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ON THE RADICAL OF BRAUER ALGEBRAS

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Abstract. The radical of the Brauer algebra $\mathcal{B}_f^{(x)}$ is known to be non-trivial when the parameter x is an integer subject to certain conditions (with respect to f). In these cases, we display a wide family of elements in the radical, which are explicitly described by means of the diagrams of the usual basis of $\mathcal{B}_f^{(x)}$. The proof is by direct approach for $x = 0$, and via classical Invariant Theory in the other cases, exploiting then the well-known representation of Brauer algebras as centralizer algebras of orthogonal or symplectic groups acting on tensor powers of their standard representation. This also gives a great part of the radical of the generic indecomposable $\mathcal{B}_f^{(x)}$–modules. We conjecture that this part is indeed the whole radical in the case of modules, and it is the whole part in a suitable step of the standard filtration in the case of the algebra. As an application, we find some more precise results for the module of pointed chord diagrams, and for the Temperley-Lieb algebra — realised inside $\mathcal{B}_f^{(1)}$ — acting on it.

"Ahi quanto a dir che sia è cosa dura
lo radical dell'algebra di Brauer
pur se'l pensier già muove a congettura"

N. Barbecue, *"Scholia"*

Introduction

The Brauer algebras first arose in Invariant Theory (cf. [Br]) in connection with the study of invariants of the action of the orthogonal or the symplectic group — call it $G(U)$ — on the tensor powers of its standard representation U. More precisely, the centralizer algebra $End_{G(U)}\big(U^{\otimes f}\big)$ of such an action can be described by generators and relations: the latter depend on the relationship among two integral parameters, f and x — the latter being related to $dim(U)$ — but when x is big enough (what is called "the stable case") the relations always remain the same. These "stable" relations then define an algebra $\mathcal{B}_f^{(x)}$ of which the centralizer one is a quotient, obtained by adding the further relations, when necessary. The abstract algebra $\mathcal{B}_f^{(x)}$ is the one which bears the name of "Brauer algebra".

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The definition of $\mathcal{B}_f^{(x)}$ still makes sense with x arbitrarily chosen in a fixed ground ring. An alternative description is possible too, by displaying an explicit basis of $\mathcal{B}_f^{(x)}$ and assigning the multiplication rules for elements in this basis.

Assume the ground field $\Bbbk$ has characteristic zero. Then $G(U)$ is linearly reductive, so by Schur duality the algebra $End_{G(U)}\big(U^{\otimes f}\big)$ is semisimple: hence in the stable case, when $\mathcal{B}_f^{(x)} \cong End_{G(U)}\big(U^{\otimes f}\big)$, the Brauer algebra is semisimple too. Otherwise, $\mathcal{B}_f^{(x)}$ may fail to be semisimple, i.e. it may have a non-trivial radical.

The most general result on $Rad\Big(\mathcal{B}_f^{(x)}\Big)$, for $Char(\Bbbk) = 0$, was found in [Wz]: for "general values" of x — i.e., all those out of a finite range (depending on f, and yielding the stable case) of values in the prime subring of $\Bbbk$ — the Brauer algebra $\mathcal{B}_f^{(x)}$ is semisimple. So the problem only remained of computing $Rad\Big(\mathcal{B}_f^{(x)}\Big)$ when x is an integer and we are not in the stable case. In this framework, the first contributions came from Brown, who reduced the task to studying the radical of "generalized matrix algebras" (cf. [Bw1–2]). In particular, this radical is strictly related with the nullspace of the matrix of structure constants of such an algebra: later authors mainly followed the same strategy, see e.g. [HW1–2]. Further results were obtained using new techniques: see [GL], [DHW], [KX], [CDM], [Hu], [DH].

In the present paper we rather come back to the Invariant Theory viewpoint. The idea we start from is a very naïve one: as the algebra $End_{G(U)}\big(U^{\otimes f}\big)$ is semisimple, we have

$$Rad\Big(\mathcal{B}_f^{(x)}\Big) \;\subseteq\; Ker\Big(\pi_U : \mathcal{B}_f^{(x)} \twoheadrightarrow End_{G(U)}\big(U^{\otimes f}\big)\Big)$$

where $\pi_U : \mathcal{B}_f^{(x)} \twoheadrightarrow End_{G(U)}\big(U^{\otimes f}\big)$ is the natural epimorphism. The second step is an intermediate result, namely a description of the kernel $Ker\Big(\pi_U : \mathcal{B}_f^{(x)} \twoheadrightarrow End_{G(U)}\big(U^{\otimes f}\big)\Big)$. Indeed, using the Second Fundamental Theorem of classical invariant theory we find a set of linear generators for it: they are explicitly written in terms of the basis of diagrams, and called *(diagrammatic) minors* or *Pfaffians*, depending on the sign of x. As $Ker(\pi_U)$ contains $Rad\Big(\mathcal{B}_f^{(x)}\Big)$, every element of $Rad\Big(\mathcal{B}_f^{(x)}\Big)$ is a linear combination of these special elements (minors or Pfaffians). As a last step, a basic knowledge of $\mathcal{B}_f^{(x)}$–modules yields some more information on the structure of the semisimple quotient of $\mathcal{B}_f^{(x)}$. Thus we determine exactly which ones among minors, or Pfaffians, belong to $Rad\Big(\mathcal{B}_f^{(x)}\Big)$: so we find a great part of $Rad\Big(\mathcal{B}_f^{(x)}\Big)$, and we conjecture that this is *all* the part of the radical inside the proper step of the standard filtration. We then find a similar result and conjecture for the generic indecomposable $\mathcal{B}_f^{(x)}$–modules too.

Our approach applies directly only in case x is an integer which is not zero nor odd negative; but *a posteriori*, we find also similar results for $x = 0$, via an *ad hoc* approach.

Also, we discuss how much of these results can be extended to the case of $Char(\Bbbk) > 0$.

Finally, we provide some more precise results for the module of pointed chord diagrams, and the Temperley-Lieb algebra — realised as a subalgebra of $\mathcal{B}_f^{(1)}$ — acting on it.

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§1 The Brauer algebra

1.1 f–diagrams. Given $f \in \mathbb{N}_+$, denote by $\mathbb{V}_f$ the datum of $2f$ vertices in a plane, arranged in two rows, one upon the other, each one of f aligned vertices. Then consider the graphs with $\mathbb{V}_f$ as set of vertices and f edges, such that each vertex belongs to exactly one edge. We call such graphs *f–diagrams*, denoting by D_f the set of all of them. In general, we shall denote them by bold roman letters, like $\mathbf{d}$. These f–diagrams are as many as the pairings of $2f$ elements, hence $\left|D_f\right| = (2f-1)!! := (2f-1)\cdot(2f-3)\cdots 5\cdot 3\cdot 1$ in number.

We shall label the vertices in $\mathbb{V}_f$ in two ways: either we label the vertices in the top row with the numbers 1^+, 2^+, $\dots$, f^+, in their natural order from left to right, and the vertices in the bottom row with the numbers 1^-, 2^-, $\dots$, f^-, again from left to right, or we label them by setting i for i^+ and $f+j$ for j^- (for all $i, j \in \{1, 2, \dots, f\}$). Thus an f–diagram can also be described by specifying its set of edges. In general, given f–tuples $\mathbf{i} := (i_1, i_2, \dots, i_f)$ and $\mathbf{j} := (j_1, j_2, \dots, j_f)$ such that $\{i_1, \dots, i_f\} \cup \{j_1, \dots, j_f\} = \mathbb{V}_f$, we call $\mathbf{d}_{\mathbf{i},\,\mathbf{j}}$ the f–diagram obtained by joining i_k to j_k, for each $k = 1, 2, \dots, f$.

When looking at the edges of an f–diagram, we shall distinguish between those which link two vertices in the same (top or bottom) row, which we call *horizontal edges* or simply *arcs*, and those which link two vertices in different rows, to be called *vertical edges*. Clearly, any f–diagram $\mathbf{d}$ has the same number of arcs in the top row and in the bottom row: if this number is k, we shall say that $\mathbf{d}$ is a k–arc (f–)diagram. Then, letting $D_{f,k} := \left\{ \mathbf{d} \in D_f \,\middle|\, \mathbf{d} \textit{ is a } k\textit{–arc diagram} \right\}$ we have $D_f = \bigcup_{k=1}^{[f/2]} D_{f,k}$; hereafter, for any $f \in \mathbb{N}$ we set $\left[f/2\right] := f/2$ if f is even and $\left[f/2\right] := (f-1)\big/2$ if f is odd.

1.2 Arc structure and permutation structure of diagrams. Let $\mathbf{d}$ be an f–diagram. With "arc structure of the top row", resp. "bottom row", of $\mathbf{d}$ we shall mean the datum of the arcs in the top, resp. bottom, row of $\mathbf{d}$, in their mutual positions. To put it in a nutshell, we shall use such terminology as "top arc structure", resp. "bottom arc structure", of $\mathbf{d}$ — to be denoted with $\mathtt{tas}(\mathbf{d})$, resp. $\mathtt{bas}(\mathbf{d})$ — and "arc structure of $\mathbf{d}$" — to be denoted with $\mathtt{as}(\mathbf{d})$ — to mean the datum of both top and bottom arc structures of $\mathbf{d}$, that is $\mathtt{as}(\mathbf{d}) := \big(\mathtt{tas}(\mathbf{d}), \mathtt{bas}(\mathbf{d})\big)$. Note that any top or bottom arc structure may be described by a one-row graph of vertices, arranged on a horizontal line, and some edges (the arcs) joining them pairwise, so that each vertex belongs to at most one edge. Following Kerov (cf. [Ke]), such a graph will be called a *k–arc f–junction*, or *(f,k)–junction*, where f is its number of vertices and k its number of edges.

We denote the set of (f,k)–junctions by $J_{f,k}$. Then clearly $\left|J_{f,k}\right| = \binom{f}{2k}(2k-1)!!$.

Any $\mathbf{d} \in D_{f,k}$ has exactly $f-2k$ vertices in its top row, and $f-2k$ vertices in its bottom row which are pairwise joined by its $f-2k$ vertical edges. Let us label with 1, 2, $\dots$, $f-2k$ from left to right the vertices in the top row, and do the same in the bottom row. Then there exists a unique permutation $\sigma = \sigma(\mathbf{d}) \in S_{f-2k}$ — to be called the "permutation structure", or "symmetric (group) part", of $\mathbf{d}$ — such that $\sigma(i)$ is the label of the bottom row vertex of the vertical edge whose top row vertex is labelled with i.

Therefore the maps $\mathbf{d} \mapsto \big(\sigma(\mathbf{d}), \mathtt{as}(\mathbf{d})\big)$ set bijections $D_{f,k} \longrightarrow S_{f-2k} \times \big(J_{f,k} \times J_{f,k}\big)$ and altogether they give a bijection $D_f \longrightarrow \bigcup_{k=1}^{[f/2]} S_{f-2k} \times \big({J_{f,k}}^{\times 2}\big)$.

1.3 The Brauer algebra. Let $\Bbbk$ be a field, $p := \mathit{Char}(\Bbbk) \geq 0$, and take $x \in \Bbbk$. Later results will require some restrictions on $\Bbbk$, but on the other hand one can also generalise, replacing $\Bbbk$ with any commutative ring with 1 — see (4.1) and the subsequent remark.

Let $\mathcal{B}_f^{(x)}$ be the $\Bbbk$–vector space with basis D_f; we introduce a product in $\mathcal{B}_f^{(x)}$ by defining the product of f–diagrams and extending by linearity. So for all $\mathbf{a}, \mathbf{b} \in D_f$ define the product $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}\mathbf{b}$ as follows. First, draw $\mathbf{b}$ below $\mathbf{a}$; second, connect the i–th bottom vertex of $\mathbf{a}$ with the i–th top vertex of $\mathbf{b}$; third, let $C(\mathbf{a}, \mathbf{b})$ be the number of cycles in the new graph thus obtained and let $\mathbf{a} * \mathbf{b}$ be this graph, *pruning out the cycles*; then $\mathbf{a} * \mathbf{b}$ is a new f–diagram, and we set $\mathbf{a}\,\mathbf{b} := x^{C(\mathbf{a},\mathbf{b})} \mathbf{a} * \mathbf{b}$. This definition makes $\mathcal{B}_f^{(x)}$ into a unital associative $\Bbbk$–algebra: this is the *Brauer algebra*, in its "abstract" form (see for instance [KX]). Its relation with Brauer's centralizer algebra is explained in §4 later on.

Note that for $\mathbf{a}, \mathbf{b} \in D_f$, the top, resp. bottom, arc structure of $\mathbf{a} * \mathbf{b}$ "contains" that of $\mathbf{a}$, resp. $\mathbf{b}$. In particular, if $\mathbf{a} \in D_{f,a}$ and $\mathbf{b} \in D_{f,b}$ this gives $\mathbf{a} * \mathbf{b} \in D_{f,\max(a,b)}$.

The symmetric group S_{2f} acts on $\mathbb{V}_f$, once a numbering of the vertices in $\mathbb{V}_f$ is fixed; so it acts on D_f, and linear extension gives a $\Bbbk$–linear action on $\mathcal{B}_f^{(x)}$ (studied in [DH], [Hu]).

By construction $D_{f,0}$ is a subset of $\mathcal{B}_f^{(x)}$, actually a subsemigroup. Now, for any $\sigma \in S_f$ let $\mathbf{d}_\sigma \in D_{f,0}$ be the f–diagram obtained by joining i^+ with $\sigma(i)^-$ (cf. §1.1). Then the map $S_f \longrightarrow D_{f,0} \subset \mathcal{B}_f^{(x)}$ is a morphism of semigroups, whose image is $D_{f,0}$; hence $\mathcal{B}_f^{(x)}$ contains a copy of S_f (namely $D_{f,0}$) and a copy of the group algebra $\Bbbk[S_f]$. Restricting the left (right) regular representation of $\mathcal{B}_f^{(x)}$ we get a left (right) action of S_f on $\mathcal{B}_f^{(x)}$.

1.4 Presentation of $\mathcal{B}_f^{(x)}$, and signs of diagrams. By §1.3, $\mathcal{B}_f^{(x)}$ contains a copy of the symmetric group S_f. Moreover, for any pair of distinct indices $i, j \in \{1, 2, \dots, f\}$ we define $\mathbf{h}_{i,j} \in D_{f,1}$ to be the f–diagram with an arc joining i^+ with j^+, an arc joining i^- with j^-, and a vertical edge joining k^+ with k^- for every $k \in \{1, 2, \dots, f\} \setminus \{i, j\}$.

The $\mathbf{h}_{i,j}$'s together with the elements $\mathbf{d}_\sigma \in D_{f,0}$ (for all $\sigma \in S_f$) generate the algebra $\mathcal{B}_f^{(x)}$; the relations among these generators are known too (see, e.g., [DP], §7). As $D_{f,1}$ is a single $D_{f,0}$–orbit (i.e. S_f–orbit), taking only one 1–arc f–diagram is enough, so $\mathcal{B}_f^{(x)}$ is generated, for instance, by $D_{f,0} \bigcup \{\mathbf{h}_{1,2}\}$. In particular, for any $\mathbf{d} \in D_{f,k}$ there exist *unique* $\mathbf{d}_\sigma, \mathbf{d}_\rho \in D_{f,0}$ such that $\mathbf{d} = \mathbf{d}_\sigma\, \mathbf{h}_{1,2} \cdots \mathbf{h}_{2k-1,2k}\, \mathbf{d}_\rho$ and moreover σ and ρ do not invert any of the pairs $(1,2)$, $(3,4)$, ..., $(2k-1, 2k)$. Then we define the *sign of* $\mathbf{d}$ to be $\varepsilon(\mathbf{d}) := \mathit{sgn}(\sigma) \cdot (-1)^k \cdot \mathit{sgn}(\rho)$, which is independent of the given factorization of $\mathbf{d}$.

