A BOUNDARY VALUE PROBLEM FOR A PDE MODEL IN MASS TRANSFER THEORY: REPRESENTATION OF SOLUTIONS AND REGULARITY RESULTS.

By
Elena Giorgieri

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE THE DEGREE OF
DOCTOR OF PHILOSOPHY IN MATHEMATICS
AT
UNIVERSITÀ DI ROMA TOR VERGATA
VIA DELLA RICERCA SCIENTIFICA
00133, ROMA, ITALIA
DECEMBER 2004

© Copyright by Elena Giorgieri, 2004

UNIVERSITÀ DI ROMA TOR VERGATA
DEPARTMENT OF
MATHEMATICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**A Boundary Value Problem for a PDE Model in Mass Transfer Theory: Representation of Solutions and Regularity Results.**" by **Elena Giorgieri** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

Dated: December 2004

Supervisor: ______________________
Prof. P. Cannarsa

Readers: ______________________
Prof. M. Quincampoix

____
Prof. G. Buttazzo

UNIVERSITÀ DI ROMA TOR VERGATA

Date: **December 2004**

Author: **Elena Giorgieri**

Title: **A Boundary Value Problem for a PDE Model in Mass Transfer Theory: Representation of Solutions and Regularity Results.**

Department: **Mathematics**

Degree: **Ph.D.** Convocation: **January** Year: **2005**

Permission is herewith granted to Università di Roma Tor Vergata to circulate and to have copied for non-commercial purposes, at its discretion, the above title upon the request of individuals or institutions.

Signature of Author

THE AUTHOR RESERVES OTHER PUBLICATION RIGHTS, AND NEITHER THE THESIS NOR EXTENSIVE EXTRACTS FROM IT MAY BE PRINTED OR OTHERWISE REPRODUCED WITHOUT THE AUTHOR'S WRITTEN PERMISSION.

THE AUTHOR ATTESTS THAT PERMISSION HAS BEEN OBTAINED FOR THE USE OF ANY COPYRIGHTED MATERIAL APPEARING IN THIS THESIS (OTHER THAN BRIEF EXCERPTS REQUIRING ONLY PROPER ACKNOWLEDGEMENT IN SCHOLARLY WRITING) AND THAT ALL SUCH USE IS CLEARLY ACKNOWLEDGED.

Table of Contents

Abstract

Given a bounded domain $\Omega \subset \mathbb{R}^n$, let us denote by $d(\cdot) : \overline{\Omega} \to \mathbb{R}$ the distance function from the boundary $\partial\Omega$. The set of points $x \in \Omega$ at which d is not differentiable is called the *singular set* of d and denoted by Σ. Its closure is often referred to as the *cut locus*. We introduce the map $\tau : \overline{\Omega} \to \mathbb{R}$, defined by $\tau(x) = \min\left\{t \geq 0 \ : \ x + tDd(x) \in \overline{\Sigma}\right\}$ for all $x \in \overline{\Omega}\backslash\overline{\Sigma}$, $\tau(x) = 0$ on $\overline{\Sigma}$, which is sometimes called the *maximal retraction length of* Ω *onto* $\overline{\Sigma}$ or *normal distance* to $\overline{\Sigma}$. The aim of this work is two–sided:

1 To present a global regularity result on the normal distance to the cut locus, showing that in the case when $n = 2$ and Ω is a bounded simply connected domain with analytic boundary, then τ is either a Lipschitz continuous or a Hölder continuous function of exponent at least 2/3. We apply this result to the study of regularity of the solutions (in a suitable sense) of system

$$(1) \qquad \begin{cases} -\operatorname{div}(vDu) = f & \text{in } \Omega \\ |Du| - 1 = 0 & \text{in } \{v > 0\} \\ |Du| \leq 1 \quad u, v \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $f : \overline{\Omega} \to \mathbb{R}$ is a non–negative Lipschitz continuous function. It turns out that the second component v of the solution of (1) is either Lipschitz continuous or Hölder continuous, as well.

2 To show an existence/uniqueness result on the solutions of system (1) in any space dimension, generalizing what recently found in dimension $n = 2$ in the context of granular matter theory.

Acknowledgements

I would like to thank Piermarco Cannarsa, my supervisor, for introducing me to the field of PDEs and Control Theory and for his many suggestions during this years. Furthermore, I would like to thank Pierre Cardaliaguet for his mathematical and human support in the development of this project. I should also thank him for inviting me in Brest and giving me the opportunity of completing this thesis in a stimulating environment.

I am also thankful to Italo Capuzzo Dolcetta, who indirectly suggested the theme of this work by attracting the attention on the paper [28].

Of course, I am deeply grateful to my husband for his love, patience and for having encouraged me anytime I felt lost. Also, I wish to thank my parents, my sister and Luigi for their constant presence and support. Without any of them this work would have been much more difficult to me.

Finally, I wish to thank all the friends that I met at the university, with a special thought to those who have been close to me during the period that I was ill.

Roma, Italia — Elena Giorgieri

December 2004

Notations

$\lvert x\rvert,\ x \in \mathbb{R}^n$	Euclidean norm of $x = (x_1, \ldots, x_n) :\ \lvert x\rvert = (x_1^2 + \cdots + x_n^2)^{1/2}$
$\langle x, y\rangle,\ x,\, y \in \mathbb{R}^n$	Euclidean scalar product : $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$
$B_r(x)$	ball of radius r and center x
$[x, y]$ $(]x, y[)$	closed (open) segment of extremes x, y
$\overline{\mathrm{co}}\, D$	closed convex hull of the set D
$\mathrm{dist}(A, B)$	distance between the sets A and B
$\mathrm{diam}(A)$	diameter of the set A
$\mathrm{int}(A)$	interior of the set A
$p \otimes q$	tensor product of p, $q \in \mathbb{R}^n :\ (p \otimes q)(x) = p\langle q, x\rangle$, $\forall x \in \mathbb{R}^n$
$\mathcal{C}^k(\Omega)$	space of k-times continuously differentiable functions
$\mathcal{C}^k_c(\Omega)$	space of $\mathcal{C}^k(\Omega)$ functions with compact support
$\mathrm{spt}(f)$	support of a function f
$\mathcal{C}^{0,\alpha}(\Omega),\ \alpha \in (0, 1)$	space of Hölder continuous functions with exponent α
$L^p(\Omega),\ p \geq 1$	space of measurable functions $f : \Omega \to \mathbb{R}$ such that $\exists \int_\Omega \lvert f\rvert^p < \infty$
$\lVert f\rVert_p$	L^p norm : $\left(\int_\Omega \lvert f\rvert^p\right)^{1/p}$
$L^\infty(\Omega)$	space of measurable functions $f : \Omega \to \mathbb{R}$ s.t. $\sup\mathrm{ess}_\Omega \lvert f\rvert < \infty$
$\lVert f\rVert_\infty$	L^∞ norm : $\sup\mathrm{ess}_\Omega \lvert f\rvert$
$\lvert f\rVert_{p,\Omega'}$ $(\lVert f\rVert_{\infty,\Omega'})$	$L^p(L^\infty)$ norm of f restricted to Ω'
$\mathcal{W}^{1,p}(\Omega)$	Sobolev space of functions $f \in L^p(\Omega)$ such that $\forall i = 1, \ldots, n$ $\exists g_i \in L^p(\Omega)$ with $\int_\Omega f \frac{\partial \phi}{\partial x_i} = -\int_\Omega g_i \phi$ $\forall$ test functions $\phi \in \mathcal{C}^\infty_c$
$\mathcal{H}^\nu(A)$	ν–Hausdorff measure of $A \subset \mathbb{R}^n$; if $C_\nu := \frac{\pi^{\nu/2} 2^{-\nu}}{\int_0^{+\infty} e^{-x} x^{\nu/2}\, dx}$, then $\mathcal{H}^\nu(A) = C_\nu \sup_{\delta > 0} \inf\{\sum_i \mathrm{diam}(A_i)^\nu : A \subseteq \cup_i A_i,\ \mathrm{diam}(A_i) \leq \delta\}$
$\delta_{x_0}(x)$	Dirac delta function at x_0

Introduction

The system of partial differential equations

$$\begin{cases} -\mathrm{div}\,(vDu) = f & \text{in } \Omega \\ |Du| - 1 = 0 & \text{in } \{v > 0\}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a given domain, arises in many different mathematical contexts, like the Monge-Kantorovich theory, shape optimization and granular matter theory. This work is concerned with its interpretation as a model for the equilibrium configurations that may occur in the case of a sandpile created by pouring dry matter onto a "table" from a constant (in time) source. The model comes essentially from the work of Hadeler and Kuttler [28], built on previous work by Boutreux and de Gennes [11]. Here, the table is represented by a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 1, 2$, and the matter source by a function $f(x) \geq 0$. The physical model presents $u(x)$ and $v(x)$, respectively, as the heights of the *standing* and *rolling layers* at a point $x \in \Omega$. Indeed, u represents the amount of matter that remains at rest, while v describes matter moving down along the surface of the standing layer and falling from the table when the base of the heap touches the boundary of Ω.

For physical reasons, the slope of the standing layer cannot exceed a given constant—typical of the matter under consideration—that we normalize to 1. Consequently, the standing layer must vanish on the boundary of the table. So, $|Du| \leq 1$ in Ω and $u = 0$ on $\partial\Omega$. Also, in the region where v is positive, the standing layer has to be "maximal", for otherwise more matter would roll down there to rest. On the other hand, the rolling layer results from transporting matter along the surface of the standing layer at a speed that is assumed proportional to the slope Du, with constant equal to 1. The above considerations lead to the boundary value problem

$$
\begin{cases}
-\mathrm{div}\,(vDu) = f & \text{in } \Omega \\
|Du| - 1 = 0 & \text{in } \{v > 0\} \\
|Du| \le 1 \quad u, v \ge 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\tag{1}
$$

A correct representation formula for the (pointwise) solution of (1) is provided in [28] in 1 space dimension, while for the 2 dimensional case the conjectured solution for the rolling layer is wrong.
Recently, Cannarsa and Cardaliaguet [13] have obtained a representation formula for the solution of problem (1) that starts from the physical considerations of Hadeler and Kuttler [28] but develops in a rigorous mathematical framework.

Due to the lack of regularity of the solutions of the eikonal equation $|Du| = 1$ and of the conservation law $-\mathrm{div}\,(vDu) = f$, the solutions of problem (1) are meant in the following sense.

Definition A pair (u, v) of *continuous* functions in Ω is a solution of problem (1) if

– $u = 0$ on $\partial\Omega$, $\|Du\|_{\infty,\Omega} \le 1$, and u is a viscosity solution of

$$|Du| = 1 \quad \text{in} \quad \{x \in \Omega \ : \ v(x) > 0\}$$

– $v \ge 0$ in Ω and, for every test function $\phi \in \mathcal{C}_c^\infty(\Omega)$,

$$\int_\Omega v(x)\langle Du(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x)dx\,.$$

The paper [13] provides a complete description of system (1) in the plane, with an explicit formula for its solutions and a uniqueness result. Indeed, in the case when $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary of class $\mathcal{C}^2$ and $f \ge 0$ is a continuous function in Ω, it is proven that the unique solution of system (1) is the pair (d, v_f), where d is the distance function from $\partial\Omega$, $v_f = 0$ on $\overline{\Sigma}$ and

$$(*) \qquad v_f(x) = \int_0^{\tau(x)} f(x + tDd(x))\, \frac{1-(d(x)+t)\kappa(x)}{1-d(x)\kappa(x)}\, dt \qquad \forall x \in \Omega\backslash\overline{\Sigma}\,.$$

Here, Σ is the set of points $x \in \Omega$ at which d is not differentiable (or equivalently the set of points with more than one projection onto $\partial\Omega$), $\kappa(x)$ denotes the curvature of $\partial\Omega$ at the projection point of x and

$$\tau(x) = \min\left\{t \geq 0 \ : \ x + tDd(x) \in \overline{\Sigma}\right\} \qquad \forall x \in \overline{\Omega}\backslash\overline{\Sigma}$$

is the so–called *normal distance* to $\overline{\Sigma}$.

Starting from the results in [13] and in particular from the representation formula that the authors provide, some questions has arisen:

(A) to determine the conditions ensuring further regularity to v_f;

(B) to extend the representation formula $(*)$ to bounded regular domains in $\mathbb{R}^n$, $n > 2$.

The aim of this work is to answer to the above problems.

In order to answer to question (A), we had to analyze first the regularity of the maximal retraction length of Ω. The first theorem below completes, for $n = 2$ and analytic boundaries, what recently obtained by Li and Nirenberg [31], who have shown–in any dimension and for $C^{2,1}$ boundaries–the Lipschitz continuity of τ when restricted to $\partial\Omega$.

Theorem 1 Let Ω be a simply connected domain with analytic boundary, different from a disc. Then there exists some $\alpha \in [2/3, 1)$ such that τ is Hölder continuous in Ω with exponent α. In particular, the map τ is at least 2/3-Hölder continuous.

THEOREM 2 Assume that f is a Lipschitz continuous function and that Ω is a bounded simply connected domain with analytic boundary, different from a disc. Then there exists some $\beta \in (0, 1/3]$ such that v_f is Hölder continuous in Ω with exponent β.

The disk is excluded in our analysis because it is the only case of analytic boundary for which τ and v_f are Lipschitz continuous (as long as f is). The proof of this fact is immediate and it is shown in two remarks. We stress that when Ω is not a disk, we cannot expect more than Hölder regularity for τ and v_f, as a suitable example will show.

The question on the extensibility of the representation formula $(*)$ to bounded regular domains in $\mathbb{R}^n$, $n > 2$, has been answered by the following result.

THEOREM 3 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$ and $f \geq 0$ be a continuous function in Ω. Then, a solution of system (1) is given by the pair (d, v_f), where d is the distance function from $\partial\Omega$, $v_f = 0$ on $\overline{\Sigma}$ and

$$v_f(x) = \begin{cases} \int_0^{\tau(x)} f(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1-(d(x)+t)\kappa_i(x)}{1-d(x)\kappa_i(x)}\, dt & \forall x \in \Omega \setminus \overline{\Sigma}, \\ \\ 0 & \forall x \in \overline{\Sigma}, \end{cases}$$

where $\kappa_i(x)$, $i \in \{1, \dots, n-1\}$ denotes the i-th principal curvature of $\partial\Omega$ at the projection point of x onto $\partial\Omega$.
Moreover, the above solution is unique in the following sense: if (u, v) is another solution of (1), then $v = v_f$ in Ω and $u = d$ in $\{x \in \Omega : v_f > 0\}$.

The thesis is organized as follows.

Chapter 1 is a survey on the main properties of the distance function from a closed set and of semiconcave functions, on the generalized gradients and on the viscosity solutions of the eikonal equation.

Some of the results in Section 1.1–which is devoted to the description of the distance function properties–are taken or are a modification of results from [27], [23], [25] and [13]. In particular, we took from [23] Proposition 1.1.2, from [27] Proposition 1.1.5, from [13] Propositions 1.1.3, 1.1.4 and 1.1.7. All the results are modified in order to fit one another and constitute a whole body. The results on the sets with piecewise $\mathcal{C}^{2,1}$ boundary and outer corners–which are sets with corners "pointing outside"–are new.

Section 1.2 is addressed to the presentation of some generalized gradients, semiconcave functions and their singularities. The results in there are taken from [15], [8] and [18], with the exception of Theorem 1.2.12–which comes form [1]–and Proposition 1.2.16–which derives from [13] and [4].

Finally, Section 1.3 is concerned with the study of the connections between viscosity solutions of the eikonal equation and the distance function. The results therein come from [8] and [15] again.

Chapter 2 is devoted to the presentation of the model problem in the setting of granular matter theory and shows the background of this work. In fact, we present some different models for growing sandpiles as well as the results obtained so far. In particular, we analyze the papers of Prigozhin [35], Aronsson [5], Aronsson–Evans–Wu [7], Hadeler–Kuttler [28] and Cannarsa–Cardaliaguet [13].

Chapter 3 and 4 constitute the original part of the thesis and deal with the analysis of problem (1) in the plane and in higher dimension, respectively.

In Chapter 3, we first extend the representation formula and the existence/uniqueness result of Cannarsa and Cardaliaguet in [13] to sets $\Omega \subset \mathbb{R}^2$ which are piecewise regular and admits corners pointing outside Ω. Section 3.2 is then devoted to the study of the regularity of the normal distance τ and of v_f, with the proof of THEOREMS 1 and 2 above.

In Chapter 4 we finally analyze problem (1) in any space dimension, providing the general representation formula and the existence/uniqueness result of THEOREM 3.

Chapter 1

Preliminaries

The purpose of this chapter is to collect some of the basic definitions and the main results that are needed throughout this work. In Section 1.1 we describe the principal properties of the distance function to a closed subset K of $\mathbb{R}^n$, proving also some regularity results that depend on the regularity of the boundary of K. In Section 1.2 we introduce some generalized gradients that replace the classical notion of gradient in the case of continuous functions and we study their properties for a special class of functions, the semiconcave functions. We also present some results on the structure of the non–differentiability set of a semiconcave function. In Section 1.3 we give the definition of viscosity solution of the eikonal equation and establish its connection with the distance function.

1.1 The Distance Function

Most of the results of this section are well–known in literature, since the distance function from a closed set arises in many different mathematical contexts. Some of them are taken or are a modification of results from [27], [23], [25] and [13].

1.1.1 General Properties of the Distance Function

Let K be a closed, nonempty, proper subset of $\mathbb{R}^n$. We denote by d_K the distance function from the set K, that is

$$d_K(x) = \min_{y \in K} |y - x| \quad x \in \mathbb{R}^n.$$

It is readily shown that d_K is Lipschitz continuous of constant 1. Indeed, take any $x,\ y \in \mathbb{R}^n$ and $z \in K$ such that $|z - y| = d_K(y)$. Then

$$d_K(x) \le |x - z| \le |x - y| + |y - z| = |x - y| + d_K(y),$$

which implies $|d_K(x) - d_K(y)| \le |x - y|$ as soon as we interchange the role of x and y. Let us denote by $\Pi_K(x)$ the set of projections of x onto K, that is

$$\Pi_K(x) = \{y \in K \ : \ d_K(x) = |y - x|\} \, .$$

It is easy to see that for any $x \in \mathbb{R}^n \setminus K$, the set $\Pi_K(x)$ is compact and that the multivalued map $x \in \mathbb{R}^n \setminus K \mapsto \Pi_K(x)$ is upper semicontinuous, i.e. if $\{x_k\}$ is a sequence in $\mathbb{R}^n \setminus K$ converging to some $x \in \mathbb{R}^n \setminus K$ as $k \to \infty$ and $y_k \in \Pi_K(x_k)$ verifies $y_k \to y \in \mathbb{R}^n$ as $k \to \infty$, then $y \in \Pi_K(x)$.

Lemma 1.1.1. *Let $x \in \mathbb{R}^n \setminus K$, $y \in \Pi_K(x)$ and $x(t) := tx + (1-t)y$, where $t \in (0,1)$. Then $\Pi_K(x(t)) = \{y\}$.*

Proof—First of all $\Pi_K(x) \neq \emptyset$, because K is closed. Suppose by contradiction that there exists some $z \in \Pi_K(x(t))$, $z \neq y$. Then z does not belong to the segment joining x and y, so that

$$|x - z| < |x - x(t)| + |x(t) - z| \le |x - x(t)| + |x(t) - y| = |x - y|,$$

against the fact that $y \in \Pi_K(x)$. $\square$

The following proposition describes further regularity properties of the distance function.

Proposition 1.1.2. *The function d_K is differentiable at $x \in \mathbb{R}^n \setminus K$ if and only if $\Pi_K(x)$ is a singleton. If d_K is differentiable at x, then $Dd_K(x) = (x-y)/|x-y|$, where $\{y\} = \Pi(x)$. Moreover, Dd_K is continuous in its domain of definition.*

Proof—Suppose that $\Pi_K(x) = \{y\}$ and for any $h \in \mathbb{R}^n$ such that $x + h \in \mathbb{R}^n \setminus K$ choose k such that $y + k \in \Pi_K(x+h)$. Then

$$\begin{aligned} & d_K(x+h)^2 - d_K(x)^2 = |y+k-x-h|^2 - |x-y|^2 \\ & = 2\langle x-y, h\rangle + 2\langle y-x, k\rangle + |h|^2 + |k|^2 - 2\langle h, k\rangle. \end{aligned}$$

Since $|y-x|^2 \le |y+k-x|^2$ and $|y+k-x-h|^2 \le |y-x-h|^2$, we obtain

$$0 \le 2\langle y-x, k\rangle + |k|^2 \quad \text{and} \quad 2\langle y-x-h, k\rangle + |k|^2 \le 0.$$

Therefore

$$2\langle x-y, h\rangle - 2\langle h, k\rangle + |h|^2 \le d_K(x+h)^2 - d_K(x)^2 \le 2\langle x-y, h\rangle + |h|^2.$$

But $k \to 0$ as $h \to 0$, as a consequence of the upper semicontinuity of Π_K and of the uniqueness of the projection of x onto K, so that we can conclude the differentiability of d_K^2 at x, with $Dd_K^2(x) = 2(x-y)$. This in turn implies that d_K is differentiable at x and $Dd_K(x) = (x-y)/|x-y|$. Suppose now that d_K is differentiable at x and take $y \in \Pi_K(x)$. Let $x(t) := tx + (1-t)y$, for $t \in (0,1)$. Then $y \in \Pi_K(x(t))$ by Lemma 1.1.1 and

$$\begin{aligned} -|x(t)-x| &= |x(t)-y| - |x-y| = d_K(x(t)) - d_K(x) \\ &= \langle Dd_K(x), x(t)-x\rangle + \mathrm{o}(|x(t)-x|). \end{aligned}$$

Hence, dividing by $1-t$ and letting $t \to 1$ we obtain

$$-|y-x| = \langle Dd_K(x), y-x\rangle.$$

Since d_K is Lipschitz continuous of constant 1, then also $|Dd_K(x)| \le 1$, which gives, together with the previous equality,

$$Dd_K(x) = \frac{(x-y)}{|x-y|} \qquad \text{and} \qquad y = x - d_K(x) Dd_K(x).$$

The representation formula for y obviously implies the uniqueness of the projection of x onto K.
The continuity of Dd_K on its domain of definition is a consequence of the upper semicontinuity of the projection. $\square$

In what follows, the set of points $x \in \mathbb{R}^n \setminus K$ at which d_K is not differentiable will be called the *singular set* of d_K and denoted by Σ. Such a set is also referred to as the *ridge*. As a consequence of Proposition 1.1.2, Σ can be viewed as the set of points x at which $\Pi(x)$ is not a singleton. All points $x \notin \Sigma$ will be called *regular*.

From now on, we will consider the case $K = \mathbb{R}^n \setminus \Omega$, where Ω is a bounded domain of $\mathbb{R}^n$. If there is no ambiguity, we will denote by d and Π the function $d_{\mathbb{R}^n\setminus\Omega}$ and the projection $\Pi_{\mathbb{R}^n\setminus\Omega}$. Furthermore, whenever x has a unique projection onto $\mathbb{R}^n \setminus \Omega$, with a minor abuse of notation, we will identify the set $\Pi(x)$ with its unique element.

Proposition 1.1.3. *Let $x \in \Omega\setminus\overline{\Sigma}$ and let $t > 0$ be such that $x + sDd(x) \notin \overline{\Sigma}$ for every $s \in [0, t)$. Then, for every $s \in [0, t)$,*

(a) $\quad d(x + sDd(x)) = d(x) + s$

(b) $\quad Dd(x + sDd(x)) = Dd(x)$

(c) $\quad \Pi(x + sDd(x)) = \Pi(x)$

Proof—Since $x \notin \overline{\Sigma}$, the gradient of the distance function exists and is continuous in a neighborhood of x. So, let $x(\cdot)$ be the unique solution of the o.d.e. $\dot{x}(s) = Dd(x(s))$ with $x(0) = x$. Notice that $x(\cdot)$ satisfies $|x(s) - x| \le s$. Since

$$\frac{d}{ds} d(x(s)) = \langle Dd(x(s)), Dd(x(s))\rangle = 1 \ ,$$

we have that $d(x(s)) = d(x) + s$, as long as $x(s) \notin \overline{\Sigma}$ and so at least for $s < \mathrm{dist}(x, \overline{\Sigma})$. On the other hand, being $x(\cdot)$ 1-Lipschitz continuous, we also have

$$|x(s) - x| \le s = d(x(s)) - d(x) \le |x(s) - x|.$$

Hence, $|x(s) - x| = s$. This in turn implies

$$d(x(s)) = d(x) + s = |x - y| + |x - x(s)| \geq |y - x(s)|,$$

where y is the unique projection of x onto $\partial\Omega$. But then $d(x(s)) = |y - x(s)|$ and $x(s) = x + sDd(x)$ as long as $x(s) \notin \overline{\Sigma}$ and so at least for $s < t$. Therefore,

$$d(x + sDd(x)) = d(x(s)) = d(x) + s \qquad \forall s \in (0, t).$$

This also proves that $y \in \Pi(x + sDd(x))$ and $Dd(x + sDd(x)) = Dd(x)$ since d is differentiable at $x + sDd(x)$. $\square$

1.1.2 The Regular Case: Sets with $\mathcal{C}^2$ Boundary

Suppose now that $\partial\Omega$ is of class $\mathcal{C}^2$. For any $y \in \partial\Omega$ let us denote by $\nu(y)$ and $T(y)$, respectively, the unit inner normal and the tangent hyperplane to $\partial\Omega$ at y. Fix some $y_0 \in \partial\Omega$ and rotate if necessary the coordinate system in order to have the x_n coordinate axis in the direction of $\nu(y_0)$. Then there exists some neighborhood $U(y_0)$ of y_0 such that $\partial\Omega \cap U(y_0)$ can be represented as

$$\partial\Omega \cap U(y_0) = \{(y', y_n) \ : \ y' \in T(y_0) \cap U(y_0), \ y_n = \phi(y')\},$$

where $\phi : T(y_0) \cap U(y_0) \to \mathbb{R}$ is a $\mathcal{C}^2$ function such that $D\phi(y_0') = 0$. The eigenvalues of the Hessian matrix $D^2\phi(y_0')$–$\kappa_1(y_0), \ldots, \kappa_{n-1}(y_0)$–are called the *principal curvatures* of $\partial\Omega$ at y_0 and the corresponding eigenvectors–$v_1(y_0), \ldots, v_{n-1}(y_0)$–are called the *principal directions* of $\partial\Omega$ at y_0. If we further rotate the coordinate system in such a way that the first $n-1$ coordinate axes are in the direction of the principal directions of $\partial\Omega$ at y_0, we obtain what is called a *principal coordinate system* at y_0.
Let us consider a principal coordinate system at y_0 and let us call

$$e_1(y_0), \ldots, e_{n-1}(y_0), e_n(y_0) := \nu(y_0)$$

the basis of unit vectors corresponding to $v_1(y_0), \ldots, v_{n-1}(y_0), \nu(y_0)$. Also denote by $N_i(y_0)$, $i = 1, \ldots, n-1$, the plane spanned by $e_i(y_0)$ and $e_n(y_0)$ and passing through

y_0. Then, $\kappa_i(y_0) \geq 0$ if the normal section of Ω along the direction $e_i(y_0)$–that is $\partial\Omega \cap N_i(y_0)$–is convex. Moreover, if $\kappa_i(y_0) \neq 0$, then $1/|\kappa_i(y_0)|$ is the radius of the osculating (2-dimensional) circle to $\partial\Omega \cap N_i(y_0)$ at y_0.

In what follows, we will extend κ_i to $\overline{\Omega} \setminus \Sigma$ setting

$$\kappa_i(x) = \kappa_i(\Pi(x)) \qquad \forall x \in \overline{\Omega} \setminus \Sigma\,. \tag{1.1.1}$$

Analogously, for any $x \in \overline{\Omega} \setminus \Sigma$ we will denote by $e_1(x), \ldots, e_{n-1}(x), e_n(x)$ the basis of the principal coordinate system at $y = \Pi(x)$.

Proposition 1.1.4. *For any $x \in \overline{\Omega}$ and any $y \in \Pi(x)$ we have*

$$\kappa_i(y)d(x) \leq 1 \qquad \forall i = 1, \ldots, n-1.$$

If, in addition, $x \in \overline{\Omega}\setminus\overline{\Sigma}$, then

$$\kappa_i(x)d(x) < 1.$$

Proof—Let $x \in \Omega$ and $y \in \Pi(x)$. Then the ball of center x and radius $d(x)$ is contained in Ω and is tangent to $\partial\Omega$ at y. Therefore, for any $i = 1, \ldots, n-1$, we have either $\kappa_i(y) \leq 0$ or $1/\kappa_i(y) \geq d(x)$. So, $\kappa_i(y)d(x) \leq 1$.
If we assume, next, that $x \notin \overline{\Sigma}$, then y belongs to the projection of $x + sDd(x)$ for $s > 0$ sufficiently small, thanks to Proposition 1.1.3. Thus, for any $i = 1, \ldots, n-1$ we have the inequality $\kappa_i(y)d(x + sDd(x)) \leq 1$. Since $d(x + sDd(x)) = d(x) + s$ and $\kappa_i(x) = \kappa_i(y)$ by definition, we conclude $\kappa_i(x)d(x) < 1$. □

The following result shows that the regularity of the distance function increases with the regularity of the boundary of Ω. In the representation formula below, $p \otimes q$ stands for the tensor product of two vectors $p, q \in \mathbb{R}^n$, defined as

$$(p \otimes q)(x) = p\,\langle q, x\rangle\,,\ \forall x \in \mathbb{R}^n.$$

Proposition 1.1.5. *Let Ω be a bounded domain with $\mathcal{C}^k$ boundary, $k \geq 2$. Then $d \in \mathcal{C}^k(\Omega \setminus \overline{\Sigma})$. Moreover, for any $x_0 \in \Omega \setminus \overline{\Sigma}$*

$$\textit{and} \qquad D^2 d(x_0) = -\sum_{i=1}^{n-1} \frac{\kappa_i(x_0)}{1 - \kappa_i(x_0)d(x_0)} e_i(x_0) \otimes e_i(x_0) \tag{1.1.2}$$

where $e_1(x_0), \ldots, e_{n-1}(x_0), e_n(x_0)$ *is the basis of the principal coordinate system at* $y_0 = \Pi(x_0)$.

Proof—We already know by Proposition 1.1.2 that for any $x \in \Omega \setminus \overline{\Sigma}$ there exists a unique $y \in \partial\Omega$ such that $d(x) = |x - y|$ and $y = x - d(x)Dd(x)$. Moreover, $Dd(x) = \nu(y)$, because the ball of center x and radius $d(x)$ must be tangent to $\partial\Omega$ at y by definition of the distance. Now, fix $x_0 \in \Omega \setminus \overline{\Sigma}$, let $y_0 = \Pi(x_0)$ and locally represent $\partial\Omega$ around y_0 as the graph of a $\mathcal{C}^2$ function ϕ as above. Also rotate the coordinate system in order to have a principal coordinate system at y_0. Next, define the map $G : (T(y_0) \cap U(y_0)) \times \mathbb{R} \to \mathbb{R}^n$ as

$$G(y', d) = y + \nu(y)d, \qquad y = (y', \phi(y')). \tag{1.1.3}$$

Since the unit inner normal vector at a point $y = (y', \phi(y')) \in \partial\Omega \cap U(y_0)$ is given by

$$\nu_j(y) = \frac{-D_j\phi(y')}{\sqrt{1 + |D\phi(y')|^2}}, \quad j = 1, \ldots, n-1, \quad \nu_n(y) = \frac{1}{\sqrt{1 + |D\phi(y')|^2}},$$

then, with respect to the principal coordinate system, we have that the function $\bar{\nu}(y') := \nu(y', \phi(y'))$ verifies

$$D_k\bar{\nu}_j(y'_0) = -\kappa_j(y_0)\delta_{jk}, \qquad j,\ k = 1, \ldots, n-1. \tag{1.1.4}$$

Therefore $G \in \mathcal{C}^{k-1}$ and the Jacobian matrix of G at $(y'_0, d(x_0))$ is a diagonal matrix, whose elements on the diagonal are $1 - \kappa_1(y_0)d(x_0), \ldots, 1 - \kappa_{n-1}(y_0)d(x_0), 1$. Since $x_0 \notin \overline{\Sigma}$, by Proposition 1.1.4

$$\det DG(y'_0, d(x_0)) = (1 - \kappa_1(y_0)d(x_0)) \ldots (1 - \kappa_{n-1}(y_0)d(x_0)) > 0$$

and by the Inverse Mapping Theorem we conclude that there exists a neighborhood $V(x_0)$ such that y' is a $\mathcal{C}^{k-1}$ function of $x \in V(x_0)$. Hence, $Dd(x) = \nu(y(x)) = \nu(y'(x), \phi(y'(x)))$ is itself in $\mathcal{C}^{k-1}(V(x_0))$, which implies $d \in \mathcal{C}^k(V(x_0))$ for any $x_0 \in \Omega \setminus \overline{\Sigma}$, i.e. $d \in \mathcal{C}^k(\Omega \setminus \overline{\Sigma})$. Notice that we have also proven that $y(x) \equiv \Pi(x) \in \mathcal{C}^{k-1}(\Omega \setminus \overline{\Sigma})$. It only remains to prove formula (1.1.2). Fix any $x_0 \in \Omega \setminus \overline{\Sigma}$ and let $y_0 = \Pi(x_0)$. Rotate again the coordinate system in order to obtain the principal coordinate system at y_0. Since $|Dd(x)| = |\nu(y(x))| = 1$ in a neighborhood of x_0 and $Dd(x_0) = \nu(y(x_0)) = (0, \dots, 0, 1)$, then

$$0 = D\Big(|Dd(x)|^2\Big)_{x_0} = 2D^2 d(x_0)\nu(x_0),$$

which implies $D^2_{in} d(x_0) = 0$ for $i = 1, \dots, n$. Furthermore, with respect to the principal coordinate system, the Jacobian $D(G^{-1})_{x_0}$ of the local inverse of the map G given in (1.1.3) is a diagonal matrix, whose elements on the diagonal are $(1 - \kappa_1(y_0)d(x_0))^{-1}, \dots, (1 - \kappa_{n-1}(y_0)d(x_0))^{-1}, 1$. Hence, for $j,\ k = 1, \dots, n-1$,

$$D_j y_k(x_0) = \begin{cases} \dfrac{1}{1 - d(x_0)\kappa_j(y_0)} & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

and, recalling also (1.1.4),

$$\begin{aligned} D^2_{ij} d(x_0) &= D_j(\nu_i \circ y)(x_0) \\ &= \textstyle\sum_{k=1}^{n-1} D_k \bar{\nu}_i(y_0') D_j y_k(x_0) \\ &= \begin{cases} \dfrac{-\kappa_i(y_0)}{1 - d(x_0)\kappa_i(y_0)} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \end{aligned}$$

Therefore, with respect to the principal coordinate system, $D^2 d(x_0)$ is itself a diagonal matrix, whose elements on the diagonal are

$$\frac{-\kappa_1(y_0)}{1 - d(x_0)\kappa_1(y_0)}, \dots, \frac{-\kappa_{n-1}(y_0)}{1 - d(x_0)\kappa_{n-1}(y_0)}, 0.$$

Formula (1.1.2) now follows by writing $D^2 d(x_0)$ with respect to the initial coordinate system, since $e_1(x_0), \dots, e_{n-1}(x_0), e_n(x_0)$ is a basis of the principal coordinate system at $y_0 = \Pi(x_0)$. $\square$

Remark 1.1.6. Since $Dd(x)$, for $x \in \Omega \setminus \Sigma$, is given by the unit inner normal to $\partial\Omega$ at $\Pi(x)$, in what follows we will often write $Dd(x)$ for $x \in \partial\Omega$ to indicate $\nu(x)$. As a matter of fact, the distance function is not differentiable at boundary points. But if we consider the *signed distance function*

$$b_\Omega(x) := d_{\mathbb{R}^n \setminus \Omega}(x) - d_{\overline{\Omega}}(x),$$

then, by using Proposition 1.1.5, it can be proven that b_Ω is of class $\mathcal{C}^k$ in a neighborhood of $\partial\Omega$ as long as $\partial\Omega$ is a $\mathcal{C}^k$ boundary. Hence, the above abuse of notation is motivated by the coincidence of d and b_Ω in Ω and by the "continuity" from inside Ω of the gradient of d. The same motivation justify the notation $D^2d(x)$ for any $x \in \partial\Omega$ in place of $D^2 b_\Omega(x)$.

Owing to our regularity assumption on $\partial\Omega$, we have

$$\sup_{\substack{y,z\in\partial\Omega \\ y\neq z}} \frac{|Dd(z) - Dd(y)|}{|z-y|} < \infty \, . \tag{1.1.5}$$

So, in view of Proposition 1.1.5, we will denote the above supremum by $\mathrm{Lip}(\kappa)$.

Let us now define the set of *regular conjugate points* Γ as

$$\Gamma = \{x \in \Omega\backslash\Sigma \ : \ d(x)\kappa_i(x) = 1 \ \text{ for some } i = 1,\dots,n-1\} \ .$$

Notice that a point $x \in \Omega\backslash\Sigma$ belongs to Γ if and only if

$$\Pi(x) = \left\{x - \frac{1}{\kappa_i(x)} Dd(x) \ \text{ for some } i = 1,\dots,n-1\right\}.$$

In other words, $x \in \Omega\backslash\Sigma$ is a regular conjugate point if and only if is the center of the osculating circle to $\partial\Omega$ at $\Pi(x)$.

Next proposition characterizes the closure of the set of singular points, that is the *cut locus* of Ω.

Proposition 1.1.7. *Suppose that Ω is a bounded domain with $\mathcal{C}^2$ boundary. Then $\overline{\Sigma} \subset \Omega$ and $\overline{\Sigma} = \Sigma \cup \Gamma$.*

Proof—Let $x \in \Sigma$ and y, z be two distinct elements of $\Pi(x)$. Then

$$x = y + d(x)Dd(y) = z + d(x)Dd(z) \, . \tag{1.1.6}$$

Therefore, recalling Remark 1.1.6,

$$|y - z| = d(x)|Dd(y) - Dd(z)| \leq d(x)K|y - z|$$

for some constant $K > 0$ independent of x. We have thus proven that $d(x) \geq 1/K$ for every $x \in \Sigma$. So, $\overline{\Sigma} \subset \Omega$. Furthermore, the inclusion $\Gamma \subset \overline{\Sigma}$ is a straightforward consequence of the strict inequality in Proposition 1.1.4.

In order to prove the inclusion $\overline{\Sigma} \subset \Sigma \cup \Gamma$, let $\{x_k\}$ be a sequence of singular points converging to a point $x \in \Omega \backslash \Sigma$. We claim that $d(x)\kappa_i(x) = 1$ for some $i = 1, \ldots, n-1$. To see this, let y_k and z_k be two distinct points in $\Pi(x_k)$. Then, both $\{y_k\}$ and $\{z_k\}$ must converge to $\Pi(x)$ as $k \to \infty$. Also, passing to a subsequence,

$$\lim_{k \to \infty} \frac{y_k - z_k}{|y_k - z_k|} = \theta$$

for some unit vector $\theta \in \mathbb{R}^n$. From identity (1.1.6) applied to x_k, y_k and z_k, we have

$$0 = \frac{y_k - z_k}{|y_k - z_k|} + d(x_k) \frac{Dd(y_k) - Dd(z_k)}{|y_k - z_k|} .$$

Hence, taking the limit as $k \to \infty$ we conclude that $0 = \theta + d(x) D^2 d(\Pi(x))\theta$. Therefore, $-1/d(x)$ is a nonzero eigenvalue of $D^2 d(\Pi(x))$. By Proposition 1.1.5 we readily conclude that $-1/d(x) = -\kappa_i(x)$ for some $i = 1, \ldots, n-1$, as claimed. $\square$

The following result ensures that segments of minimal length joining a point to $\partial\Omega$, contain no singular or conjugate points in their interior.

Proposition 1.1.8. *Let $x \in \Omega$ and $y \in \Pi(x)$. Then $\overline{\Sigma} \cap]y, x[= \emptyset$.*

Proof—We already know that $\Sigma \cap]y, x[= \emptyset$ by Lemma 1.1.1, and that $\kappa_i(y) d(x) \leq 1$ for all $i = 1, \ldots, n-1$ by Proposition 1.1.4. Since $\kappa_i(y) d(z) < 1$ for every $z \in]y, x[$ and all $i = 1, \ldots, n-1$, we conclude that $z \notin \Gamma$. $\square$

1.1.3 A Special Case: Sets with Piecewise $\mathcal{C}^{2,1}$ Boundary and Outer Corners in the Plane.

In this section we restrict our attention to sets Ω contained in $\mathbb{R}^2$. In this setting, we can analyze the structure of the singular set of the distance function even in the case of sets that are only piecewise regular, provided that regular components join in corners pointing “outside” the set Ω. The precise requirements on Ω are the following.

Definition 1.1.9. Let Ω be a connected bounded open subset of $\mathbb{R}^2$. We will say that Ω has *piecewise* $\mathcal{C}^{2,1}$ *boundary and outer corners* if it satisfies the following conditions:

(H1) $\partial\Omega = \cup_{i=1}^m \Gamma_i$, $m \in \mathbb{N}$, where

$$\Gamma_i \cap \Gamma_j = \begin{cases} \{x_i\} & \text{if } 1 \le i \le m-1,\ j = i+1 \\ \{x_m\} & \text{if } i = m,\ j = 1 \\ \emptyset & \text{if } 1 \le i \le m,\ j \neq i, i \pm 1 \end{cases}$$

and for any $i = 1, \ldots, m$ Γ_i is a $\mathcal{C}^{2,1}$ curve up to the endpoints x_i and x_{i+1} (x_m and x_1 when $i = m$);

(H2) there exists some $0 < \theta < 1$ such that for any $i = 1, \ldots, m-1$,

$$\langle \nu_i(x_i), \nu_{i+1}(x_i)\rangle \le \theta \quad \langle \nu_m(x_m), \nu_1(x_m)\rangle \le \theta,$$

where ν_i stands for the unit inner normal to the boundary component Γ_i and where

$$\nu_i(x_j) := \lim_{\substack{y \to x_j \\ y \in \Gamma_i}} \nu_i(y).$$

(H3) Ω satisfies a uniform exterior sphere condition of radius $r > 0$, that is for any $x \in \partial\Omega$ there exists some $y \in \mathbb{R}^n \setminus \Omega$ such that $B_r(y) \subset \mathbb{R}^n \setminus \overline{\Omega}$ and $x \in \partial B_r(y)$.

Notice that assumptions (H1)–(H3) imply that the boundary of Ω has a Lipschitz regularity.

In what follows we denote by $\mathcal{C}$ the set of corners of $\partial\Omega$, that is

$$\mathcal{C} = \{x_1, \ldots, x_m\}.$$

Remark 1.1.10. Notice that our assumptions on Ω guarantee that for any $x \in \Omega$ the set of projections on $\partial\Omega$ has empty intersection with $\mathcal{C}$. Indeed, let $x \in \Omega$ and $y \in \Pi(x)$ be fixed. Then, the open ball of radius $d(x)$ centered in x is contained in Ω. On the other hand, by the exterior sphere condition, there exists a point $z \in \mathbb{R}^n \setminus \Omega$ such that the open ball of radius r centered in z is contained in $\mathbb{R}^n \setminus \overline{\Omega}$ and $y \in \partial B_r(z)$. Hence, the two balls must be tangent in y and the points x, y, z are colinear. It is now easy to prove that y must be a regular boundary point. Indeed, let us suppose that $y = (0, 0)$, $x = (0, d(x))$ and $z = (0, -r)$. Fix an open neighborhood U of y such that $\partial\Omega \cap U$ can be represented as the trace of a (at least) Lipschitz continuous curve $s \in (-\varepsilon, \varepsilon) \mapsto (\alpha(s), \beta(s))$ such that $\alpha(0) = \beta(0) = 0$. Then, for s sufficiently small,

$$\sqrt{r^2 - \alpha(s)^2} - r \leq \beta(s) \leq \sqrt{d(x)^2 - \alpha(s)^2} - d(x)$$

because the balls must be separated by the boundary. Moreover, we can suppose that $\alpha(s) \neq 0$ for $s \neq 0$, since otherwise this would force $\beta(s) = 0$ by the above inequality and we could avoid these stationary points by changing the parametrization of $\partial\Omega$. Hence, for s sufficiently small, we have

$$\frac{\sqrt{r^2 - \alpha(s)^2} - r}{|\alpha(s)|} \leq \frac{\beta(s)}{|\alpha(s)|} \leq \frac{\sqrt{d(x)^2 - \alpha(s)^2} - d(x)}{|\alpha(s)|}.$$

Since $\alpha(s) \to 0$ as $s \to 0$, taking the limit as $s \to 0$ in the above inequality we obtain that

$$\lim_{s \to 0} \frac{\beta(s)}{\alpha(s)} = 0,$$

which indeed implies the differentiability of the boundary at $(0, 0)$.

Hereafter, for any $y \in \partial\Omega \setminus \mathcal{C}$, we denote by $\kappa(y)$ the curvature of $\partial\Omega$ at y. Also, using the fact that $\Pi(x) \cap \mathcal{C} = \emptyset$ for any $x \in \Omega$, we will label in the same way the extension of κ to $\Omega \setminus \Sigma$ given by

$$\kappa(x) = \kappa(\Pi(x)) \qquad \forall x \in \overline{\Omega} \setminus \Sigma\,. \tag{1.1.7}$$

Another consequence of the fact that $\mathcal{C}$ and $\Pi(x)$ are disjoint for any $x \in \Omega$ is that the conclusions of Propositions 1.1.4 and 1.1.5 still hold true. Indeed, we can prove in the same way the following results.

Proposition 1.1.11. *Suppose that Ω has a piecewise $\mathcal{C}^{2,1}$ boundary and outer corners. Thus, for any $x \in \overline{\Omega}$ and any $y \in \Pi(x)$ we have $\kappa(y)d(x) \leq 1$. If, in addition, $x \in \overline{\Omega} \setminus \overline{\Sigma}$, then*

$$\kappa(x)d(x) < 1.$$

Proposition 1.1.12. *Suppose that Ω has a piecewise $\mathcal{C}^{2,1}$ boundary and outer corners. Then $d \in \mathcal{C}^{2,1}(\Omega \setminus \overline{\Sigma})$. Moreover, for any $x \in \Omega \setminus \overline{\Sigma}$*

$$\text{and} \qquad D^2 d(x) = -\frac{\kappa(x)}{1-\kappa(x)d(x)} q \otimes q \tag{1.1.8}$$

where q is any unit vector such that $\langle q, Dd(x)\rangle = 0$.

Let us call again *regular conjugate points* the elements of the set

$$\Gamma = \{x \in \Omega \backslash \Sigma \ : \ d(x)\kappa(x) = 1\} \ .$$

In the case of sets having outer corners, the inner normal to the boundary is not continuous at corner points. So it is reasonable to argue that the singularities of the distance function reach the boundary exactly at those points. This is indeed the case, as the next proposition shows.

