf{k}')s_h] \; ; \tag{2.84}$$

moreover

$$\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) = \mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) - s_h Z_h F_\sigma^{(\le h)} - z_h Z_h [F_\zeta^{(\le h)} + v_0^* F_\alpha^{(\le h)}] \tag{2.85}$$

and the factor $\exp(-L\beta t_h)$ in (2.82) takes into account the different normalization of the two measures, so that

$$t_h = -\frac{1}{L\beta} \sum_{\mathbf{k}':C_h^{-1}(\mathbf{k}')>0} \log\left\{[1 + z_h C_h^{-1}(\mathbf{k}')]^2 \frac{k_0^2 + E(k')^2 + \sigma_{h-1}(\mathbf{k}')^2}{k_0^2 + E(k')^2 + \sigma_h(\mathbf{k}')^2}\right\} \; . \tag{2.86}$$

Note that

$$\mathcal{L}\tilde{\mathcal{V}}^{(h)}(\psi) = \gamma^h n_h F_\nu^{(\le h)} + (a_h - z_h v_0^*) F_\alpha^{(\le h)} + l_h F_\lambda^{(\le h)} \; . \tag{2.87}$$

The r.h.s of (2.82) can be written as

$$e^{-L\beta t_h} \int P_{Z_{h-1},\sigma_{h-1},C_{h-1}}(d\psi^{(\le h-1)}) \int P_{Z_{h-1},\sigma_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)})\, e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} \; , \tag{2.88}$$

where

$$Z_{h-1} = Z_h(1+z_h) \; , \qquad \tilde{f}_h(\mathbf{k}') = Z_{h-1}[\frac{C_h^{-1}(\mathbf{k}')}{\tilde{Z}_{h-1}(\mathbf{k}')} - \frac{C_{h-1}^{-1}(\mathbf{k}')}{Z_{h-1}}] \; . \tag{2.89}$$

Note that $\tilde{f}_h(\mathbf{k}')$ has the same support of $f_h(\mathbf{k}')$; in fact, by using (2.38), it is easy to see that

$$\tilde{f}_h(\mathbf{k}') = f_h(\mathbf{k}') \left[1 + \frac{z_h f_{h+1}(\mathbf{k}')}{1 + z_h f_h(\mathbf{k}')}\right] \; . \tag{2.90}$$

Moreover, by (2.49),

$$\int P_{Z_{h-1},\sigma_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)})\, \psi^{(h)-}_{\mathbf{x},\omega} \psi^{(h)+}_{\mathbf{y},\omega'} = \frac{g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})}{Z_{h-1}} \; , \tag{2.91}$$

where

$$g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}'} e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})} \tilde{f}_h(\mathbf{k}')[T_h^{-1}(\mathbf{k}')]_{\omega,\omega'} \; , \tag{2.92}$$

and $T_h^{-1}(\mathbf{k}')$ is the inverse of the $T_h(\mathbf{k}')$ defined in (2.69).

$T_h^{-1}(\mathbf{k}')$ is well defined on the support of $\tilde{f}_h(\mathbf{k}')$ and, if we set

$$A_h(\mathbf{k}') = \det T_h(\mathbf{k}') = -k_0^2 - E(k')^2 - [\sigma_{h-1}(\mathbf{k}')]^2 \; , \tag{2.93}$$

then

$$T_h^{-1}(\mathbf{k}') = \frac{1}{A_h(\mathbf{k}')} \begin{pmatrix} -ik_0 - E(k') & -i\sigma_{h-1}(\mathbf{k}') \\ i\sigma_{h-1}(\mathbf{k}') & -ik_0 + E(k') \end{pmatrix} \; . \tag{2.94}$$

The propagator $g^{(h)}_{\omega,\omega'}(\mathbf{x})$ is an antiperiodic function of x and x_0, of period L and β, respectively. Its large distance behaviour is given by the following lemma (see also [BM2]), where we use the definitions

$$\sigma_h \equiv \sigma_h(0) \; , \tag{2.95}$$

$$d_L(x) = \frac{L}{\pi}\sin(\frac{\pi x}{L}) \ , \quad d_\beta(x_0) = \frac{\beta}{\pi}\sin(\frac{\pi x_0}{\beta}) \ , \tag{2.96}$$

$$\mathbf{d}(\mathbf{x}-\mathbf{y}) = (d_L(x-y), d_\beta(x_0-y_0)) \ . \tag{2.97}$$

2.6 LEMMA. *Let us suppose that $h_{L,\beta} \le h \le 0$ and*

$$|z_h| \le \frac{1}{2} \ , \quad |s_h| \le \frac{1}{2}|\sigma_h| \ , \quad |\delta^*| \le \frac{1}{2} \ . \tag{2.98}$$

We can write

$$g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) = g^{(h)}_{L,\omega}(\mathbf{x}-\mathbf{y}) + r^{(h)}_1(\mathbf{x}-\mathbf{y}) + r^{(h)}_2(\mathbf{x}-\mathbf{y}) \ , \tag{2.99}$$

where

$$g^{(h)}_{L,\omega}(\mathbf{x}-\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}'} \frac{e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})}}{-ik_0 + \omega v_0^* k'} \tilde f_h(\mathbf{k}') \ . \tag{2.100}$$

Moreover, given the positive integers N, n_0, n_1 and putting $n = n_0+n_1$, there exist a constant $C_{N,n}$ such that

$$\begin{aligned} &|\partial^{n_0}_{x_0}\bar\partial^{n_1}_x r^{(h)}_1(\mathbf{x}-\mathbf{y})| \le C_{N,n}\frac{\gamma^{2h+n}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y}))|^N} \ , \\ &|\partial^{n_0}_{x_0}\bar\partial^{n_1}_x r^{(h)}_2(\mathbf{x}-\mathbf{y})| \le C_{N,n}|\frac{\sigma^h}{\gamma^h}|^2\frac{\gamma^{h+n}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \ , \end{aligned} \tag{2.101}$$

$$|\partial^{n_0}_{x_0}\bar\partial^{n_1}_x g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})| \le C_{N,n}|\frac{\sigma^h}{\gamma^h}|\frac{\gamma^{h+n}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \ . \tag{2.102}$$

where $\bar\partial_x$ denotes the discrete derivative.

Note that $g^{(h)}_{L,\omega}(\mathbf{x}-\mathbf{y})$ coincides, in the limit $\beta\to\infty$, with the propagator "at scale γ^h" of the Luttinger model, see [BGM], with $\tilde f_h$ in place of f_h. This remark will be crucial for studying the renormalization group flow in [BeM].

2.7 Proof of Lemma 2.6.

By using (2.38), it is easy to see that $\sigma_h(\mathbf{k}') = \sigma_h(0)$ on the support of $f_h(\mathbf{k}')$; hence, by (2.83) and (2.84), we have

$$\sigma_{h-1}(\mathbf{k}') = \frac{\sigma_h + C_h(\mathbf{k}')^{-1}s_h}{1+z_hC_h(\mathbf{k}')^{-1}} \ , \tag{2.103}$$

implying, together with (2.98), that there exist two constants c_1, c_2 such that:

$$c_1|\sigma_h| \le |\sigma_{h-1}(\mathbf{k}')| \le c_2|\sigma_h| \ . \tag{2.104}$$

Let us now consider two integers $N_0, N_1 \ge 0$, such that $N = N_0+N_1$, and note that

$$\begin{aligned} &d_L(x-y)^{N_1}d_\beta(x_0-y_0)^{N_0}g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) = \\ &e^{-i\pi(xL^{-1}N_1+x_0\beta^{-1}N_0)}(-i)^{N_0+N_1}\frac{1}{L\beta}\sum_{\mathbf{k}'}e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})}\partial^{N_1}_{k'}\partial^{N_0}_{k_0}\left[\tilde f_h(\mathbf{k}')[T_h^{-1}(\mathbf{k}')]_{\omega,\omega'}\right] \ , \end{aligned} \tag{2.105}$$

where $\partial_{k'}$ and ∂_{k_0} denote the discrete derivatives.

If $\omega = \omega'$, the decomposition (2.99) is related to the following identity:

$$[T_h^{-1}(\mathbf{k}')]_{\omega,\omega} = \frac{1}{-ik_0 + \omega v_0^* k'} + \left[\frac{1}{-ik_0 + \omega E(k')} - \frac{1}{-ik_0 + \omega v_0^* k'}\right] + \\ + \left[\frac{ik_0 + \omega E(k')}{k_0^2 + E(k')^2 + [\sigma_{h-1}(\mathbf{k}')]^2} - \frac{1}{-ik_0 + \omega E(k')}\right] . \tag{2.106}$$

The bounds (2.101) and (2.102) easily follow from (2.98), (2.104), the support properties of $f_h(\mathbf{k}')$ and the observation that $\tilde f_h(\mathbf{k}')$ and $\sigma_h(\mathbf{k}')$ are smooth functions of $\mathbf{k}'$ in R^2, in the support of $f_h(\mathbf{k}')$, so that the discrete derivatives can be bounded as the continuous derivatives. The main point is of course the fact that, on the support of $f_h(\mathbf{k}')$, $|-ik_0 + \omega E(k')|$, $|-ik_0 + \omega v_0^* k'|$ and $\sqrt{k_0^2 + E(k')^2 + [\sigma_{h-1}(\mathbf{k}')]^2}$ are of order γ^h.

2.8 We now *rescale* the field so that

$$\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) = \hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}) ; \tag{2.107}$$

it follows that

$$\mathcal{L}\tilde{\mathcal{V}}^{(h)}(\psi) = \gamma^h \nu_h F_\nu^{(\le h)} + \delta_h F_\alpha^{(\le h)} + \lambda_h F_\lambda^{(\le h)} , \tag{2.108}$$

where

$$\nu_h = \frac{Z_h}{Z_{h-1}} n_h , \quad \delta_h = \frac{Z_h}{Z_{h-1}}(a_h - v_0^* z_h) , \quad \lambda_h = (\frac{Z_h}{Z_{h-1}})^2 l_h . \tag{2.109}$$

We call the set $\vec v_h = (\nu_h, \delta_h, \lambda_h)$ the *running coupling constants.*

If we now define

$$e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}) - L\beta \tilde E_h} = \int P_{Z_{h-1},\sigma_{h-1},\tilde f_h^{-1}}(d\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})} , \tag{2.110}$$

it is easy to see that $\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})$ is of the form (2.70) and that

$$E_{h-1} = E_h + t_h + \tilde E_h . \tag{2.111}$$

It is sufficient to use the well known identity

$$L\beta \tilde E_h + \mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}) = \sum_{n=1}^\infty \frac{1}{n!}(-1)^{n+1}\mathcal{E}_h^{T,n}(\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})) , \tag{2.112}$$

where $\mathcal{E}_h^{T,n}$ denotes the *truncated expectation of order* n with propagator $Z_{h-1}^{-1} g^{(h)}_{\omega,\omega'}$, see (2.91), and observe that $\psi^{(\le h)} = \psi^{(\le h-1)} + \psi^{(h)}$.

Moreover, the symmetry relations (2.71) are still satisfied, because the symmetry properties of the free measure are not modified by the renormalization procedure, so that the effective potential on scale h has the same symmetries as the effective potential on scale 0.

Let us now define $\tilde E_{h_{L,\beta}}$, so that

$$e^{-L\beta \tilde E_{h_{L,\beta}}} = \int P_{Z_{h_{L,\beta}-1},\sigma_{h_{L,\beta}-1},\tilde f^{-1}_{h_{L,\beta}}}(d\psi^{(h_{L,\beta})})\, e^{-\hat{\mathcal{V}}^{(h_{L,\beta})}(\sqrt{Z_{h_{L,\beta}-1}}\psi^{(h_{L,\beta})})} . \tag{2.113}$$

We have

$$E_{L,\beta} = \sum_{h=h_{L,\beta}}^{1} [\tilde{E}_h + t_h] \; . \tag{2.114}$$

Note that the above procedure allows us to write the running coupling constants $\vec{v}_h$, $h \le 0$, in terms of $\vec{v}_{h'}$, $0 \ge h' \ge h+1$, and λ, ν, u:

$$\vec{v}_h = \vec{\beta}(\vec{v}_{h+1}, ..., \vec{v}_0, \lambda, \nu, u, \delta^*) \; . \tag{2.115}$$

The function $\vec{\beta}(\vec{v}_{h+1}, ..., \vec{v}_0, \lambda, \nu, u, \delta^*)$ is called the *Beta function*.

2.9 Let us now explain the main motivations of the integration procedure discussed above. In a renormalization group approach one has to identify the relevant, marginal and irrelevant interactions. By a power counting argument one sees that the terms bilinear in the fields are relevant, hence one should extract from them the relevant and marginal local contributions by a Taylor expansion of the kernel up to order 1 in the external momenta. Since $\sigma_1 + \sigma_2 = 0$ by the remark following (2.62), we have to consider only two kinds of bilinear terms: those with $\omega_1 = \omega_2$ and those with $\omega_1 = -\omega_2$. It turns out that, for the bilinear terms with $\omega_1 = -\omega_2$, a Taylor expansion up to order 0 is sufficient; the reason is that the Feynman graphs contributing to such terms contain at least one non diagonal propagator and, by lemma 2.6, such propagators are smaller than the diagonal ones by a factor $\sigma_h \gamma^{-h}$; as we shall see, this is sufficient to improve the power counting by 1.

The previous discussion implies that the regularization of the bilinear terms produces four local terms. One of them, that proportional to F_ν, is relevant; it reflects the renormalization of the Fermi momentum and is faced in a standard way [BG], by fixing properly the counterterm ν in the Hamiltonian, *i.e.* by fixing properly the chemical potential, so that the corresponding running coupling ν_h goes to 0 for $h \to -\infty$.

The term proportional to F_α is marginal, but, as we shall see, stays bounded and of order λ as $h \to -\infty$, if δ^* is of order λ; hence the convergence of the flow is not related to the exact value of δ^*. However, in order to get a detailed description of the spin correlation function asymptotic behaviour, it is convenient to choose δ^* so that $\delta_h \to 0$ as $h \to -\infty$. This choice implies that $v_0^* = v_0(1+\delta^*)$ is the "effective" Fermi velocity of the fermion system.

The other two terms are marginal, but have to be treated in different ways. The term proportional to F_ζ is absorbed in the free measure and produces a field renormalization, as in the Luttinger liquid (which is indeed obtained for $u = 0$). The term proportional to F_σ, related to the presence of a gap in the spectrum, is also absorbed in the free measure, since there is no free parameter in the Hamiltonian to control its flow, as for F_ζ. This operation can be seen as the application of a sequence of *different Bogoliubov transformations at each integration step*, to compare with the single Bogoliubov transformation that it is sufficient to see a gap $O(u)$ at the Fermi surface, in the XY model ($\lambda = 0$). It turns out that the gap is deeply renormalized by the interaction, since σ_h is a sort of "mass terms" with a non trivial renormalization group flow.

Let us now consider the quartic terms, which are all marginal. Since there are many of them, depending on the labels ω_i and σ_i of each field, their renormalization group flow seems difficult to study. However, as we have explained in §2.5, the running couplings corresponding to the quartic terms are all exactly equal to 0 for trivial reasons, unless, after a suitable permutation of the fields, $\underline{\sigma} = (+,-,+,-)$, $\underline{\omega} = (+1,-1,-1,+1)$. Hence, by a Taylor expansion of the kernel up to order 0 in the external momenta, all quartic terms can be regularized, by introducing only one running coupling, λ_h.

As in the Luttinger liquid [BGPS, BM1], the flow of λ_h and δ_h can be controlled by using some cancellations, due to the fact that the Beta function is "close" (for small u) to the Luttinger model Beta function. In lemma 2.6 we write the propagator as the Luttinger model propagator plus a remainder, so that the Beta function is equal to the Luttinger model Beta function plus a "remainder", which is small if $\sigma_h\gamma^{-h}$ is small.

Let us define

$$h^* = \inf\{h : 0 \geq h \geq h_{L,\beta}, a_0 v_0^* \gamma^{\bar{h}-1} \geq 4|\sigma_{\bar{h}}|, \forall \bar{h} : 0 \geq \bar{h} \geq h\} \ . \tag{2.116}$$

Of course this definition is meaningful only if $a_0 v_0^* \gamma^{-1} \geq 4|\sigma_0| = 4|u|v_0$ (see (2.64)), that is if

$$|u| \leq \frac{a_0}{4\gamma}(1+\delta^*) \ . \tag{2.117}$$

If the condition (2.117) is not satisfied, we shall put $h^* = 1$.

Lemma 2.6, (2.86) and the definition of h^* easily imply this other Lemma.

2.10 LEMMA. *If $h > h^* \geq 0$ and the conditions (2.98) are satisfied, there is a constant C such that*

$$|t_h| \leq C\gamma^{2h} \ . \tag{2.118}$$

Moreover, given the positive integers N, n_0, n_1 and putting $n = n_0 + n_1$, there exist a constant $C_{N,n}$ such that

$$|\partial_{x_0}^{n_0} \bar{\partial}_x^{n_1} g_{\omega,\omega'}^{(h)}(\mathbf{x};\mathbf{y})| \leq C_{N,n} \frac{\gamma^{h+n}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \ . \tag{2.119}$$

2.11 In §3 we will see that, using the above lemmas and assuming that the running coupling constants are bounded, the integration of the field $\psi^{(h)}$ in (2.88) is well defined in the limit $L, \beta \to \infty$, for $0 \geq h > h^*$.

The integration of the scales from h^* to $h_{L,\beta}$ will be performed "in a single step". This is possible because we shall prove in §3 that the integration in the r.h.s. in (2.82) is well defined in the limit $L, \beta \to \infty$, for $h = h^*$. In order to do that, we shall use the following lemma, whose proof is similar to the proof of lemma 2.6.

2.12 LEMMA. *Assume that h^* is finite uniformly in L, β, so that $|\sigma_{h^*-1}\gamma^{-h^*}| \geq \bar{\kappa}$, for a suitable constant $\bar{\kappa}$ and define*

$$\frac{\bar{g}^{(\leq h^*)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})}{Z_{h^*-1}} \equiv \int P_{\tilde{Z}_{h^*-1},\sigma_{h^*-1},C_{h^*}}(d\psi^{(\leq h^*)})\psi^{(\leq h^*)-}_{\mathbf{x},\omega}\psi^{(\leq h^*)+}_{\mathbf{y},\omega'} \; . \tag{2.120}$$

Then, given the positive integers N, n_0, n_1 and putting $n = n_0 + n_1$, there exist a constant $C_{N,n}$ such that

$$|\partial^{n_0}_{x_0}\bar{\partial}^{n_1}_{x} g^{(\leq h^*)}_{\omega,\omega'}(\mathbf{x};\mathbf{y})| \leq C_{N,n}\frac{\gamma^{h^*+n}}{1+(\gamma^{h^*}|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \; . \tag{2.121}$$

2.13 Comparing Lemma 2.10 and Lemma 2.12, we see that the propagator of the integration of all the scales between h^* and $h_{L,\beta}$ has the same bound as the propagator of the integration of a single scale greater than h^*; this property is used to perform the integration of all the scales $\leq h^*$ in a single step. In fact γ^{h^*} is a momentum scale and, roughly speaking, for momenta bigger than γ^{h^*} the theory is "essentially" a massless theory (up to $O(\sigma_h\gamma^{-h})$ terms), while for momenta smaller than γ^{h^*} it is a "massive" theory with mass $O(\gamma^{h^*})$.

3. Analyticity of the effective potential

3.1 We want to study the expansion of the effective potential, which follows from the renormalization procedure discussed in §2. In order to do that, we find it convenient to write $\mathcal{V}^{(h)}$, $h \le 1$, in terms of the variables $\psi^{(\le h)\sigma}_{\mathbf{x},\omega}$. The two contributions to $\mathcal{V}^{(1)}(\psi^{(\le 1)})$, see (2.58) and (2.14), become

$$\begin{aligned}\lambda V_\lambda(\psi^{\le 1}) &= \sum_{\underline{\sigma}} \int d\mathbf{x} d\mathbf{y}\, \lambda\, v_\lambda(\mathbf{x}-\mathbf{y}) e^{i\mathbf{p}_F\mathbf{x}(\sigma_1+\sigma_4)+i\mathbf{p}_F\mathbf{y}(\sigma_2+\sigma_3)} \cdot \\ &\cdot \psi^{(\le 1)\sigma_1}_{\mathbf{x},\sigma_1}\psi^{(\le 1)\sigma_2}_{\mathbf{y},\sigma_2}\psi^{(\le 1)\sigma_3}_{\mathbf{y},-\sigma_3}\psi^{(\le 1)\sigma_4}_{\mathbf{x},-\sigma_4}\;, \\ \nu N(\psi^{\le 1}) &= \sum_{\sigma_1,\sigma_2}\int d\mathbf{x} e^{i\mathbf{p}_F\mathbf{x}(\sigma_1+\sigma_2)}\,\nu\,\psi^{(\le 1)\sigma_1}_{\mathbf{x},\sigma_1}\psi^{(\le 1)\sigma_2}_{\mathbf{x},-\sigma_2}\;,\end{aligned} \tag{3.1}$$

where $\int d\mathbf{x}$ is a shorthand for $\sum_{x\in\Lambda}\int_{-\beta/2}^{\beta/2} dx_0$.

If we define

$$\begin{aligned}&W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n}) = \\ &= \frac{1}{(L\beta)^{2n}}\sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n}} e^{-i\sum_{r=1}^{2n}\sigma_r\mathbf{k}'_r\mathbf{x}_r}\hat W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{k}'_1,\ldots,\mathbf{k}'_{2n-1})\delta\Big(\sum_{i=1}^{2n}\sigma_i(\mathbf{k}'_i+\mathbf{p}_F)\Big)\;,\end{aligned} \tag{3.2}$$

we can write (2.70) as

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) = \sum_{n=1}^{\infty}\sum_{\underline{\sigma},\underline{\omega}}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2n}\left[\prod_{i=1}^{2n}\psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i}\right]W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})\;. \tag{3.3}$$

Note that

$$W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1+\mathbf{x},\ldots,\mathbf{x}_{2n}+\mathbf{x}) = e^{i\mathbf{p}_F\mathbf{x}\sum_{r=1}^{2n}\sigma_r}W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})\;, \tag{3.4}$$

hence $W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_{2n})$ is translation invariant if and only if $\sum_{r=1}^{2n}\sigma_r = 0$.

The representation of $\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)})$ in terms of the $\psi^{(\le h)\sigma}_{\mathbf{x},\omega}$ variables is obtained by substituting in the r.h.s. of (2.79) the $\mathbf{x}$-space representations of the definitions (2.80). We have

$$\begin{aligned}F^{(\le h)}_\nu &= \sum_{\omega=\pm1}\omega\int d\mathbf{x}\,\psi^{(\le h)+}_{\mathbf{x},\omega}\psi^{(\le h)-}_{\mathbf{x},\omega}\;,\\ F^{(\le h)}_\sigma &= \sum_{\omega=\pm1} i\omega\int d\mathbf{x}\,\psi^{(\le h)+}_{\mathbf{x},\omega}\psi^{(\le h)-}_{\mathbf{x},-\omega}\;,\\ F^{(\le h)}_\alpha &= \sum_{\omega=\pm1} i\omega\int d\mathbf{x}\,\psi^{(\le h)+}_{\mathbf{x},\omega}[\bar\partial_1\psi^{(\le h)-}_{\mathbf{x},\omega}+\frac{i\cos p_F}{2v_0}\bar\partial_1^2\psi^{(\le h)-}_{\mathbf{x},\omega}] = \\ &= \sum_{\omega=\pm1} i\omega\int d\mathbf{x}\,[-\bar\partial_1\psi^{(\le h)+}_{\mathbf{x},\omega}+\frac{i\cos p_F}{2v_0}\bar\partial_1^2\psi^{(\le h)+}_{\mathbf{x},\omega}]\psi^{(\le h)-}_{\mathbf{x},\omega}\;,\\ F^{(\le h)}_\zeta &= \sum_{\omega=\pm1}\int d\mathbf{x}\,\psi^{(\le h)+}_{\mathbf{x},\omega}\partial_0\psi^{(\le h)-}_{\mathbf{x},\omega} = -\sum_{\omega=\pm1}\int d\mathbf{x}\,\partial_0\psi^{(\le h)+}_{\mathbf{x},\omega}\psi^{(\le h)-}_{\mathbf{x},\omega}\;,\\ F^{(\le h)}_\lambda &= \int d\mathbf{x}\,\psi^{(\le h)+}_{\mathbf{x},+1}\psi^{(\le h)-}_{\mathbf{x},-1}\psi^{(\le h)+}_{\mathbf{x},-1}\psi^{(\le h)-}_{\mathbf{x},+1}\;,\end{aligned} \tag{3.5}$$

where ∂_0 is the derivative w.r.t. x_0, $\bar\partial_1$ is the symmetric discrete derivative w.r.t. x, that is, given a function $f(\mathbf{x})$,

$$\bar\partial_1 f(\mathbf{x}) = [f(x+1,x_0) - f(x-1,x_0)]/2 \;, \tag{3.6}$$

and $\bar\partial_1^2$ (which is not the square of $\bar\partial_1$, but has the same properties) is defined by the equation

$$\bar\partial_1^2 f(\mathbf{x}) = f(x+1,x_0) + f(x-1,x_0) - 2f(x,x_0) \;. \tag{3.7}$$

Let us now discuss the action of the operator $\mathcal{L}$ and $\mathcal{R} = 1 - \mathcal{L}$ on the effective potential in the x-space representation, by considering the terms for which $\mathcal{L} \neq 0$.

1)If $2n = 4$, by (2.72),

$$\mathcal{L}\int d\underline{\mathbf{x}} W(\underline{\mathbf{x}}) \prod_{i=1}^4 \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i} = \int d\underline{\mathbf{x}} W(\underline{\mathbf{x}}) \prod_{i=1}^4 \left[G_{\sigma_i}(\mathbf{x}_i - \mathbf{x}_4) \psi^{(\le h)\sigma_i}_{\mathbf{x}_4,\omega_i} \right] \;, \tag{3.8}$$

where $\underline{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_4)$, $W(\underline{\mathbf{x}}) = W^{(h)}_{4,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)$ and

$$G_\sigma(\mathbf{x}) = e^{i\sigma\bar{\mathbf{k}}_{++}\mathbf{x}} = e^{i\sigma\pi(\frac{x}{L} + \frac{x_0}{\beta})} \;. \tag{3.9}$$

Note that, as we have discussed in §2.5, the r.h.s. of (3.8) is always equal to 0, unless, after a suitable permutation of the fields, $\underline{\sigma} = (+,-,+,-)$, $\underline{\omega} = (+1,-1,-1,+1)$. In this last case the function $W(\underline{\mathbf{x}})\prod_{i=1}^4 G_{\sigma_i}(\mathbf{x}_i - \mathbf{x}_4) = W(\underline{\mathbf{x}}) G_+(\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{x}_4)$ is translation invariant and periodic in the space and time components of all variables $\mathbf{x}_k$, of period L and β, respectively. It follows that the quantities $G_{\sigma_i}(\mathbf{x}_i - \mathbf{x}_4)\psi^{(\le h)\sigma_i}_{\mathbf{x}_4,\omega_i}$ in the r.h.s. of (3.8) can be substituted with $G_{\sigma_i}(\mathbf{x}_i - \mathbf{x}_k)\psi^{(\le h)\sigma_i}_{\mathbf{x}_k,\omega_i}$, $k = 1,2,3$. Hence we have four equivalent representations of the localization operation, which differ by the choice of the *localization point*. The freedom in the choice of the localization point will be useful in the following.

If the localization point is chosen as in (3.8), we have

$$\begin{aligned} &\mathcal{R}\int d\underline{\mathbf{x}} W(\underline{\mathbf{x}}) \prod_{i=1}^4 \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i} = \\ &= \int d\underline{\mathbf{x}} W(\underline{\mathbf{x}}) \left[\prod_{i=1}^4 \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i} - \prod_{i=1}^4 G_{\sigma_i}(\mathbf{x}_i - \mathbf{x}_4)\psi^{(\le h)\sigma_i}_{\mathbf{x}_4,\omega_i} \right] \;. \end{aligned} \tag{3.10}$$

The term in square brackets in the above equation can be written as

$$\begin{aligned} &\psi^{(\le h)\sigma_1}_{\mathbf{x}_1,\omega_1}\psi^{(\le h)\sigma_2}_{\mathbf{x}_2,\omega_2} D^{1,1(\le h)\sigma_3}_{\mathbf{x}_3,\mathbf{x}_4,\omega_3}\psi^{(\le h)\sigma_4}_{\mathbf{x}_4,\omega_4} + \\ &+ G_{\sigma_3}(\mathbf{x}_3 - \mathbf{x}_4)\psi^{(\le h)\sigma_1}_{\mathbf{x}_1,\omega_1} D^{1,1(\le h)\sigma_2}_{\mathbf{x}_2,\mathbf{x}_4,\omega_2}\psi^{(\le h)\sigma_3}_{\mathbf{x}_4,\omega_3}\psi^{(\le h)\sigma_4}_{\mathbf{x}_4,\omega_4} + \\ &+ G_{\sigma_3}(\mathbf{x}_3 - \mathbf{x}_4) G_{\sigma_2}(\mathbf{x}_2 - \mathbf{x}_4) D^{1,1(\le h)\sigma_1}_{\mathbf{x}_1,\mathbf{x}_4,\omega_1}\psi^{(\le h)\sigma_2}_{\mathbf{x}_4,\omega_2}\psi^{(\le h)\sigma_3}_{\mathbf{x}_4,\omega_3}\psi^{(\le h)\sigma_4}_{\mathbf{x}_4,\omega_4} \;, \end{aligned} \tag{3.11}$$

where

$$D^{1,1(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega} = \psi^{(\le h)\sigma}_{\mathbf{y},\omega} - G_\sigma(\mathbf{y} - \mathbf{x})\psi^{(\le h)\sigma}_{\mathbf{x},\omega} \;. \tag{3.12}$$

Similar expressions can be written, if the localization point is chosen in a different way.

Note that the decomposition (3.11) corresponds to the following identity:

$$\begin{aligned}\mathcal{R}\hat{W}^{(h)}_{\tau,\mathbf{P}}(\mathbf{k}'_1,\mathbf{k}'_2,\mathbf{k}'_3) = &\left[\hat{W}^{(h)}_{\tau,\mathbf{P}}(\mathbf{k}'_1,\mathbf{k}'_2,\mathbf{k}'_3) - \hat{W}^{(h)}_{\tau,\mathbf{P}}(\mathbf{k}'_1,\mathbf{k}'_2,\bar{\mathbf{k}}_{++})\right] + \\ &+ \left[\hat{W}^{(h)}_{\tau,\mathbf{P}}(\mathbf{k}'_1,\mathbf{k}'_2,\bar{\mathbf{k}}_{++}) - \hat{W}^{(h)}_{\tau,\mathbf{P}}(\mathbf{k}'_1,\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++})\right] + \\ &+ \left[\hat{W}^{(h)}_{\tau,\mathbf{P}}(\mathbf{k}'_1,\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++}) - \hat{W}^{(h)}_{\tau,\mathbf{P}}(\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++})\right] ,\end{aligned} \tag{3.13}$$

and that the i-th term in the r.h.s. of (3.13) is equal to 0 for $\mathbf{k}'_i = \bar{\mathbf{k}}_{++}$.

The field $D^{1,1(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$ is antiperiodic in the space and time components of $\mathbf{x}$ and $\mathbf{y}$, of period L and β, and is equal to 0 if $\mathbf{x} = \mathbf{y}$ modulo (L,β). This means that it is dimensionally equivalent to the product of $d(\mathbf{x},\mathbf{y})$ (see (2.97)) and the derivative of the field, so that the bound of its contraction with another field variable on a scale $h' < h$ will produce a "gain" $\gamma^{-(h-h')}$ with respect to the contraction of $\psi^{(\le h)\sigma}_{\mathbf{y},\omega}$.

If we insert (3.11) in the r.h.s. of (3.10), we can decompose the l.h.s in the sum of three terms, which differ from the term which $\mathcal{R}$ acts on mainly because one $\psi^{(\le h)}$ field is substituted with a $D^{1,1(\le h)}$ field and some of the other $\psi^{(\le h)}$ fields are "translated" in the localization point. All three terms share the property that the field whose $\mathbf{x}$ coordinate is equal to the localization point is not affected by the action of $\mathcal{R}$.

In our approach, the regularization effect of $\mathcal{R}$ will be exploited trough the decomposition (3.11). However, for reasons that will become clear in the following, it is convenient to start the analysis by using another representation of the expression resulting from the insertion of (3.11) in (3.10). If $\psi_{\mathbf{x}_i} \equiv \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i}$, we can write, if the localization point is $\mathbf{x}_4$,

$$\begin{aligned}&\mathcal{R}\int d\underline{\mathbf{x}}\prod_{i=1}^4 \psi_{\mathbf{x}_i} W(\underline{\mathbf{x}}) = \\ &= \int d\underline{\mathbf{x}}\prod_{i=1}^4 \psi_{\mathbf{x}_i}\left[W(\underline{\mathbf{x}}) - \delta(\mathbf{x}_3-\mathbf{x}_4)\int d\mathbf{y}_3 W(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_3,\mathbf{x}_4)G_{\sigma_3}(\mathbf{y}_3-\mathbf{x}_4)\right] + \\ &+ \int d\underline{\mathbf{x}}\prod_{i=1}^4 \psi_{\mathbf{x}_i}\delta(\mathbf{x}_3-\mathbf{x}_4)\int d\mathbf{y}_3\Big[W(\mathbf{x}_1,\mathbf{x}_2,\mathbf{y}_3,\mathbf{x}_4)G_{\sigma_3}(\mathbf{y}_3-\mathbf{x}_4) - \\ &- \delta(\mathbf{x}_2-\mathbf{x}_4)\int d\mathbf{y}_2 W(\mathbf{x}_1,\mathbf{y}_2,\mathbf{y}_3,\mathbf{x}_4)G_{\sigma_3}(\mathbf{y}_3-\mathbf{x}_4)G_{\sigma_2}(\mathbf{y}_2-\mathbf{x}_4)\Big] + \\ &+ \int d\underline{\mathbf{x}}\prod_{i=1}^4 \psi_{\mathbf{x}_i}\delta(\mathbf{x}_2-\mathbf{x}_4)\delta(\mathbf{x}_3-\mathbf{x}_4)\int d\mathbf{y}_2\int d\mathbf{y}_3\Big[W(\mathbf{x}_1,\mathbf{y}_2,\mathbf{y}_3,\mathbf{x}_4)G_{\sigma_3}(\mathbf{y}_3-\mathbf{x}_4)\cdot \\ &\cdot G_{\sigma_2}(\mathbf{y}_2-\mathbf{x}_4) - \delta(\mathbf{x}_1-\mathbf{x}_4)\int d\mathbf{y}_1 W(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3,\mathbf{x}_4)\prod_{i=1}^3 G_{\sigma_i}(\mathbf{y}_i-\mathbf{x}_4)\Big] ,\end{aligned} \tag{3.14}$$

where $\delta(\mathbf{x})$ is the antiperiodic delta function, that is

$$\delta(\mathbf{x}) = \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}'_{L,\beta}} e^{i\sigma\mathbf{k}'\mathbf{x}} . \tag{3.15}$$

Similar expressions are obtained, if the localization point is chosen in a different way.

In the new representation, the action of $\mathcal{R}$ is seen as the decomposition of the original term in the sum of three terms, which are still of the form (3.3), but with a different kernel, containing suitable delta functions.

2)If $2n = 2$ and, possibly after a suitable permutation of the fields, $\underline{\sigma} = (+,-)$, $\omega_1 = \omega_2 = \omega$, by (2.74),

$$\begin{aligned}
&\mathcal{L}\int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)\psi^{(\le h)+}_{\mathbf{x}_1,\omega}\psi^{(\le h)-}_{\mathbf{x}_2,\omega} = \\
&= \int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)\psi^{(\le h)+}_{\mathbf{x}_1,\omega}T^{1(\le h)-}_{\mathbf{x}_2,\mathbf{x}_1,\omega} \\
&= \int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)T^{1(\le h)+}_{\mathbf{x}_1,\mathbf{x}_2,\omega}\psi^{(\le h)-}_{\mathbf{x}_2,\omega}\ ,
\end{aligned}\tag{3.16}$$

with

$$\begin{aligned}
&T^{1(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega} = \psi^{(\le h)\sigma}_{\mathbf{x},\omega}c_\beta(y_0-x_0)[c_L(y-x)+b_L d_L(y-x)]+ \\
&+[\bar\partial_1\psi^{(\le h)\sigma}_{\mathbf{x},\omega}+\frac{i\cos p_F}{2v_0}\bar\partial_1^2\psi^{(\le h)\sigma}_{\mathbf{x},\omega}]c_\beta(y_0-x_0)a_L d_L(y-x)+ \\
&+\partial_0\psi^{(\le h)\sigma}_{\mathbf{x},\omega}d_\beta(y_0-x_0)c_L(y-x)\ ,
\end{aligned}\tag{3.17}$$

where $d_L(x)$ and $d_\beta(x_0)$ are defined as in (2.96) and

$$c_L(x) = \cos(\pi x L^{-1})\ ,\quad c_\beta(x_0) = \cos(\pi x_0\beta^{-1})\ .\tag{3.18}$$

As in the item 1), we define the localization point as the $\mathbf{x}$ coordinate of the field which is left unchanged $\mathcal{L}$. We are free to choose it equal to $\mathbf{x}_1$ or $\mathbf{x}_2$. This freedom affects also the action of $\mathcal{R}$, which can be written as

$$\begin{aligned}
&\mathcal{R}\int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)\psi^{(\le h)+}_{\mathbf{x}_1,\omega}\psi^{(\le h)-}_{\mathbf{x}_2,\omega} = \\
&= \int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)\psi^{(\le h)+}_{\mathbf{x}_1,\omega}D^{2(\le h)-}_{\mathbf{x}_2,\mathbf{x}_1,\omega} \\
&= \int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)D^{2(\le h)+}_{\mathbf{x}_1,\mathbf{x}_2,\omega}\psi^{(\le h)-}_{\mathbf{x}_2,\omega}\ ,
\end{aligned}\tag{3.19}$$

with

$$D^{2(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega} = \psi^{(\le h)\sigma}_{\mathbf{y},\omega} - T^{1(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}\ .\tag{3.20}$$

Hence the effect of $\mathcal{R}$ can be described as the replacement of a $\psi^{(\le h)\sigma}$ field with a $D^{2(\le h)\sigma}$ field, with a gain in the bounds (see discussion in item 1) above) of a factor $\gamma^{-2(h-h')}$.

Also in this case, it is possible to write the regularized term in the form (3.3). We get

$$\begin{aligned}
&\mathcal{R}\int d\mathbf{x}d\mathbf{y}W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}-\mathbf{y})\psi^{(\le h)+}_{\mathbf{x},\omega}\psi^{(\le h)-}_{\mathbf{y},\omega} = \int d\mathbf{x}d\mathbf{y}\psi^{(\le h)+}_{\mathbf{x},\omega}\psi^{(\le h)-}_{\mathbf{y},\omega}\Big\{W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}-\mathbf{y})- \\
&-\delta(\mathbf{y}-\mathbf{x})\int d\mathbf{z}W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}-\mathbf{z})c_\beta(z_0-x_0)[c_L(z-x)+b_L d_L(z-x)]- \\
&-[-\bar\partial_1\delta(\mathbf{y}-\mathbf{x})+\frac{i\cos p_F}{2v_0}\bar\partial_1^2\delta(\mathbf{y}-\mathbf{x})]\int d\mathbf{z}W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}-\mathbf{z})c_\beta(z_0-x_0)a_L d_L(z-x)- \\
&+\partial_0\delta(\mathbf{y}-\mathbf{x})\int d\mathbf{z}W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}-\mathbf{z})d_\beta(z_0-x_0)c_L(z-x)\Big\}\ .
\end{aligned}\tag{3.21}$$

3)If $2n = 2$ and, possibly after a suitable permutation of the fields, $\underline{\sigma} = (+,-)$, $\omega_1 = -\omega_2 = \omega$, by (2.74),

$$\begin{aligned}
&\mathcal{L}\int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)\psi^{(\le h)+}_{\mathbf{x}_1,\omega}\psi^{(\le h)-}_{\mathbf{x}_2,-\omega} = \\
&= \int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)\psi^{(\le h)+}_{\mathbf{x}_1,\omega}T^{0(\le h)-}_{\mathbf{x}_2,\mathbf{x}_1,-\omega} \\
&= \int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)T^{0(\le h)+}_{\mathbf{x}_1,\mathbf{x}_2,\omega}\psi^{(\le h)-}_{\mathbf{x}_2,-\omega}\;,
\end{aligned}\tag{3.22}$$

where

$$T^{0(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega} = c_\beta(y_0-x_0)c_L(y-x)\psi^{(\le h)\sigma}_{\mathbf{x},\omega}\;. \tag{3.23}$$

Therefore

$$\begin{aligned}
&\mathcal{R}\int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)\psi^{(\le h)+}_{\mathbf{x}_1,\omega}\psi^{(\le h)-}_{\mathbf{x}_2,-\omega} = \\
&= \int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)\psi^{(\le h)+}_{\mathbf{x}_1,\omega}D^{1,2(\le h)-}_{\mathbf{x}_2,\mathbf{x}_1,-\omega} \\
&= \int d\mathbf{x}_1 d\mathbf{x}_2 W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1-\mathbf{x}_2)D^{1,2(\le h)+}_{\mathbf{x}_1,\mathbf{x}_2,\omega}\psi^{(\le h)-}_{\mathbf{x}_2,-\omega}\;,
\end{aligned}\tag{3.24}$$

where

$$D^{1,2(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega} = \psi^{(\le h)\sigma}_{\mathbf{y},\omega} - T^{0(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}\;. \tag{3.25}$$

Hence the effect of $\mathcal{R}$ can be described as the replacement of a $\psi^{(\le h)\sigma}$ field with a $D^{1,2(\le h)\sigma}$ field, with a gain in the bounds (see discussion in item 1) above) of a factor $\gamma^{-(h-h')}$. As before, we can also write

$$\begin{aligned}
&\mathcal{R}\int d\mathbf{x}d\mathbf{y}W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}-\mathbf{y})\psi^{(\le h)+}_{\mathbf{x},\omega}\psi^{(\le h)-}_{\mathbf{y},-\omega} = \int d\mathbf{x}d\mathbf{y}\psi^{(\le h)+}_{\mathbf{x},\omega}\psi^{(\le h)-}_{\mathbf{y},-\omega}\cdot \\
&\cdot\left\{W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}-\mathbf{y}) - \delta(\mathbf{y}-\mathbf{x})\int d\mathbf{z}W^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{x}-\mathbf{z})c_\beta(z_0-x_0)c_L(z-x)\right\}\;.
\end{aligned}\tag{3.26}$$

3.2 By using iteratively the "single scale expansion" (2.112), starting from $\hat{\mathcal{V}}^{(1)} = \mathcal{V}^{(1)}$, we can write the effective potential $\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})$, for $h \le 0$, in terms of a *tree expansion*, similar to that described, for example, in [BGPS].

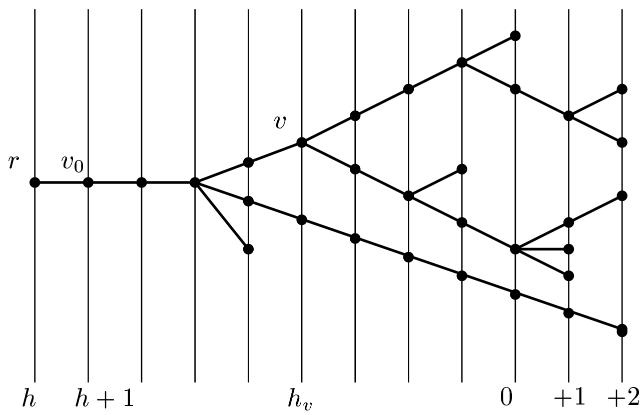

Fig. 1

We need some definitions and notations.