§2 The standard series and $\mathcal{B}_f^{(x)}$–modules

2.1 The standard series. For any $k \in \{1, 2, \dots [f/2]\}$, let $\mathcal{B}_f^{(x)}\langle k\rangle$ be the vector subspace of $\mathcal{B}_f^{(x)}$ spanned by $D_{f,k}$. We define $\mathcal{B}_f^{(x)}(k) := \bigoplus_{h=k}^{[f/2]} \mathcal{B}_f^{(x)}\langle h\rangle$, so $\mathcal{B}_f^{(x)}(k)$ has $\Bbbk$–basis $\bigcup_{h=k}^{[f/2]} D_{f,h}$. By definition, the $\mathcal{B}_f^{(x)}(k)$'s form a chain of subspaces

$$\mathcal{B}_f^{(x)} = \mathcal{B}_f^{(x)}(0) \supset \mathcal{B}_f^{(x)}(1) \supset \cdots \supset \mathcal{B}_f^{(x)}(k) \supset \cdots \supset \mathcal{B}_f^{(x)}\big([f/2]\big) \supset \{0\} \tag{2.1}$$

which we call "standard series". We denote by $\mathcal{B}_f^{(x)}[k] := \mathcal{B}_f^{(x)}(k) \Big/ \mathcal{B}_f^{(x)}(k{+}1)$ the $(k{+}1)$-th factor (a quotient space) of this series, setting also $\mathcal{B}_f^{(x)}\big([f/2]+1\big) := \{0\}$.

By construction each $\mathcal{B}_f^{(x)}(k)$ is a (two-sided) ideal of $\mathcal{B}_f^{(x)}$: thus every $\mathcal{B}_f^{(x)}[k]$ inherits a structure of associative $\Bbbk$–algebra, one of left $\mathcal{B}_f^{(x)}$–module, and one of right $\mathcal{B}_f^{(x)}$–module. Moreover $\mathcal{B}_f^{(x)}(k) = \mathcal{B}_f^{(x)}\langle k\rangle \oplus \mathcal{B}_f^{(x)}(k{+}1)$, so any basis for $\mathcal{B}_f^{(x)}\langle k\rangle$, taken modulo $\mathcal{B}_f^{(x)}(k{+}1)$, serves as basis for $\mathcal{B}_f^{(x)}[k]$; in particular, we shall use $D_{f,k}$ as a basis of $\mathcal{B}_f^{(x)}[k]$. Note that, since the $\mathcal{B}_f^{(x)}(k)$'s are two-sided ideals of $\mathcal{B}_f^{(x)}$, the $\mathcal{B}_f^{(x)}[k]$'s are $\mathcal{B}_f^{(x)}$–bimodules too.

2.2 The structure of $\mathcal{B}_f^{(x)}[k]$. Let us fix some more notation. Given $h \in \mathbb{N}$, we write $\lambda \vdash h$ to mean that λ is a partition of h ; then for $\lambda \vdash h$ we denote by λ^t the dual partition. Also, if $\lambda \vdash h$ we denote by M_λ the unique associated simple S_h–module, with the assumption that $M_{(h)}$ is the trivial S_h–module and $M_{\underbrace{(1,1,\dots,1)}_{h}}$ is the alternating one.

Now let $k \in \{1,2,\dots[f/2]\}$ be fixed. Consider the set $J_{f,k}$ of (f,k)–junctions defined in §1.2, and define $H_{f,k}$ to be the $\Bbbk$–vector space with basis $J_{f,k}$. In particular, one has that $dim(H_{f,k}) = \big|J_{f,k}\big| = \binom{f}{2k}(2k-1)!!$. Inverting the map $D_{f,k} \longrightarrow S_{f-2k} \times \big(J_{f,k} \times J_{f,k}\big)$ (cf. §1.2) and extending by linearity we define two linear isomorphisms

$$\begin{aligned} \langle\boxtimes\rangle :\ \Bbbk\big[S_{f-2k}\big] \otimes \big(H_{f,k} \otimes H_{f,k}\big) &\longrightarrow \mathcal{B}_f^{(x)}\langle k\rangle \\ \boxtimes :\ \Bbbk\big[S_{f-2k}\big] \otimes \big(H_{f,k} \otimes H_{f,k}\big) &\longrightarrow \mathcal{B}_f^{(x)}[k] \end{aligned} \tag{2.2}$$

By Young's theory, $\Bbbk\big[S_{f-2k}\big]$ splits into $\Bbbk\big[S_{f-2k}\big] = \bigoplus_{\mu\vdash(f-2k)} I_\mu$. Hereafter, each I_μ is a two-sided ideal of $\Bbbk\big[S_f\big]$, and a simple algebra, namely the algebra of linear endomorphisms of the simple S_{f-2k}–module M_μ, which is a full matrix algebra over $\Bbbk$. Then we set

Definition 2.3. *For every $\mu \vdash (f-2k)$ we define $\mathcal{B}_f^{(x)}[k\,;\mu] := \boxtimes\big(I_\mu \otimes (H_{f,k} \otimes H_{f,k})\big)$ and $\mathcal{B}_f^{(x)}\langle k\,;\mu\rangle := \langle\boxtimes\rangle\big(I_\mu \otimes (H_{f,k} \otimes H_{f,k})\big)$. Moreover, we denote by $\mathcal{B}_f^{(x)}(k\,;\mu)$ the preimage of $\mathcal{B}_f^{(x)}[k\,;\mu]$ in $\mathcal{B}_f^{(x)}(k)$.*

2.4 Generalized matrix algebras. We recall (from [Bw1]) the notion of *generalized matrix algebra*: this is any associative $\Bbbk$–algebra A with a finite basis $\{\, e_{ij} \,\}_{i,j\in I}$ for which the multiplication table looks like $e_{ij} \cdot e_{pq} = \sigma^*_{jp}\, e_{iq}$ for some $\sigma^*_{jp} \in \Bbbk$ $\big(\forall\, i,j,p,q \in I\,\big)$. Then we set $\Phi(A) := \big\{\, \sigma^*_{ij} \big\}_{i,j\in I}$. For such an A, the following hold (cf. [Bw1]):

(1) either A is simple, or A has non-zero radical $Rad(A)$, and $A\big/Rad(A)$ is simple;
(2) A is simple if and only if it has an identity element;
(3) $dim_{\Bbbk}(A) = h^2$, for some $h \in \mathbb{N}$, and $dim_{\Bbbk}\big(Rad(A)\big) = h^2 - rk\big(\Phi(A)\big)^2$;
(4) the nilpotency degree of $Rad(A)$ is at most 3 .

Hereafter, *by "radical" $Rad(\mathfrak{A})$ of any (possibly non-unital) algebra $\mathfrak{A}$* we shall mean the intersection of the annihilators of all its simple left modules (Brown's definition is the set of properly nilpotent elements: for generalized matrix algebras, the two definitions coincide).

The most general result about the structure of $\mathcal{B}_f^{(x)}[k]$ is the next one:

Theorem 2.5 *([Bw2], §§2.2–3; [KX], §§3–5).* *For any $\mu \vdash (f–2k)$, the subspace $\mathcal{B}_f^{(x)}[k\,;\mu]$ is a two-sided ideal of $\mathcal{B}_f^{(x)}[k]$, the algebra $\mathcal{B}_f^{(x)}[k]$ splits as $\mathcal{B}_f^{(x)}[k] = \bigoplus_{\mu\vdash(f-2k)} \mathcal{B}_f^{(x)}[k\,;\mu]$ and the $\mathcal{B}_f^{(x)}[k\,;\mu]$'s are pairwise non-isomorphic generalized matrix algebras.*

Moreover, every $\mathcal{B}_f^{(x)}(k\,;\mu)$ is a two-sided ideal of $\mathcal{B}_f^{(x)}(k)$, and every $\mathcal{B}_f^{(x)}[k\,;\mu]$ is a $\mathcal{B}_f^{(x)}$–sub-bimodule of $\mathcal{B}_f^{(x)}[k]$, for any $\mu \vdash (f-2k)$.

2.6 Representations of $\mathcal{B}_f^{(x)}$. Let $0 \leq k \leq f/2$, and let $H_{f,k}$ be the vector space defined in §2.2. For any $\mu \vdash (f-2k)$ $(k \in \{0,1,\dots,[f/2]\})$ we define $H^{\mu}_{f,k} := M_\mu \otimes H_{f,k}$. We endow $H^{\mu}_{f,k}$ with a structure of $\mathcal{B}_f^{(x)}$–module, following Kerov (cf. [Ke], [HW1–2], [GP]).

Let $\mathbf{d}$ be an f–diagram, and let v be an (f,k)–junction. For all $i = 1,\dots,f$, connect the i–th bottom vertex of $\mathbf{d}$ with the i–th vertex of v, let $C(\mathbf{d},v)$ be the number of loops occurring in the new graph $\Gamma(\mathbf{d},v)$ obtained in this way, and let $a \star v$ be the graph made of the vertices of the top line of $\mathbf{d}$, connected by an edge iff they are connected (by an edge or a path) in the new graph $\Gamma(\mathbf{d},v)$. Then $\mathbf{d} \star v \in J_{f,k'}$, with $k' \geq k$ and $k' = k$ iff each pair of vertices of v which are connected by a path in $\Gamma(\mathbf{d},v)$ are in fact joined by an edge in v: in this case we say that the junction v is *admissible* for the diagram $\mathbf{d}$. We set

$$\mathbf{d}.v \;:=\; x^{C(\mathbf{d},v)}\, \mathbf{d} \star v \quad \textit{if } v \textit{ is admissible for } \mathbf{d}\,, \qquad \mathbf{d}.v \;:=\; 0 \quad \textit{otherwise}\,.$$

See [Ga], §2.11 for some simple examples.

To any pair $(\mathbf{d},v) \in D_f \times J_{f,k}$ we can also attach an element $\pi(\mathbf{d},v) \in S_{f-2k}$: this is the permutation which carries — through the graph $\Gamma(\mathbf{d},v)$ — the isolated vertices of v into the isolated vertices of $\mathbf{d} \star v$ (one takes into account only the relative position of the isolated vertices in v, $\mathbf{d} \star v$) in case v is admissible for a, otherwise it is id.

Proposition 2.7 *(cf. [Ke], [HW1], [KX], [CDM]).* *Assume $Char(\Bbbk) = 0$ or $Char(\Bbbk) > f$.*

(a) Linear extension of the rule $\mathbf{d}.(u\otimes v) := \pi(\mathbf{d},v).u \otimes \mathbf{d}.v$ for every $(\mathbf{d},v) \in D_f \times J_{f,k}$ endows $H^{\mu}_{f,k}$ with a well-defined structure of module over $\mathcal{B}_f^{(x)}$. Then $H^{\mu}_{f,k}$ is also a module over $\mathcal{B}_f^{(x)} / \mathcal{B}_f^{(x)}(k+1)$ and over $\mathcal{B}_f^{(x)}[k] := \mathcal{B}_f^{(x)}(k) / \mathcal{B}_f^{(x)}(k+1)$.

(b) The various modules $H^{\mu}_{f,k}$, for different pairs (k,μ) — over any of the previous algebras — are pairwise non-isomorphic.

(c) If $\mathcal{B}_f^{(x)}$ is semisimple, then every $\mathcal{B}_f^{(x)}$–module $H^{\mu}_{f,k}$ is simple and, conversely, any simple $\mathcal{B}_f^{(x)}$–module is isomorphic to one of the $H^{\mu}_{f,k}$'s.

(d) If every $\mathcal{B}_f^{(x)}$–module $H^{\mu}_{f,k}$ is simple, then the algebra $\mathcal{B}_f^{(x)}$ is semisimple.

(e) Every simple $\mathcal{B}_f^{(x)}$–module is a quotient of some $H^{\mu}_{f,k}$. Conversely, each $H^{\mu}_{f,k}$ has a simple quotient, but for the case of even f, $k = f/2$ and $x = 0$. Indeed, these simple quotients are in bijection with the isoclasses of simple $\mathcal{B}_f^{(x)}$–modules.

§3 Semisimple quotients of $\mathcal{B}_f^{(x)}$ and of $\mathcal{B}_f^{(x)}$–modules

3.1 Splitting the semisimple quotient of $\mathcal{B}_f^{(x)}$. Let $Rad\Big(\mathcal{B}_f^{(x)}\Big)$, resp. $\mathcal{S}_f^{(x)} := \mathcal{B}_f^{(x)}\Big/Rad\Big(\mathcal{B}_f^{(x)}\Big)$, resp. $\pi_1 : \mathcal{B}_f^{(x)} \twoheadrightarrow \mathcal{S}_f^{(x)}$ denote respectively the radical of $\mathcal{B}_f^{(x)}$, its semisimple quotient, and the canonical epimorphism. By general theory, $\mathcal{S}_f^{(x)}$ has a direct sum decomposition $\mathcal{S}_f^{(x)} = \bigoplus_{i\in I} S_i$ in which the S_i's are two-sided ideals which are simple algebras. Of course, $\mathcal{S}_f^{(x)}$ is a $\mathcal{B}_f^{(x)}$–left/right/bi-module, so the S_i's are left/right/bi-submodules over $\mathcal{B}_f^{(x)}$, and each S_i is simple as a $\mathcal{B}_f^{(x)}$-bimodule. In this section we collect some information about what the set I has to be and what the blocks S_i's arise from.

Define $\mathcal{S}_f^{(x)}(k) := \pi_1\left(\mathcal{B}_f^{(x)}(k)\right)$ for all $k = 1, 2, \dots, [f/2]$: then each $\mathcal{S}_f^{(x)}(k)$ is a two-sided ideal of $\mathcal{S}_f^{(x)}$, hence also a left/right/bi-submodule over $\mathcal{B}_f^{(x)}$. In particular, there exists $I_k \subseteq I$ such that $\mathcal{S}_f^{(x)}(k) = \bigoplus_{i\in I_k} S_i$. Applying π_1 to (2.1), one gets a series

$$\mathcal{S}_f^{(x)} = \mathcal{S}_f^{(x)}(0) \supseteq \mathcal{S}_f^{(x)}(1) \supseteq \cdots \supseteq \mathcal{S}_f^{(x)}(k) \supseteq \cdots \supseteq \mathcal{S}_f^{(x)}([f/2]) \supseteq \{0\}$$

which corresponds to the chain of inclusions $I = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_k \supseteq \cdots \supseteq I_{[f/2]} \supseteq \emptyset$. So the algebra $\mathcal{S}_f^{(x)}[k] := \mathcal{S}_f^{(x)}(k)\Big/\mathcal{S}_f^{(x)}(k+1)$ (for all k, setting also $\mathcal{S}_f^{(x)}\big([f/2]+1\big) := 0$) is well defined, and splits (up to isomorphisms) as $\mathcal{S}_f^{(x)}[k] = \bigoplus_{j\in I_k\setminus I_{k+1}} S_j$, a direct sum of simple algebras. Moreover (up to isomorphisms), $\mathcal{S}_f^{(x)} = \bigoplus_{k=0}^{[f/2]} \mathcal{S}_f^{(x)}[k]$.

There is an algebra epimorphism $\pi_1^* : \mathcal{B}_f^{(x)}[k] \twoheadrightarrow \mathcal{S}_f^{(x)}[k]$ which together with π_1 and the canonical projections from $\mathcal{B}_f^{(x)}(k)$ to $\mathcal{B}_f^{(x)}[k]$ and from $\mathcal{S}_f^{(x)}(k)$ to $\mathcal{S}_f^{(x)}[k]$ forms a commutative diagram. Finally, define $\mathcal{S}_f^{(x)}[k\,;\mu] := \pi_1^*\left(\mathcal{B}_f^{(x)}[k\,;\mu]\right)$, for all k and all $\mu \vdash \big(f-2k\big)$.

The next result gives us the required information about the splitting of $\mathcal{S}_f^{(x)}$.