Proposition 1.1.13. *If Ω has piecewise $\mathcal{C}^{2,1}$ boundary and outer corners, we have*

$$\overline{\Sigma} = \Sigma \cup \Gamma \cup \mathcal{C} \ .$$

Proof—In order to prove the inclusion $\overline{\Sigma} \subset \Sigma \cup \Gamma \cup \mathcal{C}$, let $\{x_k\}$ be a sequence of singular points converging to a point $x \in \overline{\Omega} \backslash \Sigma$. We claim that either $x \in \mathcal{C}$ or $d(x)\kappa(x) = 1$. To see this, let y_k and z_k be two distinct points in $\Pi(x_k)$. Then, both $\{y_k\}$ and $\{z_k\}$ must converge to $\Pi(x)$ as $k \to \infty$. Also, passing to a subsequence,

$$\lim_{k\to\infty} \frac{y_k - z_k}{|y_k - z_k|} = \theta$$

for some unit vector $\theta \in \mathbb{R}^2$. From the identity

$$x_k = y_k + d(x_k)Dd(y_k) = z_k + d(x_k)Dd(z_k)$$

we have

$$0 = \frac{y_k - z_k}{|y_k - z_k|} + d(x_k)\frac{Dd(y_k) - Dd(z_k)}{|y_k - z_k|} \ . \tag{1.1.9}$$

Now, if $x \in \Omega$, then $\Pi(x) \notin \mathcal{C}$ and taking the limit as $k \to \infty$ we conclude that $0 = \theta + d(x)D^2 d(\Pi(x))\theta$. Therefore, $-1/d(x)$ is a nonzero eigenvalue of $D^2 d(\Pi(x))$,

a matrix of the form $-\kappa(x)q \otimes q$ by Proposition 1.1.11. So, $-1/d(x) = -\kappa(x)$. On the contrary, if $x \in \partial\Omega$, then $x = \Pi(x)$ must be a corner point, since otherwise the limit as $k \to \infty$ in (1.1.9) would produce $\theta = 0$, a contradiction.
Let us now prove the reverse inclusion $\Sigma \cup \Gamma \cup \mathcal{C} \subset \overline{\Sigma}$. The fact that $\Gamma \subset \overline{\Sigma}$ is a straightforward consequence of the strict inequality in Proposition 1.1.11. So it only remains to show that $\mathcal{C} \subset \overline{\Sigma}$. To this end, let us fix some $\rho \in (0, r/2)$ and consider the ρ-neighborhood of Ω,

$$\Omega_\rho := \{x \in \mathbb{R}^n : d_{\overline{\Omega}}(x) < \rho\}.$$

Let us first prove that

$$d_\rho(x) = d(x) + \rho, \qquad \text{for any } x \in \Omega, \tag{1.1.10}$$

where d_ρ denotes the distance function from $\mathbb{R}^n \setminus \Omega_\rho$. Indeed, for any $x \in \Omega$, let $y \in \partial\Omega$ be any projection of x onto $\partial\Omega$ and define $y_\rho = y + \rho\frac{y-x}{|y-x|}$. Since $y \notin \mathcal{C}$, then $y - x$ is normal to $\partial\Omega$ at y and the center of the exterior sphere of radius r which is tangent to $\partial\Omega$ at y must be colinear with y and y_ρ. We easily deduce that $y_\rho \in \partial\Omega_\rho$. Hence

$$d_\rho(x) \leq |x - y_\rho| = |x - y| + |y - y_\rho| = d(x) + \rho.$$

On the other hand, for any $x \in \Omega$ let $y_\rho \in \partial\Omega_\rho$ be any projection of x onto $\partial\Omega_\rho$ and call y the intersection point of the segment $[x, y_\rho]$ and $\partial\Omega$. Then we readily conclude

$$d_\rho(x) = |x - y_\rho| = |x - y| + |y - y_\rho| \geq d(x) + \rho.$$

Now, let Σ_ρ stand for the singular set of d_ρ. From (1.1.10) we deduce

$$\Sigma = \Sigma_\rho \cap \Omega,$$

and then $\overline{\Sigma} = \overline{\Sigma_\rho \cap \Omega}$. We now claim that $\mathcal{C} \subset \Sigma_\rho$. Indeed, fix some $x_j \in \mathcal{C}$, $j \in \{1, \ldots, m\}$, and consider any two sequences of points $\{x_j^k\} \subset \Gamma_j$ and $\{x_{j+1}^k\} \subset \Gamma_{j+1}$ converging to x_j as $k \to \infty$ and definitely different from x_j. Using the existence of an exterior sphere of radius $r > \rho$ to Ω at any point of the sequences, it is easy to prove that the points $y_j^k := x_j^k - \rho\nu_j(x_j^k)$ and $y_{j+1}^k := x_{j+1}^k - \rho\nu_{j+1}(x_{j+1}^k)$ are on $\partial\Omega_\rho$ and are projections of x_j^k and x_{j+1}^k onto $\partial\Omega_\rho$, respectively. By the assumptions made on Ω in Definition 1.1.9 we have that

$$\nu_{j+1}(x_{j+1}^k) \to \nu(x_j) \qquad \text{and} \qquad \nu_{j+1}(x_{j+1}^k) \to \nu_{j+1}(x_j),$$

where $\langle \nu_j(x_j), \nu_{j+1}(x_j)\rangle < 1$. Hence, y_j^k and y_{j+1}^k converge, respectively, to $x_j - \rho\nu_j(x_j)$ and $x_j - \rho\nu_{j+1}(x_j)$, which have to be distinct projections of x_j onto $\partial\Omega_\rho$ by the upper semicontinuity of the distance function. We have then proven that $\mathcal{C} \subset \Sigma_\rho$. Since also $\Sigma_\rho \cap \partial\Omega = \mathcal{C}$ by the exterior sphere condition, we conclude that $\overline{\Sigma}_\rho \cap \overline{\Omega} = \overline{\Sigma_\rho \cap \Omega}$ and then $\mathcal{C} \subset \overline{\Sigma}$. $\square$

An easy consequence of the previous result is the analogue of Proposition 1.1.8, which can be proven exactly in the same way of that proposition due to the property $\mathcal{C} \cap \Pi(x) = \emptyset$ for any $x \in \Omega$.

Proposition 1.1.14. *Let Ω have piecewise $\mathcal{C}^{2,1}$ boundary and outer corners. Take any $x \in \Omega$ and $y \in \Pi(x)$. Then $\overline{\Sigma} \cap]y, x[= \emptyset$.*

1.2 Generalized Differentials, Semiconcave Functions and Singularities

In this section we introduce two different generalizations of the concept of gradient for functions that are not differentiable in the classical sense. In particular, in Definition 1.2.1 we introduce the notion of (Frechét) superdifferential, which can also be seen as an extension to nonconcave functions of the inequality

$$f(x+h) \le f(x) + \langle v, h\rangle, \qquad \forall h,$$

associated with a concave function f. Then, in Definition 1.2.17 we introduce the so–called *proximal superdifferential*, whose definition relies in the existence of a parabola touching from above the function under consideration instead of an affine function as in the case of the Frechét superdifferential. Definition 1.2.1 was put to use mainly by L. C. Evans, M. G. Crandall and P. L. Lions in the early 80's, while Definition 1.2.17 was given by F. H. Clarke in the seventies in his dissertation. For a detailed analysis on this topic and on the relation among the definitions we refer the reader to [36], [18] and the references therein. In this section we will also introduce the class of semiconcave functions (with linear modulus), which can be seen as the set of regular perturbations of concave functions. As we will see later, the distance function from $\mathbb{R}^n \setminus \Omega$ is itself a locally semiconcave function in Ω and many properties that it has, besides Lipschitz continuity, derive from this fact.

Let $A \subset \mathbb{R}^n$ be an open set and $u : A \to \mathbb{R}$ be a continuous function. We begin with the definition of (Frechét) superdifferential and subdifferential. Most of the results of this section and of the following one are taken from [15], [8] and [18].

Definition 1.2.1. For any $x \in A$, the sets

$$\begin{aligned} D^+u(x) &= \left\{ p \in \mathbb{R}^n \mid \limsup_{h\to 0} \tfrac{u(x+h)-u(x)-\langle p,h\rangle}{|h|} \leq 0 \right\} \\ D^-u(x) &= \left\{ p \in \mathbb{R}^n \mid \liminf_{h\to 0} \tfrac{u(x+h)-u(x)-\langle p,h\rangle}{|h|} \geq 0 \right\}. \end{aligned} \tag{1.2.1}$$

are called, respectively, the (Frechét) *superdifferential* and the *subdifferential* of u at a point $x \in A$.

In order to describe the properties of these sets of generalized gradients, we first introduce the directional derivatives known as Dini derivatives.

Definition 1.2.2. Let $x \in A$ and $\theta \in \mathbb{R}^n$. The *upper* and *lower Dini derivatives* of u at x in the direction of θ are defined as

$$\partial^+u(x,\theta) = \limsup_{\substack{h\to 0^+ \\ \theta'\to\theta}} \frac{u(x+h\theta')-u(x)}{h}$$

and

$$\partial^-u(x,\theta) = \liminf_{\substack{h\to 0^+ \\ \theta'\to\theta}} \frac{u(x+h\theta')-u(x)}{h}.$$

Lemma 1.2.3. *Let $x \in A$ be fixed. Then,*

(a) *$D^+u(x)$ and $D^-u(x)$ are closed convex (possibly empty) subsets of* $\mathbb{R}^n$*;*

(b) *if u is differentiable at x, then $D^+u(x) = D^-u(x) = \{Du(x)\}$;*

(c) *if both $D^+u(x)$ and $D^-u(x)$ are nonempty, then u is differentiable at x.*

(d)

$$D^+u(x) = \{p \in \mathbb{R}^n \ : \ \partial^+u(x,\theta) \leq \langle p, \theta\rangle, \ \forall \theta \in \mathbb{R}^n\},$$

$$D^-u(x) = \{p \in \mathbb{R}^n \ : \ \partial^-u(x,\theta) \geq \langle p, \theta\rangle, \ \forall \theta \in \mathbb{R}^n\}.$$

Proof—(a) The convexity of $D^+u(x)$ and $D^-u(x)$ is a direct consequence of the properties of liminf and limsup. Let us prove that $D^+u(x)$ is closed. Let $\{p_k\} \subset D^+u(x)$ be a sequence converging to some point p and assume by contradiction that

$$\lim_{m\to\infty} \frac{u(y_m) - u(x) - \langle p, y_m - x\rangle}{|y_m - x|} = \alpha > 0$$

for some $y_m \to x$. Take k sufficiently large so that $|p_k - p| \leq \alpha/2$. Then,

$$\begin{aligned}
&\limsup_{m\to\infty} \frac{u(y_m) - u(x) - \langle p_k, y_m - x\rangle}{|y_m - x|} \\
&= \limsup_{m\to\infty} \left(\frac{u(y_m) - u(x) - \langle p, y_m - x\rangle}{|y_m - x|} - \langle p_k - p, \frac{y_m - x}{|y_m - x|}\rangle \right) \geq \frac{\alpha}{2},
\end{aligned}$$

against the fact that $p_k \in D^+u(x)$.

(b) If u is differentiable at x, then both $D^+u(x)$ and $D^-u(x)$ are nonempty, since they contain $Du(x)$. Moreover, for any $p,\ q \in \mathbb{R}^n$, taking $y_m := x + \frac{1}{m}(p - q)$, we have

$$\begin{aligned}
|p - q| \quad &= \quad \frac{u(y_m) - u(x) - \langle q, y_m - x\rangle}{|y_m - x|} - \frac{u(y_m) - u(x) - \langle p, y_m - x\rangle}{|y_m - x|} \\
&\leq \quad \limsup_{A \ni y \to x} \frac{u(y) - u(x) - \langle q, y - x\rangle}{|y - x|} \\
&- \quad \liminf_{A \ni y \to x} \frac{u(y) - u(x) - \langle p, y - x\rangle}{|y - x|}.
\end{aligned} \tag{1.2.2}$$

Hence, for $p = Du(x)$ and $q \in D^+u(x)$ we obtain

$$\begin{aligned} |Du(x) - q| &\leq \limsup_{A \ni y \to x} \frac{u(y) - u(x) - \langle q, y - x \rangle}{|y - x|} \\ &- \liminf_{A \ni y \to x} \frac{u(y) - u(x) - \langle Du(x), y - x \rangle}{|y - x|} \leq 0, \end{aligned}$$

which yields $D^+u(x) = \{Du(x)\}$. The coincidence $D^-u(x) = \{Du(x)\}$ is then obtained as above by taking $q = Du(x)$ and $p \in D^-u(x)$.

(c) Suppose that both $D^+u(x)$ and $D^-u(x)$ are nonempty. Then, by (1.2.2) with $q \in D^+u(x)$ and $p \in D^-u(x)$ we obtain that $D^+u(x) = D^-u(x)$ is a singleton. This means that u is differentiable at x.

(d) For any $p \in D^+u(x)$ and $\theta \in \mathbb{R}^n$ we have $\partial^+u(x, \theta) \leq \langle p, \theta \rangle$ as a direct consequence of the definition. The converse can be proven by contradiction. Suppose that a vector $p \in \mathbb{R}^n$ satisfies $\partial^+u(x, \theta) \leq \langle p, \theta \rangle$ for all $\theta \in \mathbb{R}^n$, but $p \notin D^+u(x)$. Then we can find $\varepsilon > 0$ and a sequence $\{x_k\} \subset A$ such that $x_k \to x$ as $k \to \infty$ and

$$u(x_k) - u(x) - \langle p, x_k - x \rangle \geq \varepsilon |x_k - x|.$$

Possibly passing to a subsequence, we can suppose that the sequence $\theta_k := \frac{x_k - x}{|x_k - x|}$ satisfies

$$\lim_{k \to \infty} \frac{x_k - x}{|x_k - x|} = \theta, \qquad \text{for some } \theta \in \partial B_1(0).$$

Therefore,

$$\begin{aligned} \varepsilon + \langle p, \theta \rangle &\leq \limsup_{k \to \infty} \frac{u(x_k) - u(x)}{|x_k - x|} \\ &= \limsup_{k \to \infty} \frac{u(x + |x_k - x|\theta_k) - u(x)}{|x_k - x|} \leq \partial^+u(x, \theta), \end{aligned}$$

against the assumptions on p. A similar reasoning can also be applied to $D^-u(x)$. □

Let us now introduce the class of semiconcave functions (with linear modulus).

Definition 1.2.4. We say that $u : A \to \mathbb{R}$ is *semiconcave (with linear modulus)* if there exists $C > 0$ such that

$$\lambda u(x) + (1 - \lambda)u(y) - u(\lambda x + (1 - \lambda)y) \leq \frac{1}{2}\lambda(1 - \lambda)C|x - y|^2 \tag{1.2.3}$$

for any pair $x, y \in A$, such that the segment $[x, y]$ is contained in A and for any $\lambda \in [0, 1]$.

A function $u : A \to \mathbb{R}$ is said to be *semiconvex (with linear modulus)* if $-u$ is semiconcave (with linear modulus).

Finally, $u : A \to \mathbb{R}$ is said to be locally semiconcave (semiconvex) if it is semiconcave (semiconvex) on any compact subset of A.

Proposition 1.2.5. *Given $u : A \to \mathbb{R}$, with $A \subset \mathbb{R}^n$ open set, and given $C \geq 0$, the following properties are equivalent:*

(a) *inequality* (1.2.3) *is satisfied;*

(b) *the function $y \to u(y) - \frac{1}{2}C|y|^2$ is concave in every convex subset of A;*

(c) *$u \in \mathcal{C}(A)$ and satisfies $u(x+h) + u(x-h) - 2u(x) \leq C|h|^2$ for any x, h such that $[x-h, x+h] \subset A$.*

Proof— We notice that, for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} & \lambda|x|^2 + (1-\lambda)|y|^2 - |\lambda x + (1-\lambda)y|^2 \\ = \; & \lambda(1-\lambda)(|x|^2 + |y|^2 - 2\langle x, y\rangle) = \lambda(1-\lambda)|x-y|^2. \end{aligned} \tag{1.2.4}$$

This shows that inequality (1.2.3) holds if and only if the function $x \to u(x) - \frac{1}{2}C|x|^2$ is concave. Therefore (a) and (b) are equivalent. A similar computation shows that, if (c) holds, then the function $v(x) = u(x) - \frac{1}{2}C|x|^2$ is continuous and satisfies

$$v(x+h) + v(x-h) - 2v(x) \leq 0 \tag{1.2.5}$$

for any x, h such that $[x-h, x+h] \subset A$. We claim that this property implies the concavity of v in the convex subsets of A. Indeed, let $x,\ y$ be such that the segment $[x, y]$ is contained in A. For any $\lambda \in [0, 1]$ set $x_\lambda := \lambda x + (1-\lambda)y$. Also define, for $k \geq 1$,

$$D_k := \{x_\lambda \ : \ \lambda = \frac{j}{2^k} \text{ for some } j = 0, 1, \ldots, 2^k\}.$$

We prove by induction that

$$\lambda v(x) + (1-\lambda)v(y) - v(x_\lambda) \leq 0, \qquad \forall x_\lambda \in D_k. \tag{1.2.6}$$

Since $\bigcup_{k\in\mathbb{N}} D_k$ is dense in $[x,y]$ and u is continuous by assumption, the previous inequality will then hold for any $\lambda \in [0,1]$, giving the concavity of v. The basis of the induction is trivially satisfied. Suppose then that (1.2.6) holds true for some k and take $x_\lambda \in D_{k+1} \setminus D_k$. If we set

$$\mu = \lambda - \frac{1}{2^{k+1}}, \qquad \nu = \lambda + \frac{1}{2^{k+1}},$$

then $x_\mu,\ x_\nu \in D_k$. Therefore, by (1.2.6) and (1.2.5) we have, respectively,

$$\begin{aligned} &2\lambda v(x) + 2(1-\lambda)v(y) - v(x_\mu) - v(x_\nu) \\ &= (\mu+\nu)v(x) + (2-\mu-\nu)v(y) - v(x_\mu) - v(x_\nu) \leq 0, \end{aligned}$$

and

$$v(x_\mu) + v(x_\nu) - 2v(x_\lambda) \leq 0,$$

which clearly imply $\lambda v(x) + (1-\lambda)v(y) - v(x_\lambda) \leq 0$. We have then proven that (c) implies (b). On the other hand, (a) obviously implies (c) and so the three properties are equivalent. □

Theorem 1.2.6. *Let $A \subset \mathbb{R}^n$ be an open set. A semiconcave function $u : A \to \mathbb{R}$ is locally Lipschitz continuous in A.*

Proof—The statement can be proven as a consequence of characterization (b) in Proposition 1.2.5, since it is well known that concave functions are locally Lipschitz continuous. But we can also give a simple direct proof. First of all, inequality (1.2.3) implies that for any $z \in [x,y] \subset A$

$$\frac{u(x)-u(z)}{|x-z|} - \frac{u(z)-u(y)}{|z-y|} \leq \frac{C}{2}|x-y|.$$

Now fix any $x_0 \in A$ and choose $r > 0$ such that $\overline{B}_r(x_0) \subset A$. Since u is continuous by assertion (b) of Proposition 1.2.5, we can find $m,\ M \in \mathbb{R}$ such that $m \leq u \leq M$

in $\overline{B}_r(x_0)$. Given any $x,\ y \in B_{\frac{r}{2}}(x_0)$, consider the straight line through $x,\ y$ and call $x',\ y'$ the points on this line having distance r from x_0 and satisfying $x \in [x', y]$, $y \in [x, y']$. Thus

$$\frac{u(x') - u(x)}{|x' - x|} - \frac{C}{2}|x' - y| \leq \frac{u(x) - u(y)}{|x - y|} \leq \frac{u(y) - u(y')}{|y - y'|} + \frac{C}{2}|y' - x|,$$

which gives

$$\frac{|u(x) - u(y)|}{|x - y|} \leq \frac{2(M - m)}{r} + C.$$

□

We already know that the distance function d_K from a closed set K is a Lipschitz continuous function. Next lemma shows that d_K is even locally semiconcave in $\mathbb{R}^n \backslash K$.

Lemma 1.2.7. *Let K be a closed set. Then the distance function d_K from K is semiconcave on any subset $A \subset \mathbb{R}^n$ such that* $\operatorname{dist}(A, K) > 0$ *with constant* $C = \frac{1}{\operatorname{dist}(A,K)}$.

Proof—First of all for any $z, h \in \mathbb{R}^n$ with $z \neq 0$ we have

$$(|z + h| + |z - h|)^2 \leq 2(|z + h|^2 + |z - h|^2) = 4(|z|^2 + |h|^2) \leq \left(2|z| + \frac{|h|^2}{|z|}\right)^2.$$

Therefore, $|z + h| + |z - h| - 2|z| \leq \frac{|h|^2}{|z|}$. Let now $A \subset \mathbb{R}^n$ such that $\operatorname{dist}(A, K) > 0$ and consider any $x,\ h \in \mathbb{R}^n$ such that $[x - h, x + h] \subset A$. If $d_K(x) = |x - y|$ for some $y \in K$, then

$$\begin{aligned} &d_K(x + h) + d_K(x - h) - 2d_K(x) \\ &\leq |x + h - y| + |x - h - y| - 2|x - y| \\ &\leq \tfrac{|h|^2}{|x-y|} \leq \tfrac{|h|^2}{\operatorname{dist}(A,K)}, \end{aligned}$$

which implies the semiconcavity of d_K in A by Proposition 1.2.5. □

The superdifferential of a semiconcave function has many properties that general Lipschitz continuous functions do not possess. Before describing these properties, let us introduce the notion of *limiting gradient*. Let A be an open set and $u : A \to \mathbb{R}$ be a Lipschitz continuous function. It is well–known, by Rademacher's Theorem, that u

is differentiable a. e. in A. Then, for any $x \in A$ the set $D^*u(x)$ of *limiting gradients* of u at x, defined as

$$D^*u(x) = \{\lim_{k\to\infty} Du(x_k) : \ A \ni x_k \to x, \ \exists Du(x_k)\} \ , \tag{1.2.7}$$

exists and is nonempty.

Proposition 1.2.8. *Let $u : A \to \mathbb{R}$ be semiconcave. Then for all $x \in A$*

(a) *a vector $p \in \mathbb{R}^n$ belongs to $D^+u(x)$ if and only if for any $y \in A$ such that $[x, y] \subset A$*
$$u(y) - u(x) - \langle p, y - x\rangle \leq \frac{C}{2}|x - y|^2,$$
where $C > 0$ is the semiconcavity constant of u in A;

(b) *if $\{x_k\} \subset A$ is a sequence converging to $x \in A$, and if $p_k \in D^+u(x_k)$ converges to some $p \in \mathbb{R}^n$, then $p \in D^+u(x)$.*

(c) *$D^+u(x) = \overline{\text{co}}\, D^*u(x)$;*

(d) *$D^*u(x) \subseteq \partial D^+u(x)$;*

(e) *either $D^-u(x) = \emptyset$ or u is differentiable at x;*

(f) *if $D^+u(x)$ is a singleton, then u is differentiable at x.*

Proof—(a) It is clear that if $p \in \mathbb{R}^n$ satisfies

$$u(y) - u(x) - \langle p, y - x\rangle \leq \frac{C}{2}|x - y|^2,$$

for any $y \in A$ such that $[x, y] \subset A$, then $p \in D^+u(x)$. On the other hand, take any $p \in D^+u(x)$. The definition of semiconcave functions directly gives

$$\frac{u(y) - u(x)}{|x - y|} \leq \frac{u(x + (1 - \lambda)(y - x)) - u(x)}{(1 - \lambda)|x - y|} + \frac{\lambda C}{2}|x - y|$$

for any $y \in A$ such that $[x, y] \subset A$ and any $\lambda \in [0, 1)$. Hence, recalling Lemma 1.2.3 (d) and taking the limit as $\lambda \to 1^-$, we conclude that

$$\frac{u(y) - u(x)}{|x - y|} \leq \frac{\langle p, y - x\rangle}{|x - y|} + \frac{C}{2}|x - y|,$$

which is the desired inequality.
(b) It is a direct consequence of (a) and of the continuity of semiconcave functions.
(c) Let $p \in D^* u(x)$ and $\{x_k\}$ be a sequence of differentiability points of u such that $x_k \to x$ and $Du(x_k) \to p$. Then $Du(x_k) \in D^+u(x_k)$ and we deduce that $p \in D^+u(x)$ by assertion (b). Being also $D^+u(x)$ convex and closed by Lemma 1.2.3, then $\overline{\mathrm{co}}\, D^*u(x) \subseteq D^+u(x)$. In order to prove the converse inclusion, remember that

$$D^+u(x) = \{p \in \mathbb{R}^n \ : \ \partial^+ u(x,\theta) \leq \langle p, \theta\rangle, \ \forall \theta \in \mathbb{R}^n\}.$$

Then, for any $\theta \in \mathbb{R}^n$

$$\partial^+ u(x,\theta) \leq \min_{p \in D^+u(x)} \langle p, \theta \rangle \leq \min_{p \in \overline{\mathrm{co}}\, D^*u(x)} \langle p, \theta\rangle.$$

We claim that also $\partial^+ u(x,\theta) \geq \min_{p\in \overline{\mathrm{co}}\, D^*u(x)} \langle p, \theta\rangle$. Indeed, let θ be a fixed unit vector. Since u is semiconcave, and then locally Lipschitz continuous in A, it is also differentiable almost everywhere by Rademacher's Theorem. Hence we can find a sequence $\{x_k\} \in A$ of differentiability points of u such that $x_k \to x$ as $k \to \infty$,

$$\theta_k := \frac{x_k - x}{|x_k - x|} \to \theta, \qquad \text{as } k \to \infty$$

and $Du(x_k)$ converges to some vector $p \in D^*u(x)$. By (a) we have

$$\langle Du(x_k), \theta_k\rangle \leq \frac{u(x + |x_k - x|\theta_k) - u(x)}{|x_k - x|} + \frac{C}{2}|x_k - x|,$$

where C is the semiconcavity constant of u. Taking the limit as $k \to \infty$ we obtain

$$\langle p, \theta \rangle \leq \partial^+ u(x,\theta),$$

which implies $\partial^+ u(x,\theta) \geq \min_{p \in D^*u(x)} \langle p,\theta\rangle \geq \min_{p\in\overline{\mathrm{co}}\, D^*u(x)}\langle p, \theta\rangle$. We have proven that

$$\min_{p \in D^+u(x)} \langle p, \theta\rangle = \min_{p \in \overline{\mathrm{co}}\, D^*u(x)} \langle p, \theta \rangle, \qquad \forall \theta \in \mathbb{R}^n.$$

The last equality means that the closed convex sets $D^+u(x)$ and $\overline{\mathrm{co}}\, D^*u(x)$ have the same support function, that is

$$\sigma_{D^+u(x)}(\theta) = \max_{p\in D^+u(x)} \langle p, \theta\rangle = \max_{p \in \overline{\mathrm{co}}\, D^*u(x)} \langle p, \theta\rangle = \sigma_{\overline{\mathrm{co}}\, D^*u(x)}(\theta), \quad \forall \theta \in \mathbb{R}^n.$$

Let us show that this is equivalent to the equality $D^+u(x) = \overline{\text{co}}\, D^*u(x)$. Suppose by contradiction that there exists $p_0 \in D^+u(x) \setminus \overline{\text{co}}\, D^*u(x)$. By the *separation theorem* for convex sets, there exist a vector $\theta \in \mathbb{R}^n$ and a number $\varepsilon > 0$ such that

$$\langle \theta, p \rangle + \varepsilon \leq \langle \theta, p_0 \rangle, \qquad \forall p \in \overline{\text{co}}\, D^*u(x).$$

Therefore, $\sigma_{\overline{\text{co}}\, D^*u(x)}(\theta) \leq \langle \theta, p_0 \rangle - \varepsilon$, which yields

$$\sigma_{D^+u(x)}(\theta) \leq \langle \theta, p_0 \rangle - \varepsilon$$

This inequality implies $p_0 \notin D^+u(x)$, a contradiction.
(d) Since $D^+u(x) = \overline{\text{co}}\, D^*u(x)$, we only have to prove that every limiting gradient is a boundary point of $D^+u(x)$. Let $p \in D^*u(x)$ and $\{x_k\}$ be a sequence of differentiability points of u such that $Du(x_k)$ converges to p. Without loss of generality we can suppose that $x_k \to x$ as $k \to \infty$ and

$$\theta_k := \frac{x_k - x}{|x_k - x|} \to \theta, \qquad \text{as } k \to \infty,$$

for some unit vector $\theta \in \mathbb{R}^n$. We will show that $p - t\theta \notin D^+u(x)$ for any $t > 0$, which is the proof of the fact that p is a boundary point of $D^+u(x)$. Indeed, by (a),

$$\begin{aligned}
&u(x_k) - u(x) - \langle p - t\theta, x_k - x \rangle \\
&= u(x_k) - u(x) - \langle Du(x_k), x_k - x \rangle \\
&+ \ \ \langle Du(x_k) - p, x_k - x \rangle + t\langle \theta, x_k - x \rangle \\
&\geq -\tfrac{C}{2}|x_k - x|^2 - |Du(x_k) - p||x_k - x| + t\langle \theta, x_k - x \rangle.
\end{aligned}$$

Thus,

$$\liminf_{k\to\infty} \frac{u(x_k) - u(x) - \langle p - t\theta, x_k - x \rangle}{|x_k - x|} \geq t,$$

and then $p - t\theta \notin D^+u(x)$ for any $t > 0$.
(e) Since $D^+u(x) = \overline{\text{co}}\, D^*u(x)$, then $D^+u(x) \neq \emptyset$. Therefore the statement follows from Lemma 1.2.3 (c).
(f) We will prove that whenever $D^+u(x)$ is a singleton the set $D^-u(x)$ is nonempty. Suppose $D^+u(x) = \{p\}$. Take any sequence $\{x_k\} \subset A$ such that $x_k \to x$ as $k \to \infty$.

Since $D^+u(x_k)$ is nonempty, we can take some $p_k \in D^+u(x_k)$ for any k. By assertion (b) we have that $\{p_k\}$ has a unique cluster point, p. Thus, $p_k \to p$ as $k \to \infty$. Furthermore, by assertion (a) we have that

$$\begin{aligned}
&u(x_k) - u(x) - \langle p, x_k - x\rangle \\
&= u(x_k) - u(x) - \langle p_k, x_k - x\rangle + \langle p_k - p, x_k - x\rangle \\
&\geq -\tfrac{C}{2}|x_k - x|^2 - |p_k - p||x_k - x|.
\end{aligned}$$

Hence,

$$\liminf_{k\to\infty} \frac{u(x_k) - u(x) - \langle p, x_k - x\rangle}{|x_k - x|} \geq 0.$$

Since $\{x_k\} \subset A$ is arbitrary we deduce that $p \in D^-u(x)$ and we conclude by using (e). $\square$

Remark 1.2.9. Notice that for a general locally Lipschitz continuous function, the coincidence $D^+u(x) = \overline{\mathrm{co}}\, D^*u(x)$ in Proposition 1.2.8 (d) does not hold, as the map $u : \mathbb{R} \to \mathbb{R}$ defined by $u(x) = |x|$ easily shows. But it can be seen from the above proof that the inclusion $D^+u(x) \subseteq \overline{\mathrm{co}}\, D^*u(x)$ is still true.

Proposition 1.2.10. *Let K be a closed set and d_K be the distance function from K. Then*

$$D^*d_K(x) = \left\{ \frac{x-y}{|x-y|} \ : \ y \in \Pi_K(x) \right\}, \tag{1.2.8}$$

where $\Pi_K(x)$ is the set of projections of x onto K.

Proof—Let us call $D^\diamond d_K(x)$ the right hand side of (1.2.8). If $p \in D^*d_K(x)$, then there exists a sequence $\{x_k\}$ of differentiability points of u such that $x_k \to x$ and $Dd_K(x_k) \to p$. By Proposition 1.1.2, we can write $Dd_K(x_k) = \frac{x_k - y_k}{|x_k - y_k|}$, where $y_k \in K$ is the unique projection of x_k onto K. Since $y_k = x_k - d_K(x_k)Dd_K(x_k)$, then y_k converges to some element $y \in K$ as $k \to \infty$. By the semicontinuity of the projection, we have that $y \in \Pi_K(x)$. Moreover, $\frac{x_k - y_k}{|x_k - y_k|} \to \frac{x-y}{|x-y|}$ as $k \to \infty$, so that $p = \frac{x-y}{|x-y|} \in D^\diamond d_K(x)$. On the other hand, if $p \in D^\diamond d_K(x)$, then $p = \frac{x-y}{|x-y|}$ for some $y \in \Pi_K(x)$. Furthermore, any $z \in]x, y[$ is a differentiability point of d_K and $\Pi_K(z) = y$ by Lemma 1.1.1 and

Proposition 1.1.2. Thus, $Dd_K(z) = \frac{z-y}{|z-y|} = \frac{x-y}{|x-y|} = p$ and we deduce $p \in D^+d_K(x)$. $\square$

Collecting together Lemma 1.2.7 and Propositions 1.2.8–1.2.10 we readily obtain the next result.

Corollary 1.2.11. *Let K be a closed set and d_K be the distance function from K. Then*

$$D^+d_K(x) = \overline{\mathrm{co}} \left\{ \frac{x-y}{|x-y|} \ : \ y \in \Pi_K(x) \right\}, \tag{1.2.9}$$

and

$$D^*d_K(x) = \begin{cases} \frac{x-y}{|x-y|} & \text{if } \Pi_K(x) = \{y\}, \\ \emptyset & \text{if } \Pi_K(x) \text{ is not a singleton.} \end{cases} \tag{1.2.10}$$

We are now in position to prove some results on the propagation of singularities of a semiconcave function $u : A \to \mathbb{R}$. As in the case of the distance function, a singular point is a point where u is not differentiable. We will denote by Σ_u the set of these points.

The following proposition, that will be crucial to our analysis, is taken from [1].

Theorem 1.2.12. *Let $u : A \to \mathbb{R}$ be a locally semiconcave function and $x_0 \in A$ be a singular point. Suppose that*

$$\partial D^+u(x_0) \setminus D^*u(x_0) \neq \emptyset. \tag{1.2.11}$$

Then there exist a Lipschitz singular arc $\zeta : [0, \rho] \to \mathbb{R}^n$ for u, with $\zeta(0) = x_0$, and a positive number δ such that

$$\lim_{s \to 0^+} \frac{\zeta(s) - x_0}{s} \neq 0 \tag{1.2.12}$$

$$\mathrm{diam}\big(D^+u(\zeta(s))\big) \geq \delta, \qquad \forall s \in [0, \rho]. \tag{1.2.13}$$

Moreover, $\zeta(s) \neq x_0$ for any $s \in]0, \rho]$.

Remark 1.2.13. Note that condition (1.2.11) is equivalent to the existence of two vectors, $p_0 \in \mathbb{R}^n$ and $q \in \mathbb{R}^n \setminus \{0\}$, such that

$$p_0 \in D^+u(x_0) \setminus D^*u(x_0) \tag{1.2.14}$$

$$\langle q, p - p_0 \rangle \geq 0 \qquad \forall p \in D^+u(x_0). \tag{1.2.15}$$

Proof—To begin with, let us choose $R > 0$ such that $\overline{B}_R(x_0) \subset A$. Let C be the semiconcavity constant and L the Lipschitz constant of u in this ball. Let p_0 and q be as in Remark 1.2.13 and for any $0 < s \leq \sigma := \min\left\{\frac{R}{4|q|}, \frac{1}{4C}\right\}$ define

$$\phi_s(x) = u(x) - u(x_0) - \langle p_0 - q, x - x_0\rangle - \frac{1}{2s}|x - x_0|^2, \qquad x \in \overline{B}_R(x_0).$$

Since u is semiconcave in $\overline{B}_R(x_0)$ with constant C and $s < 1/4C$, then ϕ_s is strictly concave. Hence, ϕ_s admits a unique maximum point in $\overline{B}_R(x_0)$, that we call x_s. Next define

$$\zeta(s) := \begin{cases} x_0 & \text{if } s = 0 \\ x_s & \text{if } s \in]0, \sigma]. \end{cases}$$

Let us show that ζ is Lipschitz continuous. Since $p_0 \in D^+u(x_0)$, then, by Proposition 1.2.8 (a) we have

$$\phi_s(x) \leq \langle q, x - x_0\rangle + \left(\frac{C}{2} - \frac{1}{2s}\right)|x - x_0|^2, \qquad x \in \overline{B}_R(x_0).$$

Moreover, there are points where ϕ_s is positive in $\overline{B}_R(x_0)$ because condition (1.2.15) implies that $p_0 - q \notin D^+u(x_0)$. Hence $\phi(\zeta(s)) > 0$ and we deduce

$$0 < \langle q, \frac{\zeta(s) - x_0}{|\zeta(s) - x_0|}\rangle + \left(\frac{C}{2} - \frac{1}{2s}\right)|\zeta(s) - x_0|,$$

which in turn gives

$$|\zeta(s) - x_0| < \frac{2|q|}{1 - Cs}s < 4|q|s, \qquad \forall s \in]0, \sigma]. \tag{1.2.16}$$

Our choice of σ and the above inequality imply that $\zeta(s) \in B_R(x_0)$ for any $s \in [0, \sigma]$. Hence, $\zeta(s)$ is also a local maximum point of ϕ_s and then

$$0 \in D^+\phi_s(\zeta(s)), \qquad \forall s \in]0, \sigma],$$

as it is easy to deduce from the definition of superdifferential. Moreover,

$$D^+\phi_s(\zeta(s)) = D^+u(\zeta(s)) - p_0 + q - \frac{\zeta(s) - x_0}{s}, \qquad \forall s \in]0, \sigma],$$

because $x \mapsto \langle p_0 - q, x - x_0 \rangle - \frac{1}{2s}|x - x_0|^2$ is a regular function in $B_R(x_0)$. If we set $p(s) := p_0 - q + \frac{\zeta(s)-x_0}{s}$, we have just proven that $p(s) \in D^+u(\zeta(s))$ for any $s \in]0, \sigma]$. Let us show that $\lim_{s\to 0^+} p(s) = p_0$. This will also imply

$$\lim_{s\to 0^+} \frac{\zeta(s) - x_0}{s} = q \tag{1.2.17}$$

and in particular (1.2.12). Applying twice Proposition 1.2.8 (a) to $p_0 \in D^+u(x_0)$ and $p(s) \in D^+u(\zeta(s))$ we obtain

$$\langle p(s) - p_0, \zeta(s) - x_0 \rangle \leq C|\zeta(s) - x_0|^2. \tag{1.2.18}$$

But by definition

$$\langle p(s) - p_0, \zeta(s) - x_0 \rangle = s\left(|p(s) - p_0|^2 + \langle p(s) - p_0, q \rangle\right). \tag{1.2.19}$$

So, take any sequence $s_k \downarrow 0$ such that $p(s_k)$ converges to some $\bar{p}$. Thus $\bar{p} \in D^+u(x_0)$ because of Proposition 1.2.8 (b) and then $\langle \bar{p} - p_0, q \rangle \geq 0$. Collecting together the last inequality and equations (1.2.18)–(1.2.19), and taking into account (1.2.16) we obtain

$$\begin{aligned}
&\lim_{k\to\infty} |p(s_k) - p_0|^2 \leq \lim_{k\to\infty} \left(|p(s_k) - p_0|^2 + \langle p(s_k) - p_0, q \rangle\right) \\
&= \lim_{k\to\infty} \langle p(s_k) - p_0, \tfrac{\zeta(s_k)-x_0}{s_k} \rangle = 0,
\end{aligned}$$

that is $\bar{p} = p_0$, as desired. Let us conclude the proof of the Lipschitz continuity of ζ. Let $s,\ r \in [0, \sigma]$. We can write $\zeta(s) - \zeta(r) = s[p(s) - p(r)] + (s - r)[p(r) - p_0 + q]$. Taking the scalar product of both sides of the above equality with $\zeta(s) - \zeta(r)$ and applying twice Proposition 1.2.8 (a) to $p(s) \in D^+u(\zeta(s))$ and $p(r) \in D^+u(\zeta(r))$, we have

$$|\zeta(s) - \zeta(r)|^2 \leq Cs|\zeta(s) - \zeta(r)|^2 + |s - r||p(r) - p_0 + q||\zeta(s) - \zeta(r)|.$$

Therefore,

$$(1 - Cs)|\zeta(s) - \zeta(r)| \leq |s - r||p(r) - p_0 + q| \leq (2L + |q|)|s - r|,$$

because L provides a bound for D^+u in $B_R(x_0)$. It remains to show (1.2.13) on a suitable sub–interval $[0,\rho]$ of $[0,\sigma]$. Suppose by contradiction that a sequence $s_k \downarrow 0$ exists such that $\operatorname{diam}(D^+u(\zeta(s_k))) \to 0$ as $k \to \infty$. We will show that $p_0 = \lim_{k\to\infty} p(s_k)$ belongs to $D^*u(x_0)$, against (1.2.14). Indeed for all $k \in \mathbb{N}$ choose any $p_k^* \in D^*u(\zeta(s_k))$. Then we can find a point $x_k^* \in A$ where u is differentiable such that

$$|\zeta(s_k) - x_k^*| + |Du(x_k^*) - p_k^*| \le \frac{1}{k}.$$

Then $x_k^* \to x_0$ and

$$\begin{aligned} |Du(x_k^*) - p_0| &\le |Du(x_k^*) - p_k^*| + |p_k^* - p(s_k)| + |p(s_k) - p_0| \\ &\le \tfrac{1}{k} + \operatorname{diam}(D^+u(\zeta(s_k))) + |p(s_k) - p_0| \to 0, \end{aligned}$$

as $k \to \infty$. Hence, $p_0 \in D^*u(x_0)$. □

Remark 1.2.14. In the proof of Theorem 1.2.12 we have also shown how to determine the direction of the singular arc starting from x_0. Indeed, we have proven that if $p_0 \in \mathbb{R}^n$ and $q \in \mathbb{R}^n \setminus \{0\}$ satisfy

$$p_0 \in D^+u(x_0) \setminus D^*u(x_0) \tag{1.2.20}$$

$$\langle q, p - p_0\rangle \ge 0 \qquad \forall p \in D^+u(x_0), \tag{1.2.21}$$

then

$$\lim_{s\to 0^+} \frac{\zeta(s) - x_0}{s} = q. \tag{1.2.22}$$

Remark 1.2.15. In the case of the distance function d from $\mathbb{R}^n \setminus \Omega$, with Ω bounded domain, condition (1.2.11) or the equivalent set of conditions (1.2.14)–(1.2.15) are automatically fulfilled whenever Ω is different from a disk and x_0 is a singular point. Indeed, elementary geometric arguments show that condition (1.2.11) holds true if and only if $D^+d(x_0)$ fails to cover the closed unit ball $\overline{B}_1$. On the other hand, in view of Proposition 1.2.8, we have that $D^+d(x_0) = \overline{B}_1$ if and only if $\partial D^+d(x_0) = D^*d(x_0)$. By Proposition 1.2.10, the last identity is necessary and sufficient for Σ to be a singleton or, equivalently, for Ω to coincide with $B_R(x_0)$, where $R = d(x_0)$. In fact,

the equivalence between Σ being a singleton and the identity $\Omega = B_R(x_0)$ follows from a classical result of Motzkin's [34].
Furthermore, $D^+d(x_0)$ fails to cover the closed unit ball $\overline{B}_1$ if and only if $D^+d(x_0)$ possesses a 1-dimensional exposed face, where an exposed face is a subset E such that for some $q \in \mathbb{R}^n$

$$E = \arg \max_{p \in D^+d(x_0)} \langle p, q \rangle .$$

Anytime Ω is different from a disk and E is an exposed face of $D^+d(x_0)$ of dimension greater than or equal to 1, Theorem 1.2.12 guarantees that singularities propagate along the direction $-q$ defining the set E.
Next proposition is a further refining of this theorem.

In order to prove Proposition 1.2.16, we will need a more detailed description of the singular set Σ. Let us recall that $\Sigma = \Sigma^1 \cup \Sigma^2$, where Σ^i $(i = 1, 2)$ is the set of points $x \in \Sigma$ with *magnitude* i, that is, such that the dimension of (the convex set) $D^+d(x)$ is equal to i. A well–known result of [2] (see also [15]) tells that Σ^i $(i = 1, 2)$ is $\mathcal{H}^{2-i}$–rectifiable, which means that Σ^i can be covered with a countable sequence of $\mathcal{C}^1$ hypersurfaces of dimension $2 - i$. In particular, the Hausdorff dimension of Σ^i does not exceed $(2 - i)$.

Proposition 1.2.16. *Let $x_0 \in \Sigma$, and let p_0, q_0 be two distinct limiting gradients at x_0 such that the segment $[p_0, q_0]$ is an exposed face of $D^+d(x_0)$. Let n_0 be a nonzero vector satisfying*

$$\langle p, n_0 \rangle \ \le \ \langle p_0, n_0 \rangle \ = \ \langle q_0, n_0 \rangle \qquad \forall p \in D^+d(x_0) \ .$$

Then, there exist a number $\eta > 0$ and a Lipschitz arc $\zeta : [0, \eta] \to \Omega$ such that

$$\zeta(0) = x_0 \,, \qquad \dot{\zeta}(0) = -n_0 \,, \qquad \zeta(s) \in \Sigma \quad \forall s \in [0, \eta] \,. \tag{1.2.23}$$

Moreover, $\zeta(s_k) \in \Sigma^1$ for some sequence $s_k \downarrow 0$, and

$$D^+d(\zeta(s_k)) = [p_k, q_k] \qquad \forall k \ge 0 \tag{1.2.24}$$

where $p_k \to p_0$ and $q_k \to q_0$ as $k \to \infty$.

Proof—The existence of a singular arc ζ satisfying (1.2.23) follows from Theorem 1.2.12 and Remark 1.2.15. Note that the direction of the singular arc is $-n_0$. Moreover, in Theorem 1.2.12 a bound of the form $\operatorname{diam}(D^+d(\zeta(s))) \geq \delta$ is also deduced for some $\delta > 0$ and every $s \in [0, \eta]$. To prove the last part of the conclusion, we note that, for any $\varepsilon > 0$, $\mathcal{H}^1(\zeta([0,\varepsilon])) > 0$ because $\dot{\zeta}(0) \neq 0$ and ζ is Lipschitz continuous. Since Σ^2 is at most countable, we conclude that $\mathcal{H}^1(\zeta([0,\varepsilon]) \cap \Sigma^1) > 0$ for any $\varepsilon > 0$. Consequently, there exists a sequence $s_k \downarrow 0$ such that $\zeta(s_k) \in \Sigma^1$ for every $k \in \mathbb{N}$. Let us set $D^+d(\zeta(s_k)) = [p_k, q_k]$, choosing p_k so that $\langle p_k, p_0\rangle \geq \langle q_k, p_0\rangle$. Also notice that $|p_k - q_k| \geq \delta$. Now, let us consider converging subsequences of $\{p_k\}$ and $\{q_k\}$ (labelled like the original sequences) and denote by p^* and q^*, respectively, their limits. We claim that

$$p^*, q^* \in \arg\max_{p \in D^+d(x_0)} \langle p, n_0\rangle = [p_0, q_0]\,. \tag{1.2.25}$$

Indeed, by Proposition 1.2.8 we have

$$d(x_0) - d(\zeta(s_k)) - \langle p_k, x_0 - \zeta(s_k)\rangle \leq \frac{C}{2}|x_0 - \zeta(s_k)|^2,$$

where C is the semiconcavity constant of d in a suitable neighborhood of x_0 as in Theorem 1.2.12. Therefore,

$$\begin{aligned}\langle p_k, \frac{\zeta(s_k) - x_0}{s_k}\rangle &\leq \frac{C}{2}\frac{|x_0 - \zeta(s_k)|^2}{s_k} + \frac{d(\zeta(s_k)) - d(x_0)}{s_k}\\ = \frac{C}{2}\frac{|x_0 - \zeta(s_k)|^2}{s_k} &+ \frac{d(\zeta(s_k)) - d(x_0 - s_k n_0)}{s_k} + \frac{d(x_0 - s_k n_0) - d(x_0)}{s_k}.\end{aligned}$$

Since

$$\lim_{k\to\infty} \frac{\zeta(s_k) - x_0}{s_k} = -n_0,$$

$$\lim_{k\to\infty} \frac{d(x_0 - s_k n_0) - d(x_0)}{s_k} \leq \partial^+ d(x_0, -n_0)$$

and

$$\frac{|d(\zeta(s_k)) - d(x_0 - s_k n_0)|}{s_k} \leq \frac{|\zeta(s_k) - (x_0 - s_k n_0)|}{s_k} = \left|\frac{\zeta(s_k) - x_0}{s_k} + n_0\right| \to 0,$$

as $k \to \infty$, we have

$$\langle p^*, -n_0\rangle \leq \partial^+ d(x_0, -n_0).$$

On the other hand, by Proposition 1.2.3,

$$\partial^+ d(x_0, -n_0) \leq \langle p, -n_0 \rangle, \ \forall p \in D^+ d(x_0),$$

so that we can conclude that $\langle p^*, -n_0 \rangle \leq \langle p, -n_0 \rangle$ for all $p \in D^+ d(x_0)$, i.e. $p^* \in \arg\max_{p \in D^+ d(x_0)} \langle p, n_0 \rangle$. The same reasoning applies to q^*. Hence, (1.2.25) is proven. Since p^* and q^* belong to $D^* d(x_0)$, we can also say that $p^*, q^* \in \{p_0, q_0\}$. Moreover, $|p^* - q^*| \geq \delta$ and $\langle p^*, p_0 \rangle \geq \langle q^*, p_0 \rangle$. This forces $(p, q) = (p_0, q_0)$. □

We now conclude this section with some results on the so–called *proximal subdifferential*, which is a crucial tool of nonsmooth analysis firstly introduced by Clarke in the seventies (see [18] for details and further references). Here we are not interested in the connections between the proximal subdifferential and the subdifferential of Definition 1.2.1, but we only present the results that we will need in Chapters 3 and 4 to show some regularity properties of the maximal retraction length (2.4.2). The results below are a simplification of the corresponding ones from [18], since here we deal with continuous functions on subsets of a finite dimensional space. In what follows we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and $f : \overline{\Omega} \to \mathbb{R}$ is a *continuous* function.

Definition 1.2.17. A vector $p \in \mathbb{R}^n$ is a *proximal subgradient* of f at $x \in \Omega$ if there exist positive numbers σ and η such that

$$f(y) \geq f(x) + \langle p, y - x \rangle - \sigma |y - x|^2, \qquad \forall y \in B(x, \eta). \tag{1.2.26}$$

The set of proximal subgradients of f at x is called *proximal subdifferential* of f at x and is denoted by $\partial_P f(x)$.