1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabeled tree* (see Fig. 1), so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order.

Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n.

We shall consider also the *labeled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label $h \leq 0$ with the root and we denote $\mathcal{T}_{h,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an an integer taking values in $[h, 2]$, and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root is on the line with index h. There is the constraint that, if v is an endpoint, $h_v > h + 1$.

The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $h + 1$.

Finally, if there is only one endpoint, its scale must be equal to $+2$ or $h + 2$.

3) With each endpoint v of scale $h_v = +2$ we associate one of the two contributions to $\mathcal{V}^{(1)}(\psi^{(\leq 1)})$, written as in (3.1) and a set $\mathbf{x}_v$ of space-time points (the corresponding integration variables), two for $\lambda V_\lambda(\psi^{(\leq 1)})$, one for $\nu N(\psi^{(\leq 1)})$; we shall say that the endpoint is of type λ or ν, respectively. With each endpoint v of scale $h_v \leq 1$ we associate one of the four local terms that we obtain if we write $\mathcal{L}V^{(h_v - 1)}$ (see (2.108)) by using the expressions (3.5) (there are four terms since F_α is the sum of two different local terms), and one space-time point $\mathbf{x}_v$; we shall say that the endpoint is of type ν, δ_1, δ_2, λ, with an obvious correspondence with the different terms.

Given a vertex v, which is not an endpoint, $\mathbf{x}_v$ will denote the family of all space-time points associated with one of the endpoints following v.

Moreover, we impose the constraint that, if v is an endpoint and $\mathbf{x}_v$ is a single space-time point (that is the corresponding term is local), $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding v.

4) If v is not an endpoint, the *cluster* L_v with frequency h_v is the set of endpoints following the vertex v; if v is an endpoint, it is itself a (*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters.

5) The trees containing only the root and an endpoint of scale $h+1$ will be called the *trivial trees*; note that they do not belong to $\mathcal{T}_{h,1}$, if $h \le 0$, and can be associated with the four terms in the local part of $\hat{\mathcal{V}}^{(h)}$.

6) We introduce a *field label* f to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, $\sigma(f)$ and $\omega(f)$ will denote the space-time point, the σ index and the ω index, respectively, of the field variable with label f.

If $h_v \le 0$, one of the field variables belonging to I_v carries also a discrete derivative $\bar{\partial}_1^m$, $m \in \{1,2\}$, if the corresponding local term is of type δ_m, see (3.5). Hence we can associate with each field label f an integer $m(f) \in \{0,1,2\}$, denoting the order of the discrete derivative. Note that $m(f)$ is not uniquely determined, since we are free to use the first or the second representation of $F_\alpha^{(\le h_v-1)}$ in (3.5); we shall use this freedom in the following.

By using (2.112), it is not hard to see that, if $h \le 0$, the effective potential can be written in the following way:

$$\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + L\beta\tilde{E}_{h+1} = \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{h,n}} V^{(h)}(\tau,\sqrt{Z_h}\psi^{(\le h)}) \qquad (3.27)\,,$$

where, if v_0 is the first vertex of τ and $\tau_1,..,\tau_s$ ($s = s_{v_0}$) are the subtrees of τ with root v_0, $V^{(h)}(\tau,\sqrt{Z_h}\psi^{(\le h)})$ is defined inductively by the relation

$$\begin{aligned} &V^{(h)}(\tau,\sqrt{Z_h}\psi^{(\le h)}) = \\ &\frac{(-1)^{s+1}}{s!}\mathcal{E}^T_{h+1}[\bar{V}^{(h+1)}(\tau_1,\sqrt{Z_h}\psi^{(\le h+1)});..;\bar{V}^{(h+1)}(\tau_s,\sqrt{Z_h}\psi^{(\le h+1)})]\,, \end{aligned} \qquad (3.28)$$

and $\bar{V}^{(h+1)}(\tau_i,\sqrt{Z_h}\psi^{(\le h+1)})$

a) is equal to $\mathcal{R}\hat{\mathcal{V}}^{(h+1)}(\tau_i,\sqrt{Z_h}\psi^{(\le h+1)})$ if the subtree τ_i is not trivial (see (2.107) for the definition of $\hat{\mathcal{V}}^{(h)}$);

b) if τ_i is trivial and $h \le -1$, it is equal to one of the terms in the r.h.s. of (2.108) with scale $h+1$ or, if $h = 0$, to one of the terms contributing to $\hat{\mathcal{V}}^{(1)}(\psi^{\le 1})$.

If $h = 0$, the r.h.s. of (3.28) can be written more explicitly in the following way. Given $\tau \in \mathcal{T}_{0,n}$, there are n endpoints of scale 2 and only another one vertex, v_0, of scale 1; let us call $v_1,\ldots,v_n$ the endpoints. We choose, in any set I_{v_i}, a subset Q_{v_i} and we define $P_{v_0} = \cup_i Q_{v_i}$; then we can write (recall that $Z_0 = 1$)

$$V^{(0)}(\tau,\sqrt{Z_0}\psi^{(\le 0)}) = \sum_{P_{v_0}} V^{(0)}(\tau,P_{v_0})\,, \qquad (3.29)$$

$$V^{(0)}(\tau,P_{v_0}) = \sqrt{Z_0}^{|P_{v_0}|}\int d\mathbf{x}_{v_0}\tilde\psi^{\le 0}(P_{v_0})K^{(1)}_{\tau,P_{v_0}}(\mathbf{x}_{v_0})\;, \tag{3.30}$$

$$K^{(1)}_{\tau,P_{v_0}}(\mathbf{x}_{v_0}) = \frac{1}{n!}\mathcal{E}_1^T[\tilde\psi^{(1)}(P_{v_1}\backslash Q_{v_1}),\ldots,\tilde\psi^{(1)}(P_{v_n}\backslash Q_{v_n})]\prod_{i=1}^n K^{(2)}_{v_i}(\mathbf{x}_{v_i})\;, \tag{3.31}$$

where we used the definitions

$$\tilde\psi^{(h)}(P_v) = \prod_{f\in P_v}\bar\partial_1^{m(f)}\psi^{(h)\sigma(f)}_{\mathbf{x}(f),\omega(f)}\;, \tag{3.32}$$

$$K^{(2)}_{v_i}(\mathbf{x}_{v_i}) = e^{i\mathbf{p}_F\sum_{f\in I_{v_i}}\mathbf{x}(f)\sigma(f)}\begin{cases}\lambda v_\lambda(\mathbf{x}-\mathbf{y}) & \text{if } v_i \text{ is of type } \lambda \text{ and } \mathbf{x}_{v_i}=(\mathbf{x},\mathbf{y}),\\ \nu & \text{if } v_i \text{ is of type } \nu,\end{cases} \tag{3.33}$$

and we suppose that the order of the (anticommuting) field variables in (3.32) is suitable chosen in order to fix the sign as in (3.31).

Note that the terms with $P_{v_0}\neq\emptyset$ in the r.h.s. of (3.29) contribute to $L\beta\tilde E_1$, while the others contribute to $\mathcal{V}^{(0)}(\sqrt{Z_0}\psi^{(\le 0)})$.

The potential $\hat{\mathcal{V}}^{(0)}(\sqrt{Z_{-1}}\psi^{(\le 0)})$, needed to iterate the previous procedure, is obtained, as explained in §2.5 and §2.8, by decomposing $\mathcal{V}^{(0)}$ in the sum of $\mathcal{L}\mathcal{V}^{(0)}$ and $\mathcal{R}\mathcal{V}^{(0)}$, by moving afterwards some local terms to the free measure and finally by rescaling the fields variables. The representation we get for $\mathcal{V}^{(-1)}(\sqrt{Z_{-1}}\psi^{(\le -1)})$ depends on the representation we use for $\mathcal{R}V^{(0)}(\tau,P_{v_0})$. We choose to use that based on (3.14), (3.21) and (3.26), where the regularization is seen, for each term in the r.h.s. of (3.29) with $P_{v_0}\neq\emptyset$, as a modification of the kernel

$$W^{(0)}_{\tau,P_{v_0}}(\mathbf{x}_{P_{v_0}}) = \int d(\mathbf{x}_{v_0}\backslash\mathbf{x}_{P_{v_0}})K^{(1)}_{\tau,P_{v_0}}(\mathbf{x}_{v_0})\;, \tag{3.34}$$

where $\mathbf{x}_{P_{v_0}} = \cup_{f\in P_{v_0}}\mathbf{x}(f)$. In order to remember this choice, we write

$$\mathcal{R}V^{(0)}(\tau,P_{v_0}) = \sqrt{Z_0}^{|P_{v_0}|}\int d\mathbf{x}_{v_0}\tilde\psi^{(\le 0)}(P_{v_0})[\mathcal{R}K^{(1)}_{\tau,P_{v_0}}(\mathbf{x}_{v_0})]\;. \tag{3.35}$$

It is then easy to get, by iteration of the previous procedure, a simple expression for $V^{(h)}(\tau,\sqrt{Z_h}\psi^{(\le h)})$, for any $\tau\in\mathcal{T}_{h,n}$.

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1,\ldots,v_{s_v}$ are the vertices immediately following it, then $P_v\subset\cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i}\backslash Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$.

Given $\tau\in\mathcal{T}_{h,n}$, there are many possible choices of the subsets P_v, $v\in\tau$, compatible with all the constraints; we shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. Then we can write

$$V^{(h)}(\tau,\sqrt{Z_h}\psi^{(\le h)}) = \sum_{\mathbf{P}\in\mathcal{P}_\tau}V^{(h)}(\tau,\mathbf{P})\;; \tag{3.36}$$

$V^{(h)}(\tau,\mathbf{P})$ can be represented as in (3.30), that is as

$$V^{(h)}(\tau,\mathbf{P}) = \sqrt{Z_h}^{|P_{v_0}|} \int d\mathbf{x}_{v_0} \tilde\psi^{(\le h)}(P_{v_0}) K_{\tau,\mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0}) \;, \tag{3.37}$$

with $K_{\tau,\mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0})$ defined inductively (recall that $h_{v_0} = h+1$) by the equation, valid for any $v \in \tau$ which is not an endpoint,

$$\begin{aligned} K_{\tau,\mathbf{P}}^{(h_v)}(\mathbf{x}_v) = \frac{1}{s_v!} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{\frac{|P_v|}{2}} \prod_{i=1}^{s_v} [K_{v_i}^{(h_v+1)}(\mathbf{x}_{v_i})] \cdot \\ \cdot\; \tilde{\mathcal{E}}_{h_v}^T [\tilde\psi^{(h_v)}(P_{v_1}\backslash Q_{v_1}), \ldots, \tilde\psi^{(h_v)}(P_{v_{s_v}}\backslash Q_{v_{s_v}})] \;, \end{aligned} \tag{3.38}$$

where $\tilde{\mathcal{E}}_h^T$ denotes the truncated expectation with propagator $g^{(h)}$ (without the scaling factor Z_{h-1}, which is present in the definition of $\mathcal{E}_h^T$ used in (2.112)) and $Z_1 \equiv 1$. Moreover, if v is an endpoint and $h_v = 2$, $K_v^{(h_v)}(\mathbf{x}_v)$ is defined by (3.33), otherwise

$$K_v^{(h_v)}(\mathbf{x}_v) = \begin{cases} \lambda_{h_v-1} & \text{if } v \text{ is of type } \lambda \;, \\ i\omega\delta_{h_v-1} & \text{if } v \text{ is of type } \delta_1,\, d_2 \text{ and } \omega(f) = \omega \text{ for both } f \in I_v, \\ \omega\gamma^{h_v-1}\nu_{h_v-1} & \text{if } v \text{ is of type } \nu \text{ and } \omega(f) = \omega \text{ for both } f \in I_v. \end{cases} \tag{3.39}$$

If v is not an endpoint, $K_v^{(h_v)} = \mathcal{R} K_{\tau_i,\mathbf{P}_i}^{(h_v)}$, where $\tau_1, \ldots, \tau_{s_v}$ are the subtrees of τ with root v, $\mathbf{P}_i = \{P_v, v \in \tau_i\}$ and the action of $\mathcal{R}$ is defined using the representation (3.14), (3.21) and (3.26) of the regularization operation, seen as a modification of the kernel

$$W_{\tau,\mathbf{P}}^{(h_v)}(\mathbf{x}_{P_v}) = \int d(\mathbf{x}_v \backslash \mathbf{x}_{P_v}) K_{\tau,\mathbf{P}}^{(h_v)}(\mathbf{x}_v) \;, \tag{3.40}$$

where $\mathbf{x}_{P_v} = \cup_{f\in P_v} \mathbf{x}(f)$. Finally we suppose again that the order of the (anticommuting) field variables is suitable chosen in order to fix the sign as in (3.37).

Remark - The definitions (3.14), (3.21) and (3.26) of $\mathcal{R}$ are sufficient, even if they are restricted to external fields with $m(f) = 0$, because we can use the freedom in the definition of $m(f)$, see item 6) above, so that the external fields of v have always $m(f) = 0$, if v is a vertex where the $\mathcal{R}$ operation is acting on. This last claim follows from the observation that, since the truncated expectation in (3.38) vanishes if $s_v > 1$ and $P_{v_i}\backslash Q_{v_1} = \emptyset$ for some i, at least one of the fields associated with the endpoints of type δ_1 or δ_2, the only ones which have fields with $m(f) > 0$, has to be an internal field; hence, if one of the two fields is external, we can put $m(f) = 0$ for it. If $s_v = 1$ the previous argument should not work, but in this case the only vertex immediately following v can be an endpoint of type δ_1 or δ_2 only if $v = v_0$, see item 2 above; however this is not a problem since the action of $\mathcal{R}$ on a local term is equal to 0.

Note also that the kernel $K_{\tau,\mathbf{P}}^{(h_v)}(\mathbf{x}_v)$ is translation invariant, if $\sum_{f\in P_v} \sigma(f) = 0$; in general, it satisfies the relation

$$K_{\tau,\mathbf{P}}^{(h_v)}(\mathbf{x}_v + \mathbf{x}) = e^{i\mathbf{p}_F \mathbf{x} \sum_{f\in P_v} \sigma(f)} K_{\tau,\mathbf{P}}^{(h_v)}(\mathbf{x}_v) \;. \tag{3.41}$$

There is a simple interpretation of $V^{(h)}(\tau,\mathbf{P})$ as the sum of a family $\mathcal{G}_\mathbf{P}$ of connected Feynman graphs build with single scale propagators of different scales, connecting the space-time points associated with the endpoints of the tree. A graph $g \in \mathcal{G}_\mathbf{P}$ is build by contracting, for any $v \in \tau$, all the internal fields in couples in all possible ways, by using the propagator g^{h_v}, so that we get a connected Feynman graph, if we represent as single points all the clusters associated with the vertices immediately following v. These graphs have the property that the set of lines connecting the endpoints of the cluster L_v and having scale $h' \geq h_v$ is a connected subgraph; by the way this property is indeed another constraint on the possible choices of $\mathbf{P}$. We shall call these graphs *compatible* with $\mathbf{P}$.

3.3 The representation (3.37) of $V^{(h)}(\tau, \mathbf{P})$ is based on the choice of representing the regularization as acting on the kernels. If we use instead the representation of $\mathcal{R}$ based on (3.10), (3.11), (3.19) and (3.24), some field variables have to be substituted with new ones, depending on two space-time points and containing possibly some derivatives. As we shall see, these new variables allow to get the right dimensional bounds, at the price of making much more involved the combinatorics. Hence, it is convenient to introduce a label $r_v(f)$ to keep trace of the regularization in the vertices of the tree where f is associated with an external field and the action of $\mathcal{R}$ turns out to be *non trivial*, that is $\mathcal{R} \neq 1$.

There are many vertices, where $\mathcal{R} = 1$ by definition, that is the vertices with more than 4 external fields, the endpoints and v_0. For these vertices all external fields will be associated with a label $r_v(f) = 0$.

Moreover, since $\mathcal{L}\mathcal{R} = 0$, the action of $\mathcal{R}$ is trivial even in most trivial vertices v with $|P_v| \leq 4$. This happens if the vertex (trivial or not) $\tilde{v}$ immediately following v has the same number of external fields as v, since then the kernels associated with v and $\tilde{v}$ are identical, up to a rescaling constant. In particular, this remark implies that, given the non trivial vertex v and the non trivial vertex v' immediately preceding v on the tree, there are at most two vertices $\bar{v}$, such that $v' < \bar{v} \leq v$ and the action of $\mathcal{R}$ is non trivial. For the same reason, given an endpoint v of scale $h_v = +2$ of type λ (hence not local), there are at most two vertices between v and the non trivial vertex v' immediately preceding v, where the action of $\mathcal{R}$ is non trivial. Since the number of endpoints is n and the number of non trivial vertices is bounded by $n-1$, the number of vertices where the action of $\mathcal{R}$ is non trivial is bounded by $2(2n-1)$.

Let us now consider one of these vertices, which all have 4 or 2 external fields. If $|P_v| = 2$ and the ω indices of the external fields are equal, we keep trace of the regularization by labeling the field variable, which is substituted with a D^2 field, see (3.19), with $r_v(f) = 2$ and the other with $r_v(f) = 0$. In principle we are free to decide which variable is labeled with $r_v(f) = 2$, that is how we fix the localization point; we make a choice in the following way. If there is no non trivial vertex v' such that $v_0 \leq v' < v$, we make an arbitrary choice, otherwise we put $r_v(f) = 2$ for the field which is an internal field in the nearest non trivial vertex preceding v. In other words, we try to avoid that a field affected by the regularization stays external in the vertices preceding v.

If $|P_v| = 2$ and the ω indices of the external fields are different, we label the field variable, which is substituted with a $D^{1,2}$ field, see (3.24), with $r_v(f) = 1$ and the other with $r_v(f) = 0$; which variable is labeled with $r_v(f) = 1$ is decided as in the previous case.

If $|P_v| = 4$, first of all we choose the localization point in the following way. If there is a vertex v' such that $v_0 \le v' < v$ and $P_{v'}$ contains one and only one $f \in P_v$, we chose $\mathbf{x}(f)$ as the localization point in v; in the other cases, we make an arbitrary choice. After that, we split the kernel associated with v into three terms as in (3.14); then we distinguish the three terms by putting $r_v(f) = 1$ for the external field which is substituted with a $D^{1,1(\le h)}$ field, when the delta functions are eliminated, and $r_v(f) = 0$ for the others.

The previous definitions imply that, given $f \in I_{v_0}$, it is possible that there are many different vertices in the tree, such that $r_v(f) \neq 0$, that is many vertices where the corresponding field variable appears as an external field and the action of $\mathcal{R}$ is non trivial. As a consequence, the expressions given in §3.1 for the regularized potentials would not be sufficient and we should consider more general expressions, containing as external fields more general variables. Even worse, there is the risk that field derivatives of arbitrary order have to be considered; this event would produce "bad" factorials in the bounds. Fortunately, we can prove that this phenomenon can be easily controlled, thanks to our choice of the localization point, see above, by a more careful analysis of the regularization procedure, that we shall keep trace of by changing the definition of the $r_v(f)$ labels.

Let us suppose first that $|P_v| = 4$ and that there is $f \in P_v$, such that $r_{\bar v}(f) \neq 0$ for some $\bar v > v$. We want to show that the action of $\mathcal{R}$ on v is indeed trivial; hence we can put $r_v(f) = 0$ for all $f \in P_v$, in agreement with the fact that the contribution to the effective potential associated with v is dimensionally irrelevant. First of all, note that it is not possible that $|P_{\bar v}| = 2$, as a consequence of the choice of the localization point in the vertices with two external fields, see above. On the other hand, if $|P_{\bar v}| = 4$, the fact that the action of $\mathcal{R}$ in the vertex v is equal to the identity follows from the observation following (3.13) and the definition (2.72).

Let us now consider the vertices v with $P_v = (f_1, f_2)$. We can exclude as before that $r_{\bar v}(f_i) \neq 0$ for $i = 1$ or $i = 2$ or both and $|P_{\bar v}| = 2$. The same conclusion can be reached, if there is no vertex $\bar v > v$, such that $|P_{\bar v}| = 4$, the action of $\mathcal{R}$ on $\bar v$ is non trivial and both f_1 and f_2 belong to the set of its external fields; this claim easily follows from the criterion for the choice of the localization point in the vertices with 4 external fields.

If, on the contrary, f_1 and f_2 are both labels of external fields of a vertex $\bar v > v$, such that $|P_{\bar v}| = 4$ and the action of $\mathcal{R}$ is non trivial, we have to distinguish two possibilities. If there is a non trivial vertex v' such that $v_0 \le v' < v$, and one of the external fields of v, let us say of label f_1, is an internal field, our choice of the localization points imply that both $r_v(f_1)$ and $r_{\bar v}(f_1)$ are different from 0, while $r_v(f_2) = r_{\bar v}(f_2) = 0$. If there is no non trivial vertex $v' < v$ with the previous property, that is if f_1 and f_2 are both labels of external fields down to v_0 (hence all vertices between v and v_0 are trivial) or they become together labels of internal fields in some vertex $v' < v$, we are still free to choose as we want the localization points in v and $\bar{v}$; we decide to choose them equal.

The previous discussion implies that, as a consequence of our prescriptions, a field variable can be affected by the regularization only once, except in the case considered in the last paragraph. However, also in this case, it is easy to see that everything works as we did not apply to the variable with label f_1 the regularization in the vertex $\bar{v}$. In fact, the first or second order zero (modulo (L,β)) in the difference $\mathbf{x}(f_1) - \mathbf{x}(f_2)$, related to the regularization in the vertex v, see §3.1, cancels the contribution of the term proportional to the delta function, related with the regularization of $\bar{v}$, see (3.14). This apparent lack of regularization in $\bar{v}$ is compensated by the fact that $\mathbf{x}(f_1) - \mathbf{x}(f_2)$ is of order $\gamma^{-h_{\bar{v}}}$, hence smaller than the factor γ^{-h_v} sufficient for the regularization of v (together with the improving effect of the field derivative). Hence there is a gain with respect to the usual bound of a factor $\gamma^{-(h_{\bar{v}}-h_v)}$, sufficient to regularize the vertex $\bar{v}$.

3.4 There is in principle another problem. Let us suppose that we decide to represent all the non trivial $\mathcal{R}$ operations as acting on the field variables. Let us suppose also that the field variable with label f is substituted, by the action of $\mathcal{R}$ on the vertex v, with a $D^{1,i}_{\mathbf{y},\mathbf{x}}$ or a $D^2_{\mathbf{y},\mathbf{x}}$ field, where $\mathbf{y} = \mathbf{x}(f)$ and $\mathbf{x} = \mathbf{x}(f')$ is the corresponding localization point. At first sight it seems possible that even the variable with label f' can be substituted with a $D^{1,i}$ or a D^2 field by the action of $\mathcal{R}$ on a vertex $\bar{v} > v$. If this happens, the point $\mathbf{x}(f')$ can not be considered as fixed and there is an "interference" between the two regularization operations, or even more than two, since this phenomenon could involve an ordered chain of vertices. This interference would not produce bad factorials in the bounds, but would certainly make more involved our expansion. However, we can show that, thanks to our localization prescription, this problem is not really present.

Let us suppose first that $|P_v| = 2$. In this case, if the field with label f' is external in some vertex $\bar{v} > v$, with $|P_{\bar{v}}|$ equal to 2 or 4, we are sure that $\mathbf{x}(f')$ is the localization point in $\bar{v}$, see §3.3, hence the corresponding filed can not be affected by the action of $\mathcal{R}$ on $\bar{v}$. The same conclusion can be reached, if $|P_v| = 4$ and $|P_{\bar{v}}| = 2$

If $|P_v| = |P_{\bar{v}}| = 4$ and the field with label f' is substituted, by the action of $\mathcal{R}$ on the vertex $\bar{v}$, with a $D^{1,i}$ or a ^{2} field, we know that the same can not be true for the field with label f, since the action of $\mathcal{R}$ on v is trivial.

The previous discussion implies that the field with label f' can be affected by the regularization (if $|P_v| = |P_{\bar{v}}| = 4$) only by changing its $\mathbf{x}$ label, but this is not a source of any problem.

3.5 In this section we want to discuss the representation of the fields $D^{1,i(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$, $i = 1,2$, and $D^{2(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$ introduced in §3.1, which allows to exploit the regularization properties of the $\mathcal{R}$ operation. In order to do that, we extend the definition of the fields $\psi^{(\le h)\sigma}_{\mathbf{x},\omega}$ to $\mathbb{R}^2$, by using (2.49); we get functions with values in the Grassmanian algebra, antiperiodic in x_0 and x with periods β and L, respectively.

Let us choose a family of positive functions $\chi_{\eta,\eta'}(\mathbf{x})$, $\eta,\eta'\in\{-1,0,+1\}$, on $\mathbb{R}^2$, such that

$$\begin{aligned}\chi_{\eta,\eta'}(\mathbf{x}) &= \begin{cases} 1 & \text{if } |x-\eta|\le 1/4 \text{ and } |x_0-\eta'|\le 1/4 \\ 0 & \text{if } |x-\eta|\ge 3/4 \text{ or } |x_0-\eta'|\ge 3/4\end{cases}\\ \sum_{\eta,\eta'}\chi_{\eta,\eta'}(\mathbf{x}) &= 1 \quad \text{if } \mathbf{x}\in[-1,1]\times[-1,1]\,.\end{aligned} \tag{3.42}$$

Given $\mathbf{x},\mathbf{y}\in\Lambda\times[-\beta/2,\beta/2]$, if $\chi_{\eta,\eta'}(\tilde{\mathbf{y}}-\tilde{\mathbf{x}})>0$, where $\tilde{\mathbf{x}}=(x/L,x_0/\beta)$ and $\tilde{\mathbf{y}}=(y/L,y_0/\beta)$, we can define $\bar{\mathbf{y}}=\mathbf{y}-(\eta L,\eta'\beta)$, so that $|x_0-\bar y_0|\le 3\beta/4$ and $|x-\bar y|\le 3L/4$. We see immediately that $D^{1,1(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}=(-1)^{\eta+\eta'}D^{1,1(\le h)\sigma}_{\bar{\mathbf{y}},\mathbf{x},\omega}$ and we can write

$$D^{1,1(\le h)\sigma}_{\bar{\mathbf{y}},\mathbf{x},\omega}=[\psi^{(\le h)\sigma}_{\bar{\mathbf{y}},\omega}-\psi^{(\le h)\sigma}_{\mathbf{x},\omega}]+[1-G_\sigma(\bar{\mathbf{y}}-\mathbf{x})]\psi^{(\le h)\sigma}_{\mathbf{x},\omega}\,. \tag{3.43}$$

It is easy to see that, if $|y_0|\le 3\beta/4$ and $|y|\le 3L/4$,

$$1-G_\sigma(\mathbf{y})=\frac{1}{L}\bar h_1(\tilde{\mathbf{y}})d_L(y)+\frac{1}{\beta}\bar h_2(\tilde{\mathbf{y}})d_\beta(y_0)\,,\quad \tilde{\mathbf{y}}=(y/L,y_0/\beta)\,, \tag{3.44}$$

where $\bar h_i(\mathbf{y})$, $i=1,2$, are suitable functions, uniformly smooth in L and β. Moreover

$$\psi^{(\le h)\sigma}_{\bar{\mathbf{y}},\omega}-\psi^{(\le h)\sigma}_{\mathbf{x},\omega}=(\bar{\mathbf{y}}-\mathbf{x})\cdot\int_0^1 dt\,\partial\,\psi^{(\le h)\sigma}_{\boldsymbol{\xi}(t),\omega}\,,\quad \boldsymbol{\xi}(t)=\mathbf{x}+t(\bar{\mathbf{y}}-\mathbf{x})\,, \tag{3.45}$$

where $\partial=(\partial_1,\partial_0)$ is the gradient, and it is easy to see that, if $|y_0|\le 3\beta/4$ and $|y|\le 3L/4$,

$$\mathbf{y}=\left(\bar h_3(\tilde{\mathbf{y}})d_L(y),\bar h_4(\tilde{\mathbf{y}})d_\beta(y_0)\right)\,, \tag{3.46}$$

where $\bar h_i(\mathbf{y})$, $i=3,4$, are other suitable functions, uniformly smooth in L and β.

Hence we can write

$$\begin{aligned}D^{1,1(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}=\sum_{\eta,\eta'}\Big\{&\left[\frac{1}{L}h_{1,\eta,\eta'}(\tilde{\mathbf{y}},\tilde{\mathbf{x}})d_L(y-x)+\frac{1}{\beta}h_{2,\eta,\eta'}(\tilde{\mathbf{y}},\tilde{\mathbf{x}})d_\beta(y_0-x_0)\right]\psi^{(\le h)\sigma}_{\mathbf{x},\omega}+ \\ &+h_{3,\eta,\eta'}(\tilde{\mathbf{y}},\tilde{\mathbf{x}})d_L(y-x)\int_0^1 dt\,\partial_1\psi^{(\le h)\sigma}_{\boldsymbol{\xi}(t),\omega}+h_{4,\eta,\eta'}(\tilde{\mathbf{y}},\tilde{\mathbf{x}})d_\beta(y_0-x_0)\int_0^1 dt\,\partial_0\psi^{(\le h)\sigma}_{\boldsymbol{\xi}(t),\omega}\Big\}\,,\end{aligned} \tag{3.47}$$

where

$$h_{i,\eta,\eta'}(\tilde{\mathbf{y}},\tilde{\mathbf{x}})=(-1)^{\eta+\eta'}\chi_{\eta,\eta'}(\tilde{\mathbf{y}}-\tilde{\mathbf{x}})\bar h_i((\bar y-x)/L,(\bar y_0-x_0)/\beta)\,,\quad i=1,4, \tag{3.48}$$

are smooth functions with support in the region $\{|y-x-\eta L|\le 3L/4, |y_0-x_0-\eta'\beta|\le 3\beta/4\}$, such that their derivatives of order n are bounded by a constant (depending on n) times $\gamma^{nh_{L,\beta}}$.

A similar expression is valid for $D^{1,2(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$. Let us now consider $D^{2(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$, see (3.20). We can write

$$D^{2(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}=(-1)^{\eta+\eta'}\tilde D^{2(\le h)\sigma}_{\bar{\mathbf{y}},\mathbf{x},\omega}+h(\tilde{\mathbf{y}}-\tilde{\mathbf{x}})d_L(y-x)\bar\partial_1^2\psi^{(\le h)\sigma}_{\mathbf{x},\omega}\,, \tag{3.49}$$

where $h(\mathbf{y}-\mathbf{x})$ is a uniformly smooth function and

$$
\begin{aligned}
\tilde{D}^{2(\leq h)\sigma}_{\bar{\mathbf{y}},\mathbf{x},\omega} &= \psi^{(\leq h)\sigma}_{\bar{\mathbf{y}},\omega} - \psi^{(\leq h)\sigma}_{\mathbf{x},\omega} - (\bar{\mathbf{y}}-\mathbf{x})\cdot\partial\,\psi^{(\leq h)\sigma}_{\mathbf{x},\omega} - \\
&- \psi^{(\leq h)\sigma}_{\mathbf{x},\omega}\{[c_\beta(\bar{y}_0-x_0)c_L(\bar{y}-x)-1] + b_L c_\beta(\bar{y}_0-x_0)d_L(\bar{y}-x)\} - \\
&- \bar{\partial}_1\psi^{(\leq h)\sigma}_{\mathbf{x},\omega}\{[c_\beta(\bar{y}_0-x_0)-1]d_L(\bar{y}-x) + [d_L(\bar{y}-x)-(\bar{y}-x)]\} - \\
&- (\bar{y}-x)[\bar{\partial}_1\psi^{(\leq h)\sigma}_{\mathbf{x},\omega} - \partial_1\psi^{(\leq h)\sigma}_{\mathbf{x},\omega}] - \partial_0\psi^{(\leq h)\sigma}_{\mathbf{x},\omega}[d_\beta(\bar{y}_0-x_0)c_L(y-x) - (\bar{y}_0-x_0)]\ .
\end{aligned}
\tag{3.50}
$$

Note that

$$
\bar{\partial}_1\psi^{(\leq h)\sigma}_{\mathbf{x},\omega} - \partial_1\psi^{(\leq h)\sigma}_{\mathbf{x},\omega} = \frac{i\sigma}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}'_{L,\beta}} e^{i\sigma\mathbf{k}'\mathbf{x}}(\sin k' - k')\hat{\psi}^{(h)\sigma}_{\mathbf{k}',\omega}
\tag{3.51}
$$

behaves dimensionally as $\partial_1^3\psi^{(\leq h)\sigma}_{\mathbf{x},\omega}$, hence we shall define

$$
\bar{\partial}_1^3\psi^{(\leq h)\sigma}_{\mathbf{x},\omega} = \bar{\partial}_1\psi^{(\leq h)\sigma}_{\mathbf{x},\omega} - \partial_1\psi^{(\leq h)\sigma}_{\mathbf{x},\omega}\ .
\tag{3.52}
$$

It is now easy to show that there exist functions $h_{\underline{n},\eta,\eta'}(\mathbf{y},\mathbf{x})$, with $\underline{n}=(n_1,\ldots,n_6)$, and $h_{i,j,\eta,\eta'}(\mathbf{y},\mathbf{x})$, $i,j=0,1$, smooth uniformly in L and β, such that

$$
\begin{aligned}
D^{2(\leq h)\sigma}_{\mathbf{y},\mathbf{x},\omega} &= \sum_{\eta,\eta'}\Big\{\sum_{\underline{n}} h_{\underline{n},\eta,\eta'}(\tilde{\mathbf{y}},\tilde{\mathbf{x}})d_L(y-x)^{n_1}d_\beta(y_0-x_0)^{n_2}L^{-n_3}\beta^{-n_4}\bar{\partial}_1^{n_5}\partial_0^{n_6}\psi^{(\leq h)\sigma}_{\mathbf{x},\omega} + \\
&+ \sum_{i,j} h_{i,j,\eta,\eta'}(\tilde{\mathbf{y}},\tilde{\mathbf{x}})d_i(\mathbf{y}-\mathbf{x})d_i(\mathbf{y}-\mathbf{x})\int_0^1 dt(1-t)\partial_i\partial_j\psi^{(\leq h)\sigma}_{\boldsymbol{\xi}(t),\omega}\Big\}\ ,
\end{aligned}
\tag{3.53}
$$

the sum over $\underline{n}$ being constrained by the conditions

$$
n_1+n_2\leq 2\ ,\quad 3\geq\sum_{i=3}^{6}n_i\geq 2\ .
\tag{3.54}
$$

3.6 In order to exploit the regularization properties of formulas like (3.47) or (3.53), one has to prove that the "zeros" $d_L(y-x)$ and $d_\beta(y_0-x_0)$ give a contribution to the bounds of order $\gamma^{-h'}$, with $h'\geq h$, if h is the scale at which the zero was produced by the action of $\mathcal{R}$. In §3.7 we shall realize this task by "distributing" the zeros along a path connecting a family of space-time points associated with a subset of field variables. Let $\mathbf{x}_0=\mathbf{x},\mathbf{x}_1,\ldots,\mathbf{x}_n=\mathbf{y}$ be the family of points connected by the path; it is easy to show that

$$
d_L(y-x) = \sum_{r=1}^{n} d_L(x_r-x_{r-1})e^{-i\frac{\pi}{L}(x_r+x_{r-1}-x_n-x_0)}\ .
\tag{3.55}
$$

A similar expression is valid for $d_\beta(y_0-x_0)$.

It can happen that one of the terms in the r.h.s. of (3.55) or the analogous expansion for $d_\beta(y_0-x_0)$ depends on the same space-time points as the integration variables in the r.h.s. of a term like (3.21) or (3.26). We want to study the effect of this event. Let us call $W(\mathbf{x}-\mathbf{y})$ the kernel appearing in the l.h.s. of (3.21) or (3.26), $W_R(\mathbf{x}-\mathbf{y})$ its regularization, that is the quantity appearing in braces in the corresponding r.h.s., and let us define

$$
I_{n_1,n_2} = \int d\mathbf{x}d\mathbf{y}\psi^{(\leq h)+}_{\mathbf{x},\omega}\psi^{(\leq h)-}_{\mathbf{y},\omega}W_R(\mathbf{x}-\mathbf{y})[e^{-i\pi\frac{y}{L}}d_L(y-x)]^{n_1}[e^{-i\pi\frac{y_0}{\beta}}d_\beta(y_0-x_0)]^{n_2}\ .
\tag{3.56}
$$

In the following we shall meet such expressions for values of n_1 and n_2, such that $1 \leq n_1+n_2 \leq 2$.

If $W(\mathbf{x}-\mathbf{y})$ is the kernel appearing in the l.h.s. of (3.26), it is easy to see that, if $n_1+n_2 \geq 1$,

$$I_{n_1,n_2} = \int d\mathbf{x}d\mathbf{y}\psi^{(\leq h)+}_{\mathbf{x},\omega}\psi^{(\leq h)-}_{\mathbf{y},\omega}W(\mathbf{x}-\mathbf{y})[e^{-i\pi\frac{y}{L}}d_L(y-x)]^{n_1}[e^{-i\pi\frac{y_0}{\beta}}d_\beta(y_0-x_0)]^{n_2} \; , \quad (3.57)$$

that is the presence of the zeros simply erases the effect of the regularization.

Let us now suppose that $W(\mathbf{x}-\mathbf{y})$ is the kernel appearing in the l.h.s. of (3.21) and $W_R(\mathbf{x}-\mathbf{y})$ its regularization. We have

$$\begin{aligned} I_{1,0} = \int & d\mathbf{x}d\mathbf{y}\psi^{(\leq h)+}_{\mathbf{x},\omega}W(\mathbf{x}-\mathbf{y})d_L(y-x)\Big\{D^{1,3(\leq h)-}_{\mathbf{y},\mathbf{x},\omega} - \\ & - c_\beta(y_0-x_0)[\frac{1}{2}\bar\partial_1^2(e^{-i\pi\frac{x}{L}}\psi^{(\leq h)-}_{\mathbf{x},\omega}) + \frac{i\cos p_F}{v_0}\bar\partial_1(e^{-i\pi\frac{x}{L}}\psi^{(\leq h)-}_{\mathbf{x},\omega})]\Big\} \; , \end{aligned} \quad (3.58)$$

$$I_{0,1} = \int d\mathbf{x}d\mathbf{y}\psi^{(\leq h)+}_{\mathbf{x},\omega}W(\mathbf{x}-\mathbf{y})d_\beta(y_0-x_0)D^{1,4(\leq h)-}_{\mathbf{y},\mathbf{x},\omega} \; , \quad (3.59)$$

where

$$D^{1,3(\leq h)-}_{\mathbf{y},\mathbf{x},\omega} = e^{-i\pi\frac{y}{L}}\psi^{(\leq h)-}_{\mathbf{y},\omega} - c_\beta(y_0-x_0)e^{-i\pi\frac{x}{L}}\psi^{(\leq h)-}_{\mathbf{x},\omega} \; . \quad (3.60)$$

$$D^{1,4(\leq h)-}_{\mathbf{y},\mathbf{x},\omega} = e^{-i\pi\frac{y_0}{\beta}}\psi^{(\leq h)-}_{\mathbf{y},\omega} - c_L(y-x)e^{-i\pi\frac{x_0}{\beta}}\psi^{(\leq h)-}_{\mathbf{x},\omega} \; . \quad (3.61)$$

Moreover

$$\begin{aligned} I_{2,0} = \int & d\mathbf{x}d\mathbf{y}\psi^{(\leq h)+}_{\mathbf{x},\omega}W(\mathbf{x}-\mathbf{y})d_L(y-x)\Big\{d_L(y-x)e^{-2i\pi\frac{y}{L}}\psi^{(\leq h)-}_{\mathbf{y},\omega} - \frac{c_\beta(y_0-x_0)}{a_L}\cdot \\ & \cdot\; [\bar\partial_1(e^{-2i\pi\frac{x}{L}}\psi^{(\leq h)-}_{\mathbf{x},\omega}) + \frac{i\cos p_F}{v_0}(e^{-2i\pi\frac{x}{L}}\psi^{(\leq h)-}_{\mathbf{x},\omega} + \frac{1}{2}\bar\partial_1^2(e^{-2i\pi\frac{x}{L}}\psi^{(\leq h)-}_{\mathbf{x},\omega}))]\Big\} \; , \end{aligned} \quad (3.62)$$

$$I_{0,2} = \int d\mathbf{x}d\mathbf{y}\psi^{(\leq h)+}_{\mathbf{x},\omega}W(\mathbf{x}-\mathbf{y})d_\beta(y_0-x_0)^2e^{-2i\pi\frac{y_0}{\beta}}\psi^{(\leq h)-}_{\mathbf{y},\omega} \; , \quad (3.63)$$

$$I_{1,1} = \int d\mathbf{x}d\mathbf{y}\psi^{(\leq h)+}_{\mathbf{x},\omega}W(\mathbf{x}-\mathbf{y})d_L(y-x)d_\beta(y_0-x_0)e^{-i\pi\frac{y}{L}-i\pi\frac{y_0}{\beta}}\psi^{(\leq h)-}_{\mathbf{y},\omega} \; . \quad (3.64)$$

Note that no cancellations are possible for $\mathbf{x}=\mathbf{y}$ modulo (L,β) between the various terms contributing to I_{n_1,n_2}; hence they will be bounded separately.

Note also that the fields $D^{1,3(\leq h)-}_{\mathbf{y},\mathbf{x},\omega}$ and $D^{1,4(\leq h)-}_{\mathbf{y},\mathbf{x},\omega}$ have a zero of first order for $\mathbf{x}=\mathbf{y}$ modulo (L,β) and can be represented by expressions analogous to the r.h.s. of (3.47). Moreover, the terms contributing to $I_{0,1}$ and $I_{1,0}$ and containing these fields can also be written in a form analogous to (3.26).

Finally, we want to stress the fact that the integrands in the previous expressions of I_{n_1,n_2}, $1 \leq n_1+n_2 \leq 2$, have a zero of order at most two for $\mathbf{x}=\mathbf{y}$ modulo (L,β), that is a zero of order not higher of the zero introduced in the r.h.s. of (3.56). As it will be more clear in §3.7, this property would be lost if one uses the representation (3.19) of the regularization operation, before performing the "decomposition of the zeros"; one should get in this case a zero of order four and the iteration of the procedure of decomposition of the zeros would produce zeros of arbitrary order and, as a consequence, bad combinatorial factors in the bounds.

3.7 We are now ready to describe in more detail our expansion. First of all, we insert the decomposition (3.14) of $V^{(h)}(\tau,\psi^{(\le h)})$ in the vertices with $|P_v|=4$, by following the prescription for the choice of the localization point described in §3.3. The discussion of §3.3 allows also to define a new label $r(f)$, to be called the $\mathcal{R}$-label, for any $f\in I_{v_0}$, by putting

(i) $r(f)=0$, if $r_v(f)=0$ for any v such that $f\in P_v$;

(ii) $r(f)=(i,v)$, if there exists one and only one vertex v, such that $f\in P_v$ and $r_v(f)=i\neq 0$;

(iii) $r(f)=(2,v,\bar v)$, if there are two vertices v and $\bar v$, such that $v<\bar v$, $f\in P_v\subset P_{\bar v}$, $|P_v|=2$, $|P_{\bar v}|=4$, $r_v(f)=2$, $r_{\bar v}(f)=1$; see discussion in the last two paragraphs of §3.3.

Then, we can write

$$V^{(h)}(\tau,\sqrt{Z_h}\psi^{(\le h)})=\sum_{\mathbf{P}\in\mathcal{P}_{\tau},\mathbf{r}}V^{(h)}(\tau,\mathbf{P},\mathbf{r})\;,\tag{3.65}$$

where $\mathbf{r}=\{r(f),f\in I_{v_0}\}$ and the sum over $\mathbf{r}$ must be understood as the sum over the possible choices of $\mathbf{r}$ compatible with $\mathbf{P}$.