Proposition 3.2. *The algebra $\mathcal{S}_f^{(x)}$ splits as* $\mathcal{S}_f^{(x)} = \bigoplus_{k=0}^{[f/2]} \bigoplus_{\mu\vdash(f-2k)} \mathcal{S}_f^{(x)}[k\,;\mu]$, *where every $\mathcal{S}_f^{(x)}[k\,;\mu]$ is a non-zero simple algebra, unless f is even and $(x,k) = \big(0, f/2\big)$: in this case, $\mathcal{S}_f^{(0)}\big[f/2\,;(0)\big] \equiv \mathcal{S}_f^{(0)}\big[f/2\big] = 0$.*

Proof. First suppose $\mathcal{S}_f^{(x)}[k\,;\mu] \neq 0$. As $\mathcal{B}_f^{(x)}[k\,;\mu]$ is a generalized matrix algebra (Theorem 2.5), the same is true for $\mathcal{S}_f^{(x)}[k\,;\mu]$ too; but $\mathcal{S}_f^{(x)}[k\,;\mu]$ is semisimple, by construction, hence — §2.4(1) — it must be simple. Second, from the construction in §3.1 we get also

$$\begin{aligned}\bigoplus_{i\in J_k} S_i \;=\; \mathcal{S}_f^{(x)}[k] \;=\; \pi_1^*\left(\mathcal{B}_f^{(x)}[k]\right) \;=\; \pi_1^*\left(\bigoplus_{\mu\vdash(f-2k)} \mathcal{B}_f^{(x)}[k\,;\mu]\right) \;=&\\ =\; \sum_{\mu\vdash(f-2k)} \pi_1^*\left(\mathcal{B}_f^{(x)}[k\,;\mu]\right) \;=\; \sum_{\mu\vdash(f-2k)} \mathcal{S}_f^{(x)}[k\,;\mu] \quad .&\end{aligned}$$

The summands in left hand sides are two-sided simple ideals, and the same is true on right hand side: but the sum on the left is direct, and this easily implies that each $\mathcal{S}_f^{(x)}[k\,;\mu]$ is one of the S_i and vice versa, so that $\mathcal{S}_f^{(x)}[k] = \bigoplus_{\mu\vdash(f-2k)} \mathcal{S}_f^{(x)}[k\,;\mu]$. Finally, since $\mathcal{S}_f^{(x)} = \bigoplus_{k=0}^{[f/2]} \mathcal{S}_f^{(x)}[k]$, we conclude that the splitting in the claim does hold.

Now we show that $\mathcal{S}_f^{(x)}[k\,;\mu] \neq 0$ for all k and μ when $(x,k) \neq \big(0, f/2\big)$. By definitions, §2.4, Theorem 2.5 and Proposition 2.7, we have that

$$\mathcal{S}_f^{(x)}[k\,;\mu] = \{0\} \iff Rad\left(\mathcal{B}_f^{(x)}[k\,;\mu]\right) = \mathcal{B}_f^{(x)}[k\,;\mu] \iff rk\Big(\Phi\big(\mathcal{B}_f^{(x)}[k\,;\mu]\big)\Big) = 0$$

and the last condition on the right clearly holds if and only if the matrix $\Phi\big(\mathcal{B}_f^{(x)}[k\,;\mu]\big)$ of all structure constants of $\mathcal{B}_f^{(x)}[k\,;\mu]$ is zero. But this occurs exactly if and only if $x=0$ and $k=f/2$, for even f. In all other cases one has $\mathcal{S}_f^{(x)}[k\,;\mu] \neq 0$, as claimed. $\square$

Corollary 3.3. *If $f \in \mathbb{N}$ is even, then* $Rad\Big(\mathcal{B}_f^{(0)}\big(f/2\big)\Big) = \mathcal{B}_f^{(0)}\big(f/2\big) = \mathcal{B}_f^{(0)}\big[f/2\big]$.

Indeed, the above also follows easily when remarking that $\mathcal{B}_f^{(0)}\big[f/2\big] = \mathcal{B}_f^{(0)}\big(f/2\big)$, and $\mathcal{B}_f^{(0)}\big(f/2\big)$ is just the $\Bbbk$–vector space $\Bbbk^{(f-1)!!^2}$ endowed with the trivial multiplication.

3.4 Semisimplicity of $\mathcal{B}_f^{(x)}$. A general criterion for the semisimplicity of $\mathcal{B}_f^{(x)}$ is given in [Ru], [RS]. For the cases we shall deal with, it reads as follows: if $x = n \in \mathbb{N}_+$, then

$$\mathcal{B}_f^{(n)} \textit{ and } \mathcal{B}_f^{(-2n)} \textit{ are semisimple } \iff n \geq f-1 \textit{ and } Char(\Bbbk) = 0 \textit{ or } Char(\Bbbk) > f$$

$$\mathcal{B}_f^{(0)} \textit{ is semisimple } \iff f \in \{1,3,5\} \textit{ and } Char(\Bbbk) = 0 \textit{ or } Char(\Bbbk) > f$$

§4 Brauer algebras in Invariant Theory

4.1 The Fundamental Theorems of Invariant Theory. Let $f \in \mathbb{N}_+$ and $n \in \mathbb{N}$. Let V be a $\Bbbk$–vector space of dimension n, endowed with a non-degenerate symmetric bilinear form $(\ ,\ )$, and let $O(V)$ be the associated orthogonal group. Also, let W be a $\Bbbk$–vector space of dimension $2n$, endowed with a non-degenerate skew-symmetric bilinear form $\langle\ ,\ \rangle$, and let $Sp(W)$ be the associated symplectic group. There exist canonical isomorphisms $V \xrightarrow{\cong} V^*, v \mapsto (v,\ \cdot\ )$, $W \xrightarrow{\cong} W^*, w \mapsto \langle w,\ \cdot\ \rangle$, which yield also isomorphisms

$$\Theta_V : V \otimes V \xrightarrow{\quad\cong\quad} End(V) \qquad\qquad \Theta_W : W \otimes W \xrightarrow{\quad\cong\quad} End(W)$$

$$v_1 \otimes v_2 \mapsto \Theta_V\,(v_1 \otimes v_2)\Big(v \mapsto \big(v_1, v\big)v_2\Big) \qquad w_1 \otimes w_2 \mapsto \Theta_W\,(w_1 \otimes w_2)\Big(w \mapsto \big\langle w_1, w\big\rangle w_2\Big)$$

In this setting, we define $\psi_V := \Theta_V^{-1}\,(id_V) \in V \otimes V$ and $\psi_W := \Theta_W^{-1}\,(id_W) \in W \otimes W$.

Definition 4.2. *Fix $f \in \mathbb{N}_+$. For each pair $p, q \in \{1,2,\dots,f\}$ with $p \neq q$ we define*

(a) a linear contraction operator $\Phi_{p,q} : V^{\otimes(f+2)} \longrightarrow V^{\otimes f}$ (for $p<q$, say), given by $\Phi_{p,q}\big(v_1 \otimes v_2 \otimes \cdots \otimes v_{f+2}\big) = \big(v_p\,,\, v_q\big) \cdot v_1 \otimes \cdots \widehat{v_p} \otimes \cdots \otimes \widehat{v_q} \otimes \cdots \otimes v_{f+2}$;

(b) a linear insertion operator $\Psi_{p,q} : V^{\otimes f} \longrightarrow V^{\otimes (f+2)}$ *, obtained by inserting the element* ψ_V *in the positions* p*,* q*;*

(c) an operator $\tau_{p,q} : V^{\otimes f} \longrightarrow V^{\otimes f}$ *defined by* $\tau_{p,q} := \Psi_{p,q} \circ \Phi_{p,q} \; \big(\in End\big(V^{\otimes f}\big) \big)$ *.*

The same definitions with $\langle \; , \; \rangle$ *instead of* $(\; , \;)$ *give operators* $\Phi_{p,q} : W^{\otimes (f+2)} \longrightarrow W^{\otimes f}$*,* $\Psi_{p,q} : W^{\otimes f} \longrightarrow W^{\otimes (f+2)}$*,* $\tau_{p,q} : W^{\otimes f} \longrightarrow W^{\otimes f}$ *in the symplectic case.*

In addition, recall that the symmetric group S_f acts on $V^{\otimes f}$ or $W^{\otimes f}$ by

$$\sigma \; : \; u_1 \otimes u_2 \otimes \cdots \otimes u_f \; \mapsto \; u_{\sigma^{-1}(1)} \otimes u_{\sigma^{-1}(2)} \otimes \cdots \otimes u_{\sigma^{-1}(f)} \qquad \forall \;\; \sigma \in S_f$$

These constructions are connected with Brauer algebras, as we now explain. The connection comes from a classical result over $\mathbb{C}$, which later has been generalized in [DP]. The technical condition required there, *for a given, fixed* $f \in \mathbb{N}_+$, is the following:

Every polynomial $p(x) \in \Bbbk[x]$ *of degree* f *which vanishes on* $\Bbbk$ *is identically 0 .* (4.1)

In this section, we assume that the field $\Bbbk$ and $f \in \mathbb{N}_+$ satisfy condition (4.1).

For instance, if $Char(\Bbbk) = 0$ or $Char(\Bbbk) > f$, then $\Bbbk$ does satisfy (4.1). Actually, thanks to [DP], we can even assume $\Bbbk$ to be any unital commutative ring satisfying (4.1).

4.3 Brauer algebras versus centralizer algebras. When the parameter x is an integer, the Brauer algebra $\mathcal{B}_f^{(x)}$ is strictly related with the invariant theory for the orthogonal or the symplectic groups. Indeed, it is a "lifting" of one of the centralizer algebras $End_{O(V)}\big(V^{\otimes f}\big)$ or $End_{Sp(W)}\big(W^{\otimes f}\big)$, in the sense of the following result:

Theorem 4.4 (cf. [Br], [DP])**.** *Let* $n \in \mathbb{N}_+$ *, and let* V *and* W *respectively be an* n*–dimensional orthogonal vector space and a* $2n$*–dimensional symplectic vector space over* $\Bbbk$ *. Then there exist well-defined* $\Bbbk$*–algebra epimorphisms, which are isomorphisms iff* $n \geq f$ *,*

$$\begin{array}{ll} \pi_V : \; \mathcal{B}_f^{(n)} \longrightarrow\!\!\!\!\!\rightarrow End_{O(V)}\big(V^{\otimes f}\big) & \qquad \pi_W : \; \mathcal{B}_f^{(-2n)} \longrightarrow\!\!\!\!\!\rightarrow End_{Sp(W)}\big(W^{\otimes f}\big) \\ \qquad \mathbf{d}_\sigma \mapsto \sigma \, , \;\; \mathbf{h}_{p,q} \mapsto \tau_{p,q} & \qquad \qquad \mathbf{d}_\sigma \mapsto sgn(\sigma)\, \sigma \, , \;\; \mathbf{h}_{p,q} \mapsto -\tau_{p,q} \end{array}$$

Moreover, $End_{\mathcal{B}_f^{(n)}}\big(V^{\otimes f}\big) = \big\langle O(V) \big\rangle$ *and* $End_{\mathcal{B}_f^{(-2n)}}\big(V^{\otimes f}\big) = \big\langle Sp\,(W) \big\rangle$ *, where* $\big\langle X \big\rangle$ *denotes the subalgebra of* $End_{\Bbbk}\big(U^{\otimes f}\big)$ *— for* $U \in \big\{V, W\big\}$ *— generated by* X *.*

The previous theorem — which follows from the First Fundamental Theorem of Invariant Theory for $O(V)$ and $Sp(V)$ — concerns either *positive* or *even negative* values of x . The case of *odd negative* parameter can be reduced to the odd positive case: see [Wz], Corollary 3.5. Finally, we shall cope with the case $x = 0$ through a direct approach.

In order to describe the kernels of π_V and π_W , we introduce some new objects.

4.5 Diagrammatic minors and diagrammatic Pfaffians. Let us consider the polynomial rings (in the symmetric or antisymmetric variables x_{ij})

$$A^O := \Bbbk[x_{ij}]_{i,j=1, i \neq j}^{2f} \Big/ \, (x_{ij} = x_{ji}) \;\; , \qquad A^{Sp} := \Bbbk[x_{ij}]_{i,j=1, i \neq j}^{2f} \Big/ \, (x_{ij} = -x_{ji})$$

For $X \in \{O, Sp\}$, define A_f^X (the space of *multilinear elements* in A^X) to be the $\Bbbk$–span of all monomials (of degree f) $x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_f j_f}$ such that $(i_1, j_1, i_2, j_2, \dots, i_f, j_f)$ is a permutation of $\{1, 2, 3, 4, \dots, 2f\}$. Clearly, A_f^X has a natural structure of S_{2f}–module.

In general, given two f–tuples $\mathbf{i} := (i_1, i_2, \dots, i_f)$ and $\mathbf{j} := (j_1, j_2, \dots, j_f)$ such that $\{i_1, \dots, i_f\} \cup \{j_1, \dots, j_f\} = \{1, 2, \dots, 2f-1, 2f\}$, we write $x_{\mathbf{i},\, \mathbf{j}} := x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_f j_f}$. Then we define $\Bbbk$–vector space isomorphisms $\Phi_V \colon A_f^O \xrightarrow{\cong} \mathcal{B}_f^{(n)}$, via $x_{\mathbf{i},\, \mathbf{j}} \mapsto \mathbf{d}_{\mathbf{i},\, \mathbf{j}}$, and $\Phi_W \colon A_f^{Sp} \xrightarrow{\cong} \mathcal{B}_f^{(-2n)}$, via $x_{\mathbf{i},\, \mathbf{j}} \mapsto \varepsilon(\mathbf{d}_{\mathbf{i},\, \mathbf{j}}) \cdot \mathbf{d}_{\mathbf{i},\, \mathbf{j}}$, where $\varepsilon(\mathbf{d}_{\mathbf{i},\mathbf{j}})$ is the "sign" of $\mathbf{d}_{\mathbf{i},\mathbf{j}}$ defined as in §1.4. Using them, an S_{2f}–action is defined on $\mathcal{B}_f^{(n)}$, resp. $\mathcal{B}_f^{(-2n)}$, based upon that on A_f^O, resp. A_f^{Sp}, letting $\sigma \in S_{2f}$ act on $\mathcal{B}_f^{(n)}$, resp. $\mathcal{B}_f^{(-2n)}$, as $\Phi_V \circ \sigma \circ \Phi_V^{-1}$, resp. $\Phi_W \circ \sigma \circ \Phi_W^{-1}$ (this action is studied in depth in [Hu] and in [DH]).

Definition 4.6

(a) We call **(diagrammatic) minor** *of order r ($\in \mathbb{N}_+$) every element of $\mathcal{B}_f^{(x)}$ which is the image through Φ_V of an element of type*

$$\textstyle\sum_{\sigma \in S_r} sgn(\sigma) \cdot x_{i_1 j_{\sigma(1)}} x_{i_2 j_{\sigma(2)}} \cdots x_{i_r j_{\sigma(r)}} \cdot x_{i_{r+1} j_{r+1}} x_{i_{r+2} j_{r+2}} \cdots x_{i_{f-1} j_{f-1}} x_{i_f j_f} \qquad (4.2)$$

with $\{i_1, \dots, i_r\} \cup \{j_1, \dots, j_r\} \cup \{i_{r+1}, \dots, i_f\} \cup \{j_{r+1}, \dots, j_f\} = \{1, 2, 3, \dots, 2f\}$.

We denote by $Min_{f;\,r}^{(x)}$ the set of all (diagrammatic) minors of order r in $\mathcal{B}_f^{(x)}$.

(b) We call **(diagrammatic) Pfaffian** *of order $2r$ ($\in 2\mathbb{N}_+$) every element of $\mathcal{B}_f^{(x)}$ which is the image through Φ_W of an element of type*

$$\textstyle\sum_{\substack{h_1<k_1,h_2<k_2,\dots \\ h_1<h_2<h_3<\cdots}} sgn \begin{pmatrix} 1 & 2 & \dots & 2r{-}1 & 2r \\ h_1 & k_1 & \dots & h_r & k_r \end{pmatrix} \cdot x_{h_1 k_1} x_{h_2 k_2} \cdots x_{h_r k_r} x_{i_{r+1} j_{r+1}} \cdots x_{i_f j_f} \qquad (4.3)$$

with $\{h_1, \dots, h_r\} \cup \{k_1, \dots, k_r\} \cup \{i_{r+1}, \dots, i_f\} \cup \{j_{r+1}, \dots, j_f\} = \{1, 2, 3, \dots, 2f\}$.

We denote by $Pf_{f;\,r}^{(x)}$ the set of all (diagrammatic) Pfaffians of order $2\,r$ in $\mathcal{B}_f^{(x)}$.

(c) If X is any given (diagrammatic) minor or Pfaffian, we call fixed edge of X *any edge which occurs the same in all diagrams occurring in the expansion of X. We call* fixed vertex of X *any vertex (in $\mathbb{V}_f$) belonging to a fixed edge of X. We call* fixed part of X *the datum of all fixed edges and all fixed vertices of X.*

(d) If X is any given (diagrammatic) minor or Pfaffian, we call moving vertex of X *any vertex (in $\mathbb{V}_f$) which is not fixed in X. We call* moving part of X *the datum of all vertices which are not fixed in X.*

Remarks 4.7. *(a)* By definitions and Proposition 4.6, any diagrammatic minor of order r is an *alternating sum of f–diagrams*: to be precise, it is an S_r–antisymmetric sum of f–diagrams. On the other hand, due to the sign entering in the definition of α_W, all diagrams occurring in the expansion of a diagrammatic Pfaffian appear there with the same sign. Thus, each diagrammatic Pfaffian is (up to sign) just a simple *sum of f–diagrams*.