Lemma 1.2.18. *Assume that f is Gâteaux differentiable at $x \in \Omega$. Then*

$$\partial_P f(x) \subseteq \{\nabla f(x)\}.$$

Proof—Suppose that f has Gâteaux derivative at x and that $p \in \partial_P f(x)$. For any $v \in \mathbb{R}^n$, the proximal subgradient inequality (1.2.26) implies that there exists a positive σ such that

$$\frac{f(x+tv)-f(x)}{t} - \langle p, v\rangle \geq -\sigma t|v|^2$$

for all sufficiently small positive t. Letting $t \downarrow 0$ we obtain

$$\langle \nabla f(x) - p, v\rangle \geq 0.$$

The conclusion follows from the arbitrariness of $v \in \mathbb{R}^n$. $\square$

Theorem 1.2.19. *Let $x_0 \in \Omega$ and $\varepsilon > 0$ be given. Then there exists a point $y \in B(x_0, \varepsilon)$ satisfying $\partial_P f(y) \neq \emptyset$. In other words, the set* $\mathrm{dom}(\partial_P f)$ *of points in Ω at which at least one proximal subgradient exists is dense in Ω.*

Proof—For $\varepsilon > 0$ fixed, let $\delta > 0$ be such that

$$|x - x_0| \leq \delta, \qquad \Rightarrow \qquad f(x_0) - \varepsilon \leq f(x).$$

Also define

$$g(x) = \begin{cases} [\delta - |x-x_0|^2]^{-1} & \text{if } |x - x_0| < \delta, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $g \in \mathcal{C}^2(B(x_0,\delta))$ and $g(x) \to \infty$ as x approaches the boundary of $B(x_0,\delta)$. Moreover, the function $f+g$ is bounded below on $\overline{B}(x_0,\delta)$, lower semicontinuous and not identically $+\infty$ there. It follows that $f+g$ attains a minimum y over $\overline{B}(x_0,\delta)$. Obviously $y \in B(x_0,\delta)$ and then $0 \in \partial_P(f+g)(y)$, as it is easy to verify. Moreover, since $g \in \mathcal{C}^2(B(x_0,\delta))$ we have that there exists a positive σ' such that

$$-g(x) + g(y) + \sigma'|x-y|^2 \geq \langle -g'(y), x - y\rangle$$

for any x close to y. Since $0 \in \partial_P(f+g)(y)$ we also have that

$$f(x) + g(x) \geq f(y) + g(y) - \sigma|y-x|^2$$

for some $\sigma > 0$ and all x near y. Adding up these inequalities we arrive at

$$f(x) \geq f(y) + \langle -g'(y), x-y\rangle - (\sigma + \sigma')|y-x|^2,$$

which holds for all x near y and says that $-g'(y) \in \partial_P f(y)$. In particular, $\partial_P f(y) \neq \emptyset$. Since δ can be made arbitrary small, it follows that $\mathrm{dom}(\partial_P f)$ is dense in Ω. $\square$

Theorem 1.2.20. *Suppose that*

$$|p| \leq L, \qquad \forall p \in \partial_P f(x), \quad \forall x \in \Omega. \tag{1.2.27}$$

Then f is locally Lipschitz continuous in Ω with Lipschitz constant L.

Proof—Fix any $x_0 \in \Omega$ and let $\varepsilon > 0$ such that $\overline{B}(x_0, 4\varepsilon) \subseteq \Omega$. Also take any $L' > L$. Denote by ϕ a function mapping the interval $[0, 3\varepsilon)$ to $[0, \infty)$ and having the following properties:
$\phi(\cdot)$ is $\mathcal{C}^2$ and strictly increasing on $(0, 3\varepsilon)$, $\phi(t) = L't$ for $t \in [0, 2\varepsilon]$, $\phi'(t) \geq L'$ for $t \in [2\varepsilon, 3\varepsilon)$, $\phi(t) \to \infty$ as $t \uparrow 3\varepsilon$.
Now consider any two points y, $z \in B(x_0, \varepsilon)$. For any $\beta \in \mathbb{R}^n$ with $|\beta| < L' - L$, the function

$$g(x) := f(y + x) + \phi(|x|) - \langle \beta, x \rangle$$

attains a minimum over $\overline{B}(0, 3\varepsilon)$ because it is bounded below, lower semicontinuous and not identically $+\infty$ there. Call u any minimizer. Clearly $|u| < 3\varepsilon$ and $0 \in \partial_P g(u)$. If $u \neq 0$, since $\phi(\cdot)$ is of class $\mathcal{C}^2$ in a neighborhood of u, we can argue as in the previous theorem to obtain that

$$\beta - \phi'(|u|)\frac{u}{|u|} \in \partial_P f(y + u).$$

But

$$\begin{aligned} \left|\beta - \phi'(|u|)\tfrac{u}{|u|}\right| &\geq \phi'(|u|) - |\beta| \\ &> L' - (L' - L) = L. \end{aligned}$$

and $y + u$ is a point in $B(x_0, 4\varepsilon) \subset \Omega$. This contradicts the given bound on $\partial_P f$ and implies that $u = 0$. But then, taking into account that $|z - y| < 2\varepsilon$ and $\phi(t) = L't$ for $t \in [0, 2\varepsilon]$, we get

$$\begin{aligned} f(y) = g(0) &\leq g(z - y) \\ &= f(z) + \phi(|z - y|) - \langle \beta, z - y \rangle \\ &\leq f(z) + L'|z - y| + (L' - L)|z - y|. \end{aligned}$$

Since y, z are arbitrary points and $L' > L$ is arbitrary too, we have shown that f is Lipschitz continuous on $B(x_0, \varepsilon)$ with Lipschitz constant L and so that it is locally Lipschitz continuous with the same constant on the whole set Ω. □

Remark 1.2.21. In the case when the proximal subdifferential is only locally bounded in Ω the same argument of Theorem 1.2.20 shows that f is still locally Lipschitz continuous, but with a Lipschitz constant that depends upon the local bound of $\partial_P f$.

1.3 Viscosity Solutions of the Eikonal Equation

In this section we will show that the distance function $d = d_{\mathbb{R}^n \setminus \Omega}$, where Ω a bounded domain of $\mathbb{R}^n$, can be also seen as the unique viscosity solution of a suitable Dirichlet problem. Viscosity solutions are a special kind of weak solutions introduced by M. G. Crandall and P. L. Lions in the early 80's. Here we are not interested in the full generality, for which we refer to [32], but we will restrict our attention to their connection with the eikonal equation

$$|Du| = 1, \qquad \text{in } \Omega.$$

Definition 1.3.1. We say that $u \in \mathcal{C}(\Omega)$ is a viscosity sub–solution of the eikonal equation $|Du| = 1$ in an open set $\Omega \subset \mathbb{R}^n$ if, for any $\phi \in \mathcal{C}^1(\Omega)$,

$$|D\phi(x_0)| \leq 1$$

at any local maximum point $x_0 \in \Omega$ of $u - \phi$. Analogously, $u \in \mathcal{C}(\Omega)$ is a viscosity super–solution of the eikonal equation $|Du| = 1$ in Ω if, for any $\phi \in \mathcal{C}^1(\Omega)$,

$$|D\phi(x_1)| \geq 1$$

at any local minimum point $x_1 \in \Omega$ of $u - \phi$. Finally, $u \in \mathcal{C}(\Omega)$ is a viscosity solution of the eikonal equation $|Du| = 1$ in Ω if it is at the same time a viscosity sub– and super–solution.

Lemma 1.3.2. *Let A be an open subset of $\mathbb{R}^n$, $u : A \to \mathbb{R}$ be a continuous function and $x \in A$ be fixed. Then,*

(a) *$p \in D^+u(x)$ if and only if there exists a function $\phi \in \mathcal{C}^1(A)$ such that $p = D\phi(x)$ and $u - \phi$ has a local maximum at x;*

(b) *$p \in D^-u(x)$ if and only if there exists a function $\phi \in \mathcal{C}^1(A)$ such that $p = D\phi(x)$ and $u - \phi$ has a local minimum at x.*

Proof—(a) Let $p \in D^+u(x)$. Then, for some $\delta > 0$,

$$u(y) \leq u(x) - \langle p, y - x\rangle + \sigma(|y - x|)|y - x|, \qquad \forall y \in B_\delta(x),$$

where σ is a continuous increasing function on $[0, +\infty)$ such that $\sigma(0) = 0$. Now define a $\mathcal{C}^1$ function ρ by

$$\rho(s) = \int_0^s \sigma(t)\ dt.$$

Then,

$$\rho(0) = \rho'(0) = 0, \qquad \rho(2s) \geq \sigma(s)s.$$

Therefore, the function

$$\phi(y) := u(x) + \langle p, y - x\rangle + \rho(2|y - x|)$$

verifies $\phi \in \mathcal{C}^1$, $D\phi(x) = p$ and

$$(u - \phi)(y) \leq \sigma(|y - x|)|y - x| - \rho(2|y - x|) \leq 0 = (u - \phi)(x).$$

On the other hand, if there exists a function $\phi \in \mathcal{C}^1(A)$ such that $p = D\phi(x)$ and $u - \phi$ has a local maximum at x, then for any y in a neighborhood of x we have

$$u(y) - u(x) - \langle D\phi(x), y - x\rangle \leq \phi(y) - \phi(x) - \langle D\phi(x), y - x\rangle,$$

which readily implies $p \in D^+u(x)$.
(b) Since $D^-u(x) = -(D^+(-u)(x))$, the statement follows from the above argument when applied to $-u$. $\square$

The previous lemma permits us to rewrite Definition 1.3.1 in the following form.

Definition 1.3.3. We say that u is a viscosity solution of the eikonal equation

$$|Du| = 1$$

in an open set $\Omega \subset \mathbb{R}^n$ if, for any $x \in \Omega$ we have

$$p \in D^-u(x) \quad \Rightarrow \quad |p| \geq 1$$
$$p \in D^+u(x) \quad \Rightarrow \quad |p| \leq 1$$

Next result is a straightforward consequence of the previous definition and Corollary 1.2.11.

Corollary 1.3.4. *The distance function $d = d_{\mathbb{R}^n \setminus \Omega}$ is a viscosity solution of the eikonal equation $|Du| = 1$ in Ω.*

Let us now prove a Comparison Principle for the eikonal equation. Such a result will guarantee that the distance function d is the unique solution of the Dirichlet problem

$$\begin{cases} |Du| = 1 & \text{in } \ \Omega, \\ u \equiv 0 & \text{on } \ \partial\Omega. \end{cases} \tag{1.3.1}$$

Theorem 1.3.5. *Suppose that $u_1,\ u_2 \in \mathcal{C}(\overline{\Omega})$ are respectively a viscosity sub–solution and super–solution of the eikonal equation $|Du| = 1$ in Ω. If $u_1 \leq u_2$ on $\partial\Omega$, then also $u_1 \leq u_2$ in Ω.*

Proof—For any $t \in]0,1[$ set

$$u^t(x) := tu_1(x) + (1-t) \min_{y \in \overline{\Omega}} u_1(y).$$

It is easy to verify that $u^t \in \mathcal{C}(\overline{\Omega})$, $u^t \leq u_1$ in $\overline{\Omega}$, u^t uniformly converges to u_1 as $t \to 1$ and u^t/t is a viscosity sub–solution of the eikonal equation in Ω. We will show that

$$u^t \leq u_2, \qquad \text{in } \ \Omega, \ \text{for any } \ t \in]0,1[. \tag{1.3.2}$$

In view of the uniform convergence of u^t to u_1, equation (1.3.2) will then imply the statement. Suppose by contradiction that there exists some $\bar{t} \in]0,1[$ such that

$$\sup_{x \in \Omega} (u^{\bar{t}} - u_2)(x) = \delta > 0.$$

Since by assumption $u_1 \leq u_2$ on $\partial\Omega$, the set

$$\Omega' := \Big\{x \in \Omega \,:\, (u^{\bar{t}} - u_2)(x) > \delta/2\Big\}$$

satisfies $\overline{\Omega'} \subset \Omega$. Now, for any $\varepsilon > 0$ consider the function

$$\Phi_\varepsilon(x, y) := u^{\bar{t}}(x) - u_2(y) - \frac{|x-y|^2}{2\varepsilon},$$

and let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega'} \times \overline{\Omega'}$ be a maximum point for Φ_ε. Then

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geq \Phi_\varepsilon(x_\varepsilon, x_\varepsilon) = u^{\bar{t}}(x_\varepsilon) - u_2(x_\varepsilon),$$

which gives

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \leq u_2(x_\varepsilon) - u_2(y_\varepsilon) \leq 2 \max_{\overline{\Omega}} u_2 =: C.$$

Therefore the previous inequality and the continuity of u_2 in $\overline{\Omega}$ imply

$$|x_\varepsilon - y_\varepsilon| \to 0 \qquad \text{and} \qquad \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \to 0 \qquad \text{as } \ \varepsilon \to 0. \tag{1.3.3}$$

Furthermore,

$$\Phi_\varepsilon(x_\varepsilon, y_\varepsilon) \geq \sup_{x \in \overline{\Omega'}} \Phi_\varepsilon(x, x) = \sup_{x \in \overline{\Omega'}} (u^{\bar{t}} - u_2)(x) = \delta. \tag{1.3.4}$$

From (1.3.3) and (1.3.4) we deduce that for ε sufficiently small $(x_\varepsilon, y_\varepsilon) \in \Omega' \times \Omega'$. But then x_ε is a local maximum point for

$$x \mapsto \frac{1}{\bar{t}} u^{\bar{t}}(x) - \frac{1}{\bar{t}} \left(u_2(y_\varepsilon) + \frac{|x - y_\varepsilon|^2}{2\varepsilon} \right).$$

Since $u^{\bar{t}}/\bar{t}$ is a viscosity sub–solution of the eikonal equation and

$$x \mapsto \frac{1}{\bar{t}} \big(u_2(y_\varepsilon) + |x - y_\varepsilon|^2/2\varepsilon \big)$$

is a $\mathcal{C}^1$ function, we obtain

$$\frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} \leq \bar{t}. \tag{1.3.5}$$

On the other hand, since y_ε is a local minimum point for

$$y \mapsto u_2(y) - \Big(u^{\bar{t}}(x_\varepsilon) - |x_\varepsilon - y|^2/2\varepsilon \Big)$$

and u_2 is a viscosity super–solution of the eikonal equation, then we also get

$$\frac{|x_\varepsilon - y_\varepsilon|}{\varepsilon} \geq 1. \tag{1.3.6}$$

Since $\bar{t} < 1$ we arrive to a contradiction. $\square$

Remark 1.3.6. The uniqueness of d as a viscosity solution of the Dirichlet problem (1.3.1) is direct consequence of the above comparison theorem. But such a result also permits us to consider the distance function as the largest element of the set

$$\mathbb{K} := \left\{ u \in \mathcal{W}^{1,\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega}) \,:\, \|Du\|_\infty \leq 1 \text{ and } u = 0 \text{ on } \partial\Omega \right\}.$$

Indeed, a simple computation shows that d belongs to $\mathbb{K}$. On the other hand, we will show that any other element u of $\mathbb{K}$ is a viscosity sub–solution of the eikonal equation. This fact and the above comparison theorem will imply that $u \leq d$ in Ω, since $u \equiv 0 = d$ on $\partial\Omega$ and d is a viscosity super–solution of the eikonal equation. So take any $u \in \mathbb{K}$. Then u is locally Lipschitz continuous in Ω and, by Remark 1.2.9, we have that $D^+u(x) \subseteq \overline{\text{co}}\, D^*u(x)$ for any $x \in \Omega$. Therefore, any $p \in D^+u(x)$ can be written as a convex sum of elements in $D^*u(x)$, say

$$p = \sum_m \lambda_m p_m, \quad 0 \leq \lambda_m \leq 1, \sum_m \lambda_m = 1, \quad p_m \in D^*u(x).$$

But then, for any m we have $p_m = \lim_{k\to\infty} Du(x_{m_k})$ for some sequence $\{x_{m_k}\}$ of differentiability points of u converging to x. Since $u \in \mathbb{K}$ we have $|Du(x_{m_k})| \leq 1$ for any k. Passing to the limit as $k \to \infty$ we obtain $|p_m| \leq 1$ for any m and then $|p| \leq 1$.

As a Corollary of the Comparison Principle 1.3.5 we can also obtain a representation formula for the Lipschitz continuous viscosity solutions of the eikonal equation which among all shows that these solutions are semiconcave functions.

Proposition 1.3.7. *Let u be a Lipschitz continuous viscosity solution of the eikonal equation $|Du| = 1$ in Ω. Then, the following statements hold true.*

(a) *For every $x \in \Omega$*

$$u(x) = \min_{y\in\partial\Omega} \left\{ u(y) + |y - x| \right\} = \min_{y\in\partial\Omega,\,]x,y[\subset\Omega} \left\{ u(y) + |y - x| \right\}. \tag{1.3.7}$$

(b) *u is locally semiconcave in Ω.*

Proof—(a) Since u is Lipschitz continuous on a bounded domain, it can be uniquely continuously prolonged up to the boundary. Let us call $\tilde{u}$ the function

$$\tilde{u}(x) := \min_{y \in \partial\Omega} \left\{ u(y) + |y - x| \right\}.$$

Since $u \leq \tilde{u}$ in $\overline{\Omega}$ and $u \equiv \tilde{u}$ on $\partial\Omega$, it suffices to show that $\tilde{u}$ is a viscosity sub–solution of the eikonal equation and apply Theorem 1.3.5. Arguing as in Remark 1.3.6 it is readily seen that being a viscosity sub–solution of the eikonal equation is equivalent to be an almost everywhere sub–solution of the equation. Hence, since $\tilde{u}$ is itself a Lipschitz continuous function with Lipschitz constant 1, the proof of (a) is complete.
(b) Let Ω' be any open convex subset compactly contained in Ω. By Proposition 1.2.5 (c) it suffices to prove that there exists some $C = C(\Omega') > 0$ such that for any $x, z \in \Omega'$

$$u(x) + u(z) - 2u(\frac{x+z}{2}) \leq C|z - x|^2.$$

By the representation formula (1.3.7) there exists some $y \in \partial\Omega$ such that

$$u(\frac{x+z}{2}) = u(y) + |y - \frac{x+z}{2}|.$$

Hence,

$$\begin{aligned}
&u(x) + u(z) - 2u(\tfrac{x+z}{2}) \\
&\leq (u(y) + |y - x|) + (u(y) + |y - z|) - 2u(y) - 2|y - \tfrac{x+z}{2}| \\
&= |y - x| + |y - z| - 2|y - \tfrac{x+z}{2}|
\end{aligned}$$

But for any $w, h \in \mathbb{R}^n$, $w \neq 0$

$$(|w + h| + |w - h|)^2 \leq 2(|w + h|^2 + |w - h|^2) = 4(|w|^2 + |h|^2) \leq \left(2|w| + \frac{|h|^2}{|w|}\right)^2.$$

Therefore, $|w + h| + |w - h| - 2|w| \leq \frac{|h|^2}{|w|}$ and with the choice $w = y - \frac{x+z}{2}$, $h = \frac{z-x}{2}$ we have

$$|y - x| + |y - z| - 2|y - \frac{x+z}{2}| \leq \frac{|z - x|^2}{4|y - \frac{x+z}{2}|}.$$

Since x, $z \in \Omega' \subset\subset \Omega$ and $y \in \partial\Omega$, $|y - \frac{x+z}{2}| \geq \mathrm{dist}(\Omega', \partial\Omega)$. We have then obtained

$$u(x) + u(z) - 2u(\frac{x+z}{2}) \leq \frac{|z-x|^2}{4\mathrm{dist}(\Omega', \partial\Omega)},$$

which is the desired semiconcavity inequality. □

Proposition 1.3.8. *Let u be a Lipschitz continuous viscosity solution of the eikonal equation $|Du| = 1$ in Ω. Also set*

$$M(x) = \{y \in \partial\Omega \,:\, u(x) = u(y) + |y - x|\}, \qquad Y(x) = \left\{\frac{(x-y)}{|x-y|} \,:\, y \in M(x)\right\}.$$

Then, for any $x \in \Omega$,

$$D^+u(x) = \overline{\mathrm{co}}\, Y(x), \tag{1.3.8}$$

$$D^-u(x) = \begin{cases} \{p\} & \textit{if } Y(x) = \{p\} \\ \emptyset & \textit{if } Y(x) \textit{ is not a singleton.} \end{cases} \tag{1.3.9}$$

Proof—From Proposition 1.3.7 we know that u is locally semiconcave in Ω and so we can use the properties stated in Proposition 1.2.8. Notice that by Proposition 1.3.7, the set $M(x)$ is nonempty for all $x \in \Omega$ and that $Y(x)$ is well posed since $|x - y| > 0$ for all $x \in \Omega$ and $y \in M(x)$.
Suppose that $\frac{(x-y)}{|x-y|} \in Y(x)$ for some $y \in M(x)$. Then the function $F : \Omega \to \mathbb{R}$ defined by $F(x) = u(y) + |y - x|$ is a $\mathcal{C}^1$ function touching u from above at x. In view of Lemma 1.3.2 we readily conclude that $\frac{(x-y)}{|x-y|} = D_xF(x) \in D^+u(x)$. Since $D^+u(x)$ is a closed convex set, we also have

$$\overline{\mathrm{co}}\, Y(x) \subseteq D^+u(x). \tag{1.3.10}$$

To prove the converse inclusion, it suffices to show that $D^*u(x) \subset Y(x)$, since $D^+u(x) = \overline{\mathrm{co}}\, D^*u(x)$ by Proposition 1.2.8. So, let $p \in D^*u(x)$. Then there exists a sequence of differentiability points $\{x_k\}$ of u converging to x as $k \to \infty$ and such that $Du(x_k) \to p$ as $k \to \infty$. If for all $k \in \mathbb{N}$ we pick $y_k \in M(x_k)$, we can suppose, up to subsequences, that $y_k \to \bar{y} \in \partial\Omega$ as $k \to \infty$. But $Du(x_k) = \frac{(x_k - y_k)}{|x_k - y_k|}$ by (1.3.10) and we deduce

$$u(x) = \lim_{k\to\infty} u(x_k) = u(\bar{y}) + |\bar{y} - x|,$$

$$p = \lim_{k\to\infty} Du(x_k) = \frac{(x-\bar{y})}{|x-\bar{y}|}.$$

We have just proven that $p \in Y(x)$ and concluded the proof of (1.3.8). To obtain (1.3.9) it suffices to recall Proposition 1.2.8 (e)–(f). □

Corollary 1.3.9. *Let u be a Lipschitz continuous viscosity solution of the eikonal equation $|Du| = 1$ in Ω. Then, the following assertions hold true.*

(a) *Let $x \in \Omega$ and $y \in \partial\Omega$ be such that $u(x) = u(y) + |x-y|$. Then, for any $z \in]x, y[$, u is differentiable at z and $Du(z) = (x-y)/|x-y|$.*

(b) *Let u be differentiable at a point $x \in \Omega$ and set*

$$\bar{t} = \inf\{t > 0 \ : \ x - tDu(x) \notin \Omega\} .$$

Then, $y := x - \bar{t}Du(x) \in \partial\Omega$ and $u(x) = u(y) + |x-y|$.

Proof—(a) In view of Proposition 1.3.8, to prove the differentiability of u at z it suffices to show that $M(z) = \{y\}$. Since $z \in]x, y[$ and $y \in M(x)$, then

$$u(z) \leq u(y) + |y-z| = u(x) - |x-y| + |y-z| = u(x) - |x-z|.$$

On the other hand, the Lipschitz continuity of u implies that $u(z) \geq u(x) - |x-z|$. Hence, the first inequality becomes an equality, yielding $u(z) = u(y) + |y-z|$, i.e. $y \in M(z)$. Moreover, no other elements different from y can be contained in $M(z)$. Indeed, if there exists $\tilde{y} \in M(z) \setminus \{y\}$, then $\tilde{y} \notin [x, y]$ and so

$$u(x) = u(z) + |z-x| = u(\tilde{y}) + |\tilde{y}-z| + |z-x| > u(\tilde{y}) + |\tilde{y}-x|,$$

contradicting the Lipschitz continuity of u with constant 1. Now, the expression of $Du(z)$ is just a consequence of the definition of $Y(z)$.
(b) It directly comes from assertion (a). □

Remark 1.3.10. As a further consequence of Corollary 1.3.7 we also have that the superdifferential of any Lipschitz continuous viscosity solution of the eikonal equation satisfies

$$D^+u(x) = \overline{\mathrm{co}}\, D^*u(x) \qquad \forall x \in \Omega \tag{1.3.11}$$

and the set-valued map $x \mapsto D^+u(x)$ is upper semicontinuous in Ω, that is, for every $x \in \Omega$,

$$\Omega \ni x_k \to x\,, \quad D^+u(x_k) \ni p_k \to p \quad (k \to \infty) \quad \Rightarrow \quad p \in D^+u(x)\,.$$

It can also be shown, but is far beyond the scope of this work, that Du is a vector-valued function of locally bounded variation in Ω, see, e. g., [22, p. 240]. Thus, Du is also approximately differentiable a.e. in Ω (see [22, p. 233]), that is, for a. e. $x \in \Omega$ there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ such that, for each $\varepsilon > 0$,

$$\lim_{r \downarrow 0} \frac{1}{r^2} \left| B_r(x) \cap \left\{ y \in \Omega : \frac{|Du(y) - Du(x) - L(y-x)|}{|y-x|} > \varepsilon \right\} \right| = 0\,.$$

Such a property will be very useful in Section 3.1.3 to prove the uniqueness of the solutions of the boundary value problem that is the basis of this thesis.

We conclude this section with a stability result for solutions of the eikonal equation.

Proposition 1.3.11. *Let $u_k \in \mathcal{C}(\Omega)$ $(k \in \mathbb{N})$ be a viscosity solution of the eikonal equation*

$$|Du| = 1 \qquad in \;\; \Omega.$$

If u_k converges to a function u locally uniformly in Ω, then also u is a viscosity solution of the same equation.

Proof—Let $\phi \in \mathcal{C}^1(\Omega)$ and x_0 be a local maximum point of $w := u - \phi$. It is not restrictive to suppose that

$$(u - \phi)(x_0) > (u - \phi)(x),$$

for any $x \neq x_0$ in a neighborhood $\overline{B}_\delta(x_0)$ of x_0. As a matter of fact, the eikonal equation depends on the gradient only and we can always replace $\phi(x)$ in the reasoning with $\phi(x) - |x - x_0|^2 + C$ for some suitable constant C. We claim that for k sufficiently large $w_k := u_k - \phi$ attains a local maximum at a point x_k close to x_0. Indeed, let x_k be a maximum point for w_k in $\overline{B}_\delta(x_0)$ and let $\{x_{k_m}\}$, $m \in \mathbb{N}$, be any converging subsequence of $\{x_k\}$, $k \in \mathbb{N}$. Call $\tilde{x}$ the limiting point of this subsequence. By uniform convergence,

$$w_{k_m}(x_{k_m}) \to w(\tilde{x}), \qquad \text{as } m \to \infty.$$

Since $w_{k_m}(x_{k_m}) \geq w_{k_m}(x)$ for any $x \in \overline{B}_\delta(x_0)$, then $w(\tilde{x}) \geq w(x)$ for any $x \in \overline{B}_\delta(x_0)$. We deduce that $w(\tilde{x}) \geq w(x_0)$. So $\tilde{x} = x_0$ because x_0 is a strict maximum point of w, and the whole sequence $\{x_k\}$ is convergent to x_0. Now, being x_k a maximum point of $w_k = u_k - \phi$, we have that

$$|D\phi(x_k)| \leq 1, \qquad \forall k \in \mathbb{N}.$$

Passing to the limit as $k \to \infty$ we obtain that

$$|D\phi(x_0)| \leq 1,$$

i.e. u is a viscosity sub–solution of the eikonal equation. A similar argument shows that it is also a viscosity super–solution. $\square$

Chapter 2

The Model Problem

The present work is mainly concerned with the study of the system of partial differential equations

$$\begin{cases} -\mathrm{div}\,(vDu) = f & \text{in } \Omega \\ |Du| - 1 = 0 & \text{in } \{v > 0\} \end{cases} \tag{2.0.1}$$

in a given domain $\Omega \subset \mathbb{R}^n$. Such a system arises in many different contexts, such as the Monge-Kantorovich theory, shape optimization theory and granular matter theory. For example, (2.0.1) can be viewed as necessary conditions to be satisfied by an optimal transfer plan, see [21], [3] and [26]. In a related framework, system (2.0.1) characterizes the limit, as $p \to \infty$, of the p-Laplace equation $-\mathrm{div}\,(|Du|^{p-2}Du) = f$, see [12], [30] and [21]. Furthermore, the above system has been applied to an idealized model for compression molding in [6], and to shape optimization in [9]. Another interesting field where system (2.0.1) applies to is granular matter theory, where it describes the equilibrium configuration that may occur to a sandpile that grows under a source constant in time.

In this chapter we will briefly presents some models for growing sandpiles and for their equilibrium configuration. We will also stress the role of system (2.0.1) in this setting, when it appears, giving also the statement of the results obtained so far. We will focus on five papers, namely,

- the paper of Prigozhin [35];

- the work of Aronsson [5] and Aronsson, Evans and Wu [7];

- the paper of Hadeler and Kuttler [28];

- the work of Cannarsa and Cardaliaguet [13].

Of course, the literature is much more widespread. But the above papers well represent the different points of view of the theory.

2.1 The Variational Model

In [35] Prigozhin presents a model to describe the growth of a sandpile under a source of density $w(x,t) \geq 0$ on a rough rigid surface $y = h_0(x)$, where x varies in a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz continuous boundary. The form of the pile thus generated is a function $y = h(x,t)$. In the paper it is assumed that the granular matter has an angle of repose γ, that is no matter can accumulate on slopes steeper than $\tan\gamma$. Moreover, following the experiments, the flow of granular material down the slopes of the growing pile is supposed to be confined in a thin boundary layer which is distinctly separated from the motionless bulk. Assuming constant the density of the sand, the free surface $h(x,t)$ of the sandpile must thus satisfy the conservation law

$$\frac{\partial h}{\partial t} + \mathrm{div}\,(\overrightarrow{q}) = w,$$

where $\overrightarrow{q}(x,t)$ is the horizontal projection of the material flux on the surface layer of the motionless bulk. It is also assumed that the surface flow is directed towards the steepest descent, i.e.

$$\overrightarrow{q} = -mDh,$$

where $m(x,t) \geq 0$ is an unknown. Hence the conservation law becomes

$$\frac{\partial h}{\partial t} - \mathrm{div}\,(mDh) = w. \tag{2.1.1}$$

At time $t = 0$ the free surface coincides with the surface h_0,

$$h(x,0) = h_0(x), \tag{2.1.2}$$

while for any $t \geq 0$ it cannot lies below the surface h_0 and must satisfy the angle of repose constraint whenever it is above h_0,

$$h(x,t) \geq h_0(x), \tag{2.1.3}$$

$$h(x,t) > h_0(x) \quad \Rightarrow \quad |Dh(x,t)| \leq \gamma. \tag{2.1.4}$$

No pouring over the motionless bulk can occur if the incline is less than the angle of repose γ,

$$|Dh(x,t)| < \gamma \quad \Rightarrow \quad m(x,t) = 0. \tag{2.1.5}$$

In the model presented by Prigozhin, the granular matter is allowed to leave Ω through a part Γ_1 of the boundary $\partial\Omega$, while on the remaining of the boundary $\Gamma_2 = \Omega \setminus \Gamma_1$, impermeable walls are set to prevent the outflow of the material,

$$h|_{\Gamma_1} = h_0|_{\Gamma_1}, \qquad m\frac{\partial h}{\partial n}|_{\Gamma_2} = 0. \tag{2.1.6}$$

The model of pile growth (2.1.1)–(2.1.6) contains two unknown, the free surface h and the function m. In his work, Prigozhin shows that actually, the function m is a Lagrange multiplier, related to the constraint (2.1.4). Moreover, he shows that m can be excluded as an unknown by transforming problem (2.1.1)–(2.1.6) into an equivalent quasi–variational problem. To be more precise, set $V = \mathcal{W}^{1,\infty}(\Omega)$, $H = L^\infty(\Omega)$, $\mathcal{V} = L^\infty(0,T,V)$ and $\mathcal{H} = L^\infty(0,T,H)$. Also denote by $\mathcal{V}'$ and $\mathcal{H}'$ the dual spaces of $\mathcal{V}$ and $\mathcal{H}$.

Let $h_0 \in V$. For every function $\phi \in \mathcal{V}$ define the map $B_\phi : \mathcal{V} \to \mathcal{H}$ by

$$B_\phi(\psi) = \frac{1}{2}(|D\psi|^2 - M(\phi)),$$

where

$$M(\phi)(x,t) = \begin{cases} \gamma^2 & \text{if } \phi(x,t) > h_0(x), \\ \max\{\gamma^2, |Dh_0(x)|^2\} & \text{if } \phi(x,t) \leq h_0(x). \end{cases}$$

Moreover, set

$$\mathcal{K}(\phi) := \{\psi \in \mathcal{V} \,:\, B_\psi(\psi) \leq 0,\, \psi|_{\Gamma_1} = h_0|_{\Gamma_1} \text{ a.e. } t\}.$$

As it can be deduced by the choice of $h_0 \in V$, no regularity is a priori required to h and m. Indeed, problem (2.1.1)–(2.1.6) is considered in the weak sense, as follows. We say that $(h, m) \in \mathcal{V} \times \mathcal{H}'$ is a weak solution of (2.1.1)–(2.1.6) if

$$\int_0^T \int_\Omega (h' - w)\psi + mDh \cdot D\psi \, dx \, dt = 0, \qquad \begin{array}{l} \forall \psi \in \mathcal{V} \text{ such that} \\ \psi|_{\Gamma_1} = 0 \text{ a.e. } x, \end{array} \tag{2.1.7}$$

$$\int_0^T \int_\Omega m\phi \, dx \, dt = 0, \qquad \begin{array}{l} \forall \phi \in \mathcal{H} \text{ with} \\ \mathrm{spt}(\phi) \subset \{(x,t) : |Dh(x,t)| \leq \gamma\}, \end{array} \tag{2.1.8}$$

$$h(x, 0) = h_0(x), \qquad \text{a.e. } x, \tag{2.1.9}$$

$$h(x, t) \geq h_0(x), \qquad \text{a.e. } (x, t), \tag{2.1.10}$$

$$h(x, t) > h_0(x) \quad \Rightarrow \quad |Dh(x, t)| \leq \gamma, \qquad \text{a.e. } (x, t), \tag{2.1.11}$$

$$h|_{\Gamma_1} = h_0|_{\Gamma_1}, \qquad \text{a.e. } x. \tag{2.1.12}$$

Notice that equation (2.1.7) takes into account condition $m\frac{\partial h}{\partial n}|_{\Gamma_2} = 0$.
The characterization of weak solutions of the model problem of sandpile growth is the following.

Theorem 2.1.1. *Suppose that $h_0 \in V$, $w \in \mathcal{V}'$, $w \geq 0$ and that there exists a function $\psi_0 \in V$ such that*

$$\psi_0|_{\Gamma_1} = h_0|_{\Gamma_1}, \qquad \|D\psi_0\|_\infty < \gamma.$$

Then a pair of functions $(h, m) \in \mathcal{V} \times \mathcal{H}'$ is a weak solution of (2.1.1)–(2.1.6) *if and* *only if h is a solution of the following quasi–variational inequality:*

$$\exists h' = \tfrac{\partial h}{\partial t} \in (L^q(0,T,\mathcal{W}^{1,q}(\Omega)))', \quad \text{for some } q \in [2,\infty) \tag{2.1.13}$$

$$h \in \mathcal{K}(h), \tag{2.1.14}$$

$$\textstyle\int_0^T \int_\Omega (h' - w)(\phi - h)(x,t)\, dx\, dt \geq 0, \quad \forall \phi \in \mathcal{K}(h), \tag{2.1.15}$$

$$h|_{t=0} = h_0. \tag{2.1.16}$$

Notice that conditions (2.1.14)–(2.1.15) are equivalent to

$$h \in \ \arg\min_{\phi \in \mathcal{K}(h)} \int_0^T \int_\Omega (h' - w)\phi\ dx\, dt. \tag{2.1.17}$$

Hence, in the above formulation, h is a solution of an optimization problem with gradient constraint. Since the linear functional

$$J_h(\phi) = \int_0^T \int_\Omega (h' - w)\phi\ dx\, dt$$

is bounded from below on $\mathcal{K}(h)$, it can be shown that the necessary and sufficient condition of optimality for (2.1.17) is the existence of a saddle point of Lagrangian

$$L(\phi, p) := J_h(\phi) + \int_0^T \int_\Omega p B_h(\phi). \tag{2.1.18}$$

This is the reason why the unknown m of problem (2.1.1)–(2.1.6) is a Lagrangian multiplier related to the gradient constraint (2.1.4). In the case when h_0 has no steep slopes, that is $|Dh_0| \leq \gamma$, problem (2.1.13)–(2.1.16) becomes a variational inequality. An existence and uniqueness result for (2.1.13)–(2.1.16) is then provided, which is an existence result for (2.1.1)–(2.1.6) as a consequence of Theorem 2.1.1.

Theorem 2.1.2. *Let $w \in (L^4(0,T,U))'$, where $U = \{\phi \in \mathcal{W}^{1,4}(\Omega) \ : \ \phi|_{\Gamma_1} = 0\}$. Moreover, take $h_0 \in V$, $h_0|_{\Gamma_1} = 0$ and $|Dh_0| \leq \gamma$. Then there exists a unique function h such that*

$$h \in L^4(0,T,U) \cap \mathcal{C}([0,T], L^2(\Omega)),$$

and h is a solution of problem (2.1.13)–(2.1.16).

As a corollary of the above theorem, it is also proven that

the solution h of the quasi–variational inequality (2.1.13)–(2.1.16) *is a non–decreasing function of time.*

In the case of $w \in L^2((0,T) \times \Omega)$, Theorem 2.1.2 can be strengthened.

Theorem 2.1.3. *Let $w \in L^2((0,T) \times \Omega)$, $h_0 \in V$ and $|Dh_0| \leq \gamma$. Then there exists a unique function h such that*

$$h \in \mathcal{V} \cap \mathcal{C}^{1/4}([0,T] \times \overline{\Omega}), \qquad h' \in L^2((0,T) \times \Omega)$$

and h solves the variational inequality

$$h(\cdot,t) \in K := \{\phi \in V \ : \ |D\phi| \leq \gamma, \ \phi|_{\Gamma_1} = h_0|_{\Gamma_1}\}, \qquad h|_{t=0} = h_0,$$

$$\int_\Omega (h' - w)(\phi - h) \ dx \geq 0, \qquad \forall \phi \in K, \ \textit{a.e. } t \in (0,T).$$

Let us remark that Theorems 2.1.2–2.1.3 do not provide a unique solution of problem (2.1.1)–(2.1.6). Indeed, while function h is uniquely determined by the above variational inequality, the function m is only a Lagrange multiplier of problem (2.1.1)–(2.1.6) and no uniqueness results are given in the paper [35]. As a matter of fact, there is probably no uniqueness for m in the dynamical case.

Assume now $h_0 \equiv 0$, $\Gamma_2 = \emptyset$ and $w = w(x)$. The equilibrium configuration of the model (2.1.1)–(2.1.6) presented by Prigozhin reads

$$\begin{cases} -\mathrm{div}\,(mDh)(x) = w(x) & \text{in } \Omega \\ |Dh(x)| - 1 = 0 & \text{in } \{m > 0\} \\ |Dh(x)| \leq 1 \quad m, h \geq 0 & \text{in } \Omega \\ h(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1.19}$$

Even in this special case, no uniqueness results can be deduced from [35], and regularity can be inferred for h only. We will see that system (2.1.19) is the only point in common between [35] and [28], where a different model for the growth of sandpiles is presented.

2.2 The PDE/ODE Model

The second set of papers we want to analyze starts with an old paper of Aronsson [5], where he proposes a model to describe the growth of sandpiles on a horizontal plane under the flow of m point sources of different intensity. As in the work of [35], it is assumed that the angle ϕ formed by the free surface of the sandpile and a horizontal plane cannot exceed a given value α, the angle of repose of the material. To describe the model more precisely, let us introduce the notations in [5].

the horizontal plane is the x, y–plane of a Cartesian 3 dimensional system;

P_k is the projection on the x, y–plane of source number k;

$\overline{AB}$ is the Euclidean distance from A to B;

P is a variable point in the x, y–plane with coordinates $(x, y, 0)$;

$f_k(t)$ is the volume of sand released per unit time via source number k, at time t.

Aronsson postulates that under m point sources pouring sand at rate $f_k(t) \geq 0$, $k = 1, \dots, m$, there is the initial formation of m cones, separated one another, until two or more of them meet. From this moment he says that the resulting sand surface is the upper envelope of the m cones. In other words, he affirms that at each moment there exist m positive numbers z_k, $k = 1, \dots, m$, such that the free surface of the sandpile is represented by the function

$$z(x, y) = \max\{\phi_1(x, y), \dots, \phi_m(x, y)\},$$

where $\phi_k(x, y) = z_k - \frac{1}{\tan\alpha}\overline{PP_k}$, for $k = 1, \dots, m$. It is clear that in the previous formula z_k can be viewed as the vertex of the cone number k. Using a mass conservation relation and the geometry of cones, Aronsson deduces that $z_k = z_k(t)$, $k = 1, \dots, m$, can be found as a solution of the differential system

$$
\begin{aligned}
\frac{dz_1}{dt} &= \frac{f_1(t)}{F_1(z_1, z_2, \dots, z_m)} \\
\frac{dz_2}{dt} &= \frac{f_2(t)}{F_2(z_1, z_2, \dots, z_m)} \\
&\vdots \\
\frac{dz_m}{dt} &= \frac{f_m(t)}{F_m(z_1, z_2, \dots, z_m)},
\end{aligned}
\tag{2.2.1}
$$

where $F_k(z_1, z_2, \dots, z_m)$, $k = 1, \dots, m$, is the area of the part of the x, y–plane where

$$
z_k - \frac{1}{\tan\alpha}\overline{PP_k} > z_i - \frac{1}{\tan\alpha}\overline{PP_i}, \qquad \forall i \neq k.
$$

A similar description is also provided in the case of m sources pouring sand on an horizontal plane, but in a region confined by vertical walls. The remaining of the paper [5] is then devoted to the analysis of the two models and in particular of functions $F_k(z_1, z_2, \dots, z_m)$.

Starting from the work [5], Aronsson, Evans and Wu, in [7], studied the system (2.2.1) in connection with the evolution governed by the p–Laplacian in the "infinitely fast/infinitely slow" diffusion limit as $p \to \infty$.
The work begins with the consideration that for large p, the p-Laplacian

$$
\Delta_p u = \operatorname{div}\left(|Du|^{p-2} Du\right)
$$

is a prototype "fast/slow diffusion" operator, because, for each $\delta > 0$ small, the diffusion coefficient $|Du|^{p-2}$ is very small in the region $\{|Du| < 1 - \delta\}$, very large in $\{|Du| > 1 + \delta\}$, while the set $\{1 - \delta \leq |Du| \leq 1 + \delta\}$ is a kind of intermediate region. Then, the authors study the behaviour of the solutions u_p of the evolution system

$$
\begin{cases}
u_{p,t} - \Delta_p u_p = f_p & \text{in } \mathbb{R}^n \times (0, \infty) \\
u_p = g & \text{on } \mathbb{R}^n \times \{t = 0\},
\end{cases}
\tag{2.2.2}
$$

where $f_p \geq 0$ represents a given source and g an initial distribution. They are mostly interested in the analysis of system (2.2.2) at the limit $p \to \infty$, because of the diffusion

effect that confines the evolution in the region where the modulus of the gradient is less than or equal to 1. They obtain what follows.
If f_p is smooth and compactly supported in $\mathbb{R}^n \times [0,T]$ for each $T > 0$, and g is a Lipschitz continuous function with compact support and $\|Dg\|_\infty \leq 1$, then for all $p \geq n+1$ the system (2.2.2) admits a unique weak solution u_p such that, for all $T > 0$

$$u_p \in L^p((0,T), \mathcal{W}^{1,p}(\mathbb{R}^n)), \qquad u_{p,t} \in L^2_{\text{loc}}((0,T), L^2(\mathbb{R}^n))$$

and

$$\int_0^T \int_{\mathbb{R}^n} (u_{p,t} v + |Du_p|^{p-2} Du_p \cdot Dv) \; dx \, dt = \int_0^T \int_{\mathbb{R}^n} v f_p \; dx \, dt.$$

Moreover, u_p has compact support in $\mathbb{R}^n \times [0,T]$ for each $T > 0$ and $u_p \in \mathcal{C}^{0,\alpha}$ in $\mathbb{R}^n \times (0,\infty)$ for some $0 < \alpha < 1$. In the case when f_p is a suitable smooth approximation to the time varying measure

$$f = \sum_{k=1}^m f_k(t) \delta_{d_k}(x),$$

where $f_k > 0$ is Lipschitz continuous for all $k = 1, \ldots, m$, the authors prove that–up to subsequences–the solutions of system (2.2.2) converge, in a suitable sense, to a function u which is Lipschitz continuous with $\|Du\|_\infty \leq 1$ and $u \in L^2(\mathbb{R}^n \times (0,T))$, $u_t \in L^2(0,T,L^2(\mathbb{R}^n))$ for every $T > 0$. Furthermore, if we set

$$I[u] = \begin{cases} 0 & \text{if } v \in L^2(\mathbb{R}^n), \ |Dv| \leq 1 \text{ a.e.} \\ \infty & \text{otherwise,} \end{cases}$$

then u satisfies

$$I[v] \geq I[u] + \int_{\mathbb{R}^n} (f(x,t) - u_t(x,t))(v - u(x,t)) \; dx, \tag{2.2.3}$$

for each $v \in L^2(\mathbb{R}^n)$ at almost every $t > 0$ and

$$\lim_{t \to 0^+} u(\cdot,t) = g, \qquad \text{in } L^2(\mathbb{R}^n). \tag{2.2.4}$$

Here $\int_{\mathbb{R}^n} f(x,t)(v - u(x,t)) \; dx$ stands for

$$\sum_{k=1}^m f_k(t)(v(d_k) - u(d_k,t)).$$

In the language of convex analysis, u solves the evolution equation

$$\begin{cases} f - u_t \in \partial I[u] & t > 0 \\ u = g & t = 0, \end{cases}$$

where $\partial I[\cdot]$ is the subdifferential of the indicator function of the set $\{v \in L^2(\mathbb{R}^n)\ :\ |Dv| \leq 1 \text{ a.e. }\}$ (in our notations $\partial I = D^- I$).
Comparing (2.2.3) with the variational inequality of Prigozhin in Theorem 2.1.3, it is not surprising that in [7] (2.2.3)–(2.2.4) is presented as a model for growing sandpiles. Moreover, in [7], the authors prove that the solution to (2.2.3)–(2.2.4) is unique and if

$$\|Dg\|_\infty < 1,$$

then its solution u is compatible with the function z constructed by Aronsson in [5]. More precisely, they prove that

$$u(x,t) = \max\{g(x), z_1(t) - |x - d_1|, \ldots, z_m(t) - |x - d_m|\}, \qquad x \in \mathbb{R}^n,\ t \geq 0,$$

where $z_1(t), \ldots, z_m(t)$ solve the ODE (2.2.1) with initial conditions

$$z_k(0) = g(d_k), \qquad k = 1, \ldots, m.$$

The results obtained so far permit the authors to analyze (2.2.3)–(2.2.4) in the case of general source term f. Indeed if

f is a nonnegative Lipschitz continuous function, with compact support in $\mathbb{R}^n \times [0,T]$ for all $T > 0$,

then (2.2.3)–(2.2.4) admits a unique solution u with $|Du| \leq 1$ almost everywhere and $u_t \in L^2(0,T,L^2(\mathbb{R}^n))$ for every $T > 0$. Furthermore, $u_t \geq 0$ almost everywhere in $\mathbb{R}^n \times (0,\infty)$ and if W is the smallest convex set such that

$$\mathrm{spt}(f)|_{\mathbb{R}^n \times [0,T]} \subset W \times [0,T],$$

then, for each $t > 0$ and $x \notin W$, u verifies

$$u(x,t) = \max\left\{0, \max_{y \in \partial W}\{u(y,t) - |x - y|\}\right\}.$$

2.3 A New Evolutionary Model

A few years ago Hadeler and Kuttler [28] proposed a new model to study the evolution of a sandpile created by pouring dry matter onto a 1 dimensional set or a 'table'. In such a model, built on previous work by Boutreux and de Gennes [11], the 1 dimensional set or the table are represented by a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 1, 2$, and the matter source by a function $f(t, x) \geq 0$. The physical description of the growing heap is based on the introduction of the so-called *standing* and *rolling layers*. The former collects the amount of matter that remains at rest, the latter represents matter moving down along the surface of the standing layer–eventually falling down when the base of the heap touches the boundary of Ω. For physical reasons, the slope of the standing layer cannot exceed a given constant–typical of the matter under consideration–that we normalize to 1. On shallow slopes matter from the rolling layer is added to the standing layer, while on steep slopes there can be abrasion from the standing layer into the rolling layer. Hence locally the dynamic is described by the local slope and the local density of the rolling layer. To be more precise, let $u(t, x)$ and $v(t, x)$, respectively, the heights of the standing and rolling layers at time t and at a point $x \in \Omega$. The system proposed in [28] with normalized constants reads

$$\begin{cases} v_t(t,x) = \operatorname{div}(vDu)(t,x) - (1 - |Du(t,x)|)v(t,x) + f(t,x) \\ u_t(t,x) = (1 - |Du(t,x)|)v(t,x). \end{cases} \tag{2.3.1}$$

The system describes that the grains enter the standing layer proportional to the thickness of v if the slope is below the critical one, while grains are pulled away from the standing layer whenever the slope exceeds the critical angle and there is *some* rolling matter.