We can also write

$$V^{(h)}(\tau,\mathbf{P},\mathbf{r})=\sqrt{Z_h}^{|P_{v_0}|}\int d\mathbf{x}_{v_0}K^{(h)}_{\tau,\mathbf{P},\mathbf{r}}(\mathbf{x}_{v_0})\tilde\psi^{(\le h)}(P_{v_0})\;,\tag{3.66}$$

with $K^{(h)}_{\tau,\mathbf{P},\mathbf{r}}(\mathbf{x}_{v_0})$ defined inductively as in (3.38).

Let us consider first the action of $\mathcal{R}$ on $V^{(h)}(\tau,\mathbf{P},\mathbf{r})$. We can write for $\mathcal{R}V^{(h)}(\tau,\mathbf{P},\mathbf{r})$ an expression similar to (3.66), if we continue to use for the $\mathcal{R}$ operation the representation based on (3.14),(3.21) and (3.26), which affects the kernels leaving the fields unchanged. We shall use the notation

$$\mathcal{R}V^{(h)}(\tau,\mathbf{P},\mathbf{r})=\int d\mathbf{x}_{v_0}\tilde\psi^{(\le h)}(P_{v_0})[\mathcal{R}K^{(h)}_{\tau,\mathbf{P},\mathbf{r}}(\mathbf{x}_{v_0})]\;.\tag{3.67}$$

Moreover, we define $\mathbf{r}'$ so that $r'(f)=r(f)$ except for the field labels $f\in P_{v_0}$, for which $r'(f)$ takes into account also the regularization acting on v_0.

However, we can use for the $\mathcal{R}$ operation also the representation based on (3.10), (3.11), (3.19) and (3.24), which can be derived from the previous one by integrating the δ-functions; the effect is to replace one of the external fields with one of the fields $D^{1,i(\le h)\sigma}$, $i=1,2$ or $D^{2(\le h)\sigma}$. We can describe the result by writing

$$\mathcal{R}V^{(h)}(\tau,\mathbf{P},\mathbf{r})=\int d\mathbf{x}_{v_0}[\mathcal{R}\tilde\psi^{(\le h)}(P_{v_0})]K^{(h)}_{\tau,\mathbf{P},\mathbf{r}}(\mathbf{x}_{v_0})\;.\tag{3.68}$$

The discussion in §3.3 and §3.5 implies that there is a finite set A_{v_0}, such that

$$[\mathcal{R}\tilde\psi^{(\le h)}(P_{v_0})] = \sum_{\alpha\in A_{v_0}} h_\alpha(\tilde{\mathbf{x}}_{P_{v_0}}) d_L^{n_1(\alpha)} d_\beta^{n_2(\alpha)} \prod_{f\in P_{v_0}} [\hat\partial^{q_\alpha(f)}_{j_\alpha(f)}\psi]^{(\le h)\sigma(f)}_{\mathbf{x}_\alpha(f),\omega(f)} \;, \tag{3.69}$$

where $\tilde{\mathbf{x}}_{P_{v_0}} = (L^{-1}x_{P_{v_0}}, \beta^{-1}x_{0P_{v_0}})$, $d_L^{n_1(\alpha)}$ and $d_\beta^{n_2(\alpha)}$ are powers of the functions (2.96), with argument the difference of two points belonging to $\mathbf{x}_{P_{v_0}}$, and $\hat\partial^q_j$, $q = 0,1,2$, $j = 1,\ldots,m_q$, is a family of operators acting on the field variables, which are dimensionally equivalent to derivatives of order q. In particular $m_0 = 1$, $\hat\partial^0_1$ is the identity and the action of $\mathcal{R}$ is trivial, that is $|A_{v_0}| = 1$, $h_\alpha = 1$, $n_1(\alpha) = n_2(\alpha) = 0$ and $q_\alpha(f) = 0$ for any $f \in P_{v_0}$, except in the following cases.

1) If $|P_{v_0}| = 4$ and $r(f) = 0$ for any $f \in P_{v_0}$, there is $\bar f \in P_{v_0}$, such that the action of $\mathcal{R}$ over the fields consists in replacing one of the field variables with a $D^{1,1(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$ field, where $\mathbf{y} = \mathbf{x}(\bar f)$ and $\mathbf{x} = \mathbf{x}(f)$ for some other $f \in P_{v_0}$, see (3.11); moreover, one or two of the other fields change their space-time point. We write $D^{1,1(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$ in the representation (3.47); the resulting expression is of the form (3.69), with A_{v_0} consisting of four different terms, such that $d_L = d_L(y-x)$, $d_\beta = d_\beta(y_0-x_0)$, $n_1(\alpha)+n_2(\alpha) = 1$ and, for all $f \ne \bar f$, $q_\alpha(f) = 0$, while $q_\alpha(\bar f) = 1$. Moreover, if $f \ne \bar f$, $\mathbf{x}_\alpha(f)$ is a single point belonging to $\mathbf{x}_{P_{v_0}}$, not necessarily coinciding with $\mathbf{x}(f)$, while, if $f = \bar f$, $\mathbf{x}_\alpha(f)$ is equal to $\mathbf{x}$ or to the couple $(\mathbf{x},\mathbf{y})$ (using the previous definitions). The precise values of $\mathbf{x}_\alpha(\bar f)$ and $[\hat\partial^1_{j_\alpha(\bar f)}\psi]^{(\le h)\sigma(\bar f)}_{\mathbf{x}_\alpha(\bar f),\omega(\bar f)}$, together with the functions h_α, can be deduced from (3.47).

2) If $P_{v_0} = (f_1, f_2)$ and $\omega(f_1) = \omega(f_2)$, the action of $\mathcal{R}$ consists in replacing one of the external fields, of label, let us say, f_1, with a $D^{2(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$ field, where $\mathbf{y} = \mathbf{x}(f_1)$ and $\mathbf{x} = \mathbf{x}(f_2)$, if f_2 is the second field label. By using the representation (3.53) of $D^{2(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$, we get an expression of the form (3.69) consisting of many different terms, such that $d_L = d_L(y-x)$, $d_\beta = d_\beta(y_0-x_0)$, $n_1(\alpha)+n_2(\alpha) \le 2$, $q_\alpha(f_1) = 2$, $q_\alpha(f_2) = 0$, $\mathbf{x}_\alpha(f_2) = \mathbf{x}(f_2)$. The values of $\mathbf{x}_\alpha(f_1)$ and $[\hat\partial^2_{j_\alpha(f_1)}\psi]^{(\le h)\sigma(f_1)}_{\mathbf{x}_\alpha(f_1),\omega(f_1)}$, together with the functions h_α, can be deduced from (3.53).

3) If $P_{v_0} = (f_1, f_2)$ and $\omega(f_1) = -\omega(f_2)$, the action of $\mathcal{R}$ consists in replacing one of the external fields, of label, let us say, f_1, with a $D^{1,2(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$ field, where $\mathbf{y} = \mathbf{x}(f_1)$ and $\mathbf{x} = \mathbf{x}(f_2)$, if f_2 is the second field label. By using the analogous of the representation (3.47) for $D^{1,2(\le h)\sigma}_{\mathbf{y},\mathbf{x},\omega}$, we get an expression of the form (3.69) consisting of four different terms, such that $n_1(\alpha)+n_2(\alpha) = 1$, $q_\alpha(f_1) = 1$, $q_\alpha(f_2) = 0$, $\mathbf{x}_\alpha(f_2) = \mathbf{x}(f_2)$.

Let us now consider the action of $\mathcal{L}$ on $V^{(h)}(\tau, \sqrt{Z_h}\psi^{(\le h)})$. We get an expansion similar to that based on (3.68), that we can write, by using (2.79), (3.65) and translation invariance, in the form

$$\begin{aligned}\mathcal{L}V^{(h)}(\tau,\sqrt{Z_h}\psi^{(\le h)}) = \gamma^h n_h(\tau) Z_h F_\nu^{(\le h)} + s_h(\tau) Z_h F_\sigma^{(\le h)} + z_h(\tau) Z_h F_\zeta^{(\le h)} + \\ + a_h(\tau) Z_h F_\alpha^{(\le h)} + l_h(\tau) Z_h^2 F_\lambda^{(\le h)} \;,\end{aligned} \tag{3.70}$$

where

$$
\begin{aligned}
n_h(\tau) &= \frac{\gamma^{-h}}{L\beta} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau,\mathbf{r} \\ P_{v_0}=(f_1,f_2),\omega(f_1)=\omega(f_2)=+1}} \int d\mathbf{x}_{v_0} h_1(\tilde{\mathbf{x}}_{P_{v_0}}) K^{(h)}_{\tau,\mathbf{P},\mathbf{r}}(\mathbf{x}_{v_0}) \;, \\
s_h(\tau) &= \frac{1}{L\beta} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau,\mathbf{r} \\ P_{v_0}=(f_1,f_2),\omega(f_1)=-\omega(f_2)=+1}} \int d\mathbf{x}_{v_0} h_2(\tilde{\mathbf{x}}_{P_{v_0}}) K^{(h)}_{\tau,\mathbf{P},\mathbf{r}}(\mathbf{x}_{v_0}) \;, \\
z_h(\tau) &= \frac{1}{L\beta} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau,\mathbf{r} \\ P_{v_0}=(f_1,f_2),\omega(f_1)=\omega(f_2)=+1}} \int d\mathbf{x}_{v_0} h_3(\tilde{\mathbf{x}}_{P_{v_0}}) d_\beta(x(f_2)-x(f_1)) K^{(h)}_{\tau,\mathbf{P},\mathbf{r}}(\mathbf{x}_{v_0}) \;, \\
a_h(\tau) &= \frac{1}{L\beta} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau,\mathbf{r} \\ P_{v_0}=(f_1,f_2),\omega(f_1)=\omega(f_2)=+1}} \int d\mathbf{x}_{v_0} h_4(\tilde{\mathbf{x}}_{P_{v_0}}) d_L(x(f_2)-x(f_1)) K^{(h)}_{\tau,\mathbf{P},\mathbf{r}}(\mathbf{x}_{v_0}) \;, \\
l_h(\tau) &= \frac{1}{L\beta} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau,\mathbf{r} \\ |P_{v_0}|=4,\underline{\sigma}=(+,-,+,-),\underline{\omega}=(+1,-1,-1,+1)}} \int d\mathbf{x}_{v_0} h_5(\tilde{\mathbf{x}}_{P_{v_0}}) K^{(h)}_{\tau,\mathbf{P},\mathbf{r}}(\mathbf{x}_{v_0}) \;,
\end{aligned}
\tag{3.71}
$$

$h_i(\tilde{\mathbf{x}}_{P_{v_0}})$, $i = 1,\ldots,5$, being bounded functions, whose expressions can be deduced from (3.8), (3.16) and (3.22), also taking into account the permutations needed to order the field variables as in the r.h.s. of (3.70).

The constants n_h, s_h, z_h, a_h and l_h, which characterize the local part of the effective potential, can be obtained from (3.71) by summing over $n \ge 1$ and $\tau \in \mathcal{T}_{h,n}$. Finally, the constant $\tilde{E}_{h+1}$ appearing in the l.h.s. of (3.27) can be written in the form

$$
\tilde{E}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}_{h,n}} \tilde{E}_{h+1}(\tau) \;, \tag{3.72}
$$

where

$$
\tilde{E}_{h+1}(\tau) = \frac{1}{L\beta} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau,\mathbf{r} \\ P_{v_0}=\emptyset}} \int d\mathbf{x}_{v_0} K^{(h)}_{\tau,\mathbf{P},\mathbf{r}}(\mathbf{x}_{v_0}) \;. \tag{3.73}
$$

3.8 We want now to iterate the previous procedure, by using equation (3.38), in order to suitably take into account the non trivial $\mathcal{R}$ operations in the vertices $v \ne v_0$. We shall focus our discussion on $\mathcal{R}V^{(h)}(\tau,\mathbf{P},\mathbf{r})$, but the following analysis applies also to $\mathcal{L}V^{(h)}(\tau,\mathbf{P},\mathbf{r})$ and $\tilde{E}_{h+1}(\tau)$.

Let us consider the truncated expectation in the r.h.s. of (3.38) and let us put $s = s_v$, $P_i \equiv P_{v_i}\backslash Q_{v_i}$. Moreover we order in an arbitrary way the sets $P_i^\pm \equiv \{f \in P_i, \sigma(f) = \pm\}$, we call $f_{ij}^\pm$ their elements and we define $\mathbf{x}^{(i)} = \cup_{f\in P_i^-}\mathbf{x}(f)$, $\mathbf{y}^{(i)} = \cup_{f\in P_i^+}\mathbf{x}(f)$, $\mathbf{x}_{ij} = \mathbf{x}(f_{i,j}^-)$, $\mathbf{y}_{ij} = \mathbf{x}(f_{i,j}^+)$. Note that $\sum_{i=1}^s |P_i^-| = \sum_{i=1}^s |P_i^+| \equiv n$, otherwise the truncated expectation vanishes. A couple $l \equiv (f_{ij}^-, f_{i'j'}^+) \equiv (f_l^-, f_l^+)$ will be called a line joining the fields with labels $f_{ij}^-, f_{i'j'}^+$ and ω indices ω_l^-, ω_l^+ and connecting the points $\mathbf{x}_l \equiv \mathbf{x}_{i,j}$ and $\mathbf{y}_l \equiv \mathbf{y}_{i'j'}$, the *endpoints* of l; moreover we shall put $m_l \equiv m(f_l^-) + m(f_l^+)$. Then, it is well known (see [Le], [BGPS], for example) that, up to a sign, if $s > 1$,

$$
\mathcal{E}_h^T(\tilde{\psi}^{(h)}(P_1),\ldots,\tilde{\psi}^{(h)}(P_s)) = \sum_T \prod_{l\in T} \bar{\partial}_1^{m_l} g^{(h)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_l - \mathbf{y}_l) \int dP_T(\mathbf{t}) \det G^{h,T}(\mathbf{t}) \;, \tag{3.74}
$$

where T is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}^{(i)} \cup \mathbf{y}^{(i)}$, that is T is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover $\mathbf{t} = \{t_{i,i'} \in [0,1], 1 \le i, i' \le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm. Finally $G^{h,T}(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by $G^{h,T}_{ij,i'j'} = t_{i,i'} \bar{\partial}_1^{m(f^-_{ij})+m(f^+_{i'j'})} g^{(h)}_{\omega^-_l,\omega^+_l}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'})$ with $(f^-_{ij}, f^+_{i'j'})$ not belonging to T.

If $s = 1$, the sum over T is empty, but we shall still use equation (3.74), by interpreting the r.h.s. as 1, if P_1 is empty (which is possible, for $s = 1$), and as $\det G^h(\mathbf{1})$ otherwise.

Inserting (3.74) in the r.h.s. of (3.38) (with $v = v_0$) we obtain, up to a sign,

$$\mathcal{R}V^{(h)}(\tau, \mathbf{P}, \mathbf{r}) = \frac{1}{s_{v_0}!} \sqrt{Z_h}^{|P_{v_0}|} \sum_{T_{v_0}} \int d\mathbf{x}_{v_0} \int dP_{T_{v_0}}(\mathbf{t}) [\mathcal{R}\tilde{\psi}^{(\le h)}(P_{v_0})] \cdot$$
$$\cdot \Big[\prod_{l \in T_{v_0}} \bar{\partial}_1^{m_l} g^{(h+1)}_{\omega^-_l,\omega^+_l}(\mathbf{x}_l - \mathbf{y}_l) \Big] \det G^{h+1,T_{v_0}}(\mathbf{t}) \sqrt{\frac{Z_{h+1}}{Z_h}}^{|P_{v_0}|} \prod_{i=1}^{s_{v_0}} [K^{(h+2)}_{v_i}(\mathbf{x}_{v_i})] \tag{3.75}$$

Let us now consider the contribution to the r.h.s. of (3.75) of one of the terms in the representation (3.69) of $\mathcal{R}\tilde{\psi}^{(\le h)}(P_{v_0})$ with $n_1(\alpha) + n_2(\alpha) > 0$. For each choice of T_{v_0}, we decompose the factors $d_L^{n_1(\alpha)}(y - x)$ and $d_\beta^{n_2(\alpha)}(y_0 - x_0)$, by using equation (3.55) and the analogous equation for $d_\beta(y_0 - x_0)$, with $\mathbf{x}_0 = \mathbf{x}$, $\mathbf{x}_n = \mathbf{y}$ and the other points $\mathbf{x}_r$, $r = 1, \ldots, n-1$, chosen in the following way.

Let us consider the unique subset $(l_1, \ldots, l_m)$ of T_{v_0}, which selects a path joining the cluster containing $\mathbf{x}_0$ with the cluster containing $\mathbf{x}_n$, if one identifies all the points in the same cluster. Let $(\bar{v}_{i-1}, \bar{v}_i)$, $i = 1, m$, the couple of vertices whose clusters of points are joined by l_i. We shall put $\mathbf{x}_{2i-1}$, $i = 1, m$, equal to the endpoint of l_i belonging to $\mathbf{x}_{\bar{v}_{i-1}}$ and $\mathbf{x}_{2i}$ equal to the endpoint of l_i belonging to $\mathbf{x}_{\bar{v}_i}$. This definition implies that there are two points of the sequence $\mathbf{x}_r$, $r = 0, \ldots, n = 2m+1$, possibly coinciding, in any set $\mathbf{x}_{\bar{v}_i}$, $i = 0, \ldots, m$; these two points are the space-time points of two different fields belonging to $P_{\bar{v}_i}$. Since $n \le 2s_{v_0} - 1$, this decomposition will produce a finite number of different terms ($\le (2s_{v_0} - 1)^2$, since $n_1(\alpha) + n_2(\alpha) \le 2$), that we shall distinguish with a label α' belonging to a set B_{v_0}, depending on $\alpha \in A_{v_0}$ and T_{v_0}. These terms can be described in the following way.

Each term is obtained from the one chosen in the r.h.s. of (3.75) by adding a factor $\exp\{i\pi L^{-1} n_1(\alpha)(x + y) + i\pi\beta^{-1} n_2(\alpha)(x_0 + y_0)\}$. Moreover each propagator $g^{(h+1)}_{\omega^-_l,\omega^+_l}(\mathbf{x}_l - \mathbf{y}_l)$ is multiplied by a factor $d^{b_{\alpha'}(l)}_{j_{\alpha'}(l)}(\mathbf{x}_l, \mathbf{y}_l)$, where d^b_j, $d = 0, 1, 2$, $j = 1, \ldots, m_b$ is a family of functions so defined. If $b = 0$, $m_0 = 1$ and $d^0_1 = 1$. If $b = 1$, $m_b = 2$ and j distinguishes, up to the sign, the two functions

$$e^{-i\frac{\pi}{L}(x_l + y_l)} d_L(x_l - y_l) , \quad e^{-i\frac{\pi}{\beta}(x_{0,l} + y_{0,l})} d_\beta(x_{l0} - y_{l0}) . \tag{3.76}$$

If $b = 2$, j distinguishes the three possibilities, obtained by taking the product of two factors equal to one of the terms in (3.76). Finally each one of the vertices $v_1, \ldots, v_{s_{v_0}}$ is multiplied by a similar factor $d^{b_{\alpha'}(v_i)}_{j_{\alpha'}(v_i)}(\mathbf{x}_i, \mathbf{y}_i)$.

Note that the definitions were chosen so that $|d^b_j(\mathbf{x}, \mathbf{y})| \le |\mathbf{d}(\mathbf{x}-\mathbf{y})|^b$. Moreover there is the constraint that

$$\sum_{l \in T_{v_0}} b_{\alpha'}(l) + \sum_{i=1}^{s_{v_0}} b_{\alpha'}(v_i) = n_1(\alpha) + n_2(\alpha) \; . \tag{3.77}$$

The previous discussion implies that (3.75) can be written in the form

$$\begin{aligned} \mathcal{R}V^{(h)}(\tau, \mathbf{P}, \mathbf{r}) = & \frac{1}{s_{v_0}!} \sqrt{Z_h}^{|P_{v_0}|} \sum_{\alpha \in A_{v_0}} \sum_{T_{v_0}} \sum_{\alpha' \in B_{v_0}} \int d\mathbf{x}_{v_0} \int dP_{T_{v_0}}(\mathbf{t}) \cdot \\ & \cdot \; h_\alpha(\tilde{\mathbf{x}}_{P_{v_0}}) \Big[\prod_{f \in P_{v_0}} (\hat\partial^{q_\alpha(f)}_{j_\alpha(f)} \psi)^{(\le h)\sigma(f)}_{\mathbf{x}_\alpha(f), \omega(f)} \Big] \Big[\prod_{l \in T_{v_0}} d^{b_{\alpha'}(l)}_{j_{\alpha'}(l)}(\mathbf{x}_l, \mathbf{y}_l) \bar\partial_1^{m_l} g^{(h+1)}_{\omega_l^-, \omega_l^+}(\mathbf{x}_l - \mathbf{y}_l) \Big] \cdot \\ & \cdot \; \det G^{h+1, T_{v_0}}(\mathbf{t}) \sqrt{\frac{Z_{h+1}}{Z_h}}^{|P_{v_0}|} \Big[\prod_{i=1}^{s_{v_0}} d^{b_{\alpha'}(v_i)}_{j_{\alpha'}(v_i)}(\mathbf{x}_i, \mathbf{y}_i) K^{(h+2)}_{v_i}(\mathbf{x}_{v_i}) \Big] \; , \end{aligned} \tag{3.78}$$

where the function $h_\alpha(\tilde{\mathbf{x}}_{P_{v_0}})$ has be redefined in order to absorb the factor $\exp\{i\pi L^{-1} n_1(\alpha)(x+y) + i\pi\beta^{-1} n_2(\alpha)(x_0 + y_0)\}$.

3.9 We are now ready to begin the iteration of the previous procedure, by considering those among the vertices $v_1, \ldots, v_{s_{v_0}}$, where the action of $\mathcal{R}$ is non trivial. It turns out that we can not simply repeat the arguments used for v_0, but we have to consider some new situations and introduce some new prescriptions, which will be however sufficient to complete the iteration up to the endpoints, without any new problem.

Let us select a term in the r.h.s. of (3.78) and one of the vertices immediately following v_0, let us say $\bar v$, where the action of $\mathcal{R}$ is non trivial. We have to consider a few different cases.

A) Suppose that $b(\bar v) = 0$ (we shall omit the dependence on α and α'). In this case the action of $\mathcal{R}$ is exploited following essentially the same procedure as for v_0. If $\mathcal{R}$ is different from the identity, we move its action on the external fields of $\bar v$, by using the analogous of (3.69), by taking into account that some of the external fields of $\bar v$ are internal fields of v_0, hence they are involved in the calculation of the truncated expectation (3.74). This means that, if f is the label of an internal field with $q(f) > 0$, the corresponding (non trivial) $\hat\partial^{q(f)}_{j(f)}$ operator acts on the quantities in the r.h.s. of (3.78), which depend on f, that is $d^{b(l)}_{j(l)}(\mathbf{x}_l, \mathbf{y}_l) g^{(h+1)}_{\omega_l^-, \omega_l^+}(\mathbf{x}_l - \mathbf{y}_l)$ or the matrix elements of $\det G^{h+1, T_{v_0}}$, which are obtained by contracting the field with label f with another internal field. For example, if $\mathbf{x}(f) = \mathbf{x}_l$ and $\hat\partial^{q(f)}_{j(f)}$ is the operator associated with the third term in the r.h.s. of (3.47), we must substitute $d^{b(l)}_{j(l)}(\mathbf{x}_l, \mathbf{y}_l) \bar\partial_1^{m_l} g^{(h+1)}_{\omega_l^-, \omega_l^+}(\mathbf{x}_l - \mathbf{y}_l)$ with

$$\int_0^1 dt \partial_1 [d^{b(l)}_{j(l)}(\boldsymbol{\xi}(t) - \mathbf{y}_l) \bar\partial_1^{m_l} g^{(h+1)}_{\omega_l^-, \omega_l^+}(\boldsymbol{\xi}(t) - \mathbf{y}_l)] \; , \tag{3.79}$$

with $\boldsymbol{\xi}(t) = \mathbf{x}' + t(\bar{\mathbf{x}}_l - \mathbf{x}')$, for some $\mathbf{x}' \in \mathbf{x}_{\bar{v}}$, $\bar{\mathbf{x}}_l$ being defined in terms of $\mathbf{x}_l$ as $\bar{y}$ is defined in terms of y in §3.5 (that is $\bar{\mathbf{x}}_l$ and $\mathbf{x}_l$ are equivalent representation of the same point on the space-time torus).

There is apparently another problem, related to the possibilities that the operators $\hat{\partial}^{q(f)}_{j(f)}$ related with the action of $\mathcal{R}$ on $\bar{v}$ do not commute with the functions h_α and the field variables introduced by the action of $\mathcal{R}$ on v_0. However, the discussion in §3.4 implies that this can not happen, because of our prescription for the choice of the localization points. This argument is of general validity, hence we will not consider anymore this problem in the following.

B) If $b(\bar{v}) > 0$, we shall proceed in a different way, in order to avoid growing powers of the factors d_L and d_β, which should produce at the end bad combinatorial factors in the bounds. We need to distinguish four different cases.

B1) If $|P_{\bar{v}}| = 4$, we do not use the decomposition (3.47) for the field changed by the action of $\mathcal{R}$ in a $D^{1,1}$ field, but we simply write it as the sum of the two terms in the r.h.s. of (3.12) (in some cases the second term does not really contributes, because the argument of the factor $d^{b(\bar{v})}_{j(\bar{v})}$ is the same as the argument of the delta function in the representation (3.14) of the $\mathcal{R}$ action, but this is not true in general). We still get a representation of the form (3.69) for $[\mathcal{R}\tilde{\psi}^{(\le h)}(P_{\bar{v}})]$, but with the property that $q(f) = 0$ for any $\alpha \in A_{\bar{v}}$ and any $f \in P_{\bar{v}}$. This procedure works, because we do not need to exploit the regularization property of $\mathcal{R}$ in this case, as the following analysis will make clear.

B2) If $|P_{\bar{v}}| = 2$, and the ω-labels of the external fields are different, the action of $\mathcal{R}$, after the insertion of the zero, is indeed trivial, as explained in §3.6, see (3.57). Hence we do not make any change in the external fields.

B3) If $|P_{\bar{v}}| = 2$, the ω-labels of the external fields are equal and $b(\bar{v}) = 2$, the presence of the factor $d^{b(\bar{v})}_{j(\bar{v})}$ does not allow to use for the action of $\mathcal{R}$ on the external fields the representation (3.69), because that factor depends on the space-time labels of the external fields. However, we can use the representation following from the equations (3.62),(3.63),(3.64), by considering the different terms in the r.h.s. as different contributions (in any case no cancellations among such terms are possible).

Note that this representation has the same properties of the representation (3.69) and can be written exactly in the same form, by suitable defining the various quantities. In particular, it is still true that $n_1(\alpha) + n_2(\alpha) \le 2$.

Of course, we have to take also into account that some of the external fields of $\bar{v}$ are internal fields of v_0, but this can be done exactly as in item A).

B4) Finally, if $|P_{\bar{v}}| = 2$, the ω-labels of the external fields are equal and $b_{\bar{v}} = 1$, we use for the action of $\mathcal{R}$ on the external fields the representation following from the equations (3.58) and (3.59), after writing for the fields $D^{1,3}$ and $D^{1,4}$ the analogous of the decomposition

(3.47).

The above procedure can be iterated, by decomposing the factors $d^{b(v)}_{j(v)}$ coming from the previous steps of the iteration along the spanning tree associated with the clusters L_v, up to the endpoints. The final result can be described in the following way.

Let us call a *zero* each factor equal to one of the two terms in (3.76). Each zero produced by the action of $\mathcal{R}$ on the vertex v is distributed along a tree graph S_v on the set x_v, obtained by putting together an anchored tree graph $T_{\bar v}$ for each non trivial vertex $\bar v \ge v$ and adding a line for the couple of space-time points belonging to the set $\mathbf{x}_{\bar v}$ for each (not local) endpoint $\bar v \ge v$ with $h_{\bar v} = 2$ of type λ or u. At the end we have many terms, which are characterized, for what concerns the zeros, by a tree graph T on the set x_{v_0} and not more than two zeros on each line $l \in T$; the very important fact that there are at most two zeros on each line follows from the considerations in item B) of §3.9.

3.10 The final result can be written in the following way:

$$\begin{aligned} \mathcal{R}V^{(h)}(\tau,\mathbf{P},\mathbf{r}) = \sqrt{Z_h}^{|P_{v_0}|} \sum_{T\in\mathbf{T}} \sum_{\alpha\in A_T} \int d\mathbf{x}_{v_0} W_{\tau,\mathbf{P},\mathbf{r},T,\alpha}(\mathbf{x}_{v_0}) \cdot \\ \cdot \left\{ \prod_{f\in P_{v_0}} [\hat\partial^{q_\alpha(f)}_{j_\alpha(f)} \psi]^{(\le h)\sigma(f)}_{\mathbf{x}_\alpha(f),\omega(f)} \right\} , \end{aligned} \tag{3.80}$$

where

$$\begin{aligned} W_{\tau,\mathbf{P},\mathbf{r},T,\alpha}(\mathbf{x}_{v_0}) = h_\alpha(\tilde{\mathbf{x}}_{v_0}) \Big[\prod_{v\,\text{not e.p.}} \Big(Z_{h_v}/Z_{h_v-1}\Big)^{|P_v|/2} \Big] \cdot \\ \cdot \Big[\prod_{i=1}^n d^{b_\alpha(v_i^*)}_{j_\alpha(v_i^*)}(\mathbf{x}_i,\mathbf{y}_i) K^{h_i}_{v_i^*}(\mathbf{x}_{v_i^*}) \Big] \Big\{ \prod_{v\,\text{not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \cdot \\ \cdot \det G^{h_v,T_v}_\alpha(\mathbf{t}_v) \Big[\prod_{l\in T_v} \hat\partial^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)} \hat\partial^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)} [d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}_l,\mathbf{y}_l) \bar\partial_1^{m_l} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_l - \mathbf{y}_l)] \Big] \Big\} , \end{aligned} \tag{3.81}$$

$\mathbf{T}$ is the set of the tree graphs on $\mathbf{x}_{v_0}$, obtained by putting together an anchored tree graph T_v for each non trivial vertex v and adding a line (which will be by definition the only element of T_v) for the couple of space-time points belonging to the set $\mathbf{x}_v$ for each (not local) endpoint v with $h_v = 2$ of type λ or u; A_T is a set of indices which allows to distinguish the different terms produced by the non trivial $\mathcal{R}$ operations and the iterative decomposition of the zeros; $v_1^*, \ldots, v_n^*$ are the endpoints of τ, f_l^- and f_l^+ are the labels of the two fields forming the line l, "e.p." is an abbreviation of "endpoint". Moreover $G^{h_v,T_v}_\alpha(\mathbf{t}_v)$ is obtained from the matrix $G^{h_v,T_v}(\mathbf{t}_v)$, associated with the vertex v and T_v, see (3.74), by substituting $G^{h_v,T_v}_{ij,i'j'} = t_{v,i,i'} \bar\partial_1^{m(f^-_{ij})+m(f^+_{i'j'})} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'})$ with

$$G^{h_v,T_v}_{\alpha,ij,i'j'} = t_{v,i,i'} \hat\partial^{q_\alpha(f^-_{ij})}_{j_\alpha(f^-_{ij})} \hat\partial^{q_\alpha(f^+_{ij})}_{j_\alpha(f^+_{ij})} \bar\partial_1^{m(f^-_{ij})+m(f^+_{i'j'})} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) . \tag{3.82}$$

Finally, $\hat\partial^q_j$, $q = 0,1,2,3$, $j = 1,\ldots,m_q$, is a family of operators, implicitly defined in the previous sections, which are dimensionally equivalent to derivatives of order q; for each $\alpha \in A_T$, there is an operator $\hat\partial^{q_\alpha(f)}_{j_\alpha(f)}$ associated with each $f \in I_{v_0}$.

It would be very difficult to give a precise description of the various contributions to the sum over A_T, but fortunately we only need to know some very general properties, which easily follows from the discussion in the previous sections.

1) There is a constant C such that, $\forall T \in \mathbf{T}_\tau$, $|A_T| \le C^n$ and, $\forall \alpha \in A_T$, $|h_a(\tilde{\mathbf{x}}_{v_0})| \le C^n$.
2) For any $\alpha \in A_T$, the following inequality is satisfied

$$\Big[\prod_{f\in I_{v_0}} \gamma^{h_\alpha(f) q_\alpha(f)}\Big]\Big[\prod_{l\in T} \gamma^{-h_\alpha(l) b_\alpha(l)}\Big] \le \prod_{v \text{ not e.p.}} \gamma^{-z(P_v)} \,, \tag{3.83}$$

where $h_\alpha(f) = h_{v_0} - 1$ if $f \in P_{v_0}$, otherwise it is the scale of the vertex where the field with label f is contracted; $h_\alpha(l) = h_v$, if $l \in T_v$ and

$$z(P_v) = \begin{cases} 1 & \text{if } |P_v| = 4, \\ 1 & \text{if } |P_v| = 2 \text{ and } \sum_{f\in P_v} \omega(f) \ne 0 \,, \\ 2 & \text{if } |P_v| = 2 \text{ and } \sum_{f\in P_v} \omega(f) = 0 \,, \\ 0 & \text{otherwise.} \end{cases} \tag{3.84}$$

3.11 In order to prove (3.83), let us suppose first that there is no vertex with two external fields and equal ω indices; hence $q_\alpha(f) \le 1$, $\forall f \in I_{v_0}$, and $b_\alpha(l) \le 1$, $\forall l \in T$. Let us choose $f \in I_{v_0}$, such that $q_\alpha(f) = 1$; by analyzing the procedure described in §3.8 and §3.9, one can easily see that there are three vertices $v' < v \le \bar v$ and a line $l \in T_{\bar v}$, such that

(i) the field with label f is affected by the action of $\mathcal{R}$ on the vertex v;
(ii) $h_{v'} = h_\alpha(f)$ and $b_\alpha(l) = 1$;
(iii) if $v \le \tilde v < \bar v$ and $\tilde l \in T_{\tilde v}$, then $b_\alpha(\tilde l) = 0$;
(iv) if $v' < \tilde v \le \bar v$ and $f \ne \tilde f \in P_{\tilde v}$, then $q_\alpha(\tilde f) = 0$.

(ii) follows from the definition of $h_\alpha(f)$ and from the remark that the zero produced by the action of $\mathcal{R}$ on v is moved by the process of distribution of the zeros along T in some vertex $\bar v \ge v$. The property (iii) characterizes $\bar v$; in fact the procedure described in item B1) and B2) of §3.9 guarantees that no zero can be produced by the action of $\mathcal{R}$ in the vertices between v and $\bar v$, if the zero in $\bar v$ "originated" from the regularization in v. (iv) follows from the previous remark and from the fact that the action of $\mathcal{R}$ is trivial in all the vertices between v' and v, see §3.3.

The previous considerations imply that we can associate each factor $\gamma^{h_\alpha(f)}$ in the l.h.s. of (3.83) with a factor $\gamma^{-b_\alpha(l)}$, by forming disjoint pairs; with each pair we can associate two vertices v' and $\bar v$ and the path on τ containing all the vertices $v' < \tilde v \le \bar v$. Since each vertex with four external fields or two external fields and different ω indices certainly belongs to one of these paths, the inequality (3.83) then follows from the trivial identity

$$\gamma^{-(b_\alpha(l) - h_\alpha(f))} = \gamma^{-(h_{\bar v} - h_{v'})} = \prod_{v' < \tilde v \le \bar v} \gamma^{-1} \,. \tag{3.85}$$

In order to complete the proof, we have now to consider also the possibility that there is some vertex with two external fields and equal ω indices, where the action of $\mathcal{R}$ is non trivial. This means that there is some $f \in I_{v_0}$, such that $q_\alpha(f) = 2$ or even (see B4) in §3.9) $q_\alpha(f) = 1$, if there is a zero associated with a line of the spanning tree related with the vertex where f is affected by the regularization. One can proceed essentially in the same way, but has to consider a few different situations, since the value of $q_\alpha(f)$ is not fixed and, if $q_\alpha(f) = 2$, there are two zeros to associate with a single factor $\gamma^{2h_\alpha(f)}$ in the l.h.s. of (3.83). We shall not give the details, which have essentially to formalize the claim that each order one derivative couples with a order one zero, so that the corresponding factors in the l.h.s. of (3.83) contribute a factor γ^{-1} to all vertices between the vertex where the derivative takes its action and the vertex where the zero is "sitting".

Let us now introduce, given any set $P \subset I_{v_0}$, the notation

$$q_\alpha(P) = \sum_{f \in P} q_\alpha(f) \ , \quad m(P) = \sum_{f \in P} m(f) \ . \tag{3.86}$$

Note that, by the remark at the end of §3.2, $m(P_v) = 0$ for any $v \neq v_0$ which is not an endpoint of type δ_1 or δ_2 and that also $m(P_{v_0}) = 0$ for all the terms in the r.h.s. of (3.80).

We also define

$$\begin{aligned} |\vec{v}_h| &= \begin{cases} \sup\{|\lambda|, |\nu|\}, & \text{if } h = +1, \\ \sup\{|\lambda_h|, |\delta_h|, |\nu_h|\}, & \text{if } h =\leq 0. \end{cases} \\ \varepsilon_h &= \sup_{h' > h} |\vec{v}_{h'}| \ . \end{aligned} \tag{3.87}$$

Moreover, we suppose that the condition (2.117) is satisfied, so that $h^* \geq 0$. We shall prove the following theorem.

3.12 THEOREM. *Let $h > h^* \geq 0$, with h^* defined by (2.116). If the bounds (2.98) are satisfied and, for some constants c_1,*

$$\sup_{h' > h} \left| \frac{Z_{h'}}{Z_{h'-1}} \right| \leq e^{c_1 \varepsilon_h^2}, \quad \sup_{h' > h} \left| \frac{\sigma_{h'}}{\sigma_{h'-1}} \right| \leq e^{c_1 \varepsilon_h}, \tag{3.88}$$

there exists a constant $\bar{\varepsilon}$ (depending on c_1) such that, if $\varepsilon_h \leq \bar{\varepsilon}$, then, for a suitable constant c_0, independent of c_1, as well as of u, L and β,

$$\begin{aligned} &\sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\substack{\mathbf{P} \\ |P_{v_0}| = 2m}} \sum_{\mathbf{r}} \sum_{T \in \mathbf{T}} \sum_{\substack{\alpha \in A_T \\ q_\alpha(P_{v_0}) = k}} \int d\mathbf{x}_{v_0} |W_{\tau, \mathbf{P}, \mathbf{r}, T, \alpha}(\mathbf{x}_{v_0})| \leq \\ &\leq L\beta \gamma^{-h D_k(P_{v_0})} (c_0 \varepsilon_h)^n \ , \end{aligned} \tag{3.89}$$

where

$$D_k(P_{v_0}) = -2 + m + k \ . \tag{3.90}$$

Moreover

$$\sum_{\tau \in \mathcal{T}_{h,n}} [|n_h(\tau)| + |z_h(\tau)| + |a_h(\tau)| + |l_h(\tau)|] \leq (c_0 \varepsilon_h)^n \ , \tag{3.91}$$

$$\sum_{\tau \in \mathcal{T}_{h,n}} |s_h(\tau)| \leq |\sigma_h| (c_0 \varepsilon_h)^n \ , \tag{3.92}$$

$$\sum_{\tau\in\mathcal{T}_{h,n}} |\tilde E_{h+1}(\tau)| \le \gamma^{2h}(c_0\varepsilon_h)^n\;. \tag{3.93}$$

3.13 An important role in the proof of Theorem 3.12 plays the estimation of $\det G_\alpha^{h_v,T_v}(\mathbf{t}_v)$, that we shall now discuss, by referring to §3.8 and §3.10 for the notation. From now on C will denote a generic constant independent of u, L and β.

Given a vertex v which is not an endpoint and an anchored tree graph T_v (empty, if v is trivial), we consider the set of internal fields which do not belong to the any line of T_v and the corresponding sets $\tilde P^{\sigma,\omega}$ of field labels with $\sigma(f)=\sigma$ and $\omega(f)=\omega$. The sets $\cup_\omega \tilde P^{-,\omega}$ and $\cup_\omega \tilde P^{+,\omega}$ label the rows and the columns, respectively, of the matrix $G_\alpha^{h_v,T_v}(\mathbf{t}_v)$, hence they contain the same number of elements; however, $|\tilde P^{-,\omega}|$ can be different from $|\tilde P^{+,\omega}|$, if $h\le 0$. We introduce an integer $\rho(T_v)$, that we put equal to 1, if $|\tilde P^{-,\omega}|\neq|\tilde P^{+,\omega}|$, equal to 0 otherwise. We want to prove that

$$\begin{aligned} |\det G_\alpha^{h_v,T_v}(\mathbf{t}_v)| \le & \left(\frac{|\sigma_{h_v}|}{\gamma^{h_v}}\right)^{\rho(T_v)} C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)}\;\cdot \\ & \cdot\;\gamma^{\frac{h_v}{2}\left(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|-2(s_v-1)\right)}\gamma^{h_v\sum_{i=1}^{s_v}\left[q_\alpha(P_{v_i}\backslash Q_{v_i})+m(P_{v_i}\backslash Q_{v_i})\right]}\;\cdot \\ & \cdot\;\gamma^{-h_v\sum_{l\in T_v}\left[q_\alpha(f_l^+)+q_\alpha(f_l^-)+m(f_l^+)+m(f_l^-)\right]}\;. \end{aligned} \tag{3.94}$$

In order to prove this inequality, we shall suppose, for simplicity, that all the operators $\hat\partial_{j(f)}^{q(f)}$ and $\hat\partial_1^{m(f)}$ acting on the fields with field label $f\in\cup_{\sigma,\omega}\tilde P^{\sigma,\omega}$ are equal to the identity. It is very easy to modify the following argument, in order to prove that each operator $\hat\partial_{j(f)}^{q(f)}$ or $\hat\partial_1^{m(f)}$ gives a contribution to the bound proportional to $\gamma^{h_v q(f)}$ or $\gamma^{h_v m(f)}$, so proving (3.94) in the general case.

The proof is based on the well known *Gram-Hadamard inequality*, stating that, if M is a square matrix with elements M_{ij} of the form $M_{ij}=<A_i,B_j>$, where A_i, B_j are vectors in a Hilbert space with scalar product $<\cdot,\cdot>$, then

$$|\det M| \le \prod_i ||A_i||\cdot||B_i||\;. \tag{3.95}$$

where $||\cdot||$ is the norm induced by the scalar product.