(b) Let δ_r be a minor of order r. Its moving vertices may be partitioned into two sets I, J (each of r elements) so that, looking at all the diagrams occurring in the expansion of δ_r, no vertex in one of these sets is ever joined to a vertex in the same set, but it is

joined to each vertex in the other set. These I and J correspond, via Φ_V, to the set of rows and the set of columns (or vice versa) in the matrix $\big(x_{ij}\big)_{i,j=1}^{2f}$ on which the minor corresponding to δ_r is computed. So in the sequel expressions like "v is a row vertex and w is a column vertex" will mean that v and w are moving vertices which belong one to I and the other to J. Similarly, by "v and w are both row vertices" or "column vertices" will mean that they are moving vertices which both belong to I or both to J. Indeed, a minor δ_r is determined, up to sign, by: *(i)* assigning its fixed part; *(ii)* assigning the sets I and J, each endowed with a labelling of its vertices by $\{1,2,\dots,r\}$; *(iii)* joining every vertex in one set — say I — to a vertex in the other set — say J — according to a permutation $\sigma \in S_r$, so to get an f–diagram $\mathbf{d}(\sigma)$; *(iv)* adding up the diagrams $\mathbf{d}(\sigma)$ with coefficient $sgn(\sigma)$, for all $\sigma \in S_r$: this eventually gives $\pm\mathbf{d}_r$ (the sign depends on the labellings).

(c) The step *(iii)* above may be better understood as follows. First, join every vertex in I with the vertex in J labelled with the same number: this gives the diagram $\mathbf{d}(id)$ which, outside the fixed part, is given by the r edges $\{i_1,j_1\},\dots,\{i_r,j_r\}$ (where $\{i_1,\dots,i_r\}=I$, $\{j_1,\dots,j_r\}=J$). Second, let S_r act on J, and let $\mathbf{d}[\sigma]$ be the diagram which is equal to $\mathbf{d}(id)$ in the fixed part and outside it is given by the r edges $\big\{i_1,\sigma(j_1)\big\},\dots,\big\{i_r,\sigma(j_r)\big\}$: then $\mathbf{d}[\sigma]=\mathbf{d}(\sigma)$. Then we can also write δ_r as an S_r–antisymmetric sum

$$\delta_r \;=\; \textstyle\sum_{\sigma\in S_r} sgn(\sigma)\,\mathbf{d}(\sigma) \;=\; \sum_{\sigma\in S_r} sgn(\sigma)\,\mathbf{d}[\sigma] \;=\; \sum_{\sigma\in S_r} sgn(\sigma)\,\sigma.\mathbf{d}[id]$$

(d) The counterpart for Pfaffians of *(b)* and *(c)* above is that every Pfaffian of order $2r$ is the sum of all diagrams obtained by assigning the fixed part and joining the $2r$ vertices in the moving part with r edges in all possible ways.

(e) Examples of diagrammatic minors or Pfaffians can be found in [Ga], Example 3.6.

The importance of diagrammatic minors and Pfaffians lies in the following:

Theorem 4.8. *([Ga], §3) Let $n \in \mathbb{N}_+$ and $f \in \mathbb{N}_+$ with (4.1) satisfied by the field $\Bbbk$.*

(a) The kernel of $\pi_V : \mathcal{B}_f^{(n)} \twoheadrightarrow End_{O(V)}\big(V^{\otimes f}\big)$ is the $\Bbbk$–span of the set of all diagrammatic minors in $\mathcal{B}_f^{(n)}$ of order $n+1$. In particular, it is an S_{2f}–submodule.

(b) The kernel of $\pi_W : \mathcal{B}_f^{(-2n)} \twoheadrightarrow End_{Sp(W)}\big(W^{\otimes f}\big)$ is the $\Bbbk$–span of the set of all diagrammatic Pfaffians in $\mathcal{B}_f^{(-2n)}$ of order $2(n+1)$. In particular, it is an S_{2f}–submodule.

4.9 Comparison with others' work. The problem of describing $Ker(\pi_W)$ is solved also by Hu in [Hu]. He studies the action of S_{2f} onto D_f, and the S_{2f}–module structure of the $\mathbb{Z}$–algebra $\mathcal{B}_f^{(x)}(\mathbb{Z})$, the $\mathbb{Z}$–span of D_f. As main results, he finds a new basis, and Specht filtrations, for $\mathcal{B}_f^{(x)}(\mathbb{Z})$, and a characteristic free description of $Ker(\pi_W)$, proving that it is an S_{2f}–submodule of $\mathcal{B}_f^{(x)}(\mathbb{Z})$. Some of his results can be compared to ours: for instance, Theorem 3.4 in [Hu], describing $Ker(\pi_W)$, coincides with our Theorem 4.8*(b)*.

Another interesting point is Lemma 3.3 in [Hu], which proves that certain sums of diagrams do belong to $Ker(\pi_W)$. Well, definitions imply that any such sum is simply (a special type of) a *Pfaffian* — of order 2(a+b) — in the sense of our Definition 4.6*(b)* (see Remarks 4.7 too). Therefore, Hu's lemma is just a special case of our Theorem 4.8*(b)*.

Similarly, an analogous solution for $Ker(\pi_V)$ is (just very recently) provided in [DH].

§5 Within the radical of $\mathcal{B}_f^{(x)}$

5.1 From Invariant Theory to $Rad\big(\mathcal{B}_f^{(x)}\big)$. In the present work, relying on the results of Invariant Theory in §§3–4, we locate a large family of elements in $Rad\big(\mathcal{B}_f^{(x)}\big)$, namely diagrammatic minors or Pfaffians, when x is an integer which is not odd negative.

Until §5.15, **we assume now** $Char(\Bbbk) = 0$. *In particular, (4.1) holds for any* $f \in \mathbb{N}_+$.

Being in characteristic zero, the orthogonal groups are linearly reductive. Hence, by general theory, $End_{O(V)}\big(V^{\otimes f}\big)$ is semisimple; so the epimorphism $\pi_V : \mathcal{B}_f^{(n)} \twoheadrightarrow End_{O(V)}\big(V^{\otimes f}\big)$ factors through $\pi_1 : \mathcal{B}_f^{(n)} \twoheadrightarrow \mathcal{S}_f^{(n)}$. In other words, π_V is the composition of maps $\pi_V = \pi_2 \circ \pi_1 : \mathcal{B}_f^{(n)} \twoheadrightarrow \mathcal{S}_f^{(n)} \twoheadrightarrow End_{O(V)}\big(V^{\otimes f}\big)$ where π_2 is the map given by the universality of the semisimple quotient. It follows that $Rad\big(\mathcal{B}_f^{(n)}\big) = Ker\big(\pi_1\big) \subseteq Ker\big(\pi_V\big)$. By Theorem 4.8*(a)*, the latter space is the $\Bbbk$–span of all (diagrammatic) minors of order $n+1$ in $\mathcal{B}_f^{(n)}$. So we shall look here for elements of $Rad\big(\mathcal{B}_f^{(n)}\big)$, in particular we shall determine (in Theorem 5.3 and Theorem 5.5) exactly which of those minors do belong to $Rad\big(\mathcal{B}_f^{(n)}\big)$.

The same arguments give also $\pi_W = \pi_2 \circ \pi_1 : \mathcal{B}_f^{(-2n)} \twoheadrightarrow \mathcal{S}_f^{(-2n)} \twoheadrightarrow End_{Sp(W)}\big(W^{\otimes f}\big)$, so $Rad\big(\mathcal{B}_f^{(-2n)}\big) = Ker\big(\pi_1\big) \subseteq Ker\big(\pi_W\big)$. The latter space is the $\Bbbk$–span of all (diagrammatic) Pfaffians of order $2(n+1)$ in $\mathcal{B}_f^{(-2n)}$, by Theorem 4.8*(b)*. Hence we shall look here for elements of $Rad\big(\mathcal{B}_f^{(-2n)}\big)$, and we shall determine exactly — in Theorem 5.3 and Theorem 5.5 again — which of those Pfaffians actually do belong to $Rad\big(\mathcal{B}_f^{(-2n)}\big)$.

Proposition 5.2. *Let* $n \in \mathbb{N}_+$, $f \in \mathbb{N}_+$. *Then*

(*a*)
$$Ker(\pi_V) \bigcap \Big(\bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu \vdash (f-2h) \\ \mu_1^t + \mu_2^t \leq n}} \mathcal{B}_f^{(n)} \langle h\,;\mu \rangle \Big) \subseteq Rad\big(\mathcal{B}_f^{(n)}\big)$$

(*b*)
$$\mathcal{B}_f^{(0)}\big(\big[(f+1)/2\big]\big) \subseteq Rad\big(\mathcal{B}_f^{(0)}\big)$$

(*c*)
$$Ker(\pi_W) \bigcap \Big(\bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu \vdash (f-2h) \\ \mu_1^t \leq n}} \mathcal{B}_f^{(-2n)} \langle h\,;\mu \rangle \Big) \subseteq Rad\big(\mathcal{B}_f^{(-2n)}\big)$$

Proof. *(a)* By [Wz], §3, we know that $End_{O(V)}\big(V^{\otimes f}\big)$ splits into a direct sum of pairwise non-isomorphic simple algebras as $End_{O(V)}\big(V^{\otimes f}\big) = \bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu \vdash (f-2h) \\ \mu_1^t + \mu_2^t \leq n}} A[h\,;\mu]$, where μ^t is the dual partition to μ, as in §2.2. Furthermore, the analysis in §§3–4 shows that $A[h\,;\mu] = \pi_2\Big(\mathcal{S}_f^{(n)}[h\,;\mu]\Big) = \pi_2\Big(\pi_1^*\Big(\mathcal{B}_f^{(n)}[h\,;\mu]\Big)\Big)$. Now, the map

$$\pi_2 \; : \bigoplus_{h=0}^{[f/2]} \bigoplus_{\mu \vdash (f-2h)} \mathcal{S}_f^{(n)}[h\,;\mu] \; = \; \mathcal{S}_f^{(n)} \twoheadrightarrow End_{O(V)}\big(V^{\otimes f}\big) \; = \; \bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu \vdash (f-2h) \\ \mu_1^t + \mu_2^t \leq n}} A[h\,;\mu]$$

(cf. Proposition 3.2) must have kernel $Ker(\pi_2) = \bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1^t+\mu_2^t>n}} \mathcal{S}_f^{(n)}[h\,;\mu]$, and it must map $\bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1^t+\mu_2^t\leq n}} \mathcal{S}_f^{(n)}[h\,;\mu]$ isomorphically onto $\bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1^t+\mu_2^t\leq n}} A[h\,;\mu] = End_{O(V)}\big(V^{\otimes f}\big)$.

Let now consider an element $y \in Ker(\pi_V) \bigcap \left(\bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1^t+\mu_2^t\leq n}} \mathcal{B}_f^{(n)}\langle h\,;\mu\rangle \right)$. Then $\pi_1(y)$ belongs to $\bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1^t+\mu_2^t\leq n}} \mathcal{S}_f^{(n)}[h\,;\mu]$. But on the latter space π_2 acts injectively, hence $\pi_2\big(\pi_1(y)\big) = \pi_V(y) = 0$ implies $\pi_1(y) = 0$, so $y \in Ker\big(\pi_1\big) \equiv Rad\Big(\mathcal{B}_f^{(n)}\Big)$, q.e.d.

(b) If f is odd the claim is empty, and there is nothing to prove. If f is even, then $\big[(f+1)/2\big] = f/2$, and the claim follows from Corollary 3.3. Indeed, the latter gives $\mathcal{B}_f^{(0)}[f/2] \equiv \mathcal{B}_f^{(0)}\big(f/2\big) = Rad\Big(\mathcal{B}_f^{(0)}\big(f/2\big)\Big) \subseteq Rad\Big(\mathcal{B}_f^{(0)}\Big)$, with the last inclusion following by easy arguments of Artinian algebras (as in [HW1], §4.B). Otherwise, one can proceed as follows. Definitions imply $\mathcal{B}_f^{(0)}\big(f/2\big) \subseteq Ann\big(H_{f,h}^{\mu}\big)$ for all h, $\mu \vdash (f-2h)$; therefore $\mathcal{B}_f^{(0)}\big(f/2\big)$ kills also all simple $\mathcal{B}_f^{(0)}$–modules, for they are quotients of the $H_{f,h}^{\mu}$'s. But the radical of a finite dimensional $\Bbbk$–algebra A is characterized (or defined) by $Rad(A) = \bigcap_{M\in Spec(A)} Ann(M)$, where $Spec(A)$ is the set of finite dimensional simple A–modules. Thus we conclude that $\mathcal{B}_f^{(0)}\big(f/2\big) \subseteq Rad\Big(\mathcal{B}_f^{(0)}\Big)$, whence the claim follows again.

(c) By [Wz], §3, there is also a splitting $End_{Sp(W)}\big(W^{\otimes f}\big) = \bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1^t\leq n}} A[h\,;\mu]$, and the analysis in §§3–4 ensures that $A[h\,;\mu] = \pi_2\left(\mathcal{S}_f^{(-2n)}[h\,;\mu]\right) = \pi_1^*\left(\mathcal{B}_f^{(-2n)}[h\,;\mu]\right)$. Like before, using the splitting of $\mathcal{S}_f^{(x)}$ in Proposition 3.2, we see that the map

$$\pi_2 \ : \bigoplus_{h=0}^{[f/2]} \bigoplus_{\mu\vdash(f-2h)} \mathcal{S}_f^{(-2n)}[h\,;\mu] = \mathcal{S}_f^{(-2n)} \relbar\joinrel\twoheadrightarrow End_{Sp(W)}\big(W^{\otimes f}\big) = \bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1^t\leq n}} A[h\,;\mu]$$

must have kernel $Ker(\pi_2) = \bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1^t>n}} \mathcal{S}_f^{(-2n)}[h\,;\mu]$; in addition, it must map $\bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2k) \\ \mu_1^t\leq n}} \mathcal{S}_f^{(-2n)}[h\,;\mu]$ isomorphically onto $\bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1^t\leq n}} A[h\,;\mu] = End_{Sp(W)}\big(W^{\otimes f}\big)$.

Let now $\eta \in Ker(\pi_W) \bigcap \left(\bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1^t\leq n}} \mathcal{B}_f^{(-2n)}\langle h\,;\mu\rangle \right)$. Then $\pi_1(\eta)$ belongs to $\bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu\vdash(f-2h) \\ \mu_1\leq n}} \mathcal{S}_f^{(-2n)}[h\,;\mu]$. As π_2 acts injectively on the latter space, we get that $\pi_2\big(\pi_1(\varpi_{n+1})\big) = \pi_W(\varpi_{n+1}) = 0$ yields $\pi_1(\eta) = 0$, so $\eta \in Ker\big(\pi_1\big) \equiv Rad\Big(\mathcal{B}_f^{(-2n)}\Big)$. $\square$

The previous statement has the following direct consequence:

Theorem 5.3. *Let $n \in \mathbb{N}_+$, $f \in \mathbb{N}_+$, $k := \left[\frac{f-n+1}{2}\right]$. Then (notation of Definition 4.6)*

(a) every minor of order $(n+1)$ in $\mathcal{B}_f^{(n)}(k)$ belongs to $Rad\left(\mathcal{B}_f^{(n)}\right)$, hence

$$\Bbbk\text{–}span\Big(Min_{f\,;\,n+1}^{(n)} \bigcap \mathcal{B}_f^{(n)}(k)\Big) \subseteq Rad\left(\mathcal{B}_f^{(n)}\right) \quad ;$$

(b) every f–diagram in $\mathcal{B}_f^{(0)}\big(\big[(f+1)/2\big]\big)$ belongs to $Rad\left(\mathcal{B}_f^{(0)}\right)$, hence

$$\mathcal{B}_f^{(0)}\big(\big[(f+1)/2\big]\big) \subseteq Rad\left(\mathcal{B}_f^{(0)}\right) \quad ;$$

(c) every Pfaffian of order $2(n+1)$ in $\mathcal{B}_f^{(-2n)}(k)$ belongs to $Rad\left(\mathcal{B}_f^{(-2n)}\right)$, hence

$$\Bbbk\text{–}span\Big(Pf_{f\,;\,n+1}^{(-2n)} \bigcap \mathcal{B}_f^{(-2n)}(k)\Big) \subseteq Rad\left(\mathcal{B}_f^{(-2n)}\right) \quad .$$

Proof. *(a)* Let δ_{n+1} be a minor of order $(n+1)$ in $\mathcal{B}_f^{(n)}(k)$, with $k := \left[\frac{f-n+1}{2}\right]$. This means that all the f–diagrams occurring (with non-zero coefficient) in the expansion of δ_{n+1} have at least k arcs, so they have at most $f-2k$ vertical edges, with $f-2k < n+1$. Therefore $\delta_{n+1} \in \bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu \vdash (f-2h) \\ \mu_1^t + \mu_2^t \leq n}} \mathcal{B}_f^{(n)}\langle h\,;\mu\rangle$, and in addition $\delta_{n+1} \in Ker(\pi_V)$, by Theorem 4.8*(a)*. Then Proposition 5.2*(a)* gives $\delta_{n+1} \in Rad\left(\mathcal{B}_f^{(n)}\right)$, as claimed.