System (2.3.1) is also associated with a boundary condition on u that takes into account the gradient constraint $|Du| \leq 1$: $u \equiv 0$ on $\partial\Omega$ for any $t \geq 0$. Of course this condition says that no matter can accumulate on the boundary of Ω. The above considerations lead to the boundary value problem

$$\begin{cases} v_t = \operatorname{div}(vDu) - (1-|Du|)v + f & \text{in } [0,\infty)\times\Omega \\ u_t = (1-|Du|)v & \text{in } [0,\infty)\times\Omega \\ |Du| \le 1 \quad u,v \ge 0 & \text{in } [0,\infty)\times\Omega \\ u = 0 & \text{on } [0,\infty)\times\partial\Omega \end{cases} \tag{2.3.2}$$

A similar problem is also presented to describe sandpiles growing in a silo. The remaining of the paper [28] is devoted to the analysis of the equilibrium configurations that may occur in the case of constant (in time) source. The boundary value problem (2.3.2) becomes

$$\begin{cases} -\operatorname{div}(vDu)(x) = f(x) & \text{in } \Omega \\ |Du(x)| - 1 = 0 & \text{in } \{v>0\} \\ |Du(x)| \le 1 \quad u,v \ge 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.3}$$

which is formally the same system (2.1.19) that can be deduced from [35]. An interesting representation formula for the (pointwise) solution of (2.3.3), in 1 space dimension, is presented in [28]. There is also an attempt to build a representation solution in the 2 dimensional case. Unfortunately, the conjectured solution for the rolling layer is wrong and indeed no rigorous mathematical proof is presented in [28]. Nevertheless, the paper underlines some interesting features of problem (2.3.3) that will lead to the correct solution in [13].

First of all, there is a discussion on the subeikonal solutions. Indeed, starting from physical considerations, Hadeler and Kuttler guess that the standing layer must have some pointwise maximality property in the class of the nonnegative Lipschitz functions of constant 1 that vanishes on the boundary of Ω. Moreover, they introduce the singular set Σ to explain the structure of the rolling layer. They assume that $\operatorname{spt}(f)$ has positive distance from the boundary and they conjecture that the "unique solution" (they do not specify in which class) of (2.3.3) is the pair (u,v) given (in our notations) by

$$u(x) = \max_{y\in\operatorname{spt}(f)} \{d(y) - |x-y|\},$$

and

$$v(x) = \begin{cases} 0 & \text{on } \Sigma \\ \int_0^{t_\Sigma(x)} f(\Pi(x) + t(x - \Pi(x))d(x)\, dt & \text{on } \Omega \setminus \Sigma, \end{cases}$$

where $d(x)$ is the usual distance function from $\mathbb{R}^2 \setminus \Omega$, $\Pi(x)$ is the unique projection of $x \notin \Sigma$ onto $\partial\Omega$ and $t_\Sigma(x) = \min\{t : \Pi(x) + t(x - \Pi(x)) \in \Sigma\}$. Throughout the paper, the regularity of the boundary $\partial\Omega$ is not specified.

As we will in the next section, the exact solution (in a suitable sense) of problem (2.3.3) can be explicitly given, provided we ask some regularity to the boundary and to f.

2.4 Representation of Solutions in the Plane

Recently, Cannarsa and Cardaliaguet [13] have obtained a representation formula for the solution of problem (2.3.3) that starts from the physical considerations of Hadeler and Kuttler [28] but develops in a rigorous mathematical framework.

It is well-known that the eikonal equation $|Du| = 1$ does not possess global smooth solutions in general, neither does the conservation law $-\mathrm{div}\,(vDu) = f$. Thus, solutions of problem (2.3.3) are meant in the following sense.

Definition 2.4.1. A pair (u, v) of *continuous* functions in Ω is a solution of problem (2.3.3) if

- $u = 0$ on $\partial\Omega$, $\|Du\|_{\infty,\Omega} \leq 1$, and u is a viscosity solution of

$$|Du| = 1 \quad \text{in} \quad \{x \in \Omega \ : \ v(x) > 0\}$$

- $v \geq 0$ in Ω and, for every test function $\phi \in \mathcal{C}_c^\infty(\Omega)$,

$$\int_\Omega v(x)\langle Du(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x)dx\,.$$

In Definition 2.4.1, continuity is required because u and v should represent the layers describing the (non microlocal) configuration of a pile of sand. Moreover, the maximality of the standing layer, justified in [28] by physical considerations, is ensured by typical properties of viscosity solutions. Finally, the requirement on v to be a weak solution of the conservation law $-\mathrm{div}\,(vDu) = f$ is a natural one, since the viscosity solutions of the eikonal equations are not globally smooth and so we cannot read $-\mathrm{div}\,(vDu) = f$ in the classical sense. In the theorem below, which is the main result in [13], we use the notations d for the distance function from $\mathbb{R}^2 \setminus \Omega$, Σ for its singular set, $\Pi(x)$ for the projection of $x \in \Omega \setminus \Sigma$ onto $\partial\Omega$ and $\kappa(x)$ for the curvature of $\partial\Omega$ at the point $\Pi(x)$ whenever $x \in \Omega \setminus \Sigma$.

Theorem 2.4.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary of class $\mathcal{C}^2$ and $f \geq 0$ be a continuous function in Ω. Then, a solution of system* (2.3.3) *is given by the pair (u, v), where $u = d$ in Ω, $v = 0$ on $\overline{\Sigma}$ and*

$$v(x) = \int_0^{\tau(x)} f(x + tDd(x))\, \frac{1 - (d(x) + t)\kappa(x)}{1 - d(x)\kappa(x)}\, dt \qquad \forall x \in \Omega\backslash\overline{\Sigma}\,, \tag{2.4.1}$$

where $\tau : \overline{\Omega} \to [0, \mathrm{diam}(\Omega)/2]$ is the maximal retraction length *of Ω, defined by*

$$\tau(x) = \begin{cases} \min\left\{t \geq 0 \;:\; x + tDd(x) \in \overline{\Sigma}\right\} & \forall x \in \overline{\Omega}\backslash\overline{\Sigma} \\ 0 & \forall x \in \overline{\Sigma}. \end{cases} \tag{2.4.2}$$

Moreover, the above solution is unique in the following sense: if (u', v') is another solution of (2.3.3)*, then $v' = v$ in Ω and $u' = u$ in $\{x \in \Omega : v > 0\}$.*

The first step of the proof of Theorem 2.4.2 is to show that the pair (u, v) is indeed a solution of the boundary value problem. This also provides an existence result for (2.3.3), as in [35], [21], [3], [9], [10], where existence results are given in any space dimension. Then the second step is to prove that the solution of (2.3.3) is unique. Uniqueness of the solution to (2.3.3) might also be deduced from a theorem by Bouchitté, Buttazzo and Seppecher [9], characterizing any solution of (2.3.3) in terms of a suitable Monge-Kantorovich problem, and from the uniqueness result by Ambrosio [3] for such a problem. This procedure, however, would use very powerful–as well as very technical–tools that are not really needed in this context. The main novelty of the paper [13] is neither existence nor uniqueness but formula (2.4.1). No other simple representation for the solution of (2.3.3) has been provided in the previous papers. Moreover, Theorem 2.4.2 produces a continuous solution, instead of just a measure or a function in $L^1(\Omega)$ as is generally expected for v. So, Theorem 2.4.2 can be also viewed as a regularity result.

Summarizing the results obtained in [35], [5]–[7], [28] and [13], we can say that for the problem of growing sandpiles, some existence and uniqueness results are provided for the free surface of the sandpile and regularity results are proven in special cases. In the case of the equilibrium configuration, existence and uniqueness results are proven for $n = 2$ as well, but also a representation formula is given.
The main difference between the "dynamical" and "stationary" case is in the role of the function that describes the "flow" of the sand on the descent of the sandpile. Indeed, in the "dynamical" model, it is ignored or considered just a Lagrange multiplier, related to the gradient constraint on the free surface. On the other hand, in the "stationary" model such a function (the "rolling layer") is explicitly used in the mathematical description and plays an important role there. But, apart from continuity, no regularity result is given for the rolling layer, while the standing layer (the function describing the profile of the sandpile) is completely described.

In this work we want to go deeper into the analysis of the regularity of the rolling layer in dimension $n = 2$ and eventually extend the representation formula of the table problem in [13] to general space dimension. Before the regularity analysis, we will presents the results of Cannarsa and Cardaliaguet [13] in slightly more general assumptions. Indeed, we will permit the "table" to have corners pointing "outside" the table, which is physically more realistic.

Chapter 3

The problem in the plane

Let us start with the analysis of the system we introduced as a model for the equilibrium configuration of a growing sandpile,

$$\begin{cases} -\mathrm{div}\,(vDu)(x) = f(x) & \text{in } \Omega \\ |Du(x)| - 1 = 0 & \text{in } \{v > 0\} \\ |Du(x)| \le 1 \quad u, v \ge 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.0.1}$$

In this chapter we restrict our attention to the the system (3.0.1) when Ω is a set in the plane. Throughout the chapter, unless otherwise specified, we will suppose that Ω has piecewise $\mathcal{C}^{2,1}$ boundary and outer corners and then that it satisfies conditions (H1)–(H3) of Definition 1.1.9:

(H1) $\partial\Omega = \cup_{i=1}^{m} \Gamma_i$, $m \in \mathbb{N}$, where

$$\Gamma_i \cap \Gamma_j = \begin{cases} \{x_i\} & \text{if } 1 \le i \le m-1,\ j = i+1 \\ \{x_m\} & \text{if } i = m,\ j = 1 \\ \emptyset & \text{if } 1 \le i \le m,\ j \ne i, i \pm 1 \end{cases}$$

and for any $i = 1, \dots, m$ Γ_i is a $\mathcal{C}^{2,1}$ curve up to the endpoints x_i and x_{i+1} (x_m and x_1 when $i = m$);

(H2) there exists some $0 < \theta < 1$ such that for any $i = 1, \dots, m-1$,

$$\langle \nu_i(x_i), \nu_{i+1}(x_i) \rangle \le \theta \quad \langle \nu_m(x_m), \nu_1(x_m) \rangle \le \theta,$$

where ν_i stands for the unit inner normal to the boundary component Γ_i and where

$$\nu_i(x_j) := \lim_{\substack{y \to x_j \\ y \in \Gamma_i}} \nu_i(y).$$

(H3) Ω satisfies a uniform exterior sphere condition of radius $r > 0$, that is for any $x \in \partial\Omega$ there exists some $y \in \mathbb{R}^n \setminus \Omega$ such that $B_r(y) \subset \mathbb{R}^n \setminus \overline{\Omega}$ and $x \in \partial B_r(y)$.

The plan of this chapter is the following.
Section 3.1 is devoted to the proof that the pair (d, v_f), where d is the distance function to the boundary $\partial\Omega$ and v_f is the function $v_f : \overline{\Omega} \to \mathbb{R}$ defined by

$$v_f(x) = \begin{cases} \displaystyle\int_0^{\tau(x)} f(x + tDd(x)) \frac{1 - (d(x) + t)\kappa(x)}{1 - d(x)\kappa(x)} \, dt & \forall x \in \overline{\Omega} \backslash \overline{\Sigma} \\ 0 & \forall x \in \overline{\Sigma}, \end{cases} \tag{3.0.2}$$

is the unique solution, in the sense of Definition 2.4.1, of problem (3.0.1).
In Section 3.2 we will face the problem of the regularity of the solution of problem (3.0.1), under the stronger assumption that Ω has analytic boundary. In particular, in Section 3.2.2 we will deeply analyze the regularity of the *maximal retraction length* of Ω. This is the first step to prove that v_f in (3.0.2) is itself Hölder continuous, which is the main result in the concluding Section 3.2.3

3.1 Solutions of the Model Problem

We recall that the solutions of system (3.0.1) are meant in the sense of [13].

Definition 3.1.1. A pair (u, v) of *continuous* functions in Ω is a solution of problem (3.0.1) if

- $u = 0$ on $\partial\Omega$, $\|Du\|_{\infty,\Omega} \leq 1$, and u is a viscosity solution of

$$|Du| = 1 \quad \text{in} \quad \{x \in \Omega \ : \ v(x) > 0\}$$

- $v \geq 0$ in Ω and, for every test function $\phi \in \mathcal{C}_c^\infty(\Omega)$,

$$\int_\Omega v(x)\langle Du(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x)dx\,.$$

Before proving our existence/uniqueness result we need some preliminary regularity results on the maximal retraction length, which are the content of the next section.

3.1.1 The maximal retraction length

Let us consider again the maximal retraction length of Ω onto $\overline{\Sigma}$

$$\tau(x) = \begin{cases} \min\left\{t \geq 0 \ : \ x + tDd(x) \in \overline{\Sigma}\right\} & \forall x \in \overline{\Omega}\backslash\overline{\Sigma} \\ 0 & \forall x \in \overline{\Sigma}. \end{cases} \tag{3.1.1}$$

We will show that if Ω has piecewise $\mathcal{C}^{2,1}$ boundary and outer corners then the map τ is Lipschitz continuous when restricted to the boundary. Such a result will permit us to show that the pair (d, v_f) is a solution of problem (3.0.1) almost everywhere, which is the first step of the existence result. It is important to stress that the regularity result for τ is inspired by recent work on the subject. Indeed, a proof of this result for $\mathcal{C}^\infty$ smooth submanifolds of an n-dimensional smooth manifold is given in [29], while [31] treats the case of the normal distance that arises in the context of Finsler geometry. The proof we present is instead modeled on the particular case of sets in $\mathbb{R}^2$ and is based on the propagation of singularities described in Proposition 1.2.16. In the case when $\partial\Omega$ is a $\mathcal{C}^{2,1}$ boundary in the plane, the results below have been proven first in [13] and our arguments are a modifications of those ones.

We start with a continuity result.

Lemma 3.1.2. *Assume that Ω has piecewise $\mathcal{C}^{2,1}$ boundary and outer corners. Then the map τ, extended to 0 on $\overline{\Sigma}$, is continuous in $\overline{\Omega}$.*

Proof—Let us first show that τ is lower semicontinuous in $\overline{\Omega}$. If $x \in \overline{\Sigma}$, then $\tau(x) = 0$, while $\tau(y) \geq 0$ for any $y \in \overline{\Omega}$. Hence $\liminf_{y \to x} \tau(y) \geq \tau(x)$ for any $x \in \overline{\Sigma}$. On the other hand, if $x \in \overline{\Omega} \setminus \overline{\Sigma}$, let $\{x_k\}$ be any sequence converging to x such that $\liminf_{y \to x} \tau(y) = \lim_{k \to \infty} \tau(x_k) =: t^*$. Then, for k sufficiently large, $x_k \notin \overline{\Sigma}$ because $\overline{\Sigma}$ is closed and $Dd(x_k) \to Dd(x)$. Hence, $x_k + \tau(x_k)Dd(x_k) \to x + t^* Dd(x) \in \overline{\Sigma}$. We readily conclude $\tau(x) \leq t* = \liminf_{y \to x} \tau(y)$ because of the definition of τ.
Let us now prove the upper semicontinuity of τ in $\overline{\Omega}$. For this purpose, consider a sequence $\{x_k\}$ in $\overline{\Omega} \setminus \overline{\Sigma}$, converging to some point $x \in \overline{\Omega}$, and suppose by contradiction $t^* := \lim_k \tau(x_k) > \tau(x)$. In particular, this implies that t^* is positive. We can also assume, without loss of generality, that the sequence $\{Dd(x_k)\}$ converges, say to p. Let $\bar{t} \in (\tau(x), t^*)$. Then, d is differentiable at $x_k + \bar{t} Dd(x_k)$ by definition. Thus, for k large enough,

$$\Pi(x_k + \bar{t} Dd(x_k)) = \Pi(x_k) = \{x_k - d(x_k) Dd(x_k)\} .$$

Taking the limit as $k \to \infty$, we obtain $x - d(x)p \in \Pi(x + \bar{t}p)$. So, $x + \tau(x)p$ belongs to the interior of the segment $[x - d(x)p, x + \bar{t}p]$. Since $x + \tau(x)p \in \overline{\Sigma}$, this contradicts Proposition 1.1.8. $\square$

The following result is a modification of the analogous result given in [13].

Theorem 3.1.3. *Let Ω be a bounded domain in $\mathbb{R}^2$ with piecewise $\mathcal{C}^{2,1}$ boundary and outer corners. Then the map τ defined in* (2.4.2) *is Lipschitz continuous on $\partial\Omega$.*

Hereafter, we will denote by $\mathrm{Lip}(\tau)$ the Lipschitz semi-norm of τ on $\partial\Omega$. Since $x \mapsto x + \tau(x)Dd(x)$ maps $\partial\Omega$ onto $\overline{\Sigma}$, a straightforward application of Theorem 3.1.3 is that the 1-dimensional Hausdorff measure of $\overline{\Sigma}$ is finite:

Corollary 3.1.4. *Let Ω be a bounded domain in $\mathbb{R}^2$ with piecewise $\mathcal{C}^{2,1}$ boundary and outer corners. Then,*

$$\mathcal{H}^1(\overline{\Sigma}) \leq k_\Omega \, \mathcal{H}^1(\partial\Omega) < \infty$$

where $k_\Omega \geq 0$ *is a constant depending on* $Lip(\tau)$ *and* Ω.

In order to prove Theorem 3.1.3 we will proceed by steps. As a first step we will prove a one–sided Lipschitz estimate of τ on any Γ_i^0, where Γ_i^0 is the relative interior of the set Γ_i in assumption (H1); then we will show that $\tau|_{\Gamma_i}$ is Lipschitz continuous with some constant K_i for any $i = 1, \ldots, m$, by using the fact that the boundary components join in outer corners; finally, we will prove the Lipschitz continuity on the whole boundary using the continuity of τ in $\overline{\Omega}$.

Lemma 3.1.5. *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with piecewise* $\mathcal{C}^{2,1}$ *boundary and outer corners. Then there exists a constant* $K_i > 0$ *such that every* $x \in \Gamma_i^0$ *has a neighborhood,* U*, where*

$$\tau(y) \leq \tau(x) + K_i|y - x| \qquad \forall y \in \Gamma_i^0 \cap U\,. \tag{3.1.2}$$

Proof—Put

$$L_i = \sup_{\substack{x,y\in\Gamma_i^0 \\ x\neq y}} \max\left\{\frac{|\kappa(y)-\kappa(x)|}{|y-x|}, \frac{|Dd(y) - Dd(x) - D^2d(x)(y-x)|}{|y-x|^2}\right\}.$$

and let $x \in \Gamma_i^0$ be fixed. We will analyze, first, the simpler case $\tau(x)\kappa(x) = 1$. Recalling that $\tau(x) \leq \mathrm{diam}(\Omega)/2$, we have $\kappa(x) \geq 2/\mathrm{diam}(\Omega)$. Let U be an open neighborhood of x such that $\kappa(y) > 1/\mathrm{diam}(\Omega)$ for every $y \in U$. Then, for every $y \in \Gamma_i^0 \cap U$,

$$\tau(y) \leq \frac{1}{\kappa(y)} \leq \frac{1}{\kappa(x)} + \frac{\kappa(y)-\kappa(x)}{\kappa(y)\kappa(x)} \leq \tau(x) + \frac{L_i}{2}\mathrm{diam}(\Omega)^2|y-x|$$

and (3.1.2) is proven with $K_i = \frac{L_i}{2}\mathrm{diam}(\Omega)^2$.

Now, suppose $\tau(x)\kappa(x) < 1$ and define $\bar{x} = x + \tau(x)Dd(x)$. We claim that $Dd(x)$ must be isolated in $D^*d(\bar{x})$. For suppose $Dd(x) = \lim_k p_k$ for some sequence $\{p_k\}$ in $D^*d(\bar{x})$ satisfying $p_k \neq Dd(x)$ for every k. Then, $p_k = Dd(x_k)$, where $x_k = \bar{x} - d(\bar{x})p_k \neq \bar{x} - d(\bar{x})Dd(x) = x$ is a sequence of boundary points $\{x_k\}$ converging to x. We can also assume, without loss of generality, that $(x_k - x)/|x_k - x|$ converges to some unit vector θ. Hence,

$$\theta = \lim_{k\to\infty} \frac{x_k - x}{|x_k - x|} = -d(\bar{x}) \lim_{k\to\infty} \frac{Dd(x_k) - Dd(x)}{|x_k - x|} = -d(\bar{x}) D^2 d(x)\theta \,.$$

Therefore, recalling that the nonzero eigenvalue of $D^2 d(x)$ is given by $-\kappa(x)$, we obtain $-\kappa(x) = -1/d(\bar{x}) = -1/\tau(x)$ in contrast with $\tau(x)\kappa(x) < 1$. So, our claim is proven.

Hereafter, we denote by $\mathcal{R}$ the rotation matrix

$$\mathcal{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and by $\{e_1, e_2\}$ the orthonormal basis of $\mathbb{R}^2$ given by

$$e_1 = \mathcal{R}^{-1} Dd(x) \qquad e_2 = Dd(x)\,.$$

We split the reasoning into several steps.

Step 1: *constructing a singular arc.*
We want to construct a Lipschitz arc $\zeta : [0, \eta] \to \Omega$ such that

$$\zeta(0) = \bar{x}\,, \qquad \langle \dot{\zeta}(0), e_1 \rangle > 0\,, \qquad \zeta(s) \in \Sigma \quad \forall s \in [0, \eta]\,. \tag{3.1.3}$$

Suppose, first, $\bar{x} \in \Sigma_2$. Since e_2 is isolated in $D^* d(\bar{x})$, there are two distinct vectors $p_1, p_2 \in D^* d(\bar{x})$ such that the segments $[p_1, e_2]$ and $[p_2, e_2]$ are contained in $\partial D^+ d(\bar{x})$. Let n_1 and n_2 be unit outward normals to $D^+ d(\bar{x})$ exposing the faces $[p_1, e_2]$ and $[p_2, e_2]$ respectively, i.e.

$$\max_{p \in D^+ d(\bar{x})} \langle p, n_i \rangle = \langle p_i, n_i \rangle = \langle e_2, n_i \rangle \qquad i = 1, 2\,.$$

We claim that

$$e_2 = \lambda_1 n_1 + \lambda_2 n_2 \tag{3.1.4}$$

for suitable numbers $\lambda_1, \lambda_2 > 0$. Indeed, the normal cone to $D^+ d(\bar{x})$ at e_2 is generated by $\{n_1, n_2\}$. Since e_2 belongs to such a cone, $e_2 = \lambda_1 n_1 + \lambda_2 n_2$ with $\lambda_1, \lambda_2 \geq 0$. If

$\lambda_1 = 0$, then $\lambda_2 = 1$ and $e_2 = n_2$. Therefore, $\langle p_2, n_2 \rangle = \langle e_2, n_2 \rangle = 1$, which implies $p_2 = n_2 = e_2$ in contrast with the definition of p_2. So, $\lambda_1 > 0$. Similarly, $\lambda_2 > 0$; our claim is thus proven. Now, observe that $0 = \lambda_1 \langle n_1, e_1 \rangle + \lambda_2 \langle n_2, e_1 \rangle$, on account of (3.1.4). So, either $\langle n_1, e_1 \rangle < 0$ or $\langle n_2, e_1 \rangle < 0$. Suppose $\langle n_1, e_1 \rangle < 0$, and apply Proposition 1.2.16 to the face $[p_1, e_2]$ of $D^+ d(\bar{x})$, with normal n_1, to construct a Lipschitz arc $\zeta : [0, \eta] \to \Omega$ such that

$$\zeta(0) = \bar{x}\,, \qquad \dot{\zeta}(0) = -n_1\,, \qquad \zeta(s) \in \Sigma \quad \forall s \in [0, \eta]\,.$$

Since $\langle n_1, e_1 \rangle < 0$, we have $\langle \dot{\zeta}(0), e_1 \rangle > 0$, which proves (3.1.3). To complete the proof of this step it suffices to note that the case $\bar{x} \in \Sigma_1$ can be treated by a similar–yet simpler–argument.

Step 2: *intersecting the singular arc.*

We want to construct a neighborhood of x, U, such that, for any boundary point $y \in \Gamma_i^0 \cap U$ satisfying $\langle y - x, e_1 \rangle > 0$, there exist $s_y, \rho_y > 0$ with

$$\zeta(s_y) = y + \rho_y Dd(y) \tag{3.1.5}$$

$$\lim_{y \to x} s_y = 0 \tag{3.1.6}$$

(where the limit is taken for $y \in \Gamma_i^0 \cap U$ such that $\langle y - x, e_1 \rangle > 0$). Let V be an open neighborhood of x such that $\Gamma_i^0 \cap V$ is the trace of a regular curve $h \in\,]-r, r[\mapsto \chi(h)$, with $\chi(0) = x$ and $\dot{\chi}(0) = e_1$. Then, $\chi(h) = x + he_1 + \mathrm{o}(h)$, where the standard notation $\mathrm{o}(h)$ denotes–hereafter–a (scalar- or vector-valued) map satisfying $\mathrm{o}(h)/h \to 0$ as $h \to 0$. Moreover, for every $y \in \Gamma_i^0 \cap V$ satisfying $\langle y - x, e_1 \rangle > 0$,

$$\exists! \quad h_y \in (0, r) \quad \text{such that} \quad y = \chi(h_y)\,. \tag{3.1.7}$$

Now, for $0 < h < r$, consider the map $\phi_h : [0, \eta] \to \mathbb{R}$

$$\phi_h(s) = \langle \zeta(s) - \chi(h), \mathcal{R} Dd(\chi(h)) \rangle \qquad \forall s \in [0, \eta]\,,$$

where ζ is the singular arc of Step 1. Since $D^2d(x) = -\kappa(x)e_1 \otimes e_1$, we have

$$\begin{aligned}\phi_h(0) &= \langle \bar{x} - \chi(h), \mathcal{R}Dd(\chi(h))\rangle \\ &= \langle x + \tau(x)e_2 - (x + he_1 + h\varepsilon(h)), -e_1 + h\mathcal{R}D^2d(x)e_1\rangle + \mathrm{o}(h) \\ &= h(1 - \tau(x)\kappa(x)) + \mathrm{o}(h)\,.\end{aligned}$$

But $1 - \tau(x)\kappa(x) > 0$. So, $\phi_h(0) > 0$ for h small enough, say $0 < h < r_0$. Moreover,

$$\begin{aligned}\phi_h(s) &= \langle \zeta(s) - \chi(h), \mathcal{R}Dd(\chi(h))\rangle \\ &= \langle \tau(x)e_2 + s\dot{\zeta}(0) - he_1, -e_1 + h\mathcal{R}D^2d(x)e_1\rangle + \mathrm{o}(h) + \mathrm{o}(s) \\ &= -s\langle \dot{\zeta}(0), e_1\rangle + h(1 - \tau(x)\kappa(x)) + \mathrm{o}(h) + \mathrm{o}(s) \qquad (3.1.8)\end{aligned}$$

Since $\langle \dot{\zeta}(0), e_1\rangle > 0$, there exists $\bar{s} \in (0, \eta]$ such that

$$\phi_h(s) \leq -\frac{s}{2}\langle \dot{\zeta}(0), e_1\rangle + h(1 - \tau(x)\kappa(x)) + \mathrm{o}(h) \qquad \forall s \in [0, \bar{s}] \tag{3.1.9}$$

So, there exists $\bar{r} \in (0, r_0]$ such that $\phi_h(\bar{s}) < 0$ for every $h \in [0, \bar{r}]$. This proves that, for any $h \in [0, \bar{r}]$, there exists $s(h) \in (0, \bar{s})$ such that

$$\phi_h(s(h)) = \langle \zeta(s(h)) - \chi(h), \mathcal{R}Dd(\chi(h))\rangle = 0\,. \tag{3.1.10}$$

Furthermore, recalling (3.1.9),

$$0 < s(h) \leq \frac{2}{\langle \dot{\zeta}(0), e_1\rangle}\Big[h(1 - \tau(x)\kappa(x)) + \mathrm{o}(h)\Big] \qquad \forall h \in [0, \bar{r}]\,, \tag{3.1.11}$$

so that $s(h) \to 0$ as $h \downarrow 0$. Next, observe that, in view of (3.1.7), equality (3.1.10) can be expressed in intrinsic terms saying that for any point $y \in \Gamma_i^0$ of a suitable neighborhood of x, say $U \subset V$, satisfying $\langle y - x, e_1\rangle > 0$, there exists $s_y := s(h_y) > 0$ such that $\langle \zeta(s_y) - y, \mathcal{R}Dd(y)\rangle = 0$. Consequently, $\zeta(s_y) = y + \rho_y Dd(y)$ for some $\rho_y \in \mathbb{R}$, and (3.1.5) will be proven if we show $\rho_y > 0$. To this end, observe that

$$h_y = |y - x| + \mathrm{o}(|y - x|) \tag{3.1.12}$$

as $\Gamma_i^0 \cap U \ni y \to x$ satisfying $\langle y - x, e_1\rangle > 0$. Also, in view of the above formula and (3.1.11),

$$0 < s_y \leq C|y - x| \tag{3.1.13}$$

for some constant $C > 0$. So, (3.1.6) is proven. Furthermore,

$$\lim_{y\to x}\zeta(s_y) = \bar{x} = x + \tau(x)Dd(x)\,,$$

so that $\rho_y \to \tau(x)$ as $y \to x$. Hence, $\rho_y > 0$ for y sufficiently close to x, which completes the proof of this step.

Step 3: *an estimate for s_y.*
We claim that

$$s_y = \frac{1-\tau(x)\kappa(x)}{\langle\dot{\zeta}(0), e_1\rangle}|y-x| + \mathrm{o}(|y-x|) \tag{3.1.14}$$

as $\Gamma_i^0 \cap U \ni y \to x$ satisfying $\langle y-x, e_1\rangle > 0$. Indeed, (3.1.8) yields

$$0 = -s_y\langle\dot{\zeta}(0), e_1\rangle + h_y(1-\tau(x)\kappa(x)) + \mathrm{o}(h_y) + \mathrm{o}(s_y)\ .$$

The above identity yields the desired result thanks to (3.1.12) and (3.1.13).

Step 4: *an upper bound for ρ_y.*
We claim that

$$\rho_y \le \tau(x) + \frac{1-\tau(x)\kappa(x)}{\langle\dot{\zeta}(0), e_1\rangle}|y-x| + \mathrm{o}(|y-x|) \tag{3.1.15}$$

as $\Gamma_i^0 \cap U \ni y \to x$ satisfying $\langle y-x, e_1\rangle > 0$. Indeed, returning to the parametric representation of Γ_i^0 introduced in Step 1, we have, for every $h \in [0, \bar{r}]$,

$$\begin{aligned}
\rho_{y(h)} &= |\zeta(s(h)) - \chi(h)| = \langle\zeta(s(h)) - \chi(h), Dd(y(h))\rangle \\
&= \langle\tau(x)e_2 + s(h)\dot{\zeta}(0) - he_1 + \mathrm{o}(h), e_2 + hD^2d(x)(e_1) + \mathrm{o}(h)\rangle \\
&= \tau(x) + s(h)\langle\dot{\zeta}(0), e_2\rangle + \mathrm{o}(h)
\end{aligned}$$

since $0 < s(h) \le Ch$. In intrinsic notation, $\rho_y = \tau(x) + s_y\langle\dot{\zeta}(0), e_2\rangle + \mathrm{o}(h_y)$ for every $y \in \Gamma_i^0 \cap U$ satisfying $\langle y-x, e_1\rangle > 0$. Since $|\langle\dot{\zeta}(0), e_2\rangle| \le 1$, our claim follows in view of (3.1.12) and (3.1.14).

Step 5: *a global bound.*

We will now derive the estimate

$$\frac{1-\tau(x)\kappa(x)}{\langle\dot{\zeta}(0),e_1\rangle} \leq \frac{L_i}{2}\mathrm{diam}(\Omega)^2 \tag{3.1.16}$$

that is a delicate one, since $\dot{\zeta}(0) = -n_1$ and $e_1 = \mathcal{R}^{-1}Dd(x)$ also depend on x. Let p_1 and n_1 be as in Step 1. Then, the point $z := \bar{x} - \tau(x)p_1$ belongs to $\Pi(\bar{x})$. Moreover, $Dd(z) = p_1$. So,

$$z - x = -\tau(x)(Dd(z) - Dd(x)) = \tau(x)(e_2 - p_1) \ .$$

We now have to distinguish two cases: $z \in \Gamma_i^0$ and $z \notin \Gamma_i^0$. In the first case, due to the regularity of Γ_i^0, we also have

$$|Dd(z) - (Dd(x) + D^2d(x)(z-x))| \leq L_i|z-x|^2 \ .$$

Therefore, recalling that $D^2d(x) = -\kappa(x)e_1 \otimes e_1$,

$$\begin{aligned} |(I - \tau(x)\kappa(x)e_1 \otimes e_1)(p_1 - e_2)| &\leq L_i\tau^2(x)|p_1 - e_2|^2 \\ &\leq \frac{L_i}{4}\mathrm{diam}(\Omega)^2|p_1 - e_2|^2 \ . \end{aligned}$$

Since the matrix $I - \tau(x)\kappa(x)e_1 \otimes e_1$ is positive definite and has eigenvalues 1 and $1 - \tau(x)\kappa(x) > 0$, this proves that

$$(1 - \tau(x)\kappa(x))|p_1 - e_2| \leq \frac{L_i}{4}\mathrm{diam}(\Omega)^2|p_1 - e_2|^2 \ .$$

Now, recall that $p_1 \neq e_2$ to conclude

$$1 - \tau(x)\kappa(x) \leq \frac{L_i}{4}\mathrm{diam}(\Omega)^2|p_1 - e_2| \ . \tag{3.1.17}$$

Next, the identity $\langle\dot{\zeta}(0), p_1 - e_2\rangle = 0$ implies that $\dot{\zeta}(0) = \lambda\mathcal{R}(p_1 - e_2)$ for some $\lambda \in \mathbb{R}$ satisfying $|\lambda| = 1/|p_1 - e_2|$. Therefore,

$$\langle\dot{\zeta}(0), e_1\rangle = |\lambda\langle\mathcal{R}(p_1 - e_2), e_1\rangle| = \frac{|\langle p_1 - e_2, e_2\rangle|}{|p_1 - e_2|} = \frac{1 - \langle p_1, e_2\rangle}{|p_1 - e_2|} \ .$$

Since $|p_1 - e_2|^2 = 2(1 - \langle p_1, e_2\rangle)$, we have $\langle \dot\zeta(0), e_1\rangle = |p_1 - e_2|/2$. Combining the last equality and (3.1.17) proves our claim (3.1.16) in the case when $z \in \Gamma_i^0$.
It remains to consider the case when $z \notin \Gamma_i^0$. We still have to prove

$$1 - \tau(x)\kappa(x) \leq \frac{K_i}{2}|p_1 - e_2| \ . \tag{3.1.18}$$

for some $K_i > 0$ independent of x, since the identity $\langle \dot\zeta(0), e_1\rangle = |p_1 - e_2|/2$ holds true as in the previous case. Suppose by contradiction that for any $k \in \mathbb{N}$ there exists a point $x_k \in \Gamma_i^0$ such that $z_k \notin \Gamma_i^0$ and

$$1 - \tau(x_k)\kappa(x_k) > n|p_{1,n} - e_{2,n}| \ .$$

Then $|p_{1,n} - e_{2,n}| \to 0$ as $k \to \infty$, which in turn gives

$$|z_k - x_k| = d(x_k)|p_{1,n} - e_{2,n}| \to 0, \qquad \text{as } k \to \infty.$$

Hence, without loss of generality we can suppose that $z_k \in \Gamma_{i+1}^0$ definitely and that $x_k,\ z_k \to x_i$ as $k \to \infty$, where $\{x_i\} = \Gamma_i \cap \Gamma_{i+1}$. But also $e_{2,n} = \nu_i(x_k)$, $p_{1,n} = \nu_{i+1}(z_k)$, where ν_i and ν_{i+1} are the unit inner normals to Γ_i and Γ_{i+1} respectively. Hence, due to hypothesis (H2),

$$e_{2,n} \to \nu_i(x_i), \qquad p_{1,n} \to \nu_{i+1}(x_i), \qquad \text{as } k \to \infty,$$

which is a contradiction to $|p_{1,n} - e_{2,n}| \to 0$, since $\nu_i(x_i) \neq \nu_{i+1}(x_i)$. This shows (3.1.18) in the case of $z \notin \Gamma_i^0$ for some $K_i > 0$ dependent on Γ_i only.
Step 6: *conclusion.* Possibly reducing the neighborhood U of x that we found in the previous steps, the above construction shows that, for every $y \in U \cap \partial\Omega$ satisfying $\langle y - x, e_1\rangle > 0$,

$$\tau(y) \leq \rho_y \leq \tau(x) + K_i|y - x| \ .$$

By a similar reasoning, there exists another neighborhood U' of x such that, for every $y \in U' \cap \partial\Omega$ satisfying $\langle y - x, e_1\rangle < 0$,

$$\tau(y) \leq \tau(x) + K_i|y - x| \ .$$

Putting these estimates together completes the proof of the lemma. $\square$

We are now ready to prove the Lipschitz continuity of the maximal retraction length on each component Γ_i.

Theorem 3.1.6. *Let Ω be a bounded domain in $\mathbb{R}^2$ with piecewise $\mathcal{C}^{2,1}$ boundary and outer corners. Then the map τ defined in* (2.4.2) *is Lipschitz continuous when restricted to any Γ_i.*

Proof—Fix a boundary component Γ_i and let C_i be the connected component of $\Omega \setminus \overline{\Sigma}$ containing Γ_i in its closure. We first claim that $\Gamma_j \not\subseteq \overline{C}_i$ for $j \neq i$. Indeed, let $t \in [a, b] \mapsto x(t)$ a $\mathcal{C}^{2,1}$ parametrization of Γ_i, with $x(a) = x_i$ and $x(b) = x_{i+1}$, and denote by $\nu_i(x)$ the unit inner normal of Ω at $x \in \Gamma_i$. By the regularity of Γ_i, we have that the limits

$$\lim_{\substack{y \to x_i \\ y \in \Gamma_i}} \nu_i(y), \qquad \lim_{\substack{y \to x_{i+1} \\ y \in \Gamma_i}} \nu_i(y)$$

exist, so that ν_i can be considered as a continuous function on Γ_i. Hence, the map $\phi : [a, b] \to \overline{\Sigma}$, defined by $\phi(t) = x(t) + \tau(x(t))\nu_i(x(t))$ is itself continuous (τ is continuous by the previous proposition). Moreover, its image is a curve in $\overline{\Sigma}$, because $\nu_i(x) = Dd(x)$ on Γ_i^0, whose extremes are the points $x_i,\ x_{i+1} \in \mathcal{C}$. Therefore, $\phi([a, b]) \cup \Gamma_i$ is a closed curve, contained in $\overline{C}_i$, which implies that no other boundary components Γ_j can be contained in $\overline{C}_i$.
Now, in order to prove the statement we have to extend estimate (3.1.2) to the ε-neighborhood $\Gamma^0_{i,\varepsilon} := \{x \in C_i \ : \ 0 < d(x) < \varepsilon\}$ of Γ_i^0. Let $\varepsilon > 0$ be such that $d \in \mathcal{C}^{2,1}(\Gamma^0_{i,\varepsilon})$. We claim that a constant $\tilde{K}_i = \tilde{K}_i(\varepsilon) > 0$ exists so that every $x \in \Gamma^0_{i,\varepsilon}$ has a ball $B_\rho(x) \subset \Omega^\varepsilon$ such that

$$\tau(y) \leq \tau(x) + \tilde{K}_i|y - x| \qquad \forall y \in B_\rho(x)\,. \tag{3.1.19}$$

To show this, observe that for every $y \in \Gamma^0_{i,\varepsilon}$ such that $\Pi(y)$ is in the neighborhood

U of $\Pi(x)$ provided by Lemma 3.1.5, we have, in view of (3.1.2),

$$\begin{aligned}\tau(y) &= \tau(\Pi(y)) - d(y) \\ &\leq \tau(\Pi(x)) + K_i|\Pi(y) - \Pi(x)| - d(y) \\ &\leq \tau(x) + K_i\|D\Pi\|_{\infty,\Gamma^0_{i,\varepsilon}}|y-x| + d(x) - d(y)\,.\end{aligned} \tag{3.1.20}$$

Our claim (3.1.19) follows with $\tilde{K}_i = K_i\|D\Pi\|_{\infty,\Gamma^0_{i,\varepsilon}} + 1$. Next, we will derive the bound

$$|p| \leq \tilde{K}_i \qquad \forall p \in \partial_P\tau(x) \quad \forall x \in \Gamma^0_{i,\varepsilon}\,, \tag{3.1.21}$$

where $\partial_P\tau(x)$ denotes the proximal subgradient of τ at x and $\tilde{K}_i$ is the constant that appears in (3.1.19). Then, by Theorem 1.2.20, such an estimate will imply that τ is locally Lipschitz continuous in $\Gamma^0_{i,\varepsilon}$ with Lipschitz constant $\tilde{K}_i$. To check (3.1.21), recall that a vector $p \in \mathbb{R}^2$ belongs to $\partial_P\tau(x)$ if and only if there exist numbers $\sigma, \eta > 0$ such that

$$\tau(y) \geq \tau(x) + \langle p, y-x\rangle - \sigma|y-x|^2 \qquad \forall y \in B_\eta(x)\,,$$

see Definition 1.2.17. Now, combine the above inequality with (3.1.19) to obtain

$$\langle p, y-x\rangle \leq \tilde{K}_i|y-x| + \sigma|y-x|^2$$

whenever $|y-x| < \min\{\rho,\eta\}$. This implies (3.1.21). In order to complete the proof we have to pass from the local Lipschitz continuity of τ in $\Gamma^0_{i,\varepsilon}$ to its Lipschitz continuity on Γ_i. First of all, we can choose $r = r(\varepsilon, \Gamma_i) > 0$ such that

(a) there exists a finite covering of Γ_i with balls $B_r(z_1), \ldots, B_r(z_{i_k})$ of radius r;

(b) $\Gamma_i \cap B_r(z_s)$, $s = 1, \ldots, i_k$, can be written as the graph of a $\mathcal{C}^{2,1}$ function;

(c) if x, $y \in \Gamma_i$ satisfy $|x-y| \leq r/2$, then x, $y \in \Gamma_i \cap B_r(z_s)$ for some $s = 1, \ldots, i_k$.

Since $\Gamma_i \cap B_r(z_s)$ is the graph of a $\mathcal{C}^{2,1}$ function, we can call $M_i = M_i(r)$ the maximal Lipschitz constant among all detected in $\Gamma_i \cap B_r(z_s)$, $s = 1, \ldots, i_k$ and without loss of generality we can suppose that $r < \varepsilon/(M_i + 2)$ ($M_i(r)$ is non–decreasing with r). So, take any x, $y \in \Gamma_i$. If $|x - y| > r/2$, then obviously

$$|\tau(x) - \tau(y)| \leq \mathrm{diam}(\Omega) < \frac{2\mathrm{diam}(\Omega)}{r}|x - y|.$$

On the other hand, if $|x - y| \leq r/2$, then x, $y \in \Gamma_i \cap B_r(z_s)$ for some $s = 1, \ldots, i_k$ and without loss of generality we can suppose that

$$\Gamma_i \cap B_r(z_s) = \{(s, \gamma(s)) \,:\, s \in [0, 1]\},$$

where $\gamma(\cdot)$ is of class $\mathcal{C}^{2,1}$ and that $\Gamma^0_{i,\varepsilon} \cap B_r(z_s)$ is contained in the epigraph of $\gamma(\cdot)$. Moreover we can take $x = (s_x, \gamma(s_x))$, $y = (s_y, \gamma(s_y))$ with $s_x < s_y$. Now consider the lines thru x and y with slope $M_i + 1$ and $-M_i - 1$ respectively and call z their intersection. By the assumption $r < \varepsilon/(M_i + 2)$ we have that $0 < d(z) < \varepsilon$. If also z, $(x, z]$, $[z, y)$ are contained in C_i, then τ is differentiable almost everywhere on $(x, z]$ and $[z, y)$ and we have

$$\begin{aligned}
&|\tau(x) - \tau(y)| \leq |\tau(x) - \tau(z)| + |\tau(z) - \tau(y)| \\
&\leq \textstyle\int_0^1 \left(|\nabla\tau(z + t(x - z))||x - z| + |\nabla\tau(y + t(z - y))||z - y|\right) dt \\
&\leq 2\tilde{K}_i(M_i + 1)|x - y|
\end{aligned}$$

In all the other cases, we claim that there exists a piecewise linear curve $\phi : [0, 1] \to \overline{\Omega}$, such that $\phi(0) = x$ and $\phi(1) = y$, $\phi(s) \in C_i$ for $s \neq 0, 1$ and whose graph is in between the graph of γ and $[x, z] \cup [z, y]$. Indeed, C_i is an open connected component of $\Omega \setminus \overline{\Sigma}$ and then it is also connected by piecewise linear curves. Also, x and y are points in its closure and the segments $[x, z]$, $[z, y]$ intersect the graph of γ in x and y, respectively, and cannot be tangent to the curve at x and y by construction. With the choice of ϕ as above, we find

$$\begin{aligned}
&|\tau(x) - \tau(y)| \leq \textstyle\int_0^1 |\nabla\tau(\phi(t))||\phi'(t)|\, dt \\
&\leq 2\tilde{K}_i(M_i + 1)|x - y|.
\end{aligned}$$

The desired Lipschitz constant of τ on Γ_i is given by $\max\{2\tilde{K}_i(M_i + 1), \frac{2\mathrm{diam}(\Omega)}{r}\}$. $\square$

We are now in position to prove Theorem 3.1.3 as a simple corollary of the previous results.

Proof of Theorem 3.1.3—It remains to prove is that we can extend the Lipschtiz property of τ from any single component $\Gamma_i \subset \partial\Omega$ to the whole boundary. To see this, just set

$$K = \max_{i\in\{1,\dots,m\}} \max\left\{2\tilde{K}_i(M_i+1), \frac{2\mathrm{diam}(\Omega)}{r}\right\}.$$

The conclusion follows from the fact that τ is continuous on $\overline{\Omega}$ and that for any $i = 1,\dots,m$ Γ_i and Γ_{i+1} join in a point. □

3.1.2 Existence

In this section we prove that the pair (d, v_f), where

$$v_f(x) = \int_0^{\tau(x)} f(x+tDd(x))\frac{1-(d(x)+t)\kappa(x)}{1-d(x)\kappa(x)}dt \qquad \forall x\in\Omega\setminus\overline{\Sigma} \tag{3.1.22}$$

and $v_f \equiv 0$ on $\overline{\Sigma}$, is a solution of system (3.0.1). We begin with two preliminary results, the former describing continuity and differentiability properties of v_f, the latter providing an approximation result for the characteristic function of a compact set, in the spirit of capacity theory.

Proposition 3.1.7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain in $\mathbb{R}^2$ with piecewise $\mathcal{C}^{2,1}$ boundary and outer corners and $f \geq 0$ be a Lipschitz continuous function in Ω. Then, v_f is a continuous function in Ω, which satisfies the bound*

$$0 \leq v_f(x) \leq \|f\|_\infty\Big[1+\|[\kappa]_-\|_*\mathrm{diam}(\Omega)\Big]\tau(x) \qquad \forall x\in\Omega_\varepsilon, \tag{3.1.23}$$

where

$$\|[\kappa]_-\|_* := \max_{i=1,\dots,m}\sup_{x\in\Gamma_i^0}[\kappa(x)]_-,$$

and $[\kappa(x)]_- = \max\{-\kappa(x),0\}$. Moreover, v_f is locally Lipschitz continuous in $\Omega\backslash\overline{\Sigma}$ and satisfies

$$-\mathrm{div}\,(v_f(x)Dd(x)) = f(x) \tag{3.1.24}$$

at each point $x\in\Omega\backslash\overline{\Sigma}$ at which v_f is differentiable.

Remark 3.1.8. Since d is $\mathcal{C}^{2,1}$ in any (open) connected component of $\Omega\backslash\overline{\Sigma}$, equality (3.1.24) reads

$$\langle Dv_f(x), Dd(x)\rangle + v_f(x)\Delta d(x) + f(x) = 0\,. \tag{3.1.25}$$

Moreover, a straightforward consequence of Proposition 3.1.7 is that the equality

$$-\mathrm{div}\,(v_f Dd) = f$$

holds in the sense of distributions in $\Omega\backslash\overline{\Sigma}$.

Proof—We note, first, that the maps Dd, τ and κ are continuous in $\Omega\backslash\overline{\Sigma}$. Hence, when f is continuous, so is v_f in $\Omega\backslash\overline{\Sigma}$.