Let $\mathcal{H}=\mathbb{R}^s\otimes\mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of complex four dimensional vectors $F(\mathbf{k}')=(F_1(\mathbf{k}'),\ldots,F_4(\mathbf{k}'))$, $F_i(\mathbf{k}')$ being a function on the set $\mathcal{D}'_{L,\beta}$, with scalar product

$$<F,G>=\sum_{i=1}^4\frac{1}{\beta L}\sum_{\mathbf{k}'}F_i^*(\mathbf{k}')G_i(\mathbf{k}')\;. \tag{3.96}$$

If $h_v\le 0$, it is easy to verify that

$$G^{h_v,T_v}_{ij,i'j'} = t_{i,i'}g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'}) = <\mathbf{u}_i\otimes A^{(h_v)}_{\mathbf{x}(f^-_{ij}),\omega(f^-_{ij})},\mathbf{u}_{i'}\otimes B^{(h_v)}_{\mathbf{x}(f^+_{i'j'}),\omega(f^+_{i'j'})}>\;, \tag{3.97}$$

where $\mathbf{u}_i\in\mathbb{R}^s$, $i=1,\ldots,s$, are the vectors such that $t_{i,i'}=\mathbf{u}_i\cdot\mathbf{u}_{i'}$, and

$$\begin{aligned} A^{(h)}_{\mathbf{x},\omega}(\mathbf{k}') &= e^{i\mathbf{k}'\mathbf{x}}\frac{\sqrt{\tilde f_h(\mathbf{k}')}}{\sqrt{-A_h(\mathbf{k}')}}\cdot\begin{cases}(-ik_0+E(k'),0,-i\sigma_{h-1}(\mathbf{k}'),0), & \text{if } \omega=+1,\\ (0,i\sigma_{h-1}(\mathbf{k}'),0,\sigma_{h-1}), & \text{if } \omega=-1,\end{cases} \\ B^{(h)}_{\mathbf{x},\omega} &= e^{i\mathbf{k}'\mathbf{y}}\frac{\sqrt{\tilde f_h(\mathbf{k}')}}{\sqrt{-A_h(\mathbf{k}')}}\cdot\begin{cases}(1,1,0,0), & \text{if } \omega=+1,\\ (0,0,1,(ik_0-E(k'))/\sigma_{h-1}), & \text{if } \omega=-1.\end{cases} \end{aligned} \tag{3.98}$$

Let us now define $n_+ = |\tilde P^{-,+}|$, $m_+ = |\tilde P^{+,+}|$, $m = |\tilde P^{-,+}| + |\tilde P^{-,-}| = |\tilde P^{+,+}| + |\tilde P^{+,-}|$; by using (3.95) and (3.98), it is easy to see, by proceeding as in §2.7, that, if the conditions (2.98) hold,

$$|\det G_\alpha^{h_v,T_v}(\mathbf{t}_v)| \le C^m \gamma^{h_v n_+} |\sigma_{h_v}|^{m-n_+} \left(\frac{\gamma^{h_v}}{|\sigma_{h_v}|}\right)^{m-m_+} = C^m \gamma^{h_v m} \left(\frac{|\sigma_{h_v}|}{\gamma^{h_v}}\right)^{m_+-n_+} . \quad (3.99)$$

Since $2m = \sum_{i=1}^{s_v} |P_{v_i}| - |P_v| - 2(s_v-1)$ and $\sum_{i=1}^{s_v} q_\alpha(P_{v_i}/Q_{v_i}) - \sum_{l\in T_v} [q_\alpha(f_l^+) + q_\alpha(f_l^-)] = 0$, we get the inequality (3.94), if $m_+ \ge n_+$, by using (2.116). The case $m_+ < n_+$ can be treated in a similar way, by exchanging the definitions of $A^{(h)}_{\mathbf{x},\omega}(\mathbf{k}')$ and $B^{(h)}_{\mathbf{x},\omega}(\mathbf{k}')$.

If $h_v = 0$, the inequality (3.95) can not be directly applied, because of the k_0^{-1} behaviour of the ultraviolet propagator for $k_0 \to \infty$; we would not get bounds uniform in the ultraviolet cutoff M (see (2.7)). However, we can make a further multiscale expansion of $g^{(+1)}(\mathbf{x})$, by an obvious smooth partition of the interval $\{|k_0| \ge 1\}$, and we can modify the trees by putting the endpoints on scale $h = M$; see [BGPS] for a similar procedure. It is easy to see that there is no relevant or marginal term on any scale $h_v > 0$, except for those which are obtained by contracting two fields associated with the same space-time point in a vertex located between an endpoint and the first non trivial vertex following it. However, the sum on the scale of this type of term (which is not absolutely convergent for $M \to \infty$) can be controlled by using the explicit expression of the single scale propagator, since there is indeed no divergence, but only a discontinuity at $x_0 = 0$ for $x = 0$. Hence, we can reduce again the calculation to the bound (3.95); we shall omit the details, which are of the same type of those used below for the infrared part of the model.

3.14 Proof of Theorem 3.12.

By using (3.81) and (3.94) we get

$$\begin{aligned}\int d\mathbf{x}_{v_0} |W_{\tau,\mathbf{P},\mathbf{r},T,\alpha}(\mathbf{x}_{v_0})| \le C^n J_{\tau,\mathbf{P},\mathbf{r},T,\alpha} \prod_{v\,\text{not e.p.}} \Big\{ \Big(Z_{h_v}/Z_{h_v-1}\Big)^{|P_v|/2} \cdot \\ \cdot\, C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v| - 2(s_v-1)} \left(\frac{|\sigma_{h_v}|}{\gamma^{h_v}}\right)^{\rho(T_v)} \gamma^{\frac{h_v}{2}\left(\sum_{i=1}^{s_v} |P_{v_i}| - |P_v| - 2(s_v-1)\right)} \cdot \\ \cdot\ \gamma^{h_v \sum_{i=1}^{s_v}[q_\alpha(P_{v_i}\backslash Q_{v_i}) + m(P_{v_i}\backslash Q_{v_i})]} \gamma^{-h_v \sum_{l\in T_v}[q_\alpha(f_l^+) + q_\alpha(f_l^-) + m(f_l^+) + m(f_l^-)]} \Big\},\end{aligned} \quad (3.100)$$

where

$$\begin{aligned}J_{\tau,\mathbf{P},\mathbf{r},T,\alpha} = \int d\mathbf{x}_{v_0} \Big| \Big[\prod_{i=1}^n d^{b_\alpha(v_i^*)}_{j_\alpha(v_i^*)}(\mathbf{x}_i, \mathbf{y}_i) K^{h_i}_{v_i^*}(\mathbf{x}_{v_i^*}) \Big] \cdot \\ \cdot \Big\{ \prod_{v\,\text{not e.p.}} \frac{1}{s_v!} \Big[\prod_{l\in T_v} \hat\partial^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)} \hat\partial^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)} [d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}_l, \mathbf{y}_l) \hat\partial_1^{m_l} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_l - \mathbf{y}_l)] \Big] \Big\} \Big| .\end{aligned} \quad (3.101)$$

In §3.15 we will prove that

$$\begin{aligned}J_{\tau,\mathbf{P},\mathbf{r},T,\alpha} \le C^n L\beta(\varepsilon_h)^n \prod_{v\,\text{not e.p.}} \Big[\frac{1}{s_v!} C^{2(s_v-1)} \gamma^{h_v n_\nu(v)} \Big(\prod_{l\in \bar T_v} |\frac{\sigma_{h_v}}{\gamma^{h_v}}| \Big) \cdot \\ \cdot\ \gamma^{-h_v \sum_{l\in T_v} b_\alpha(l)} \gamma^{-h_v(s_v-1)} \gamma^{h_v \sum_{l\in T_v}[q_\alpha(f_l^+) + q_\alpha(f_l^-) + m(f_l^+) + m(f_l^-)]} \Big] ,\end{aligned} \quad (3.102)$$

where $n_\nu(v)$ is the number of vertices of type ν with scale h_v+1 and $\bar T_v$ is the subset of the lines of T_v corresponding to *non diagonal* propagators, that is propagators with different ω indices.

It is easy to see that

$$\sum_{v \text{ not e.p.}} h_v \sum_{i=1}^{s_v} q_\alpha(P_{v_i}\backslash Q_{v_i}) + h\, q_\alpha(P_{v_0}) = \sum_{f\in I_{v_0}} h_\alpha(f) q_\alpha(f) \tag{3.103}$$

and, by using also the remark after (3.86), that

$$\begin{aligned}&\sum_{\bar v\ge v}\left\{\frac12\Big(\sum_{i=1}^{s_{\bar v}}|P_{\bar v_i}|-|P_{\bar v}|\Big)-2(s_{\bar v-1})+n_\nu(\bar v)+\sum_{i=1}^{s_{\bar v}} m(P_{\bar v_i}\backslash Q_{\bar v_i})\right\}=\\ &=\frac12(|I_v|-|P_v|)+m(I_v\backslash P_v)+\sum_{\bar v\ge v} n_\nu(\bar v)-2(n_v-1)=-\frac12|P_v|+2\,.\end{aligned} \tag{3.104}$$

By inserting (3.102) in (3.100) and using (3.83), (3.103), (3.104), we find

$$\begin{aligned}&\int d\mathbf{x}_{v_0}|W_{\tau,\mathbf{P},\mathbf{r},T,\alpha}(\mathbf{x}_{v_0})|\le C^n L\beta\varepsilon_h^n\gamma^{-hD_k(P_{v_0})}\prod_{v\in V_2}\frac{|\sigma_{h_v}|}{\gamma^{h_v}}\cdot\\ &\cdot\prod_{v \text{ not e.p.}}\left\{\frac{1}{s_v!}C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}\Big(Z_{h_v}/Z_{h_v-1}\Big)^{|P_v|/2}\gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]}\right\}\,,\end{aligned} \tag{3.105}$$

where V_2 is the set of vertices, which are not endpoints, such that $\rho(T_v)+|\tilde T_v|>0$, while the vertices $\bar v>v$ do not enjoy this property.

Let us now consider a vertex v, which is not an endpoint, such that $|P_v|=2$ and $\sum_{f\in P_v}\omega(f)\neq 0$. We want to show that there is a vertex $\bar v\ge v$, such that $\bar v\in V_2$. In order to prove this claim, we note that, if v^* is an endpoint, then $\sum_{f\in P_{v^*}}\sigma(f)\omega(f)=0$, while $\sum_{f\in P_v}\sigma(f)\omega(f)\neq 0$. Since all diagonal propagators join two fields with equal ω indices and opposite σ indices, given any Feynman graph connecting the endpoints of the cluster L_v, at least one of its lines has to be a non diagonal propagator, so that at least one of the vertices $\bar v\ge v$ must belong to V_2.

Moreover, if $v\in V_2$,

$$\frac{|\sigma_{h_v}|}{\gamma^{h_v}}=\frac{|\sigma_h|}{\gamma^h}\frac{|\sigma_{h_v}|}{|\sigma_h|}\gamma^{h-h_v}\le\frac{|\sigma_h|}{\gamma^h}\gamma^{(h-h_v)(1-c_1\varepsilon_h)}\le C\gamma^{(h-h_v)(1/2)}\,, \tag{3.106}$$

if $\varepsilon_h\le\bar\varepsilon$ and $\bar\varepsilon\le 1/(2c_1)$. We have used the second inequality in (3.88) and the definition (2.116), implying that $|\sigma_h|\le\frac{a_0}{4\gamma}\gamma^h$, if $h\ge h^*$.

It follows that

$$\prod_{v\in V_2}\frac{|\sigma_{h_v}|}{\gamma^{h_v}}\le C^n\prod_{v \text{ not e.p.}}\gamma^{-\frac12\tilde z(P_v)}\,, \tag{3.107}$$

where

$$\tilde z(P_v)=\begin{cases}1 & \text{if } |P_v|=2 \text{ and } \sum_{f\in P_v}\omega(f)\neq 0\,,\\ 0 & \text{otherwise,}\end{cases} \tag{3.108}$$

so that

$$-2+\frac{|P_v|}{2}+z(P_v)+\frac{\tilde z(P_v)}{2}\ge\frac12\,,\quad \forall v \text{ not e.p.}\,. \tag{3.109}$$

Hence (3.105) can be changed in

$$\int d\mathbf{x}_{v_0} |W_{\tau,\mathbf{P},\mathbf{r},T,\alpha}(\mathbf{x}_{v_0})| \le C^n L\beta \varepsilon_h^n \gamma^{-hD_k(P_{v_0})} \cdot$$
$$\cdot \prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} \left(Z_{h_v}/Z_{h_v-1} \right)^{|P_v|/2} \gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)+\frac{\tilde z(P_v)}{2}]} \right\} , \tag{3.110}$$

In order to complete the proof of the bound (3.89), we have to perform the sums in the r.h.s. of (3.89). The number of unlabeled trees is $\le 4^n$; fixed an unlabeled tree, the number of terms in the sum over the various labels of the tree is bounded by C^n, except the sums over the scale labels and the sets $\mathbf{P}$. The number of addenda in the sums over α and $\mathbf{r}$ is again bounded by C^n, since the action of $\mathcal{R}$ can be non trivial at most two times between two consecutive non trivial vertices (see §3.3) and the number of non trivial vertices is of order n.

Regarding the sum over T, it is empty if $s_v = 1$. If $s_v > 1$ and $N_{v_i} \equiv |P_{v_i}| - |Q_{v_i}|$, the number of anchored trees with d_i lines branching from the vertex v_i can be bounded, by using Caley's formula, by

$$\frac{(s_v-2)!}{(d_1-1)!...(d_{s_v}-1)!} N_{v_1}^{d_1} ... N_{v_{s_v}}^{d_{s_v}} \; ;$$

hence the number of addenda in $\sum_{T\in\mathbf{T}}$ is bounded by $\prod_{v \text{ not e.p.}} s_v! \; C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|}$.

In order to bound the sums over the scale labels and $\mathbf{P}$ we first use the inequality, following from (3.109) and the first inequality in (3.88), if $c_1 \varepsilon_h^2 \le 1/16$,

$$\prod_{v \text{ not e.p.}} \left(Z_{h_v}/Z_{h_v-1} \right)^{|P_v|/2} \gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)+\frac{\tilde z(P_v)}{2}]} \le$$
$$\le [\prod_{\tilde v} \gamma^{-\frac{1}{40}(h_{\tilde v} - h_{\tilde v'})}][\prod_{v \text{ not e.p.}} \gamma^{-\frac{|P_v|}{40}}] , \tag{3.111}$$

where $\tilde v$ are the non trivial vertices, and $\tilde v'$ is the non trivial vertex immediately preceding $\tilde v$ or the root. The factors $\gamma^{-\frac{1}{40}(h_{\tilde v} - h_{\tilde v'})}$ in the r.h.s. of (3.111) allow to bound the sums over the scale labels by C^n.

Finally the sum over $\mathbf{P}$ can be bound by using the following combinatorial inequality, trivial for γ large enough, but valid for any $\gamma > 1$ (see [BGPS], \$3). Let $\{p_v, v \in \tau\}$ a set of integers such that $p_v \le \sum_{i=1}^{s_v} p_{v_i}$ for all $v \in \tau$ which are not endpoints; then

$$\prod_{v \text{ not e.p.}} \sum_{p_v} \gamma^{-\frac{p_v}{40}} \le C^n . \tag{3.112}$$

It follows that

$$\sum_{\substack{\mathbf{P} \\ |P_{v_0}|=2m}} \prod_{v \text{ not e.p.}} \gamma^{-\frac{|P_v|}{40}} \le \prod_{v \text{ not e.p.}} \sum_{p_v} \gamma^{-\frac{p_v}{40}} \le C^n . \tag{3.113}$$

The proof of the bounds (3.91) and (3.93) is very similar. For the terms contributing to n_h one gets a bound like (3.89), with $m = 1$ and $k = 0$, but the factor $\gamma^{-hD_k(P_{v_0})} = \gamma^h$

is compensated by the factor γ^{-h} appearing in the definition of $n_h(\tau)$, see (3.71). For the terms contributing to z_h and a_h $D_k(P_{v_0}) = 0$ ($m = k = 1$), as well as for those contributing to l_h ($m = 2$, $k = 0$). Finally, for the terms contributing to $\tilde{E}_{h+1}$, $D_k(P_{v_0}) = 2$. For the terms contributing to s_h, $D_k(P_{v_0}) = -1$, but each term has also at least one small factor $|\sigma_h|\gamma^{-h}$ in its bound, since $|V_2| \geq 1$, see (3.106); so we get the bound (3.92).

3.15 Proof of (3.102).

We shall refer to the definitions and the discussion in §3.7 and §3.9. Let us consider the factor in the r.h.s. of (3.101) associated with the line $l \in T_v$ and let us suppose that $\mathbf{x}_l \in \mathbf{x}^{(i)}$, $\mathbf{y}_l \in \mathbf{x}^{(i')}$. By using (3.47), (3.53) and the similar expressions for the other difference fields produced by the regularization, we can write

$$\begin{aligned}
&\hat{\partial}^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)}\hat{\partial}^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)}[d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}_l,\mathbf{y}_l)\bar{\partial}_1^{m_l} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_l-\mathbf{y}_l)] = \\
&= \int_0^1 dt_l \int_0^1 ds_l \tilde{\partial}^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)}\tilde{\partial}^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)}[d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}'_l(t_l),\mathbf{y}'_l(s_l))\bar{\partial}_1^{m_l} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}'_l(t_l)-\mathbf{y}'_l(s_l))]\ ,
\end{aligned} \tag{3.114}$$

where, depending on α, there are essentially two different possibilities for the operators $\tilde{\partial}^{q_\alpha}_{j_\alpha}$ and the space-time points $\mathbf{x}'_l(t_l)$, $\mathbf{y}'_l(s_l)$. Let us consider, for example, f_l^-; then the first possibility is that $\tilde{\partial}^{q_\alpha}_{j_\alpha}$ is a derivative of order q_α and

$$\mathbf{x}'_l(t_l) = \tilde{\mathbf{x}}_l + t_l(\bar{\mathbf{x}}_l - \tilde{\mathbf{x}}_l)\ \text{,for some}\ \tilde{\mathbf{x}}_l \in \mathbf{x}^{(i)}\ , \tag{3.115}$$

$\bar{\mathbf{x}}_l$ being defined in terms of $\mathbf{x}_l$ as $\bar{y}$ is defined in terms of y in §3.5 (that is $\bar{\mathbf{x}}_l$ and $\mathbf{x}_l$ are equivalent representation of the same point on the space-time torus). The second possibility is that $\tilde{\partial}^{q_\alpha}_{j_\alpha}$ is a local operator of the form $L^{-n_1}\beta^{-n_2}\bar{\partial}_1^{n_3}\partial_0^{n_4}$, with $q_\alpha \leq \sum_{i=1}^4 n_i \leq q_\alpha + 1$, and $\mathbf{x}'_l(t_l) = \tilde{\mathbf{x}}_l \in \mathbf{x}^{(i)}$. Note that, by (2.40), $L^{-n_1}\beta^{-n_2} \leq \gamma^{h_{L,\beta}(n_1+n_2)} \leq \gamma^{h_v(n_1+n_2)}$.

By proceeding as in the proof of lemma (2.6) and using (2.105) it is very easy to show that, for any $N > 1$,

$$\begin{aligned}
&\left|\tilde{\partial}^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)}\tilde{\partial}^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)}[d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}'_l(t_l),\mathbf{y}'_l(s_l))\bar{\partial}_1^{m_l} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}'_l(t_l)-\mathbf{y}'_l(s_l))]\right| \leq \\
&\leq C\frac{\gamma^{h_v[1+q_\alpha(f_l^+)+q_\alpha(f_l^-)+m(f_l^-)+m(f_l^+)-b_\alpha(l)]}}{1+[\gamma^{h_v}|\mathbf{d}(\mathbf{x}'_l(t_l)-\mathbf{y}'_l(s_l))|]^N}\left(\frac{|\sigma_{h_v}|}{\gamma^{h_v}}\right)^{\rho_l}\ ,
\end{aligned} \tag{3.116}$$

where $\mathbf{d}(\mathbf{x})$ is defined in (2.97) and $\rho_l = 1$ if $\omega(f_l^-) \neq \omega(f_l^+)$, $\rho_l = 0$ otherwise. We used here the fact that, if $h_v = +1$, then $q_\alpha(f_l^-) = q_\alpha(f_l^+) = 0$, which allows to avoid the problems connected with the singularity of the time derivatives of the scale 1 propagator at $x'_{l,0}(t_l) - y'_{l,0}(s_l) = 0$.

Let us now consider the contribution of the endpoints to the r.h.s. of (3.101) and recall (see §3.10) that $T_{v_i^*}$ is empty, if $|\mathbf{x}_{v_i^*}| = 1$, hence $b_\alpha(v_i^*) = 0$, while, if $\mathbf{x}_{v_i^*} = (\mathbf{x}_i, \mathbf{y}_i)$, $T_{v_i^*}$ contains the line l_i connecting $\mathbf{x}_i$ with $\mathbf{y}_i$ and $h_{v_i^*} = 2$. By using (3.33) and (3.39), we get, if $h_i \equiv h_{v_i^*}$ and $S_\nu \equiv \{i : v_i^* \text{ is of type } \nu\}$,

$$\begin{aligned} &\left| \Big[\prod_{i=1}^n d_{j_\alpha(v_i^*)}^{b_\alpha(v_i^*)}(\mathbf{x}_i, \mathbf{y}_i) K_{v_i^*}^{h_i}(\mathbf{x}_{v_i^*}) \Big] \right| \le \\ &\le C^n \varepsilon_h^n \prod_{i:|\mathbf{x}_{v_i^*}|=2} \frac{1}{[1+|\mathbf{d}(\mathbf{x}_i - \mathbf{y}_i)|]^N} \prod_{i\in S_\nu} \gamma^{(h_i-1)} \; . \end{aligned} \tag{3.117}$$

Let us now remark that, after the insertion of the bounds (3.116) and (3.117) in the r.h.s. of (3.101), by possibly changing the constant C, we can substitute $\int d\mathbf{x}_{v_0}$, which is there a shorthand for $\prod_{\mathbf{x}\in\mathbf{x}_{v_0}} \sum_{x\in\Lambda} \int dx_0$, with the real integral over $(\mathbb{T}_{L,\beta})^{|\mathbf{x}_{v_0}|}$, where $\mathbb{T}_{L,\beta}$ is the space-time torus $[-L/2, L/2]\times[-\beta/2,\beta/2]$. Moreover, equation (3.115) can be thought, and we shall do that, as defining an interval on $\mathbb{T}_{L,\beta}$, when t_l spans the interval $[0,1]$; this is possible thanks to the introduction of the partition (3.42) in §3.5.

Hence, in order to complete the proof of (3.102), we have to show that, fixed a point $\bar{\mathbf{x}} \in \mathbf{x}_{v_0}$, the interpolation parameters associated with the regularization operations and an integer $N \ge 3$,

$$\int_\Xi d(\mathbf{x}_{v_0}\backslash\bar{\mathbf{x}}) \prod_{v\in\tau}\prod_{l\in T_v} \frac{1}{1+[\gamma^{h_v}|\mathbf{d}(\mathbf{x}'_l(t_l) - \mathbf{y}'_l(s_l))|]^N} \le \prod_{v\in\tau} C\gamma^{-h_v(s_v-1)} \; , \tag{3.118}$$

where Ξ denotes the subset of $(\mathbb{T}_{L,\beta})^{|\mathbf{x}_{v_0}\backslash\bar{\mathbf{x}}|}$ satisfying all the constraints associated with the interpolated points of the form (3.115).

Let us call $\tilde{T} = \cup_v \tilde{T}_v$, where $\tilde{T}_v$ is the set of lines connecting $\mathbf{x}'_l(t_l)$ with $\mathbf{y}'_l(s_l)$, for any $l \in T_v$. $\tilde{T}$ is not a tree in general; however, for any v, $\tilde{T}_v$ is still an anchored tree graph between the clusters of points $\mathbf{x}^{(i)}$, $i = 1,\ldots,s_v$. Hence, the proof of (3.118) becomes trivial, if we can show that

$$d(\mathbf{x}_{v_0}\backslash\bar{\mathbf{x}}) = \prod_{l\in\tilde{T}} d\mathbf{r}_l \; , \tag{3.119}$$

where $\mathbf{r}_l = \mathbf{x}'_l(t_l) - \mathbf{y}'_l(s_l)$.

In order to prove (3.119), we can proceed, for example, as in [BM1]. Let us consider first a vertex v with $|T_v| > 0$, which is maximal with respect to the tree order; hence either v is a non local endpoint with $h_v = 2$ or it is a non trivial vertex with no vertex v' with $|T_{v'}| > 0$ following it. In this case $\tilde{T}_v = T_v$, that is no line depends on the interpolation parameters, and $\tilde{T}_v$ is a tree on the set $\mathbf{x}_v$, so that we get immediately the identity

$$d\mathbf{x}_v = d\bar{\mathbf{x}}^{(v)} \prod_{l\in\tilde{T}_v} d\mathbf{r}_l \; , \tag{3.120}$$

where $\bar{\mathbf{x}}^{(v)}$ is an arbitrary point of $\mathbf{x}_v$. If we use (3.120) for the family S_0 of all maximal vertices with $|T_v| > 0$, we get

$$d\mathbf{x}_{v_0} = \prod_{v\in S_0} \Big[d\bar{\mathbf{x}}^{(v)} \prod_{l\in\tilde{T}_v} d\mathbf{r}_l \Big] \; . \tag{3.121}$$

Let us now consider a line $\bar{l} \in \tilde{T}$, which connects two clusters of points $\mathbf{x}_{v_1}$ and $\mathbf{x}_{v_2}$, with $v_i \in S_0$, $i = 1, 2$. By (3.115)

$$\mathbf{r}_{\bar{l}} = \mathbf{x}'_{\bar{l}}(t_{\bar{l}}) - \mathbf{y}'_{\bar{l}}(s_{\bar{l}}) = t_{\bar{l}}\mathbf{x}_{\bar{l}} + (1 - t_{\bar{l}})\bar{\mathbf{x}}_{\bar{l}} - \mathbf{y}'_{\bar{l}}(s_{\bar{l}}) \; , \tag{3.122}$$

implying that

$$\bar{\mathbf{x}}^{(v_1)} = \mathbf{r}_{\bar{l}} + \bar{\mathbf{x}}^{(v_1)} - \mathbf{r}_{\bar{l}} = \mathbf{r}_{\bar{l}} + t_{\bar{l}}(\bar{\mathbf{x}}^{(v_1)} - \mathbf{x}_{\bar{l}}) + (1 - t_{\bar{l}})(\bar{\mathbf{x}}^{(v_1)} - \bar{\mathbf{x}}_{\bar{l}}) + \mathbf{y}'_{\bar{l}}(s_{\bar{l}}) \; . \tag{3.123}$$

Since $\mathbf{y}'_{\bar{l}}(s_{\bar{l}})$ depends only on the variables $\mathbf{x}_{v_2}$ and $(\bar{\mathbf{x}}^{(v_1)} - \mathbf{x}_{\bar{l}})$ and $(\bar{\mathbf{x}}^{(v_1)} - \bar{\mathbf{x}}_{\bar{l}})$ both depend only on $\{\mathbf{r}_l, l \in \tilde{T}_{v_1}\}$, we get

$$\prod_{i=1}^{2} \Big[d\bar{\mathbf{x}}^{(v_i)} \prod_{l \in \tilde{T}_{v_i}} d\mathbf{r}_l \Big] = d\mathbf{r}_{\bar{l}} d\bar{\mathbf{x}}^{(v_2)} \prod_{i=1}^{2} \prod_{l \in \tilde{T}_{v_i}} d\mathbf{r}_l \; . \tag{3.124}$$

By iterating this procedure, one gets (3.119).

3.16 As we have discussed in §2.13, it is not necessary to perform the scale decomposition of the Grassmanian integration up to the last scale $h_{L,\beta}$, but we can stop it to the scale h^*, defined in (2.116). Hence, we redefine $\tilde{E}_{h^*}$, so that

$$e^{-L\beta \tilde{E}_{h^*}} = \int P_{Z_{h^*-1}, \sigma_{h^*-1}, C_{h^*}}(d\psi^{(\le h^*)})\, e^{-\hat{\mathcal{V}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\le h^*)})} \; , \tag{3.125}$$

implying that

$$E_{L,\beta} = \sum_{h=h^*}^{1} [\tilde{E}_h + t_h] \; . \tag{3.126}$$

Thanks to Lemma 2.12, we can proceed as in the proof of Theorem 3.12 to prove the following Theorem.

3.17 THEOREM. *There exists a constant $\bar{\varepsilon}$ such that, if $\varepsilon_{h^*} \le \bar{\varepsilon}$ and, for $h = h^*$, (2.98) holds and the bounds (3.88) are satisfied, then*

$$\sum_{\tau \in \mathcal{T}_{h^*-1,n}} |\tilde{E}_{h^*}(\tau)| \le \gamma^{2h^*} (C\varepsilon_{h^*})^n \; . \tag{3.127}$$

3.18 Theorems 3.12 and 3.17, together with (3.126) and (2.118), imply that the expansion defining $E_{L,\beta}$ is convergent, uniformly in L, β. With some more work (essentially trivial, but cumbersome to describe) one can also prove that $\lim_{L,\beta\to\infty} E_{L,\beta}$ does exist.

4. The flow of the running coupling constants

4.1 The convergence of the expansion for the effective potential is proved by theorems 3.12, 3.17 under the hypothesis that, uniformly in $h \ge h^*$, the running coupling constants are small enough and the bounds (2.98) and (3.88) are satisfied. In this section we prove that, if $|\lambda|$ is small enough and ν is properly chosen, the above conditions are indeed verified.

Let us consider first the bounds in (2.98). They immediately follow from (3.91) and (3.92), by a simple inductive argument, if the bounds (3.88) are verified and

$$\varepsilon_h \le \bar{\varepsilon}_0 \le \bar{\varepsilon} \;, \quad \text{for } h > h^* \;, \tag{4.1}$$

with $\bar{\varepsilon}_0$ small enough.

Let us now consider the bounds (3.88). By (2.83), (2.84), the first of (2.89) and the third of (2.98), we get

$$\frac{Z_{h-1}}{Z_h} = 1 + z_h \;, \tag{4.2}$$

$$\frac{\sigma_{h-1}}{\sigma_h} = 1 + \frac{s_h/\sigma_h - z_h}{1 + z_h} \;. \tag{4.3}$$

By explicit calculation of the lower order non zero terms contributing to z_h and s_h/σ_h, one can prove that

$$\begin{aligned} z_h &= b_1 \lambda_h^2 + O(\varepsilon_h^3) \;, \quad b_1 > 0 \;, \\ s_h/\sigma_h &= -b_2 \lambda_h + O(\varepsilon_h^2) \;, \quad b_2 > 0 \;, \end{aligned} \tag{4.4}$$

which imply (3.88), if $\bar{\varepsilon}_0$ is small enough, with a suitable constant c_1 depending on the constant c_0 appearing in Theorem 3.12, since the value of c_0 is independent of c_1.

The equation (4.2) and the definitions (2.109) allow to get the following representation of the Beta function in terms of the tree expansions (3.71):

$$\lambda_h = \lambda_{h+1} + \left(\frac{1}{1+z_h}\right)^2 \left[-\lambda_{h+1}(z_h^2 + 2z_h) + \sum_{n=2}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} l_h(\tau) \right] \;, \tag{4.5}$$

$$\delta_h = \delta_{h+1} + \frac{1}{1+z_h} \left[-\delta_{h+1} z_h + c_0^\delta \lambda_1 \delta_{h,0} + \sum_{n=2}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \Big(a_h(\tau) - z_h(\tau) \Big) \right] \;, \tag{4.6}$$

$$\nu_h = \gamma \nu_{h+1} + \frac{1}{1+z_h} \left[-\gamma \nu_{h+1} z_h + c_h^\nu \gamma^h \lambda_{h+1} + \sum_{n=2}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} n_h(\tau) \right] \;, \tag{4.7}$$

where we have extracted the terms of first order in the running couplings and we have extended to $h = +1$ the definition of λ_h and δ_h, so that, see (2.81),

$$\lambda_1 = 4\lambda \sin^2(p_F + \pi/L) \;, \quad \delta_1 = -v_0 \delta^* \;. \tag{4.8}$$

Note that the first order term proportional to λ_{h+1} in the equation for ν_h is of size γ^h, while the similar term in the equation for δ_h is equal to zero, if $h < 0$; moreover the constants c_0^ν and c_h^λ are bounded uniformly in L, β.

Hence, if we put $\vec{a}_h = (\delta_h, \lambda_h)$, the Beta function can be written, if condition (4.1) is satisfied, with $\bar{\varepsilon}_0$ small enough, in the form

$$\lambda_{h-1} = \lambda_h + \beta_h^\lambda(\vec{a}_h, \nu_h; \ldots; \vec{a}_1, \nu_1; u, \delta^*) \ , \tag{4.9}$$

$$\delta_{h-1} = \delta_h + \beta_h^\delta(\vec{a}_h, \nu_h; \ldots; \vec{a}_1, \nu_1; u, \delta^*) \ , \tag{4.10}$$

$$\nu_{h-1} = \gamma\nu_h + \beta_h^\nu(\vec{a}_h, \nu_h; \ldots; \vec{a}_1, \nu_1; u, \delta^*) \ , \tag{4.11}$$

where β_h^λ, β_h^δ and β_h^ν are functions of $\vec{a}_h, \nu_h, \ldots, \vec{a}_1, \nu_1, u$, which can be easily bounded, by using Theorem 3.12, if the condition (4.1) is verified. Note that these functions depend on $\vec{a}_h, \nu_h, \ldots, \vec{a}_1, \nu_1, u$, directly trough the endpoints of the trees, indirectly trough z_h and the quantities $Z_{h'}/Z_{h'-1}$ and $\sigma_{h'-1}(\mathbf{k}')$, $h < h' \le 0$, appearing in the tree expansions.

Let us define

$$\mu_h = \sup_{k\ge h} \max\{|\lambda_k|, |\delta_k|\} \ , \quad \bar{\lambda}_h = \sup_{k\ge h} |\lambda_k| \ . \tag{4.12}$$

We want to prove the following Lemma.

4.2 LEMMA. *Suppose that u satisfies the condition (2.117) and let us consider the equation (4.11) for fixed values of $\vec{a}_h$, Z_{h-1} and $\sigma_{h-1}(\mathbf{k}')$, $\tilde{h} \le h \le 1$, satisfying the conditions*

$$\mu_h \le \bar{\varepsilon}_1 \le \bar{\varepsilon}_0 \ , \tag{4.13}$$

$$a_0\gamma^{h-1} \ge 4|\sigma_h| \ , \tag{4.14}$$

$$\gamma^{-c_0\mu_h} \le \frac{\sigma_{h-1}}{\sigma_h} \le \gamma^{+c_0\mu_h} \ , \tag{4.15}$$

$$\gamma^{-c_0\mu_h^2} \le \frac{Z_{h-1}}{Z_h} \le \gamma^{+c_0\mu_h^2} \ , \tag{4.16}$$

for some constant c_0.

Then, if $\bar{\varepsilon}_0$ is small enough, there exist some constants $\bar{\varepsilon}_1$, η, γ', c_1, B, and a family of intervals $I^{(\bar{h})}$, $\tilde{h} \le \bar{h} \le 0$, such that $\bar{\varepsilon}_1 \le \bar{\varepsilon}_0$, $0 < \eta < 1$, $1 < \gamma' < \gamma$, $I^{(\bar{h})} \subset I^{(\bar{h}+1)}$, $|I^{(\bar{h})}| \le c_1\bar{\varepsilon}_1(\gamma')^{\bar{h}}$ and, if $\nu = \nu_1 \in I^{(\bar{h})}$,

$$|\nu_h| \le B\bar{\varepsilon}_1 \left[\gamma^{-\frac{1}{2}(h-\bar{h})} + \gamma^{\eta h}\right] \le \bar{\varepsilon}_0 \ , \quad \bar{h} \le h \le 1 \ . \tag{4.17}$$

4.3 *Proof.* Let us consider (4.11), for fixed values of $\vec{a}_h$, Z_h/Z_{h-1} (hence of z_h) and $\sigma_{h-1}(\mathbf{k}')$, $\tilde{h} \le h \le 1$, satisfying (4.13)-(4.16).

Note that, if $|\nu_h| \le \bar{\varepsilon}_0$ for $\bar{h} \le h \le 1$ and $\bar{\varepsilon}_0$ is small enough, the r.h.s. of (4.11) is well defined for $h = \bar{h}$ and we can write, by using (4.7),

$$\nu_{\bar{h}-1} = \gamma\nu_{\bar{h}} + b_{\bar{h}} + r_{\bar{h}} \ , \tag{4.18}$$

where $b_{\bar{h}} = c_{\bar{h}-1}^\nu \gamma^{\bar{h}-1}\lambda_{\bar{h}}$ and $r_{\bar{h}}$ collects all terms of second or higher order in $\bar{\varepsilon}_0$.

Note also that, in the tree expansion of $n_h(\tau)$, the dependence on $\nu_h, \ldots, \nu_1$ appears only in the endpoints of the trees and there is no contribution from the trees with $n \geq 2$ endpoints, which are only of type ν or δ, because of the support properties of the single scale propagators. It follows, by using (3.91) and (4.14)-(4.16), that

$$|r_{\bar h}| \leq c_2 \mu_{\bar h} \bar\varepsilon_0 \; . \tag{4.19}$$

Let us now fix a positive constant c, consider the intervals

$$J^{(h)} = [-\frac{b_h}{\gamma-1} - c\bar\varepsilon_1, -\frac{b_h}{\gamma-1} + c\bar\varepsilon_1] \; . \tag{4.20}$$

and suppose that there is an interval $I^{(\bar h)}$ such that, if ν_1 spans $I^{(\bar h)}$, then $\nu_{\bar h}$ spans the interval $J^{(\bar h+1)}$ and $|\nu_h| \leq \bar\varepsilon_0$ for $\bar h \leq h \leq 1$. Let us call $\tilde J^{(\bar h)}$ the interval spanned by $\nu_{\bar h-1}$ when ν_1 spans $I^{(\bar h)}$. Equation (4.18) can be written in the form

$$\nu_{\bar h-1} + \frac{b_{\bar h}}{\gamma-1} = \gamma\Big(\nu_{\bar h} + \frac{b_{\bar h}}{\gamma-1}\Big) + r_{\bar h} \; . \tag{4.21}$$

Hence, by using also the definition of b_h and (4.19), we see that

$$\begin{aligned} &\min_{\nu_1 \in I^{(\bar h)}} \left[\nu_{\bar h-1} + \frac{b_{\bar h}}{\gamma-1}\right] = \\ &= \gamma \min_{\nu_{\bar h} \in J^{(\bar h+1)}} \left[\nu_{\bar h} + \frac{b_{\bar h+1}}{\gamma-1}\right] + \min_{\nu_1 \in I^{(\bar h)}} \left[r_{\bar h} + \frac{\gamma}{\gamma-1}(b_{\bar h} - b_{\bar h+1})\right] \leq \\ &\leq -\gamma c\bar\varepsilon_1 + c_2\bar\varepsilon_1\bar\varepsilon_0 + c_3\gamma^{\bar h}\bar\varepsilon_1 \; , \end{aligned} \tag{4.22}$$

for some constant c_3. In a similar way we can show that

$$\max_{\nu_1 \in I^{(\bar h)}} \left[\nu_{\bar h-1} + \frac{b_{\bar h}}{\gamma-1}\right] \geq \gamma c\bar\varepsilon_1 - c_2\bar\varepsilon_1\bar\varepsilon_0 - c_3\gamma^{\bar h}\bar\varepsilon_1 \; . \tag{4.23}$$

It follows that, if c is large enough and $\bar\varepsilon_0$ is small enough, $J^{(\bar h)}$ is strictly contained in $\tilde J^{(\bar h)}$. On the other hand, it is obvious that there is a one to one correspondence between ν_1 and the sequence ν_h, $\bar h - 1 \leq h \leq 1$. Hence there is an interval $I^{(\bar h-1)} \subset I^{(\bar h)}$, such that, if ν_1 spans $I^{(\bar h-1)}$, then $\nu_{\bar h-1}$ spans the interval $J^{(\bar h)}$ and, if $\bar\varepsilon_1$ is small enough, $|\nu_h| \leq \bar\varepsilon_0$ for $\bar h - 1 \leq h \leq 1$.

The previous calculations also imply that the inductive hypothesis is verified for $\bar h = 0$, so that we have proved that there exists a decreasing family of intervals $I^{(\bar h)}$, $\tilde h \leq \bar h \leq 0$, such that, if $\nu = \nu_1 \in I^{(\bar h)}$, then the sequence ν_h is well defined for $h \geq \bar h$ and satisfies the bound $|\nu_h| \leq \bar\varepsilon_0$.

The bound on the size of $I^{(\bar h)}$ easily follows (4.18) and (4.19). Let us denote by ν_h and ν'_h, $\bar h \leq h \leq 1$, the sequences corresponding to $\nu_1, \nu'_1 \in I^{(\bar h)}$. We have

$$\nu_{h-1} - \nu'_{h-1} = \gamma(\nu_h - \nu'_h) + r_h - r'_h \; , \tag{4.24}$$

where r'_h is a shorthand for the value taken from r_h in correspondence of the sequence ν'_h. Let us now observe that $r_h - r'_h$ is equal to $\gamma z_{h-1}(1+z_{h-1})^{-1}(\nu'_h - \nu_h)$ plus a sum of terms, associated with trees, containing at least one endpoint of type ν, with a difference $\nu_k - \nu'_k$, $k \ge h$, in place of the corresponding running coupling, and one endpoint of type λ. Then, if $|\nu_k - \nu'_k| \le |\nu_h - \nu'_h|$, $k \ge h$, we have

$$|\nu_h - \nu'_h| \le \frac{|\nu_{h-1} - \nu'_{h-1}|}{\gamma} + C\bar\varepsilon_1 |\nu_h - \nu'_h| \; . \tag{4.25}$$

On the other hand, if $h = 1$, this bound implies that $|\nu_1 - \nu'_1| \le |\nu_0 - \nu'_0|$, if $\bar\varepsilon_1$ is small enough; hence it allows to show inductively that, given any γ', such that $1 < \gamma' < \gamma$, if $\bar\varepsilon_1$ is small enough, then

$$|\nu_1 - \nu'_1| \le \gamma'^{(\bar h - 1)} |\nu_{\bar h} - \nu'_{\bar h}| \; . \tag{4.26}$$

Since, by definition, if ν_1 spans $I^{(\bar h)}$, then $\nu_{\bar h}$ spans the interval $J^{(\bar h+1)}$, of size $2c\bar\varepsilon_1$, the size of $I^{(\bar h)}$ is bounded by $2c\bar\varepsilon_1\gamma'^{(\bar h-1)}$.

In order to complete the proof of Lemma 4.2, we have still to prove the bound (4.17). Note that, if we iterate (4.11), we can write, if $\bar h \le h \le 0$ and $\nu_1 \in I^{(\bar h)}$,

$$\nu_h = \gamma^{-h+1} \left[\nu_1 + \sum_{k=h+1}^{1} \gamma^{k-2} \beta^\nu_k(\nu_k, \ldots, \nu_1) \right] \; , \tag{4.27}$$

where now the functions β^k_ν are thought as functions of $\nu_k, \ldots, \nu_1$ only.

If we put $h = \bar h$ in (4.27), we get the following identity:

$$\nu_1 = - \sum_{k=\bar h+1}^{1} \gamma^{k-2} \beta^\nu_k(\nu_k, \ldots, \nu_1) + \gamma^{\bar h - 1} \nu_{\bar h} \; . \tag{4.28}$$

(4.27) and (4.28) are equivalent to

$$\nu_h = -\gamma^{-h} \sum_{k=\bar h+1}^{h} \gamma^{k-1} \beta^\nu_k(\nu_k, \ldots, \nu_1) + \gamma^{-(h-\bar h)} \nu_{\bar h} \; , \qquad \bar h < h \le 1 \; . \tag{4.29}$$

The discussion following (4.18) implies that

$$|\beta^\nu_k(\nu_k, \ldots, \nu_1)| \le C\mu_k \; , \tag{4.30}$$

if $\bar\varepsilon_0$ is small enough. However this bound it is not sufficient and we have to analyze in more detail the structure of the functions β^ν_h, by looking in particular to the trees in the expansion of $n_h(\tau)$, which have no endpoint of type ν. Let us suppose that, given a tree with this property, we decompose the propagators by using (2.99); we get a family of C^n different contributions, which can be bounded as before, by using an argument similar to that used in §3.13. However, the terms containing only the propagators $g^{(h')}_{L,\omega}$ cancel out, for simple parity properties. On the other hand, the terms containing at least one propagator $r^{(h_v)}_2$ or two propagators $g^{(h_v)}_{\omega,-\omega}$ (the number of such propagators has to be even) can be bounded by $(C\varepsilon_h)^n(|\sigma_h|/\gamma^h)^2$, by using (2.101) and (3.106). Analogously the terms with at least one propagator $r^{(h_v)}_1$ can be bounded by $(C\varepsilon_h)^n\gamma^{\eta h}$, with some positive $\eta < 1$. In fact, for these terms, by using (2.101), the bound can be improved by a factor $\gamma^{h_v} \le \gamma^{\eta h}\gamma^{\eta(h_v-h)}$, for any positive $\eta \le 1$, and the bad factor $\gamma^{\eta(h_v-h)}$ can be controlled by the sum over the scales, if η is small enough, thanks to (3.111). Finally, the parity properties of the propagators imply that the only term linear in the running couplings, which contributes to ν_h, is of order γ^h. Hence, we can write

$$\beta_h^\nu = \mu_h \sum_{k=h}^{1} \nu_k \tilde\beta_{h,k}^\nu \gamma^{-2\eta(k-h)} + \mu_h \varepsilon_h \left(\frac{|\sigma_h|}{\gamma^h}\right)^2 \hat\beta_h^\nu + \gamma^{\eta h}\mu_h R_h^\nu \ , \tag{4.31}$$

where $|R_h^\nu|, |\hat\beta_h^\nu|, |\tilde\beta_{h,k}^\nu| \le C$.