(b) This is obvious from Proposition 5.2*(b)*.

(c) Let ϖ_{n+1} be a Pfaffian of order $2(n+1)$ in $\mathcal{B}_f^{(-2n)}(k)$, with $k := \left[\frac{f-n+1}{2}\right]$. Then, as in *(a)*, all the f–diagrams occurring (with non-zero coefficient) in the expansion of ϖ_{n+1} have at most $f-2k$ vertical edges, with $f-2k < n+1$. It follows that $\varpi_{n+1} \in \bigoplus_{h=0}^{[f/2]} \bigoplus_{\substack{\mu \vdash (f-2h) \\ \mu_1^t \leq n}} \mathcal{B}_f^{(-2n)}\langle h\,;\mu\rangle$, and moreover $\varpi_{n+1} \in Ker(\pi_W)$, by Theorem 4.8*(b)*. Thus Proposition 5.2*(c)* gives $\varpi_{n+1} \in Rad\left(\mathcal{B}_f^{(-2n)}\right)$, as expected. $\square$

Note that in the statement above, part *(b)* is an improvement of Corollary 3.3. Moreover, it can be formulated as in *(a)* or in *(c)*, with $n=0$; indeed, a diagram is just a minor of order 1, or a Pfaffian of order 2, and vice versa.

For the next step, we need a technical (combinatorial) result about minors and Pfaffians:

Lemma 5.4. *Let $n \in \mathbb{N}_+$, $f \in \mathbb{N}_+$, and let $\mathcal{B}_f^{(n)}$, $\mathcal{B}_f^{(-2n)}$ be defined over any ring $\Bbbk$.*

(a) Let $\mathbf{d}$ be an f–diagram, and δ_{n+1} a minor of order $n+1$ in $\mathcal{B}_f^{(n)}$. If $\mathbf{d}$ has an arc $r^- \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\, s^-$, resp. $r^+ \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\, s^+$, and r^+ and s^+, resp. r^- and s^-, are moving vertices in δ_{n+1}, then $\mathbf{d} \cdot \delta_{n+1} = 0$, resp. $\delta_{n+1} \cdot \mathbf{d} = 0$. Otherwise, $\mathbf{d} \cdot \delta_{n+1}$, resp. $\delta_{n+1} \cdot \mathbf{d}$, is a power of n times a minor of order $n+1$.

Similarly, if $j \in J_{f,k}$ is an (f,k)–junction (for some k) having an arc $r \circ\!\!-\!\!\!-\!\!\!-\!\!\circ\, s$, and r^- and s^- are moving vertices in δ_{n+1}, then $\delta_{n+1}.j = 0$ in $H_{f,k}^{\mu}$ for all $\mu \vdash (f-2k)$.

(b) Let $\mathbf{d}$ *be an* f*–diagram, and* ϖ_{n+1} *a Pfaffian of order* $2(n+1)$ *in* $\mathcal{B}_f^{(-2n)}$ *. If* $\mathbf{d}$ *has an arc* $r^- \circ\!\!-\!\!\!-\!\!\circ\, s^-$ *, resp.* $r^+ \circ\!\!-\!\!\!-\!\!\circ\, s^+$ *, and* r^+ *and* s^+*, resp.* r^- *and* s^-*, are moving vertices in* ϖ_{n+1} *, then* $\mathbf{d} \cdot \varpi_{n+1} = 0$ *, resp.* $\varpi_{n+1} \cdot \mathbf{d} = 0$ *. Otherwise,* $\mathbf{d} \cdot \varpi_{n+1}$ *, resp.* $\varpi_{n+1} \cdot \mathbf{d}$ *, is a power of* $(-2n)$ *times a Pfaffian of order* $2(n+1)$ *.*

Similarly, if $j \in J_{f,k}$ *is an* (f,k)*–junction (for some* k*) having an arc* $r \circ\!\!-\!\!\!-\!\!\circ\, s$ *, and* r^- *and* s^- *are moving vertices in* ϖ_{n+1} *, then* $\varpi_{n+1}.j = 0$ *in* $H^{\mu}_{f,k}$ *for all* $\mu \vdash (f-2k)$ *.*

Proof. Lemma 3.9 in [Ga] proves almost all of the present claim. What is missing is only the parts which start with “Otherwise”. Now, checking also these facts is immediate from definitions. Here we just mention explicitly that a coefficient n, or $(-2n)$, will pop up — so its exponent will increase to give a power of n, or $(-2n)$ — whenever an arc in $\mathbf{d}$ matches a *fixed* arc in the minor δ_{n+1} — for *(a)* — or in the Pfaffian ϖ_{n+1} — for *(b)*. □

Theorem 5.3 gives a sufficient condition for a minor, resp. a Pfaffian (of the proper order) to belong to $Rad\left(\mathcal{B}_f^{(n)}\right)$, resp. to $Rad\left(\mathcal{B}_f^{(-2n)}\right)$. The next result claims that this is necessary too. Note that again claim *(b)* may be enclosed in *(a)* or in *(c)*, as case $n=0$.

Theorem 5.5. *Let* $x = n \in \mathbb{N}$ *, and let* $f \in \mathbb{N}_+$ *,* $k := \left[\frac{f-n+1}{2}\right]$ *. Then*

(a) no minor of order $(n+1)$ *in* $\mathcal{B}_f^{(n)} \setminus \mathcal{B}_f^{(n)}(k)$ *belongs to* $Rad\left(\mathcal{B}_f^{(n)}\right)$ *,*

(b) no f*–diagram in* $\mathcal{B}_f^{(0)} \setminus \mathcal{B}_f^{(0)}\big([(f+1)/2]\big)$ *belongs to* $Rad\left(\mathcal{B}_f^{(0)}\right)$ *,*

(c) no Pfaffian of order $2(n+1)$ *in* $\mathcal{B}_f^{(-2n)} \setminus \mathcal{B}_f^{(-2n)}(k)$ *belongs to* $Rad\left(\mathcal{B}_f^{(-2n)}\right)$ *.*

Proof. (a) Let δ_{n+1} be a minor of order $(n+1)$ in $\mathcal{B}_f^{(n)}(h) \setminus \mathcal{B}_f^{(n)}(h+1)$, with $h < k$; then δ_{n+1} is an alternating sum of diagrams $\mathbf{d}_i$ ($i = 1, 2, \dots, (n+1)!$), and at least one of these diagrams — say $\mathbf{d}_1$ — has h arcs, so it has at least $n+1$ vertical edges. Let v_1, v_2, …, $v_{2(n+1)}$ be the moving vertices of δ_{n+1} : then some of them, say v_1, v_2, …, v_r, lay on the top row, and the others, namely v_{r+1}, v_{r+2}, …, $v_{2(n+1)}$, lay on the bottom row. Here $0 \leq r \leq 2(n+1)$; setting $s := 2(n+1) - r$, we can assume (by symmetry) $s \geq r$. As $s + r = 2(n+1)$ is even, $s - r$ is even too, say $s - r = 2t$, $t \in \mathbb{N}$.

We assumed that $s \geq r$, but we reduce at once to the case $s = r$. In fact, suppose $s \gneqq r$, so $t \geq 1$: then in $\mathbf{d}_1$ there are at least t arcs on the bottom row which pairwise join $2t$ vertices among v_{r+1}, v_{r+2}, …, $v_{2(n+1)}$. Moreover, there are at least r vertical edges joining r vertices among v_1, v_2, …, v_r with r vertices among v_{r+1}, v_{r+2}, …, $v_{2(n+1)}$. Since $\mathbf{d}_1$ has exactly $f - 2h$ vertical edges, we get that at least $f - 2h - r$ of these vertical edges do not involve v_1, v_2, …, $v_{2(n+1)}$, hence they are fixed in δ_{n+1}, i.e. they also appear in all the other summands $\mathbf{d}_i$ of δ_{n+1} . Now, $r + s = 2(n+1)$ and $s - r = 2t$ imply $r + t = n + 1$; since $f - 2h \geq n + 1$, we find $f - 2h - r \geq n + 1 - r = t$; therefore $\mathbf{d}_1$ has at least t vertical edges which also appear as well in all the other summands $\mathbf{d}_i$.

Let now $\mathbf{h}$ be a diagram selected as follows: pick t vertices w_1, w_2, …, w_t in the bottom row which in δ_{n+1} belong to fixed vertical edges, and pick t vertices v_{i_1}, v_{i_2}, …, v_{i_t} among v_{r+1}, v_{r+2}, …, $v_{2(n+1)}$ which are not joined with each other in $\mathbf{d}_1$. Then let $\mathbf{h}$ be any f–diagram which has t arcs in the top row joining each w_i to a v_{i_j} , has t arcs in the top row in the same positions than in the bottom one, and whose remaining $f-2t$ vertices are joined by straight-vertical edges (such an $\mathbf{h}$ is nothing but a suitable product of $\mathbf{h}_{i,j}$'s).

Now consider $\delta_{n+1} \cdot \mathbf{h}$. By construction (see Lemma 5.4 and its proof) we have that $\delta'_{n+1} := \delta_{n+1} \cdot \mathbf{h}$ is a new minor (of order $n+1$), with again $\delta'_{n+1} := \delta_{n+1} \cdot \mathbf{h} \in \mathcal{B}_f^{(n)}(h)$. But now δ'_{n+1} has $r' = r + t$ moving vertices in the top row, and $s' = s - t$ moving vertices in the bottom row, thus $r' = n+1 = s'$. Since $\mathit{Rad}\Big(\mathcal{B}_f^{(n)}\Big)$ is an ideal, proving $\delta'_{n+1} \notin \mathit{Rad}\Big(\mathcal{B}_f^{(n)}\Big)$ will also imply $\delta_{n+1} \notin \mathit{Rad}\Big(\mathcal{B}_f^{(n)}\Big)$, as we wish to show.

Let us consider the $(n{+}1)$ moving vertices in the top row of the minor δ'_{n+1}. As explained in Remarks 4.7*(b)*, we can split them into two disjoint subsets, that of *row vertices*, say p in number, and that of *column vertices*, say q in number (with $p+q = n+1$). In the bottom row of course we have instead exactly q row (moving) vertices and p column (moving) vertices. Among the f–diagrams in the expansion of δ'_{n+1}, we collect those attached to permutations in $S_p \times S_q$ $(\subseteq S_{p+q} = S_{n+1})$, i.e. those in which the edges in the moving part are all vertical (which act separately on the row and the column vertices in top row), and we denote by $\varDelta$ their sum (with — alternating — signs). Thus we find that $\delta'_{n+1} = \varDelta + \varGamma$, where $\varGamma$ is an algebraic sum of diagrams which all have the same fixed part as $\varDelta$, and moving part having at least one arc more than $\varDelta$. Namely, we have $\varDelta \in \mathcal{B}_f^{(n)}\langle h\rangle$ and $\varGamma \in \mathcal{B}_f^{(n)}(h+1)$. Therefore

$$\delta'_{n+1} \equiv \varDelta \mod \mathcal{B}_f^{(n)}(h+1)$$

so $\overline{\delta'_{n+1}} = \overline{\varDelta}$ as cosets in $\mathcal{B}_f^{(n)}[h]$. Furthermore, if $\pi_1\big(\delta'_{n+1}\big) = 0$ — i.e. $\delta'_{n+1} \in \mathit{Rad}\Big(\mathcal{B}_f^{(n)}\Big)$ — then clearly $\pi_1^*\big(\overline{\varDelta}\big) = \overline{0}$, so it is enough to show that the latter cannot occur.

By construction, the "symmetric group part" (i.e. the one made of vertical edges) of $\varDelta$ is just a product of some $\sigma\,(\in S_f)$ times a product of antisymmetrizers $\mathit{Alt}_p \cdot \mathit{Alt}_q$. Since $\mathit{Alt}_p \cdot \mathit{Alt}_q$ inside the group algebra $\Bbbk\big[S_{f-2h}\big]$ generates the two-sided ideal $\bigoplus_{\substack{\mu\vdash(f-2h)\\ \mu_1^t\geq p\,,\,\mu_2^t\geq q}} I_\mu$ (assuming $p \geq q$, say), we can conclude that $\overline{\varDelta}$ generates the whole $\mathcal{B}_f^{(n)}$–bimodule $\bigoplus_{\substack{\mu\vdash(f-2h)\\ \mu_1^t\geq p\,,\,\mu_2^t\geq q}} \mathcal{B}_f^{(n)}[h\,;\mu]$ (cf. §2). Therefore, all of the $\mathcal{B}_f^{(n)}$–bimodule $\bigoplus_{\substack{\mu\vdash(f-2h)\\ \mu_1^t\geq p\,,\,\mu_2^t\geq q}} \mathcal{S}_f^{(n)}[h\,;\mu]$ is generated by $\pi_1^*\big(\overline{\varDelta}\big)$. Then if $\pi_1^*\big(\overline{\varDelta}\big) = \overline{0}$ we have also $\bigoplus_{\substack{\mu\vdash(f-2h)\\ \mu_1^t\geq p\,,\,\mu_2^t\geq q}} \mathcal{S}_f^{(n)}[h\,;\mu] = 0$; as the latter is false — thanks to Proposition 3.2 — we must have $\pi_1^*\big(\overline{\varDelta}\big) \neq \overline{0}$, q.e.d.

(b) We can repeat the proof of the previous case, with the obvious, wide simplifications (only the last third of that proof is still necessary — and sufficient!).

(c) The proof is similar to that of case *(a)*. Let ϖ_{n+1} be a Pfaffian of order $2(n+1)$ in $\mathcal{B}_f^{(-2n)}(h) \setminus \mathcal{B}_f^{(-2n)}(h+1)$, for some $h < k$; then ϖ_{n+1} is a sum of diagrams $\mathbf{d}_i$ $(i = 1, 2, \dots, (n+1)!)$, and at least one of them — say $\mathbf{d}_1$ — has h arcs, hence it has at least $n+1$ vertical edges. Let $v_1, v_2, \dots, v_{2(n+1)}$ be the moving vertices of ϖ_{n+1}: then some of them, say $v_1, v_2, \dots, v_r$, lay on the top row, and the others, namely $v_{r+1}, v_{r+2}, \dots, v_{2(n+1)}$, lay on the bottom row (with $0 \leq r \leq 2(n+1)$, $s := 2(n+1) - r$). Exactly as in *(a)*, we can reduce to the case $s = r\ (= n+1)$.

Now the Pfaffian ϖ has $n+1$ moving vertices up and $n+1$ down. Hence among the diagrams in the sum expressing ϖ there are some whose moving edges are all vertical: namely, those corresponding to the terms in (4.3) with $\{h_i \mid i=1,2,\dots,n+1\} \subseteq \{1,2,\dots,f\}$ (the top row) and $\{k_1,k_2,\dots,k_{n+1}\} \subseteq \{f+1,f+2,\dots,2f\}$ (the bottom row). But h_1, h_2, ..., h_{n+1} are fixed by the condition $h_1 < h_2 < \cdots < h_{n+1}$, whilst there is no condition on the ordering of k_1, k_2, ..., k_{n+1}. Thus all diagrams of the previous type are obtained by fixing the sets $\{h_1,h_2,\dots,h_{n+1}\}$ and $\{k_1,k_2,\dots,k_{n+1}\}$ and joining h_i to $k_{\sigma(i)}$ for all $i=1,2,\dots,n+1$, for all $\sigma \in S_{n+1}$. We denote by $\mathit{\Pi}$ the sum of these diagrams, and we note that the S_{n+1}–action on $\{h_1,h_2,\dots,h_{n+1}\}$ or $\{k_1,k_2,\dots,k_{n+1}\}$ turns ϖ into itself.