Let us now prove that v_f is continuous on $\overline{\Sigma}$. Observe that, for any $x \notin \overline{\Sigma}$, the term

$$\frac{1-(d(x)+t)\kappa(x)}{1-d(x)\kappa(x)} = \frac{1-d(x+tDd(x))\kappa(x)}{1-d(x)\kappa(x)} \qquad 0 < t < \tau(x)$$

is nonnegative by Proposition 1.1.4. A simple computation shows that it is also bounded by $1 + \|[\kappa]_-\|_*\tau(x)$. This proves (3.1.23). The continuity of v_f on $\overline{\Sigma}$ is an immediate consequence of (3.1.23). Next, since $\partial\Omega$ is a piecewise $\mathcal{C}^{2,1}$ boundary with outer corners, then Theorem 3.1.3 ensures that τ is Lipschitz on $\partial\Omega$. Therefore, $\tau = \tau\circ\Pi$ is locally Lipschitz in $\overline{\Omega}\backslash\overline{\Sigma}$, as well as v_f. Finally, let us check the validity of (3.1.24) at every differentiability point x for v_f in the open set $\Omega\backslash\overline{\Sigma}$. We note that, at any such point x,

$$\langle Dv_f(x), Dd(x)\rangle = \frac{d}{d\lambda} v_f(x+\lambda Dd(x))_{|\lambda=0}\,.$$

But $\tau(x+\lambda Dd(x)) = \tau(x) - \lambda$ and $d(x+\lambda Dd(x)) = d(x) + \lambda$ for $\lambda > 0$ sufficiently small. So,

$$\begin{aligned} v_f(x+\lambda Dd(x)) &= \int_0^{\tau(x)-\lambda} f\Big(x+(t+\lambda)Dd(x)\Big)\frac{1-(d(x)+\lambda+t)\kappa(x)}{1-(d(x)+\lambda)\kappa(x)}\,dt \\ &= \int_\lambda^{\tau(x)} f(x+tDd(x))\frac{1-(d(x)+t)\kappa(x)}{1-(d(x)+\lambda)\kappa(x)}\,dt\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle Dv_f(x), Dd(x)\rangle &= -f(x) + \int_0^{\tau(x)} f(x+tDd(x))\frac{1-(d(x)+t)\kappa(x)}{(1-d(x)\kappa(x))^2}\,\kappa(x)\,dt \\ &= -f(x) - v_f(x)\Delta d(x) \end{aligned}$$

where we have taken into account the identity

$$\Delta d(x) = -\frac{\kappa(x)}{1 - d(x)\kappa(x)} \qquad \forall x \in \Omega\backslash\overline{\Sigma}\,,$$

that follows from Proposition 1.1.12. We have thus obtained (3.1.25)–an equivalent version of (3.1.24)–and completed the proof. $\square$

Proposition 3.1.9. *Let K be a compact subset of $\mathbb{R}^2$ such that $\mathcal{H}^1(K) < \infty$. Then, there exists a sequence $\{\xi_k\}$ of functions in $W^{1,1}(\mathbb{R}^2)$ with compact support, such that*

(a) $0 \leq \xi_k \leq 1$ *for every* $k \in \mathbb{N}$*;*

(b) $\operatorname{dist}(\operatorname{spt}(\xi_k), K) \to 0$ *as* $k \to \infty$*;*

(c) $K \subset \operatorname{int}\{x \in \mathbb{R}^2 : \xi_k(x) \geq 1\}$ *for every* $k \in \mathbb{N}$*;*

(d) $\xi_k \to 0$ *in* $L^1(\mathbb{R}^2)$ *as* $k \to \infty$*;*

(e) $\int_{\mathbb{R}^2} |D\xi_k| dx \leq \frac{3}{2}\pi(\mathcal{H}^1(K) + 1/k)$ *for every* $k \in \mathbb{N}$.

The standard notations dist, spt and int stand for distance (between two sets), support (of a function) and interior (of a set), respectively. We give a proof of the proposition for the reader's convenience.

Proof—Since $\mathcal{H}^1(K) < \infty$, for any fixed $k \in \mathbb{N}$ there exists a sequence of points $\{x_i^{(k)}\}_{i\in\mathbb{N}}$ in K and a sequence of radii $\{r_i^{(k)}\}_{i\in\mathbb{N}}$ such that

- $0 < r_i^{(k)} \leq \frac{1}{k}$ and $\sum_i r_i^{(k)} \leq \frac{1}{2}\left(\mathcal{H}^1(K) + \frac{1}{k}\right)$;
- $K \subset \operatorname{int}\Big(\bigcup_i B_{r_i^{(k)}}(x_i^{(k)})\Big)$.

Now, define, for any $x \in \mathbb{R}^2$,

$$\xi_i^{(k)}(x) = \left[1 - \frac{1}{r_i^{(k)}}\left(|x - x_i^{(k)}| - r_i^{(k)}\right)_+\right]_+$$

$$\xi_k(x) = \sup_{i\in\mathbb{N}} \xi_i^{(k)}(x)$$

and observe that

$$\operatorname{spt}(\xi_i^{(k)}) = \overline{B}_{2r_i^{(k)}}(x_i^{(k)})$$
$$\operatorname{spt}(D\xi_i^{(k)}) = \overline{B}_{2r_i^{(k)}}(x_i^{(k)}) \setminus B_{r_i^{(k)}}(x_i^{(k)})$$

Then, $\xi_k \in L^1(\mathbb{R}^2)$ since $0 \leq \xi_k \leq 1$ and ξ_k has compact support. Also, $\xi_i^{(k)} \in \mathcal{W}^{1,1}(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} |D\xi_i^{(k)}|dx = 3\pi r_i^{(k)}$. In order to prove that $\xi_k \in \mathcal{W}^{1,1}(\mathbb{R}^2)$, let us set for all $l \in \mathbb{N}$

$$f_{k,l} = \max_{1\leq i\leq l} \xi_i^{(k)}, \qquad h_k = \sup_{1\leq i<\infty} |D\xi_i^{(k)}|.$$

Notice that h_k can be seen as the limit of a non–decreasing family of L^1 functions $\sup_{1\leq i\leq l} |D\xi_i^{(k)}|$, $l \in \mathbb{N}$, and ξ_k is the limit of the monotone family $\{f_{k,l}\}_l$. Direct computations using Beppo Levi's Theorem show that

$$\int_{\mathbb{R}^2} h_k dx = \int_{\mathbb{R}^2} \lim_{l\to\infty} \sup_{1\leq i\leq l} |D\xi_i^{(k)}|dx$$
$$\leq \lim_{l\to\infty} \int_{\mathbb{R}^2} \sum_{i=1}^l |D\xi_k^{(k)}|dx \leq \tfrac{3}{2}\pi\Big(\mathcal{H}^1(K) + \tfrac{1}{k}\Big).$$

and $f_{k,l} \in \mathcal{W}^{1,1}(\mathbb{R}^2)$ with

$$|Df_{k,l}| \leq \sup_{1\leq i\leq l} |D\xi_i^{(k)}| \leq h_k, \qquad \text{a.e.}$$

Now, for each $\phi \in \mathcal{C}_c^1(\mathbb{R}^2)$ and $m = 1, 2$,

$$\int_{\mathbb{R}^2} \xi_k \tfrac{\partial}{\partial x_m}\phi dx = \lim_{l\to\infty} \int_{\mathbb{R}^2} f_{k,l} \tfrac{\partial}{\partial x_m}\phi dx$$
$$= -\lim_{l\to\infty} \int_{\mathbb{R}^2} \phi \tfrac{\partial}{\partial x_m} f_{k,l} dx \leq \int_{\mathbb{R}^2} |\phi| h_k dx.$$

The linear functional $L_k^m(\phi) := \int_{\mathbb{R}^2} \xi_k \frac{\partial}{\partial x_m}\phi dx$, with $\phi \in \mathcal{C}_c^1(\mathbb{R}^2)$, can be uniquely extended to a linear functional $\bar{L}_k^m : \mathcal{C}_c(\mathbb{R}^2) \to \mathbb{R}$ such that

$$\bar{L}_k^m(\phi) \leq \int_{\mathbb{R}^2} |\phi| h_k dx, \qquad \forall \phi \in \mathcal{C}_c(\mathbb{R}^2). \tag{3.1.26}$$

By Riesz Representation Theorem we conclude that there exits a Radon measure μ_k^m on $\mathbb{R}^2$ such that

$$\bar{L}_k(\phi) = \int_{\mathbb{R}^2} \phi d\mu_k^m, \qquad \forall \phi \in \mathcal{C}_c(\mathbb{R}^2).$$

But (3.1.26) implies that for any Lebesgue measurable set $A \subset \mathbb{R}^2$

$$\mu_k^m(A) \leq \int_A h_k dx.$$

Hence, μ_k^m is absolutely continuous with respect to the Lebesgue measure and by Radon–Nikodym Theorem there exists a function $g_k^m \in L^1(\mathbb{R}^2)$, $|g_k^m| \leq h_k$ almost everywhere, such that for any Lebesgue measurable set $A \subset \mathbb{R}^2$

$$\mu_k^m(A) = \int_A g_k^m(x) dx.$$

In particular, we deduce that for $m = 1, 2$

$$\int_{\mathbb{R}^2} \xi_k \frac{\partial}{\partial x_m} \phi dx = \int_{\mathbb{R}^2} \phi g_k^m dx, \qquad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^2),$$

which gives $\xi_k \in \mathcal{W}^{1,1}(\mathbb{R}^2)$ for any $k \in \mathbb{N}$. The above computations also show (e), because

$$\int_{\mathbb{R}^2} |D\xi_k| dx \leq \lim_{l \to \infty} \int_{\mathbb{R}^2} \sup_{1 \leq i \leq l} |D\xi_i^{(k)}| dx \leq \frac{3}{2}\pi\Big(\mathcal{H}^1(K) + \frac{1}{k}\Big).$$

Properties (b) and (c) are true by construction. Finally, (d) follows by Lebesgue's Theorem because $0 \leq \xi_k \leq 1$ and $\xi_k(x) = 0$ for any point $x \notin K$ and k large enough. $\square$

Proof of Theorem 2.4.2[Part 1: Existence]—We will prove that the pair (d, v_f), with v_f defined by (3.1.22), is a solution of system (3.0.1). Let us point out, to begin with, that d is a viscosity solution of the eikonal equation in Ω, and so, a fortiori, in the open set $\{x \in \Omega : v_f(x) > 0\}$. Therefore, what actually remains to be shown is that

$$\int_\Omega f\phi \, dx = \int_\Omega v_f \langle Dd, D\phi \rangle dx \qquad \forall \phi \in \mathcal{C}_c^\infty(\Omega)\,. \tag{3.1.27}$$

Since $\mathcal{H}^1(\overline{\Sigma}) < \infty$ by Proposition 3.1.4, we can apply Proposition 3.1.9 with $K = \overline{\Sigma}$ to construct a sequence $\{\xi_k\}$ enjoying properties (a), (b), (c) and (d). Choose any test function $\phi \in \mathcal{C}_c^\infty(\Omega)$ and set $\phi_k = \phi(1 - \xi_k)$. Notice that, for k large enough, $\mathrm{spt}(\phi_k) \subset\subset \Omega \backslash \overline{\Sigma}$. This follows from (a), (b) and from the fact that $\overline{\Sigma} \cap \partial\Omega = \mathcal{C}$ (see Proposition 1.1.13). Then, Proposition 3.1.7 and Rademacher's Theorem imply that

$-\mathrm{div}\,(v_f Dd) = f$ a. e. in $\Omega\backslash\overline{\Sigma}$. So, multiplying this equation by ϕ_k and integrating by parts, we obtain

$$\int_\Omega f\phi_k dx = \int_\Omega v_f(1-\xi_k)\langle Dd, D\phi\rangle dx - \int_\Omega v_f\phi\langle Dd, D\xi_k\rangle dx\,. \tag{3.1.28}$$

We claim that the right–most term above goes to 0 as $k\to\infty$. Indeed,

$$\begin{aligned}\Big|\int_\Omega v_f\phi\langle Dd, D\xi_k\rangle dx\Big| &\le \|\phi\|_{\infty,\Omega}\,\|v_f\|_{\infty,\mathrm{spt}(\xi_k)}\int_\Omega |D\xi_k|dx\\ &\le C\,\|\phi\|_{\infty,\Omega}\,\|v_f\|_{\infty,\mathrm{spt}(\xi_k)}\end{aligned}$$

where C is the constant provided by Proposition 3.1.9 (d). Now, using property (a) of that proposition and the fact that v_f is a continuous function vanishing on $\overline{\Sigma}$, we conclude that $\|v_f\|_{\infty,\mathrm{spt}(\xi_k)}\to 0$ as $k\to\infty$. This proves our claim. The conclusion (3.1.27) immediately follows since, in view of (a) and (c), the integrals $\int_\Omega f\phi_k dx$ and $\int_\Omega v_f(1-\xi_k)\langle Dd, D\phi\rangle dx$ converge to $\int_\Omega f\phi dx$ and $\int_\Omega v_f\langle Dd, D\phi\rangle dx$–respectively–as $k\to\infty$.

3.1.3 Uniqueness

In this section we will prove that, if (u,v) is a solution of system (3.0.1), then v is given by (2.4.1) and $u\equiv d$ in $\Omega_v := \{x\in\Omega\,:\,v(x)>0\}$. We begin by showing the last statement.

Proposition 3.1.10. *If* (u,v) *is a solution of system* (3.0.1)*, then* $u\equiv d$ *in* Ω_v*.*

Proof—Since $\|Du\|_{\infty,\Omega}\le 1$ and $u=0$ on $\partial\Omega$, we have that $u\le d$ in Ω because, in view of Remark 1.3.6 , d is the largest function with such properties. Moreover, since u solves the eikonal equation in Ω_v, Proposition 1.3.7 (a) ensures that

$$u(x) = \min_{y\in\partial\Omega_v,\,]x,y[\subset\Omega_v}\Big\{\,u(y)+|y-x|\,\Big\}\qquad \forall x\in\Omega_v\,.$$

We will argue by contradiction, supposing $u(x_0)<d(x_0)$ for some point $x_0\in\Omega_v$. Without loss of generality, x_0 may be assumed to be a point of differentiability of u,

and of approximate differentiability of Du (see Remark 1.3.10). Let then $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that, for any $\varepsilon > 0$,

$$\lim_{r\downarrow 0} \frac{1}{r^2}\Big|B_r(x_0) \cap \Big\{x \in \Omega : \frac{|Du(x) - Du(x_0) - L(x - x_0)|}{|x - x_0|} > \varepsilon\Big\}\Big| = 0\,. \quad (3.1.29)$$

Moreover, let $y_0 \in \partial\Omega_v$ be such that

$$]x_0, y_0[\subset \Omega_v \quad \text{and} \quad u(x_0) = u(y_0) + |y_0 - x_0|\,.$$

Notice that $y_0 \notin \partial\Omega$ because, otherwise, one would have $u(y_0) = 0$, and in turn $u(x_0) = |y_0 - x_0|$, in contrast with $u(x_0) < d(x_0)$. Next, let us fix

$$0 < \varepsilon < \min\Big\{1, \frac{v(x_0)}{16[1 + (1 + \|L\|)\mathrm{diam}(\Omega)]}\Big\}\,, \quad (3.1.30)$$

where $\|L\| = \sup_{|x|=1} |L(x)|$. We claim that there exists $\rho > 0$ such that the balls $B_\rho(x_0)$ and $B_\rho(y_0)$ are both contained in Ω, and

$$|p - Du(x_0)| \le 1/2 \qquad \forall p \in D^+u(x),\ \forall x \in B_\rho(x_0) \quad (3.1.31)$$

$$v(x) \ge v(x_0)/2 \qquad \forall x \in B_\rho(x_0) \quad (3.1.32)$$

$$v(y) \le \varepsilon \qquad \forall y \in B_\rho(y_0) \quad (3.1.33)$$

Indeed, (3.1.31) follows from the upper semicontinuity of D^+u (see Remark 1.3.10), while (3.1.32) and (3.1.33) can be obtained by a simple continuity argument since $v(x_0) > 0$ and $v(y_0) = 0$. Let us set, for the sake of brevity, $e_2 = Du(x_0)$ and let $e_1 \in \mathbb{R}^2$ be such that $\{e_1, e_2\}$ is a positively oriented orthonormal basis of $\mathbb{R}^2$. From (3.1.29) it follows that, for every sufficiently small $r > 0$, there exists a point $x_r \in B_r(x_0)$ of differentiability for u such that

$$\begin{array}{ll} (i) & |Du(x_r) - Du(x_0) - L(x_r - x_0)| \le \varepsilon r \\ (ii) & \langle e_2, x_r - x_0\rangle \ < \ 0 \\ (iii) & \langle e_1, x_r - x_0\rangle \ > \ r/2. \end{array} \quad (3.1.34)$$

Now, fix $y_1 \in]x_0, y_0[\cap B_{\rho/2}(y_0)$, and let $r > 0$ and be so small that

$$y_r := x_r - |x_0 - y_1|Du(x_r) \in B_\rho(y_0) \quad \text{and} \quad \overline{\mathrm{co}}\,\{x_0, x_r, y_1, y_r\} \subset \Omega_v\,.$$

Such a number r exists because $[x_0, y_1] \subset \Omega_v$ and

$$\lim_{r\downarrow 0} y_r \;=\; x_0 - |x_0 - y_1| Du(x_0) = y_1\,,$$

since $x_r \to x_0$ and $Du(x_r) \to Du(x_0)$ as $r \downarrow 0$. Finally, let us set $x_1 = \Pi_{[x_0,y_1]}(x_r)$ and $\mathcal{Q} = \overline{\mathrm{co}}\,\{x_1, y_1, x_r, y_r\}$. We point out that, because of (3.1.34)(ii), x_1 belongs to the open segment $]x_0, y_1[$. The convex set $\mathcal{Q}$ is a quadrilateral with sides $[x_1, x_r]$, $[x_r, y_r]$, $[y_r, y_1]$ and $[y_1, x_1]$. Moreover, u is differentiable at any point $x \in [y_1, x_1]$ and $Du(x) = Du(x_0)$, as guaranteed by Corollary 1.3.9. Similarly, combining properties (b) and (c) of the same corollary shows that u is differentiable at any point $x \in [x_r, y_r]$ and $Du(x) = Du(x_r)$. Our next step would be to integrate the equation $-\mathrm{div}\,(vDu) = f$ over $\mathcal{Q}$ and apply the Divergence Theorem. This reasoning needs the following approximation argument to be made rigorous. For any $\sigma > 0$, consider the test function

$$\psi_\sigma(x) := \Big[1 - \frac{1}{\sigma} d_{\mathcal{Q}}(x)\Big]_+ \qquad x \in \mathbb{R}^2\,,$$

an element of $W^{1,\infty}(\mathbb{R}^2)$ with support $\mathcal{Q}_\sigma := \{x \in \mathbb{R}^2 : d_{\mathcal{Q}}(x) \le \sigma\}$. Observe that, for σ sufficiently small, $\psi_\sigma \in W^{1,\infty}_c(\Omega)$. Also, $\mathrm{spt}(D\psi_\sigma) = \overline{\mathcal{Q}_\sigma \backslash \mathcal{Q}}$. Thus,

$$\int_\Omega f\psi_\sigma dx = \int_\Omega v\langle Du, D\psi_\sigma\rangle dx = \int_{\mathcal{Q}_\sigma\backslash\mathcal{Q}} v\langle Du, D\psi_\sigma\rangle dx\,. \tag{3.1.35}$$

In the right-hand side of the above equality, we split the integration domain as $\mathcal{Q}_\sigma \backslash \mathcal{Q} = E_1(\sigma) \cup E_2(\sigma) \cup E_3(\sigma) \cup E_4(\sigma)$, where

$$\begin{aligned}
E_1(\sigma) &= \{x \in E : \Pi_{\mathcal{Q}}(x) \in]x_1, y_1[\}\\
E_2(\sigma) &= \{x \in E : \Pi_{\mathcal{Q}}(x) \in]x_r, y_r[\}\\
E_3(\sigma) &= \{x \in E : \Pi_{\mathcal{Q}}(x) \in [y_1, y_r]\}\\
E_4(\sigma) &= \{x \in E : \Pi_{\mathcal{Q}}(x) \in [x_1, x_r]\}\quad,
\end{aligned}$$

and proceed to estimate the integrals

$$\mathcal{E}_i(\sigma) := \int_{E_i(\sigma)} v\langle Du, D\psi_\sigma\rangle dx \qquad i = 1, \dots, 4\,.$$

To find an upper bound for $\mathcal{E}_1(\sigma)$, observe that $|E_1(\sigma)| \leq \sigma|y_1 - x_1|$ and $D\psi_\sigma = -e_1/\sigma$ on $E_1(\sigma)$. Therefore, recalling that $Du(x_0) = e_2$,

$$
\begin{aligned}
|\mathcal{E}_1(\sigma)| = \frac{1}{\sigma}\Big|\int_{E_1(\sigma)} v'\langle Du, e_1\rangle dx\Big| &= \frac{1}{\sigma}\Big|\int_{E_1(\sigma)} v'\langle Du - Du(x_0), e_1\rangle dx\Big| \\
&\leq \frac{1}{\sigma}\,|E_1(\sigma)|\,\|v\|_{\infty,\mathcal{Q}_\sigma}\|Du - Du(x_0)\|_{\infty,E_1(\sigma)} \\
&\leq |y_1 - x_1|\,\|v\|_{\infty,\mathcal{Q}_\sigma}\|Du - Du(x_0)\|_{\infty,E_1(\sigma)}. \qquad (3.1.36)
\end{aligned}
$$

Moreover, since u is continuously differentiable at every point $x \in]y_0, x_0[$ and satisfies $Du(x) = Du(x_0)$, we have

$$
\omega_1(\sigma) := \|Du - Du(x_0)\|_{\infty,E_1(\sigma)} \to 0 \quad \text{as} \quad \sigma \downarrow 0\,.
$$

Similarly,

$$
|\mathcal{E}_2(\sigma)| \leq |y_r - x_r|\,\|v'\|_{\infty,\mathcal{Q}_\sigma}\,\omega_2(\sigma)\,, \qquad (3.1.37)
$$

where $\omega_2(\sigma) := \|Du - Du(x_0)\|_{\infty,E_2(\sigma)} \to 0$ as $\sigma \downarrow 0$. Next, to bound $\mathcal{E}_3(\sigma)$ we note that $E_3(\sigma) \subset B_\rho(y_0)$ for $\sigma > 0$ small enough. So, in view of (3.1.33), $|\mathcal{E}_3(\sigma)| \leq \varepsilon|E_3(\sigma)|/\sigma$ because $|D\psi_\sigma| \leq 1/\sigma$ and $|Du| \leq 1$. Since $|E_3(\sigma)| \leq 2\sigma(|y_1 - y_r| + 2\sigma)$, we finally get the estimate

$$
|\mathcal{E}_3(\sigma)| \leq 2\varepsilon(|y_1 - y_r| + 2\sigma)\,. \qquad (3.1.38)
$$

The reasoning we need to estimate $\mathcal{E}_4(\sigma)$ is just slightly longer than the previous ones. Let us split $E_4(\sigma)$ in two parts, $E_4'(\sigma)$ and $E_4''(\sigma)$, where

$$
\begin{aligned}
E_4'(\sigma) &= \{x \in E_4(\sigma) : \Pi_{\mathcal{Q}}(x) \in]x_1, x_r[\} \\
E_4''(\sigma) &= \{x \in E_4(\sigma) : \Pi_{\mathcal{Q}}(x) \in \{x_1, x_2\}\}\,.
\end{aligned}
$$

By choosing $\sigma > 0$ so small that $E_4(\sigma) \subset B_\rho(x_0)$, we have $|Du - e_2| \leq 1/2$ a.e. in $E_4(\sigma)$ owing to (3.1.31). Therefore,

$$
\langle Du, D\psi_\sigma\rangle \leq \langle e_2, D\psi_\sigma\rangle + \frac{1}{2\sigma} \leq -\frac{1}{2\sigma} \quad \text{a. e. in} \quad E_4'(\sigma)
$$

because, on such a set, $D\psi_\sigma = -e_2/\sigma$. Now, by (3.1.32),

$$\begin{aligned}\mathcal{E}_4(\sigma) &\le \int_{E_4'(\sigma)} v\langle Du, D\psi_\sigma\rangle\,dx + \frac{|E_4''(\sigma)|}{\sigma}\,\|v\|_{\infty,\mathcal{Q}_\sigma} \\ &\le -\frac{1}{2\sigma}\,\frac{v(x_0)}{2}|E_4''(\sigma)| + 2\pi\sigma\|v\|_{\infty,\mathcal{Q}_\sigma} \\ &\le -\frac{v(x_0)}{4}\,|x_1 - x_r| + 2\pi\sigma\|v\|_{\infty,\mathcal{Q}_\sigma} \qquad (3.1.39)\end{aligned}$$

Now, plugging estimates (3.1.36), (3.1.37), (3.1.38) and (3.1.39) into (3.1.35), we obtain

$$\begin{aligned}0 \le \int_\Omega f\psi_\sigma\,dx &\le 2\varepsilon\Big(|y_1 - y_r| + 2\sigma\Big) - \frac{v(x_0)}{4}\,|x_1 - x_r| \\ &+\|v\|_{\infty,\mathcal{Q}_\sigma}\Big[\,|y_1 - x_1|\omega_1(\sigma) + |y_r - x_r|\omega_2(\sigma) + 2\pi\sigma\,\Big]\end{aligned}$$

Hence, letting $\sigma \downarrow 0$,

$$0 \le 2\varepsilon|y_1 - y_r| - \frac{v(x_0)}{4}|x_1 - x_r|\;. \qquad (3.1.40)$$

Since, owing to (3.1.34)(i), $|Du(x_r) - Du(x_0)| \le \varepsilon r + \|L\||x_r - x_0|$, we have

$$\begin{aligned}|y_1 - y_r| &= \Big|\,x_r - |x_0 - y_1|Du(x_r) - \Big(x_0 - |x_0 - y_1|Du(x_0)\Big)\,\Big| \\ &\le |x_0 - y_1|\,(\varepsilon r + \|L\||x_r - x_0|) + |x_r - x_0|\end{aligned}$$

But $|x_r - x_0| \le r$ and, by (3.1.34)(iii), $|x_1 - x_r| \ge r/2$. So,

$$\varepsilon r + \|L\||x_r - x_0| \le 2(\varepsilon + \|L\|)|x_1 - x_r|$$

and

$$|y_1 - y_r| \le 2\,\left[\,1 + |x_0 - y_1|\,(\varepsilon + \|L\|)\,\right]\,|x_1 - x_r|\;. \qquad (3.1.41)$$

Combining (3.1.40) and (3.1.41), we obtain

$$0 \le \Big\{4\varepsilon\Big[\,1 + |x_0 - y_1|\,(\varepsilon + \|L\|)\Big] - \frac{v(x_0)}{4}\,\Big\}|x_1 - x_r|\;,$$

which is in contrast with (3.1.30). We have reached a contradiction assuming that $u(x_0) < d(x_0)$. So, $u \equiv d$ and the proof is complete. $\square$

Our next task is to show that v is given by the representation formula (2.4.1). We will do this in the next two propositions: the first one computes v away from the singular set, the second one on $\overline{\Sigma}$.

Proposition 3.1.11. *Let* (d, v) *be a solution of system* (3.0.1)*. Then, for any* $z_0 \in \Omega \setminus \overline{\Sigma}$ *and* $\theta \in (0, \tau(z_0))$*, we have*

$$v(z_0) - \frac{1-(d(z_0)+\theta)\kappa(z_0)}{1-d(z_0)\kappa(z_0)} v(z_0+\theta Dd(z_0)) \\ = \int_0^\theta f(z_0+tDd(z_0)) \frac{1-(d(z_0)+t)\kappa(z_0)}{1-d(z_0)\kappa(z_0)} dt\,.$$

Proof—Let $z_0 \in \Omega \setminus \overline{\Sigma}$, $\theta \in (0, \tau(z_0))$ and set $x_0 = z_0 + \theta Dd(z_0)$. Notice that $[z_0, x_0] \subset \Omega \backslash \overline{\Sigma}$ and $Dd(z) = Dd(z_0)$ for $z \in [z_0, x_0]$ by Proposition 1.1.3. Let us use–once again–a coordinate system that simplifies the notation: we set $e_2 = Dd(z_0)$ and choose e_1 such that $\{e_1, e_2\}$ is a positively oriented orthonormal basis of $\mathbb{R}^2$. Also, fix $r > 0$ so small that $x_r := x_0 + re_1 \notin \overline{\Sigma}$ and $\langle Dd(x_r), e_2\rangle > 0$. Let then $\bar{t} > 0$ be such that the point $z_r := x_r - \bar{t}Dd(x_r)$ satisfies $\langle z_r - z_0, e_2\rangle = 0$. We note that $\bar{t}$ is given by

$$\bar{t} = \frac{\langle x_r - z_0, e_2\rangle}{\langle Dd(x_r), e_2\rangle} = \frac{|x_0 - z_0|}{\langle Dd(x_r), e_2\rangle}\,. \tag{3.1.42}$$

Finally, let us possibly reduce $r > 0$ in order to ensure that $D_r := \overline{\text{co}}\,\{x_0, x_r, z_r, z_0\}$ is contained in $\Omega \backslash \overline{\Sigma}$ and d be of class $\mathcal{C}^{2,1}$ in a neighborhood of D_r.

Integrating by parts the equation $-\text{div}\,(vDd) = f$ on D_r, we obtain

$$\int_{D_r} f dx = -\int_{\partial D_r} v\langle Dd, \nu\rangle d\mathcal{H}^1 \tag{3.1.43}$$

where ν is the unit outward normal to ∂D_r. The above right-hand side amounts to

$$\int_{\partial D_r} v\langle Dd, \nu\rangle d\mathcal{H}^1 = \int_{[x_0,x_r]} v\langle Dd, e_2\rangle d\mathcal{H}^1 + \int_{[z_0,z_r]} v\langle Dd, -e_2\rangle d\mathcal{H}^1 \tag{3.1.44}$$

because

$$\int_{[z_0,x_0]} v\langle Dd, \nu\rangle d\mathcal{H}^1 = \int_{[z_0,x_0]} v\langle e_2, -e_1\rangle d\mathcal{H}^1 = 0$$

and, similarly, $\langle Dd, \nu\rangle = 0$ on $[z_r, x_r]$. Moreover, we have

$$\int_{D_r} f dx = \int_0^{|z_0-x_0|} dt \int_0^{l_t} f(z_0 + te_2 + se_1) ds \tag{3.1.45}$$

where

$$l_t = \Big(1 - \frac{t}{|z_0 - x_0|}\Big)|z_0 - z_r| + \frac{t}{|z_0 - x_0|}|x_0 - x_r|\,.$$

Our next step will be to compute $\lim_{r\downarrow 0}\frac{1}{r}\int_{D_r} f dx$. Aiming at this, let us recall that, in view of Proposition 1.1.12,

$$D^2 d(x_0) = \gamma_0 (e_1 \otimes e_1) \quad \text{ where } \quad \gamma_0 = -\frac{\kappa(x_0)}{1-\kappa(x_0)d(x_0)} .$$

Hence,

$$\frac{1}{r}\,\frac{\langle Dd(x_r), e_1\rangle}{\langle Dd(x_r), e_2\rangle} = \frac{1}{r}\,\frac{\langle Dd(x_0) + rD^2 d(x_0)e_1 + \mathrm{o}(r), e_1\rangle}{\langle Dd(x_0) + rD^2 d(x_0)e_1 + \mathrm{o}(r), e_2\rangle} = \frac{\gamma_0 + \varepsilon(r)}{1+\varepsilon(r)}$$

where $\varepsilon(r) \to 0$ as $r \downarrow 0$. Since

$$\frac{|z_0 - z_r|}{r} = 1 - |x_0 - z_0|\,\frac{1}{r}\,\frac{\langle Dd(x_r), e_1\rangle}{\langle Dd(x_r), e_2\rangle} = 1 - |x_0 - z_0|\,\frac{\gamma_0 + \varepsilon(r)}{1+\varepsilon(r)}\,, \qquad (3.1.46)$$

we obtain

$$\begin{aligned} \lim_{r\to 0^+}\frac{l_t}{r} &= \Big(1 - \frac{t}{|z_0 - x_0|}\Big)(1 - \gamma_0|x_0 - z_0|) + \frac{t}{|z_0 - x_0|} \\ &= 1 - \gamma_0|x_0 - z_0| + t\gamma_0 \end{aligned}$$

Therefore, in view of (3.1.45), we conclude that

$$\lim_{r\downarrow 0}\frac{1}{r}\int_{D_r} f dx = \int_0^{|z_0 - x_0|} f(z_0 + te_2)\Big(1 - \gamma_0|x_0 - z_0| + t\gamma_0\Big) dt\,. \qquad (3.1.47)$$

We now turn to the evaluation of $\lim_{r\downarrow 0}\frac{1}{r}\int_{\partial D_r} v\langle Dd, \nu\rangle$. Since Dd is continuous at x_0 and $Dd(x_0) = e_2$, we have

$$\lim_{r\downarrow 0}\frac{1}{r}\int_{[x_0, x_r]} v\langle Dd, e_2\rangle d\mathcal{H}^1 = v(x_0)\,.$$

A similar continuity argument and (3.1.46) show that

$$\lim_{r\downarrow 0}\frac{1}{r}\int_{[z_0, z_r]} v\langle Dd, -e_2\rangle d\mathcal{H}^1 = -v(z_0)(1 - \gamma_0|x_0 - z_0|)\,.$$

Then, recalling (3.1.43), (3.1.44) and (3.1.47), we conclude that

$$\begin{aligned} \lim_{r\downarrow 0} -\frac{1}{r}\int_{\partial D_r} v\langle Dd, \nu\rangle d\mathcal{H}^1 &= v(z_0)(1 - \gamma_0|x_0 - z_0|) - v(x_0) \\ &= \int_0^{|z_0 - x_0|} f(z_0 + te_2)\Big(1 - \gamma_0|x_0 - z_0| + t\gamma_0\Big) dt\,, \end{aligned}$$

whence, since $|z_0 - x_0| = \theta$,

$$v(z_0) - \frac{v(x_0)}{1-\gamma_0\theta} = \int_0^\theta f(z_0 + te_2)\Big(1 + \frac{t\gamma_0}{1-\gamma_0\theta}\Big)dt\,. \tag{3.1.48}$$

Finally, recalling the definition of γ_0 and using the equality $d(x_0) = d(z_0)+\theta$, we have

$$1 - \gamma_0\theta = 1 + \frac{\kappa(x_0)\theta}{1-d(x_0)\kappa(x_0)} = \frac{1-d(z_0)\kappa(x_0)}{1-d(x_0)\kappa(x_0)}$$

and

$$\frac{\gamma_0}{1-\gamma_0\theta} = -\frac{\kappa(x_0)}{1-d(z_0)\kappa(x_0)}\,.$$

In view of the above identities and of the fact that $\kappa(x_0) = \kappa(z_0)$, (3.1.48) can be recasted as

$$v(z_0) - \frac{1-d(x_0)\kappa(z_0)}{1-d(z_0)\kappa(z_0)}v(x_0) = \int_0^\theta f(z_0+te_2)\frac{1-(d(z_0)+t)\kappa(z_0)}{1-d(z_0)\kappa(z_0)}dt\,.$$

The last formula yields the conclusion. $\square$

Proposition 3.1.12. *If* (d, v) *is a solution of system* (3.0.1)*, then* $v = 0$ *on* $\overline{\Sigma}$*.*

Proof—Let us assume, first, that Σ is a singleton, say $\{x_0\}$. Then, by a classical result of Motzkin's [34] (see also Remark 1.2.15), Ω is the disk $B_R(x_0)$ with $R = d(x_0)$. Also $\mathcal{C} = \emptyset$. Integrating the equation $-\mathrm{div}\,(vDd) = f$ on $B_r(x_0)$, for $0 < r < R$, gives

$$\int_{B_r(x_0)} f dx = -\int_{\partial B_r(x_0)} v\langle Dd, \nu\rangle d\mathcal{H}^1\,,$$

where ν is the unit outward normal to $\partial B_r(x_0)$. Since $\langle Dd, \nu\rangle = -1$, we have

$$0 = \lim_{r\downarrow 0}\frac{1}{r}\int_{B_r(x_0)} f dx = \lim_{r\downarrow 0}\frac{1}{r}\int_{\partial B_r(x_0)} v d\mathcal{H}^1 = 2\pi v(x_0)\,.$$

Thus, $v(x_0) = 0$. Suppose, next, that Σ is not a singleton. Then, again by Remark 1.2.15, the set Σ^1 of singular points with magnitude 1 is dense in Σ. Since v is continuous, it suffices to prove that v vanishes on Σ^1. So, suppose that $x_0 \in \Sigma^1$ and let $D^+d(x_0) = [p_0, q_0]$ with $p_0 \neq q_0$. Then, by Proposition 1.2.16, there exists a Lipschitz arc $\zeta : [0, \eta] \to \Sigma$ such that $\zeta(0) = x_0$, $\dot\zeta(0) \neq 0$, and

$$\langle\dot\zeta(0), p_0 - q_0\rangle = 0\,. \tag{3.1.49}$$

Moreover, $\zeta(s_k) \in \Sigma^1$ for some sequence $s_k \downarrow 0$, and

$$D^+d(\zeta(s_k)) = [p_k, q_k] \quad \text{with} \quad p_k \to p_0\,,\ q_k \to q_0\,. \tag{3.1.50}$$

Since Σ has Lebesgue measure zero, we have, by Fubini's Theorem,

$$\mathcal{H}^1\Big([x_0 - \alpha s_k p_0, \zeta(s_k) - \alpha s_k p_k] \cap \Sigma\Big) = 0 \quad \text{for a. e. } \alpha \in [1,2]\,,$$

provided k is sufficiently large. Let $\alpha_k \in [1,2]$ be such that

$$\mathcal{H}^1\Big([x_0 - \alpha_k s_k p_0, \zeta(s_k) - \alpha_k s_k p_k] \cap \Sigma\Big) = 0\,.$$

In the same way, let $\beta_k \in [1,2]$ be such that

$$\mathcal{H}^1\Big([x_0 - \beta_k s_k q_0, \zeta(s_k) - \beta_k s_k q_k] \cap \Sigma\Big) = 0\,.$$

Let us set, for every $k \in \mathbb{N}$,

$$I_p^k :=]x_0 - \alpha_k s_k p_0, \zeta(s_k) - \alpha_k s_k p_k[\,, \quad I_q^k :=]x_0 - \beta_k s_k q_0, \zeta(s_k) - \beta_k s_k q_k[\,,$$

and let us denote by $\overline{I}_p^k$ (resp. $\overline{I}_q^k$) the closure of I_p^k (resp. I_q^k). Now, for $k \in \mathbb{N}$ large enough define the domain

$$D_k := \overline{\mathrm{co}}\,([x_0, \zeta(s_k)] \cup \overline{I}_p^k) \cup \overline{\mathrm{co}}\,([x_0, \zeta(s_k)] \cup \overline{I}_p^k)$$

and consider, for $\sigma > 0$, the function

$$\psi_\sigma^k(x) = \Big[1 - \frac{1}{\sigma} d_{D_k}(x)\Big]_+ \qquad x \in \Omega\,.$$

Notice that, for k large enough, $\psi_\sigma^k \in W_c^{1,\infty}(\Omega)$. Therefore, using ψ_σ^k as test function for the equation $-\mathrm{div}\,(vDd) = f$, we have

$$\int_\Omega f\psi_\sigma^k dx = \int_\Omega v\langle Dd, D\psi_\sigma^k\rangle dx\ .$$

In order to estimate the right-hand side, observe that the support of $D\psi_\sigma^k$ is given by the closure of the set $A^k(\sigma) := \{x \in \Omega\backslash D_k : d_{D_k}(x) < \sigma\}$. This set can be represented as the disjoint union $A_p^k(\sigma) \cup A_q^k(\sigma) \cup \widetilde{A}^k(\sigma)$, where

$$A_p^k(\sigma) = \{x \in A(\sigma) : \Pi_{D_k}(x) \in I_p^k\}\,, \quad A_q^k(\sigma) = \{x \in A(\sigma) : \Pi_{D_k}(x) \in I_p^k\}\,.$$

Then, the gradient of d_{D_k} is constant on $A_p^k(\sigma)$, say $Dd_{D_k} \equiv \nu_p^k$. Similarly, $Dd_{D_k} \equiv \nu_q^k$ on $A_q^k(\sigma)$. Now, observe that

$$\int_\Omega f\psi_\sigma dx \tag{3.1.51}$$
$$= \int_{A_p^k(\sigma)} \frac{v}{\sigma}\langle Dd, \nu_p^k\rangle dx + \int_{A_q^k(\sigma)} \frac{v}{\sigma}\langle Dd, \nu_q^k\rangle dx + \int_{\widetilde{A}^k(\sigma)} v\langle Dd, D\psi_\sigma^k\rangle dx$$

We will pass to the limit as $\sigma \downarrow 0$ in the above identity. We have

$$\lim_{\sigma\downarrow 0} \int_\Omega f\psi_\sigma^k dx = \int_{D_k} f dx\,.$$

Moreover, arguing as in the proof of Proposition 3.1.10, we find

$$\lim_{\sigma\downarrow 0} \int_{\widetilde{A}^k(\sigma)} v\langle Dd, D\psi_\sigma^k\rangle dx = 0\,.$$

In order to estimate the term

$$\int_{A_p^k(\sigma)} \frac{v}{\sigma}\langle Dd, \nu_p^k\rangle dx = \frac{1}{\sigma}\int_{I_p^k} d\mathcal{H}^1(y)\int_0^\sigma v(y+t\nu_p^k)\langle Dd(y+t\nu_p^k), \nu_p^k\rangle dt\,,$$

recall that $\mathcal{H}^1(I_p^k \cap \Sigma) = 0$, and so Dd is continuous at $\mathcal{H}^1$-almost every point of I_p^k. Therefore,

$$\lim_{\sigma\downarrow 0} \int_{A_p^k(\sigma)} \frac{v}{\sigma}\langle Dd, \nu_p^k\rangle dx = \int_{I_p^k} v(y)\langle Dd(y), \nu_p^k\rangle d\mathcal{H}^1(y)\,.$$

Similarly,

$$\lim_{\sigma\downarrow 0} \int_{A_q^k(\sigma)} \frac{v}{\sigma}\langle Dd, \nu_q^k\rangle dx = \int_{I_q^k} v(y)\langle Dd(y), \nu_q^k\rangle d\mathcal{H}^1(y)\,.$$

Thus, passing to the limit as $\sigma \downarrow 0$ in (3.1.51), we conclude that

$$\int_{D_k} f = \int_{I_p^k} v(y)\langle Dd(y), \nu_p^k\rangle d\mathcal{H}^1(y) + \int_{I_q^k} v(y)\langle Dd(y), \nu_q^k\rangle d\mathcal{H}^1(y)\,. \tag{3.1.52}$$

Our final step will be to divide both sides of (3.1.52) by s_k and to take the limit as $k \to \infty$. For this we need two preliminary remarks. The first one is that, for every sequence $\{y_k\}_k$ such that $y_k \in I_p^k$ and d is differentiable at y_k, $Dd(y_k)$ converges to p_0 as $k \to \infty$. For let $\lambda_k \in [0,1]$ be such that

$$\begin{aligned} y_k &= \lambda_k(x_0 - \alpha_k s_k p_0) + (1-\lambda_k)(\zeta(s_k) - \alpha_k s_k p_k) \\ &= \lambda_k(x_0 - \alpha_k s_k p_0) + (1-\lambda_k)(x_0 + s_k\dot\zeta(0) + \mathrm{o}(s_k) - \alpha_k s_k p_k) \end{aligned}$$

and suppose $\lambda_k \to \lambda^* \in [0,1]$, $\alpha_k \to \alpha^* \in [1,2]$ and $Dd(y_k) \to p^*$ as $k \to \infty$ (which always holds, up to subsequences). Then,

$$\lim_{k\to\infty} \frac{y_k - x_0}{s_k} = -\alpha^* p_0 + (1-\lambda^*)\dot\zeta(0) =: \theta^* \, .$$

But $\min\{\langle p, \theta^*\rangle : p \in D^+d(x_0)\}$ is attained at p_0, since $\langle\dot\zeta(0), p\rangle = \langle\dot\zeta(0), p_0\rangle$ for every $p \in [p_0, q_0]$, in view of (3.1.49). Now, arguing as in Proposition 1.2.16 for the proof of (1.2.25), we deduce from the semiconcavity of d that

$$p^* \in \arg\min_{p\in D^+d(x_0)} \langle p, \theta^*\rangle.$$

Hence $p^* = p_0$, since $p^* \in D^*d(x_0) = \{p_0, q_0\}$ and $q_0 \notin \arg\min_{p\in D^+d(x_0)}\langle p, \theta^*\rangle$. The second remark we need, to proceed with our computation, is that

$$\lim_{k\to\infty} \nu_p^k = -\frac{p_0 - q_0}{|p_0 - q_0|} \, . \tag{3.1.53}$$

Indeed, by definition,

$$\langle \nu_p^k, \zeta(s_k) - \alpha_k s_k p_k - (x_0 - \alpha_k s_k p_0)\rangle = 0$$

where

$$\zeta(s_k) - x_0 + \alpha_k s_k(p_0 - p_k) = s_k\dot\zeta(0) + \mathrm{o}(s_k) \tag{3.1.54}$$

in view of (3.1.50). Thus, ν_p^k is nearly orthogonal to $\dot\zeta(0)$, and so

$$\nu_p^k = \rho_0(p_0 - q_0) + \varepsilon_k \quad \text{with} \quad \lim_{k\to\infty}\varepsilon_k = 0 \quad \text{and} \quad |\rho_0| = \frac{1}{|p_0 - q_0|} \, .$$

Moreover, $\langle \nu_p^k, p_0\rangle \le 0$ for k large enough, because ν_p^k is an outward normal to the set $\overline{\mathrm{co}}\,([x_0, \zeta(s_k)] \cup \overline{I}_p^k)$ at the point $x_0 - \alpha_k s_k p_0$ and x_0 belongs to such a set. Therefore,

$\rho_0 < 0$ and (3.1.53) follows. We are now ready for our final step. Dividing both sides of (3.1.52) by s_k and taking the limit as $k \to \infty$, we obtain

$$0 = \lim_{k\to\infty} \frac{1}{s_k}\Big\{ \int_{I_p^k} v(y)\langle Dd(y), \nu_p^k\rangle d\mathcal{H}^1(y) + \int_{I_q^k} v(y)\langle Dd(y), \nu_q^k\rangle d\mathcal{H}^1(y)\Big\}\,.$$

Since $\mathcal{H}^1(I_p^k) = |\zeta(s_k) - \alpha_k s_k p_k - (x_0 - \alpha_k s_k p_0)| = s_k|\dot\zeta(0)| + \mathrm{o}(s_k)$ on account of (3.1.54), we have

$$\lim_{k\to\infty} \frac{1}{s_k} \int_{I_p^k} v(y)\langle Dd(y), \nu_p^k\rangle d\mathcal{H}^1(y) = -v(x_0)|\dot\zeta(0)|\Big\langle p_0, \frac{p_0 - q_0}{|p_0 - q_0|}\Big\rangle\,.$$

By a similar argument,

$$\lim_{k} \frac{1}{s_k} \int_{I_q^k} v(y)\langle Dd(y), \nu_q^k\rangle d\mathcal{H}^1(y) = v(x_0)|\dot\zeta(0)|\Big\langle q_0, \frac{p_0 - q_0}{|p_0 - q_0|}\Big\rangle\,.$$

Thus,

$$\begin{aligned} 0 &= v(x_0)|\dot\zeta(0)|\Big\{ -\Big\langle p_0, \frac{p_0 - q_0}{|p_0 - q_0|}\Big\rangle + \Big\langle q_0, \frac{p_0 - q_0}{|p_0 - q_0|}\Big\rangle\Big\} \\ &= -v(x_0)|\dot\zeta(0)|\,|p_0 - q_0| \end{aligned}$$

Since $\dot\zeta(0) \neq 0$ and $p_0 \neq q_0$, we have finally obtained that $v(x_0) = 0$. $\square$

We are now ready to complete the proof of our main result.