The factor $\gamma^{-2\eta(k-h)}$ in the r.h.s. of (4.31) follows from the simple remark that the bound over all the trees contributing to ν_h, which have at least one endpoint of fixed scale $k > h$, can be improved by a factor $\gamma^{-\eta'(k-h)}$, with η' positive but small enough. It is sufficient to use again (3.111), which allows to extract such factor from the r.h.s. before performing the sum over the scale indices, and to choose $\eta' = 2\eta$, which is possible if η is small enough.

Let us now observe that the sequence ν_h, $\bar h < h \le 1$, satisfying (4.29) can be obtained as the limit as $n \to \infty$ of the sequence $\{\nu_h^{(n)}\}$, $\bar h < h \le 1$, $n \ge 0$, parameterized by $\nu_{\bar h} \in J^{(\bar h+1)}$ and defined recursively in the following way:

$$\begin{aligned} &\nu_h^{(0)} = 0 \ , \\ &\nu_h^{(n)} = -\gamma^{-h} \sum_{k=\bar h+1}^{h} \gamma^{k-1}\beta_k^\nu(\nu_k^{(n-1)}, \ldots, \nu_1^{(n-1)}) + \gamma^{-(h-\bar h)}\nu_{\bar h} \ , \qquad n \ge 1 \ . \end{aligned} \tag{4.32}$$

In fact, it is easy to show inductively, by using (4.30), that, if $\bar\varepsilon_1$ is small enough, $|\nu_h^{(n)}| \le C\bar\varepsilon_1 \le \bar\varepsilon_0$, so that (4.32) is meaningful, and

$$\max_{h^* < h \le 1} |\nu_h^{(n)} - \nu_h^{(n-1)}| \le (C\bar\varepsilon_1)^n \ . \tag{4.33}$$

In fact this is true for $n = 1$ by (4.30) and the fact that $\nu_h^{(0)} = 0$; for $n > 1$ it follows trivially by the fact that $\beta_k^\nu(\nu_k^{(n-1)}, \ldots, \nu_1^{(n-1)}) - \beta_k^\nu(\nu_k^{(n-2)}, \ldots, \nu_1^{(n-2)})$ can be written as a sum of terms in which there are at least one endpoint of type ν, with a difference $\nu_{h'}^{n-1} - \nu_{h'}^{n-2}$, $h' \ge k$, in place of the corresponding running coupling, and one endpoint of type λ. Then $\nu_h^{(n)}$ converges as $n \to \infty$, for $\bar h < h \le 1$, to a limit ν_h, satisfying (4.29) and the bound $|\nu_h| \le \bar\varepsilon_0$, if $\bar\varepsilon_1$ is small enough. Since the solution of the equations (4.29) is unique, it must coincide with the previous one.

Conditions (4.14) and (4.15) imply that

$$\frac{|\sigma_h|}{\gamma^h} = \frac{|\sigma_{\bar h}|}{\gamma^{\bar h}}\frac{|\sigma_h|}{|\sigma_{\bar h}|}\gamma^{\bar h-h} \le C\gamma^{-(h-\bar h)(1-c_0\bar\varepsilon_1)} \ . \tag{4.34}$$

Hence, if $\bar\varepsilon_1$ is small enough, by (4.31),

$$|\beta_k^\nu| \le C\bar\varepsilon_1 \left[\sum_{m=k}^{1} |\nu_m|\gamma^{-2\eta(m-k)} + \bar\varepsilon_0\gamma^{-\frac12(h-\bar h)} + \gamma^{\eta k}\right] \ . \tag{4.35}$$

Hence, it is easy to show that there exists a constant $\bar{c}$ such that

$$|\nu_h^{(n)}| \le \bar{c}\bar{\varepsilon}_1 \Big[\sum_{m=\bar{h}+1}^{h} |\nu_m^{(n-1)}|\gamma^{-(h-m)} + \sum_{m=h+1}^{1} |\nu_m^{(n-1)}|\gamma^{-2\eta(m-h)} + + \bar{\varepsilon}_0 \gamma^{-\frac{1}{2}(h-\bar{h})} + \gamma^{\eta h} + \gamma^{-(h-\bar{h})} \Big] . \tag{4.36}$$

Let us now suppose that, for some constant c_{n-1},

$$|\nu_m^{(n-1)}| \le c_{n-1}\bar{\varepsilon}_1 \left[\gamma^{-\frac{1}{2}(h-\bar{h})} + \gamma^{\eta h} \right] \le \bar{\varepsilon}_0 , \tag{4.37}$$

which is true for $n = 1$, since $\nu_m^{(0)} = 0$, if $\bar{\varepsilon}_1$ is small enough. Then, it is easy to verify that the same bound is verified by $\nu_m^{(n)}$, if c_{n-1} is substituted with

$$c_n = \bar{c}(1 + c_4 c_{n-1}\bar{\varepsilon}_1) , \tag{4.38}$$

where c_4 is a suitable constant. Hence, we can easily prove the bound (4.17) for $\nu_h = \lim_{n\to\infty} \nu_h^{(n)}$, for $\bar{\varepsilon}_1$ small enough.

4.4 Let us now consider the equations (4.9) and (4.10), for a fixed, arbitrary, sequence ν_h, $\bar{h} \le h \le 1$, satisfying the bound (4.17). In order to study the corresponding flow, we compare our model with an approximate model, obtained by putting $u = \nu = 0$ and by substituting all the propagators with the Luttinger propagator $g_{L,\omega}^{(k)}(\mathbf{x};\mathbf{y})$, see (2.100). It is easy to see that, in this model, $\sigma_h(\mathbf{k}') = \nu_h = 0$, for any $h \le 1$, so that the flow of the running couplings is described only by the equations

$$\begin{aligned} \lambda_{h-1}^{(L)} &= \lambda_h^{(L)} + \beta_h^{\lambda,L}(\vec{a}_h^{(L)}, \ldots, \vec{a}_1^{(L)}; \delta^*) , \\ \delta_{h-1}^{(L)} &= \delta_h^{(L)} + \beta_h^{\delta,L}(\vec{a}_h^{(L)}, \ldots, \vec{a}_1^{(L)}; \delta^*) , \end{aligned} \tag{4.39}$$

where the functions $\beta_h^{\lambda,L}$ and $\beta_h^{\delta,L}$ can be represented as in (4.5) and (4.6), by suitably changing the definition of the trees and of the related quantities $l_h(\tau)$, $a_h(\tau)$, $z_h(\tau)$, which we shall distinguish by a superscript L. Of course Theorem 3.12 applies also to the new model, which differs from the well known Luttinger model only because the space variables are restricted to the unit lattice, instead of the real axis.

Let us define, for $\alpha = \lambda, \delta$,

$$r_h^\alpha(\vec{a}_h, \nu_h; \ldots; \vec{a}_1, \nu_1; u, \delta^*) = \beta_h^\alpha(\vec{a}_h, \nu_h; \ldots; \vec{a}_1, \nu_1; u, \delta^*) - \beta_h^{\alpha,L}(\vec{a}_h, \ldots, \vec{a}_1; \delta^*) . \tag{4.40}$$

Note that, in the r.h.s. of (4.40), the function $\beta_h^{\alpha,L}$ is calculated at the values of $\vec{a}_{h'}$, $h \le h' \le 1$, which are the solutions of the equations (4.9) and (4.10); these values are of course different from those satisfying the equations (4.39). We shall prove the following Lemma.

4.5 LEMMA. *Suppose that u satisfies the condition (2.117), the sequence ν_h, $\bar{h} \le h \le 1$, satisfies the bound (4.17) and δ^* satisfies the condition*

$$| -\delta^* v_0 + c_0^\delta \lambda_1| \le |\lambda_1| , \tag{4.41}$$

c_0^δ being the constant appearing in the r.h.s. of (4.6),

Then, if η is defined as in Lemma 4.2 and $\mu_h \le \bar\varepsilon_0$ (hence (4.1) is satisfied) and $\bar\varepsilon_0$ is small enough,

$$|r_h^\lambda| + |r_h^\delta| \le C\bar\lambda_h^2[\gamma^{-\frac{1}{2}(h-\bar h)} + \gamma^{\eta h}]\ , \quad \bar h \le h \le 0\ ; \tag{4.42}$$

$$|r_1^\lambda| \le C\lambda_1^2\ , \quad |r_1^\delta| \le C|\lambda_1|\ . \tag{4.43}$$

4.6 *Sketch of the proof.* Note that all trees with $n \ge 2$ endpoints, contributing to the expansions in the r.h.s. of the equations (4.5)-(4.7), may have an endpoint of type ν or δ only if there are at least two endpoints of type λ; this claim follows from the definition of localization and the support properties of the single scale propagators. The bound (4.43) is an easy consequence of this remark, equations (4.5), (4.6), condition (4.41) and Theorem 3.12.

We then consider $h \le 0$ and we define

$$\Delta z_h = z_h - z_h^L = \frac{Z_{h-1}}{Z_h} - \frac{Z_{h-1}^L}{Z_h^L}\ . \tag{4.44}$$

Remember that all quantities in (4.44) have to be considered as functions of the same running couplings. Suppose now that

$$|\Delta z_k| \le c_0\mu_k^2[\gamma^{-\frac{1}{2}(k-\bar h)} + \gamma^{\eta k}]\ , \quad h < k \le 0\ . \tag{4.45}$$

We want to prove that this bound is verified also for $k = h$, together with (4.42). Since the proof will also imply that (4.45) is verified for $k = 0$, we shall achieve the proof of Lemma 4.5.

By using the decomposition (2.99) of the propagator, it is easy to see that

$$r_h^\alpha = \sum_{i=1}^3 r_h^{\alpha,i}\ , \tag{4.46}$$

where the quantities $r_h^{\alpha,i}$ are defined in the following way.

1) $r_h^{\alpha,1}$ is obtained from β_h^α by restricting the sum over the trees in the r.h.s. of (4.5) and (4.6) to those containing at least one endpoint of type ν.

2) $r_h^{\alpha,2}$ is obtained from β_h^α by restricting the sum over the trees to those containing no endpoint of type ν, and by substituting, in each term contributing to the expansions appearing in the r.h.s. of (4.5) and (4.6), at least one propagator with a propagator of type $r_1^{(h')}$ or $r_2^{(h')}$ (see (2.99)), $h \le h' \le 1$. Note that z_h and all the ratios Z_k/Z_{k-1}, $k > h$, appearing in the expansions are left unchanged.

3) $r_h^{\alpha,3}$ is obtained by subtracting $\beta_h^{\alpha,L}$ from the expression we get, if we substitute all propagators appearing in the expansions contributing to β_h^α with Luttinger propagators and if we eliminate all trees containing endpoints of type ν.

By using (4.17), (2.101) and (4.34), it is easy to prove that $r_h^{\alpha,1}$ and $r_h^{\alpha,2}$ satisfy a bound like (4.42). The main point is the remark, already used in the proof of Lemma 4.2, that there is an improvement of order $\gamma^{-\eta'(k-h)}$, $0<\eta'<1$, in the bound of the sum over the trees with a vertex of fixed scale $k>h$. One has also to use a trick similar to that of §3.13, in order to keep the bound (3.94) on the determinants, after the decomposition of the propagators. Finally, one has to use the remark made at the beginning of this section in order to justify the presence of $\bar\lambda_h^2$, instead of ε_h^2, in the r.h.s. of (4.42).

In order to prove that a bound like (4.42) is satisfied also by $r_h^{\alpha,3}$, one must first prove that the bound in (4.45) is valid for $k=h$, with the same constant c_0. This result can be achieved by decomposing Δz_h in a way similar to that used for r_h^α; let us call $\Delta_i z_h$ the three corresponding terms. By proceeding as before, we can show that

$$|\Delta_1 z_h| + |\Delta_2 z_h| \le C\bar\lambda_h^2[\gamma^{-\frac{1}{2}(h-\bar h)} + \gamma^{\eta h}] \; . \tag{4.47}$$

Let us now consider $\Delta_3 z_h$; we can write $\Delta_3 z_h = \sum_{n=2}^\infty \sum_{\tau\in\mathcal{T}_{h,n}} \Delta_3 z_h(\tau)$, with $\Delta_3 z_h(\tau) = 0$, if τ contains endpoints of type ν, and $\Delta_3 z_h(\tau) = \sum_{v\in\tau} \bar z_h(\tau,v)$, where $\bar z_h(\tau,v)=0$, if v is an endpoint, otherwise $\bar z_h(\tau,v)$ is obtained from $z_h(\tau)$ by selecting a family V vertices, which are not endpoints, containing v, and by substituting, for each $v'\in V$, the factor $Z_{h_{v'}}/Z_{h_{v'}-1}$ with $Z_{h_{v'}}/Z_{h_{v'}-1} - Z^L_{h_{v'}}/Z^L_{h_{v'}-1}$. By using (4.2), we have

$$|Z_{h_v}/Z_{h_v-1} - Z^L_{h_v}/Z^L_{h_v-1}| \le C|\Delta z_{h_v}|^2 \; ; \tag{4.48}$$

hence it is easy to show that $\Delta_3 z_h$ can be bounded as in the proof of Theorem 3.12, by adding a sum over the non trivial vertices (whose number is proportional to n) and, for each term of this sum, a factor

$$Cc_0\bar\lambda_h^2[\gamma^{-\frac{1}{2}(h-\bar h)} + \gamma^{\eta h}]\gamma^{\eta(h_{\tilde v}-h)}(h_{\tilde v} - h_{\tilde v'}) \; , \tag{4.49}$$

where $\tilde v$ is the non trivial vertex corresponding to the selected term and $\tilde v'$ is the non trivial vertex immediately preceding $\tilde v$ or the root. Hence, we get

$$|\Delta_3 z_h| \le Cc_0\varepsilon_h^2\bar\lambda_h^2[\gamma^{-\frac{1}{2}(h-\bar h)} + \gamma^{\eta h}] \; , \tag{4.50}$$

implying, together with (4.47), the bound (4.45) for $k=h$, if $\bar\varepsilon_0$ is small enough and c_0 is large enough.

Given this result, it is possible to prove in the same manner that $r_h^{\alpha,3}$ satisfies a bound like (4.42). This completes the proof of Lemma 4.5.

4.7 Lemma 4.5 allows to reduce the study of running couplings flow to the same problem for the flow (4.39). This one, in its turn, can be reduced to the study of the beta function for the *Luttinger model*, see [BGM]. This model is exactly solvable, see [ML], and the Schwinger functions can be exactly computed, see [BGM]. It is then possible to show, see [BGM], [BGPS], [GS], [BM1], that there exists $\bar\varepsilon>0$, such that, if $|\vec a_h|\le\bar\varepsilon$,

$$|\bar\beta_h^{\alpha,L}(\vec a_h,\ldots,\vec a_h)| \le C\mu_h^2\gamma^{\eta' h} \; , \tag{4.51}$$

where $\bar{\beta}_h^{\alpha,L}(\vec{a}_h,...,\vec{a}_1)$, $\alpha = \lambda, \delta$, denote the analogous of the functions $\beta_h^{\alpha,L}(\vec{a}_h,...,\vec{a}_1)$ for this model and $0 < \eta' < 1$. Note that in the l.h.s. of (4.51) all running couplings $\vec{a}_k$, $h \le k \le 1$, are put equal to $\vec{a}_h$ and that $\vec{a}_h$ can take any value such that $|\vec{a}_h| \le \bar{\varepsilon}$, since $\vec{a}_h$ is a continuous function of $\vec{a}_0$ and $\vec{a}_h = \vec{a}_0 + O(\mu_h^2)$, see [BGPS].

We argue now that a bound like (4.51) is valid also for the functions $\beta_h^{\alpha,L}$. In fact the Luttinger model differs from our approximate model only because the space coordinates take values on the real axis, instead of the unit lattice. This implies, in particular, that we have to introduce a scale decomposition with a scale index h going up to $+\infty$. However, as it has been shown in [GS], the effective potential on scale $h = 0$ is well defined; on the other hand, it differs from the effective potential on scale $h = 0$ of our approximate model only for the non local part of the interaction. In terms of the representation (2.61) of $\mathcal{V}^{(0)}(\psi^{(\le 0)})$, this difference is the same we would get, by changing the kernels of the non local terms (without qualitatively affecting their bounds) and the delta function, which in the Luttinger model is defined as $L\beta\delta_{k,0}\delta_{k_0,0}$, instead of as in (2.62).

Note that the difference of the two delta functions has no effect on the local part of $\mathcal{V}^{(0)}(\psi^{(\le 0)})$, because of the support properties of $\hat{\psi}^{(\le 0)}$, but it slightly affects the non local terms on any scale, hence it affects the beta function; however, it is easy to show that this is a negligible phenomenon. Let us consider in fact a particular tree τ and a vertex $v \in \tau$ of scale h_v with $2n$ external fields of space momenta k'_r, $r = 1, \ldots, 2n$; the conservation of momentum implies that $\sum_{r=1}^{2n} \sigma_r k'_r = 2\pi m$, with $m = 0$ in the continuous model, but m arbitrary integer for the lattice model. On the other hand, k'_r is of order γ^{h_v} for any r, hence m can be different from 0 only if n is of order γ^{h_v}. Since the number of endpoints following a vertex with $2n$ external fields is greater or equal to $n-1$ and there is a small factor (of order μ_h) associated with each endpoint, we get an improvement, in the bound of the terms with $|m| > 0$, with respect to the others, of a factor $\exp(-C\gamma^{-h_v})$. Hence, by using the usual arguments, it is easy to show that the difference between the two beta functions is of order $\mu_h^2 \gamma^{\eta h}$.

The previous considerations prove the following, very important, Lemma.

4.8 LEMMA. *There are $\bar{\varepsilon}_0$ and $\eta' > 0$, such that, if $|\mu_h| \le \bar{\varepsilon}_0$, $\alpha = \lambda, \delta$ and $h \le 0$,*

$$|\beta_h^{\alpha,L}(\vec{a}_h, \ldots, \vec{a}_h)| \le C\bar{\lambda}_h^2 \gamma^{\eta' h} . \tag{4.52}$$

We are now ready to prove the following main Theorem on the running couplings flow.

4.9 THEOREM. *If $u \ne 0$ satisfies the condition (2.117) and δ^* satisfies the condition (4.41), there exist $\bar{\varepsilon}_3$ and a finite integer $h^* \le 0$, such that, if $|\lambda_1| \le \bar{\varepsilon}_3$ and ν belongs to a suitable interval $I^{(h^*)}$, of size smaller than $c|\lambda_1|\gamma'^{h^*}$ for some constants c and γ', $1 < \gamma' < \gamma$, then the running coupling constants are well defined for $h^* - 1 \le h \le 0$ and h^* satisfies the* *definition (2.116). Moreover, there exist positive constants c_i, $i = 1, \ldots, 5$, such that*

$$|\lambda_h - \lambda_1| \le c_1 |\lambda_1|^{3/2} , \qquad |\delta_h| \le c_1 |\lambda_1| , \tag{4.53}$$

$$\gamma^{\lambda_1 c_2 h} < \frac{\sigma_h}{\sigma_0} < \gamma^{\lambda_1 c_3 h} , \tag{4.54}$$

$$\gamma^{-c_4 \lambda_1^2 h} < Z_h < \gamma^{-c_5 \lambda_1^2 h} , \tag{4.55}$$

$$\max\left\{ h_{L,\beta}, \frac{\log_\gamma\left(\frac{4\gamma a_0^{-1}}{1+\delta^*}|\sigma_0|\right)}{1-\lambda_1 c_2} \right\} \le h^* \le \max\left\{ h_{L,\beta}, \frac{\log_\gamma\left(\frac{4\gamma a_0^{-1}}{1+\delta^*}|\sigma_0|\right) + 1 - \lambda_1 c_3}{1-\lambda_1 c_3} \right\} . \tag{4.56}$$

Finally, it is possible to choose δ^* so that, for a suitable $\eta > 0$,

$$|\delta_h| \le C|\lambda_1|^{3/2} [\gamma^{-\eta(h-h^*)} + \gamma^{\eta h}] . \tag{4.57}$$

4.10 *Proof.* We shall proceed by induction. Equations (4.5), (4.6) and Lemma 4.2 imply that, if λ_1 is small enough, there exists an interval $I^{(0)}$, whose size is of order λ_1, such that, if $\nu \in I^{(0)}$, then the bound (4.17) is satisfied, together with

$$|\lambda_0 - \lambda_1| \le C|\lambda_1|^2 , \quad |\delta_0 - \delta_1| = |\delta_0| \le C|\lambda_1| . \tag{4.58}$$

Let us now suppose that the solution of (4.9)-(4.11) is well defined for $\bar{h} \le h \le 0$ and satisfies the conditions (4.14)-(4.17), for any ν belonging to an interval $I^{(\bar{h})}$, defined as in Lemma 4.2. This implies, in particular, that $h^* \le \bar{h}$, see (4.14) and (2.116). Suppose also that there exists a constant c_0, such that

$$\bar{\lambda}_{\bar{h}} \le 2|\lambda_1| . \tag{4.59}$$

We want to prove that all these conditions are verified also if $\bar{h}$ is substituted with $\bar{h}-1$, if λ_1 is small enough. The induction will be stopped as soon as the condition (4.14) is violated for some $\nu \in I^{(\bar{h})}$. We shall put ν equal to one of these values, so defining h^* as equal to $\bar{h}+1$.

The fact that the condition on ν_1 and the bound (4.17) are verified also if $\bar{h}-1$ takes the place of $\bar{h}$, follows from Lemma 4.2, since the condition (4.13) follows from (4.59), if λ_1 is small enough. There is apparently a problem in using this Lemma, since in its proof we used the hypothesis that the values of $\vec{a}_h$, Z_{h-1} and $\sigma_{h-1}(\mathbf{k}')$, $\bar{h} \le h \le 1$, are independent of ν_1. This is not true for the full flow, but the proof of Lemma 4.5 can be easily extended to cover this case. In fact, the only part of the proof, where we use the fact that $\vec{a}_h$ is constant, is the identity (4.24), which should be corrected by adding to the r.h.s. the difference $b_h - b'_h$. However, since λ_1 is independent of ν_1, it is not hard to prove that $|b_h - b'_h| \le C|\nu_h - \nu'_h|$ and that the bound on $r_h - r'_h$ does not change (qualitatively), if we take into account also the dependence on ν_1 of the various quantities, before considered as constant. Hence, the bound (4.25) is left unchanged.

The conditions (4.15) and (4.16) follow immediately from (4.59) and (4.2)-(4.4). Hence, we still have to show only that (4.59) is verified also if $\bar{h}$ is substituted with $\bar{h}-1$, if λ_1 is small enough.

By using (4.39) and (4.40), we have, if $\alpha = \lambda, \delta$,

$$\alpha_{\bar{h}-1} = \alpha_{\bar{h}} + \beta_{\bar{h}}^{\alpha,L}(\vec{a}_{\bar{h}},\ldots,\vec{a}_{\bar{h}}) + \sum_{k=\bar{h}+1}^{1} D_{\bar{h},k}^{\alpha} + r_{\bar{h}}^{\alpha}(\vec{a}_{\bar{h}},\nu_{\bar{h}};\ldots;\vec{a}_1,\nu_1;u) , \tag{4.60}$$

where

$$D_{h,k}^{\alpha} = \beta_h^{\alpha,L}(\vec{a}_h,\ldots,\vec{a}_h,\vec{a}_k,\vec{a}_{k+1},\ldots,\vec{a}_1) - \beta_h^{\alpha,L}(\vec{a}_h,\ldots,\vec{a}_h,\vec{a}_h,\vec{a}_{k+1},\ldots,\vec{a}_1) . \tag{4.61}$$

On the other hand, it is easy to see that $D_{h,k}^{\alpha}$ admits a tree expansion similar to that of $\beta_h^{\alpha,L}(\vec{a}_h,\ldots,\vec{a}_1)$, with the property that all trees giving a non zero contribution must have an endpoint of scale h, associated with a difference $\lambda_k - \lambda_h$ or $\delta_k - \delta_h$. Hence, if η is the same constant of Lemma 4.2 and Lemma 4.5 and $h \le 0$,

$$|D_{h,k}^{\alpha}| \le C|\bar{\lambda}_h|\gamma^{-\eta(k-h)}|\vec{a}_k - \vec{a}_h| . \tag{4.62}$$

Let us now suppose that $\bar{h} \le h \le 0$ and that there exists a constant c_0, such that

$$|\vec{a}_{k-1} - \vec{a}_k| \le c_0|\lambda_1|^{3/2}[\gamma^{-\frac{1}{2}(k-\bar{h})} + \gamma^{\vartheta k}] , \quad h < k \le 0 . \tag{4.63}$$

where $\vartheta = \min\{\eta/2, \eta'\}$, η' being defined as in Lemma 4.8. (4.63) is certainly verified for $k=0$, thanks to (4.5), (4.6); we want to show that it is verified also if h is substituted with $h-1$, if λ_1 is small enough.

By using (4.60), (4.62), (4.42), (4.52) and (4.63), we get

$$\begin{aligned} |\vec{a}_{h-1} - \vec{a}_h| \le\ & C\bar{\lambda}_h^2\gamma^{\eta' h} + C|\bar{\lambda}_h|^2[\gamma^{-\frac{1}{2}(h-\bar{h})} + \gamma^{\eta h}] + \\ & + Cc_0|\bar{\lambda}_h|^{5/2}\sum_{k=h+1}^{1}\gamma^{-\eta(k-h)}\sum_{h'=h+1}^{k}[\gamma^{-\frac{1}{2}(h'-h^*)} + \gamma^{\vartheta h'}] , \end{aligned} \tag{4.64}$$

which immediately implies (4.63) with $h \to h-1$ and (4.59) with $\bar{h} \to \bar{h}-1$.

The bound (4.64) implies also (4.53), while the bounds (4.54) and (4.55) are an immediate consequence of (4.15), (4.16) and an explicit calculation of the leading terms; (4.56) easily follows from (4.54) and the definition (2.116) of h^*.

All previous results can be obtained uniformly in the value of δ^*, under the condition (4.41). However, by using (4.63) with $\bar{h} = h^*$, it is not hard to prove, by an implicit function theorem argument (we omit the details, which are of the same type of those used many times before), that one can choose δ^* so that

$$|\delta_0| \le C|\lambda_1|^2 , \quad \delta_{h^*/2} = 0 , \tag{4.65}$$

which easily implies (4.57), for a suitable value of η.

5. The Correlation function

5.1 The correlation function $\Omega^3_{L,\beta,\mathbf{x}}$, in terms of fermionic operators, is given by

$$\Omega^3_{L,\beta,\mathbf{x}} =< a^+_{\mathbf{x}} a^-_{\mathbf{x}} a^+_0 a^-_0 >_{L,\beta} - < a^+_{\mathbf{x}} a^-_{\mathbf{x}} >_{L,\beta}< a^+_0 a^-_0 >_{L,\beta}= \frac{\partial^2 \mathcal{S}(\phi)}{\partial\phi(\mathbf{x})\partial\phi(\mathbf{0})}|_{\phi=0} \; , \tag{5.1}$$

where $\phi(\mathbf{x})$ is a bosonic external field, periodic in x and x_0, of period L and β, respectively, and

$$e^{\mathcal{S}(\phi)} = \int P(d\psi^{(\le 1)}) e^{-\mathcal{V}^{(1)}(\psi^{(\le 1)}) + \int d\mathbf{x}\phi(\mathbf{x})\psi^{(\le 1)+}_{\mathbf{x}}\psi^{(\le 1)-}_{\mathbf{x}}} \; . \tag{5.2}$$

Note that, because of the discontinuity at $x_0 = 0$ of the scale 1 free measure propagator $\tilde g^{(1)}_{\omega,\omega}$ in the limit $M \to \infty$ (see §2.3, the product $\psi^{(\le 1)+}_{\mathbf{x}}\psi^{(\le 1)-}_{\mathbf{x}}$ has to be understood as $\psi^{(\le 0)+}_{\mathbf{x}}\psi^{(\le 0)-}_{\mathbf{x}} + \lim_{\varepsilon\to 0^+} \psi^{(1)+}_{(x,x_0+\varepsilon)}\psi^{(\le 1)-}_{(x,x_0)}$. Since this remark is important only in the explicit calculation of some physical quantities, but does not produce any problem in the analysis of this section, we shall in general forget it in the notation.

We shall evaluate the integral in the r.h.s. of (5.2) in a way which is very close to that used for the integration in (2.13). We introduce the scale decomposition described in §2.3 and we perform iteratively the integration of the single scale fields, starting from the field of scale 1. The main difference is of course the presence in the interaction of a new term, that we shall call $\mathcal{B}^{(1)}(\psi^{(\le 1)},\phi)$; in terms of the fields $\psi^{(\le 1)\sigma}_{\mathbf{x},\omega}$, it can be written as

$$\mathcal{B}^{(1)}(\psi^{(\le 1)},\phi) = \sum_{\sigma_1,\sigma_2} \int d\mathbf{x} e^{i\mathbf{p}_F \mathbf{x}(\sigma_1+\sigma_2)} \phi(\mathbf{x}) \psi^{(\le 1)\sigma_1}_{\mathbf{x},\sigma_1} \psi^{(\le 1)\sigma_2}_{\mathbf{x},-\sigma_2} \; . \tag{5.3}$$

After integrating the fields $\psi^{(1)}, \ldots \psi^{(h+1)}$, $0 \ge h \ge h^*$, we find

$$e^{\mathcal{S}(\phi)} = e^{-L\beta E_h + S^{(h+1)}(\phi)} \int P_{Z_h,\sigma_h,C_h}(d\psi^{\le h}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)} \; , \tag{5.4}$$

where $P_{Z_h,\sigma_h,C_h}(d\psi^{(\le h)})$ and $\mathcal{V}^h$ are given by (**I**2.66) and (3.3), respectively, while $S^{(h+1)}$ (ϕ), which denotes the sum over all the terms dependent on ϕ but independent of the ψ field, and $\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)$, which denotes the sum over all the terms containing at least one ϕ field and two ψ fields, can be represented in the form

$$S^{(h+1)}(\phi) = \sum_{m=1}^\infty \int d\mathbf{x}_1 \cdots d\mathbf{x}_m S^{(h+1)}_m(\mathbf{x}_1,\ldots,\mathbf{x}_m) \Big[\prod_{i=1}^m \phi(\mathbf{x}_i)\Big] \tag{5.5}$$

$$\begin{aligned} \mathcal{B}^{(h)}(\psi^{(\le h)},\phi) = \sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{\underline{\sigma},\underline{\omega}} \int d\mathbf{x}_1 \cdots d\mathbf{x}_m d\mathbf{y}_1 \cdots d\mathbf{y}_{2n} \cdot \\ \cdot\; B^{(h)}_{m,2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n}) \Big[\prod_{i=1}^m \phi(\mathbf{x}_i)\Big] \Big[\prod_{i=1}^{2n} \psi^{(\le h)\sigma_i}_{\mathbf{y}_i,\omega_i}\Big] \; . \end{aligned} \tag{5.6}$$

Since the field ϕ is equivalent, from the point of view of dimensional considerations, to two ψ fields, the only terms in the r.h.s. of (5.6) which are not irrelevant are those with $m = 1$ and $n = 1$, which are marginal. However, if $\sum_{i=1}^2 \sigma_i\omega_i \ne 0$, also these terms are indeed irrelevant, since the dimensional bounds are improved by the presence of a non diagonal propagator, as for the analogous terms with no ϕ field and two ψ fields, see §3.14. Hence we extend the definition of the localization operator $\mathcal{L}$, so that its action on $\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)$ in described in the following way, by its action on the kernels $B^{(h)}_{m,2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n})$:

1) if $m=1$, $n=1$ and $\sum_{i=1}^2\sigma_i\omega_i=0$, then

$$\begin{aligned}\mathcal{L}B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{y}_1,\mathbf{y}_2) &= \sigma_1\omega_1\delta(\mathbf{y}_1-\mathbf{x}_1)\delta(\mathbf{y}_2-\mathbf{x}_1)\cdot\\ &\cdot\int d\mathbf{z}_1 d\mathbf{z}_2 c_\beta(2x_0-z_{10}-z_{20})c_L(z_1-z_2)B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{z}_1,\mathbf{z}_2)\;;\end{aligned}\tag{5.7}$$

2) in all the other cases

$$\mathcal{L}B^{(h)}_{m,2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,...\mathbf{x}_m;\mathbf{y}_1,...,\mathbf{y}_{2n})=0\;.\tag{5.8}$$

Let us define, in analogy to definition (3.2), the Fourier transform of $B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{y}_1,\mathbf{y}_2)$ by the equation

$$\begin{aligned}&B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{y}_1,\mathbf{y}_2)=\\ &=\frac{1}{(L\beta)^3}\sum_{\mathbf{p},\mathbf{k}'_1,\mathbf{k}'_2} e^{i\mathbf{p}\mathbf{x}-i\sum_{r=1}^2\sigma_r\mathbf{k}'_r\mathbf{y}_r}\hat B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{p},\mathbf{k}'_1)\delta(\sum_{r=1}^2\sigma_r(\mathbf{k}'_r+\mathbf{p}_F)-\mathbf{p})\;,\end{aligned}\tag{5.9}$$

where $\mathbf{p}=(p,p_0)$ is summed over momenta of the form $(2\pi n/L, 2\pi m/\beta)$, with n,m integers. Hence (5.7) can be written in the form

$$\begin{aligned}\mathcal{L}B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{y}_1,\mathbf{y}_2) &= \sigma_1\omega_1\delta(\mathbf{y}_1-\mathbf{x}_1)\delta(\mathbf{y}_2-\mathbf{x}_1)e^{i\mathbf{p}_F\mathbf{x}(\sigma_1+\sigma_2)}\cdot\\ &\cdot\frac14\sum_{\eta,\eta'=\pm1}\hat B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{p}}_{\eta'}+2\mathbf{p}_F(\sigma_1+\sigma_2),\bar{\mathbf{k}}_{\eta,\eta'})\;,\end{aligned}\tag{5.10}$$

where $\bar{\mathbf{k}}_{\eta,\eta'}$ is defined as in (2.73) and

$$\bar{\mathbf{p}}_{\eta'}=\left(0,\eta'\frac{2\pi}{\beta}\right)\;.\tag{5.11}$$

By using the symmetries of the interaction, as in §2.4, it is easy to show that

$$\mathcal{L}\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)=\frac{Z_h^{(1)}}{Z_h}F_1^{(\le h)}+\frac{Z_h^{(2)}}{Z_h}F_2^{(\le h)}\;,\tag{5.12}$$

where $Z_h^{(1)}$ and $Z_h^{(2)}$ are real numbers, such that $Z_1^{(1)}=Z_1^{(2)}=1$ and

$$F_1^{(\le h)}=\sum_{\sigma=\pm1}\int d\mathbf{x}\phi(\mathbf{x})e^{2i\sigma\mathbf{p}_F\mathbf{x}}\psi^{(\le h)\sigma}_{\mathbf{x},\sigma}\psi^{(\le h)\sigma}_{\mathbf{x},-\sigma}\;,\tag{5.13}$$

$$F_2^{(\le h)}=\sum_{\sigma=\pm1}\int d\mathbf{x}\phi(x)\psi^{(\le h)\sigma}_{\mathbf{x},\sigma}\psi^{(\le h)-\sigma}_{\mathbf{x},\sigma}\;.\tag{5.14}$$

By using the notation of §2.5, we can write the integral in the r.h.s. of (5.4) as

$$\begin{aligned}&e^{-L\beta t_h}\int P_{\tilde Z_{h-1},\sigma_{h-1},C_h}(d\psi^{(\le h)})e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})+\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)}=\\ &=e^{-L\beta t_h}\int P_{Z_{h-1},\sigma_{h-1},C_{h-1}}(d\psi^{(\le h-1)})\cdot\\ &\cdot\int P_{Z_{h-1},\sigma_{h-1},\tilde f_h^{-1}}(d\psi^{(h)})e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi)}\;,\end{aligned}\tag{5.15}$$

where $\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})$ is defined as in (2.107) and

$$\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)},\phi) = \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},\phi) \;. \tag{5.16}$$

$\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)},\phi)$ and $S^{(h)}(\phi)$ are then defined through the analogous of (2.110), that is

$$\begin{aligned} &e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})+\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)},\phi)-L\beta\tilde{E}_h+\tilde{S}^{(h)}(\phi)} = \\ &= \int P_{Z_{h-1},\sigma_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)})e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)},\phi)} \;. \end{aligned} \tag{5.17}$$

The definitions (5.16) and (5.12) easily imply that

$$\frac{Z^{(i)}_{h-1}}{Z^{(i)}_h} = 1 + z^{(i)}_h \;, \quad i=1,2 \;, \tag{5.18}$$

where $z^{(1)}_h$ and $z^{(2)}_h$ are some quantities of order ε_h, which can be written in terms of a tree expansion similar to that described in §3, as we shall explain below.

As in §3, the fields of scale between h^* and $h_{L,\beta}$ are integrated in a single step, so we define, in analogy to (3.125),

$$\begin{aligned} &e^{\tilde{S}^{(h^*)}(\phi)-L\beta\tilde{E}_{h^*}} = \\ &\int P_{Z_{h^*-1},\sigma_{h^*-1},C_{h^*}}(d\psi^{(\leq h^*)})e^{-\hat{\mathcal{V}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\leq h^*)})+\hat{\mathcal{B}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\leq h^*)},\phi)} \;. \end{aligned} \tag{5.19}$$

It follows, by using (3.126), that

$$S(\phi) = -L\beta E_{L,\beta} + S^{(h)}(\phi) = -L\beta E_{L,\beta} + \sum_{h=h^*}^{1} \tilde{S}^{(h)}(\phi) \;; \tag{5.20}$$

hence, by (5.1)

$$\Omega^3_{L,\beta,\mathbf{x}} = S^{(h)}_2(\mathbf{x},0) = \sum_{h=h^*}^{1} \tilde{S}^{(h)}_2(\mathbf{x},0) \;. \tag{5.21}$$

5.2 The functionals $\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)},\phi)$ and $S^{(h)}(\phi)$ can be written in terms of a tree expansion similar to the one described in §(3.2). We introduce, for each $n \geq 0$ and each $m \geq 1$, a family $\mathcal{T}^m_{h,n}$ of trees, which are defined as in §(3.2), with some differences, that we shall explain.

1) First of all, if $\tau \in \mathcal{T}^m_{h,n}$, the tree has $n+m$ (instead of n) endpoints. Moreover, among the $n+m$ endpoints, there are n endpoints, which we call *normal endpoints*, which are associated with a contribution to the effective potential on scale $h_v - 1$. The m remaining endpoints, which we call *special endpoints*, are associated with a local term of the form (5.13) or (5.14); we shall say that they are of type $Z^{(1)}$ or $Z^{(2)}$, respectively.

2) We associate with each vertex v a new integer $l_v \in [0,m]$, which denotes the number of special endpoints following v, *i.e.* contained in L_v.

3) We introduce an *external field label* f^ϕ to distinguish the different ϕ fields appearing in the special endpoints. I_v^ϕ will denote the set of external field labels associated with the endpoints following the vertex v; of course $l_v = |I_v^\phi|$ and $m = |I_{v_0}^\phi|$.

These definitions allow to represent $\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}, \phi) + S^{(h+1)}(\phi)$ in a way similar to that described in detail in §3.3-3.11. It is sufficient to extend in an obvious way some notations and some procedures, in order to take into account the presence of the new terms depending on the external field and the corresponding localization operation.

In particular, if $l_v \neq 0$, the $\mathcal{R}$ operation associated with the vertex v can be deduced from (5.7) and (5.8) and can be represented as acting on the kernels or on the fields in a way similar to what we did in §3.1. We will not write it in detail; we only remark that such definition is chosen so that, when $\mathcal{R}$ is represented as acting on the fields, no derivative is applied to the ϕ field.

All the considerations in §3.2, up to the modifications listed above, can be trivially repeated. The same is true for the definition of the labels $r_v(f)$, described in §3.3. One has only to consider, in addition to the cases listed there, the case in which $|P_v| = 2$ and $l_v = 1$; in such a case, if there is no non trivial vertex v' such that $v_0 \le v' < v$, we make an arbitrary choice, otherwise we put $r_v(f) = 1$ for the ψ field which is an internal field in the nearest non trivial vertex preceding v. As in §3.2, this is sufficient to avoid the proliferation of r_v indices.

Also the considerations in §3.4-3.7 can be adjusted without any difficulty. It is sufficient to add to the three items listed after (3.69) the case $l_{v_0} = 1$, $P_{v_0} = (f_1, f_2)$, by noting that in this case the action of $\mathcal{R}$ consists in replacing one external ψ field with a $D^{11}_{\mathbf{y},\mathbf{x}}$ field.

5.3 Let us consider in more detail the representation we get for the constants $z_h^{(l)}$, $l = 1, 2$, defined in (5.18). We have

$$z_h^{(l)} = \sum_{n=1}^{\infty} \sum_{\substack{\tau\in\mathcal{T}^1_{h,n}, \mathbf{P}\in\mathcal{P}_\tau, \mathbf{r}: P_{v_0}=(f_1,f_2), \\ \sigma_1=\omega_1=(-1)^{l-1}\, \sigma_2=(-1)^l\, \omega_2=+1}} \sum_{T\in\mathbf{T}} \sum_{\substack{\alpha\in A_T \\ q_\alpha(P_{v_0})=0}} z_h^{(l)}(\tau, \mathbf{P}, \mathbf{r}, T, \alpha) \;, \tag{5.22}$$

where, if $\mathbf{x}$ is the space time point associated with the special endpoint,

$$\begin{aligned} z_h^{(l)}(\tau, \mathbf{P}, \mathbf{r}, T, \alpha) = &\Big[\prod_{v \text{ not e.p.}} \Big(Z_{h_v}/Z_{h_v-1} \Big)^{|P_v|/2} \Big] \cdot \\ \cdot &\int d(\mathbf{x}_{v_0}\backslash\mathbf{x}) h_\alpha(\mathbf{x}_{v_0}) \Big[\prod_{i=1}^n d^{b_\alpha(v_i^*)}_{j_\alpha(v_i^*)}(\mathbf{x}_i, \mathbf{y}_i) K^{h_i}_{v_i^*}(\mathbf{x}_{v_i^*}) \Big] \Big\{ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \cdot \\ \cdot &\det G^{h_v,T_v}_\alpha(\mathbf{t}_v) \Big[\prod_{l\in T_v} \hat\partial^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)} \hat\partial^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)} [d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}_l, \mathbf{y}_l) \bar\partial_1^{m_l} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_l - \mathbf{y}_l)] \Big] \Big\} \;. \end{aligned} \tag{5.23}$$

The notations are the same as in §3.10 and we can derive for $z_h^{(l)}(\tau, \mathbf{P}, \mathbf{r}, T, \alpha)$ a bound similar to (3.110), without the volume factor $L\beta$ (the integration over x_{v_0} is done keeping $\mathbf{x}$ fixed). The only relevant difference is that the bounds (3.83) and (3.107) have to be modified, in order to take into account the properties of the extended localization operation, by substituting $z(P_v)$ and $\tilde z(P_v)$ with $z(P_v,l_v)$ and $\tilde z(P_v,l_v)$, respectively, with

$$z(P_v,l_v)=\begin{cases}1 & \text{if } |P_v|=4,\ l_v=0\\ 1 & \text{if } |P_v|=2,\ l_v=0 \text{ and } \sum_{f\in P_v}\omega(f)\neq 0\ ,\\ 2 & \text{if } |P_v|=2,\ l_v=0 \text{ and } \sum_{f\in P_v}\omega(f)=0\ ,\\ 1 & \text{if } |P_v|=2,\ l_v=1 \text{ and } \sum_{f\in P_v}\sigma(f)\omega(f)=0\ ,\\ 0 & \text{otherwise.}\end{cases} \tag{5.24}$$

$$\tilde z(P_v,l_v)=\begin{cases}1 & \text{if } |P_v|=2,\ l_v=0 \text{ and } \sum_{f\in P_v}\omega(f)\neq 0\ ,\\ 1 & \text{if } |P_v|=2,\ l_v=1 \text{ and } \sum_{f\in P_v}\sigma(f)\omega(f)\neq 0\ ,\\ 0 & \text{otherwise.}\end{cases} \tag{5.25}$$

It follows that

$$\begin{aligned}|z_h^{(l)}(\tau,\mathbf{P},\mathbf{r},T,\alpha)| &\le C^n\varepsilon_h^n\gamma^{-h[D_0(P_{v_0})+l_{v_0}]}\prod_{v \text{ not e.p.}}\Big\{C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}\cdot\\ &\cdot\frac{1}{s_v!}\Big(Z_{h_v}/Z_{h_v-1}\Big)^{|P_v|/2}\gamma^{-[-2+\frac{|P_v|}{2}+l_v+z(P_v,l_v)+\frac{\tilde z(P_v,l_v)}{2}]}\Big\}\ ,\end{aligned} \tag{5.26}$$

with

$$-2+\frac{|P_v|}{2}+l_v+z(P_v,l_v)+\frac{\tilde z(P_v,l_v)}{2}\ge\frac12\ ,\quad \forall v \text{ not e.p.}\ . \tag{5.27}$$

Hence, we can proceed as in §3.14 and, since $D_0(P_{v_0})+l_{v_0}=0$, we can easily prove the following Theorem.