So far we found that $\varpi_{n+1} = \mathit{\Pi} + \mathit{\Gamma}$, where $\mathit{\Gamma}$ is a sum of diagrams which all have the same fixed part as $\mathit{\Pi}$ and moving part having at least one arc more than $\mathit{\Pi}$. That is, we have $\mathit{\Pi} \in \mathcal{B}_f^{(-2n)}\langle h\rangle$ and $\mathit{\Gamma} \in \mathcal{B}_f^{(-2n)}(h+1)$. Thus

$$\varpi_{n+1} \equiv \mathit{\Pi} \mod \mathcal{B}_f^{(-2n)}(h+1)$$

and so $\overline{\varpi_{n+1}} = \overline{\mathit{\Pi}} \in \mathcal{B}_f^{(-2n)}[h]$. Moreover, here again $\varpi'_{n+1} \in \mathit{Rad}\left(\mathcal{B}_f^{(-2n)}\right)$ would imply $\pi_1^*\left(\overline{\mathit{\Pi}}\right) = \overline{0}$ as well, thus we have to show that the latter does not occur.

By construction, the "symmetric group part" of $\mathit{\Pi}$ is just the product of some $\sigma\ (\in S_f)$ times a symmetrizer Sym_{n+1}, whence it follows that $\overline{\mathit{\Pi}}$ generates the whole $\mathcal{B}_f^{(-2n)}$–bimodule $\bigoplus_{\substack{\mu\vdash(f-2h)\\ \mu_1\geq(n+1)}} \mathcal{B}_f^{(-2n)}[h\,;\mu]$. Therefore, $\pi_1^*\left(\overline{\mathit{\Pi}}\right)$ in turn generates the $\mathcal{B}_f^{(-2n)}$–bimodule $\bigoplus_{\substack{\mu\vdash(f-2t)\\ \mu_1\geq(n+1)}} \mathcal{S}_f^{(-2n)}[h\,;\mu]$, hence one has that $\pi_1\big(\mathit{\Pi}\big) = 0$ would imply also that $\bigoplus_{\substack{\mu\vdash(f-2t)\\ \mu_1\geq(n+1)}} \mathcal{S}_f^{(-2n)}[h\,;\mu] = 0$; the latter is false, by Proposition 3.2, so $\pi_1^*\left(\overline{\mathit{\Pi}}\right) \neq 0$. □

Theorems 5.3 and 5.5 have many consequences for the radicals of the various algebras $\mathcal{B}_f^{(x)}(h)$ and $\mathcal{B}_f^{(x)}[h]$, which we collect in the following two statements. Here again, we wish to point out that, altogether, these corollaries give a necessary and sufficient condition for the coset of a minor — of order $(n+1)$ — respectively of an f–diagram, respectively of a Pfaffian — of order $2(n+1)$ — to belong to the radical of the suitable quotient algebra.

Corollary 5.6. *Let $n \in \mathbb{N}_+$, $f \in \mathbb{N}_+$, $k := \left[\frac{f-n+1}{2}\right]$, and $h \geq k$. Then*

(a) every minor of order $(n+1)$ in $\mathcal{B}_f^{(n)}(h)$ belongs to $\mathit{Rad}\left(\mathcal{B}_f^{(n)}(h)\right)$, and its coset either in $\mathcal{B}_f^{(n)}\Big/\mathcal{B}_f^{(n)}(h+1)$ or in $\mathcal{B}_f^{(n)}[h] := \mathcal{B}_f^{(n)}(h)\Big/\mathcal{B}_f^{(n)}(h+1)$ belongs to $\mathit{Rad}\left(\mathcal{B}_f^{(n)}\Big/\mathcal{B}_f^{(n)}(h+1)\right)$ or $\mathit{Rad}\left(\mathcal{B}_f^{(n)}[h]\right)$, respectively;

(b) every f–diagram in $\mathcal{B}_f^{(0)}\big([(f+1)/2]\big)$ belongs to $\mathit{Rad}\left(\mathcal{B}_f^{(0)}\big([(f+1)/2]\big)\right)$, hence $\mathcal{B}_f^{(0)}\big([(f+1)/2]\big) = \mathit{Rad}\left(\mathcal{B}_f^{(0)}\big([(f+1)/2]\big)\right)$, so that $\mathcal{S}_f^{(0)}\big([(f+1)/2]\big) = 0$;

(c) every Pfaffian of order $2(n+1)$ in $\mathcal{B}_f^{(-2n)}(h)$ belongs to $\mathit{Rad}\left(\mathcal{B}_f^{(-2n)}(h)\right)$, and its coset either in $\mathcal{B}_f^{(-2n)}\Big/\mathcal{B}_f^{(-2n)}(h+1)$ or in $\mathcal{B}_f^{(-2n)}[h] := \mathcal{B}_f^{(-2n)}(h)\Big/\mathcal{B}_f^{(-2n)}(h+1)$ belongs to $\mathit{Rad}\left(\mathcal{B}_f^{(-2n)}\Big/\mathcal{B}_f^{(-2n)}(h+1)\right)$ or $\mathit{Rad}\left(\mathcal{B}_f^{(-2n)}[h]\right)$, respectively.

Proof. Everything follows directly from Theorem 5.2 once we remind that

$$Rad(R) \bigcap I \;=\; Rad(I) \quad , \qquad \big(Rad(R)+I\big)\Big/I \;=\; Rad\Big(R/I\Big)$$

for every ideal I in an Artinian ring R — as observed in [HW1] — and we apply this fact to the cases $R = \mathcal{B}_f^{(x)}$ or $R = \mathcal{B}_f^{(x)}(h)$ and $I = \mathcal{B}_f^{(x)}(h+1)$. $\square$

Corollary 5.7. *Let* $n \in \mathbb{N}_+$, $f \in \mathbb{N}_+$, $k := \left[\frac{f-n+1}{2}\right]$, *and* $h < k$.

(a) let δ_{n+1} *be a minor of order* $(n+1)$ *in* $\mathcal{B}_f^{(n)}(h)$. *Then the coset of* δ_{n+1} *either in* $\mathcal{B}_f^{(n)}\Big/\mathcal{B}_f^{(n)}(h+1)$ *or in* $\mathcal{B}_f^{(n)}[h] := \mathcal{B}_f^{(n)}(h)\Big/\mathcal{B}_f^{(n)}(h+1)$ *does not belong to the radical* $Rad\Big(\mathcal{B}_f^{(n)}\Big/\mathcal{B}_f^{(n)}(h+1)\Big)$ *nor to* $Rad\Big(\mathcal{B}_f^{(n)}[h]\Big)$ *respectively;*

(b) let $\mathbf{d}$ *be an* f*–diagram in* $D_{f,h} \setminus D_{f,[(f+1)/2]}$. *Then the coset of* $\mathbf{d}$ *either in* $\mathcal{B}_f^{(0)}\Big/\mathcal{B}_f^{(0)}([f/2]+1)$ *or in* $\mathcal{B}_f^{(0)}[h] := \mathcal{B}_f^{(0)}(h)\Big/\mathcal{B}_f^{(0)}(h+1)$ *does not belong to the radical* $Rad\Big(\mathcal{B}_f^{(0)}\Big/\mathcal{B}_f^{(0)}([f/2]+1)\Big)$ *nor to* $Rad\Big(\mathcal{B}_f^{(0)}[h]\Big)$ *respectively;*

(c) let ϖ_{n+1} *be a Pfaffian of order* $2(n+1)$ *in* $\mathcal{B}_f^{(-2n)}(h)$. *Then the coset of* ϖ_{n+1} *either in* $\mathcal{B}_f^{(-2n)}\Big/\mathcal{B}_f^{(-2n)}(h+1)$ *or in* $\mathcal{B}_f^{(-2n)}[h] := \mathcal{B}_f^{(-2n)}(h)\Big/\mathcal{B}_f^{(-2n)}(h+1)$ *does not belong to the radical* $Rad\Big(\mathcal{B}_f^{(-2n)}\Big/\mathcal{B}_f^{(-2n)}(h+1)\Big)$ *nor to* $Rad\Big(\mathcal{B}_f^{(-2n)}[h]\Big)$ *respectively.*

Proof. The claim follows easily from the analysis carried out to prove Theorem 5.5. $\square$

5.8 An overall conjecture about the radical. Theorem 5.3 provides the following global information about the radical of Brauer algebras:

— *a)* $\quad Rad\Big(\mathcal{B}_f^{(n)}\Big) \supseteq R_f^{(n)} := \Bbbk\text{–}span\Big(Min_{f;\,n+1}^{(n)} \bigcap \mathcal{B}_f^{(n)}\Big(\big[(f-n+1)/2\big]\Big)\Big)$

— *b)* $\quad Rad\Big(\mathcal{B}_f^{(0)}\Big) \supseteq R_f^{(0)} := \Bbbk\text{–}span\Big(D_{f,\,[(f+1)/2]}\Big) = \mathcal{B}_f^{(0)}\big(\big[(f+1)/2\big]\big)$

— *c)* $\quad Rad\Big(\mathcal{B}_f^{(-2n)}\Big) \supseteq R_f^{(-2n)} := \Bbbk\text{–}span\Big(Pf_{f;\,n+1}^{(-2n)} \bigcap \mathcal{B}_f^{(-2n)}\Big(\big[(f-n+1)/2\big]\Big)\Big)$

In principle, $Rad\Big(\mathcal{B}_f^{(x)}\Big)$ might be greater than the space $R_f^{(x)}$ considered above. Nevertheless, at least in some cases we are able to leave out of the radical some elements which, *a priori*, might belong to it. We shall now briefly sketch what we mean.

Let us consider for instance the case of $\mathcal{B}_f^{(n)}$, with $n \in \mathbb{N}_+$. Pick $\eta \in Rad\Big(\mathcal{B}_f^{(n)}\Big)$. Resuming notations of §5.1, we have $\eta \in Rad\Big(\mathcal{B}_f^{(n)}\Big) \subseteq Ker\big(\pi_V\big) = \Bbbk\text{–}span\Big(Min_{f;\,n+1}^{(n)}\Big)$. Now, fix a subset B of $Min_{f;\,n+1}^{(n)}$ which is a $\Bbbk$–basis of $Ker\big(\pi_V\big)$, and expand η as a $\Bbbk$–linear combination of elements of B . Since $Min_{f;\,n+1}^{(n)} \bigcap \mathcal{B}_f^{(n)}\Big(\big[(f-n+1)/2\big]\Big)$ is contained in $Rad\Big(\mathcal{B}_f^{(n)}\Big)$, we can even reduce to the case where in such an expansion of η there occur (with non-zero coefficient) *only elements of* $B \setminus \mathcal{B}_f^{(n)}\Big(\big[(f-n+1)/2\big]\Big)$.

In particular, let us assume, for instance, that $\eta = c_1\,\delta_1 + c_2\,\delta_2$ *,* where we have $\delta_1, \delta_2 \in B \setminus \mathcal{B}_f^{(n)}\Big(\big[(f-n+1)/2\big]\Big)$, $\delta_1 \neq \delta_2$ and $c_1, c_2 \in \Bbbk$; *in addition, we assume that* δ_2 *has two distinct moving vertices on a single row, say* i^+ *and* j^+*, which are not both moving for* δ_1 *, nor both always on (fixed or "moving") vertical edges.*

In this situation, we consider $\mathbf{h}_{i,j} \cdot \eta$, which clearly also belongs to $Rad\Big(\mathcal{B}_f^{(n)}\Big)$. Then Lemma 5.4 gives $\mathbf{h}_{i,j} \cdot \delta_2 = 0$ and $\mathbf{h}_{i,j} \cdot \delta_1 = n^e\,\delta'_1$ with $e \in \mathbb{N}$ and $\delta'_1 \in Min_{f;\,n+1}^{(n)}$, so

$$Rad\Big(\mathcal{B}_f^{(n)}\Big) \ni \mathbf{h}_{i,j} \cdot \eta \;=\; c_1\,\mathbf{h}_{i,j} \cdot \delta_1 + c_2\,\mathbf{h}_{i,j} \cdot \delta_2 \;=\; c_1\,n^e \delta'_1 \quad .$$

But then it follows that $\delta'_1 \in Rad\Big(\mathcal{B}_f^{(n)}\Big)$, and also $\delta'_1 \notin \mathcal{B}_f^{(n)}\Big(\big[(f-n+1)/2\big]\Big)$, by construction, which by Theorem 5.5*(a)* is impossible.

Similar results can be obtained (via Theorem 5.5*(c)*) in the case of $\mathcal{B}_f^{(-2n)}$ $(n\in\mathbb{N}_+)$ too.

On the other hand, in some cases the inclusion of $R_f^{(z)}$ in $Rad\Big(\mathcal{B}_f^{(z)}\Big)$ is strict; for instance, this is the case for $z = 0$ and $f \geq 7$, by definitions and by the results in [Ru] and [RS], for which $\mathcal{B}_f^{(0)}$ is semisimple iff $f \in \{1, 3, 5\}$.

However, we can also remark that (cf. the proof of Corollary 5.6)

$$R_f^{(n)} \;\subseteq\; Rad\Big(\mathcal{B}_f^{(n)}\Big) \bigcap \mathcal{B}_f^{(n)}\Big(\big[(f-n+1)/2\big]\Big) \;=\; Rad\bigg(\mathcal{B}_f^{(n)}\Big(\big[(f-n+1)/2\big]\Big)\bigg)$$

$$R_f^{(0)} \;\subseteq\; Rad\Big(\mathcal{B}_f^{(0)}\Big) \bigcap \mathcal{B}_f^{(0)}\Big(\big[(f+1)/2\big]\Big) \;=\; Rad\bigg(\mathcal{B}_f^{(0)}\Big(\big[(f+1)/2\big]\Big)\bigg)$$

$$R_f^{(-2n)} \;\subseteq\; Rad\Big(\mathcal{B}_f^{(-2n)}\Big) \bigcap \mathcal{B}_f^{(n)}\Big(\big[(f-n+1)/2\big]\Big) \;=\; Rad\bigg(\mathcal{B}_f^{(-2n)}\Big(\big[(f-n+1)/2\big]\Big)\bigg)$$

Even more, the mid-line inclusion is indeed an identity, namely

Proposition 5.9. *Let* $f \in \mathbb{N}_+$ *. Then* $Rad\bigg(\mathcal{B}_f^{(0)}\Big(\big[(f+1)/2\big]\Big)\bigg) \;=\; R_f^{(0)}$.

Proof. If f is odd, then $\mathcal{B}_f^{(0)}\Big(\big[(f+1)/2\big]\Big) = 0$ and $R_f^{(0)} = 0$ by definition. If f is even, then $R_f^{(0)} := \mathcal{B}_f^{(0)}(f/2) = \mathcal{B}_f^{(0)}\Big(\big[(f+1)/2\big]\Big)$, and the claim follows from Corollary 3.3. □

Thus, inspired by these experimental evidences, we are lead to formulate the following

Conjecture 5.10. *Let* $f \in \mathbb{N}_+$ *,* $n \in \mathbb{N}_+$ *. Then*

$$Rad\bigg(\mathcal{B}_f^{(n)}\Big(\big[(f-n+1)/2\big]\Big)\bigg) \;=\; R_f^{(n)} \quad , \qquad Rad\bigg(\mathcal{B}_f^{(-2n)}\Big(\big[(f-n+1)/2\big]\Big)\bigg) \;=\; R_f^{(-2n)}$$

5.11 Inheriting the radical. In [HW2], §3.2, an algorithm is described for constructing a part of $Rad\Big(\mathcal{B}_f^{(x)}\Big)$, called "the Inherited Piece of the Radical", out of $Rad\Big(\mathcal{B}_{f-2}^{(x)}\Big)$.