Proof of Theorem 2.4.2[Part 2: Uniqueness]—Let (u, v) is a solution of system (3.0.1). Then, $u \equiv d$ in $\Omega_v := \{x \in \Omega : v(x) > 0\}$ by Proposition 3.1.10. In particular, (d, v) is also a solution of (3.0.1). So, owing to Proposition 3.1.12, $v = 0$ on $\overline{\Sigma}$. Now, let $x \in \Omega \setminus \overline{\Sigma}$. In view of Proposition 3.1.11, we have

$$\begin{aligned} &v(x) - \frac{1 - (d(x) + \theta)\kappa(x)}{1 - d(x)\kappa(x)}\, v(x + \theta Dd(x)) \\ &\qquad = \int_0^\theta f(x + tDd(x))\,\frac{1 - (d(x) + t)\kappa(x)}{1 - d(x)\kappa(x)}\, dt \end{aligned}$$

for each $\theta \in (0, \tau(x))$. Since v is continuous and vanishes on $\overline{\Sigma}$, the left-hand side above converges to $v(x)$ as $\theta \uparrow \tau(x)$. So, $v(x)$ coincides with $v_f(x)$, given by (2.4.1), and the proof is complete. $\square$

3.2 Analytic Boundaries: Regularity results

We have seen in Section 3.1.1 that if $\partial\Omega$ is a piecewise $\mathcal{C}^{2,1}$ boundary with outer corners, then τ has the following properties:

(i) τ is continuous in $\overline{\Omega}$;

(ii) τ is Lipschitz continuous on $\partial\Omega$;

(iii) τ is locally Lipschitz continuous in $\Omega \setminus \overline{\Sigma}$.

Also, the regularity of τ affects the regularity of v_f too, since it appears in its representation formula (3.1.22). Our aim is to show that under suitable hypotheses on $\partial\Omega$ and on f, then τ and v_f are Hölder continuous functions in $\overline{\Omega}$. In our analysis we will exclude the case when Ω is a disk. For suppose that $\Omega = B_R(0)$, for some $R > 0$. Then τ is trivially Lipschitz continuous in $\overline{B}_R(0)$, since $\tau(x) = |x|$ for $x \in \overline{B}_R(0)$. The same can be proven for the map v_f (see Section 3.2.3). But in general, if Ω is not a disk, Lipschitz continuity may fail, even if the boundary is very smooth, as the next example shows.

Example 3.2.1 (The parabola case). In the cartesian plane consider the set

$$\Omega := \{(x, y) \in \mathbb{R}^2 \,:\, y > x^2\},$$

whose boundary is a parabola with vertex $(0, 0)$. The graph of the map $s \mapsto s^2$ is a regular parametrization of the boundary and the vector

$$\nu(s) = \frac{1}{\sqrt{1+4s^2}} \begin{pmatrix} -2s, \\ 1 \end{pmatrix}$$

is the inward unit normal to $\partial\Omega$ at the point (s, s^2). By the symmetry of $\partial\Omega$ with respect to the vertical axis we deduce that $\overline{\Sigma}$ must be contained in such an axis. Moreover, an easy calculation shows that for any $s \neq 0$, the line through (s, s^2) with direction $\nu(s)$ intersects the vertical axis in the point $(0, s^2 + 1/2)$. Hence,

$$\overline{\Sigma} = \{(0, y) \,:\, y \geq 1/2\}$$

and

$$\tau((s,s^2)) = \frac{1}{2}\sqrt{1+4s^2}.$$

Taking into account that the curvature at the point (s,s^2) is given by

$$\kappa((s,s^2)) = \frac{2}{(1+4s^2)^{3/2}},$$

we deduce that the unique conjugate point of Ω is $(0,1/2)$, which is also regular.
Let us prove that the map τ cannot be Lipschitz continuous in the whole set Ω by showing that for any a small enough

$$|\tau((a,1/2)) - \tau((0,1/2))| \geq M|(a,1/2)-(0,1/2)|^{2/3},$$

for some constant $M > 0$.
For any fixed $a \in (-\sqrt{2}/2, 0) \cup (0, \sqrt{2}/2)$, the unique projection on the boundary of $(a,1/2)$ is the point (s_a, s_a^2) where s_a satisfies $s_a = \frac{a^{1/3}}{2^{1/3}}$. Indeed, for any s, the line through (s,s^2) with direction $\nu(s) = Dd((s,s^2))$ has equation

$$y - s^2 = -\frac{1}{2s}(x-s).$$

Hence, $(a,1/2)$ belongs to this line if and only if $s^2 = \frac{a}{2s}$, i.e. $s_a = \frac{a^{1/3}}{2^{1/3}}$. We deduce that

$$\begin{aligned}
\tau((a,1/2)) &= |(a,1/2)-(0,s_a^2+1/2)| = \left(a^2 + \frac{a^{4/3}}{2^{4/3}}\right)^{1/2} \\
&= a^{2/3}\left(a^{2/3} + \frac{1}{2^{4/3}}\right)^{1/2} \geq \frac{1}{2^{2/3}}\, a^{2/3},
\end{aligned}$$

which proves the claim.

This example shows that even in the case of analytic boundaries, τ cannot be Lipschitz continuous around a regular conjugate point. Indeed, as we will see in Theorem 3.2.6, the only obstruction to Lipschitz regularity is the presence of conjugate points. On the other hand, such points necessarily occur in the case of simply connected domains with analytic boundary.

In what follows our standing assumption will be that

$$\Omega \text{ is a simply connected domain in } \mathbb{R}^2, \\ \text{with analytic boundary, different from a disk.}$$

The main motivation for this strong requirement on $\partial\Omega$ is that the knowledge of the structure of $\overline{\Sigma}$ is essential in the analysis of the regularity of the maximal retraction length τ, and only in the case of analytic boundaries a complete description is available.

3.2.1 The Cut Locus of Analytic Sets

In this section we collect together some known and new results on $\overline{\Sigma}$ in the case of analytic boundary $\partial\Omega$. Before starting our analysis we have to introduce some more definitions.

Definition 3.2.2. We call a geometric graph any closed connected set which consists of a finite number of disjoint vertices and edges, where a vertex is a point in $\mathbb{R}^2$ and an edge is a regular curve with finite length whose limits of tangents at the end points exist.

Also, let us introduce singular analogous of the set Γ, that is

$$\tilde{\Gamma} = \{x \in \Sigma \ : \ d(x)\kappa(y) = 1 \ \text{ for some } y \in \Pi(x)\,\} \ . \tag{3.2.1}$$

The following result can be deduced from [17] or [33].

Proposition 3.2.3. *Let Ω be a bounded simply connected domain with analytic boundary, different from a disk. Then Γ, $\tilde{\Gamma}$ and Σ^2 are finite sets and Γ is nonempty. Moreover, $\overline{\Sigma}$ is a geometric graph. The edges of the graph are real analytic curves and the vertices are precisely the points of $\Gamma \cup \tilde{\Gamma} \cup \Sigma^2$. The number of analytic arcs emanating from a vertex equals the number of projections of the vertex on the boundary.*

By Proposition 1.2.16 we also know how singular arcs propagate from singular points. In order to complete the knowledge of $\overline{\Sigma}$ we need a description of the propagation of singularities from regular conjugate points as well. We have the following result.

Lemma 3.2.4. *Let Ω be a bounded domain with analytic boundary and x_0 be a regular conjugate point of the distance function. Then the analytic singular arc propagating from x_0 coincides in a suitable neighborhood of x_0 with the unique solution of the differential inclusion*

$$\begin{cases} \dot{\zeta}(s) \in D^+ d(\zeta(s)) \\ \zeta(0) = x_0 \end{cases} \tag{3.2.2}$$

Proof– For any starting point $x_0 \in \Omega$, the existence of a global solution of (3.2.2) is a classical result in the theory of differential inclusions. The solution of (3.2.2) is at least Lipschitz continuous as a consequence of the inclusion $D^+d(x) \subseteq \overline{B}_1(0)$ for any $x \in \Omega$. The fact that such a solution is unique is a consequence of the semiconcavity property of the distance function. Indeed, if ζ_i, for $i = 1, 2$ are two distinct solutions of (3.2.2), then we can apply Proposition 1.2.8 (a) twice to get

$$\langle \dot{\zeta}_1(s) - \dot{\zeta}_2(s), \zeta_1(s) - \zeta_2(s) \rangle \leq C|\zeta_1(s) - \zeta_2(s)|^2$$

for all s sufficiently small, being C the semiconcavity constant of d in a neighborhood of x_0. This means that

$$\frac{1}{2}\frac{d}{ds}|\zeta_1(s) - \zeta_2(s)|^2 \leq |\zeta_1(s) - \zeta_2(s)|^2$$

and the coincidence of ζ_1 and ζ_2 follows from Gronwall's Lemma. So let us denote by $\zeta(\cdot)$ the unique global Lipschitz solution of (3.2.2) with $x_0 \in \Gamma$. We will first prove that, at least for small times, arc $\zeta(\cdot)$ cannot consist of regular points only. Indeed, if there exists $s_0 > 0$ such that $\zeta(s) \notin \Sigma$ for any $s \in (0, s_0)$, the differential inclusion reduces to the equation $\dot{\zeta}(s) = Dd(\zeta(s))$ for $s \in (0, s_0)$. Moreover, being Γ finite in the case of analytic boundary, we can suppose that $\zeta(s) \notin \Gamma$ for all $s \in (0, s_0)$. Hence differentiating the equation we obtain

$$\ddot{\zeta}(s) = D^2 d(\zeta(s))\dot{\zeta}(s) = D^2 d(\zeta(s)) Dd(\zeta(s)) = 0, \qquad s \in (0, s_0).$$

But then $\zeta(s) = x_0 + sDd(x_0)$, $Dd(\zeta(s)) = Dd(x_0)$ for any $s \in (0, s_0)$ and we have $d(\zeta(s)) = d(x_0) + s = 1/\kappa(x_0) + s$, i.e.

$$d(\zeta(s))\kappa(\zeta(s)) = 1 + s\kappa(x_0) > 1,$$

against Proposition 1.1.4. Hence we have proven that there exists a sequence $\{s_k\}$ converging to 0 such that $\zeta(s_k)$ is singular. From the upper semicontinuity of the superdifferential $D^+d(\cdot)$ we get that there exists $\delta > 0$ such that $D^+d(x) \subseteq Dd(x_0) + \frac{1}{2}B_1(0)$ for any $x \in B_\delta(x_0)$; since $|\zeta(s_k) - x_0| \leq s_k$ we deduce that $0 \notin D^+d(\zeta(s_k))$ for k sufficiently large. In light of Proposition 1.2.16 we then have that $\zeta(\cdot)$ is locally singular around each point $\zeta(s_k)$. For any k set

$$\sigma_k := \sup\{t \geq 0 \ : \ \zeta(s_k + t) \in \Sigma \cap B_\delta(x_0)\}.$$

In order to complete the proof we need to show that σ_k does not shrink to 0 as $k \to \infty$. So, suppose by contradiction that $\sigma_k \to 0$ as $k \to \infty$. By definition, $\zeta(s_k + \sigma_k)$ is either a regular conjugate point or $|\zeta(s_k + \sigma_k) - x_0| = \delta$. In the latter case, $\delta = |\zeta(s_k + \sigma_k) - x_0| \leq s_k + \sigma_k$ and then $\sigma_k \geq \delta/2$ for k large. The former case is excluded by Theorem 3.2.3, because the number of regular conjugate points is finite in the case of analytic boundary.
Hence, we have found a singular Lipschitz arc propagating from x_0; such an arc must coincide with the unique analytic arc with vertex x_0 given by Proposition 3.2.3. □

Remark 3.2.5. Collecting together the previous results, we can say that if $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain with analytic boundary, different from a disk, then for any $x_0 \in \overline{\Sigma}$ there exist exactly m analytic singular arcs starting from x_0, where m is the number of elements of $D^*d(x_0)$, say $D^*d(x_0) = \{p_1, \ldots, p_m\}$. When x_0 is singular, the initial directions of these arcs are given by the opposite of the unit outward normal vectors to the exposed faces of $D^+d(x_0)$. More precisely, for any $p_i \neq p_j$ such that $[p_i, p_j] \subset \partial D^+d(x_0)$ let n_{ij} be defined by

$$\max_{p \in D^+d(\bar{x})} \langle p, n_{ij} \rangle = \langle p_i, n_{ij} \rangle = \langle p_j, n_{ij} \rangle.$$

Then, $-n_{ij}$ gives the initial direction of a singular arc emanating from x_0. In the case when x_0 is regular and conjugate, the initial direction of the unique singular arc starting from x_0 is $Dd(x_0)$. Moreover, being Σ^2 finite, any analytic singular arc ζ starting from a point $x_0 \in \overline{\Sigma}$ is locally made of points of Σ^1 only. Hence,

$D^+d(\zeta(s)) = [p(s), q(s)]$, with $p(s),\ q(s) \in D^*d(\zeta(s))$. Also, when $x_0 \in \Sigma$, Proposition 1.2.16 gives that there exist $\delta_0 > 0$ and $s_0 > 0$ such that

$$\operatorname{diam}(D^+d(\zeta(s)) = |p(s) - q(s)| \geq \delta_0, \qquad \forall s \in (0, s_0).$$

Finally, as a consequence of the fact that $\Gamma \cup \tilde{\Gamma}$ is finite, we deduce the following property. For any $x_0 \in \Gamma \cup \tilde{\Gamma}$, let $\mathcal{S}(x_0)$ be the line segment $[x_0, x_0 - d(x_0)p_0]$, where $p_0 = Dd(x_0)$ if x_0 is a regular point and $p_0 \in D^*d(x_0)$ satisfies $d(x_0)\kappa(x_0 - d(x_0)p_0) = 1$ if x_0 is singular. Then, there exists an open cone $\mathcal{C}_0$, with apex x_0 and symmetry axis containing the segment $\mathcal{S}(x_0)$ such that $\mathcal{C}_0 \cap \Sigma = \emptyset$. Such a property will be useful in the sequel when we study the behaviour of τ and of v_f, respectively, on the sets

$$\Sigma \cup \left(\bigcup_{x_0 \in \Gamma} \mathcal{S}(x_0) \right), \qquad \Sigma \cup \left(\bigcup_{x_0 \in \tilde{\Gamma} \cup \Gamma} \mathcal{S}(x_0) \right).$$

3.2.2 Regularity of the Maximal Retraction Length of Ω

The main result of this section is the proof of the Hölder continuity of τ in the whole set Ω. We already know that τ is locally Lipschitz continuous in $\Omega \setminus \overline{\Sigma}$, but, as the next theorem will show, the (local) Lipschitz constant of τ explodes near the set of regular conjugate points. On the other hand, Example 3.2.1 gives an indication on how things go in the case of a regular boundary, suggesting the idea that a local analysis near the set of regular conjugate points can produce the right estimates to get the Hölder regularity of τ.

The formal proof turns out to be surprisingly long, so that it must be divided in several steps. As a first step, we will provide an estimate for the local Lipschitz constant of τ in the set $\Omega \setminus \mathcal{S}$, where

$$\mathcal{S} := \bigcup_{x_0 \in \Gamma} \{x_0 - tDd(x_0)\ :\ t \in [0, d(x_0)]\}.$$

Afterwards, we will compare it with the distance to the set $\Sigma \cup \mathcal{S}$. This is the crucial part of the proof, where we will make use of local coordinates for the boundary $\partial\Omega$

and the set $\overline{\Sigma}$. At the very end, we will be able to conclude the Hölder continuity of τ by means of a simple regularity lemma.
Let us start with the local Lipschitz estimate. We recall that our standing assumption is the following:

$$\Omega \text{ is a simply connected bounded domain of } \mathbb{R}^2$$
$$\text{with analytic boundary, different from a disk.}$$

We will omit the above assumption in the sequel.

Theorem 3.2.6. *Set* $\mathcal{S} := \bigcup_{x_0 \in \Gamma} \{x_0 - tDd(x_0) \ : \ t \in [0, d(x_0)]\}$. *Then, for any* $x \in \Omega \setminus (\Sigma \cup \mathcal{S})$ *there exists an open ball* $B_r(x)$, $r = r(x) > 0$, *such that for all* $y \in B_r(x)$

$$\tau(y) \leq \tau(x) + C(x)|x - y|, \tag{3.2.3}$$

where

$$C(x) = 2\left(1 + \frac{1 - (d(x) + \tau(x))\kappa(x)}{1 - d(x)\kappa(x)} \frac{1}{\delta(\bar{x})}\right), \tag{3.2.4}$$

$\bar{x} = x + \tau(x)Dd(x)$ *is the singular point corresponding to* x *and*

$$\delta(\bar{x}) = \min\left\{|p - q| \ : \ p, q \in D^*d(\bar{x}), \ [p, q] \subseteq \partial D^+ d(\bar{x})\right\}. \tag{3.2.5}$$

Proof– Fix any $x \in \Omega \setminus (\Sigma \cup \mathcal{S})$ and set $\bar{x} = x + \tau(x)Dd(x)$. Moreover, let $e_2 = Dd(x)$ and e_1 such that $\{e_1, e_2\}$ is a positively oriented orthonormal basis of $\mathbb{R}^2$. We will first prove the theorem in the case $\bar{x} \in \Sigma^1$. Under this assumption, there exists a limiting gradient p such that $D^+d(\bar{x}) = [e_2, p]$. Moreover, by Theorem 3.2.3 and Proposition 1.2.16 there exists an analytic singular arc (except maybe for the point $\bar{x}$), say $\zeta(\cdot)$, passing through $\bar{x}$ with direction $-n$, where n is defined by

$$\max_{q \in D^+d(\bar{x})} \langle q, n \rangle = \langle p, n \rangle = \langle e_2, n \rangle$$

and $\langle n, e_1 \rangle < 0$. As a matter of fact, there are exactly two nonzero vectors satisfying the previous equality, both orthogonal to $p - e_2$ and then having opposite direction. So, let us locally represent the arc $\zeta(\cdot)$ above as

$$\zeta(s) = \bar{x} - ns + \mathrm{o}(s), \tag{3.2.6}$$

where $s \in (-s_0, s_0)$ and $s_0 > 0$. Now, take $r > 0$ sufficiently small such that the ball $B_r(x)$ is contained in $\Omega \setminus (\Sigma \cup \mathcal{S})$ and consider any point $y \in B_r(x)$. Define the continuous map $\phi : B_r(x) \times (-s_0, s_0) \to \mathbb{R}$ by

$$\phi(y, s) := \langle \zeta(s) - y, \mathcal{R}Dd(y) \rangle, \tag{3.2.7}$$

where $\mathcal{R}$ is the rotation matrix

$$\mathcal{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.2.8}$$

Recalling (1.1.2), we have for any $y \in B_r(x)$

$$\begin{aligned} Dd(y) &= Dd(x) + D^2 d(x)(y - x) + \mathrm{o}(|y - x|) \\ &= e_2 - \frac{\kappa(x)}{1 - \kappa(x)d(x)} \left(e_1 \otimes e_1\right) (y - x) + \mathrm{o}(|y - x|). \end{aligned}$$

Then,

$$\begin{aligned} &\phi(y, s) = \left\langle x - y, -e_1 - \frac{\kappa(x)}{1 - \kappa(x)d(x)} \langle y - x, e_1 \rangle e_2 + \mathrm{o}(|y - x|) \right\rangle \\ &+\left\langle \tau(x)e_2 - ns + \mathrm{o}(s), -e_1 - \frac{\kappa(x)}{1 - \kappa(x)d(x)} \langle y - x, e_1 \rangle e_2 + \mathrm{o}(|y - x|) \right\rangle \\ &= \langle y - x, e_1 \rangle \left(1 - \frac{\tau(x)\kappa(x)}{1 - \kappa(x)d(x)}\right) + \frac{\kappa(x)}{1 - \kappa(x)d(x)} \langle y - x, e_1 \rangle \langle y - x, e_2 \rangle \\ &+\langle n, e_1 \rangle s + \frac{\kappa(x)s}{1 - \kappa(x)d(x)} \langle y - x, e_1 \rangle \langle n, e_2 \rangle + \mathrm{o}(s) + \mathrm{o}(|y - x|). \end{aligned} \tag{3.2.9}$$

Hence, there exist $\bar{s} = \bar{s}(x) > 0$ and $r = r(x) > 0$ such that

$$\phi(y, \bar{s}) < 0, \qquad \phi(y, -\bar{s}) > 0, \qquad \forall\, y \in B_r(x). \tag{3.2.10}$$

Therefore, we conclude that for any $y \in B_r(x)$ we can find some $s_y \in (-\bar{s}, \bar{s})$ such that $\phi(y, s_y) = 0$, i.e.

$$\zeta(s_y) = y + \rho_y Dd(y), \qquad \text{for some } \rho_y \in \mathbb{R}. \tag{3.2.11}$$

Notice that $s_y \to 0$ as $y \to x$. So, $\zeta(s_y) \to \bar{x} = x + \tau(x)Dd(x)$ as $y \to x$ and then $\rho_y \to \tau(x)$ as $y \to x$. Possibly reducing again r we can then assume that $\rho_y > 0$

for any $y \in B_r(x)$. Let us estimate s_y. Since $s_y \to 0$ as $y \to x$, we have that $\frac{\kappa(x)}{1-\kappa(x)d(x)} s_y \langle y - x, e_1 \rangle \langle n, e_2 \rangle = \mathrm{o}(|y-x|)$, so that

$$0 = s_y \langle n, e_1 \rangle + \langle y - x, e_1 \rangle \left(\frac{1 - (\tau(x) + d(x))\kappa(x)}{1 - \kappa(x)d(x)} \right) + \mathrm{o}(s_y) + \mathrm{o}(|y-x|) \quad (3.2.12)$$

and then

$$s_y + \mathrm{o}(s_y) = -\frac{\langle y - x, e_1 \rangle}{\langle n, e_1 \rangle} \left(\frac{1 - (\tau(x) + d(x))\kappa(x)}{1 - \kappa(x)d(x)} \right) + \mathrm{o}(|y-x|), \quad (3.2.13)$$

which gives

$$s_y = -\frac{\langle y - x, e_1 \rangle}{\langle n, e_1 \rangle} \left(\frac{1 - (\tau(x) + d(x))\kappa(x)}{1 - \kappa(x)d(x)} \right) + \mathrm{o}(|y-x|). \quad (3.2.14)$$

Using (3.2.14) we can actually estimate ρ_y, which is an upper bound for $\tau(y)$. Indeed,

$$\begin{aligned}
\rho_y &= \langle \zeta(s_y) - y, Dd(y) \rangle \\
&= \left\langle x + \tau(x)e_2 - n s_y - y, e_2 - \frac{\kappa(x)}{1 - \kappa(x)d(x)} \langle y - x, e_1 \rangle e_1 \right\rangle \\
&\quad + \mathrm{o}(s_y) + \mathrm{o}(|y-x|) \\
&= -\langle y - x, e_2 \rangle + \tau(x) - s_y \langle n, e_2 \rangle + \frac{\kappa(x) s_y}{1 - \kappa(x)d(x)} \langle y - x, e_1 \rangle \langle n, e_1 \rangle \\
&\quad + \mathrm{o}(s_y) + \mathrm{o}(|y-x|) \\
&= \tau(x) - \langle y - x, e_2 \rangle - \left(\frac{1 - (\tau(x) + d(x))\kappa(x)}{1 - \kappa(x)d(x)} \right) \frac{\langle n, e_2 \rangle}{\langle n, e_1 \rangle} \langle y - x, e_1 \rangle \\
&\quad + \mathrm{o}(|y-x|) \\
&\le \tau(x) + \left(2 + \frac{1 - (\tau(x) + d(x))\kappa(x)}{1 - \kappa(x)d(x)} \frac{|\langle n, e_2 \rangle|}{|\langle n, e_1 \rangle|} \right) |y - x|, \quad \forall y \in B_r(x),
\end{aligned}$$

provided we take r small enough. Now, recalling that n is orthogonal to $p - e_2$, we deduce that

$$\begin{aligned}
|\langle n, e_1 \rangle| &= \frac{|n|}{|p - e_2|} |\langle \mathcal{R}(p - e_2), e_1 \rangle| = \frac{|n|}{|p - e_2|} |\langle p - e_2, e_2 \rangle| \\
&= \frac{|n|}{|p - e_2|} (1 - \langle p, e_2 \rangle) = \frac{|n|}{2} |p - e_2|.
\end{aligned}$$

Therefore,

$$\tau(y) \le \tau(x) + 2 \left(1 + \frac{1 - (\tau(x) + d(x))\kappa(x)}{1 - \kappa(x)d(x)} \right) \frac{1}{|p - e_2|} |y - x|, \quad (3.2.15)$$

which is the desired inequality, since in this case $\delta(\bar{x}) = |p - e_2|$.
Next, let us suppose that $\bar{x} \in \Sigma^2$. By Remark 3.2.5 we already know that $D^*d(\bar{x})$ is finite. Then, there exist two limiting gradients, say $p_1, p_2 \neq e_2$, such that $[p_i, e_2]$ $(i = 1, 2)$ is an exposed face of $D^+d(\bar{x})$, that is $[p_i, e_2] \subset \partial D^+d(\bar{x})$. Moreover, there exist two analytic (except for the starting point) arcs propagating from $\bar{x}$ with initial direction given by the opposite of the unit outward normals n_1 and n_2 to the faces $[p_1, e_2]$ and $[p_2, e_2]$ of $D^+d(\bar{x})$ respectively. As in Lemma 3.1.5 we can prove that

$$e_2 = \lambda_1 n_1 + \lambda_2 n_2 \tag{3.2.16}$$

for suitable numbers $\lambda_1, \lambda_2 > 0$. Taking into account that e_1 and e_2 are mutually orthogonal, we have $0 = \lambda_1\langle n_1, e_1\rangle + \lambda_2\langle n_2, e_1\rangle$. So, either $\langle n_1, e_1\rangle < 0$ and $\langle n_2, e_1\rangle > 0$ or viceversa. Suppose $\langle n_1, e_1\rangle < 0$. Then the arc

$$\zeta(s) = \begin{cases} \bar{x} - n_1 s + \mathrm{o}(s), & \text{for } s \in [0, s_0) \\ \bar{x} + n_2 s + \mathrm{o}(s), & \text{for } s \in (-s_0, 0) \end{cases} \tag{3.2.17}$$

is the local representation of the singular arc mentioned above. By repeating the argument of the case $\bar{x} \in \Sigma^1$, we obtain that there exists a ball $B_r(x)$ such that, for any $y \in B_r(x)$

$$\tau(y) \leq \tau(x) + 2\left(1 + \frac{1 - (\tau(x) + d(x))\kappa(x)}{1 - \kappa(x)d(x)}\right)\frac{1}{\min\{|p_1 - e_2|, |p_2 - e_2|\}}|y - x|.$$

The general inequality is now a straightforward consequence of the previous computations. □

Lemma 3.2.7. *For any ball B compactly embedded in $\Omega \setminus \mathcal{S}$ there exists a positive constant δ_B such that*

$$\delta(\bar{x}) \geq \delta_B \qquad \textit{for any} \qquad x \in B,$$

where $\bar{x} = x + \tau(x)Dd(x)$.

Proof– First of all, $\delta(\cdot)$ is strictly positive on $\Omega \setminus \mathcal{S}$ (see Remark 3.2.5). Moreover, let $\overline{B} \subset \Omega \setminus \mathcal{S}$ be any ball and suppose, by contradiction, that there exists a sequence

$\{x_k\} \subset B$ such that $\delta(\bar{x}_k) \to 0$ as $k \to \infty$. We can assume, without loss of generality, that $\bar{x}_k \in \Sigma^1$. Hence, for any k, there exist $p_k, q_k \in D^* d(\bar{x}_k)$ with $[p_k, q_k] = \partial D^+ d(\bar{x}_k)$ and $\delta(\bar{x}_k) = |p_k - q_k| \to 0$ as $k \to \infty$. Consider now the projections y_k and z_k corresponding to p_k and q_k respectively, that is $y_k = \bar{x}_k - d(\bar{x}_k)p_k$ and $z_k = \bar{x}_k - d(\bar{x}_k)q_k$. Then,

$$\begin{aligned}\frac{y_k - z_k}{|y_k - z_k|} &= -d(\bar{x}_k)\frac{(p_k - q_k)}{|y_k - z_k|} = -b_\Omega(\bar{x}_k)\frac{(Db_\Omega(y_k) - Db_\Omega(z_k))}{|y_k - z_k|} \\ &= -b_\Omega(\bar{x}_k)D^2 b_\Omega(z_k + \lambda_k(y_k - z_k)) \cdot \frac{(y_k - z_k)}{|y_k - z_k|},\end{aligned} \tag{3.2.18}$$

for some $\lambda_k \in (0,1)$, where $b_\Omega(\cdot)$ denotes the signed distance from $\partial\Omega$ (see also Remark 1.1.6)

$$b_\Omega(x) = d_{\mathbb{R}^2 \setminus \Omega}(x) - d_{\overline{\Omega}}(x).$$

Choosing appropriate subsequences, still called $\{x_k\}$, $\{p_k\}$, $\{q_k\}$, we can suppose that $x_k \to x_0 \in \Omega \setminus \mathcal{S}$, $p_k, q_k \to e_0 \in D^* d(\bar{x}_0)$ and $\frac{y_k - z_k}{|y_k - z_k|} \to \theta_0$ as $k \to \infty$. Thus $\bar{x}_k \to \bar{x}_0$ and, passing to the limit in (3.2.18), we obtain

$$\theta_0 = -d(\bar{x}_0)D^2 d(\hat{x}_0) \cdot \theta_0, \tag{3.2.19}$$

where $\hat{x}_0 \in \partial\Omega$ is the limiting point of both y_k and z_k. Recalling the structure of the Hessian matrix $D^2 d(\hat{x}_0)$ (see (1.1.2)), we conclude that $d(\bar{x}_0)\kappa(\hat{x}_0) = 1$. Therefore, $\bar{x}_0$ belongs to $\tilde{\Gamma} \cup \Gamma$. But $\bar{x}_0$ cannot be a regular conjugate point because $x_0 \notin \mathcal{S}$ by construction; on the other hand, $\bar{x}_0$ cannot be a singular point either, for otherwise $\{\bar{x}_k\}$ would be a sequence of singular points approaching x_0 with

$$\operatorname{diam}(D^+ d(\bar{x}_k)) = |p_k - q_k| \to 0, \qquad \text{as } k \to \infty,$$

in contrast with Remark 3.2.5 on the structure of $\overline{\Sigma}$. This contradiction prove the assertion of the lemma. $\square$

Proposition 3.2.8. *The map τ is locally Lipschitz continuous on the set $\Omega \setminus \mathcal{S}$. Moreover, τ is differentiable a.e. in $\Omega \setminus (\Sigma \cup \mathcal{S})$ and*

$$|\nabla\tau(x)| \leq C(x) \qquad x \in \Omega \setminus (\Sigma \cup \mathcal{S}) \quad a.e.,$$

where $C(x)$ is given by (3.2.4).

Proof– We will first prove that for any $x \in \Omega \setminus (\Sigma \cup \mathcal{S})$ we have

$$|p| \leq C(x) \qquad \forall\, p \in \partial_P \tau(x) \quad \forall\, x \in \Omega \setminus (\Sigma \cup \mathcal{S})\,, \tag{3.2.20}$$

where $\partial_P \tau(x)$ denotes the proximal subgradient of τ at x. Indeed, recall that a vector $p \in \mathbb{R}^2$ belongs to $\partial_P \tau(x)$ if and only if there exist numbers $\sigma, \eta > 0$ such that

$$\tau(y) \geq \tau(x) + \langle p, y - x \rangle - \sigma |y - x|^2 \qquad \forall\, y \in B_\eta(x)\,,$$

see Definition 1.2.17. Now, combine the above inequality with (3.2.3) to obtain

$$\langle p, y - x \rangle \leq C(x)|y - x| + \sigma |y - x|^2$$

whenever $|y - x| < \min\{r, \eta\}$. The last inequality implies (3.2.20).
Now, we note that $C(\cdot)$ is locally bounded on $\Omega \setminus (\Sigma \cup \mathcal{S})$ by Lemma 3.2.7 and the inequality

$$\frac{1 - (\tau(x) + d(x))\kappa(x)}{1 - \kappa(x) d(x)} \leq 1 + \mathrm{diam}(\Omega) \max_{x \in \partial\Omega} [\kappa(x)]_-,$$

where $[\kappa(x)]_- := \max\{0, -\kappa(x)\}$. Owing to Theorem 1.2.20, τ is locally Lipschitz in $\Omega \setminus (\Sigma \cup \mathcal{S})$. Thus, τ is also differentiable a.e. on such a set. Moreover, by Theorem 1.2.19 and Lemma 1.2.18 we have respectively that $\partial_P \tau(x) \neq \emptyset$ a.e. and $\partial_P \tau(x) \subseteq \{\nabla \tau(x)\}$ at any differentiability point of τ. Collecting together all these properties we obtain

$$|\nabla \tau(x)| \leq C(x) \qquad x \in \Omega \setminus (\Sigma \cup \mathcal{S}) \quad \text{a.e.}$$

In order to complete the proof, we need to bound $C(x)$ from above when x approaches Σ. The expression of $C(x)$ given by (3.2.4) for any $x \in \Omega \setminus (\Sigma \cup \mathcal{S})$ is meaningful also on the set of singular points, provided we define

$$C(x) = 2\left(1 + \frac{1}{\delta(x)}\right), \qquad \text{for all} \quad x \in \Sigma.$$

Taking into account Lemma 3.2.7 we easily deduce the local Lipschitz continuity of τ on $\Omega \setminus \mathcal{S}$. $\square$

Remark 3.2.9. At this point of our reasoning it is important to stress again that the loss of Lipschitz regularity for τ (and then for v_f) depends on the presence of conjugate points only. When Ω is a bounded domain with no conjugate points (both regular and singular) and $\mathcal{C}^{2,1}$ boundary, then it can be shown that the results obtained so far still hold true. In particular, it turns out that τ is Lipschitz continuous on the whole set $\overline{\Omega}$. Indeed, if Ω has no conjugate points, it can be proven that any $x \in \Omega$ has a finite number of projections onto $\partial\Omega$, which is one of the main properties we need in the proof of those results. On the other hand, if Ω is a simply connected domain with analytic boundary, different from a disk, then the set of regular conjugate points is nonempty, so that we cannot avoid the loss of regularity they produce.

The second step of our argument is to estimate $C(x)$ in (3.2.4) in terms of $d_{\tilde{\mathcal{S}}}(x)$, which is the distance of x from the set $\tilde{\mathcal{S}} := \mathcal{S} \cup \Sigma$. Aiming at this, we need some deeper results on the behaviour of the singular arcs starting from a regular conjugate point. In what follows we choose the reference system so that the regular conjugate point coincides with the point $x_0 = (0, r)$, $r > 0$, being $(0, 0)$ its projection on the boundary. Moreover, we locally represent $\partial\Omega$ as the graph of an analytic function $\alpha : (-s_0, s_0) \to \mathbb{R}$, $0 < s_0 < r$, such that $\alpha(0) = 0$, $\alpha'(0) = 0$ and $\alpha''(0) = \frac{1}{r}$. We claim that there exist $n \geq 2$, $a = a(n) > 0$ and an analytic function $b(\cdot)$ in $(-s_0, s_0)$ satisfying

$$b(s) = \sum_{i \geq 2n+1} b_i s^i, \tag{3.2.21}$$

such that

$$\alpha(s) = \left[r - \left(r^2 - s^2 \right)^{1/2} \right] - a s^{2n} + b(s) \qquad \forall s \in (-s_0, s_0). \tag{3.2.22}$$

Indeed, one of the main properties of analytic boundaries is that the curvature κ has a maximum at the projection of a conjugate point (see [17, Theorem 3.1]). More precisely, given a local representation of the boundary as above, the curvature, whose expression is

$$\kappa(s) := \kappa((s, \alpha(s)) = \frac{\alpha''(s)}{(1 + \alpha'(s)^2)^{3/2}}, \tag{3.2.23}$$

satisfies $\kappa(s) \leq \kappa(0)$ for any s in a neighborhood of 0. In particular, being κ analytic and nonconstant (Ω is not a disk), we obtain that there exists $n \geq 2$ such that, for any $1 \leq m \leq 2n-3$,

$$\kappa^{(m)}(0) = 0$$

and

$$\kappa^{(2n-2)}(0) < 0.$$

Writing the above relations in terms of the derivatives of α, and taking into account that $\beta(s) := r - (r^2 - s^2)^{1/2}$ is a local representation of the circle of centre $(0, r)$ and radius r (the unique analytic curve with constant curvature $1/r$), we obtain that the difference $\alpha(s) - \beta(s)$ is not identically zero and its Taylor expansion at 0 is of the form $-as^{2n} + b(s)$, where a is $\frac{1}{(2n)!}$ times the difference between the $2n$-th derivatives of the functions $\beta(s)$ and of $\alpha(s)$ at $s = 0$, and $b(s)$ is the remainder of the difference of the Taylor expansions in 0 of α and β. Being $b(s)$ of the form $\sum_{i \geq 2n+1} b_i s^i$, we will say that it is a series of valuation $\mathrm{Val}(b) \geq 2n+1$, meaning that the first index with nonzero coefficient is $2n+1$.

Our next lemma provides a description of the singular arc emanating from x_0 with respect to the boundary parameter s.

Lemma 3.2.10. *There exist $\varepsilon > 0$ and two analytic functions $t : (-\varepsilon, \varepsilon) \to \mathbb{R}$ and $\rho : (-\varepsilon, \varepsilon) \to \mathbb{R}$, with $t(0) = 0$, $\rho(0) = r$ and*

$$\begin{cases} t(s) = s + o(s), \\ \rho(s) = r + 2nar^2 s^{2n-2} + o(s^{2n-2}), \end{cases} \tag{3.2.24}$$

such that for any $s \in (0, \varepsilon)$

$$A(s) + \rho(s)\ \nu(s) = A(-t(s)) + \rho(s)\ \nu(-t(s)), \tag{3.2.25}$$

where

$$A(s) = (s, \alpha(s))$$

and

$$\nu(s) := \left(\frac{-\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}}, \frac{1}{(1+\alpha'(s)^2)^{1/2}} \right).$$

Moreover, there exist $\eta > 0$ *such that*

$$\Sigma \cap B_\eta((0,r)) = \{\xi(s) \mid s \in (0,\varepsilon)\},$$

where

$$\xi(s) = A(s) + \rho(s)\ \nu(s).$$

Proof– Our first step is to find, for any $s > 0$ sufficiently small, numbers $t < 0$ and $\rho > 0$ satisfying

$$s + \rho \left[\frac{-\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}} \right] = -t + \rho \left[\frac{-\alpha'(-t)}{(1+\alpha'(-t)^2)^{1/2}} \right], \tag{3.2.26}$$

$$\alpha(s) + \frac{\rho}{(1+\alpha'(s)^2)^{1/2}} = \alpha(-t) + \frac{\rho}{(1+\alpha'(-t)^2)^{1/2}}. \tag{3.2.27}$$

Since $\alpha''(0) > 0$, if we choose s and t sufficiently small, we have $\alpha'(-t) < 0$ and $\alpha'(s) > 0$. Hence, (3.2.26) gives

$$\rho = -\frac{s+t}{\dfrac{\alpha'(-t)}{(1+\alpha'(-t)^2)^{1/2}} - \dfrac{\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}}}. \tag{3.2.28}$$

Now we want to simplify this expression by using (3.2.22). Since the map $x \mapsto \frac{x}{(1+x^2)^{1/2}}$ is analytic, for any x and y we can write

$$\frac{y}{(1+y^2)^{1/2}} = \frac{x}{(1+x^2)^{1/2}} + \frac{y-x}{(1+x^2)^{3/2}} + C(y-x),$$

where $\mathrm{Val}(C) \geq 2$. Substituting $x = \beta'(s)$ and $y = \alpha'(s) = \beta'(s) - 2nas^{2n-1} + b'(s)$ in the above expression, we obtain

$$\frac{\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}} = \frac{\beta'(s)}{(1+\beta'(s)^2)^{1/2}} + \frac{-2nas^{2n-1} + b'(s)}{(1+\beta'(s)^2)^{3/2}} + C_1(s),$$

where $\mathrm{Val}(C_1) \geq 2(2n-1)$. Moreover,

$$\beta'(s) = \frac{s}{(r^2-s^2)^{1/2}},$$

$$\frac{\beta'(s)}{(1+\beta'(s)^2)^{1/2}} = \frac{s}{(r^2-s^2)^{1/2}} \cdot \left[1+\frac{s^2}{r^2-s^2}\right]^{-1/2} = \frac{s}{r},$$

while

$$\frac{1}{(1+\beta'(s)^2)^{3/2}} = \left[1+\frac{s^2}{r^2-s^2}\right]^{-3/2} = \frac{(r^2-s^2)^{3/2}}{r^3} = 1 + C_2(s),$$

with $\mathrm{Val}(C_2) \geq 2$. So,

$$\frac{\alpha'(s)}{(1+\alpha'(s)^2)^{1/2}} = \frac{s}{r} - 2nas^{2n-1} + C_3(s),$$

where $\mathrm{Val}(C_3) \geq 2n$, since $\mathrm{Val}(b') \geq 2n$. Substituting the last expression into (3.2.28), we then deduce that

$$\begin{aligned} \rho = \rho(s,t) &= -\frac{s+t}{-\dfrac{s+t}{r} + 2na(t^{2n-1}+s^{2n-1}) + C_3(s) - C_3(-t)} \\ &= \frac{r}{1 - 2nar\dfrac{t^{2n-1}+s^{2n-1}}{s+t} + r\dfrac{C_3(-t)-C_3(s)}{s+t}}. \end{aligned} \tag{3.2.29}$$

Notice that $\frac{t^{2n-1}+s^{2n-1}}{s+t}$ is a polynomial of degree $2n-2$, while $C_3(s,t) := \frac{C_3(-t)-C_3(s)}{s+t}$ is analytic of valuation greater than or equal to $2n-1$.

We will now try to solve (3.2.27), which is equivalent to

$$\frac{\alpha(-t)-\alpha(s)}{\rho} + \frac{1}{(1+\alpha'(-t)^2)^{1/2}} - \frac{1}{(1+\alpha'(s)^2)^{1/2}} = 0. \tag{3.2.30}$$

Reasoning as above we find that

$$\frac{1}{(1+\alpha'(s)^2)^{1/2}} = \frac{1}{(1+\beta'(s)^2)^{1/2}} - \frac{\beta'(s)}{(1+\beta'(s)^2)^{3/2}}(-2nas^{2n-1} + b'(s)) + C_4(s),$$

with $\mathrm{Val}(C_4) \geq 2(2n-1)$. On the other hand,

$$\frac{\beta'(s)}{(1+\beta'(s)^2)^{3/2}} = \frac{(r^2-s^2)}{r^2} \cdot \frac{s}{r} = \frac{s}{r} + C_5(s)$$

where $\mathrm{Val}(C_5) \geq 3$. Thus,

$$\frac{1}{(1+\alpha'(s)^2)^{1/2}} = \left(1 - \frac{s^2}{r^2}\right)^{1/2} + \frac{2na}{r}s^{2n} + C_6(s),$$

where $\mathrm{Val}(C_6) \geq 2n+1$. Using the previous computations and taking into account the expression of ρ in (3.2.29), we can finally estimate (3.2.30) as

$$\begin{aligned}
&0 = \left(\frac{1}{r} - 2na\frac{t^{2n-1}+s^{2n-1}}{s+t} + C_3(s,t)\right)\cdot \\
&\left(r - (r^2-t^2)^{1/2} - at^{2n} + b(-t) - r + (r^2-s^2)^{1/2} + as^{2n} - b(s)\right) \\
&+\left(1-\frac{t^2}{r^2}\right)^{1/2} + \frac{2na}{r}t^{2n} + C_6(-t) - \left(1-\frac{s^2}{r^2}\right)^{1/2} - \frac{2na}{r}s^{2n} - C_6(s),
\end{aligned} \tag{3.2.31}$$

that is

$$\begin{aligned}
&0 = (2na-a)(t^{2n}-s^{2n}) \\
&-2nar\frac{t^{2n-1}+s^{2n-1}}{s+t}\left((r^2-s^2)^{1/2} - (r^2-t^2)^{1/2}\right) + M(s,t),
\end{aligned} \tag{3.2.32}$$

with $\mathrm{Val}(M) \geq 2n+1$. Furthermore,

$$\left((r^2-s^2)^{1/2} - (r^2-t^2)^{1/2}\right) = \frac{t^2-s^2}{2r} + P(s,t), \qquad \mathrm{Val}(P) \geq 4.$$

This gives

$$(2na-a)(t^{2n}-s^{2n}) - na(t^{2n}-s^{2n}) - na(s^{2n-1}t - st^{2n-1}) + Q(s,t) = 0, \tag{3.2.33}$$

being $Q(s,t)$ an analytic function of valuation $\mathrm{Val}(Q) \geq 2n+1$ of the form

$$Q(s,t) = \sum_{k\geq 2n+1} q_k(s^k - (-1)^k t^k). \tag{3.2.34}$$

Now, let us set $u := \dfrac{t}{s}$. Since $s > 0$, then (3.2.33) becomes

$$(na-a)(u^{2n}-1) - na(u-u^{2n-1}) + \frac{1}{s^{2n}}Q(s,su) = 0. \tag{3.2.35}$$

Exploiting the structure of Q in (3.2.34) we see that $\frac{1}{s^{2n}}Q(s,su)$ is equal to $sR(s,u)$, where R is an analytic function of valuation $\mathrm{Val}(R) \geq 2n+1$. So, at the end of these computations we can say that finding $t(\cdot)$ and $\rho(\cdot)$ that verify (3.2.24) and (3.2.25) is equivalent to find, for any s sufficiently small, some $u = u(s)$ which solves (3.2.35). To this end, we apply the Implicit Function Theorem to the analytic function

$$\phi(s,u) = (na-a)(u^{2n}-1) - na(u-u^{2n-1}) + sR(s,u)$$

at the point $(\bar{s}, \bar{u}) = (0, 1)$. Since

$$\phi(0,1) = 0, \qquad \text{and} \qquad \frac{\partial \phi}{\partial u}(0,1) = 2na(n-1) - na(1-2n+1) = 4na(n-1) \neq 0,$$

the existence of $t(\cdot)$ and $\rho(\cdot)$ is proven.

Now, let us recover the local representation of Σ in terms of the above maps. Since $\Gamma \cup \Sigma^2$ is finite and $(0, r) \notin \Sigma^2$, we can find some $\eta > 0$ such that

$$\Sigma \cap B_\eta((0,r)) \subset \Sigma^1, \quad \overline{\Sigma} \cap B_\eta((0,r)) \subset \Sigma^1 \cup \{(0,r)\}. \tag{3.2.36}$$

Moreover, by Lemma 3.2.4, there is an analytic arc $\zeta : [0, \varepsilon_0) \to \mathbb{R}^2$ such that $\zeta(0) = (0, r)$ and $\Sigma \cap B_\eta((0,r)) = \{\zeta(r) \mid r \in (0, \varepsilon_0)\}$. Possibly reducing ε, we can suppose that for any $s \in (0, \varepsilon)$

$$\begin{aligned} &A(s) + \tau(A(s))\ \nu(s) \in B_\eta((0,r)), \\ &A(-t(s)) + \tau(A(-t(s)))\ \nu(-t(s)) \in B_\eta((0,r)). \end{aligned} \tag{3.2.37}$$

Then, for any $s \in (0, \varepsilon)$ there exist θ_s and $\tilde{\theta}_s$ satisfying, respectively,

$$A(s) + \tau(A(s))\ \nu(s) = \zeta(\theta_s)$$

and

$$A(-t(s)) + \tau(A(-t(s)))\ \nu(-t(s)) = \zeta(\tilde{\theta}_s).$$

Suppose that $\theta_s < \tilde{\theta}_s$. Then, $\zeta(\theta_s)$ belongs to the interior of the Jordan curve delimited by the segments $\left[A(-t(s)), \zeta(\tilde{\theta}_s)\right]$, $\left[p(s), \zeta(\tilde{\theta}_s)\right]$ and the curve joining $p(s)$ and $A(t(s))$ on the graph of α, being $p(s)$ the other projection of $\zeta(\tilde{\theta}_s)$. On the other hand, $A(s)$ does not belong to the interior of this curve on the graph of α because otherwise the point $A(s) + \rho(s)\ \nu(s)$ would lie on the segment $\left[A(-t(s)), \zeta(\tilde{\theta}_s)\right]$ and have two projections, namely $A(s)$ and $A(-t(s))$; a contradiction. Since $A(s)$ does not belong to the interior of the curve $(p(s), A(-t(s)))$, we have that either $A(s) = p(s)$ or

$$|\zeta(\theta_s) - p(s)| < |\zeta(\theta_s) - A(s)|,$$

which is again a contradiction. Hence, $\theta_s \geq \tilde{\theta}_s$. By the same argument, $\tilde{\theta}_s \geq \theta_s$. Therefore, $\theta_s = \tilde{\theta}_s$ and $\tau(A(s)) = \tau(A(-t(s))) = \rho(s)$. $\square$

Lemma 3.2.11. *There exists $\varepsilon > 0$ such that for any $s \in (-\varepsilon, \varepsilon)$ the curvature of $\partial\Omega$ at the point $A(s) = (s, \alpha(s))$ is given by*

$$\kappa(s) = \frac{1}{r} - 2n(n-1)as^{2n-2} + o(s^{2n-2}). \tag{3.2.38}$$

Moreover, $\kappa(s) \geq \frac{1}{2r}$ for any $s \in (-\varepsilon, \varepsilon)$.