5.4 THEOREM. *Suppose that $u\neq 0$ satisfies the condition (2.117), δ^* satisfies the condition (4.41), $\bar\varepsilon_3$ is defined as in Theorem 4.9 and $\nu\in I^{(h^*)}$. Then, there exist two constants $\bar\varepsilon_4\le\bar\varepsilon_3$ and c, independent of u, L, β, such that, if $|\lambda_1|\le\bar\varepsilon_4$, then*

$$|z_h^{(l)}|\le c|\lambda_1|\ ,\quad 0\le h\le h^*\ . \tag{5.28}$$

5.5 Theorem 5.4, the bound (4.55) on Z_h, the definition (5.18) and an explicit first order calculation of $z_h^{(1)}$ imply that there exist two positive constants c_1 and c_2, such that

$$\gamma^{-c_2\lambda_1 h}\le\frac{Z_h^{(1)}}{Z_h}\le\gamma^{-c_1\lambda_1 h}\ . \tag{5.29}$$

A similar bound is in principle valid also for $Z_h^{(2)}/Z_h$, but we shall prove that a much stronger bound is verified, by comparing our model with the Luttinger model. First of all, we consider an approximated Luttinger model, which is similar to that introduced in §4.4. It is obtained from the original model by substituting the free measure and the potential with the following expressions, where we use the notation of §2:

$$\begin{aligned}P^{(L)}(d\psi^{(\le 0)}) &= \prod_{\mathbf{k}':C_0^{-1}(\mathbf{k}')>0}\prod_{\omega=\pm1}\frac{d\hat\psi^{(\le0)+}_{\mathbf{k}',\omega}d\hat\psi^{(\le0)-}_{\mathbf{k}',\omega}}{\mathcal{N}_L(\mathbf{k}')}\cdot\\ &\cdot\exp\Big\{-\frac{1}{L\beta}\sum_{\omega=\pm1}\sum_{\mathbf{k}':C_0^{-1}(\mathbf{k}')>0}C_0(\mathbf{k}')\big(-ik_0+\omega v_0^* k'\big)\hat\psi^{(\le0)+}_{\mathbf{k}',\omega}\hat\psi^{(\le0)-}_{\mathbf{k}',\omega}\Big\}\ ,\end{aligned} \tag{5.30}$$

$$V^{(L)}(\psi^{(\le 0)}) = \lambda_0^{(L)} \int_{\mathbb{T}_{L,\beta}} d\mathbf{x}\, \psi^{(\le 0)+}_{\mathbf{x},+1} \psi^{(\le 0)-}_{\mathbf{x},-1} \psi^{(\le 0)+}_{\mathbf{x},-1} \psi^{(\le 0)-}_{\mathbf{x},+1} + \\ + \delta_0^{(L)} \sum_{\omega=\pm 1} i\omega \int_{\mathbb{T}_{L,\beta}} d\mathbf{x}\, \psi^{(\le h)+}_{\mathbf{x},\omega} \partial_x \psi^{(\le h)-}_{\mathbf{x},\omega} \;, \tag{5.31}$$

where $\mathcal{N}_L(\mathbf{k}') = C_0(\mathbf{k}')(L\beta)^{-1}[k_0^2 + (v_0^* k')^2]^{1/2}$, $\lambda_0^{(L)}$ and $\delta_0^{(L)}$ have the role of the running couplings on scale 0 of the original model, but are not necessarily equal to them, $\mathbb{T}_{L,\beta}$ is the (continuous, as in §3.15) torus $[0, L] \times [0, \beta]$ and $\psi^{(\le 0)}$ is the (continuous) Grassmanian field on $\mathbb{T}_{L,\beta}$ with antiperiodic boundary conditions. Moreover, the interaction with the external field $\mathcal{B}^{(1)}(\psi^{(\le 1)}, \phi)$ is substituted with the corresponding expression on scale 0, deprived of the irrelevant terms, that is

$$\mathcal{B}^{(0)}(\psi^{(\le 0)}, \phi) = \sum_{\sigma=\pm 1} \int d\mathbf{x} \phi(\mathbf{x}) \left(e^{2i\sigma \mathbf{p}_F \mathbf{x}} \psi^{(\le h)\sigma}_{\mathbf{x},\sigma} \psi^{(\le h)\sigma}_{\mathbf{x},-\sigma} + \psi^{(\le h)\sigma}_{\mathbf{x},\sigma} \psi^{(\le h)-\sigma}_{\mathbf{x},\sigma} \right) . \tag{5.32}$$

We shall call $Z_h^{(2,L)}$, $z_h^{(2,L)}$, $Z_h^{(L)}$ and $z_h^{(L)}$ the analogous of $Z_h^{(2)}$, $z_h^{(2)}$, Z_h and z_h for this approximate Luttinger model.

We want to compare the flow of $Z_h^{(2,L)}/Z_h^{(L)}$ with the flow of $Z_h^{(2)}/Z_h$; hence we write

$$\frac{Z_{h-1}^{(2)}}{Z_{h-1}} = \frac{Z_h^{(2)}}{Z_h} \left[1 + \beta^{(2)}(\vec{a}_h, \nu_h; \ldots; \vec{a}_1, \nu_1; u, \delta^*) \right] , \tag{5.33}$$

$$\frac{Z_{h-1}^{(2,L)}}{Z_{h-1}^{(L)}} = \frac{Z_h^{(2,L)}}{Z_h^{(L)}} \left[1 + \beta^{(2,L)}(\vec{a}_h^{(L)}, \ldots; \vec{a}_0^{(L)}, \delta^*) \right] , \tag{5.34}$$

where $a_h^{(L)}$ are the running couplings in the approximated Luttinger model (by symmetry $\nu_h^{(L)} = 0$, since $\nu = 0$, see §4.4), $1+\beta^{(2)} = (1+z_h^{(2)})/(1+z_h)$ and $1+\beta^{(2,L)} = (1+z_h^{(2,L)})/(1+z_h^{(L)})$.

The Luttinger model has a special symmetry, the *local gauge invariance*, which allows to prove many *Ward identities*. As we shall prove in §7, the approximate Luttinger model satisfies some approximate version of these identities and one of them implies that, if $|\delta_* + (\delta_0^{(L)}/v_0)| \le 1/2$,

$$\gamma^{-C|\lambda_0^{(L)}|} \le \frac{Z_h^{(2,L)}}{Z_h^{(L)}} \le \gamma^{C|\lambda_0^{(L)}|} . \tag{5.35}$$

By proceeding as in the proof of (4.51) (see [BGPS], §7), one can show that (5.35) implies that there exists $\bar{\varepsilon} > 0$ and $\eta' < 1$, such that, if $|\vec{a}_h| \le \bar{\varepsilon}$,

$$|\beta_h^{(2,L)}(\vec{a}_h, \ldots, \vec{a}_h, \delta^*)| \le C \mu_h^2 \gamma^{\eta' h} . \tag{5.36}$$

Remark - The analogous bound (4.51) was obtained in [BGPS] by a comparison with the exact solution of the Luttinger model; this was possible, thanks to the proof given in [GS] that the effective potential on scale 0 is well defined also in the Luttinger model, a non trivial result because of the ultraviolet problem. This procedure would be much harder in the case of the bound (5.36), because the density is not well defined in the Luttinger model, see §1.3. In any case, the bound (5.35), whose proof is relatively simple, allows to get very easily the same result.

One can also show, as in the proof of Lemma 4.5, that

$$|\beta^{(2)}(\vec{a}_h,\nu_h;\ldots;\vec{a}_1,\nu_1;u,\delta^*)-\beta^{(2,L)}(\vec{a}_h,,\ldots;\vec{a}_0,\delta^*)|\le C\bar\lambda_h^2[\gamma^{-\frac{1}{2}(h-h^*)}+\gamma^{\eta h}]\;, \tag{5.37}$$

for any $h\ge h^*$ and for some $\eta<1$.

Note that, in (5.37), $\beta^{(2,L)}$ is evaluated at the values of the running couplings $\vec{a}_h$ of the original model; this is meaningful, since in (5.36) $\vec{a}_h$ can take any value such that $|\vec{a}_h|\le\bar\varepsilon$; this follows from the remark, already used in §4.7, that $\vec{a}_h^{(L)}$ is a continuous function of $\vec{a}_0^{(L)}$ and $\vec{a}_h^{(L)}=\vec{a}_0^{(L)}+O(\mu_h^2)$, see also [BGPS].

By using (5.36) and (5.37) and proceeding as in the proof of Theorem 4.9, one can easily prove the following Theorem.

5.6 THEOREM. *If the hypotheses of Theorem 5.4 are verified, there exists a positive constant c_1, independent of u, L, β, such that*

$$\gamma^{-c_1|\lambda_1|}\le\frac{Z_h^{(2)}}{Z_h}\le\gamma^{c_1|\lambda_1|}\;. \tag{5.38}$$

5.7 We are now ready to study the expansion of the correlation function $\Omega^3_{L,\beta}(\mathbf{x})$, which follows from (5.21) and the considerations of §5.2. We have to consider the trees with two special endpoints, whose space-points we shall denote $\mathbf{x}$ and $\mathbf{y}=\mathbf{0}$; moreover, we shall denote by $h_\mathbf{x}$ and $h_\mathbf{y}$ the scales of the two special endpoints and by $h_{\mathbf{x},\mathbf{y}}$ the scale of the smallest cluster containing both special endpoints. Finally $\mathcal{T}^2_{h,n,l}$ will denote the family of all trees belonging to $\mathcal{T}^2_{h,n}$, such that the two special endpoints are both of type $Z^{(1)}$, if $l=1$, both of type $Z^{(2)}$, if $l=2$, one of type $Z^{(1)}$ and the other of type $Z^{(2)}$, if $l=3$.

If we extract from the expansion the contribution of the trees with one special endpoint and no normal endpoints, we can write

$$\begin{aligned}\Omega^3_{L,\beta}(\mathbf{x})=&\sum_{h,h'=h^*}^{1}\sum_{\sigma=\pm1}\Big\{e^{2i\sigma p_F x}\cdot\\ &\cdot\frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}}[g^{(h)}_{\sigma,\sigma}(-\sigma\mathbf{x})g^{(h')}_{-\sigma,-\sigma}(-\sigma\mathbf{x})-g^{(h)}_{+1,-1}(-\sigma\mathbf{x})g^{(h')}_{-1,+1}(-\sigma\mathbf{x})]+\\ &+\frac{(Z^{(2)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}}[-g^{(h)}_{\sigma,\sigma}(-\sigma\mathbf{x})g^{(h')}_{\sigma,\sigma}(\sigma\mathbf{x})+g^{(h)}_{-1,+1}(-\sigma\mathbf{x})g^{(h')}_{+1,-1}(\sigma\mathbf{x})]\Big\}+\\ &+\sum_{h=h^*}^{1}\left\{\left(\frac{Z_h^{(1)}}{Z_h}\right)^2G^{(h)}_{1,L,\beta}(\mathbf{x})+\left(\frac{Z_h^{(2)}}{Z_h}\right)^2G^{(h)}_{2,L,\beta}(\mathbf{x})+\frac{Z_h^{(1)}Z_h^{(2)}}{Z_h^2}G^{(h)}_{3,L,\beta}(\mathbf{x})\right\}\;,\end{aligned} \tag{5.39}$$

where $h\vee h'=\max\{h,h'\}$ and $g^{(h^*)}_{\omega_1,\omega_2}(\mathbf{x})$ has to be understood as $g^{(\le h^*)}_{\omega_1,\omega_2}(\mathbf{x})$; moreover,

$$G^{(h)}_{l,L,\beta}(\mathbf{x})=\sum_{n=1}^{\infty}\sum_{h_r=h^*-1}^{h-1}\sum_{\substack{\tau\in\mathcal{T}^2_{h_r,n,l}\\ h_{\mathbf{x},\mathbf{y}}=h}}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau,\mathbf{r}\\ P_{v_0}=\emptyset}}\sum_{T\in\mathbf{T}}\sum_{\alpha\in A_T}G^{(h,h_r)}_{l,L,\beta}(\mathbf{x},\tau,\mathbf{P},\mathbf{r},T,\alpha)\;, \tag{5.40}$$

where, if $\hat{\mathbf{x}}_{v_0}$ denotes the set of space-time points associated with the normal endpoints and $i_{\mathbf{x}} = i$, if the corresponding special endpoint is of type $Z^{(i)}$,

$$\begin{aligned}
&G^{(h,h_r)}_{l,L,\beta}(\mathbf{x},\tau,\mathbf{P},\mathbf{r},T,\alpha) = \\
&= \left(\frac{Z^{(i_{\mathbf{x}})}_{h_{\mathbf{x}}} Z_h}{Z_{h_{\mathbf{x}}-1} Z^{(i_{\mathbf{x}})}_h}\right)\left(\frac{Z^{(i_{\mathbf{y}})}_{h_{\mathbf{y}}} Z_h}{Z_{h_{\mathbf{y}}-1} Z^{(i_{\mathbf{y}})}_h}\right)\Big[\prod_{v \text{ not e.p.}} \Big(Z_{h_v}/Z_{h_v-1}\Big)^{|P_v|/2}\Big] \cdot \\
&\cdot \int d\hat{\mathbf{x}}_{v_0} h_\alpha(\hat{\mathbf{x}}_{v_0}) \Big[\prod_{i=1}^n d^{b_\alpha(v_i^*)}_{j_\alpha(v_i^*)}(\mathbf{x}_i,\mathbf{y}_i) K^{(h_i)}_{v_i^*}(\mathbf{x}_{v_i^*})\Big]\Big\{\prod_{v \text{ not e.p.}} \frac{1}{s_v!}\int dP_{T_v}(\mathbf{t}_v) \cdot \\
&\cdot \det G^{h_v,T_v}_\alpha(\mathbf{t}_v)\Big[\prod_{l\in T_v} \hat{\partial}^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)} \hat{\partial}^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)} [d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}_l,\mathbf{y}_l) \bar{\partial}_1^{m_l} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_l-\mathbf{y}_l)]\Big]\Big\} .
\end{aligned} \tag{5.41}$$

In the r.h.s. of (5.41) all quantities are defined as in §3, except the kernels $K^{(h_i)}_{v_i^*}(\mathbf{x}_{v_i^*})$ associated with the special endpoints. If v is one of these endpoints, $\mathbf{x}_v$ is always a single point and

$$K^{(h_v)}_v(\mathbf{x}_v) = e^{i\mathbf{p}_F\mathbf{x}_v \sum_{f\in I_v}\sigma(f)} . \tag{5.42}$$

We want to prove the following Theorem.

5.8 THEOREM. *Suppose that the conditions of Theorem 5.4 are verified, that $\bar\varepsilon_4$ is defined as in that theorem and that δ^* is chosen so that condition (4.57) is satisfied. Then, there exist positive constants $\vartheta < 1$ and $\bar\varepsilon_5 \le \bar\varepsilon_4$, independent of u, L, β, such that, if $|\lambda_1| \le \bar\varepsilon_5$ and $\gamma \ge 1+\sqrt{2}$, given any integer $N \ge 0$,*

$$|G^{(h)}_{1,L,\beta}(\mathbf{x})| + |G^{(h)}_{2,L,\beta}(\mathbf{x})| + \gamma^{-\vartheta h}|G^{(h)}_{3,L,\beta}(\mathbf{x})| \le C_N|\lambda_1|\frac{\gamma^{2h}}{1+[\gamma^h|\mathbf{d}(\mathbf{x})|]^N} , \tag{5.43}$$

for a suitable constant C_N.

Moreover, if $h \le 0$, we can write

$$\begin{aligned}
G^{(h)}_{1,L,\beta}(\mathbf{x}) &= \cos(2p_F x)\bar G^{(h)}_{1,L,\beta}(\mathbf{x}) + \sum_{\sigma=\pm1} e^{ip_F\sigma x} s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x}) + r^{(h)}_{1,L,\beta}(\mathbf{x}) , \\
G^{(h)}_{2,L,\beta}(\mathbf{x}) &= \bar G^{(h)}_{2,L,\beta}(\mathbf{x}) + s^{(h)}_{2,L,\beta}(\mathbf{x}) + r^{(h)}_{2,L,\beta}(\mathbf{x}) ,
\end{aligned} \tag{5.44}$$

so that

$$\bar G^{(h)}_{l,L,\beta}(\mathbf{x}) = \bar G^{(h)}_{l,L,\beta}(-\mathbf{x}) , \quad l=1,2 , \tag{5.45}$$

$$|r^{(h)}_{1,L,\beta}(\mathbf{x})| + |r^{(h)}_{2,L,\beta}(\mathbf{x})| \le C_N|\lambda_1|\gamma^{2h}\frac{\gamma^{\vartheta h}}{1+[\gamma^h|\mathbf{d}(\mathbf{x})|]^N} , \tag{5.46}$$

and, if we define $D_{m_0,m_1} = \partial_0^{m_0}\bar\partial_1^{m_1}$, given any integers $m_0, m_1 \ge 0$, there exists a constant C_{N,m_0,m_1}, such that

$$\sum_{l=1,2} |D_{m_0,m_1}\bar G^{(h)}_{l,L,\beta}(\mathbf{x})| \le C_{N,m_0,m_1}|\lambda_1|\frac{\gamma^{2h}\gamma^{h(m_0+m_1)}}{1+[\gamma^h|\mathbf{d}(\mathbf{x})|]^N} , \tag{5.47}$$

$$\begin{aligned}
&\sum_{\sigma=\pm1} |D_{m_0,m_1} s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})| + |D_{m_0,m_1} s^{(h)}_{2,L,\beta}(\mathbf{x})| \le \\
&\le C_{N,m_0,m_1}|\lambda_1|\frac{\gamma^{2h}\gamma^{h(m_0+m_1)}}{1+[\gamma^h|\mathbf{d}(\mathbf{x})|]^N}[\gamma^{-\vartheta(h-h^*)} + \gamma^{\vartheta h}] .
\end{aligned} \tag{5.48}$$

$\Omega^3_{L,\beta}(\mathbf{x})$, *as well as the functions* $\bar G^{(h)}_{l,L,\beta}(\mathbf{x})$, $r^{(h)}_{l,L,\beta}(\mathbf{x})$, $s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})$ *and* $s^{(h)}_{2,L,\beta}(\mathbf{x})$ *converge, as* $L,\beta\to\infty$, *to continuous bounded functions on* $\mathbb{Z}\times\mathbb{R}$, *that we shall denote* $\Omega^3(\mathbf{x})$, $\bar G^{(h)}_l(\mathbf{x})$, $r^{(h)}_l(\mathbf{x})$, $s^{(h)}_{1,\sigma}(\mathbf{x})$ *and* $s^{(h)}_2(\mathbf{x})$, *respectively.* $\bar G^{(h)}_1(\mathbf{x})$ *and* $\bar G^{(h)}_2(\mathbf{x})$ *are the restrictions to* $\mathbb{Z}\times\mathbb{R}$ *of two even functions on* $\mathbb{R}^2$ *satisfying the bound (5.47) with the continuous derivative* ∂_1 *in place of the discrete one and* $|\mathbf{x}|$ *in place of* $|\mathbf{d}(\mathbf{x})|$.

Finally, $\bar G^{(h)}_1(\mathbf{x})$, *as a function on* $\mathbb{R}^2$, *satisfies the symmetry relation*

$$\bar G^{(h)}_1(x,x_0)=\bar G^{(h)}_1(x_0v_0^*,\frac{x}{v_0^*})\;. \tag{5.49}$$

5.9 *Proof.* As in the proof of Theorem 5.4, we shall try to mimic as much as possible the proof of the bound (3.110), by only remarking the relevant differences. Since $D_0(P_{v_0})+l_{v_0}=0$, if the integral in the r.h.s. of (5.41) were over the set of variables $x_{v_0}\backslash\mathbf{x}$, we should get for $G^{(h,h_r)}_{l,L,\beta}(\mathbf{x},\tau,\mathbf{P},\mathbf{r},T,\alpha)$ the same bound we derived in §5.3 for $z^{(l)}_h(\tau,\mathbf{P},\mathbf{r},T,\alpha)$. However, in this case, we have to perform the integration over the set x_{v_0} by keeping fixed two points ($\mathbf{x}$ and $\mathbf{y}$), instead of one; hence we have to modify the bound (3.102) in a way different from what we did in the proof of Theorem 5.4.

Let us call $\bar v_0$ the higher vertex $v\in\tau$, such that both $\mathbf{x}$ and $\mathbf{y}$ belong to $\mathbf{x}_v$; by the definition of h, it is a non trivial vertex and its scale is equal to h. Moreover, given the tree graph T on x_{v_0}, let us call $T_{\mathbf{x},\mathbf{y}}$ its subtree connecting the points of $\mathbf{x}_{\bar v_0}$ and $\tilde T_{\mathbf{x},\mathbf{y}}=\cup_{v\ge\bar v_0}\tilde T_v$, $\tilde T_v$ being defined as §3.15, after (3.118). We want to bound $\mathbf{d}(\mathbf{x}-\mathbf{y})$ in terms of the distances between the points connected by the lines $l\in\tilde T_{\mathbf{x},\mathbf{y}}$.

Let us call $\bar v^{(i)}$, $i=1,\ldots,s_{\bar v_0}$ the non trivial vertices or endpoints following $\bar v_0$. The definition of $\bar v_0$ implies that $s_{\bar v_0}>1$ and that $\mathbf{x}$ and $\mathbf{y}$ belong to two different sets $\mathbf{x}_{\bar v^{(i)}}$; note also that $\tilde T_{\bar v_0}$ is an anchored tree graph between the sets of points $\mathbf{x}_{\bar v^{(i)}}$. Hence there is an integer r, a family $l_1,\ldots,l_r$ of lines belonging to $\tilde T_{\bar v_0}$ and a family $v^{(1)},\ldots,v^{(r+1)}$ of vertices to be chosen among $\bar v^{(1)},\ldots,\bar v^{(s_{\bar v_0})}$, such that $1\le r\le s_{\bar v_0}-1$ and

$$\begin{aligned}|\mathbf{d}(\mathbf{x}-\mathbf{y})|&\le\sum_{j=1}^r|\mathbf{d}(\mathbf{x}'_{l_j}-\mathbf{y}'_{l_j})|+\sum_{j=1}^{r+1}|\mathbf{d}(\mathbf{x}^{(j)}-\mathbf{y}^{(j)})|\;\le\\&\le\sum_{l\in\tilde T_{\bar v_0}}|\mathbf{d}(\mathbf{x}'_l-\mathbf{y}'_l)|+\sum_{j=1}^{r+1}|\mathbf{d}(\mathbf{x}^{(j)}-\mathbf{y}^{(j)})|\;,\end{aligned} \tag{5.50}$$

where $\mathbf{x}^{(1)}=\mathbf{x}$, $\mathbf{y}^{(r+1)}=\mathbf{y}$, $\mathbf{x}'_{l_j}$ and $\mathbf{y}'_{l_j}$ are defined as in (3.114) and, finally, the couple of points $(\mathbf{x}'_{l_j},\mathbf{y}'_{l_j})$ coincide, up to the order, with the couple $(\mathbf{y}^{(j)},\mathbf{x}^{j+1})$.

If no propagator associated with a line $l\in\tilde T_{\mathbf{x},\mathbf{y}}$ is affected by the regularization, we can iterate in an obvious way the previous considerations, so getting the bound

$$|\mathbf{d}(\mathbf{x}-\mathbf{y})|\le\sum_{l\in\tilde T_{\mathbf{x},\mathbf{y}}}|\mathbf{d}(\mathbf{x}'_l-\mathbf{y}'_l)|\;. \tag{5.51}$$

However, this is not in general true and we have to consider in more detail the subsequent steps of the iteration.

Let us consider one of the vertices $\mathbf{x}_{v^{(j)}}$; if $\mathbf{x}^{(j)} = \mathbf{y}^{(j)}$, there is nothing to do. Hence we shall suppose that $\mathbf{x}^{(j)} \neq \mathbf{y}^{(j)}$ and we shall say that the propagators associated with the lines l_j, if $1 \le j \le r$, and l_{j-1}, if $2 \le j \le r+1$, are *linked* to $v^{(j)}$. There are two different cases to consider.

1) $\mathbf{x}^{(j)}$ and $\mathbf{y}^{(j)}$ belong to two different non trivial vertices or endpoints following $v^{(j)}$ and the propagators linked to $v^{(j)}$ are not affected by action of $\mathcal{R}$ on the vertex $v^{(j)}$ or some trivial vertex v, such that $\bar{v}_0 < v < v^{(j)}$. In this case, we iterate the previous procedure without any change.

2) One of the propagators linked to $v^{(j)}$ is affected by action of $\mathcal{R}$ on the vertex $v^{(j)}$ or some trivial vertex v, such that $\bar{v}_0 < v < v^{(j)}$; note that, if there are two linked propagators, only one may have this property, as a consequence of the regularization procedure described in §3. This means that $\mathbf{x}^{(j)}$ or $\mathbf{y}^{(j)}$, let us say $\mathbf{x}^{(j)}$, is of the form (3.115), with $t_l \neq 1$, that is there are two points $\tilde{\mathbf{x}}_l, \mathbf{x}_l \in \mathbf{x}_{v^{(j)}}$ and a point $\bar{\mathbf{x}}_l \in \mathbb{R}^2$, coinciding with $\mathbf{x}_l$ modulo (L, β), such that

$$\mathbf{x}^{(j)} = \tilde{\mathbf{x}}_l + t_l(\bar{\mathbf{x}}_l - \tilde{\mathbf{x}}_l) \ , \quad |\bar{x}_l - x_l| \le 3L/4 \ , |\bar{x}_{l,0} - x_{l,0}| \le 3\beta/4 \ . \tag{5.52}$$

By using (2.96), (5.52), the fact that $0 \le |t_l| \le 1$ and the remark that $\mathbf{d}(\bar{\mathbf{x}}_l - \tilde{\mathbf{x}}_l) = \mathbf{d}(\mathbf{x}_l - \tilde{\mathbf{x}}_l)$, we get

$$|\mathbf{d}(\mathbf{x}^{(j)} - \mathbf{y}^{(j)})| \le |\mathbf{d}(\tilde{\mathbf{x}}_l - \mathbf{y}^{(j)})| + \sqrt{2}\, |\mathbf{d}(\mathbf{x}_l - \tilde{\mathbf{x}}_l)| \ . \tag{5.53}$$

We can now bound $|\mathbf{d}(\tilde{\mathbf{x}}_l - \mathbf{y}^{(j)})|$ and $|\mathbf{d}(\mathbf{x}_l - \tilde{\mathbf{x}}_l)|$, by proceeding as in the proof of (5.50), since the points $\tilde{\mathbf{x}}_l$, $\mathbf{x}_l$ and $\mathbf{y}^{(j)}$ all belong to $v^{(j)}$. We get

$$|\mathbf{d}(\mathbf{x}^{(j)} - \mathbf{y}^{(j)})| \le (1 + \sqrt{2}) \left[\sum_{l \in \tilde{T}_{v^{(j)}}} |\mathbf{d}(\mathbf{x}'_l - \mathbf{y}'_l)| + \sum_{m=1}^{r_j} |\mathbf{d}(\mathbf{x}'^{(m)} - \mathbf{y}'^{(m)})| \right] \ , \tag{5.54}$$

where $2 \le r_j \le s_{v^{(j)}}$ and the points $\mathbf{x}'^{(m)}$, $\mathbf{y}'^{(m)}$ are endpoints of propagators linked to some non trivial vertex or endpoint following $v^{(j)}$.

By iterating the previous procedure we get, instead of (5.51), the bound

$$|\mathbf{d}(\mathbf{x} - \mathbf{y})| \le \sum_{l \in \tilde{T}_{\mathbf{x},\mathbf{y}}} (1 + \sqrt{2})^{p_l} |\mathbf{d}(\mathbf{x}'_l - \mathbf{y}'_l)| \ , \tag{5.55}$$

where, if $l \in T_{v_l}$, p_l is an integer less or equal to the number of non trivial vertices v such that $\bar{v}_0 \le v < v_l$; note that

$$p_l \le h_{v_l} - h \ . \tag{5.56}$$

Let us now suppose that

$$\gamma \ge 1 + \sqrt{2} \ . \tag{5.57}$$

Since there are at most $2n+1$ lines in T, (5.55), (5.56) and (5.57) imply that there exists at least one line $l \in T_{\mathbf{x},\mathbf{y}}$, such that

$$\gamma^{h_{v_l}} |\mathbf{d}(\mathbf{x}'_l - \mathbf{y}'_l)| \geq \frac{\gamma^h |\mathbf{d}(\mathbf{x}-\mathbf{y})|}{2n+1} \ . \tag{5.58}$$

It follows that, given any $N \geq 0$, for the corresponding propagator we can use, instead of the bound (3.116), the following one:

$$\begin{aligned} &\left| \tilde{\partial}^{q_\alpha(f_l^-)}_{j_\alpha(f_l^-)} \tilde{\partial}^{q_\alpha(f_l^+)}_{j_\alpha(f_l^+)} [d^{b_\alpha(l)}_{j_\alpha(l)}(\mathbf{x}'_l(t_l), \mathbf{y}'_l(s_l)) \bar{\partial}_1^{m_l} g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}'_l(t_l) - \mathbf{y}'_l(s_l))] \right| \leq \\ &\leq \frac{\gamma^{h_v[1+q_\alpha(f_l^+)+q_\alpha(f_l^-)+m(f_l^-)+m(f_l^+)-b_\alpha(l)]}}{1+[\gamma^{h_v}|\mathbf{d}(\mathbf{x}'_l(t_l)-\mathbf{y}'_l(s_l))|]^3} \Big(\frac{|\sigma_{h_v}|}{\gamma^{h_v}}\Big)^{\rho_l} \frac{C_N(2n+1)^N}{1+[\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|]^N} \ . \end{aligned} \tag{5.59}$$

For all others propagators we use again the bound (3.116) with $N=3$ and we proceed as in §3.15, recalling that we have to substitute in (3.118) $d(\mathbf{x}_{v_0} \backslash \bar{\mathbf{x}})$ with $d\hat{\mathbf{x}}_{v_0}$. This implies that, in the r.h.s. of (3.119), one has to eliminate one $d\mathbf{r}_l$ factor and, of course, this can be done in an arbitrary way. We choose to eliminate the integration over the $\mathbf{r}_l$ corresponding to a propagator of scale h (there is at least one of them), so that the bound (3.118) is improved by a factor γ^{2h}.

At the end, we get

$$\begin{aligned} &|G^{(h,h_r)}_{l,L,\beta}(\mathbf{x},\tau,\mathbf{P},\mathbf{r},T,\alpha)| \leq (C\varepsilon_h)^n C_N (2n+1)^N \frac{\gamma^{2h}}{1+[\gamma^h \mathbf{d}(\mathbf{x})]^N} \cdot \\ &\cdot \left(\frac{Z^{(i_\mathbf{x})}_{h_\mathbf{x}} Z_h}{Z_{h_\mathbf{x}-1} Z^{(i_\mathbf{x})}_h}\right) \left(\frac{Z^{(i_\mathbf{y})}_{h_\mathbf{y}} Z_h}{Z_{h_\mathbf{y}-1} Z^{(i_\mathbf{y})}_h}\right) \prod_{v \text{ not e.p.}} \Big\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} \cdot \\ &\cdot \Big(Z_{h_v}/Z_{h_v-1}\Big)^{|P_v|/2} \gamma^{-[-2+\frac{|P_v|}{2}+l_v+z(P_v,l_v)+\frac{\tilde{z}(P_v,l_v)}{2}]} \cdot \Big\} \end{aligned} \tag{5.60}$$

We can now perform as in §3.14 the various sums in the r.h.s. of (5.40). There are some differences in the sum over the scale labels, but they can be easily treated. First of all, one has to take care of the factors $(Z^{(i_\mathbf{x})}_{h_\mathbf{x}} Z_h)/(Z_{h_\mathbf{x}-1} Z^{(i_\mathbf{x})}_h)$ and $(Z^{(i_\mathbf{y})}_{h_\mathbf{y}} Z_h)/(Z_{h_\mathbf{y}-1} Z^{(i_\mathbf{y})}_h)$. However, by using (5.29) and (5.38), it is easy to see that these factors have the only effect to add to the final bound a factor $\gamma^{C|\lambda_1|(h_v - h_{v'})}$ for each non trivial vertex v containing one of the special endpoints and strictly following the vertex $v_{\mathbf{x},\mathbf{y}}$; this has a negligible effect, thanks to analogous of the bound (3.111), valid in this case. The other difference is in the fact that, instead of fixing the scale of the root, we have now to fix the scale of $v_{\mathbf{x},\mathbf{y}}$. However, this has no effect, since we bound the sum over the scales with the sum over the the differences $h_v - h_{v'}$.

The previous considerations are sufficient to get the bound (5.43) for $G^{(h)}_{1,L,\beta}(\mathbf{x})$ and $G^{(h)}_{2,L,\beta}(\mathbf{x})$. In order to explain the factor $\gamma^{\vartheta h}$ multiplying $G^{(h)}_{3,L,\beta}(\mathbf{x})$, one has to note that the trees whose normal endpoints are all of scale lower than 2 give no contribution to $G^{(h)}_{3,L,\beta}(\mathbf{x})$. In fact, these endpoints have the property that $\sum_{f \in P_v} \sigma(f) = 0$, while this condition is satisfied from one of the special endpoints but not from the other, in any tree contributing to $G^{(h)}_{3,L,\beta}(\mathbf{x})$. It follows, since any propagator couples two fields with different σ indices, that it is possible to produce a non zero contribution to $G^{(h)}_{3,L,\beta}(\mathbf{x})$, only if there is at least one endpoint of scale 2; this allows to extract from the bound a factor $\gamma^{\vartheta h}$, with $0<\vartheta<1$, as remarked many times before.

We now want to show that $G^{(h)}_{1,L,\beta}(\mathbf{x})$ and $G^{(h)}_{2,L,\beta}(\mathbf{x})$ can be decomposed as in (5.44), so that the bounds (5.46), (5.47) and (5.45) are satisfied. To begin with, we define $r^{(h)}_{i,L,\beta}(\mathbf{x})$, $i=1,2$, by using the definition (5.40) of $G^{(h)}_{i,L,\beta}(\mathbf{x})$, with the constraint that the sum is restricted to the trees, which contain at least one endpoint of scale $h_v=2$; this implies, in particular, that $G^{(+1)}_{i,L,\beta}(\mathbf{x})-r^{(+1)}_{i,L,\beta}(\mathbf{x})=0$. Moreover, in the remaining trees, we decompose the propagators in the following way:

$$g^{(h)}_{\omega,\omega'}(\mathbf{x})=\bar g^{(h)}_{\omega,\omega'}(\mathbf{x})+\delta g^{(h)}_{\omega,\omega'}(\mathbf{x})\;, \tag{5.61}$$

where $\bar g^{(h)}_{\omega,\omega'}(\mathbf{x})$ is defined by putting, in the r.h.s. of (2.94), $(v_0^* k')$ in place of $E(k')$, and we absorb in $r^{(h)}_{i,L,\beta}(\mathbf{x})$ the terms containing at least one propagator $\delta g^{(h)}_{\omega,\omega'}(\mathbf{x})$, which is of size γ^{2h}. The substitution of $(v_0^* k')$ in place of $E(k')$ is done also in the definition of the $\mathcal{R}$ operator, so producing other "corrections", to be added to $r^{(h)}_{i,L,\beta}(\mathbf{x})$. An argument similar to that used for $G^{(h)}_{3,L,\beta}(\mathbf{x})$ easily allows to prove the bound (5.46).

$\sum_{\sigma=\pm1}\exp(i\sigma p_F x)s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})$ and $s^{(h)}_{2,L,\beta}(\mathbf{x})$ will denote the sum of the trees contributing to $G^{(h)}_{1,L,\beta}(\mathbf{x})$ $-r^{(h)}_{1,L,\beta}(\mathbf{x})$ and $G^{(h)}_{2,L,\beta}(\mathbf{x})-$ $r^{(h)}_{2,L,\beta}(\mathbf{x})$, respectively, which have at least one endpoint of type ν or δ.

Let us now consider the "leading" contribution to $G^{(h)}_{2,L,\beta}(\mathbf{x})$, which is defined by the second of the equations (5.44) as $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$ and is obtained by using again (5.40), but with the constraint that the sum over the trees is restricted to those having only endpoints with scale $h_v\le 1$ and only normal endpoints of type λ. Moreover we have to use everywhere the propagator $\bar g^{(h)}_{\omega,\omega'}(\mathbf{x})$, which has well defined parity properties in the $\mathbf{x}$ variables; it is odd, if $\omega=\omega'$, and even, if $\omega=-\omega'$.

Note that all the normal endpoints with $h_v\le 1$ are such that $\sum_{f\in I_v}\sigma(f)=0$ and that this property is true also for the special endpoints, which have to be of type $Z^{(2)}$; hence there is no oscillating factor in the kernels associated with the endpoints, which are suitable constants (the associated effective potential terms are local). It follows that any graph contributing to $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$ is given, up to a constant, by an integral over the product of an even number of propagators (we are using here the fact that there is no endpoint of type ν or δ). Moreover, since all the endpoints satisfy also the condition $\sum_{f\in I_v}\sigma(f)\omega(f)=0$, which is violated by the set of two lines connected by a non diagonal propagator, the number of non diagonal propagators has to be even. These remarks immediately imply that $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})=\bar G^{(h)}_{2,L,\beta}(-\mathbf{x})$.

In order to prove the bound (5.47) for $\bar G^{(h)}_{2,L,\beta}(\mathbf{x})$, we observe that, since the propagators only couple fields with different σ indices and $\sum_{f\in I_v}\sigma(f)=0$, given any tree τ contributing to $\bar{G}^{(h)}_{2,L,\beta}(\mathbf{x})$ and any $v \in \tau$, we must have

$$\sum_{f \in P_v} \sigma(f) = 0 \,. \tag{5.62}$$

Let us now consider the vertex $\bar{v}_0$, defined as in §5.9, that is the higher vertex $v \in \tau$, such that both $\mathbf{x}$ and $\mathbf{y} = \mathbf{0}$ belong to $\mathbf{x}_v$, and let $v_\mathbf{x}$ be the vertex immediately following $\bar{v}_0$, such that $\mathbf{x} \in v_\mathbf{x}$. We can associate with $v_\mathbf{x}$ a contribution to $\mathcal{B}^h(\psi^{(\le h)}, \phi)$ (recall that h is the scale of $\bar{v}_0$ and hence the scale of the external fields of $v_\mathbf{x}$), with $m = 1$ and $2n = P_{v_\mathbf{x}}$ (see (5.6)), whose kernel is of the form, thanks to (5.62)

$$\begin{aligned} B(\mathbf{x}; \mathbf{y}_1, \ldots, \mathbf{y}_{2n}) = \frac{1}{(L\beta)^{2n+1}} \sum_{\mathbf{p}, \mathbf{k}'_1, \ldots, \mathbf{k}'_{2n}} e^{i\mathbf{p}\mathbf{x} - i\sum_{r=1}^{2n} \sigma_r \mathbf{k}'_r \mathbf{y}_r} \cdot \\ \cdot \; \hat{B}(\mathbf{p}; \mathbf{k}'_1, \ldots, \mathbf{k}'_{2n-1}) \delta(\sum_{r=1}^{2n} \sigma_r \mathbf{k}'_r - \mathbf{p}) \,. \end{aligned} \tag{5.63}$$

If we apply the differential operator $\partial_0^{m_0}$ to $\bar{G}^{(h)}_{2,L,\beta}(\mathbf{x})$, this operator acts on $B(\mathbf{x}; \mathbf{y}_1, \ldots, \mathbf{y}_{2n})$, so that its Fourier transform is multiplied by $(ip_0)^{m_0}$; since $p_0 = \sum_{r=1}^{2n} \sigma_r k_{r0}$ and the external fields of $v_\mathbf{x}$ are contracted on a scale smaller or equal to h, it is easy to see that there is an improvement on the bound of $\partial_0 \bar{G}^{(h)}_{2,L,\beta}(\mathbf{x})$, with respect to the bound of $\bar{G}^{(h)}_{2,L,\beta}(\mathbf{x})$, of a factor $c_{m_0}\gamma^{hm_0}$, for a suitable constant c_{m_0}. We are using here the fact that $\bar{G}^{(+1)}_{i,L,\beta}(\mathbf{x}) = 0$, so that we can suppose $h \le 0$, otherwise we would be involved with the singularity of the scale 1 propagator $g^{(1)}_{\omega_l^-, \omega_l^+}(\mathbf{x}_l - \mathbf{y}_l)$ at $x_l - y_l = 0$, which allows to get uniform bounds on the derivatives only for $|x_l - y_l|$ bounded below, a condition not verified in general.

Let us now consider $\bar{\partial}_1^{m_1} \bar{G}^{(h)}_{2,L,\beta}(\mathbf{x})$ (see (3.6) for the definition of $\bar{\partial}_1$). By using (2.62) and the conservation of the spatial momentum, we find that $\bar{\partial}_1^{m_1}$ acts on $B(\mathbf{x}; \mathbf{y}_1, \ldots, \mathbf{y}_{2n})$, so that its Fourier transform is multiplied by $\sin(px)^{m_1}$, with $p = \sum_{r=1}^{2n} \sigma_r k'_r + 2\pi m$, where m is an arbitrary integer and p is chosen so that $|p| \le \pi$. If $m = 0$, we proceed as in the case of the time derivative, otherwise we note that the support properties of the external fields, see §2.2, implies that $|\sum_{r=1}^{2n} \sigma_r k'_r| \le 2na_0\gamma^h$; hence, if $|m| > 0$, $2n \ge (\pi/a_0)\gamma^{-h}$. Since the number of endpoints following $v_\mathbf{x}$ is proportional to $2n$ and each endpoint carries a small factor of order λ_1, it is clear that, if λ_1 is small enough, we get an improvement in the bound of the terms with $|m| > 0$, with respect to the corresponding contributions to $\bar{G}^{(h)}_{2,L,\beta}(\mathbf{x})$, of a factor $\exp(-C\gamma^{-h}) \le c_{m_1}\gamma^{hm_1}$, for some constant c_{m_1}. In the same manner, we can treat the operator D_{m_0,m_1}, so proving the bound (5.47) for $D_{m_0,m_1}\bar{G}^{(h)}_{2,L,\beta}(\mathbf{x})$.