The construction is the following. Take an $(f-2)$–diagram $\mathbf{d} \in D_{f-2}$, and let (i, j) and (h, k) be two pairs of numbers such that $1 \leq i < j \leq f$ and $1 \leq h < k \leq f$.

Define an f–diagram $\mathbf{d}^{i,j}_{h,k} \in D_f$ to be the diagram obtained from $\mathbf{d}$ by inserting a new arc $i^+ \circ\!\!-\!\!\circ\, j^+$ in the top row and a new arc $h^- \circ\!\!-\!\!\circ\, k^-$ in the bottom row. Then let $Ex^{i,j}_{h,k} : \mathcal{B}^{(x)}_{f-2} \longrightarrow \mathcal{B}^{(x)}_f$ be the unique linear embedding defined by $Ex^{i,j}_{h,k}(\mathbf{d}) := \mathbf{d}^{i,j}_{h,k}$ for all $\mathbf{d} \in D_{f-2}$. Given a subspace $\mathcal{I}$ of $\mathcal{B}^{(x)}_{f-2}$, define $\mathcal{I}^{(1)} := \sum_{(i,j),(h,k)} Ex^{i,j}_{h,k}(\mathcal{I})$, the $\Bbbk$–span of all the $Ex^{i,j}_{h,k}(\mathcal{I})$'s. Then Theorem 3.2.9 in [HW2] claims that

$$\Big(Rad\Big(\mathcal{B}^{(x)}_{f-2}\Big)\Big)^{(1)} \textit{ is a two-sided ideal of } \mathcal{B}^{(x)}_f \textit{, and it is contained in } Rad\Big(\mathcal{B}^{(x)}_f\Big) .$$

Actually, this also follows (see [KX], Lemma 3.1) because $\mathcal{B}^{(x)}_f$ is a cellular algebra.

Now the remark is the following. Let η be an element of $Rad\Big(\mathcal{B}^{(x)}_{f-2}\Big)$ of the type given in Theorem 5.3 (i.e. a minor or a Pfaffian). By the very definitions, $Ex^{i,j}_{h,k}(x)$ is again an element of the same type, so Theorem 5.3 applies to give $Ex^{i,j}_{h,k}(\eta) \in Rad\Big(\mathcal{B}^{(x)}_f\Big)$. Therefore, if $\mathcal{I}$ is the span of the previous elements in $\mathcal{B}^{(x)}_{f-2}$, then we see directly that $\mathcal{I}^{(1)} \subseteq Rad\Big(\mathcal{B}^{(x)}_f\Big)$, which is a nice way to verify the "inheritance phenomenon" of [HW2].

Note also that this remark is fully consistent with Proposition 5.9 and Conjecture 5.10.

5.12 The radical of the $\mathcal{B}^{(x)}_f$–modules $H^\mu_{f,k}$. The information about the radical of a Brauer algebra $\mathcal{B}^{(x)}_f$ given in Theorem 5.3 yield also information about the radical of the $\mathcal{B}^{(x)}_f$–modules $H^\mu_{f,k}$. Namely, the first, immediate result is (with notation of §5.8)

Theorem 5.13. *Let $n \in \mathbb{N}_+$, $f \in \mathbb{N}_+$, $k \in \big\{0,1,\dots,[f/2]\big\}$, and $\mu \vdash (f-2k)$. Then*

$$R^{(n)}_f . H^\mu_{f,k} \subseteq Rad\big(H^\mu_{f,k}\big) \quad , \quad R^{(0)}_f . H^\mu_{f,k} \subseteq Rad\big(H^\mu_{f,k}\big) \quad , \quad R^{(-2n)}_f . H^\mu_{f,k} \subseteq Rad\big(H^\mu_{f,k}\big)$$

where in each case $H^\mu_{f,k}$ stands for the suitable module for the algebra $\mathcal{B}^{(n)}_f$, $\mathcal{B}^{(0)}_f$ or $\mathcal{B}^{(-2n)}_f$.

Proof. The claim follows immediately from the standard inclusion $Rad(A).M \subseteq Rad(M)$, which holds for every ring A and every A–module M. $\square$

In addition, Conjecture 5.10 about the radical of the Brauer algebras also involve a similar conjecture about the radicals of the $H^\mu_{f,k}$'s, namely that the inclusions in Theorem 5.13 actually be identities. We prove this conjecture in some special cases, see §6 below; also, the anlogous claim for $\mathcal{B}^{(0)}_f$ is easily seen to be true. The complete statement is

Conjecture 5.14. *If $n \in \mathbb{N}_+$, $f \in \mathbb{N}_+$, $k \in \big\{[(f-n+1)/2],\dots,[f/2]\big\}$, $\mu \vdash (f-2k)$, then*

$$Rad\big(H^\mu_{f,k}\big) = R^{(n)}_f . H^\mu_{f,k} \ , \qquad\qquad Rad\big(H^\mu_{f,k}\big) = R^{(-2n)}_f . H^\mu_{f,k}$$

where in each case $H^\mu_{f,k}$ stands for the suitable module for the algebra $\mathcal{B}^{(n)}_f$ or $\mathcal{B}^{(-2n)}_f$.

Proof. If Conjecture 5.10 holds true, then the claim follows immediately from the standard identity $Rad(M) = Rad(A).M$, which holds for every ring A and every A–module M such that $M/Rad(M)$ is simple, which is the case for $M = H^\mu_{f,k}$, by Proposition 2.7. $\square$

Remark: in §6 below we shall see some cases in which Conjecture 5.10 and Conjecture 5.14 do hold true, namely when $n = 1$ and f is even. For the sake of completeness, we report also that [Ga], Corollary 4.6, gives other results about $Rad\big(\mathcal{B}_f^{(x)}\big)$ and $Rad\big(H_{f,k}^{\mu}\big)$.

5.15 The case of positive characteristic. Let $p := Char(\Bbbk)$. All our results about $Rad\big(\mathcal{B}_f^{(x)}\big)$ in this section are based on the assumption $p = 0$. We shall now discuss to what extent these results might hold for $p > 0$ as well.

First of all, the results of §2 and §3 about the $\mathcal{B}_f^{(x)}$–modules and the semisimple quotient $\mathcal{S}_f^{(x)}$ of $\mathcal{B}_f^{(x)}$ only require $p > f$. On the other hand, the results of §4 assume condition (4.1) to be satisfied. The latter does hold, in particular, whenever $p > f$; therefore, under the latter, stronger assumption all results of §§2, 3 and 4 are available.

On the other hand, from §5.1 on we assumed $p = 0$. This ensures that the group $O(V)$, resp. $Sp(W)$, is linearly reductive: so $V^{\otimes f}$, resp. $W^{\otimes f}$, is a semisimple module for $O(V)$, resp. for $Sp(W)$. Then, by general theory, *the centralizer algebra* $End_{O(V)}\big(V^{\otimes f}\big)$, *resp.* $End_{Sp(W)}\big(W^{\otimes f}\big)$, *is semisimple too.* This last fact is the basis to obtain Proposition 5.2; the precise block decomposition of this semisimple algebra then can be recovered from the results of §§2–4 — see the proof of Proposition 5.2, where we summarized all this quoting [Wz]. Then Theorem 5.3, our first main result, follows as a direct consequence.

We must point out a key fact. Let $U \in \{V, W\}$, let $G(V) := O(V)$, $G(W) := Sp(W)$. For $E \subseteq End_{G(U)}\big(U^{\otimes f}\big)$ let $\langle E \rangle$ be the subalgebra of $End_{\Bbbk}\big(U^{\otimes f}\big)$ generated by U. Then

Lemma 5.16. *Assume $p > f$. Then the following are equivalent:*

(a) $V^{\otimes f}$, resp. $W^{\otimes f}$, is a semisimple module for $O(V)$, resp. for $Sp(W)$;

(b) the $\Bbbk$–algebra $\langle O(V) \rangle$, resp. $\langle Sp(W) \rangle$, is semisimple;

(c) the $\Bbbk$–algebra $\pi_V\big(\mathcal{B}_f^{(n)}\big)$, resp. $\pi_W\big(\mathcal{B}_f^{(-2n)}\big)$, is semisimple;

(d) $V^{\otimes f}$, resp. $W^{\otimes f}$, is a semisimple module for $\pi_V\big(\mathcal{B}_f^{(n)}\big)$, resp. for $\pi_W\big(\mathcal{B}_f^{(-2n)}\big)$.

Proof. By Theorem 4.4, $\langle O(V) \rangle$ and $\pi_V\big(\mathcal{B}_f^{(n)}\big)$ are the centralizer of each other, for their action on $V^{\otimes f}$, and similarly for $\langle Sp(W) \rangle$ and $\pi_W\big(\mathcal{B}_f^{(-2n)}\big)$ acting on $W^{\otimes f}$. Therefore, Schur duality tells us that *(b)* and *(c)* are equivalent.

On the other hand, the equivalences *(a)* $\Longleftrightarrow$ *(b)* and *(c)* $\Longleftrightarrow$ *(d)* are obvious. $\square$

To sum up, the above analysis proves the following result:

Theorem 5.17. *Let $p := Char(\Bbbk) > f$. If any one of the (equivalent) conditions in Lemma 5.16 is satisfied, then Proposition 5.2 and Theorem 5.3 still hold true.* $\square$

On the other hand, the proof of Theorem 5.5 (our second main result) only exploits combinatorial techniques and some results of §3. Thus, its proof still is valid if $p > f$; so

Theorem 5.18. *Let $p := Char(\Bbbk) > f$. Then Theorem 5.5 still holds true.* $\square$

Finally, we extend Conjectures 5.10 and 5.14 to the case of $p := Char(\Bbbk) > f$, too.

§6 Applications: the Temperley-Lieb algebra and pointed chord diagrams

6.1 Temperley-Lieb algebra and pointed chord diagrams. Let $f \in \mathbb{N}_+$ be *even*. In this section we study the cases of $\mathcal{B}_f^{(1)}$ and $H_{f,f/2}^{(0)}$: in particular, we compute $Rad\Big(\mathcal{B}_f^{(1)}\big(f/2\big)\Big)$ and $Rad\Big(H_{f,f/2}^{(0)}\Big)$, proving that Conjecture 5.10 and Conjecture 5.14 (respectively) do hold true for them. To this end, we shall not make use of Theorem 5.3; in particular, we need no special assumptions on the ground field.

Indeed, **in this section we assume that $\Bbbk$ is any field.**

First, a terminological remark. The *unital* subalgebra of $\mathcal{B}_f^{(1)}$ generated by $\mathcal{B}_f^{(1)}\big(f/2\big)$ is usually called *Temperley-Lieb algebra* (possibly defined in other ways, usually involving some parameter too: see e.g. [DN], and references therein), call it $\mathcal{TL}_f$. The restriction functor yields an equivalence between the category of all $\mathcal{TL}_f$–modules and the category of all $\mathcal{B}_f^{(1)}\big(f/2\big)$–modules, so studying the latter we are studying the former too.

Second, the set $J_{f,f/2}$ of all $\big(f, f/2\big)$–junctions (a $\Bbbk$–basis of $H_{f,f/2}^{(0)}$) can be represented by *pointed chord diagrams*, as follows. Given $j \in J_{f,f/2}$, let us lay the f vertices of j on the interior of a (horizontal) segment, and draw the arcs of j as arcs above this segment. Now close up the segment into a circle (like winding up the segment around a circle of same length), gluing together the vertices of the segment and sealing them with a special, marking dot. Then the arcs of j are turned into *chords* of the circle. This (up to details) sets a bijection from $J_{f,f/2}$ to the set of pointed chord diagrams on the circle with f chords.

We'd better point out that ours are *pointed* chord diagrams. Indeed, usually chord diagrams are considered up to rotations, which is not the present case — roughly, the marked point forbids rotations. Our previous construction shows that one can identify $H_{f,f/2}^{(0)}$ with the $\Bbbk$–span of the set of all pointed chord diagrams on the circle with f chords: so this $\Bbbk$–span is a module for $\mathcal{B}_f^{(1)}\big(f/2\big)$ (hence for $\mathcal{B}_f^{(1)}$ as well) or the Temperley-Lieb algebra. It is customary to drop the "pointed" datum, considering chord diagrams on the circle up to rotations: this amounts to look at the vertices of a junction up to cyclical permutations. In this case, the $\Bbbk$–span of the set of all *non-pointed* chord diagrams on the circle with f chords (or of the set of all $\big(f, f/2\big)$–junctions up to cyclical permutations of their vertices) bears a natural structure of module for the quotient algebra of $\mathcal{TL}_f$ — or of $\mathcal{B}_f^{(1)}\big(f/2\big)$, or of $\mathcal{B}_f^{(1)}$ too — given by taking f–diagrams up to simultaneous cyclic equivalence of their top and bottom vertices. This algebra is sometimes called *Temperley-Lieb* (or, respectively, *Brauer*) *loop — or affine — algebra*. Indeed, this is the most common framework where chord diagrams, and Temperley-Lieb or Brauer algebras acting on them, do appear in literature: see, for instance, [DN], or [Jo], and references therein. The results we find below for modules built upon pointed chord diagrams and for Temperley-Lieb or Brauer algebras can be easily adapted to the non-pointed and the loop/affine case as well.

6.2 Special features in $\mathcal{B}_f^{(1)}$ and $H_{f,f/2}^{(0)}$. The choice of parameter $x = 1$ has two important consequences. Namely, the set $D_{f,f/2}$ of $\big(f/2\big)$–arc f–diagrams is just a (multiplicative) *submonoid* of $\mathcal{B}_f^{(1)}$, and the action of $\mathcal{B}_f^{(1)}$ onto $H_{f,f/2}^{(0)}$ restricts to an action of $D_{f,f/2}$ onto the set $J_{f,f/2}$ of $(f, f/2)$–junctions. Both facts follow directly from definitions, which in fact give even more precise results, as follows.

First observe that every $\mathbf{d} \in D_{f,f/2}$ is uniquely determined by its arc structure $\mathtt{as}(\mathbf{d})$ (notation of §1.2): in short we write $\mathbf{d} \cong \mathtt{as}(\mathbf{d})$. Second, let $\mathbf{d}_1$, $\mathbf{d}_2 \in D_{f,f/2}$. Then by §1.3 one has $\mathbf{d}_1\mathbf{d}_2 = \mathbf{d}_1 * \mathbf{d}_2 \in D_{f,f/2}$ $\Big(\subset \mathcal{B}_f^{(1)}(f/2) = \mathcal{B}_f^{(1)}[f/2] \Big)$, and this product is uniquely characterized by $\mathbf{d}_1\mathbf{d}_2 \cong \mathtt{as}(\mathbf{d}_1\mathbf{d}_2) = \big(\mathtt{tas}(\mathbf{d}_1), \mathtt{bas}(\mathbf{d}_2)\big)$.

Third, let $\mathbf{d} \in D_{f,f/2}$ $\Big(\subset \mathcal{B}_f^{(1)}(f/2) \Big)$ and $j \in J_{f,f/2}$ $\Big(\subset H_{f,f/2}^{(0)} \Big)$. Then by definition (§2.6) the $\mathcal{B}_f^{(1)}$–action on $H_{f,f/2}^{(0)}$ yields $\mathbf{d}.j = \mathtt{tas}(\mathbf{d}) \in J_{f,f/2}$ $\Big(\subset H_{f,f/2}^{(0)} \Big)$.

We still need two more tools: the unique $\Bbbk$–linear map $Tr_{\mathcal{B}} : \mathcal{B}_f^{(1)}(f/2) \longrightarrow \Bbbk$ such that $Tr_{\mathcal{B}}(\mathbf{d}) = 1$ for all $\mathbf{d} \in D_{f,f/2}$, and the unique $\Bbbk$–linear map $Tr_H : H_{f,f/2}^{(0)} \longrightarrow \Bbbk$ such that $Tr_H(j) = 1$ for all $j \in J_{f,f/2}$ (where notation *Tr* should be a reminder for "trace").