Proof– Fix $\varepsilon > 0$ as in Lemma 3.2.10. Using (3.2.23), (3.2.22) and arguing as in the previous lemma, we have

$$\begin{aligned} \kappa(s) &= \left(\beta''(s) - 2n(n-1)as^{2n-2} + \mathrm{o}(s^{2n-2})\right) \\ &\cdot \left(\frac{1}{(1+\beta'(s)^2)^{3/2}} + \frac{3\beta'(s)}{(1+\beta'(s)^2)^{5/2}} 2nas^{2n-1} + \mathrm{o}(s^{2n-1})\right), \end{aligned}$$

where

$$\beta'(s) = \frac{s}{(r^2 - s^2)^{1/2}}, \quad \beta''(s) = \frac{r^2}{(r^2 - s^2)^{3/2}}$$

and

$$\frac{1}{(1+\beta'(s)^2)^{3/2}} = \frac{(r^2 - s^2)^{3/2}}{r^3}.$$

Substituting β', β'' in the expression of $\kappa(s)$ we easily obtain

$$\kappa(s) = \frac{1}{r} - 2n(n-1)as^{2n-2} + \mathrm{o}(s^{2n-2})$$

and, possibly reducing ε, $\kappa(s) \geq \frac{1}{2r}$. $\square$

Now, we proceed to estimate $C(x)$, given in (3.2.4), with respect to $d_{\tilde{\mathcal{S}}}(x)$, where $\tilde{\mathcal{S}} := \mathcal{S} \cup \Sigma$ and

$$\mathcal{S} := \bigcup_{x_0 \in \Gamma} \{x_0 - tDd(x_0) \ : \ t \in [0, d(x_0)]\}.$$

For any $h_0 > 0$ sufficiently small set $\mathcal{S}_{h_0} := \{x \in \Omega \mid d_{\mathcal{S}}(x) < h_0\}$. By Proposition 3.2.8 we deduce that, outside $\mathcal{S}_{h_0}$, the L^∞ norm of $C(x)$ is bounded by some constant C_{h_0}. Hence, for any $x \in \Omega \setminus (\mathcal{S}_{h_0} \cup \Sigma)$ we have that

$$C(x) \leq C_{h_0} \frac{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}} \leq C_{h_0} \frac{\mathrm{diam}(\Omega)^{\frac{1}{2n-1}}}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}}.$$

It remains to estimate $C(x)$ in $\mathcal{S}_{h_0} \setminus \tilde{\mathcal{S}}$.

Lemma 3.2.12. *Let x_0 be a regular conjugate point. For any $h_0 > 0$ let $\mathcal{S}_{h_0}(x_0)$ be the connected component of $\mathcal{S}_{h_0}$ containing x_0. Then, for h_0 sufficiently small and for any $x \in \mathcal{S}_{h_0}(x_0) \setminus \tilde{\mathcal{S}}$, we have*

$$C(x) \leq \frac{K}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}}, \tag{3.2.39}$$

where n is the integer given in (3.2.22)) *and K is a constant depending on h_0 and Ω only.*

Proof– To begin with, let us fix the coordinates so that $x_0 = (0, r)$, $r > 0$, and $(0, 0)$ is the projection of x_0 onto $\partial\Omega$. Moreover, let $\partial\Omega$ be represented, in a neighborhood of $(0, 0)$, by the graph of an analytic function $\alpha(\cdot)$, defined in $(-s_0, s_0)$, such that $\alpha(0) = 0$, $\alpha'(0) = 0$ and $\alpha''(0) = \frac{1}{r}$. Let us call again

$$A(s) = (s, \alpha(s)) \quad \text{and} \quad \nu(s) := \left(\frac{-\alpha'(s)}{(1 + \alpha'(s)^2)^{1/2}}, \frac{1}{(1 + \alpha'(s)^2)^{1/2}} \right).$$

Now, take $\varepsilon > 0$ as in Lemma 3.2.10. Choose h_0 sufficiently small such that the projection onto $\partial\Omega$ of any $x \in \mathcal{S}_{h_0}(x_0) \setminus (\Sigma \cup \mathcal{S})$ is given by $A(s_x)$ for some $s_x \in (-\varepsilon, \varepsilon)$. Actually $s_x \neq 0$ because $x \notin \mathcal{S}$ and $\mathcal{S}_{h_0}(x_0) \setminus (\Sigma \cup \mathcal{S})$ is a two-connected-components set, contained in the disjoint union of the sets $\mathcal{C}^-$ and $\mathcal{C}^+$, where

$$\mathcal{C}^+ = \cup_{s \in (0,\varepsilon)} T_s, \qquad \mathcal{C}^- = \cup_{s \in (-\varepsilon, 0)} T_s$$

and

$$T_s :=]A(s); A(s) + \tau(A(s)) Dd(A(s))[.$$

Let us fix our attention on the connected component

$$\mathcal{S}^+ := (\mathcal{S}_{h_0}(x_0) \setminus (\Sigma \cup \mathcal{S})) \cap \mathcal{C}^+.$$

In light of Lemma 3.2.10, we have

$$\Sigma \cap \mathcal{S}_{h_0}(x_0) = \{\xi(s) : s \in (0, \varepsilon)\},$$

where

$$\xi(s) = A(s) + \rho(s)\nu(s)$$

and $\rho(s) = r + 2nar^2 s^{2n-2} + \mathrm{o}(s^{2n-2})$. Hence, for any $s \in (0, \varepsilon)$,

$$T_s =]A(s), \xi(s)[.$$

Let us first estimate $C(\cdot)$ on any of these rays. Fix a ray T_s, with $s \in (0, \varepsilon)$. Then, from (3.2.4) we have that for any $x \in T_s$

$$C(x) = 2\left(1 + \frac{1 - d(\xi(s))\kappa(s)}{1 - d(x)\kappa(s)} \frac{1}{\delta(\xi(s))}\right),$$

where $\kappa(s)$ stands for the curvature at the boundary point $A(s)$. Since s is fixed, in order to estimate $C(x) d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}$ on T_s let us consider first the ratio

$$\frac{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}}}{1 - d(x)\kappa(s)}. \tag{3.2.40}$$

We claim that for $x \in T_s$ an upper bound for $d_{\tilde{\mathcal{S}}}(x)$ is given by

$$\frac{(d(\xi(s)) - d(x))\,|A(s) - A(-t(s))|}{d(\xi(s))}, \tag{3.2.41}$$

with $t(s)$ as il Lemma 3.2.10. Indeed, $\xi(s) \in \Sigma^1$; so, consider the other projecting line from $\xi(s)$, which is $T_{-t(s)}$, and the point $\tilde{x}$ on this line satisfying the condition $d(x) = d(\tilde{x})$; then we have

$$d_{\tilde{\mathcal{S}}}(x) \le |x - \tilde{x}| = \frac{(d(\xi(s)) - d(x))\,|A(s) - A(-t(s))|}{d(\xi(s))}.$$

Hence, the ratio in (3.2.40) is bounded from above by

$$\sup_{x \in T_s} \frac{(d(\xi(s)) - d(x))^{\frac{1}{2n-1}}\,|A(s) - A(-t(s))|^{\frac{1}{2n-1}}}{d(\xi(s))^{\frac{1}{2n-1}}(1 - d(x)\kappa(s))}. \tag{3.2.42}$$

On the other hand, since $d(\cdot)$ is linearly increasing on T_s, the above supremum is attained at the (unique) point $x \in T_s$ satisfying

$$d(x) = \frac{2n-1}{2n-2} d(\xi(s)) - \frac{1}{(2n-2)\kappa(s)}.$$

In conclusion, for any $x \in T_s$,

$$
\begin{aligned}
&C(x)d_{\tilde{S}}(x)^{\frac{1}{2n-1}} \leq 2\mathrm{diam}(\Omega)^{\frac{1}{2n-1}}\\
&+\frac{2\left((-\frac{1}{2n-2}d(\xi(s))+\frac{1}{(2n-2)\kappa(s)})\,|A(s)-A(-t(s))|\right)^{\frac{1}{2n-1}}(1-d(\xi(s))\kappa(s))}{d(\xi(s))^{\frac{1}{2n-1}}\left(\frac{2n-1}{2n-2}-\frac{2n-1}{2n-2}d(\xi(s))\kappa(s)\right)\delta(\xi(s))}\\
&=2\mathrm{diam}(\Omega)^{\frac{1}{2n-1}} \qquad (3.2.43)\\
&+\frac{2(2n-2)^{\frac{2n-2}{2n-1}}}{2n-1}\frac{[1-\kappa(s)d(\xi(s))]^{\frac{1}{2n-1}}\,|A(s)-A(-t(s))|^{\frac{1}{2n-1}}}{(\kappa(s)d(\xi(s)))^{\frac{1}{2n-1}}\delta(\xi(s))}.
\end{aligned}
$$

In order to finish the estimate of $C(x)d_{\tilde{S}}(x)^{\frac{1}{2n-1}}$ it remains to bound from above the last term in (3.2.43)–call it E_s–on $(0,\varepsilon)$. We will complete the reasoning in three steps.

Step 1: *Estimate of* $\delta(\xi(s))$ *and* $|A(s)-A(-t(s))|$.

Since $\xi(s) \in \Sigma^1$, we have by (1.2.10)

$$\delta(\xi(s)) = \frac{|A(s)-A(-t(s))|}{d(\xi(s))}. \qquad (3.2.44)$$

Moreover, Lemma 3.2.10 gives that

$$|A(s)-A(-t(s))| = 2s + \mathrm{o}(s). \qquad (3.2.45)$$

Also, $|A(s)-A(-t(s))| \geq s$.

Step 2: *Estimate of* $1-d(\xi(s))\kappa(s)$.

Recalling that $d(\xi(s)) = \rho(s)$, with ρ given by (3.2.24) and that κ satisfies (3.2.38), we easily derive

$$
\begin{aligned}
&\quad 1-d(\xi(s))\kappa(s)\\
&= 1-[r+2nar^2s^{2n-2}+\mathrm{o}(s^{2n-2})]\ \left[\tfrac{1}{r}-2n(n-1)as^{2n-2}+\mathrm{o}(s^{2n-2})\right]\\
&= [2n(2n-1)-2n]ras^{2n-2}+\mathrm{o}(s^{2n-2})\\
&= 4n(n-1)ras^{2n-2}+\mathrm{o}(s^{2n-2}).
\end{aligned} \qquad (3.2.46)
$$

Step 3: *Estimate of* E_s.

If we collect together (3.2.44), (3.2.45) and (3.2.46), and take into account the bounds

$\kappa(s) \geq \frac{1}{2r}$ from Lemma 3.2.11 and $r \leq d(\xi(s)) \leq \mathrm{diam}(\Omega)/2$, which follows from the identity $d(\xi(s)) = \rho(s)$ and from (3.2.24) with s small enough, we readily obtain

$$\begin{aligned} E_s &\leq \frac{2}{(k(s)d(\xi(s)))^{\frac{1}{2n-1}}} \frac{[4n(n-1)ras^{2n-2} + \mathrm{o}(s^{2n-2})]^{\frac{1}{2n-1}} d(\xi(s))}{|A(s) - A(-t(s))|^{\frac{2n-2}{2n-1}}} \\ &\leq \left([4n(n-1)a]^{\frac{1}{2n-1}} \mathrm{diam}(\Omega)^{\frac{2n}{2n-1}}\right)(1 + \mathrm{o}(1_s)). \end{aligned}$$

Possibly reducing again ε (and then h_0) we get that

$$C(x) d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2n-1}} \leq 2\mathrm{diam}(\Omega)^{\frac{1}{2n-1}} + 2[4n(n-1)a]^{\frac{1}{2n-1}} \mathrm{diam}(\Omega)^{\frac{2n}{2n-1}}, \tag{3.2.47}$$

for any $x \in (\mathcal{S}_{h_0}(x_0) \setminus (\Sigma \cup \mathcal{S})) \cap \mathcal{C}^+$, which is an upper bound with an absolute constant, depending on Ω and on the conjugate point x_0. Since the above estimate can be proven with the same reasoning on $(\mathcal{S}_{h_0}(x_0) \setminus (\Sigma \cup \mathcal{S})) \cap \mathcal{C}^-$, the result is complete. □

Corollary 3.2.13. *There exist a constant $C > 0$ and an integer $\bar{n} \geq 2$ such that for any $x \in \Omega \setminus \tilde{\mathcal{S}}$ we have*

$$C(x) \leq \frac{C}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2\bar{n}-1}}}. \tag{3.2.48}$$

Proof– This is an immediate consequence of the previous lemma and of the finiteness of the set of regular conjugate points Γ. In particular, $\bar{n}$ is the smallest n arisen in the previous local estimates. □

Roughly speaking, the previous result is an estimate of the "explosion speed" of the local Lipschitz constant of τ when approaching the set $\tilde{\mathcal{S}}$. In the following computations, it will be important to know also the behaviour of τ when restricted to $\tilde{\mathcal{S}}$.

Lemma 3.2.14. *The restriction of τ to $\tilde{\mathcal{S}}$ is Lipschitz continuous.*

Proof–By definition τ is zero on the (closure of the) ridge set and linear, with rate 1, on any ray $[x_0, x_0 - d(x_0)Dd(x_0)]$, when $x_0 \in \Gamma$. So, in order to find the (global)

Lipschitz constant of τ on the set $\tilde{\mathcal{S}}$, it suffices to estimate $|\tau(x) - \tau(y)|$ in the case when

$$x \in [x_0, x_0 - d(x_0)Dd(x_0)] =: \mathcal{S}(x_0)$$

and

$$y \in [y_0, y_0 - d(y_0)Dd(y_0)] =: \mathcal{S}(y_0)$$

for some $x_0, y_0 \in \Gamma$, $x_0 \neq y_0$ and when

$$x \in [x_0, x_0 - d(x_0)Dd(x_0)] =: \mathcal{S}(x_0) \qquad \text{and} \qquad y \in \Sigma.$$

In the former case we have

$$\begin{aligned} |\tau(x) - \tau(y)| &= ||x - x_0| - |y - y_0|| \leq |x - y| + |y_0 - x_0| \\ &\leq |x - y| \left(1 + \frac{\operatorname{diam}(\Omega)}{\min_{x_0 \neq y_0 \in \Gamma} \operatorname{dist}(\mathcal{S}(x_0), \mathcal{S}(y_0))}\right), \end{aligned}$$

since the set of conjugate points is finite and $\operatorname{dist}(\mathcal{S}(x_0), \mathcal{S}(y_0)) > 0$ for $x_0 \neq y_0$. In the latter case, the special structure of the ridge set for an analytic boundary, and in particular the finiteness of the set of conjugate points, guarantees that there exists a cone $\mathcal{C}_0$, with apex x_0, semi–vertex angle $\theta_0 = \theta(x_0) > 0$ and symmetry axis containing the segment $\mathcal{S}(x_0)$ such that $\mathcal{C}_0 \cap \Sigma = \emptyset$. Hence, $|x - y| > d_{\mathcal{C}_0}(x) = |x - x_0| \sin\theta_0$, which gives

$$|\tau(x) - \tau(y)| = |\tau(x)| = |x - x_0| < \frac{1}{\sin\theta_0}|x - y|.$$

Defining γ as the maximum of $1/\sin\theta(x_0)$ over all $x_0 \in \Gamma$, the Lipschitz constant of τ over $\tilde{\mathcal{S}}$ is given by

$$L := \max\left\{\gamma, 1 + \frac{\operatorname{diam}(\Omega)}{\min_{x_0 \neq y_0 \in \Gamma} \operatorname{dist}(\mathcal{S}(x_0), \mathcal{S}(y_0))}\right\}. \tag{3.2.49}$$

□

Summarizing the properties of the map τ obtained so far in this section and in Section 3.1.1, we can say that in the case of a bounded simply connected domain with analytic boundary, different from a disk,

1. τ is locally Lipschitz continuous on $\Omega \setminus \tilde{\mathcal{S}}$ and almost everywhere

$$|\nabla \tau(x)| \leq \frac{C}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2\bar{n}-1}}},$$

where $C > 0$ and $\bar{n} \in \mathbb{N}$ are the ones of Corollary 3.2.13.

2. τ is continuous on Ω.

3. τ is Lipschitz continuous on $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}$ has empty interior.

Now we are going to show that 1–3 are enough to conclude the Hölder continuity of τ on the whole set Ω. Aiming at this, we need another technical lemma.

Lemma 3.2.15. *Let $\phi : [0,1] \to \mathbb{R}$ be a locally Lipschitz continuous function on $(0,1]$ such that for some $\alpha \in (0,1)$ $|\phi'(t)| \leq \frac{C}{t^\alpha}$ almost everywhere and ϕ in continuous on $[0,1]$. Then $|\phi(1) - \phi(0)| \leq \frac{C}{1-\alpha}$.*

Proof– For any $s \in (0,1)$

$$|\phi(1) - \phi(s)| \leq \left| \int_s^1 \phi'(u)\, du \right| \leq \int_s^1 \frac{C}{u^\alpha}\, du = \frac{C}{1-\alpha} \left[u^{1-\alpha}\right]_s^1 \leq \frac{C}{1-\alpha}.$$

Letting $s \to 0^+$ and using the continuity of ϕ the previous inequality gives the result. □

Theorem 3.2.16. *Let Ω be a bounded simply connected domain with analytic boundary, different from a disk. Then τ is Hölder continuous in Ω with exponent $\frac{2\bar{n}-2}{2\bar{n}-1}$, being $\bar{n}$ as in Corollary 3.2.13. In particular, the map τ is at least $2/3$-Hölder continuous.*

Proof– Since $\tilde{\mathcal{S}}$ has empty interior and τ in continuous on Ω, it is enough to show that there exists some constant $C' > 0$ such that

$$|\tau(x) - \tau(y)| \leq C'|x - y|^{\frac{2\bar{n}-2}{2\bar{n}-1}} \qquad \forall x, y \in \Omega \setminus \tilde{\mathcal{S}}. \tag{3.2.50}$$

We distinguish two cases.

Case 1: Assume that $\max\{d_{\tilde{\mathcal{S}}}(x), d_{\tilde{\mathcal{S}}}(y)\} \le 2|x-y|$. Then

$$|\tau(x)-\tau(y)| \le |\tau(x)-\tau(x_1)| + |\tau(x_1)-\tau(y_1)| + |\tau(y_1)-\tau(y)|,$$

where x_1 and y_1 belong to the projection set of x and y on $\tilde{\mathcal{S}}$ respectively. Now set

$$\phi(s) := \tau(x_1 + s(x-x_1)), \qquad \text{for } s \in [0,1].$$

Since $x_1 + s(x-x_1) \notin \tilde{\mathcal{S}}$ for $s \in (0,1]$ and $\phi'(s) = \nabla\tau(x_1+s(x-x_1)) \cdot (x-x_1)$ almost everywhere, we have by property 1. above that

$$|\phi'(s)| \le \frac{C|x-x_1|}{d_{\tilde{\mathcal{S}}}(x_1+s(x-x_1))^{\frac{1}{2\bar{n}-1}}} \qquad \text{a.e. } s \in (0,1],$$

where $C > 0$ and $\bar{n} \in \mathbb{N}$ are the ones of Corollary 3.2.13. Also notice that ϕ is continuous on $[0,1]$ because τ is continuous on Ω and that $d_{\tilde{\mathcal{S}}}(x_1+s(x-x_1)) = s|x-x_1|$. Hence we can apply Lemma 3.2.15 to ϕ, obtaining

$$|\tau(x)-\tau(x_1)| = |\phi(1)-\phi(0)| \le \frac{C(2\bar{n}-1)}{2\bar{n}-2}|x-x_1|^{\frac{2\bar{n}-2}{2\bar{n}-1}} \le 2C|x-x_1|^{\frac{2\bar{n}-2}{2\bar{n}-1}}. \quad (3.2.51)$$

Arguing in the same way for y we get $|\tau(y)-\tau(y_1)| \le 2C|y-y_1|^{\frac{2\bar{n}-2}{2\bar{n}-1}}$. Moreover, being τ Lipschitz continuous of constant L on $\tilde{\mathcal{S}}$ (see (3.2.49)), then

$$|\tau(x)-\tau(y)| \le 2C\left[|x-x_1|^{\frac{2\bar{n}-2}{2\bar{n}-1}} + |y-y_1|^{\frac{2\bar{n}-2}{2\bar{n}-1}}\right] + L|y_1-x_1|.$$

By assumption $|x-x_1| = d_{\tilde{\mathcal{S}}}(x) \le 2|x-y|$ and $|y-y_1| = d_{\tilde{\mathcal{S}}}(y) \le 2|x-y|$. Thus $|y_1-x_1| \le |y-y_1| + |x-y| + |x-x_1| \le 5|x-y|$. Therefore, setting

$$C' := 2 \cdot 2^{\frac{2\bar{n}-2}{2\bar{n}-1}}C + 5L\mathrm{diam}(\Omega)^{\frac{1}{2\bar{n}-1}},$$

we conclude that $|\tau(x)-\tau(y)| \le C'|x-y|^{\frac{2\bar{n}-2}{2\bar{n}-1}}$ in the above hypotheses.

Case 2: Suppose now that $\max\{d_{\tilde{\mathcal{S}}}(x), d_{\tilde{\mathcal{S}}}(y)\} > 2|x-y|$. Without loss of generality we can assume that $d_{\tilde{\mathcal{S}}}(x) > 2|x-y|$. Then for any $z \in [x,y]$ we have

$$d_{\tilde{\mathcal{S}}}(z) \ge d_{\tilde{\mathcal{S}}}(x) - |z-x| \ge 2|x-y| - |y-x| = |y-x|.$$

Hence the map $\phi(s) := \tau(x + s(y - x))$ is well defined and satisfies

$$|\phi'(s)| \leq \frac{C|x-y|}{d_{\tilde{\mathcal{S}}}(x + s(y-x))^{\frac{1}{2\bar{n}-1}}} \leq \frac{C|x-y|}{|x-y|^{\frac{1}{2\bar{n}-1}}} = C|x-y|^{\frac{2\bar{n}-2}{2\bar{n}-1}}$$

almost everywhere. Hence

$$|\tau(x) - \tau(y)| = |\phi(1) - \phi(0)| \leq C|x-y|^{\frac{2\bar{n}-2}{2\bar{n}-1}}.$$

Since $C' > C$, (3.2.50) is proven. $\square$

3.2.3 The Regularity of v_f

In this section we will analyze the regularity of the map $v_f : \overline{\Omega} \to \mathbb{R}$ defined by

$$v_f(x) = \begin{cases} \displaystyle\int_0^{\tau(x)} f(x + tDd(x)) \frac{1 - (d(x) + t)\kappa(x)}{1 - d(x)\kappa(x)}\, dt & \forall x \in \overline{\Omega} \backslash \overline{\Sigma} \\ 0 & \forall x \in \overline{\Sigma}, \end{cases} \tag{3.2.52}$$

where $f : \overline{\Omega} \to \mathbb{R}$ is a non–negative continuous function. The aim of this section is to show that v_f is a Hölder continuous function on Ω under the standing assumptions that

f is a Lipschitz continuous function in Ω

and

Ω is a simply connected bounded domain of $\mathbb{R}^2$

with analytic boundary, different from a disk.

Remark 3.2.17. As in the case of the maximal retraction length of Ω onto $\overline{\Sigma}$, we exclude a priori the case when Ω is a disk, because otherwise v_f is Lipschitz continuous as soon as f is. Indeed, let $\Omega = B_R(0)$, for some $R > 0$. Then,

$$d(x) = R - |x|, \qquad \tau(x) = |x|, \qquad \kappa(x) = 1/R, \qquad \forall x \in \overline{B}_R(0).$$

Hence, for any choice of x, $y \in \overline{B}_R(0) \setminus \{0\}$ with $|x| \geq |y|$ we have

$$\begin{aligned}
|v_f(y) - v_f(x)| = &\left| \int_0^{\tau(y)} f(y + tDd(y)) \frac{1 - (d(y)+t)\kappa(y)}{1 - d(y)\kappa(y)} \, dt \right. \\
&\left. - \int_0^{\tau(x)} f(x + tDd(x)) \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)} \, dt \right| \\
\leq & \int_0^{|y|} \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)} \, |f(y + tDd(y)) - f(x + tDd(x))| \, dt \\
&+ \int_0^{|y|} f(y + tDd(y)) \left| \frac{1 - (d(y)+t)\kappa(y)}{1 - d(y)\kappa(y)} - \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)} \right| \, dt \\
&+ \int_{|y|}^{|x|} f(x + tDd(x)) \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)} \, dt \\
=: & \, I_1 + I_2 + I_3,
\end{aligned}$$

But,

$$\begin{aligned}
|f(y + tDd(y)) - f(x + tDd(x))| &\leq \|f\|_{Lip} \left| x - t\tfrac{x}{|x|} - y + t\tfrac{y}{|y|} \right| \\
&\leq \|f\|_{Lip} \left\{ |x - y| + t \left| \tfrac{|x|y - |y|x}{|x||y|} \right| \right\}, \\
&\leq \|f\|_{Lip} |x - y| \left\{ 1 + t \left(\tfrac{1}{|y|} + \tfrac{1}{|x|} \right) \right\}
\end{aligned}$$

$$\left| \frac{1 - (d(y)+t)\kappa(y)}{1 - d(y)\kappa(y)} - \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)} \right| = t \left| \frac{|y| - |x|}{|x||y|} \right|$$

and

$$\frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)} = \frac{|x| - t}{|x|} \leq 1.$$

Hence,

$$I_1 \leq \|f\|_{Lip} |x - y| \left\{ |y| + \frac{|y|^2}{2} \left(\frac{1}{|y|} + \frac{1}{|x|} \right) \right\} \leq \|f\|_{Lip} (R+1) |x - y|,$$

$$I_2 \leq \|f\|_\infty \frac{|y|^2}{2} \left| \frac{|y| - |x|}{|x||y|} \right| \leq \frac{\|f\|_\infty}{2} |x - y|$$

and

$$I_3 \leq \|f\|_\infty (|x| - |y|) \leq \|f\|_\infty |x - y|,$$

which gives the Lipschitz continuity of v_f in $\overline{B}_R(0)$.
We also remark that if Ω is not a disk, Lipschitz continuity may fail, as it can be seen by considering the parabola case together with the choice $f \equiv 1$ in $\overline{\Omega}$.

In what follows, we will denote by $\tilde{\Sigma}$ the set

$$\tilde{\Sigma} = \Sigma \cup \left(\bigcup_{\substack{x \in \partial\Omega \\ \tau(x)\kappa(x)=1}} [x, x + \tau(x) Dd(x)] \right). \tag{3.2.54}$$

The precise regularity statement for v_f is the following.

Theorem 3.2.18. *Assume that f is a Lipschitz continuous function and that Ω is a simply connected bounded domain of $\mathbb{R}^2$ with analytic boundary, different from a disk. Then v_f is a Hölder continuous function with exponent $\frac{1}{2m-1}$ for some suitable $m \in \mathbb{N}$, $m \geq 2$ depending on the geometry of $\partial\Omega$.*

Proof– The regularity result on v_f will be proven by an argument similar to the one used to prove the Hölder continuity of the normal distance τ. We will then divide the proof in several steps, aiming at the following goals:
step 1 We prove that for all $x \in \Omega \setminus \tilde{\Sigma}$ there exists a ball $B_r(x) \subset \Omega \setminus \tilde{\Sigma}$, with $r = r(x) > 0$, such that for any $y \in B_r(x)$

$$v_f(y) - v_f(x) \leq C \left(1 + \frac{\tau(x)^2}{(1 - d(x)\kappa(x))^3} + \frac{1}{d_{\tilde{\Sigma}}(x)^{\frac{1}{2\bar{n}-1}}} \right) |x - y|, \tag{3.2.55}$$

where $C > 0$ depends on f and Ω only, and $\bar{n} \in \mathbb{N}$ is the integer that appears in Theorem 3.2.6.
step 2 We show that for all $x \in \Omega \setminus \tilde{\Sigma}$

$$\frac{\tau(x)^2}{(1 - d(x)\kappa(x))^3} \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}} \tag{3.2.56}$$

for some constant $C > 0$ independent of x and some $m \in \mathbb{N}$, $2 \leq \bar{n} \leq m$. In this way we can rewrite (3.2.55) as

$$v_f(y) - v_f(x) \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}} |x - y|, \tag{3.2.57}$$

where $C > 0$ is some constant independent on x.
step 3 We show that v_f is differentiable almost everywhere in $\Omega \setminus \tilde{\Sigma}$, with

$$|\nabla v_f(x)| \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}}, \qquad \text{a.e.} \tag{3.2.58}$$

step 4 We prove the Lipschitz continuity of v_f when restricted to $\tilde{\Sigma}$.

step 5 We conclude the proof as in Theorem 3.2.6 by applying Lemma 3.2.15.

Let us start our argument.

step 1 Consider any $x \in \Omega \setminus \tilde{\Sigma}$ and set $\eta(x) := \frac{\kappa(x)}{1-d(x)\kappa(x)}$, so that we can write $\frac{1-(d(x)+t)\kappa(x)}{1-d(x)\kappa(x)} = 1 - t\eta(x)$ in (3.2.52). Notice that

$$1 - t\eta(x) \le 1 + \operatorname{diam}(\Omega) \max_{x \in \partial\Omega} [\kappa(x)]_-, \qquad \forall\, x \in \Omega \setminus \tilde{\Sigma},\, t \in [0, \tau(x)) \tag{3.2.59}$$

and

$$|\eta(x)|\tau(x) \le \max\left\{1; \operatorname{diam}(\Omega) \max_{x \in \partial\Omega} [\kappa(x)]_-\right\}, \qquad \forall\, x \in \Omega \setminus \tilde{\Sigma}, \tag{3.2.60}$$

where $[\kappa(x)]_- := \max\{0, -\kappa(x)\}$. Now, for any $x \in \Omega \setminus \tilde{\Sigma}$ choose $r > 0$ such that $B_r(x) \subset \Omega \setminus \tilde{\Sigma}$ and

$$\begin{cases} |\eta(y)| \le |\eta(x)| + 1, \\ \dfrac{1}{1 - d(y)\kappa(y)} \le \dfrac{2}{1 - d(x)\kappa(x)}, \end{cases} \qquad \text{for all } y \in B_r(x). \tag{3.2.61}$$

Suppose first that $\tau(y) \le \tau(x)$. Thus, for any $y \in B_r(x)$

$$\begin{aligned} v_f(y) - v_f(x) &= \int_0^{\tau(y)} f(y + tDd(y))(1 - t\eta(y))\, dt \\ &\quad - \int_0^{\tau(x)} f(x + tDd(x))(1 - t\eta(x))\, dt \\ &\le \int_0^{\tau(y)} [f(y + tDd(y))(1 - t\eta(y)) - f(x + tDd(x))(1 - t\eta(x))]\, dt \\ &\le \int_0^{\tau(x)} |(1 - t\eta(x))\,(f(y + tDd(y)) - f(x + tDd(x)))|\, dt \\ &\quad + \|f\|_\infty \frac{\tau(x)^2}{2} |\eta(y) - \eta(x)| \\ &=: I_1 + I_2. \end{aligned} \tag{3.2.62}$$

Observe that (3.2.59) and (3.2.61) give

$$\begin{aligned} I_1 &\le \|f\|_{Lip} \int_0^{\tau(x)} (1 - t\eta(x))\,[|x - y| + t|Dd(x) - Dd(y)|]\, dt \\ &\le \|f\|_{Lip}(1 + \operatorname{diam}(\Omega) \max\nolimits_{x \in \partial\Omega} [\kappa(x)]_-)\tau(x)|x - y| \\ &+ \|f\|_{Lip}(1 + \operatorname{diam}(\Omega) \max_{x \in \partial\Omega} [\kappa(x)]_-)\frac{\tau(x)^2}{2}|Dd(x) - Dd(y)|, \end{aligned}$$

where

$$\begin{aligned}
|Dd(x) - Dd(y)| &= \left| \int_0^1 D^2 d(y + t(x-y)) \cdot (x-y)\, dt \right| \\
&= \left| \int_0^1 -\eta(x_t)(\mathcal{R}Dd(x_t) \otimes \mathcal{R}Dd(x_t)) \cdot (x-y)\, dt \right| \\
&\leq \int_0^1 |\eta(x_t)|\, dt\, |x-y| \\
&\leq (|\eta(x)| + 1)|x-y|, \qquad (3.2.63)
\end{aligned}$$

(in the above inequalities x_t stands for $y + t(x-y)$). Applying (3.2.60), we can write

$$I_1 \leq C_1 |x-y|$$

where C_1 is a constant depending on the data f and Ω, but independent of the choice of x. So it remains to estimate I_2. We have

$$\begin{aligned}
&\tau(x)^2 |\eta(x) - \eta(y)| = \tau(x)^2 \frac{|\kappa(x) - \kappa(y) - (d(x) - d(y))\kappa(x)\kappa(y)|}{(1 - d(x)\kappa(x))(1 - d(y)\kappa(y))} \\
&\leq \frac{\tau(x)^2 |\kappa(x) - \kappa(y)|}{(1 - d(x)\kappa(x))(1 - d(y)\kappa(y))} + \frac{\tau(x)^2 |d(x) - d(y)|\, |\kappa(x)\kappa(y)|}{(1 - d(x)\kappa(x))(1 - d(y)\kappa(y))} \\
&=: I_{21} + I_{22}.
\end{aligned}$$

Now, exploiting (3.2.60), (3.2.61) and (3.2.63) we get

$$\begin{aligned}
\frac{\tau(x)^2 |\kappa(x)\kappa(y)|}{(1 - d(x)\kappa(x))(1 - d(y)\kappa(y))} &= \tau(x)^2 |\eta(x)\eta(y)| \\
&\leq \tau(x)^2 |\eta(x)|(1 + |\eta(x)|) \leq C_{22},
\end{aligned}$$

and then

$$I_{22} \leq C_{22} |x-y|, \qquad (3.2.64)$$

where C_{22} is a constant depending on Ω and independent of x. On the other hand, denoting by $\|\kappa\|_{Lip}$ the Lipschitz constant of κ over $\partial\Omega$, we have

$$\begin{aligned}
I_{21} &= \tau(x)^2 \frac{|\kappa(x)-\kappa(y)|}{(1-d(x)\kappa(x))(1-d(y)\kappa(y))} \\
&= \tau(x)^2 \frac{|\kappa(x-d(x)Dd(x))-\kappa(y-d(y)Dd(y))|}{(1-d(x)\kappa(x))(1-d(y)\kappa(y))} \\
&\leq \tau(x)^2 \frac{\|\kappa\|_{Lip}\,(2|x-y|+\operatorname{diam}(\Omega)|Dd(x)-Dd(y)|)}{(1-d(x)\kappa(x))(1-d(y)\kappa(y))} \\
&\leq 2\tau(x)^2 \frac{\|\kappa\|_{Lip}\,(2|x-y|+\operatorname{diam}(\Omega)(|\eta(x)|+1)|x-y|)}{(1-d(x)\kappa(x))^2} \\
&\leq C_{21} \frac{\tau(x)^2}{(1-d(x)\kappa(x))^3}|x-y|,
\end{aligned}$$

where $C_{21}=C_{21}(\Omega)$ is independent of x. Summarizing the previous computations we can write

$$v_f(y)-v_f(x) \leq \tilde{C}_1 \left(1+\frac{\tau(x)^2}{(1-d(x)\kappa(x))^3}\right)|x-y|, \tag{3.2.65}$$

where $\tilde{C}_1$ is a constant depending on the data f and Ω, but independent of the choice of x. If we consider the case of $\tau(y)>\tau(x)$, it is easy to see that inequality (3.2.62) becomes

$$\begin{aligned}
v_f(y)-v_f(x) &= \int_0^{\tau(y)} f(y+tDd(y))(1-t\eta(y))\,dt \\
&\quad - \int_0^{\tau(x)} f(x+tDd(x))(1-t\eta(x))\,dt \\
&\leq \left| \int_0^{\tau(x)} [f(y+tDd(y))(1-t\eta(y))-f(x+tDd(x))(1-t\eta(x))]\,dt \right| \\
&\quad + \left| \int_{\tau(x)}^{\tau(y)} f(y+tDd(y))(1-t\eta(y))\,dt \right| \\
&\leq \int_0^{\tau(x)} |(1-t\eta(x))\,(f(y+tDd(y))-f(x+tDd(x)))|\,dt \\
&\quad + \|f\|_\infty \frac{\tau(x)^2}{2}|\eta(y)-\eta(x)| + \int_{\tau(x)}^{\tau(y)} f(y+tDd(y))(1-t\eta(y))\,dt \\
&=: I_1+I_2+I_3,
\end{aligned} \tag{3.2.66}$$

where I_1 and I_2 are exactly the same of inequality (3.2.62). Moreover,

$$I_3 \leq (1 + \mathrm{diam}(\Omega) \max_{x \in \partial\Omega} [\kappa(x)]_-) \|f\|_\infty (\tau(y) - \tau(x)).$$

Hence, by Theorem 3.2.6, Corollary 3.2.13 and inclusion $\tilde{\mathcal{S}} \subset \tilde{\Sigma}$ we deduce that

$$I_3 \leq \frac{C_3}{d_{\tilde{\mathcal{S}}}(x)^{\frac{1}{2\bar{n}-1}}} |x - y| \leq \frac{C_3}{d_{\tilde{\Sigma}}(x)^{\frac{1}{2\bar{n}-1}}} |x - y|, \tag{3.2.67}$$

for some constants $C_3 > 0$, $\bar{n} \in \mathbb{N}$ which depend on the data f and Ω, but are independent of the choice of x. Therefore if $\tau(y) > \tau(x)$ we have

$$v_f(y) - v_f(x) \leq \tilde{C}_2 \left(1 + \frac{\tau(x)^2}{(1 - d(x)\kappa(x))^3} + \frac{1}{d_{\tilde{\Sigma}}(x)^{\frac{1}{2\bar{n}-1}}} \right) |x - y|, \qquad \tilde{C}_2 > 0. \tag{3.2.68}$$

The first step is then concluded with the choice of $C = \tilde{C}_2$ in (3.2.55).

step 2 Let us now estimate

$$E(x) := \frac{\tau(x)^2}{(1 - d(x)\kappa(x))^3}.$$

We can restrict our attention to a neighborhood of the set of conjugate points, since we can globally bound $E(x)$ on the complement of such a set. In order to do so, we first give some local estimates around each conjugate point, giving next a global estimate on the basis of the finiteness of the set of conjugate points (regular and singular). Let us start considering the case of a regular conjugate point x_0. Once again suppose that $x_0 = (0, r)$, $r > 0$, and $\Pi(x_0) = \{(0, 0)\}$. Moreover, suppose that $\partial\Omega$ is locally the graph of an analytic function $\alpha : (-s_0, s_0) \to \mathbb{R}$ such that $\alpha(0) = 0$, $\alpha'(0) = 0$ and $\alpha''(0) = \frac{1}{r}$ and let $n = n(x_0) \geq 2$ be the integer such that representation (3.2.22) holds true. We claim that for some h_0 sufficiently small there exists a constant $C_0 > 0$ such that

$$d_{\tilde{\Sigma}}(x) \leq C_0 \tau(x) s, \qquad \forall\, x \in B_{h_0}(x_0) \setminus \tilde{\Sigma}, \tag{3.2.69}$$

where s is the parameter defining the projection $(s, \alpha(s))$ of the point x. Indeed, let $t(s)$ and $\xi(s)$ be as in Lemma 3.2.10, and y be the point on the line segment

$[(-t(s), \alpha(-t(s))); \xi(s)]$ satisfying $|y-\xi(s)| = |x-\xi(s)|$. Then, from Thales Theorem

$$\frac{|y-x|}{((s+t(s))^2 + (\alpha(s) - \alpha(-t(s)))^2)^{1/2}} = \frac{\tau(x)}{d(\xi(s))}.$$

Recalling that by definition $d(\xi(s)) \geq d(x_0)$ and that

$$\left((s+t(s))^2 + (\alpha(s) - \alpha(-t(s)))^2\right)^{1/2} = 2s + \mathrm{o}(s),$$

we get

$$|y-x| \leq C_0 \tau(x) s, \qquad \forall\, x \in B_{h_0}(x_0) \setminus \tilde{\Sigma},$$

for some $C_0 > 0$, provided that s is sufficiently small and a fortiori for h_0 small enough. Thus (3.2.69) follows, since the segment $[x; y]$ intersect $\tilde{\Sigma}$ and then $d_{\tilde{\Sigma}}(x) \leq |y-x|$. Now, let us conclude the estimate of $E(x)$ in the set $B_{h_0}(x_0) \setminus \tilde{\Sigma}$. We distinguish two cases: $\tau(x) > |\xi(s) - x_0|$ and $\tau(x) \leq |\xi(s) - x_0|$. In what follows we will assume (eventually reducing h_0) that $\kappa(x) \geq \frac{\kappa(x_0)}{2} = \frac{1}{2r}$ in $B_{h_0}(x_0) \setminus \tilde{\Sigma}$, since κ is a continuous function on $\Omega \setminus \Sigma$. Suppose first that $\tau(x) > |\xi(s) - x_0|$. Then, taking into account that $\xi(s) = \left(\mathrm{o}(s^{2n-2}), r + 2nr^2 a s^{2n-2} + \mathrm{o}(s^{2n-2})\right)$, we have

$$|\xi(s) - x_0| = 2nr^2 a s^{2n-2} + \mathrm{o}(s^{2n-2}) \leq C s^{2n-2} \tag{3.2.70}$$

provided h_0 is small enough. Thus, since $1 - d(x)\kappa(x) \geq \tau(x)\kappa(x)$ (because $(\tau(x) + d(x))\kappa(x) \leq 1)$,

$$\begin{aligned} d_{\tilde{\Sigma}}(x)^\alpha \frac{\tau(x)^2}{(1-d(x)\kappa(x))^3} &\leq \frac{C_0^\alpha \tau(x)^{2+\alpha} s^\alpha}{(\tau(x)\kappa(x))^3} \\ \leq \frac{C_1 s^\alpha}{|\xi(s) - x_0|^{1-\alpha}} &\leq C_2 s^{\alpha(2n-1)-2n+2}. \end{aligned} \tag{3.2.71}$$

On the other hand, when $\tau(x) \leq |\xi(s) - x_0|$, taking into account that by (3.2.46)

$$1 - d(\xi(s))\kappa(\xi(s)) = 4n(n-1)ras^{2n-2} + \mathrm{o}(s^{2n-2}) \geq \tilde{C} s^{2n-2}$$

(for h_0 is sufficiently small and some $\tilde{C} > 0$), we have

$$\begin{aligned} d_{\tilde{\Sigma}}(x)^\alpha \frac{\tau(x)^2}{(1-d(x)\kappa(x))^3} &\leq \frac{C_0^\alpha \tau(x)^{2+\alpha} s^\alpha}{(1-d(\xi(s))\kappa(\xi(s)))^3} \\ \leq \frac{C_3 s^\alpha |\xi(s) - x_0|^{\alpha+2}}{s^{3(2n-2)}} &\leq C_4 s^{\alpha(2n-1)-2n+2}. \end{aligned} \tag{3.2.72}$$

In both inequalities (3.2.71) and (3.2.72) the exponent $\alpha = \frac{2n-2}{2n-1} < 1$ guarantees that there is some $h_0 > 0$ and a constant $C > 0$ such that

$$E(x) \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2n-2}{2n-1}}}, \qquad \forall\, x \in B_{h_0}(x_0) \setminus \tilde{\Sigma}. \tag{3.2.73}$$

Now, let us estimate $E(x)$ near any singular conjugate point $\tilde{x}_0$. Recall that $\tilde{x}_0$ is singular and conjugate if $\tau(y)\kappa(y) = 1$ for some $y \in \Pi(\tilde{x}_0)$. Hence it is easy to see that $E(x)$ can explode only if x approaches $\tilde{x}_0$ near the projecting line on y. But then the local behaviour of the boundary $\partial\Omega$ around y is not different from the case of the regular conjugate point. In particular, the local representation of $\partial\Omega$ via an analytic map α holds true as before. Moreover, take any analytic curve starting from $\tilde{x}_0$ with direction $Dd(y)$ and let Σ^* be the union of the trace of such a curve with the line segment $[\tilde{x}_0; y]$. Then, $d_{\tilde{\Sigma}}(x) \leq d_{\Sigma^*}(x)$ for any x near the projecting line on y, because the singular arcs emanating from $\tilde{x}_0$ have initial directions that are transversal to $Dd(y)$. Notice that if $\tilde{x}_0$ was a regular conjugate point, then $d_{\Sigma^*}(x)$ would coincide with the distance $d_{\tilde{\Sigma}}(x)$. Therefore, the previous inequality and (3.2.71)–(3.2.72) give

$$d_{\tilde{\Sigma}}(x)^{\frac{2n-2}{2n-1}} \frac{\tau(x)^2}{(1-d(x)\kappa(x))^3} \leq d_{\Sigma^*}(x)^{\frac{2n-2}{2n-1}} \frac{\tau(x)^2}{(1-d(x)\kappa(x))^3} \leq C, \tag{3.2.74}$$

where $n = n(\tilde{x}_0) \geq 2$ is the integer that appears in (3.2.22). Now, being $\Gamma \cup \tilde{\Gamma}$ a finite set, the maximum $m \geq 2$ of all integers n selected in the previous computations fits for the estimates (3.2.73) and (3.2.74) in a suitable neighborhood of $\tilde{\Sigma}$. Moreover, $2 \leq \bar{n} \leq m$, where $\bar{n}$ is the integer that appears in (3.2.55). Hence,

$$\frac{1}{2\bar{n}-1} \leq \frac{2\bar{n}-2}{2\bar{n}-1} \leq \frac{2m-2}{2m-1},$$

which means that actually the estimate (3.2.55) reads

$$v_f(y) - v_f(x) \leq \tilde{C}\left(1 + \frac{1}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}}\right)|x-y|, \qquad \text{for some } \tilde{C} > 0. \tag{3.2.75}$$

Since $d_{\tilde{\Sigma}}(x) \leq \operatorname{diam}(\Omega)$, we finally get (3.2.57) by taking $C = \tilde{C}\left(\operatorname{diam}(\Omega)^{\frac{2m-2}{2m-1}} + 1\right)$.

step 3 As in Proposition 3.2.8 it suffices to prove that for any $x \in \Omega \setminus \tilde{\Sigma}$ we have

$$|p| \leq \frac{C}{d_{\tilde{\Sigma}}(x)^{\frac{2m-2}{2m-1}}} \qquad \forall\, p \in \partial_P v_f(x) \quad \forall\, x \in \Omega \setminus \tilde{\Sigma}, \tag{3.2.76}$$

where $\partial_P v_f(x)$ denotes the proximal subgradient of v_f at x. By definition, a vector $p \in \mathbb{R}^2$ belongs to $\partial_P v_f(x)$ if and only if there exist numbers $\sigma, \eta > 0$ such that

$$v_f(y) \geq v_f(x) + \langle p, y-x\rangle - \sigma|y-x|^2 \qquad \forall\, y \in B_\eta(x)\,,$$

see Definition 1.2.17. Now, combine the above inequality with (3.2.57) to obtain

$$\langle p, y-x\rangle \leq \frac{C}{d_{\tilde\Sigma}(x)^{\frac{2m-2}{2m-1}}}|y-x| + \sigma|y-x|^2$$

whenever $|y-x| < \min\{r, \eta\}$. The last inequality implies (3.2.76). The differentiability almost everywhere of v_f in $\Omega \setminus \tilde\Sigma$ and (3.2.58) follow from (3.2.76), combined with Theorem 1.2.19 and Lemma 1.2.18.

step 4 Let us prove the Lipschitz continuity of v_f on $\tilde\Sigma$. By definition, $v_f \equiv 0$ on $\overline{\Sigma}$. Moreover, by Proposition 3.1.7 and Proposition 3.1.11, for any $x \in \Omega \setminus \overline{\Sigma}$ and $\theta \in (0, \tau(x))$, we have

$$v_f(x) \leq \|f\|_\infty \left[1 + \max_{x\in\partial\Omega}[\kappa(x)]_-\right]\tau(x) =: K_-\tau(x) \tag{3.2.77}$$

and

$$\begin{aligned} &v_f(x) - \frac{1-(d(x)+\theta)\kappa(x)}{1-d(x)\kappa(x)} v_f(x+\theta Dd(x)) \\ &= \int_0^\theta f(x+tDd(x))\frac{1-(d(x)+t)\kappa(x)}{1-d(x)\kappa(x)}\,dt. \end{aligned}$$

Hence,

(i) if $x \in\,]x_0, x_0+\tau(x_0)Dd(x_0)]$ for some $x_0 \in \partial\Omega$ such that $\kappa(x_0)\tau(x_0) = 1$ and $y = x_0 + \tau(x_0)Dd(x_0)$, then

$$|v_f(x) - v_f(y)| = |v_f(x)| \leq K_-\tau(x) = K_-|x-y|; \tag{3.2.78}$$

(ii) if $x, y \in\,]x_0, x_0+\tau(x_0)Dd(x_0)]$ for some $x_0 \in \partial\Omega$ such that $\kappa(x_0)\tau(x_0) = 1$, then we can suppose without loss of generality that $\tau(x) > \tau(y)$, obtaining

$$\begin{aligned}
&|v_f(x) - v_f(y)| = |v_f(x) - v_f(x + |x-y|Dd(x))| = \\
&v_f(x + |x-y|Dd(x)) \left| 1 - \frac{1 - (d(x) + |x-y|)\kappa(x)}{1 - d(x)\kappa(x)} \right| \\
&+ \int_0^{|x-y|} f(x + tDd(x)) \frac{1 - (d(x)+t)\kappa(x)}{1 - d(x)\kappa(x)}\, dt \\
&\leq K_- \tau(y) \frac{|x-y|\kappa(x)}{1 - d(x)\kappa(x)} + K_-|x-y| \qquad (3.2.79) \\
&\leq K_-|x-y| \frac{\tau(x)\kappa(x)}{1 - d(x)\kappa(x)} + K_-|x-y| \\
&\leq K_- \left(1 + \frac{\mathrm{diam}(\Omega)}{2} \max_{x \in \partial\Omega} [\kappa(x)]_- \right) |x-y| =: L_1|x-y|
\end{aligned}$$

(iii) if

$$x \in\ [x_0, x_0 + \tau(x_0)Dd(x_0)] =: \mathcal{S}(x_0)$$

and

$$y \in\ [y_0, y_0 + \tau(y_0)Dd(y_0)] =: \mathcal{S}(y_0)$$

for some $x_0 \neq y_0 \in \partial\Omega$ with $\kappa(x_0)\tau(x_0) = \kappa(y_0)\tau(y_0) = 1$, then

$$\begin{aligned}
&|v_f(x) - v_f(y)| \leq |v_f(x)| + |v_f(y)| \leq K_-\mathrm{diam}(\Omega) \\
&\leq \frac{K_-\mathrm{diam}(\Omega)}{\min_{x_0 \neq y_0 \in \Gamma \cup \tilde{\Gamma}} \mathrm{dist}(\mathcal{S}(x_0), \mathcal{S}(y_0))} |x-y| =: L_2|x-y|
\end{aligned} \qquad (3.2.80)$$

(iv) if $x \in \mathcal{S}(x_0)$ for some $x_0 \in \partial\Omega$ with $\kappa(x_0)\tau(x_0) = 1$ and $y \in \Sigma$, we have to distinguish to sub-cases.
If $x_0 + \tau(x_0)Dd(x_0) \in \Gamma$, then the Lipschitz continuity of τ on $\Sigma \cup \mathcal{S}$ (see Lemma 3.2.14) gives

$$|v_f(x) - v_f(y)| = |v_f(x)| \leq K_-\tau(x) \leq K_-L|x-y| =: L_3|x-y|. \qquad (3.2.81)$$

On the other hand, when $x_0 + \tau(x_0)Dd(x_0) \in \tilde{\Gamma}$, as in estimate (3.2.81) we find

$$|v_f(x) - v_f(y)| = |v_f(x)| \leq K_-\tau(x) = K_-|x - x_0|,$$

but we cannot directly apply Lemma 3.2.14, since it does not cover the case of segments starting from singular conjugate points. However, the finiteness of the set of conjugate points guarantees again (as in the case of regular points) that there exists a cone $\tilde{\mathcal{C}}_0$, with apex $x_0+\tau(x_0)Dd(x_0)$, semi–vertex angle $\tilde{\theta}_0 = \tilde{\theta}(x_0+\tau(x_0)Dd(x_0)) > 0$ and symmetry axis containing the segment $\mathcal{S}(x_0)$, such that $\tilde{\mathcal{C}}_0 \cap \Sigma = \emptyset$. Hence, $|x-y| > d_{\tilde{\mathcal{C}}_0}(x) = |x-x_0| \sin\tilde{\theta}_0$, which gives

$$|v_f(x) - v_f(y)| = |v_f(x)| \leq K_-\tau(x) = K_-|x-x_0| < \frac{K_-}{\sin\tilde{\theta}_0}|x-y|.$$

Defining $\tilde{\gamma}$ as the maximum of $1/\sin\tilde{\theta}_0$ over all $\tilde{\theta}_0$ related to singular conjugate points, we then obtain that

$$|v_f(x) - v_f(y)| = |v_f(x)| \leq K_-\tilde{\gamma}|x-y| =: L_4|x-y|. \tag{3.2.82}$$

The (global) Lipschitz continuity of v over all $\tilde{\Sigma}$ follows by taking as Lipschitz constant of v_f the maximal constant L_i, $i = 1, \ldots, 4$, arisen in the above inequalities (3.2.79)–(3.2.82).

step 5 The proof is really the same of Theorem 3.2.6. Since $\tilde{\Sigma}$ has empty interior and v_f in continuous on Ω, it is enough to show that there exists some constant $C' > 0$ such that

$$|v_f(x) - v_f(y)| \leq C'|x-y|^{\frac{1}{2m-1}} \qquad \forall x, y \in \Omega \setminus \tilde{\Sigma}. \tag{3.2.83}$$

We distinguish two cases.