Let us now consider $G^{(h)}_{1,L,\beta}(\mathbf{x}-\mathbf{y}) - r^{(h)}_{1,L,\beta}(\mathbf{x}-\mathbf{y})$. In this case the kernels of the two special endpoints $\mathbf{x}$ and $\mathbf{y}$ are equal to $\exp(2i\sigma_\mathbf{x} p_F x)$ and $\exp(2i\sigma_\mathbf{y} p_F y)$, respectively. However, since the propagators couple fields with different σ indices and all the other endpoints satisfy the condition $\sum_{f \in I_v} \sigma(f) = 0$, $\sigma_\mathbf{x} = -\sigma_\mathbf{y}$ and we can write

$$G^{(h)}_{1,L,\beta}(\mathbf{x}-\mathbf{y}) - r^{(h)}_{1,L,\beta}(\mathbf{x}-\mathbf{y}) = \frac{1}{2} \sum_{\sigma=\pm 1} e^{2i\sigma p_F(x-y)} \Big[\bar{G}^{(h)}_{1,\sigma}(\mathbf{x}-\mathbf{y}) + 2s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x}-\mathbf{y}) \Big] \,, \tag{5.64}$$

with $\bar{G}^{(h)}_{1,\sigma}(\mathbf{x})$ having the same properties as $\bar{G}^{(h)}_2(\mathbf{x})$; in particular it is an even function of $\mathbf{x}$ and satisfies the bound (5.47). Moreover, it is easy to see that $\bar{G}^{(h)}_{1,+}(\mathbf{x}-\mathbf{y})$ is equal to $\bar{G}^{(h)}_{1,-}(\mathbf{y}-\mathbf{x}) = \bar{G}^{(h)}_{1,-}(\mathbf{x}-\mathbf{y})$, hence $\bar{G}^{(h)}_{1,\sigma}(\mathbf{y}-\mathbf{x})$ is independent of σ and we get the decomposition in the first line of (5.44), with $\bar{G}^{(h)}_1(\mathbf{x}-\mathbf{y})$ satisfying (5.47) and (5.45).

The bound (5.48) is proved in the same way as the bound (5.47). The factor $[\gamma^{-\vartheta(h-h^*)} + \gamma^{\vartheta h}]$ in the r.h.s. comes from the fact that the trees contributing to $s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})$ and $s^{(h)}_{2,L,\beta}(\mathbf{x})$ have at least one vertex of type ν or δ, whose running constants satisfy (4.17) and (4.57).

Note that $s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})$ and $s^{(h)}_{2,L,\beta}(\mathbf{x})$ are not even functions of $\mathbf{x}$ and that $s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x})$ is not independent of σ.

In order to complete the proof of Theorem 5.8, we observe that all the functions appearing in the r.h.s. of (5.39), as well as those defined in (5.44), clearly converge, as $L,\beta\to\infty$, and that their limits can be represented in the same way as the finite L and β quantities, by substituting all the propagators with the corresponding limits. This follows from the tree structure of our expansions and some straightforward but lengthy standard arguments; we shall omit the details.

Let us consider, in particular, the limits $G^{(h)}_i(\mathbf{x})$ of the functions $G^{(h)}_{i,L,\beta}(\mathbf{x})$. Their tree expansions contain only trees with endpoints of scale $h_v \le 1$, which are associated with local terms of type λ or of the form (5.13) and (5.14), whose ψ fields are of scale less or equal to 0. The support properties of the field Fourier transform imply that the local terms of type λ can be rewritten by substituting the sum over the corresponding lattice space point with a continuous integral over $\mathbb{R}^1$. We can of course use these new expressions to build the expansions, since the propagators of scale $h\le 0$, in the limit $L,\beta\to\infty$, are well defined smooth functions on $\mathbb{R}^2$. For the same reason, the tree expansions are well defined also if the space points associated with the special endpoints vary over $\mathbb{R}^1$, instead of $\mathbb{Z}^1$; therefore there is a natural way to extend to $\mathbb{R}^2$ the functions $G^{(h)}_i(\mathbf{x})$, which of course satisfy the bound (5.47), with the continuous derivative ∂_1 in place of the discrete one and $|\mathbf{x}|$ in place of $|\mathbf{d}(\mathbf{x})|$, as well as the analogous of identity (5.45).

The function $G^{(h)}_{1,L,\beta}(\mathbf{x})$ satisfies also another symmetry relation, related with a remarkable property of the propagators $\bar{g}^{(h)}_{\omega,\omega'}$, see (5.61), appearing in its expansion, that is

$$\begin{aligned} \bar{g}^{(h)}_{\omega,\omega}(x,x_0) &= -i\omega \bar{g}^{(h)}_{-\omega,-\omega}\left(v_0^* x_0, \frac{x}{v_0^*}\right) , \\ \bar{g}^{(h)}_{\omega,-\omega}(x,x_0) &= -\bar{g}^{(h)}_{-\omega,+\omega}\left(v_0^* x_0, \frac{x}{v_0^*}\right) . \end{aligned} \tag{5.65}$$

On the other hand, each tree contributing to $G^{(h)}_{1,L,\beta}(\mathbf{x})$ with n normal endpoints (which are all of type λ) can be written as a sum of Feynman graphs (if we use the representation of the regularization operator as acting on the kernels, see §3), built by using $4n+4$ ψ fields, $2n+2$ with $\omega=+1$ and $2n+2$ with $\omega=-1$, hence containing the same number of propagators $\bar{g}^{(h)}_{+1,+1}$ and $\bar{g}^{(h)}_{-1,-1}$ and, by the argument used in the proof of (5.45), an even number of non diagonal propagators. Then, by using (5.65), we can easily show that the value of any graph, calculated at (x, x_0), is equal to the value at $(v_0^* x_0, x/v_0^*)$ of the graph with the same structure but opposite values for the ω-indices of all propagators, which implies (5.49).

6. Proof of Theorem 1.5

6.1 Theorem 3.12 and the analysis performed in §4 and §5 imply immediately the statements in item a) of Theorem 1.5, except the continuity of $\Omega^3_{L,\beta}(\mathbf{x})$ in $x_0 = 0$, which will be briefly discussed below. Hence, from now on we shall suppose that all parameters are chosen as in item a).

Let us define

$$\eta = \log_\gamma(1+z^*) \;, \qquad z^* = z_{[h^*/2]} \;, \tag{6.1}$$

z_h being defined as in (4.2). The analysis performed in §4 allows to show (we omit the details) that there exists a positive $\vartheta < 1$, such that

$$|z_h - z_{h+1}| \le C\lambda_1^2 [\gamma^{-\vartheta(h-h^*)} + \gamma^{\vartheta h}] \;, \quad h^* \le h \le 0 \;. \tag{6.2}$$

We can write

$$\log_\gamma Z_h = \sum_{h'=h+1}^{0} \log_\gamma [1 + z^* + (z_{h'} - z^*)] = -\eta h + \sum_{h'=h+1}^{0} r_{h'} \;. \tag{6.3}$$

On the other hand, if $h > [h^*/2]$, thanks to (6.2), $|r_h| \le C \sum_{h'=[h^*/2]}^{h-1} |z_{h'} - z_{h'+1}| \le C\lambda_1^2 \gamma^{\vartheta h}$ and, if $h \le [h^*/2]$, $|r_h| \le C\lambda_1^2 \gamma^{-\vartheta(h-h^*)}$; it follows that

$$|r_h| \le C\lambda_1^2 [\gamma^{-\vartheta(h-h^*)} + \gamma^{\vartheta h}] \;. \tag{6.4}$$

Hence, if we define

$$c_h = \frac{\gamma^{-\eta h}}{Z_{h-1}} \;, \tag{6.5}$$

we get immediately the bound

$$|c_h - 1| \le C\lambda_1^2 \;. \tag{6.6}$$

In a similar way, if we define

$$\tilde\eta_1 = \log_\gamma (1 + z^{(1)}_{[h^*/2]}) \;, \quad c^{(1)}_h = \frac{\gamma^{-\tilde\eta_1 h}}{Z^{(1)}_h} \;, \tag{6.7}$$

$z^{(1)}_h$ being defined by (5.18), we get the bound

$$|c^{(1)}_h - 1| \le C|\lambda_1| \;. \tag{6.8}$$

Bounds similar to (6.7) and (6.8) are valid also for the constants $Z^{(2)}_h$, but in this case Theorem 5.6 implies a stronger result; if we define

$$c^{(2)}_h = \frac{Z^{(2)}_h}{Z_{h-1}} \;, \tag{6.9}$$

then

$$|c^{(2)}_h - 1| \le C|\lambda_1| \;. \tag{6.10}$$

Let us now consider the terms in the first three lines of the r.h.s. of (5.39) and let us call $\Omega^{3,0}_{L,\beta}$ their sum; we can write

$$\Omega^{3,0}_{L,\beta}(\mathbf{x}) = \bar{\Omega}^{3,0}_{L,\beta}(\mathbf{x}) + \delta\Omega^{3,0}_{L,\beta}(\mathbf{x}) \;, \tag{6.11}$$

where $\bar{\Omega}^{3,0}_{L,\beta}$ is obtained from $\Omega^{3,0}_{L,\beta}$ by restricting the sums over h and h' to the values ≤ 0 and by substituting the propagators $g^{(h)}_{\omega,\omega'}$ with the propagators $\bar{g}^{(h)}_{\omega,\omega'}$, defined in (5.61). By using the symmetry relations

$$\begin{aligned} \bar{g}^{(h)}_{\omega,\omega}(x,x_0) &= -\bar{g}^{(h)}_{\omega,\omega}(-x,-x_0) = \bar{g}^{(h)}_{+,+}(\omega x,x_0) \;, \\ \bar{g}^{(h)}_{\omega,-\omega}(\mathbf{x}) &= \bar{g}^{(h)}_{\omega,-\omega}(-\mathbf{x}) = \omega\bar{g}^{(h)}_{+,-}(\mathbf{x}) \;, \end{aligned} \tag{6.12}$$

it is easy to show that we can write

$$\bar{\Omega}^{3,0}_{L,\beta}(\mathbf{x}) = \cos(2p_F x)\bar{\Omega}_{1,L,\beta}(\mathbf{x}) + \bar{\Omega}_{2,L,\beta}(\mathbf{x}) \;, \tag{6.13}$$

$$\begin{aligned} \bar{\Omega}_{1,L,\beta}(\mathbf{x}) = 2 \sum_{h^*\le h,h'\le 0} \frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} \Big[&\bar{g}^{(h)}_{+,+}(x,x_0)\bar{g}^{(h')}_{+,+}(-x,x_0) + \\ &+ \bar{g}^{(h)}_{+,-}(x,x_0)\bar{g}^{(h')}_{+,-}(x,x_0)\Big] \;, \end{aligned} \tag{6.14}$$

$$\begin{aligned} \bar{\Omega}_{2,L,\beta}(\mathbf{x}) = \sum_{h,h'\le 0} \frac{(Z^{(2)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} \Big[&\sum_\omega \bar{g}^{(h)}_{+,+}(\omega x,x_0)\bar{g}^{(h')}_{+,+}(\omega x,x_0) - \\ &- 2\bar{g}^{(h)}_{+,-}(x,x_0)\bar{g}^{(h')}_{+,-}(x,x_0)\Big] \;. \end{aligned} \tag{6.15}$$

By using (5.39), (5.44), (6.13) and the fact that $G^{(+1)}_{i,L,\beta}(\mathbf{x}) - r^{(+1)}_{i,L,\beta}(\mathbf{x}) = 0$ for $i = 1,2$, we can decompose $\Omega^3_{L,\beta}$ as in (1.14), by defining

$$\Omega^{3,a}_{L,\beta}(\mathbf{x}) = \bar{\Omega}_{1,L,\beta}(\mathbf{x}) + \sum_{h=h^*}^{0} \left(\frac{Z^{(1)}_h}{Z_h}\right)^2 \bar{G}^{(h)}_{1,L,\beta}(\mathbf{x}) \;, \tag{6.16}$$

$$\Omega^{3,b}_{L,\beta}(\mathbf{x}) = \bar{\Omega}_{2,L,\beta}(\mathbf{x}) + \sum_{h=h^*}^{0} \left(\frac{Z^{(2)}_h}{Z_h}\right)^2 \bar{G}^{(h)}_{2,L,\beta}(\mathbf{x}) \;, \tag{6.17}$$

$$\begin{aligned} \Omega^{3,c}_{L,\beta}(\mathbf{x}) = \delta\Omega^{3,0}_{L,\beta}(\mathbf{x}) + \sum_{h=h^*}^{1} \Bigg\{ &\left(\frac{Z^{(1)}_h}{Z_h}\right)^2 r^{(h)}_{1,L,\beta}(\mathbf{x}) + \left(\frac{Z^{(2)}_h}{Z_h}\right)^2 r^{(h)}_{2,L,\beta}(\mathbf{x}) + \\ &+ \frac{Z^{(1)}_h Z^{(2)}_h}{Z^2_h} G^{(h)}_{3,L,\beta}(\mathbf{x}) \Bigg\} + s_{L,\beta}(\mathbf{x}) \;, \end{aligned} \tag{6.18}$$

$$s_{L,\beta}(\mathbf{x}) = \sum_{h=h^*}^{0} \left\{ \sum_{\sigma=\pm1} e^{2i\sigma p_F x} \left(\frac{Z^{(1)}_h}{Z_h}\right)^2 s^{(h)}_{1,\sigma,L,\beta}(\mathbf{x}) + \left(\frac{Z^{(2)}_h}{Z_h}\right)^2 s^{(h)}_{2,L,\beta}(\mathbf{x}) \right\} \;. \tag{6.19}$$

Theorem 5.8 implies that $\Omega^{3,a}_{L,\beta}(\mathbf{x})$, $\Omega^{3,b}_{L,\beta}(\mathbf{x})$ and $s_{L,\beta}(\mathbf{x})$ are smooth functions of x_0, essentially because their expansions do not contain any graph with a propagator of scale $+1$ (this propagator has a discontinuity at $x_0 = 0$). The function $\Omega^{3,c}_{L,\beta}(\mathbf{x})$ is not differentiable at $x_0 = 0$, but it is in any case continuous, since all graphs contributing to it have a Fourier transform decaying at least as k_0^{-2} as $k_0 \to \infty$.

6.2 We want now to prove the bounds in item b) of Theorem 1.5. To start with, we consider the function $\bar{\Omega}_{1,L,\beta}(\mathbf{x})$ defined in (6.14) and note that it can be written in the form

$$\bar{\Omega}_{1,L,\beta}(\mathbf{x}) = \sum_{h=h^*}^{0} \left(\frac{Z_h^{(1)}}{Z_h} \right)^2 \bar{\Omega}_{1,L,\beta}^{(h)}(\mathbf{x}) \ , \tag{6.20}$$

with $\bar{\Omega}_{1,L,\beta}^{(h)}(\mathbf{x})$ satisfying a bound similar to that proved for $\bar{G}_{1,L,\beta}^{(h)}(\mathbf{x})$, see (5.47), that is

$$|D_{m_0,m_1} \bar{\Omega}_{1,L,\beta}^{(h)}(\mathbf{x})| \le C_{N,m_0,m_1} \frac{\gamma^{2h} \gamma^{h(m_0+m_1)}}{1 + [\gamma^h |\mathbf{d}(\mathbf{x})|]^N} \ . \tag{6.21}$$

This claim easily follows from Lemma 2.6, together with (6.5) and (6.6). Hence we can write, by using (5.47), (6.6), (6.8) and (6.21), given any positive integers n_0, n_1 and putting $n = n_0 + n_1$,

$$\begin{aligned} |\partial_{x_0}^{n_0} \bar{\partial}_x^{n_1} \Omega_{L,\beta}^{3,a}(\mathbf{x})| &\le C_{N,n} \sum_{h=h^*}^{0} \frac{\gamma^{(2+2\eta_1+n)h}}{[1 + (\gamma^h |\mathbf{d}(\mathbf{x})|)^N]} \le \\ &\le \frac{C_{N,n}}{|\mathbf{d}(\mathbf{x})|^{2+2\eta_1+n}} H_{N,2+2\eta_1+n}(|\mathbf{d}(\mathbf{x})|) \ , \end{aligned} \tag{6.22}$$

where

$$\eta_1 = \eta - \tilde{\eta}_1 \ , \tag{6.23}$$

$$H_{N,\alpha}(r) = \sum_{h=h^*}^{0} \frac{(\gamma^h r)^\alpha}{1 + (\gamma^h r)^N} \ . \tag{6.24}$$

By using the second of the definitions (2.2), the definition (4.8) and the bounds (4.16), (5.29), one can see that the constant η_1 can be represented as in (1.15).

On the other hand, it is easy to see that, if $\alpha \ge 1/2$ and $N - \alpha \ge 1$, there exists a constant $C_{N,\alpha}$ such that

$$H_{N,\alpha}(r) \le \frac{C_{N,\alpha}}{1 + (\Delta r)^{N-\alpha}} \ , \quad \Delta = \gamma^{h^*} \ . \tag{6.25}$$

The definition (2.40), the first of definitions (2.33), the second bound in (2.34) and the bound (4.56) easily imply that Δ can be represented as in (1.20), with η_2 satisfying the second of equations (1.15).

By using (6.22) and (6.25), one immediately gets the bound (1.17). A similar procedure allows to get also the bound (1.18), by using (6.10).

Let us now consider $\Omega_{L,\beta}^{3,c}(\mathbf{x})$. By using (5.43) and (5.46), as well as the remark that one gains a factor γ^h in the bound of $g_{\omega,\omega'}^{(h)}(\mathbf{x}) - \bar{g}_{\omega,\omega'}^{(h)}(\mathbf{x})$ with respect to the bound of $\bar{g}_{\omega,\omega'}^{(h)}(\mathbf{x})$, we get

$$|\Omega_{L,\beta}^{3,c}(\mathbf{x}) - s_{L,\beta}(\mathbf{x})| \le \frac{C_N}{|\mathbf{d}(\mathbf{x})|^2} \left[\frac{H_{N,2+2\eta_1+\vartheta}(|\mathbf{d}(\mathbf{x})|)}{|\mathbf{d}(\mathbf{x})|^{\vartheta+2\eta_1}} + \frac{H_{N,2+\vartheta}(|\mathbf{d}(\mathbf{x})|)}{|\mathbf{d}(\mathbf{x})|^{\vartheta}} \right] \ , \tag{6.26}$$

for some positive $\vartheta < 1$.

The bound of $s_{L,\beta}(\mathbf{x})$ is slightly different, because of the $\gamma^{-\vartheta(h-h^*)}$ in the r.h.s. of (5.48). We get, in addition to a term of the same form as the r.h.s. of (6.26), another term of the form

$$\frac{C_N}{|\mathbf{d}(\mathbf{x})|^2}(\Delta|\mathbf{d}(\mathbf{x})|)^\vartheta \left[\frac{H_{N,2+2\eta_1-\vartheta}(|\mathbf{d}(\mathbf{x})|)}{|\mathbf{d}(\mathbf{x})|^{2\eta_1}} + H_{N,2-\vartheta}(|\mathbf{d}(\mathbf{x})|)\right] . \tag{6.27}$$

The bounds (6.26) and (6.27) immediately imply (1.19), if λ is so small that, for example, $2|\eta_1| \le \vartheta/2$.

6.3 We want now to prove the statements in item c) of Theorem 1.5. The existence of the limit as $L,\beta \to \infty$ of all functions follows from Theorem 5.8. The claim that $\Omega^{3,a}(\mathbf{x})$ and $\Omega^{3,b}(\mathbf{x})$ are even as functions of $\mathbf{x}$ follows from (5.45) and (6.14)-(6.18). Moreover $\Omega^{3,a}(\mathbf{x})$ and $\Omega^{3,b}(\mathbf{x})$ are the restriction to $\mathbb{Z}\times\mathbb{R}$ of two functions on $\mathbb{R}^2$, that we shall denote by the same symbols, and $\Omega^{3,a}(\mathbf{x})$ satisfies the symmetry relation (1.23), since this is true for $\lim_{L,\beta\to\infty}\bar\Omega_{1,L,\beta}(\mathbf{x})$, as it is easy to check by using (5.65), and for $\bar G_1^{(h)}(\mathbf{x})$, see (5.49).

In order to prove (1.21), we suppose that $|\mathbf{x}| \ge 1$ and we put $\bar\Omega_i(\mathbf{x}) = \lim_{L,\beta\to\infty}\bar\Omega_{i,L,\beta}(\mathbf{x})$; then we define $\tilde\Omega_i(\mathbf{x})$, $i=1,2$, as the functions which are obtained by making in the r.h.s. of (6.14) and (6.15), evaluated in the limit $L,\beta\to\infty$, the substitutions

$$\frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} \to [x^2+(v_0^*x_0)^2]^{-\eta_1} , \qquad \frac{(Z^{(2)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} \to 1 . \tag{6.28}$$

Note that the choice of $x^2+(v_0^*x_0)^2$, instead of $x^2+x_0^2$, which is equivalent for what concerns the following arguments, was done only in order to have a function $\tilde\Omega_1(\mathbf{x})$ satisfying the same symmetry relation as $\bar\Omega_1(\mathbf{x})$ in the exchange of (x,x_0) with $(v_0^*x_0, x/v_0^*)$.

It is easy to see that

$$\begin{aligned} |\bar\Omega_1(\mathbf{x})-\tilde\Omega_1(\mathbf{x})| \le \frac{C_N}{|\mathbf{x}|^{2+2\eta_1}} \sum_{h^*\le h,h'\le 0} \frac{\gamma^h|\mathbf{x}|}{1+(\gamma^h|\mathbf{x}|)^N}\frac{\gamma^{h'}|\mathbf{x}|}{1+(\gamma^{h'}|\mathbf{x}|)^N} \cdot \\ \cdot \left| \left(\frac{x^2+x_0^2}{x^2+(v_0^*x_0)^2}\right)^{\eta_1} (\gamma^h|\mathbf{x}|)^\eta(\gamma^{h'}|\mathbf{x}|)^\eta(\gamma^{h\vee h'}|\mathbf{x}|)^{-2\tilde\eta_1}\frac{c_hc_{h'}}{(c^{(1)}_{h\vee h'})^2} - 1\right| . \end{aligned} \tag{6.29}$$

Note that, if $r>0$ and $\alpha\in\mathbb{R}$

$$|r^\alpha - 1| \le |\alpha\log r|\left(r^\alpha + r^{-\alpha}\right) ; \tag{6.30}$$

Hence, by using (6.6), (6.8), (6.25) and (1.15), we get

$$|\bar\Omega_1(\mathbf{x})-\tilde\Omega_1(\mathbf{x})| \le \frac{|J_3|}{|\mathbf{x}|^{2+2\eta_1}}\frac{C_N}{1+(\Delta|\mathbf{x}|)^N} . \tag{6.31}$$

In the same way, one can show that

$$|\bar\Omega_2(\mathbf{x})-\tilde\Omega_2(\mathbf{x})| \le \frac{|J_3|}{|\mathbf{x}|^2}\frac{C_N}{1+(\Delta|\mathbf{x}|)^N} . \tag{6.32}$$

Let us now define

$$\Omega_1^*(\mathbf{x}) = \frac{2}{[x^2+(v_0^*x_0)^2]^{\eta_1}}\frac{1}{(v_0^*)^2} g_{\mathcal{L}}(x/v_0^*, x_0) g_{\mathcal{L}}(-x/v_0^*, x_0) , \tag{6.33}$$

$$\Omega_2^*(\mathbf{x}) = \frac{1}{(v_0^*)^2} \sum_{\omega=\pm1} g_{\mathcal{L}}(\omega x/v_0^*, x_0) g_{\mathcal{L}}(\omega x/v_0^*, x_0) \;, \tag{6.34}$$

where

$$g_{\mathcal{L}}(\mathbf{x}) = \frac{1}{(2\pi)^2} \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_0(\mathbf{k})}{-ik_0 + k} \;, \tag{6.35}$$

$\chi_0(\mathbf{k})$ being a smooth function of $\mathbf{k}$, which is equal to 1, if $|\mathbf{k}| \le t_0$, and equal to 0, if $|\mathbf{k}| \ge \gamma t_0$ (see §2.3 for the definition of t_0).

It is easy to check that $\Omega_i^*(\mathbf{x})$, $i = 1, 2$, is obtained from $\tilde{\Omega}_i(\mathbf{x})$ by making in the $L, \beta = \infty$ expression of the propagators $\bar{g}^{(h)}_{\omega,\omega'}(\mathbf{x})$, which are evaluated from (2.92), if $h^* < h \le 0$, and (2.121), if $h = h^*$, the following substitutions:

$$\sigma_{h-1}(\mathbf{k}') \to 0 \;, \qquad \tilde{f}_h(\mathbf{k}') \to f_h(\mathbf{k}') \;. \tag{6.36}$$

Hence, by using also the remark that, by (2.116) and (4.54), $|\sigma_h/\gamma^h| \le C\gamma^{-(h-h^*)/2}$, it is easy to show that

$$|\Omega_1^*(\mathbf{x}) - \tilde{\Omega}_1(\mathbf{x})| \le \frac{C_N}{|\mathbf{x}|^{2+2\eta_1}} H_{N,1}(\Delta|\mathbf{x}|) \left[\lambda_1^2 H_{N,1}(\Delta|\mathbf{x}|) + (\Delta|\mathbf{x}|)^{1/2} H_{N,1/2}(\Delta|\mathbf{x}|)\right] \;. \tag{6.37}$$

In a similar way, one can show also that

$$|\Omega_2^*(\mathbf{x}) - \tilde{\Omega}_2(\mathbf{x})| \le \frac{C_N}{|\mathbf{x}|^2} H_{N,1}(\Delta|\mathbf{x}|) \left[\lambda_1^2 H_{N,1}(\Delta|\mathbf{x}|) + (\Delta|\mathbf{x}|)^{1/2} H_{N,1/2}(\Delta|\mathbf{x}|)\right] \;. \tag{6.38}$$

Moreover, by an explicit calculation, one finds that, if $|\mathbf{x}| \ge 1$,

$$g_{\mathcal{L}}(\mathbf{x}) = \frac{x_0 - ix}{2\pi|\mathbf{x}|^2} F(\mathbf{x}) \;, \tag{6.39}$$

where $F(\mathbf{x})$ is a smooth function of $\mathbf{x}$, satisfying the bound

$$|F(\mathbf{x}) - 1| \le \frac{C_N}{1 + |\mathbf{x}|^N} \;. \tag{6.40}$$

The bounds (6.31) and (6.32), the similar bounds satisfied by $|\Omega^{3,a}(\mathbf{x}) - \bar{\Omega}_1(\mathbf{x})|$ and $|\Omega^{3,b}(\mathbf{x}) - \bar{\Omega}_2(\mathbf{x})|$ and the equations (6.37)-(6.40) allow to prove very easily (1.21) and (1.22).

6.4 We still have to prove the statements in items d) and e) of Theorem 1.5. By using (1.14), (6.18) and (6.19), we see that

$$\begin{aligned} \hat{\Omega}^3(\mathbf{k}) = & \sum_{\sigma=\pm1} \left[\frac{1}{2}\hat{\Omega}^{3,a}(k + 2\sigma p_F, k_0) + \hat{s}_{1,\sigma}(k + 2\sigma p_F, k_0)\right] + \\ & + \hat{\Omega}^{3,b}(\mathbf{k}) + \hat{s}_2(\mathbf{k}) + \delta\hat{\Omega}^{3,c}(\mathbf{k}) \;, \end{aligned} \tag{6.41}$$

where we used the definitions

$$s_{1,\sigma}(\mathbf{x}) = \sum_{h=h^*}^{0} \left(\frac{Z_h^{(1)}}{Z_h}\right)^2 s_{1,\sigma}^{(h)}(\mathbf{x}) \;, \quad s_2(\mathbf{x}) = \sum_{h=h^*}^{0} \left(\frac{Z_h^{(2)}}{Z_h}\right)^2 s_2(h)(\mathbf{x}) \;, \tag{6.42}$$

$$\delta\Omega^{3,c}(\mathbf{x}) = \Omega^{3,c}(\mathbf{x}) - s(\mathbf{x}) \;. \tag{6.43}$$

Since any graph contributing to the expansion of $\Omega^{3,a}(\mathbf{x}-\mathbf{y})$ has only two propagators of scale ≤ 0 connected to $\mathbf{x}$ or $\mathbf{y}$, $\hat{\Omega}^{3,a}(\mathbf{k})$ has support on a set of value of $\mathbf{k}$ such that $|k| \leq 2\gamma t_0 < \pi$; hence we can calculate $\hat{\Omega}^{3,a}(\mathbf{k})$ by thinking $\Omega^{3,a}(\mathbf{x})$ as a function on $\mathbb{R}^2$. Let us suppose that $|\mathbf{k}| > 0$ and $|k| \geq |\mathbf{k}|/2$; then

$$\hat{\Omega}^{3,a}(\mathbf{k}) = \int d\mathbf{x} e^{i\mathbf{k}\mathbf{x}} \Omega^{3,a}(\mathbf{x}) = \frac{i}{k} \int d\mathbf{x} \left[e^{i\mathbf{k}\mathbf{x}} - 1\right] \partial_x \Omega^{3,a}(\mathbf{x}) \;, \tag{6.44}$$

since $\Omega^{3,a}(\mathbf{x})$, by (1.17), is a smooth function of fast decrease as $|\mathbf{x}| \to \infty$. If $|k| < |\mathbf{k}|/2$, it has to be true that $|k_0| \geq |\mathbf{k}|/2$ and we write a similar identity, with k_0 in place of k and ∂_{x_0} in place of ∂_x. In both case we can write, by using (1.17),

$$|\hat{\Omega}^{3,a}(\mathbf{k})| \leq \frac{C}{|\mathbf{k}|} \int_{|\mathbf{x}| \geq |\mathbf{k}|^{-1}} \frac{d\mathbf{x}}{1 + |\mathbf{x}|^{3+2\eta_1}} + C \int_{|\mathbf{x}| \leq |\mathbf{k}|^{-1}} d\mathbf{x} \frac{|\mathbf{x}|}{1 + |\mathbf{x}|^{3+2\eta_1}} \;. \tag{6.45}$$

A even better bound can be proved for $|\hat{s}_{1,\sigma}(\mathbf{k})|$, $\sigma = \pm 1$, by using (5.48). Hence, uniformly for $u \to 0$, $|\hat{\Omega}^{3,a}(\mathbf{k})| + |\hat{s}_{1,\sigma}(\mathbf{k})| \leq C|\mathbf{k}|^{-1}$ for $|\mathbf{k}| \geq 1$ and

$$\frac{1}{2}|\hat{\Omega}^{3,a}(\mathbf{k})| + |\hat{s}_{1,\sigma}(\mathbf{k})| \leq C\left[1 + \frac{1 - |\mathbf{k}|^{2\eta_1}}{2\eta_1}\right] \;, \quad 0 < |\mathbf{k}| \leq 1 \;. \tag{6.46}$$

This bound is divergent for $|\mathbf{k}| \to 0$, if $\eta_1 < 0$, that is if $J_3 < 0$; however, if $u \neq 0$ and $|\mathbf{k}| \leq \Delta$, we easily get from (1.17) (with $n = 0$) the better bound

$$\frac{1}{2}|\hat{\Omega}^{3,a}(\mathbf{k})| + |\hat{s}_{1,\sigma}(\mathbf{k})| \leq C\left[1 + \frac{1 - \Delta^{2\eta_1}}{2\eta_1}\right] \;. \tag{6.47}$$

In a similar way, by using (1.18), one can prove that

$$|\hat{\Omega}^{3,b}(\mathbf{k})| + |\hat{s}_2(\mathbf{k})| \leq C\left[1 + \log|\mathbf{k}|^{-1}\right] \;, \quad 0 < |\mathbf{k}| \leq 1 \;, \tag{6.48}$$

$$|\hat{\Omega}^{3,b}(\mathbf{k})| + |\hat{s}_2(\mathbf{k})| \leq C\left[1 + \log \Delta^{-1}\right] \;. \tag{6.49}$$

However, a more careful analysis of the Fourier transform of the leading contribution to $\Omega^{3,b}(\mathbf{x})$, given by $\Omega_2^*(\mathbf{x})$ (see (6.34)), which takes into account the oddness in the exchange $(x, x_0) \to (x_0 v_0^*, x/v_0^*)$, shows that $|\hat{\Omega}_2^*(\mathbf{k})| \leq C$. One can show that a similar bound is satisfied by the Fourier transform of the terms contributing to $\tilde{\Omega}_2(\mathbf{x})$ and proportional to σ_h/γ^h. Therefore, in the bounds (6.48) and (6.49), we can multiply by J_3 both $\log|\mathbf{k}|^{-1}$ and $\log \Delta^{-1}$.

Let us now consider $\hat{\delta\Omega}^{3,c}(\mathbf{k})$. By using (6.26), we see immediately that, uniformly in $\mathbf{k}$ and u,

$$|\hat{\delta\Omega}^{3,c}(\mathbf{k}) - \hat{s}(\mathbf{k})| \leq C \;. \tag{6.50}$$

The bounds (6.46)-(6.50), together with the positivity of the leading term in (1.21) and the remark after (6.49), immediately imply all the claims in item d) of Theorem 1.5.

Let us now consider $G(x) \equiv \Omega^3(x, 0)$, $x \in \mathbb{Z}$. It is easy to see, by using the previous results and the fact that also $s_{1,\sigma}(\mathbf{x})$ and $s_2(\mathbf{x})$ are even functions of $\mathbf{x}$, that $G(x)$ can be written in the form

$$G(x) = \sum_{\sigma=\pm 1} e^{2i\sigma p_F x} G_{1,\sigma}(x) + G_2(x) + \delta G(x) \;, \tag{6.51}$$

where $G_{1,\sigma}(x)$ and $G_2(x)$ are the restrictions to $\mathbb{Z}$ of some even smooth functions on $\mathbb{R}$, satisfying, for any integers $n, N \geq 0$, the bounds

$$|\partial_x^n G_{1,\sigma}(x)| \leq \frac{C_{n,N}}{[1+|x|^{2+n+2\eta_1}][1+(\Delta|x|)^N]} , \tag{6.52}$$

$$|\partial_x^n G_2(x)| \leq \frac{C_{n,N}}{1+|x|^{2+n}[1+(\Delta|x|)^N]} , \tag{6.53}$$

while $\delta G(x)$ satisfies the bound

$$|\delta G(x)| \leq \frac{C}{[1+|x|^{2+\vartheta}][1+(\Delta|x|)^N]} , \tag{6.54}$$

with some $\vartheta > 0$.

These properties immediately imply that, uniformly in k and u,

$$|\hat{G}(k)| + |\partial_k \delta \hat{G}(k)| \leq C . \tag{6.55}$$

Let us now consider $\partial_k \hat{G}_{1,\sigma}(k)$ and note that, if $|k| > 0$,

$$\begin{aligned} \partial_k \hat{G}_{1,\sigma}(k) &= -\frac{1}{k}\int dx[e^{ikx}-1]\partial_x[xG_{1,\sigma}(x)] = \\ &= -\frac{1}{k}\int_{|x|\geq|k|^{-1}} dx[e^{ikx}-1]\partial_x[xG_{1,\sigma}(x)] - \\ &\quad -\frac{1}{k}\int_{|x|\leq|k|^{-1}} dx[e^{ikx}-1-ikx]\partial_x[xG_{1,\sigma}(x)] , \end{aligned} \tag{6.56}$$

where we used the fact that $\partial_x[xG_{1,\sigma}(x)]$ is an even function of x, since $G_{1,\sigma}(x)$ is even, see (5.45). Hence, if $|k| \geq 1$, $|\partial_k \hat{G}_{1,\sigma}(k)| \leq C|k|^{-1}$, while, if $0 < |k| \leq 1$, uniformly in u,

$$|\partial_k \hat{G}_{1,\sigma}(k)| \leq C[1+|k|^{2\eta_1}] . \tag{6.57}$$

In a similar way, we can prove that, uniformly in k and u,

$$|\partial_k \hat{G}_2(k)| \leq C . \tag{6.58}$$

The bound (6.57) is divergent for $k \to 0$, if $J_3 < 0$; however, if $|u| > 0$ and $|k| \leq \Delta$, one can get a better bound, by using the identity

$$\partial_k \hat{G}_{1,\sigma}(k) = i\int_{|x|\geq\Delta^{-1}} dx e^{ikx}[xG_{1,\sigma}(x)] + i\int_{|x|\leq\Delta^{-1}} dx[e^{ikx}-1][xG_{1,\sigma}(x)] , \tag{6.59}$$

together with (6.52). One finds

$$|\partial_k \hat{G}_{1,\sigma}(k)| \leq C[1+\Delta^{2\eta_1}] . \tag{6.60}$$

The bounds (6.55), (6.58) and (6.60), together with the identity (6.51), imply (1.24). The statements about the discontinuities of $\partial_k \hat{G}(k)$ at $u=0$ and $k=0,\pm 2p_F$ follow from an explicit calculation involving the leading contribution, obtained by putting $A_1(\mathbf{x}) = A_2(\mathbf{x}) = 0$ in (1.21).

7. Proof of the approximate Ward identity (5.35)

7.1 In this section we prove the relation (5.35) between the quantities $Z_h^{(L)}$ and $Z_h^{(2,L)}$, related to the approximate Luttinger model defined by (5.30) and (5.31).

First of all, we move from the interaction to the free measure (4.30) the term proportional to $\delta_0^{(L)}$ and we redefine correspondingly the interaction. This can be realized by slightly changing the free measure normalization (which has no effect on the problem we are studying), by putting $\delta_0^{(L)} = 0$ in (4.31) and by substituting, in (4.30), v_0^* with $\bar v_0(\mathbf{k}') = v_0^* + \delta_0^{(L)} C_0^{-1}(\mathbf{k}')$. However, since $C_0^{-1}(\mathbf{k}') = 1$ on all scales $h < 0$, $Z_h^{(2,L)}$ and $Z_h^{(L)}$ may be modified only by a factor $\gamma^{C|\lambda_0^{(L)}|}$, if we substitute $\bar v_0(\mathbf{k}')$ with $\bar v_0 \equiv \bar v_0(\mathbf{0})$. It follows that it is sufficient to prove the bound (5.35) for the corresponding free measure

$$
\begin{aligned}
P^{(L)}(d\psi^{(\le 0)}) = &\prod_{\mathbf{k}': C_0^{-1}(\mathbf{k}')>0} \prod_{\omega=\pm 1} \frac{d\hat\psi^{(\le 0)+}_{\mathbf{k}',\omega} d\hat\psi^{(\le 0)-}_{\mathbf{k}',\omega}}{\mathcal{N}_L(\mathbf{k}')} \cdot \\
&\cdot \exp\left\{ -\frac{1}{L\beta} \sum_{\omega=\pm 1} \sum_{\mathbf{k}': C_0^{-1}(\mathbf{k}')>0} C_0(\mathbf{k}')\big(-ik_0 + \omega \bar v_0 k'\big) \hat\psi^{(\le 0)+}_{\mathbf{k}',\omega} \hat\psi^{(\le 0)-}_{\mathbf{k}',\omega} \right\} ,
\end{aligned}
\tag{7.1}
$$

by using as interaction the function

$$
V^{(L)}(\psi^{(\le 0)}) = \lambda_0^{(L)} \int_{\mathbb{T}_{L,\beta}} d\mathbf{x}\, \psi^{(\le 0)+}_{\mathbf{x},+1} \psi^{(\le 0)-}_{\mathbf{x},-1} \psi^{(\le 0)+}_{\mathbf{x},-1} \psi^{(\le 0)-}_{\mathbf{x},+1} \; . \tag{7.2}
$$

Let us consider, instead of the free measure (7.1), the corresponding measure with *infrared cutoff on scale* h, $h \le 0$, given by

$$
\begin{aligned}
P^{(L,h)}(d\psi^{[h,0]}) = &\prod_{\mathbf{k}': C_{h,0}^{-1}(\mathbf{k}')>0} \prod_{\omega=\pm 1} \frac{d\hat\psi^{[h,0]+}_{\mathbf{k}',\omega} d\hat\psi^{[h,0]-}_{\mathbf{k}',\omega}}{\mathcal{N}_L(\mathbf{k}')} \cdot \\
&\cdot \exp\left\{ -\frac{1}{L\beta} \sum_{\omega=\pm 1} \sum_{\mathbf{k}': C_{h,0}^{-1}(\mathbf{k}')>0} C_{h,0}(\mathbf{k}')\big(-ik_0 + \omega \bar v_0 k'\big) \hat\psi^{[h,0]+}_{\mathbf{k}',\omega} \hat\psi^{[h,0]-}_{\mathbf{k}',\omega} \right\} ,
\end{aligned}
\tag{7.3}
$$

where $C_{h,0}^{-1} = \sum_{k=h}^0 f_k$.

We will find convenient to write the above integration in terms of the space-time field variables; if we put

$$
\mathcal{D}\psi^{[h,0]} = \prod_{\mathbf{k}': C_{h,0}^{-1}(\mathbf{k}')>0} \prod_{\omega=\pm 1} \frac{d\hat\psi^{[h,0]+}_{\mathbf{k}',\omega} d\hat\psi^{[h,0]-}_{\mathbf{k}',\omega}}{\mathcal{N}_L(\mathbf{k}')} , \tag{7.4}
$$

we can rewrite (7.3) as

$$
P^{(L,h)}(d\psi^{[h,0]}) = \mathcal{D}\psi^{[h,0]} \exp\Big[-\sum_\omega \int_{\mathbb{T}_{L,\beta}} d\mathbf{x}\, \psi^{[h,0]+}_{\mathbf{x},\omega} D^{[h,0]}_\omega \psi^{[h,0]-}_{\mathbf{x},\omega} \Big] , \tag{7.5}
$$

where

$$
D^{[h,0]}_\omega \psi^{[h,0]\sigma}_{\mathbf{x},\omega} = \frac{1}{L\beta} \sum_{\mathbf{k}': C_{h,0}^{-1}(\mathbf{k}')>0} e^{i\sigma\mathbf{k}'\mathbf{x}} C_{h,0}(\mathbf{k}')(i\sigma k_0 - \omega\sigma \bar v_0 k') \hat\psi^{[h,0]\sigma}_{\mathbf{k}',\omega} \; . \tag{7.6}
$$

$D_\omega^{[h,0]}$ has to be thought as a "regularization" of the linear differential operator

$$D_\omega = \frac{\partial}{\partial x_0} + i\omega \bar v_0 \frac{\partial}{\partial x} \; . \tag{7.7}$$

Let us now introduce the external field variables $\phi^\sigma_{\mathbf{x},\omega}$, $\mathbf{x} \in \mathbb{T}_{L,\beta}$, $\omega = \pm 1$, antiperiodic in x_0 and x and anticommuting with themselves and $\psi^{[h,0]\sigma}_{\mathbf{x},\omega}$, and let us define

$$U(\phi) = -\log \int P^{(L,h)}(d\psi^{[h,0]}) e^{-V^{(L)}(\psi^{[h,0]}+\phi)} \; . \tag{7.8}$$

If we perform the *gauge transformation*

$$\psi^{[h,0]\sigma}_{\mathbf{x},\omega} \to e^{i\sigma\alpha_{\mathbf{x}}} \psi^{[h,0]\sigma}_{\mathbf{x},\omega} \; , \tag{7.9}$$

and we define $(e^{-i\alpha}\phi)^\sigma_{\mathbf{x},\omega} = e^{-i\sigma\alpha_{\mathbf{x}}}\phi^\sigma_{\mathbf{x},\omega}$, we get

$$\begin{aligned} U(\phi) = -\log \int P^{(L,h)}(d\psi^{[h,0]}) \exp\Big\{ - V^{(L)}(\psi^{[h,0]} + e^{-i\alpha}\phi) - \\ - \sum_\omega \int d\mathbf{x}\; \psi^{[h,0]+}_{\mathbf{x},\omega}\Big(e^{i\alpha_{\mathbf{x}}} D^{[h,0]}_\omega e^{-i\alpha_{\mathbf{x}}} - D^{[h,0]}_\omega \Big)\psi^{[h,0]-}_{\mathbf{x},\omega} \Big\} \; . \end{aligned} \tag{7.10}$$

Since $U(\phi)$ is independent of α, the functional derivative of the r.h.s. of (7.10) w.r.t. $\alpha_{\mathbf{x}}$ is equal to 0 for any $\mathbf{x} \in \mathbb{T}_{L,\beta}$. Hence, we find the following identity:

$$\sum_\omega \left[-\phi^+_{\mathbf{x},\omega} \frac{\partial U}{\partial \phi^+_{\mathbf{x},\omega}} + \frac{\partial U}{\partial \phi^-_{\mathbf{x},\omega}} \phi^-_{\mathbf{x},\omega} + \frac{1}{Z(\phi)} \int P^{(L,h)}(d\psi^{[h,0]}) T_{\mathbf{x},\omega}\, e^{-V^{(L)}(\psi^{[h,0]}+\phi)} \right] = 0 \; , \tag{7.11}$$

where

$$Z(\phi) = \int P^{(L,h)}(d\psi^{[h,0]}) e^{-V^{(L)}(\psi^{[h,0]}+\phi)} \; , \tag{7.12}$$

$$\begin{aligned} T_{\mathbf{x},\omega} &= \psi^{[h,0]+}_{\mathbf{x},\omega} [D^{[h,0]}_\omega \psi^{[h,0]-}_{\mathbf{x},\omega}] + [D^{[h,0]}_\omega \psi^{[h,0]+}_{\mathbf{x},\omega}] \psi^{[h,0]-}_{\mathbf{x},\omega} = \\ &= \frac{1}{(L\beta)^2} \sum_{\mathbf{p},\mathbf{k}} e^{-i\mathbf{p}\mathbf{x}} \hat\psi^{[h,0],+}_{\mathbf{k},\omega} [C_{h,0}(\mathbf{p}+\mathbf{k}) D_\omega(\mathbf{p}+\mathbf{k}) - C_{h,0}(\mathbf{k}) D_\omega(\mathbf{k})] \hat\psi^{[h,0],-}_{\mathbf{p}+\mathbf{k},\omega} \; , \end{aligned} \tag{7.13}$$

$$D_\omega(\mathbf{k}) = -ik_0 + \omega \bar v_0 k \; . \tag{7.14}$$

Moreover, the sum over $\mathbf{p}$ and $\mathbf{k}$ in (7.13) is restricted to the momenta of the form $\mathbf{p} = (2\pi n/L, 2\pi m/\beta)$ and $\mathbf{k} = (2\pi(n+1/2)/L, 2\pi(m+1/2)/\beta)$, with n and m integers, such that $|p|$, $|p_0|$, $|k_0|$, $|k|$ are all smaller or equal to π and satisfy the constraints $C^{-1}_{h,0}(\mathbf{p}+\mathbf{k}) > 0$, $C^{-1}_{h,0}(\mathbf{k}) > 0$.