Theorem 6.3. *Let* $R_f^{(1)} := \Bbbk\text{–}span\Big(Min_{f;2}^{(1)} \bigcap \mathcal{B}_f^{(1)}(f/2) \Big)$ *like in §5.8(a). Then*

(a) $Rad\Big(\mathcal{B}_f^{(1)}(f/2)\Big) \;=\; R_f^{(1)} \;=\; Ker\big(Tr_{\mathcal{B}}\big) \;=\; \Bbbk\text{–}span\Big(\big\{ \mathbf{d} - \mathbf{d}' \,\big|\, \mathbf{d}, \mathbf{d}' \in D_{f,f/2} \big\} \Big)$

hence in particular Conjecture 5.10 holds true in this case ($n = 1$, $f \in 2\mathbb{N}_+$ *).*

In addition, the semisimple quotient $\mathcal{S}_f^{(1)}[f/2]$ *of* $\mathcal{B}_f^{(1)}[f/2]$ *is simple of dimension 1.*

(b) $Rad\Big(H_{f,f/2}^{(0)}\Big) \;=\; R_f^{(1)}.\,H_{f,f/2}^{(0)} \;=\; Ker\big(Tr_H\big) \;=\; \Bbbk\text{–}span\Big(\big\{ j - j' \,\big|\, j, j' \in J_{f,f/2} \big\} \Big)$

In particular, Conjecture 5.14 is true for $n = 1$, $f \in 2\mathbb{N}_+$ *and* $(k\,, \mu) = \big(f/2\,, (0)\big)$.

In addition, the semisimple quotient of $H_{f,f/2}^{(0)}$ *is simple of dimension 1.*

Proof. *(a)* Let us write $N(f) := \big|D_{f,f/2}\big| = 2(f-1)!!$, and let $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_{N(f)}$ be a numbering of the elements of $D_{f,f/2}$. It is clear that $\big\{ \mathbf{d}_i - \mathbf{d}_{i+1} \,\big|\, i = 1, 2, \dots, N(f){-}1 \big\}$ is a $\Bbbk$–basis of $Ker\big(Tr_{\mathcal{B}}\big)$, and also that $Ker\big(Tr_{\mathcal{B}}\big) = \Bbbk\text{–}span\Big(\big\{ \mathbf{d} - \mathbf{d}' \,\big|\, \mathbf{d}, \mathbf{d}' \in D_{f,f/2} \big\} \Big)$.

Now let $\mathbf{d}, \mathbf{d}' \in D_{f,f/2}$. Then there exist permutations $\sigma_+, \sigma_- \in S_f$ such that $\mathtt{tas}\big(\mathbf{d}'\big) = \mathbf{d}_{\sigma_+}.\mathtt{tas}(\mathbf{d})$ and $\mathtt{bas}\big(\mathbf{d}'\big) = \mathbf{d}_{\sigma_-}.\mathtt{bas}(\mathbf{d})$. Let us set $\mathbf{d}^* := \mathbf{d}_{\sigma_+} \cdot \mathbf{d}$ — so that $\mathbf{d}^* \cong \big(\mathbf{d}_{\sigma_+}.\mathtt{tas}(\mathbf{d})\,, \mathtt{bas}(\mathbf{d})\big)$ — let $\sigma_\pm = \big(h_1^\pm\, k_1^\pm\big)\,\big(h_2^\pm\, k_2^\pm\big) \cdots \big(h_{\ell(\sigma^\pm)}\, k_{\ell(\sigma^\pm)}\big)$ be a factorisation of $\sigma_\pm$ into a product of transpositions, and define

$$\mathbf{d}_0 := \mathbf{d}\,, \quad \mathbf{d}_s := \mathbf{d}_{(h_{\ell(\sigma_+)-s+1}\, k_{\ell(\sigma_+)-s+1})}.\,\mathbf{d}_{s-1} \qquad \big(\, s = 1, 2, \dots, \ell(\sigma_+) \,\big)$$

$$\mathbf{d}^*_0 := \mathbf{d}^*\,, \quad \mathbf{d}^*_s := \mathbf{d}_{(h_{\ell(\sigma_-)-s+1}\, k_{\ell(\sigma_-)-s+1})}.\,\mathbf{d}^*_{s-1} \qquad \big(\, s = 1, 2, \dots, \ell(\sigma_-) \,\big)$$

(in particular, $\mathbf{d}_{\ell(\sigma_+)} = \mathbf{d}_{\sigma_+}.\,\mathbf{d} =: \mathbf{d}^* =: \mathbf{d}^*_0$). Then we can expand $\mathbf{d} - \mathbf{d}'$ as

$$\mathbf{d} - \mathbf{d}' \;=\; \textstyle\sum_{s=0}^{\ell(\sigma_+)-1} \big(\mathbf{d}_s - \mathbf{d}_{s+1}\big) \;+\; \sum_{s=0}^{\ell(\sigma_-)-1} \big(\mathbf{d}^*_s - \mathbf{d}^*_{s+1}\big) \tag{6.1}$$

By definition, any minor in $R_f^{(1)}$ is of the form $\mathbf{d} - \mathbf{d}'$, with $\mathbf{d}, \mathbf{d}' \in D_{f,f/2}$ which differ from each other only for a transposition of two bottom or two top vertices. In other words, there are indices h, $k \in \{1, \dots, f\}$ such that — in notation of §1.3 — we have

$\mathbf{d}' = \mathbf{d}\,\mathbf{d}_{(h\,k)}$ (for *bottom* vertices) or $\mathbf{d}' = \mathbf{d}_{(h\,k)}\,\mathbf{d}$ (for *top* vertices). Therefore, each summand in the right-hand side of (6.1) above belongs to $R_f^{(1)}$, hence $\big(\mathbf{d}-\mathbf{d}'\big) \in R_f^{(1)}$ too. By the previous analysis, it follows that

$$R_f^{(1)} \;=\; \Bbbk\text{–}span\Big(\big\{\, \mathbf{d}-\mathbf{d}' \,\big|\, \mathbf{d},\mathbf{d}' \in D_{f,f/2} \,\big\}\Big) \;=\; Ker\big(Tr_{\mathcal{B}}\big) \quad .$$

For every $\bar{\mathbf{d}}\,,\mathbf{d}\,,\mathbf{d}' \in D_{f,f/2}$, we have also $\bar{\mathbf{d}}\,\big(\mathbf{d}-\mathbf{d}'\big) = \bar{\mathbf{d}}\,\mathbf{d} - \bar{\mathbf{d}}\,\mathbf{d}' \in Ker\big(Tr_{\mathcal{B}}\big)$, so that $Ker\big(Tr_{\mathcal{B}}\big)$ is a $\mathcal{B}_f^{(1)}\big(f/2\big)$–submodule of $\mathcal{B}_f^{(1)}\big(f/2\big)$ itself. Then the quotient $\mathcal{B}_f^{(1)}\big(f/2\big)\Big/Ker\big(Tr_{\mathcal{B}}\big)$, which has dimension 1, is a *simple* module for $\mathcal{B}_f^{(1)}\big(f/2\big) = \mathcal{B}_f^{(1)}\big[f/2\big]$.

In addition, it is easy to see by direct computation that $\big(\mathbf{d}_1-\mathbf{d}'_1\big)\big(\mathbf{d}_2-\mathbf{d}'_2\big)\big(\mathbf{d}_3-\mathbf{d}'_3\big) = 0$ for every $\mathbf{d}_i\,,\mathbf{d}'_i \in D_{f,f/2}$ $(i=1,2,3)$. This implies that

$$\Big(Ker\big(Tr_{\mathcal{B}}\big)\Big)^3 \;=\; \Big(R_f^{(1)}\Big)^3 \;=\; \Big(\Bbbk\text{–}span\Big(\big\{\, \mathbf{d}-\mathbf{d}' \,\big|\, \mathbf{d},\mathbf{d}' \in D_{f,f/2} \,\big\}\Big)\Big)^3 \;=\; 0$$

so that $R_f^{(1)} = Ker\big(Tr_{\mathcal{B}}\big)$ is contained in $Rad\Big(\mathcal{B}_f^{(1)}\big(f/2\big)\Big)$. But then $\mathcal{B}_f^{(1)}\big(f/2\big)\Big/Ker\big(Tr_{\mathcal{B}}\big)$ must be isomorphic to the semisimple quotient $\mathcal{S}_f^{(1)}\big[f/2\big]$ of $\mathcal{B}_f^{(1)}\big[f/2\big] = \mathcal{B}_f^{(1)}\big(f/2\big)$; therefore $\mathcal{S}_f^{(1)}\big[f/2\big]$ itself is simple of dimension 1, and

$$Rad\Big(\mathcal{B}_f^{(1)}\big(f/2\big)\Big) \;=\; Ker\big(Tr_{\mathcal{B}}\big) \;=\; \Bbbk\text{–}span\Big(\big\{\, \mathbf{d}-\mathbf{d}' \,\big|\, \mathbf{d},\mathbf{d}' \in D_{f,f/2} \,\big\}\Big) \;=\; R_f^{(1)} \;, \quad \text{q.e.d.}$$

(b) Let us write $J_{f,f/2} = \big\{\, j_1,\, j_2\,,\, \dots\,,\, j_{n(f)} \,\big\}$, with $n(f) := \big|J_{f,f/2}\big| = (f-1)!!$. Then, like in the case of $Ker\big(Tr_{\mathcal{B}}\big)$, it is clear that $\big\{\, j_s - j_{s+1} \,\big|\, s = 1,2,\dots,n(f)-1 \big\}$ is a $\Bbbk$–basis of $Ker\big(Tr_H\big)$, and $Ker\big(Tr_H\big) = \Bbbk\text{–}span\Big(\big\{\, j-j' \,\big|\, j,j' \in J_{f,f/2} \,\big\}\Big)$. Since $\mathbf{d}.j = \mathtt{tas}(\mathbf{d})$ for all $\mathbf{d} \in D_{f,f/2}$ and all $j \in J_{f,f/2}$ (see §6.2), we have also $\mathbf{d}.\big(j-j'\big) = 0$ (for all $j,j' \in J_{f,f/2}$) and so $Ker\big(Tr_H\big)$ is a $\mathcal{B}_f^{(1)}\big(f/2\big)$–submodule of $H_{f,f/2}^{(0)}$. As the quotient $H_{f,f/2}^{(0)}\Big/Ker\big(Tr_H\big)$ is 1–dimensional, we conclude that it is a *simple* $\mathcal{B}_f^{(1)}\big(f/2\big)$–module.

Now pick any $j,\, j' \in J_{f,f/2}$: then there exists a permutation $\sigma \in S_f$ such that $j' = \mathbf{d}_\sigma.j$ (notation of §1.3). Let $\sigma = (h_1\,k_1)\,(h_2\,k_2)\cdots(h_{\ell(\sigma)}\,k_{\ell(\sigma)})$ be a factorisation of σ into a product of transpositions, and let $j_0 := j$, $j_s := \mathbf{d}_{(h_{\ell(\sigma)-s+1}\,k_{\ell(\sigma)-s+1})}.j_{s-1}$ for all $s = 1,2,\dots,\ell(\sigma)$. Then we can expand $j-j'$ as $j-j' = j - \mathbf{d}_\sigma.j = \sum_{s=0}^{\ell(\sigma)-1}\big(j_s - j_{s+1}\big)$.

Now take any $y \in J_{f,f/2}$ and let $\mathbf{d}_s \in D_{f,f/2}$ be characterized by $\mathtt{as}(\mathbf{d}_s) \cong \big(j_s\,,y\big)$. Recalling that $\mathbf{d}.y' = \mathtt{tas}(\mathbf{d})$, for all $\mathbf{d} \in D_{f,f/2}$ and $y' \in J_{f,f/2}$ (see §6.2), we have

$$\Big(\sum_{s=0}^{\ell(\sigma)-1}\big(\mathbf{d}_s - \mathbf{d}_{s+1}\big)\Big).\,y \;=\; \sum_{s=0}^{\ell(\sigma)-1}\big(\mathtt{tas}(\mathbf{d}_s) - \mathtt{tas}(\mathbf{d}_{s+1})\big) \;=\; \sum_{s=0}^{\ell(\sigma)-1}\big(j_s - j_{s+1}\big) \;=\; j - j' \,.$$

By the previous description of $R_f^{(1)}$ (see part *(a)*) we have $\sum_{s=0}^{\ell(\sigma)-1}\big(\mathbf{d}_s - \mathbf{d}_{s+1}\big) \in R_f^{(1)}$, hence we can conclude that $\big(j-j'\big) \in R_f^{(1)}.\,H_{f,f/2}^{(0)}$. As we have already shown that $\Bbbk\text{–}span\Big(\big\{\, j-j' \,\big|\, j,j' \in J_{f,f/2} \,\big\}\Big) = Ker\big(Tr_H\big)$ and $R_f^{(1)} = Rad\Big(\mathcal{B}_f^{(1)}\big(f/2\big)\Big)$, we have

$$Ker\big(Tr_H\big) \;=\; Rad\Big(\mathcal{B}_f^{(1)}\big(f/2\big)\Big).\,H_{f,f/2}^{(0)} \;\subseteq\; Rad\Big(H_{f,f/2}^{(0)}\Big)$$

where the last inclusion follows from the standard inclusion $Rad(A).M \subseteq Rad(M)$, which holds for every ring A and every A–module M. Finally, as we saw that the quotient $H^{(0)}_{f,f/2}\Big/ Ker\big(Tr_H\big)$ is a (1–dimensional) *simple* $\mathcal{B}^{(1)}_f(f/2)$–module, we can conclude that $Ker\big(Tr_H\big) \supseteq Rad\Big(H^{(0)}_{f,f/2}\Big)$ as well, thus eventually $Ker\big(Tr_H\big) = R^{(1)}_f . H^{(0)}_{f,f/2} = Rad\Big(H^{(0)}_{f,f/2}\Big)$, and the semisimple quotient of $H^{(0)}_{f,f/2}$ is simple of dimension 1, q.e.d. $\square$

References

[Br] R. Brauer, *On algebras which are concerned with semisimple continuous groups*, Ann. of Math. **38** (1937), 854–872.

[Bw1] Wm. P. Brown, *Generalized matrix algebras*, Canadian J. Math. **7** (1955), 188–190.

[Bw2] ———, *An algebra related to the orthogonal group*, Mich. Math. Jour. **3** (1955–56), 1–22.

[CDM] A. Cox, M. De Visscher, P. Martin, *The blocks of the Brauer algebra in characteristic zero*, `arXiv:math/0601387v2` [math.RT] (2006).

[DH] S. Doty, J. Hu, *Schur-Weyl duality for orthogonal groups*, `arXiv:0712.0944v1` [math.RT] (2007).

[DHW] W. F. Doran, P. Hanlon, D. Wales, *On the semisemplicity of Brauer centralizer algebras*, J. Algebra **211** (1999), 647–685.

[DN] J. De Gier, B. Nienhuis, *Brauer loops and the commuting variety*, J. Stat. Mech. Theory Exp. (2005), 10 pages (electronic) — see also `arXiv:math/0410392v2` [math.AG].

[DP] C. De Concini, C. Procesi, *A characteristic free approach to invariant theory*, Adv. Math. **21** (1976), 330–354.

[Ga] F. Gavarini, *A Brauer algebra theoretic proof of Littlewood's restriction rules*, J. Algebra **212** (1999), 240–271.

[GL] J. J. Graham, G. I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), 1–34.

[GP] F. Gavarini, P. Papi, *Representations of the Brauer algebra and Littlewood's restriction rules*, J. Algebra **194** (1997), 275–298.

[Hu] J. Hu, *Specht filtrations and tensor spaces for the Brauer algebra*, `arXiv:math/0604577v2` [math.RT] (2006).

[HW1] P. Hanlon, D. Wales, *On the decomposition of Brauer's centralizer algebras*, J. Algebra **121** (1989), 409–445.

[HW2] ———, *A tower construction for the radical in Brauer's centralizer algebras*, J. Algebra **164** (1994), 773–830.

[Jo] V. Jones, *A quotient of the affine Hecke algebra in the Brauer algebra*, Enseign. Math. **40** (1994), 313–344.

[Ke] S. V. Kerov, *Realizations of representations of the Brauer semigroup*, J. of Soviet Math. **47** (1989), 2503–2507.

[KX] S. König, C. Xi, *A characteristic free approach to Brauer algebras*, Trans. Amer. Math. Soc. **353** (2000), 1489–1505.

[Ru] H. Rui, *A criterion on the semisimple Brauer algebras*, J. Combin. Theory Ser. A **111** (2005), 78–88.

[RS] H. Rui, M. Si, *A criterion on the semisimple Brauer algebras. II*, J. Combin. Theory Ser. A **113** (2006), 1199–1203.

[Wz] H. Wenzl, *On the structure of Brauer's centralizer algebras*, Ann. Math. **188** (1988), 173–193.

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