Note that (7.13) can be rewritten as

$$T_{\mathbf{x},\omega} = D_\omega [\psi^{[h,0]+}_{\mathbf{x},\omega} \psi^{[h,0]-}_{\mathbf{x},\omega}] + \delta T_{\mathbf{x},\omega} \; , \tag{7.15}$$

where

$$\begin{aligned} \delta T_{\mathbf{x},\omega} = \frac{1}{(L\beta)^2} \sum_{\mathbf{p},\mathbf{k}} e^{-i\mathbf{p}\mathbf{x}} \hat\psi^{[h,0],+}_{\mathbf{k},\omega} \cdot \\ \cdot \; \big\{ [C_{h,0}(\mathbf{p}+\mathbf{k}) - 1] D_\omega(\mathbf{p}+\mathbf{k}) - [C_{h,0}(\mathbf{k}) - 1] D_\omega(\mathbf{k}) \big\} \hat\psi^{[h,0],-}_{\mathbf{p}+\mathbf{k},\omega} \; . \end{aligned} \tag{7.16}$$

It follows that, if $C_{h,0}$ is substituted with 1, that is if we consider the formal theory without any ultraviolet and infrared cutoff, $T_{\mathbf{x},\omega} = D_\omega[\psi^{[h,0]+}_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}]$ and we would get the usual Ward identities. As we shall see, the presence of the cutoffs make the analysis a bit more involved and adds some corrections to the Ward identities, which however, for λ_0 small enough, can be controlled by the same type of multiscale analysis, that we used in §5.

7.2 Let us introduce a new external field $J_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{T}_{L,\beta}$, periodic in x_0 and x and commuting with the fields ϕ^σ and $\psi^{[h,0]\sigma}$, and let us consider the functional

$$\mathcal{W}(\phi, J) = -\log \int P^{(L,h)}(d\psi^{[h,0]}) e^{-V^{(L)}(\psi^{[h,0]}+\phi)+\int d\mathbf{x} J_{\mathbf{x}} \sum_\omega \psi^{[h,0]+}_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}} . \tag{7.17}$$

We also define the functions

$$\Sigma_{h,\omega}(\mathbf{x}-\mathbf{y}) = \frac{\partial^2}{\partial\phi^+_{\mathbf{x},\omega}\partial\phi^-_{\mathbf{y},\omega}} U(\phi)\Big|_{\phi=0} = \frac{\partial^2}{\partial\phi^+_{\mathbf{x},\omega}\partial\phi^-_{\mathbf{y},\omega}} \mathcal{W}(\phi, J)\Big|_{\phi=J=0} , \tag{7.18}$$

$$\Gamma_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{\partial}{\partial J_{\mathbf{x}}} \frac{\partial^2}{\partial\phi^+_{\mathbf{y},\omega}\partial\phi^-_{\mathbf{z},\omega}} \mathcal{W}(\phi, J)|_{\phi=J=0} . \tag{7.19}$$

These functions have here the role of the *self-energy* and the *vertex part* in the usual treatment of the Ward identities. However, they do not coincide with them, because the corresponding Feynman graphs expansions are not restricted to the one particle irreducible graphs. However, their Fourier transforms at zero external momenta, which are the interesting quantities in the limit $L, \beta \to \infty$, are the same; in fact, because of the support properties of the fermion fields, the propagators vanish at zero momentum, hence the one particle reducible graphs give no contribution at that quantities.

In the language of this paper, if we did not perform any free measure regularization, $\Sigma_{h,\omega}(\mathbf{x}-\mathbf{y})$ would coincide with the kernel of the contribution to the effective potential on scale $h-1$ with two external fields, that is the function $W^{(h-1)}_{2,(+,-),(\omega,\omega)}$ of equation (3.3). Analogously, $1 + \Gamma_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z})$ would coincide with the kernel $B^{(h-1)}_{1,2,(+,-),(\omega,\omega)}$ of equation (5.6).

Note that

$$\Gamma_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = \sum_{\tilde\omega} \Gamma_{h,\omega,\tilde\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) , \tag{7.20}$$

where $\Gamma_{h,\omega,\tilde\omega}(\mathbf{x};\mathbf{y},\mathbf{z})$ is defined as in (7.17), by substituting $J_{\mathbf{x}} \sum_\omega \psi^{[h,0]+}_{\mathbf{x},\omega}\psi^{[h,0]-}_{\mathbf{x},\omega}$ with $J_{\mathbf{x}}$ $\psi^{[h,0]+}_{\mathbf{x},\tilde\omega}$ $\psi^{[h,0]-}_{\mathbf{x},\tilde\omega}$.

If we derive the l.h.s. of (7.11) with respect to $\phi^+_{\mathbf{y},\omega}$ and to $\phi^-_{\mathbf{z},\omega}$ and we put $\phi = 0$, we get

$$\begin{aligned} 0 = &-\delta(\mathbf{x}-\mathbf{y})\Sigma_{h,\omega}(\mathbf{x}-\mathbf{z}) + \delta(\mathbf{x}-\mathbf{z})\Sigma_{h,\omega}(\mathbf{y}-\mathbf{x}) - \\ &< \left[\frac{\partial^2 V}{\partial\psi^{[h,0]+}_{\mathbf{y},\omega}\partial\psi^{[h,0]-}_{\mathbf{z},\omega}} - \frac{\partial V}{\partial\psi^{[h,0]+}_{\mathbf{y},\omega}}\frac{\partial V}{\partial\psi^{[h,0]-}_{\mathbf{z},\omega}}\right] ; \sum_{\tilde\omega}\left[D_{\tilde\omega}(\psi^{[h,0]+}_{\mathbf{x},\tilde\omega}\psi^{[h,0]-}_{\mathbf{x},\tilde\omega}) + \delta T_{\mathbf{x},\tilde\omega}\right] >^T , \end{aligned} \tag{7.21}$$

where $< \cdot ; \cdot >^T$ denotes the truncated expectation w.r.t. the measure $Z(0)^{-1}P^{(L,h)}(d\psi^{[h,0]})$ $e^{-V^{(L)}(\psi^{[h,0]})}$.

By using the definitions (7.18) and (7.19), equation (7.21) can be rewritten as

$$\begin{aligned} 0 = &-\delta(\mathbf{x}-\mathbf{y})\Sigma_{h,\omega}(\mathbf{x}-\mathbf{z}) + \delta(\mathbf{x}-\mathbf{z})\Sigma_{h,\omega}(\mathbf{y}-\mathbf{x}) - \\ &- \sum_{\tilde\omega} D_{\mathbf{x},\tilde\omega}\Gamma_{h,\omega,\tilde\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) - \Delta_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) \;, \end{aligned} \tag{7.22}$$

where

$$\Delta_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = < \left[\frac{\partial^2 V}{\partial\psi^+_{\mathbf{y},\omega}\partial\psi^-_{\mathbf{z},\omega}} - \frac{\partial V}{\partial\psi^+_{\mathbf{y},\omega}}\frac{\partial V}{\partial\psi^-_{\mathbf{z},\omega}} \right] \;;\; \sum_{\tilde\omega} \delta T_{\mathbf{x},\tilde\omega} >^T \;. \tag{7.23}$$

In terms of the Fourier transforms, defined so that, in agreement with (3.2) and (5.9),

$$\Sigma_{h,\omega}(\mathbf{x}-\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\hat\Sigma_{h,\omega}(\mathbf{k}) \;, \tag{7.24}$$

$$\Gamma_{h,\omega,\tilde\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{1}{(L\beta)^2}\sum_{\mathbf{p},\mathbf{k}} e^{i\mathbf{p}(\mathbf{x}-\mathbf{z})}e^{-i\mathbf{k}(\mathbf{y}-\mathbf{z})}\hat\Gamma_{h,\omega,\tilde\omega}(\mathbf{p},\mathbf{k}) \;, \tag{7.25}$$

$$\Delta_{h,\omega}(\mathbf{x};\mathbf{y},\mathbf{z}) = \frac{1}{(L\beta)^2}\sum_{\mathbf{p},\mathbf{k}} e^{i\mathbf{p}(\mathbf{x}-\mathbf{z})}e^{-i\mathbf{k}(\mathbf{y}-\mathbf{z})}\hat\Delta_{h,\omega}(\mathbf{p},\mathbf{k}) \;, \tag{7.26}$$

(7.22) can be written as

$$0 = \hat\Sigma_{h,\omega}(\mathbf{k}-\mathbf{p}) - \hat\Sigma_{h,\omega}(\mathbf{k}) - \sum_{\tilde\omega}(-ip_0 + \tilde\omega\bar v_0 p)\hat\Gamma_{h,\omega,\tilde\omega}(\mathbf{p},\mathbf{k}) + \hat\Delta_{h,\omega}(\mathbf{p},\mathbf{k}) \;. \tag{7.27}$$

Let us now define

$$\tilde Z^{(2)}_h = 1 - \frac{1}{4}\sum_{\eta,\eta'=\pm1} \hat\Gamma_{h,\omega}(\bar{\mathbf{p}}_{\eta'},\bar{\mathbf{k}}_{\eta,\eta'}) \;, \tag{7.28}$$

$$\tilde Z_h = 1 + \frac{i}{4}\sum_{\eta,\eta'=\pm1} \eta'\frac{\beta}{\pi}\hat\Sigma_{h,\omega}(\bar{\mathbf{k}}_{\eta,\eta'}) \;, \tag{7.29}$$

where $\bar{\mathbf{p}}_{\eta'}$ is defined as in (5.11) and $\bar{\mathbf{k}}_{\eta,\eta'}$ as in (2.73).

If we put in (7.27) $\mathbf{p} = \bar{\mathbf{p}}_{\eta'}$ and $\mathbf{k} = \bar{\mathbf{k}}_{\eta,\eta'}$, multiply both sides by $(i\eta'\beta)/(2\pi)$ and sum over η,η', we get

$$\tilde Z_h = \tilde Z^{(2)}_h + \delta\tilde Z^{(2)}_h \;, \tag{7.30}$$

where

$$\delta\tilde Z^{(2)}_h = \frac{1}{4}\sum_{\eta,\eta'=\pm1} \frac{\hat\Delta_{h,\omega}(\bar{\mathbf{p}}_{\eta'},\bar{\mathbf{k}}_{\eta,\eta'})}{-i\bar p_{\eta' 0}} \;. \tag{7.31}$$

7.3 The considerations preceding (7.21) suggest that $\tilde Z_h$ and $\tilde Z^{(2)}_h$ are "almost equal" to the quantities $Z^{(L)}_h$ and $Z^{(2,L)}_h$, related to the full approximate Luttinger model and defined analogously to Z_h and $Z^{(2)}_h$ for the original model, on the base of a multiscale analysis. In order to clarify this point, we consider the measure $P^{(L,h)}(d\psi^{[h,0]})e^{-V^{(L)}(\psi^{[h,0]}+\psi^{(<h)})}$, where $\psi^{(<h)}$ is fixed and has the same role of the external field ϕ in (7.17), and define $\bar E_{h-1}$ and $\bar{\mathcal{V}}^{(h-1)}(\psi^{(<h)})$, the *one step effective potential* on scale $h-1$, so that $\bar{\mathcal{V}}^{(h-1)}(0) = 0$ and

$$e^{-\bar{\mathcal{V}}^{(h-1)}(\psi^{(<h)}) - L\beta\bar E_{h-1}} = \int P^{(L,h)}(d\psi^{[h,0]})e^{-V^{(L)}(\psi^{[h,0]}+\psi^{(<h)})} \;. \tag{7.32}$$

We want to calculate this quantity, by extending to it the definitions of effective potentials and running couplings, given in §2 for the original model.

We start from the scale 0 with potential $\mathcal{V}'^{(0)}(\psi^{[h,0]},\psi^{(<h)}) = V^{(L)}(\psi^{[h,0]}+\psi^{(<h)})$ and we introduce, in analogy to the procedure described in §2.5, for each $\tilde h$ such that $h \le \tilde h \le 0$, two constants $Z'_{\tilde h}$, $E'_{\tilde h}$ and an effective potential $\mathcal{V}'^{(\tilde h)}(\psi,\psi^{(<h)})$, so that $Z'_0 = 1$, $E'_0 = 0$ and

$$e^{-\bar{\mathcal{V}}^{(h-1)}(\psi^{(<h)})-L\beta\bar E_{h-1}} = \int P_{Z'_{\tilde h},C_{h,\tilde h}}(d\psi^{[h,\tilde h]}) e^{-\mathcal{V}'^{(\tilde h)}(\sqrt{Z'_{\tilde h}}\psi^{[h,\tilde h]},\psi^{(<h)})-E'_{\tilde h}} , \qquad (7.33)$$

where $P_{Z'_{\tilde h},C_{h,\tilde h}}(d\psi^{[h,\tilde h]})$ is obtained from the analogous definition (2.66), by putting $\sigma_{\tilde h}(\mathbf{k}') = 0$, $E(\mathbf{k}') = \bar v_0 \sin k'$, and by substituting $C_{\tilde h}^{-1}$ with $C_{h,\tilde h}^{-1} = \sum_{k=h}^{\tilde h} f_k$. Moreover, we suppose that the localization procedure is applied also to the field $\psi^{(<h)}$, even if it does appear in the integration measure and, therefore, can not be involved in the free measure renormalization.

We want to compare these effective potentials with the potentials $\mathcal{V}^{(\tilde h)}(\psi^{(\le \tilde h)})$, related to the approximate Luttinger model without any infrared cutoff and defined following again the procedure described in §2. We shall use for the various objects related to this model the same notation of §2, while the corresponding objects of the model with infrared cutoff will be distinguished with a superscript $'$. The definitions are such that $\mathcal{V}^{(0)}(\psi^{(\le 0)}) = \mathcal{V}'^{(0)}(\psi^{[h,0]},\psi^{(<0)})$, $Z_0 = 1$ and

$$\int P_{Z_0,C_0}(d\psi^{(\le 0)})\, e^{-\mathcal{V}^{(0)}(\sqrt{Z_0}\psi^{(\le 0)})} = \int P_{Z_{\tilde h},C_{\tilde h}}(d\psi^{(\le \tilde h)})\, e^{-\mathcal{V}^{(\tilde h)}(\sqrt{Z_{\tilde h}}\psi^{(\le \tilde h)})-L\beta E_{\tilde h}} . \qquad (7.34)$$

Note that the single scale propagators involved in the calculation of $\mathcal{V}^{(\tilde h)}(\sqrt{Z_{\tilde h}}\psi^{(\le \tilde h)})$ and $\mathcal{V}'^{(\tilde h)}(\sqrt{Z'_{\tilde h}}\psi^{[h,\tilde h]},\psi^{(<h)})$, that is those with scale $\bar h \ge \tilde h + 1$, may differ only if $Z_{\bar h} \ne Z'_{\bar h}$ or $z_{\bar h} \ne z'_{\bar h}$. This immediately follows from the observation that, if $h+1 \le \tilde h \le 0$, the identity (2.90) is satisfied even if we substitute in (2.89) $C_{\tilde h}$ with $C_{h,\tilde h}$. This implies, in particular, since $z_0 = z'_0 = 0$, that (see (2.110) and (2.107))

$$\mathcal{V}'^{(-1)}(\sqrt{Z'_{-1}}\psi^{[h,-1]},\psi^{(<h)}) = \mathcal{V}^{(-1)}[\sqrt{Z_{-1}}(\psi^{[h,-1]}+\psi^{(<h)})] , \qquad (7.35)$$

with $Z'_{-1} = Z_{-1} = 1$, and that $z_{-1} = z'_{-1}$, $\delta_{-1} = \delta'_{-1}$, $\lambda_{-1} = \lambda'_{-1}$, $Z'_{-2} = Z_{-2}$.

Let us now compare the effective potentials on scale -2. The fact that the free measure in (7.33) does not depend on the fields with scale less than h implies that the free measure renormalization does not use all the local part of $\mathcal{V}'^{(-1)}$ proportional to z_{-1}. Therefore, the analogous of the potential $\hat{\mathcal{V}}^{(-1)}(\sqrt{Z_{-2}}\,\psi^{(\le -1)})$ for the model with infrared cutoff has to be defined so that (see (2.107))

$$\begin{aligned}
&\hat{\mathcal{V}}'^{(-1)}(\sqrt{Z_{-2}}\psi^{[h,-1]},\psi^{(<h)}) = \hat{\mathcal{V}}^{(-1)}[\sqrt{Z_{-2}}(\psi^{[h,-1]}+\psi^{(<h)})]+\\
&+ z_{-1}Z_{-1}\sum_{\omega=\pm1}\int d\mathbf{x}\ \Big[-\Big(D_\omega\psi^{[h,-1]+}_{\mathbf{x},\omega}\Big)\psi^{(<h)-}_{\mathbf{x},\omega} + \psi^{(<h)+}_{\mathbf{x},\omega}\Big(D_\omega\psi^{[h,-1]-}_{\mathbf{x},\omega}\Big)\Big]+\\
&+ z_{-1}Z_{-1}\sum_{\omega=\pm1}\int d\mathbf{x}\ \psi^{(<h)+}_{\mathbf{x},\omega}D_\omega\psi^{(<h)-}_{\mathbf{x},\omega} .
\end{aligned} \qquad (7.36)$$

It follows, by using also the remark on the single scale propagators following (7.34), that $\mathcal{V}'^{(-2)}(\sqrt{Z_{-2}}\psi^{[h,-2]},\psi^{(<h)})$, calculated through the analogous of (2.110), can be obtained from $\mathcal{V}^{(-2)}[\sqrt{Z_{-2}}(\psi^{[h,-2]}+\psi^{(<h)})]$ by adding some new terms. First of all, there is the term in the third line of (7.36), which is independent of the integration variables, and the two terms in the second line with $\psi^{[h,-2]\sigma}_{\mathbf{x},\omega}$ in place of $\psi^{[h,-1]\sigma}_{\mathbf{x},\omega}$. Moreover, in the Feynman graph expansion, we have to add the graphs which are obtained by inserting, in the external lines of a graph contributing to $\mathcal{V}^{(-2)}$, one or more vertices corresponding to the two terms in the second line of (7.36). These new terms are not irrelevant, if the number of external lines is 2 or 4; hence one could worry about the need of new running couplings in order to regularize the expansion. However, because of the support properties of the propagators, these new terms do not give any contribution to the local part (which is calculated by putting equal to $\bar{\mathbf{k}}_{\eta,\eta'}$ the external momenta, hence also the momenta of the internal line propagators of the insertions in the external lines), so that the only running couplings to consider are those related with $\mathcal{V}^{(-2)}$ and their values are the same, that is $z_{-2}=z'_{-2}$, $\delta_{-2}=\delta'_{-2}$, $\lambda_{-2}=\lambda'_{-2}$, $Z'_{-3}=Z_{-3}$.

By iterating the previous considerations, it is easy to show that, if $h\le\tilde{h}\le-2$, one can calculate $\mathcal{V}'^{(\tilde{h})}(\sqrt{Z_{\tilde{h}}}\psi^{[h,\tilde{h}]},\psi^{(<h)})$ by adding to $\mathcal{V}^{(\tilde{h})}[\sqrt{Z_{\tilde{h}}}(\psi^{[h,\tilde{h}]}+\psi^{(<h)})]$ some new terms. First of all, there are the local terms of the form of that in the second and the third line of (7.36), with $\psi^{[h,\tilde{h}]\sigma}_{\mathbf{x},\omega}$ in place of $\psi^{[h,-1]\sigma}_{\mathbf{x},\omega}$ and $z_{\bar{h}}Z_{\bar{h}}$, $\tilde{h}\le\bar{h}\le-1$ in place of $z_{-1}Z_{-1}$. Moreover, in the Feynman graph expansion, we have to add the graphs, which are obtained by inserting, in the external lines of a graph contributing to $\mathcal{V}^{(\tilde{h})}$, one or more vertices corresponding to terms similar to those in the second line of (7.36), with $\psi^{[h,\tilde{h}]\sigma}_{\mathbf{x},\omega}$ in place of $\psi^{[h,-1]\sigma}_{\mathbf{x},\omega}$ and $z_{\bar{h}}Z_{\bar{h}}$, $\tilde{h}\le\bar{h}\le-1$ in place of $z_{-1}Z_{-1}$. Finally

$$\begin{aligned}&\mathcal{L}\mathcal{V}'^{(\tilde{h})}(\sqrt{Z_{\tilde{h}}}\psi^{[h,\tilde{h}]},\psi^{(<h)})=\mathcal{L}\mathcal{V}^{(\tilde{h})}[\sqrt{Z_{\tilde{h}}}(\psi^{[h,\tilde{h}]}+\psi^{(<h)})]+\\&+\sum_{\bar{h}=\tilde{h}+1}^{-1}z_{\bar{h}}Z_{\bar{h}}\sum_{\omega=\pm1}\int d\mathbf{x}\ \Big[-\Big(D_\omega\psi^{[h,\tilde{h}]+}_{\mathbf{x},\omega}\Big)\psi^{(<h)-}_{\mathbf{x},\omega}+\psi^{(<h)+}_{\mathbf{x},\omega}\Big(D_\omega\psi^{[h,\tilde{h}]-}_{\mathbf{x},\omega}\Big)\Big]+\\&+\sum_{\bar{h}=\tilde{h}+1}^{-1}z_{\bar{h}}Z_{\bar{h}}\sum_{\omega=\pm1}\int d\mathbf{x}\ \psi^{(<h)+}_{\mathbf{x},\omega}D_\omega\psi^{(<h)-}_{\mathbf{x},\omega}\ ,\end{aligned}\tag{7.37}$$

and all the running couplings, as well as the renormalization constants, are the same as those defined through $\mathcal{V}^{(\tilde{h})}(\sqrt{Z_{\tilde{h}}}\psi^{(\le\tilde{h})})$.

Equations (7.33) and (7.37) also imply that

$$\begin{aligned}&\mathcal{L}\bar{\mathcal{V}}^{(h-1)}(\psi^{(<h)})=\mathcal{L}\mathcal{V}'^{(h-1)}(\psi^{(<h)})+\\&+\sum_{\bar{h}=h+1}^{-1}z_{\bar{h}}Z_{\bar{h}}\sum_{\omega=\pm1}\int d\mathbf{x}\ \psi^{(<h)+}_{\mathbf{x},\omega}D_\omega\psi^{(<h)-}_{\mathbf{x},\omega}+z'_hZ_h\sum_{\omega=\pm1}\int d\mathbf{x}\ \psi^{(<h)+}_{\mathbf{x},\omega}D_\omega\psi^{(<h)-}_{\mathbf{x},\omega}\ ,\end{aligned}\tag{7.38}$$

where $\mathcal{V}'^{(h-1)}(\psi^{(<h)})$ is obtained from $\mathcal{V}^{(h-1)}(\psi^{(<h)})$ "almost" as before. We still have to add some new graphs with suitable insertions on the external lines, which do not affect the

local part, but we also have to change the propagators of scale h, since the function $\tilde f'_h(\mathbf{k}')$, calculated as $\tilde f_h(\mathbf{k}')$, see (2.90), with $C_{h,h}^{-1} = f_h$ in place of C_h^{-1}, is different from $\tilde f_h(\mathbf{k}')$.

The definition (7.29) of $\tilde Z_h$ and the definition of $\mathcal{L}$, together with (7.38), imply that

$$\tilde Z_h = 1 + \sum_{\bar h = h+1}^{-1} z_{\bar h} Z_{\bar h} + z'_h Z_h = Z_h(1 + z'_h) \; . \tag{7.39}$$

Since $Z_h^{(L)} = Z_h$ and $|z'_h| \le C|\lambda_0|^2$, if λ_0 is small enough, as one can show by using the arguments of §4, we get the bound

$$\left| \frac{\tilde Z_h}{Z_h^{(L)}} - 1 \right| \le C|\lambda_0| \; . \tag{7.40}$$

A similar argument can be used for $Z_h^{(2,L)}$, by using the results of §5, and we get the similar bound

$$\left| \frac{\tilde Z_h^{(2)}}{Z_h^{(2,L)}} - 1 \right| \le C|\lambda_0| \; . \tag{7.41}$$

We will prove in §7.4 that

$$|\delta \tilde Z_h^{(2)}| \le C Z_h^{(2,L)} |\lambda_0| \; , \tag{7.42}$$

so that we finally get

$$| \frac{Z_h^{(L)}}{Z_h^{(2,L)}} - 1| \le C|\lambda_0| \; , \tag{7.43}$$

implying (5.35).

Remark (7.42) shows that the corrections to the *exact* Ward identity $Z_h^{(L)} = Z_h^{(2,L)}$ could diverge as $h \to -\infty$. This is not important in our proof, since we are only interested in the ratio $Z_h^{(L)}/Z_h^{(2,L)}$, which is near to 1, but suggests that it would be difficult to prove the approximate Ward identity, by directly looking at the cancellations in presence of the cutoffs.

7.4 In order to prove (7.42), we note that

$$\begin{aligned} &[C_{h,0}(\mathbf{p}+\mathbf{k}) - 1] D_\omega(\mathbf{p}+\mathbf{k}) - [C_{h,0}(\mathbf{k}) - 1] D_\omega(\mathbf{k}) = \\ &D_\omega(\mathbf{p})[C_{h,0}(\mathbf{p}+\mathbf{k}) - 1] + C_{h,0}(\mathbf{p}+\mathbf{k}) D_\omega(\mathbf{k}) C_{h,0}(\mathbf{k}) [C_{h,0}^{-1}(\mathbf{k}) - C_{h,0}^{-1}(\mathbf{p}+\mathbf{k})] \; , \end{aligned} \tag{7.44}$$

and that

$$\begin{aligned} C_{h,0}(\bar{\mathbf{p}}_{\eta'} + \mathbf{k}) \frac{[C_{h,0}^{-1}(\mathbf{k}) - C_{h,0}^{-1}(\bar{\mathbf{p}}_{\eta'} + \mathbf{k})]}{-i\bar p_{\eta' 0}} &= \\ C_{h,0}(\mathbf{p} + \mathbf{k}) \frac{[C_{h,0}^{-1}(\mathbf{k}) - C_{h,0}^{-1}(\bar{\mathbf{p}}_{\eta'} + \mathbf{k})]}{-i\bar p_{\eta' 0}} \Bigg|_{\mathbf{p} = \bar{\mathbf{p}}_{\eta'}} &\; , \end{aligned} \tag{7.45}$$

$$C_{h,0}(\mathbf{p}+\mathbf{k}) = 1 + [C_{h,0}(\mathbf{p}+\mathbf{k}) - 1] \; . \tag{7.46}$$

Hence, by using (7.16) and (7.23), we can write

$$\frac{\hat\Delta_{h,\omega}(\bar{\mathbf{p}}_{\eta'}, \bar{\mathbf{k}}_{\eta,\eta'})}{-i\bar p_{\eta' 0}} = \hat\Delta^{(1)}_{h,\omega,\eta'}(\bar{\mathbf{p}}_{\eta'}, \bar{\mathbf{k}}_{\eta,\eta'}) \; , \tag{7.47}$$

where

$$\Delta^{(1)}_{h,\omega,\eta'}(\mathbf{x};\mathbf{y},\mathbf{z}) =< \left[\frac{\partial^2 V}{\partial\psi^+_{\mathbf{y},\omega}\partial\psi^-_{\mathbf{z},\omega}} - \frac{\partial V}{\partial\psi^+_{\mathbf{y},\omega}}\frac{\partial V}{\partial\psi^-_{\mathbf{z},\omega}}\right] \; ; \; \sum_{\tilde\omega} \delta^{(1)} T_{\mathbf{x},\tilde\omega,\eta'} >^T , \tag{7.48}$$

with

$$\delta^{(1)} T_{\mathbf{x},\omega,\eta'} = \psi^{[h,0]+}_{\mathbf{x},\omega}\delta\psi^{[h,0]-}_{\mathbf{x},\omega} + \delta\tilde\psi^{[h,0]+}_{\mathbf{x},\omega,\eta'}\delta\psi^{[h,0]-}_{\mathbf{x},\omega} + \delta\tilde\psi^{[h,0]+}_{\mathbf{x},\omega,\eta'}\psi^{[h,0]-}_{\mathbf{x},\omega} , \tag{7.49}$$

$$\delta\psi^{[h,0]-}_{\mathbf{x},\omega} = \frac{1}{L\beta}\sum_{\mathbf{k}:C^{-1}_{h,0}(\mathbf{k})>0} e^{-i\mathbf{k}\mathbf{x}} C_{h,0}(\mathbf{k})(1 - C^{-1}_{h,0}(\mathbf{k}))\hat\psi^{[h,0],-}_{\mathbf{k},\omega} , \tag{7.50}$$

$$\delta\tilde\psi^{[h,0]+}_{\mathbf{x},\omega,\eta'} = \frac{1}{L\beta}\sum_{\mathbf{k}:C^{-1}_{h,0}(\mathbf{k})>0} e^{i\mathbf{k}\mathbf{x}} D_\omega(\mathbf{k}) C_{h,0}(\mathbf{k})\frac{[C^{-1}_{h,0}(\mathbf{k}) - C^{-1}_{h,0}(\bar{\mathbf{p}}_{\eta'} + \mathbf{k})]}{-i\bar p_{\eta' 0}}\hat\psi^{[h,0],+}_{\mathbf{k},\omega} . \tag{7.51}$$

Note that there is no divergence, in the limit $L, \beta \to \infty$, associated with the fields $\delta\psi^{[h,0]-}$ and $\delta\tilde\psi^{[h,0]+}$, even if the function $C_{h,0}(\mathbf{k})$ diverges on the boundary of the set $\{\mathbf{k} : C^{-1}_{h,0}(\mathbf{k}) > 0\}$. In fact, the integration of these fields on scale $\bar h$, with $h \le \bar h \le 0$, yields a factor $\tilde f'_{\bar h}(\mathbf{k})$ proportional to $f_{\bar h}(\mathbf{k})$ (see (2.90) and the considerations after (7.38)), and the functions $f_{\bar h}(\mathbf{k})$ are non negative, if we suitably choose the function (2.30); therefore $C_{h,0}(\mathbf{k})\tilde f'_{\bar h}(\mathbf{k})$ is bounded.

Note also that, $[C^{-1}_{h,0}(\mathbf{k}) - C^{-1}_{h,0}(\bar{\mathbf{p}}_{\eta'} + \mathbf{k})]/ - i\bar p_{\eta' 0}$ is bounded, uniformly in β, and is equal to 0, at least if $|\mathbf{k}|$ belongs to the interval $[a_0\gamma^h + 2\pi/\beta, a_0 - 2\pi/\beta]$ (see §**I**2.3). However, the interval where this function vanishes can contain the interval $[a_0\gamma^h, a_0]$, if the function (2.30) is suitably chosen (by slightly broadening the regions where it has to be equal to 1 or 0) and β is large enough, which is not of course an important restriction (the real problem is the uniformity of the bounds in the limit $\beta \to \infty$, and in any case the following arguments could be easily generalized to cover the general case). Hence, it is easy to show that

$$1 - C^{-1}_{h,0}(\mathbf{k}) = \frac{C^{-1}_{h,0}(\mathbf{k}) - C^{-1}_{h,0}(\bar{\mathbf{p}}_{\eta'} + \mathbf{k})}{-i\bar p_{\eta' 0}} = 0 , \quad \text{if } \tilde f'_{\bar h}(\mathbf{k}) \ne 0 \quad h < \bar h < 0 , \tag{7.52}$$

so that we can write

$$\delta\psi^{[h,0]-}_{\mathbf{x},\omega} = \delta\psi^{(0)-}_{\mathbf{x},\omega} + \delta\psi^{(h)-}_{\mathbf{x},\omega} , \quad \delta\tilde\psi^{[h,0]+}_{\mathbf{x},\omega,\eta'} = \delta\tilde\psi^{(0)+}_{\mathbf{x},\omega,\eta'} + \delta\tilde\psi^{(h)+}_{\mathbf{x},\omega,\eta'} , \tag{7.53}$$

where the fields $\delta\psi^{(h')-}_{\mathbf{x},\omega}$ and $\delta\tilde\psi^{(h')+}_{\mathbf{x},\omega,\eta'}$ are defined by substituting, in (7.50) and (7.51), $\hat\psi^{[h,0],+}_{\mathbf{k},\omega}$ with $\hat\psi^{(h'),+}_{\mathbf{k},\omega}$.

Let us now consider the functional

$$e^{\mathcal{S}_{h,\eta'}(\psi^{(<h)},J)} = \int P^{(L,h)}(d\psi^{[h,0]}) e^{-V^{(L)}(\psi^{[h,0]}+\psi^{(<h)}) + \sum_{\tilde\omega}\int d\mathbf{x} J_\mathbf{x}\delta^{(1)}T_{\mathbf{x},\tilde\omega,\eta'}} . \tag{7.54}$$

We can write for $\mathcal{S}_{h,\eta'}(\psi^{(<h)}, J)$ an expansion similar to that used in §5 to study the correlation function of the original model. We introduce, for any $\tilde h$ such that $h \le \tilde h \le -1$, an effective potential $\mathcal{V}'^{(\tilde h)}(\psi, \psi^{(<h)})$, defined as in §7.3, and two functionals $S'^{(\tilde h+1)}(J)$, $\mathcal{B}'^{(\tilde h)}(\psi, \psi^{(<h)}, J)$, so that, by using the notation of §7.4,

$$\begin{aligned} e^{\mathcal{S}_{h,\eta'}(\psi^{(<h)},J)} &= e^{-L\beta E'_{\tilde h} + S'^{(\tilde h+1)}(J)}\int P_{Z'_{\tilde h}, C_{h,\tilde h}}(d\psi^{[h,\tilde h]}) \cdot \\ &\quad \cdot e^{-\mathcal{V}'^{(\tilde h)}(\sqrt{Z'_{\tilde h}}\psi^{[h,\tilde h]}, \psi^{(<h)}) + \mathcal{B}'^{(\tilde h)}(\sqrt{Z'_{\tilde h}}\psi^{[h,\tilde h]}, \psi^{(<h)}, J)} . \end{aligned} \tag{7.55}$$

We introduce also the functionals $S^{'(h)}(J)$, $\mathcal{V}^{'(h-1)}(\psi^{(<h)})$ and $\mathcal{B}^{'(h-1)}(\psi^{(<h)}, J)$, such that

$$\mathcal{S}_{h,\eta'}(\psi^{(<h)}, J) = S^{'(h)}(J) - \mathcal{V}^{'(h-1)}(\psi^{(<h)}) + \mathcal{B}^{'(h-1)}(\psi^{(<h)}, J) \; . \tag{7.56}$$

We can write for $\mathcal{B}^{'(h-1)}(\psi^{(<h)}, J)$ a representation similar to (5.6), with J in place of ϕ and $\psi^{(<h)}$ in place of $\psi^{(\le h)}$. By (7.48)

$$\Delta^{(1)}_{h,\omega,\eta'}(\mathbf{x};\mathbf{y},\mathbf{z}) = B^{'(h-1)}_{1,2,(+,-),(\omega,\omega)}(\mathbf{x};\mathbf{y},\mathbf{z}) \; ; \tag{7.57}$$

hence, in order to prove (7.42), we have to study the flow of the local part of $\mathcal{B}^{'(\tilde h)}(Z^{'-1/2}_{\tilde h}$ $\psi^{[h,\tilde h]}, \psi^{(<h)}, J)$.

To start with, let us consider $\mathcal{B}^{'(-1)}(Z^{'-1/2}_{-1}\psi^{[h,-1]}, \psi^{(<h)}, J)$. By (7.53), the graphs contributing to it may have an external line of type $\delta\psi$ or $\delta\tilde\psi$ only if that line is of scale h and $h < -1$. Moreover, if the graph has an external line of this type and it is not trivial, that is if it has more than one vertex, the corresponding local part, defined as in §5, is 0, even if there are only two external lines, because of the support properties of the propagators, since there is at least one internal line with momentum equal to one of the external momenta, which are of order β^{-1} for the local part. It follows that these graphs do not participate in any manner to the flow of $\mathcal{L}\mathcal{B}^{'(-1)}(Z^{'-1/2}_{-1}\psi^{[h,-1]}, \psi^{(<h)}, J)$, up to the scale h; therefore we modify the definition of $\mathcal{L}$, so that they are not included.

This modification of the definition of $\mathcal{L}$ allows to study the flow of $\mathcal{L}\mathcal{B}^{'(\tilde h)}(Z'_{\tilde h-1}\psi^{[h,\tilde h]}$, $\psi^{(<h)}, J)$ essentially as in §5, since, as we have explained in §7.3, the infrared cutoff has no influence on the other local terms, except on the last scale, so that, if $h \le \tilde h \le -1$,

$$\mathcal{L}\mathcal{B}^{'(\tilde h)}(\sqrt{Z'_{\tilde h}}\psi^{[h,\tilde h]}, \psi^{(<h)}, J) = \mathcal{L}\mathcal{B}^{(\tilde h)}(\sqrt{Z_{\tilde h}}(\psi^{[h,\tilde h]} + \psi^{(<h)}), J) \; , \tag{7.58}$$

where $\mathcal{B}^{(\tilde h)}(\sqrt{Z_{\tilde h}}\psi^{(\le \tilde h)}, J)$ is the expression we should get in absence of infrared cutoff and we used the fact, proved in §7.3, that $Z'_{\tilde h} = Z_{\tilde h}$. We can write

$$\mathcal{L}\mathcal{B}^{(\tilde h)}(\sqrt{Z_{\tilde h}}(\psi^{(\le\tilde h)}), J) = \frac{Z^{(3)}_{\tilde h}}{Z_{\tilde h}} \sum_{\omega=\pm1} \int d\mathbf{x} J_{\mathbf{x}} \psi^{(\le\tilde h)+}_{\mathbf{x},\omega} \psi^{(\le\tilde h)-}_{\mathbf{x},\omega} \; . \tag{7.59}$$

The flow of $Z^{(3)}_{\tilde h}$ can be studied, starting from the scale $h = -1$, as the flow of the renormalization constants $Z^{(2)}_{\tilde h}$ related to the analogous of the functional (5.2) for the model defined by (7.1) and (7.2), that is

$$e^{\mathcal{S}(J)} = \int P^{(L)}(d\psi^{(\le 0)}) e^{-V^{(L)}(\psi^{(\le 0)}) + \sum_{\omega=\pm1}\int d\mathbf{x} J_{\mathbf{x}} \psi^{(\le 0)+}_{\mathbf{x},\omega}\psi^{(\le 0)-}_{\mathbf{x},\omega}} \; . \tag{7.60}$$

Note that the values of $Z^{(3)}_{-1}$ and $Z^{(2)}_{-1}$ are very different; in fact, the previous considerations imply that

$$|Z^{(2)}_{-1} - 1| \le C|\lambda_0| \; . \quad |Z^{(3)}_{-1}| \le C|\lambda_0| \; . \tag{7.61}$$

However, since the local part on scale -1 is of the same form and the contribution of the non local terms on scale -1 to $Z^{(3)}_{\tilde{h}}/Z^{(3)}_{\tilde{h}+1}$ or $Z^{(2)}_{\tilde{h}}/Z^{(2)}_{\tilde{h}+1}$ is exponentially depressed, as $\tilde{h}$ decreases, it is easy to show, by using the arguments of §4.4-§4.7, that

$$Z^{(3)}_h = \frac{Z^{(3)}_h}{Z^{(3)}_{-1}} Z^{(3)}_{-1} = \frac{Z^{(2)}_h}{Z^{(2)}_{-1}}[1+O(\lambda_0)]Z^{(3)}_{-1} \; . \tag{7.62}$$

The integration of the fields of scale h can only change this identity by a factor $[1+O(\lambda_0)]$, hence (7.61) and (7.62) imply that

$$\left| \frac{Z^{(3)}_{h-1}}{Z^{(2)}_h} \right| \le C|\lambda_0| \; . \tag{7.63}$$

If $\Delta^{(1)}_{h,\omega,\eta'}(\mathbf{x};\mathbf{y},\mathbf{z})$ were independent of η', $\delta\tilde{Z}^{(2)}_h$ would be exactly equal to $Z^{(3)}_{h-1}$ and (7.42) would have been proved. Since this is true only in the limit $\beta \to \infty$, we have to bound $\hat{\Delta}^{(1)}_{h,\omega,\eta'}(\bar{\mathbf{p}}_{\eta'}, \bar{\mathbf{k}}_{\eta,\eta'})$ for each η, η'. This means that we have to bound even the Fourier transform at momenta of order β^{-1} of $\mathcal{R}B^{'(h-1)}_{1,2,(+,-),(\omega,\omega)}(\mathbf{x};\mathbf{y},\mathbf{z})$, see (7.57). However, it is easy to see that we still get the bound (7.42), on the base of a simple dimensional argument (we skip the details, which should be by now obvious). In fact, if we consider a term contributing to the expansion of $\mathcal{R}B^{'(h-1)}_{1,2,(+,-),(\omega,\omega)}(\mathbf{x};\mathbf{y},\mathbf{z})$ described in §5, whose external fields are affected by the regularization so that some derivative acts on them, the corresponding bound differs from the bound of a generic term contributing to $\mathcal{L}B^{'(h-1)}_{1,2,(+,-),(\omega,\omega)}(\mathbf{x};\mathbf{y},\mathbf{z})$ in the following way. One has to add a factor γ^{-h_v}, for each "zero" produced by the regularization and, at the same time, a factor β^{-1} produced by the corresponding derivative on the external momenta. Since $\beta^{-1}\gamma^{-h_v} \le 1$, we get the same result.

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