CASE 1: Assume that $\max\{d_{\tilde{\Sigma}}(x), d_{\tilde{\Sigma}}(y)\} \leq 2|x-y|$. Then

$$|v_f(x) - v_f(y)| \leq |v_f(x) - v_f(x_1)| + |v_f(x_1) - v_f(y_1)| + |v_f(y_1) - v_f(y)|,$$

where x_1 and y_1 belong to the projection set of x and y on $\tilde{\Sigma}$ respectively. Now set

$$\phi(s) := v_f(x_1 + s(x-x_1)), \qquad \text{for } s \in [0,1].$$

Since $x_1 + s(x-x_1) \notin \tilde{\Sigma}$ for $s \in (0,1]$ and $\phi'(s) = \nabla v_f(x_1 + s(x-x_1)) \cdot (x-x_1)$ almost everywhere, we have by (3.2.58) that

$$|\phi'(s)| \leq \frac{C|x-x_1|}{d_{\tilde{\mathcal{S}}}(x_1 + s(x-x_1))^{\frac{2m-2}{2m-1}}} \qquad \text{a.e. } s \in (0,1],$$

where $C > 0$ and $m \in \mathbb{N}$ are the ones of step 3. Moreover, $d_{\tilde{\Sigma}}(x_1+s(x-x_1)) = s|x-x_1|$ and ϕ is continuous on $[0,1]$ because v_f is continuous on Ω. Hence we can apply Lemma 3.2.15 to ϕ, obtaining

$$|v_f(x) - v_f(x_1)| = |\phi(1) - \phi(0)| \leq C(2m-1)|x - x_1|^{\frac{1}{2m-1}}. \tag{3.2.84}$$

Arguing in the same way for y we get $|v_f(y) - v_f(y_1)| \leq C(2m-1)|y - y_1|^{\frac{1}{2m-1}}$. Moreover, being v_f Lipschitz continuous–say of constant L–on $\tilde{\mathcal{S}}$ by step 4, then

$$|v_f(x) - v_f(y)| \leq C(2m-1)\left[|x - x_1|^{\frac{1}{2m-1}} + |y - y_1|^{\frac{1}{2m-1}}\right] + L|y_1 - x_1|.$$

By assumption $|x - x_1| = d_{\tilde{\Sigma}}(x) \leq 2|x-y|$ and $|y - y_1| = d_{\tilde{\Sigma}}(y) \leq 2|x-y|$. Thus $|y_1 - x_1| \leq |y - y_1| + |x - y| + |x - x_1| \leq 5|x - y|$. Therefore, setting

$$C' := 2(2m-1) \cdot 2^{\frac{1}{2m-1}} C + 5L\mathrm{diam}(\Omega)^{\frac{2m-2}{2m-1}},$$

we conclude that $|v_f(x) - v_f(y)| \leq C'|x-y|^{\frac{1}{2m-1}}$ in the above hypotheses.
CASE 2: Suppose now that $\max\{d_{\tilde{\Sigma}}(x), d_{\tilde{\Sigma}}(y)\} > 2|x-y|$. Without loss of generality we can assume that $d_{\tilde{\Sigma}}(x) > 2|x-y|$. Then for any $z \in [x,y]$ we have

$$d_{\tilde{\Sigma}}(z) \geq d_{\tilde{\Sigma}}(x) - |z - x| \geq 2|x-y| - |y-x| = |y-x|.$$

Hence the map $\phi(s) := v_f(x + s(y-x))$ is well defined and satisfies

$$|\phi'(s)| \leq \frac{C|x-y|}{d_{\tilde{\Sigma}}(x + s(y-x))^{\frac{2m-2}{2m-1}}} \leq \frac{C|x-y|}{|x-y|^{\frac{2m-2}{2m-1}}} = C|x-y|^{\frac{1}{2m-1}}$$

almost everywhere. Hence

$$|v_f(x) - v_f(y)| = |\phi(1) - \phi(0)| \leq C|x-y|^{\frac{1}{2m-1}}.$$

Since $C' > C$, (3.2.83) is proven. □

Remark 3.2.19. As a concluding remark, we observe once again that when Ω is a bounded domain with no conjugate points (both regular and singular) and $\mathcal{C}^{2,1}$ boundary, then v_f is Lipschitz continuous on the whole set $\overline{\Omega}$, as long as f is, because of the Lipschitz continuity of τ (see Remark 3.2.9).

Chapter 4

The n-dimensional Problem: Existence and Uniqueness

In this chapter we present an extension, to an arbitrary space dimension, of the existence and uniqueness result of [13] for the solutions of system of partial differential equations

$$\begin{cases} -\mathrm{div}\,(vDu) = f & \text{in } \Omega \\ v \geq 0,\ |Du| \leq 1 & \text{in } \ \Omega \\ |Du| - 1 = 0 & \text{in } \ \{v > 0\}, \end{cases} \tag{4.0.1}$$

complemented with the conditions

$$\begin{cases} u \geq 0, & \text{in } \ \Omega \\ u \equiv 0 & \text{on } \ \partial\Omega. \end{cases} \tag{4.0.2}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\mathcal{C}^2$ boundary and $f \geq 0$ is a continuous function in Ω.

We will prove that, in arbitrary space dimension, the unique solution of system (2.0.1)–(4.0.2), in the sense of Definition 3.1.1, is given by the pair (d, v_f), where d is the distance function from the boundary $\partial\Omega$ and

$$v_f(x) = \begin{cases} \int_0^{\tau(x)} f(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1-(d(x)+t)\kappa_i(x)}{1-d(x)\kappa_i(x)}\, dt & \forall x \in \Omega \setminus \overline{\Sigma}, \\ 0 & \forall x \in \overline{\Sigma}, \end{cases} \tag{4.0.3}$$

where $\kappa_i(x)$, $i \in \{1, \dots, n-1\}$ stand for the principal curvatures of $\partial\Omega$ at the (unique) projection $\Pi(x)$ of x onto $\partial\Omega$.

Let us first compare the present work with its two–dimensional analogue [13]. On the one hand, showing that (d, v_f) is a solution of (4.0.1)–(4.0.2) follows the same lines as in dimension two. On the other hand, the proof of the fact that (d, v_f) is the unique solution to (4.0.1)–(4.0.2) requires completely different arguments. In fact, in dimension two one can exploit the relatively simple structure of $\overline{\Sigma}$ to show–by a direct argument–that any solution (u, v) of (4.0.1)–(4.0.2) satisfies $u \equiv d$ in the set $\{x \in \Omega \, : \, v_f(x) > 0\}$ and $v \equiv 0$ on $\overline{\Sigma}$. Such a technique cannot be extended to higher space dimension due to obvious topological obstructions.

So uniqueness is obtained as follows. To see that the first component of a solution of (4.0.1)–(4.0.2) is given by the distance function, we adapt an idea of [35], showing that (u, v) is a saddle point of a suitable integral functional. Then, to identify the second component of (u, v) with the function v_f, we compute the variation of v along all rays $x + tDd(x)$, $0 < t < \tau(x)$–which cover the set $\Omega \setminus \overline{\Sigma}$–as follows:

$$\begin{aligned} & v(x) - \prod_{i=1}^{n-1} \frac{1-(d(x)+t)\kappa_i(x)}{1-d(x)\kappa_i(x)} v(x+tDd(x)) \\ & = \int_0^t f(x+sDd(x)) \prod_{i=1}^{n-1} \frac{1-(d(x)+s)\kappa_i(x)}{1-d(x)\kappa_i(x)} \, ds. \end{aligned} \tag{4.0.4}$$

Finally, using the fact that $v \equiv 0$ on $\overline{\Sigma}$–which can be proven by a blow up argument as in [21]–we easily deduce that $v \equiv v_f$ in Ω.

As in the 2–dimensional case, a major role in our analysis will be played by the following regularity result of [29] (also [31]) on the maximal retraction length of Ω onto $\overline{\Sigma}$.

Theorem 4.0.20. *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary of class $\mathcal{C}^{2,1}$. Then the map τ defined in* (2.4.2) *is Lipschitz continuous on $\partial\Omega$.*

In the 2-dimensional case, we could give a proof of this result even for piecewise $\mathcal{C}^{2,1}$ boundaries (see Theorem 3.1.3), by using, besides integration by parts, the analysis of [4] and [1], describing the propagation of singularities of semi-concave functions. In the general case, such a result seems to be more difficult to obtain with the same techniques and so we refer to [29] or [31] for its proof.

Hereafter, we will denote by $\mathrm{Lip}(\tau)$ the Lipschitz semi-norm of τ on $\partial\Omega$. Since $x \mapsto x+\tau(x)Dd(x)$ maps $\partial\Omega$ onto $\overline{\Sigma}$, a straightforward application of Theorem 4.0.20 is that the (n-1)-dimensional Hausdorff measure of $\overline{\Sigma}$ is finite:

Corollary 4.0.21. *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary of class $\mathcal{C}^{2,1}$. Then,*

$$\mathcal{H}^{n-1}(\overline{\Sigma}) \leq k_\Omega \, \mathcal{H}^{n-1}(\partial\Omega) < \infty$$

where $k_\Omega \geq 0$ is a constant depending on $Lip(\tau)$ and Ω.

For less regular domains the Lipschitz continuity of τ may fail, but continuity is preserved. The proof of the lemma below is the same of Lemma 3.1.2.

Lemma 4.0.22. *Assume that Ω is a connected bounded open subset of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary. Then the map τ, extended to 0 on $\overline{\Sigma}$, is continuous in $\overline{\Omega}$.*

We now give an approximation result that guarantees the stability of the singular set and of the maximal retraction length with respect to the convergence in the $\mathcal{C}^2$ topology. Recall that the signed distance function of Ω is defined as

$$b_\Omega(x) := d_{\mathbb{R}^n \setminus \Omega}(x) - d_{\overline{\Omega}}(x).$$

We say that a sequence of bounded domains $\{\Omega_k\}$ with $\mathcal{C}^2$ boundary converges to Ω in the $\mathcal{C}^2$ topology if b_{Ω_k}, Db_{Ω_k} and $D^2 b_{\Omega_k}$ converge to b_Ω, Db_Ω and $D^2 b_\Omega$, uniformly in a neighborhood of $\partial\Omega$. Also we say that a sequence of closed nonempty sets $\{A_k\}_k$ converges to a closed nonempty set A in the Hausdorff topology if

$$\sup_{x \in \mathbb{R}^n} |d_{A_k}(x) - d_A(x)| \to 0 \qquad \text{as } k \to \infty.$$

Whenever A_k, A are contained in a bounded set of $\mathbb{R}^n$ for all $k \in \mathbb{N}$, it can be shown that the Hausdorff convergence is equivalent to the conditions

$$\limsup_{k\to\infty} A_k \subseteq A, \qquad \liminf_{k\to\infty} A_k \supseteq A,$$

where

$$\limsup_{k\to\infty} A_k := \Big\{x \;:\; \forall\{A_{k_m}\}_m \subseteq \{A_k\}_k\, \exists x_{k_m} \in A_{k_m} \text{ such that } \lim_m x_{k_m} = x\Big\},$$

$$\liminf_{k\to\infty} A_k := \Big\{x \;:\; \exists\{A_{k_m}\}_m \subseteq \{A_k\}_k\, \exists x_{k_m} \in A_{k_m} \text{ such that } \lim_m x_{k_m} = x\Big\}.$$

Proposition 4.0.23. *Let $\{\Omega_k\}$ be a sequence of bounded domains $\{\Omega_k\}$ with $\mathcal{C}^2$ boundary. For any $k \in \mathbb{N}$, denote by Σ_k and τ_k, respectively, the singular set and maximal retraction length of Ω_k. If $\{\Omega_k\}$ converges to Ω in the $\mathcal{C}^2$ topology, then $\{\overline{\Sigma}_k\}$ converges to $\overline{\Sigma}$ in the Hausdorff topology, and $\{\tau_k\}$ converges to τ uniformly on all compact subsets of Ω.*

Proof—Let us prove, first, that the upper limit of $\{\overline{\Sigma}_k\}$ is contained in $\overline{\Sigma}$. For this it suffices to show that, if a sequence $\{x_k\}$ in Σ_k converges to a point $x \in \overline{\Omega}$, then x belongs to $\overline{\Sigma}$. Indeed, let y_k and z_k be two distinct projections of x_k onto $\partial\Omega_k$. Without loss of generality we can assume that both $\{y_k\}$ and $\{z_k\}$ converge to points of $\Pi(x)$, say y and z respectively. If $y \neq z$, then x belongs to Σ and our claim follows. So, suppose $x \in \overline{\Omega}\setminus\Sigma$ and $y = z$. Since $y_k + b_{\Omega_k}(x_k)Db_{\Omega_k}(y_k) = z_k + b_{\Omega_k}(x_k)Db_{\Omega_k}(z_k)$, we have

$$\frac{y_k - z_k}{|y_k - z_k|} = -b_{\Omega_k}(x_k)\frac{Db_{\Omega_k}(y_k) - Db_{\Omega_k}(z_k)}{|y_k - z_k|}\,. \tag{4.0.5}$$

The sequence in the left-hand side above will converge, up to replacement with a subsequence, to some unit vector $\theta \in \mathbb{R}^n$. Then, passing to the limit in (4.0.5) we obtain $\theta = -b_\Omega(x)D^2b_\Omega(y)\theta$. Hence, recalling the structure of the hessian matrix $D^2d(y)$ in Proposition 1.1.5, we conclude that $d(x)\kappa_i(x) = d(x)\kappa_i(y) = 1$ for some $i = 1, \ldots, n-1$. Therefore, x belongs to $\overline{\Sigma}$.

Now, let us prove that the lower limit of the sequence $\{\overline{\Sigma}_k\}$ contains $\overline{\Sigma}$. For this, it suffices to show that $\Sigma \subset \liminf \Sigma_k$. Let $x \in \Omega \setminus \liminf \Sigma_k$. Then, there exists a subsequence $\{\Sigma_{k_m}\}_m$ of $\{\Sigma_k\}$ such that, for some $\varepsilon > 0$, $B_\varepsilon(x) \subset \Omega \setminus \Sigma_{k_m}$. We claim that $B_{\varepsilon/2}(x) \cap \Sigma = \emptyset$. For let $z \in B_{\varepsilon/2}(x)$ and set $y_m = \Pi_{\partial\Omega_{k_m}}(z)$. Since

$$z + \varepsilon\frac{z - y_m}{2b_{\Omega_{k_m}}(z)} \in B_{\varepsilon/2}(z) \subset B_\varepsilon(x) \subset \Omega \setminus \Sigma_{k_m}\,,$$

y_m is also the unique projection of $z + \varepsilon(z - y_m)/2b_{\Omega_{k_m}}(z)$ onto $\partial\Omega_{k_m}$. Now, a subsequence of $\{y_m\}$ will converge to some point $y \in \partial\Omega$ belonging to both $\Pi(z)$ and $\Pi(z + \varepsilon(z - y)/2b_{\Omega}(z))$. Therefore, $z \notin \overline{\Sigma}$ owing to Proposition 1.1.8, and our claim is proved as well as the convergence of $\overline{\Sigma}_k$ to $\overline{\Sigma}$.

We omit the proof that $\{\tau_k\}$ converges to τ, because the reasoning has much in common with the proof Lemma 3.1.2. $\square$

4.1 Existence

In this section we prove that the pair (d, v_f), where d is the distance function from $\partial\Omega$ and

$$v_f(x) = \begin{cases} \int_0^{\tau(x)} f(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1-(d(x)+t)\kappa_i(x)}{1-d(x)\kappa_i(x)}\, dt & \forall x \in \Omega \setminus \overline{\Sigma}, \\ 0 & \forall x \in \overline{\Sigma}, \end{cases} \tag{4.1.1}$$

is a solution of system (4.0.1)–(4.0.2) in the sense of Definition 3.1.1. More precisely, we will prove the following result.

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$ and $f \geq 0$ be a continuous function in Ω. Then, the pair (d, v_f) defined above satisfies* (4.0.1)–(4.0.2) *in the following sense:*

1. *(d, v_f) is a pair of continuous functions*

2. *$d = 0$ on $\partial\Omega$, $\|Dd\|_{\infty,\Omega} \leq 1$, and d is a viscosity solution of*

$$|Du| = 1 \quad \text{in} \quad \{x \in \Omega \ : \ v_f(x) > 0\}$$

3. *$v_f \geq 0$ in Ω and, for every test function $\phi \in \mathcal{C}_c^\infty(\Omega)$,*

$$\int_\Omega v_f(x)\langle Dd(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x) dx\,. \tag{4.1.2}$$

We begin with two preliminary results, the former describing continuity and differentiability properties of v_f, the latter providing an approximation result for the characteristic function of a compact set, in the spirit of capacity theory.

Proposition 4.1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$ and $f \geq 0$ be a continuous function in Ω. Then, v_f is a locally bounded continuous function in Ω. Moreover, in any set $\Omega_\varepsilon := \{x \in \Omega \ : \ d(x) > \varepsilon\}$, $\varepsilon > 0$, v_f satisfies the bound*

$$0 \leq v_f(x) \leq \|f\|_{\infty,\Omega_\varepsilon} \prod_{i=1}^{n-1} \Big[1 + \|[\kappa_i]_-\|_{\mathcal{C}(\partial\Omega)} \mathrm{diam}(\Omega)\Big] \tau(x) \qquad \forall x \in \Omega_\varepsilon, \tag{4.1.3}$$

where $\|[\kappa_i]_-\|_{\mathcal{C}(\partial\Omega)} := \max_{x\in\partial\Omega}[\kappa_i(x)]_-$. If, in addition, $\partial\Omega$ is of class $\mathcal{C}^{2,1}$ and f is Lipschitz continuous in Ω, then v_f is locally Lipschitz continuous in $\Omega\backslash\overline{\Sigma}$ and satisfies

$$-\mathrm{div}\,(v_f(x)Dd(x)) = f(x) \tag{4.1.4}$$

at each point $x \in \Omega\backslash\overline{\Sigma}$ at which v_f is differentiable.

Remark 4.1.3. Since d is $\mathcal{C}^2$ in $\Omega\backslash\overline{\Sigma}$, equality (4.1.4) reads, as in the 2 dimensional case,

$$\langle Dv_f(x), Dd(x)\rangle + v_f(x)\Delta d(x) + f(x) = 0\,. \tag{4.1.5}$$

Also, Proposition 4.1.2 directly implies that equality $-\mathrm{div}\,(v_f Dd) = f$ holds in the sense of distributions in $\Omega\backslash\overline{\Sigma}$ as soon as f is Lipschitz and $\partial\Omega$ of class $\mathcal{C}^{2,1}$.

Proof—Since Ω has a $\mathcal{C}^2$ boundary, then the maps Dd, τ and κ_i are continuous in $\Omega\backslash\overline{\Sigma}$. Hence, v_f is continuous in $\Omega\backslash\overline{\Sigma}$ as soon as f is. Let us now prove that v_f is continuous on $\overline{\Sigma}$. Observe that, for any $x \notin \overline{\Sigma}$ and $i = 1, \ldots, n-1$, the term

$$\frac{1-(d(x)+t)\kappa_i(x)}{1-d(x)\kappa_i(x)} = \frac{1-d(x+tDd(x))\kappa_i(x)}{1-d(x)\kappa_i(x)} \qquad 0 < t < \tau(x)$$

is nonnegative by Proposition 1.1.4. Also, the product of the $n-1$ terms above is bounded by $\prod_{i=1}^{n-1}\Big[1 + \|[\kappa_i]_-\|_{\mathcal{C}(\partial\Omega)}\Big]\tau(x)$. Recalling that $x + tDd(x) \in \Omega_\varepsilon$ whenever $x \in \Omega_\varepsilon$ and $0 \leq t \leq \tau(x)$, we have just proven (4.1.3) . The continuity of v_f on $\overline{\Sigma}$ is an immediate consequence of (4.1.3). Next, let $\partial\Omega$ be of class $\mathcal{C}^{2,1}$ and f be Lipschitz continuous. Then, Theorem 4.0.20 ensures that τ is Lipschitz continuous on $\partial\Omega$. Therefore, $\tau = \tau \circ \Pi$ is locally Lipschitz continuous in $\overline{\Omega}\backslash\overline{\Sigma}$, as well as v_f. Finally, let us check the validity of (4.1.4) at every differentiability point x for v_f in the open set $\Omega\backslash\overline{\Sigma}$. Set $e_n := Dd(x)$ and consider $\{e_1, \ldots, e_n\}$ as a coordinate system, where $e_i = e_i(x)$, $i = 1, \ldots, n-1$ are the unit eigenvectors corresponding to the principal curvatures of $\partial\Omega$ at the projection point of x on the boundary. We note that, at any such point x,

$$\langle Dv_f(x), Dd(x)\rangle = \frac{d}{d\lambda} v_f(x + \lambda Dd(x))\Big|_{\lambda=0}.$$

But $\tau(x + \lambda Dd(x)) = \tau(x) - \lambda$ and $d(x + \lambda Dd(x)) = d(x) + \lambda$ for $\lambda > 0$ sufficiently small. So,

$$\begin{aligned}
&v_f(x + \lambda Dd(x)) \\
&= \int_0^{\tau(x)-\lambda} f\Big(x + (t+\lambda)Dd(x)\Big) \prod_{i=1}^{n-1} \frac{1 - (d(x) + \lambda + t)\kappa_i(x)}{1 - (d(x) + \lambda)\kappa_i(x)}\, dt \qquad (4.1.6) \\
&= \int_\lambda^{\tau(x)} f(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x) + t)\kappa_i(x)}{1 - (d(x) + \lambda)\kappa_i(x)}\, dt\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\langle Dv_f(x), Dd(x)\rangle \\
&= \int_0^{\tau(x)} f(x + tDd(x)) \sum_{i=1}^{n-1} \Bigg[\frac{1 - (d(x) + t)\kappa_i(x)^2}{(1 - d(x)\kappa_i(x))^2} \\
&\qquad\qquad \cdot \prod_{\substack{j=1 \\ j\neq i}}^{n-1} \frac{1 - (d(x) + t)\kappa_j(x)}{1 - d(x)\kappa_j(x)} \Bigg]\, dt - f(x) \\
&= \int_0^{\tau(x)} f(x + tDd(x)) \left[\sum_{i=1}^{n-1} \frac{\kappa_i(x)}{(1 - d(x)\kappa_i(x))} \right] \prod_{i=1}^{n-1} \frac{1 - (d(x) + t)\kappa_i(x)}{1 - d(x)\kappa_i(x)}\, dt \\
&-f(x) \\
&= -v_f(x)\Delta d(x) - f(x)
\end{aligned}$$

where we have taken into account the identity

$$\Delta d(x) = -\sum_{i=1}^{n-1} \frac{\kappa_i(x)}{(1 - d(x)\kappa_i(x))} \qquad \forall x \in \Omega \backslash \overline{\Sigma}\,,$$

that follows from Proposition 1.1.4. We have thus obtained (4.1.5)–an equivalent version of (4.1.4)–and completed the proof. □

Proposition 4.1.4. *Let K be a compact subset of $\mathbb{R}^n$ such that $\mathcal{H}^{n-1}(K) < \infty$. Then, there exists a sequence $\{\xi_k\}$ of functions in $W^{1,1}(\mathbb{R}^n)$ with compact support, such that*

(a) $0 \leq \xi_k \leq 1$ *for every* $k \in \mathbb{N}$*;*

(b) $\mathrm{dist}(\mathbb{R}^n \setminus \mathrm{spt}(\xi_k), K) \to 0$ *as* $k \to \infty$*;*

(c) $K \subset \mathrm{int}\{x \in \mathbb{R}^n : \xi_k(x) \geq 1\}$ *for every* $k \in \mathbb{N}$*;*

(d) $\xi_k \to 0$ *in* $L^1(\mathbb{R}^n)$ *as* $k \to \infty$*;*

(e) $\int_{\mathbb{R}^n} |D\xi_k| dx \leq C$ *for every* $k \in \mathbb{N}$ *and some constant* $C > 0$*.*

The proof of this lemma is very similar to the one of Lemma 3.1.9 and will be omitted.

Proof of Theorem 4.1.1—Let us first suppose that $\partial\Omega$ is of class $\mathcal{C}^{2,1}$ and f is Lipschitz continuous in Ω. We will prove that the pair (d, v_f), with v_f defined by (4.1.1), is a solution of system (4.0.1)–(4.0.2). Since d is a viscosity solution of the eikonal equation in Ω, then so is in the open set $\{x \in \Omega : v_f(x) > 0\}$. Therefore, it remains to show is that

$$\int_\Omega f\phi\, dx = \int_\Omega v_f \langle Dd, D\phi\rangle dx \qquad \forall \phi \in \mathcal{C}_c^\infty(\Omega)\,. \tag{4.1.7}$$

Since $\mathcal{H}^{n-1}(\overline{\Sigma}) < \infty$ by Proposition 4.0.21, we can apply Proposition 4.1.4 with $K = \overline{\Sigma}$ to construct a sequence $\{\xi_k\}$ enjoying properties (a), (b), (c) and (d). Let $\phi \in \mathcal{C}_c^\infty(\Omega)$ be a test function, and set $\phi_k = \phi(1 - \xi_k)$. By (a), (b) and the inclusion $\overline{\Sigma} \subset \Omega$ (see Proposition 1.1.7), we have that $\mathrm{spt}(\phi_k) \subset\subset \Omega \backslash \overline{\Sigma}$ for k large enough.

Then, Proposition 4.1.2 and Rademacher's Theorem imply that $-\mathrm{div}\,(v_f Dd) = f$ almost everywhere in $\Omega\backslash\overline{\Sigma}$. Multiplying this equation by ϕ_k and integrating by parts, we obtain

$$\int_\Omega f\phi_k dx = \int_\Omega v_f(1-\xi_k)\langle Dd, D\phi\rangle dx - \int_\Omega v_f\phi\langle Dd, D\xi_k\rangle dx\,. \tag{4.1.8}$$

We claim that the rightmost term above goes to 0 as $k\to\infty$. Indeed,

$$\begin{aligned}\Big|\int_\Omega v_f\phi\langle Dd, D\xi_k\rangle dx\Big| \;&\leq\; \|\phi\|_{\infty,\Omega}\,\|v_f\|_{\infty,\mathrm{spt}(\xi_k)}\int_\Omega |D\xi_k|dx\\ &\leq\; C\,\|\phi\|_{\infty,\Omega}\,\|v_f\|_{\infty,\mathrm{spt}(\xi_k)}\end{aligned}$$

where C is the constant provided by Proposition 4.1.4 (d). Now, using property (a) of the proposition and the fact that v_f is a continuous function vanishing on $\overline{\Sigma}$, we conclude that $\|v_f\|_{\infty,\mathrm{spt}(\xi_k)}\to 0$ as $k\to\infty$. This proves our claim. The conclusion (4.1.7) immediately follows since, in view of (a) and (c), the integrals $\int_\Omega f\phi_k dx$ and $\int_\Omega v_f(1-\xi_k)\langle Dd, D\phi\rangle dx$ converge to $\int_\Omega f\phi dx$ and $\int_\Omega v_f\langle Dd, D\phi\rangle dx$–respectively–as $k\to\infty$. Finally, the extra assumptions that $\partial\Omega$ be of class $\mathcal{C}^{2,1}$ and f be Lipschitz in Ω, can be easily removed by an approximation argument based on the lemma below. Let $\{\Omega_k\}$ be a sequence of open domains, with $\mathcal{C}^{2,1}$ boundary, converging to Ω in the $\mathcal{C}^2$ topology, and let $\{f_k\}$ be a sequence of Lipschitz functions in Ω_k converging to f, uniformly on all compact subsets of Ω. Denote by Σ_k and τ_k, respectively, the singular set and maximal retraction length of Ω_k. Define $v_k(x)=0$ for every $x\in\overline{\Sigma}_k$ and

$$v_k(x) = \int_0^{\tau_k(x)} f_k(x+tDb_{\Omega_k}(x))\prod_{i=1}^{n-1}\frac{1-(b_{\Omega_k}(x)+t)\kappa_{i,k}(x)}{1-b_{\Omega_k}(x)\kappa_{i,k}(x)}dt \quad \forall x\in\Omega_k\setminus\overline{\Sigma}_k\,,$$

where $\kappa_{i,k}(x)$ stands for the i-th principal curvature of $\partial\Omega_k$ at the projection of x.

Lemma 4.1.5. $\{v_k\}$ *converges to* v_f *in* $L^1_{loc}(\Omega)$.

Proof—Since, owing to (4.1.3), the sequence $\{v_k\}$ is locally uniformly bounded in Ω, it suffices to prove that it converges uniformly to v_f on every compact subset of Ω. For this, recall that, on account of Proposition 4.0.23, $\{\overline{\Sigma}_k\}$ converges to $\overline{\Sigma}$ in the

Hausdorff topology and $\{\tau_k\}$ converges to τ uniformly on all compact subsets of Ω. Then, our assumptions imply that $\{\kappa_{i,k}\}$ converges to κ_i uniformly on every compact subset of $\Omega\backslash\overline{\Sigma}$ for any $i \in \{1, \ldots, n-1\}$, and so does $\{v_k\}$ to v_f. To complete the proof it suffices to combine the above local uniform convergence in $\Omega\backslash\overline{\Sigma}$ with the estimate

$$0 \leq v_k(x) \leq \|f_k\|_{\infty,\Omega_\varepsilon} \prod_{i=1}^{n-1} \big(1 + \|[\kappa_{i,k}]_-\|_{\mathcal{C}(\partial\Omega_k)} \mathrm{diam}(\Omega_k)\big)\, \tau_k(x) \qquad \forall x \in \Omega_\varepsilon,$$

that allows to estimate v_k on any neighborhood of $\overline{\Sigma}$. $\square$

4.2 Uniqueness

In this section we will prove the following uniqueness result.

Theorem 4.2.1. *If* (u, v) *is a solution of system* (4.0.1)–(4.0.2)*, in the sense of Theorem 4.1.1, then* v *is given by* (4.1.1) *and* $u \equiv d$ *in* $\Omega_{v_f} := \{x \in \Omega \ : \ v_f(x) > 0\}$.

The techniques used in this section come essentially from three papers, [35], [13] and [21]. In particular, functional Φ below is the "stationary" version of the Lagrangian L introduced by Prigozhin in [35] in order to study the evolving shape of a sandpile. The idea of linking solutions of system (4.0.1)–(4.0.2) and saddle points of Φ also comes from his work. Moreover, Proposition 4.2.5 is the generalization to the n-dimensional case of the representation formula given in the plane by Cannarsa and Cardaliaguet in [13]. Finally, Proposition 4.2.6 is a modification of [21, Proposition 7.1, step 6.]; actually, Evans and Gangbo prove there the vanishing property of the transport density a at the ends of transport rays, which is the analogue of the vanishing of our v_f on $\overline{\Sigma}$ in the different framework of the Monge–Kantorovich mass transfer problem.

In order to prove Theorem 4.2.1, let us start by considering the lower semicontinuous functional $\Phi : H_0^1(\Omega) \times L_+^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ defined by

$$\Phi(w, r) = -\int_\Omega f(x)w(x)\, dx + \int_\Omega \frac{r(x)}{2}\big(|Dw(x)|^2 - 1\big)\, dx. \qquad (4.2.1)$$

We will first prove the uniqueness of the first component of the solution of system (4.0.1)–(4.0.2). More precisely, we will show that if (u, v) is a solution of system (4.0.1)–(4.0.2), then $u \equiv d$ in $\Omega_{v_f} := \{x \in \Omega \ : \ v_f(x) > 0\}$.

Lemma 4.2.2. *If* (u, v) *is a solution of system* (4.0.1)–(4.0.2)*, then* (u, v) *is a saddle point of* Φ*, in the sense that*

$$\Phi(u, r) \leq \Phi(u, v) \leq \Phi(w, v) \qquad \forall (w, r) \in H^1_0(\Omega) \times L^2_+(\Omega).$$

Proof—Since (u, v) is a solution of (4.0.1)–(4.0.2), then

$$\int_\Omega \frac{v(x)}{2}\big(|Du(x)|^2 - 1\big)\, dx = 0$$

and

$$\int_\Omega \frac{r(x)}{2}\big(|Du(x)|^2 - 1\big)\, dx \leq 0, \quad \forall r \in L^2_+(\Omega).$$

Hence, for any $r \in L^2_+(\Omega)$ we have

$$\begin{aligned} &\Phi(u, v) = -\int_\Omega f(x)u(x)\, dx \geq \\ &-\int_\Omega f(x)u(x)\, dx + \int_\Omega \frac{r(x)}{2}\big(|Du(x)|^2 - 1\big)\, dx = \Phi(u, r). \end{aligned} \tag{4.2.2}$$

Moreover, for any $w \in H^1_0(\Omega)$ we have $\int_\Omega \frac{v(x)}{2}|Dw(x) - Du(x)|^2\, dx \geq 0$ and

$$-\int_\Omega f(x)(w(x) - u(x))\, dx + \int_\Omega v(x)\langle Du(x), Dw(x) - Du(x)\rangle\, dx = 0$$

as a consequence of the fact that for every $\phi \in C^\infty_c(\Omega)$ (actually for every $\phi \in H^1_0(\Omega)$, including the case $\phi := w - u$),

$$\int_\Omega v(x)\langle Du(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x)\, dx\,.$$

Thus, for any $w \in H^1_0(\Omega)$,

$$\begin{aligned} &\Phi(w, v) = \Phi(u, v) - \int_\Omega f(x)(w(x) - u(x))\, dx \\ &+ \int_\Omega v(x)\langle Du(x), Dw(x) - Du(x)\rangle\, dx + \int_\Omega \frac{v(x)}{2}|Dw(x) - Du(x)|^2\, dx \\ &\geq \Phi(u, v). \end{aligned} \tag{4.2.3}$$

Collecting together (4.2.2) and (4.2.3) we get the conclusion. □

Lemma 4.2.3. *If* (u, v) *is a solution of system* (4.0.1)–(4.0.2)*, then also* (d, v) *is a solution of* (4.0.1)–(4.0.2).

Proof—First of all, we claim that $u \equiv d$ in the set $\mathrm{spt}(f)$ as a consequence of Lemma 4.2.2. In fact, if we consider the set of functions

$$\mathbb{K} := \{w \in W_0^{1,\infty}(\Omega) \ : \ \|Dw\|_\infty \leq 1\},$$

then for any $w \in \mathbb{K}$ we have

$$\int_\Omega f(x)w(x)\ dx \ \leq \int_\Omega f(x)u(x)\ dx, \tag{4.2.4}$$

because

$$\begin{aligned}&-\int_\Omega f(x)w(x)\ dx \geq -\int_\Omega f(x)w(x)\ dx + \int_\Omega \frac{v(x)}{2}\big(|Dw(x)|^2 - 1\big)\ dx\\ &= \Phi(w, v) \geq \Phi(u, v) = -\int_\Omega f(x)u(x)\ dx.\end{aligned}$$

On the other hand, by Remark 1.3.6, $d \in \mathbb{K}$ is the largest element of $\mathbb{K}$, meaning that $w \leq d$ for any $w \in \mathbb{K}$. Since $f \geq 0$, the maximality of d implies that

$$\int_\Omega f(x)u(x)\ dx \ \leq \int_\Omega f(x)d(x)\ dx.$$

Thus $\int_\Omega f(x)u(x)\ dx = \int_\Omega f(x)d(x)\ dx$, yielding $u \equiv d$ in the set $\mathrm{spt}(f)$. As an easy consequence of the previous equality we also get that (d, v) is a saddle point of functional Φ. Indeed, the coincidence $u \equiv d$ on $\mathrm{spt}(f)$ gives $\Phi(d, v) = \Phi(u, v)$ and then for any $w \in H_0^1(\Omega)$ we have

$$\Phi(d, v) = \Phi(u, v) \leq \Phi(w, v).$$

Moreover, for any choice of $r \in L^2_+(\Omega)$, $\int_\Omega \frac{r(x)}{2}\big(|Dd(x)|^2 - 1\big)\ dx = 0$; therefore

$$\Phi(d, r) = -\int_\Omega f(x)d(x)\ dx = \Phi(d, v).$$

Now, let us conclude the proof. Consider any $\phi \in C_c^\infty(\Omega)$. Since (d,v) is a saddle point of Φ, then for any $h>0$

$$\begin{aligned}\Phi(d,v) &\le \Phi(d+h\phi,v)\\ &= -\int_\Omega f(x)\big(d(x)+h\phi(x)\big)\,dx + \int_\Omega \frac{v(x)}{2}\big(|D\big(d+h\phi\big)(x)|^2-1\big)\,dx\\ &= \Phi(d,v) + h\left(-\int_\Omega f(x)\phi(x)\,dx + \int_\Omega v(x)\langle Dd(x), D\phi(x)\rangle dx\right)\\ &\quad + \frac{h^2}{2}\int_\Omega v(x)|D\phi(x)|^2\,dx,\end{aligned}$$

which gives

$$\begin{aligned}&h\left(-\int_\Omega f(x)\phi(x)\,dx + \int_\Omega v(x)\langle Dd(x), D\phi(x)\rangle dx\right)\\ &+\frac{h^2}{2}\int_\Omega v(x)|D\phi(x)|^2\,dx \ge 0.\end{aligned}$$

Dividing by h and letting $h\to 0^+$ we obtain

$$-\int_\Omega f(x)\phi(x)\,dx + \int_\Omega v(x)\langle Dd(x), D\phi(x)\rangle dx \ge 0.$$

Replacing ϕ by $-\phi$ we also get the opposite inequality. □

Proposition 4.2.4. *If (u,v) is a solution of system* (4.0.1)–(4.0.2)*, then $u \equiv d$ in the set $\{x\in\Omega \mid v_f(x)>0\}$, where v_f is the function defined by* (4.1.1).

Proof—By definition of v_f, it is readily seen that

$$\operatorname{spt}(v_f) := \overline{\{x\in\Omega \mid v_f(x)>0\}}$$

$$= \{x\in\overline{\Omega} \mid \exists p\in D^*d(x) \text{ s.t. } [x, x+\tau(x)p]\cap \operatorname{spt}(f)\ne\emptyset\}.$$

Hence, for any $y\in\{x\in\Omega \mid v_f(x)>0\}$ we can find $x\in\operatorname{spt}(f)$ such that

$$d(y) = d(x) - |x-y|.$$

Now, $u\equiv d$ in $\operatorname{spt}(f)$–as shown in the proof of Lemma 4.2.3–u is 1-Lipschitz continuous and d is the unique viscosity solution of the eikonal equation with Dirichlet boundary conditions, that is the largest function such that $\|Du\|_{\infty,\Omega} \leq 1$ and $u = 0$ on $\partial\Omega$ (see Remark 1.3.6). Therefore we conclude

$$d(y) = d(x) - |x - y| = u(x) - |x - y| \leq u(y) \leq d(y),$$

i.e. $u(y) = d(y)$. $\square$

Now that we have proven the uniqueness of the first component of the solution of system (4.0.1)–(4.0.2), it remains to prove the uniqueness of the second one. In order to do so, we will first exhibit for such a function a representation formula on the set $\Omega \setminus \overline{\Sigma}$ and then analyze its behaviour on $\overline{\Sigma}$.

Proposition 4.2.5. *If (d, v) is a solution of system* (4.0.1)–(4.0.2)*, then for any choice of $z_0 \in \Omega \setminus \overline{\Sigma}$ and $\theta \in (0, \tau(z_0))$ we have*

$$\begin{aligned} &v(z_0) - \prod_{i=1}^{n-1} \frac{1 - (d(z_0) + \theta)\kappa_i(z_0)}{1 - d(z_0)\kappa_i(z_0)} v(z_0 + \theta Dd(z_0)) \\ &= \int_0^\theta f(z_0 + tDd(z_0)) \prod_{i=1}^{n-1} \frac{1 - (d(z_0) + t)\kappa_i(z_0)}{1 - d(z_0)\kappa_i(z_0)}\, dt \end{aligned} \tag{4.2.5}$$

Proof—Set $e_n = Dd(z_0)$ and choose $e_1, \ldots, e_{n-1}$ such that $\{e_1, \ldots, e_n\}$ is a positively oriented orthonormal basis of $\mathbb{R}^n$ and, for any $i = 1, \ldots, n-1$, the vector e_i is a principal direction whose corresponding principal curvature is $\kappa_i(z_0)$. Moreover, let $x_0 = z_0 + \theta Dd(z_0)$, with $\theta \in (0, \tau(z_0))$, and fix $r > 0$ sufficiently small such that $S_0(r) := \{y \in \mathbb{R}^n \mid |y - x_0| \leq r, \langle y - x_0, e_n\rangle = 0\} \subset \Omega \setminus \overline{\Sigma}$ and for any $y \in S_0(r)$ we have $\langle Dd(y), e_n\rangle > 0$. Finally, denote by $S_i(r)$, $i = 1, 2$, the sets

$$\begin{aligned} S_1(r) &:= \left\{ y - \frac{\theta Dd(y)}{\langle Dd(y), e_n\rangle} \;\Big|\; y \in S_0(r) \right\} \\ S_2(r) &:= \left\{ y - \frac{tDd(y)}{\langle Dd(y), e_n\rangle} \;\Big|\; |y - x_0| = r,\ \langle y - x_0, e_n\rangle = 0,\ t \in [0, \theta] \right\}. \end{aligned}$$

and let $D(r)$ be the set enclosed by $S_0(r) \cup S_1(r) \cup S_2(r)$. So,

$$D(r) = \left\{ y - \frac{tDd(y)}{\langle Dd(y), e_n\rangle} \;\Big|\; y \in S_0(r),\ t \in [0, \theta] \right\} \subset \Omega \setminus \overline{\Sigma}$$

is set with piecewise regular boundary, because of the regularity of d. Thus we can integrate by parts the identity

$$\int_{D(r)} f(x)\,dx = -\int_{D(r)} \operatorname{div}(vDd)(x)\,dx,$$

obtaining

$$\int_{D(r)} f(x)\,dx = -\int_{\partial D(r)} v(x)\langle Dd(x),\nu(x)\rangle\,d\mathcal{H}^{n-1}(x), \tag{4.2.6}$$

where $\nu(x)$ is the unit outward normal to $\partial D(r)$. Now,

$$\begin{aligned}
&\int_{\partial D(r)} v(x)\langle Dd(x),\nu(x)\rangle\,d\mathcal{H}^{n-1}(x)\\
&= \int_{S_0(r)\cup S_1(r)\cup S_2(r)} v(x)\langle Dd(x),\nu(x)\rangle\,d\mathcal{H}^{n-1}(x)\\
&= \int_{S_0(r)} v(x)\langle Dd(x),\nu(x)\rangle\,d\mathcal{H}^{n-1}(x)\\
&+ \int_{S_1(r)} v(x)\langle Dd(x),\nu(x)\rangle\,d\mathcal{H}^{n-1}(x)
\end{aligned}$$

because by construction $\nu(x)$ is orthogonal to $Dd(x)$ on $S_2(r)$. Moreover,

$$\begin{aligned}
&\int_{S_0(r)} v(x)\langle Dd(x),\nu(x)\rangle\,d\mathcal{H}^{n-1}(x)\\
&= \int_{S_0(r)} v(x)\langle Dd(x),e_n\rangle\,d\mathcal{H}^{n-1}(x).
\end{aligned} \tag{4.2.7}$$

Since Dd and v are continuous functions in $S_0(r)$ and $Dd(x)\to e_n$ as $x\to x_0$, then

$$\lim_{r\to 0}\frac{1}{\omega_{n-1}r^{n-1}}\int_{S_0(r)} v(x)\langle Dd(x),e_n\rangle\,d\mathcal{H}^{n-1}(x) = v(x_0), \tag{4.2.8}$$

where ω_{n-1} is the area of the unit ball in $\mathbb{R}^{n-1}$. On the other hand,

$$\begin{aligned}
&\int_{S_1(r)} v(x)\langle Dd(x),\nu(x)\rangle\,d\mathcal{H}^{n-1}(x)\\
&= -\int_{S_0(r)} v(g(x))\langle Dd(g(x)),e_n\rangle|Jg(x)|\,d\mathcal{H}^{n-1}(x),
\end{aligned} \tag{4.2.9}$$

where

$$g(x) = x - \frac{\theta Dd(x)}{\langle Dd(x),e_n\rangle}, \qquad x\in S_0(r)$$

and $|Jg(x)|$ is the modulus of the determinant of the Jacobian matrix of g

$$Jg(x) = I + \frac{\theta}{\langle Dd(x), e_n\rangle} \sum_{i=1}^{n-1} \frac{\kappa_i(x)}{1 - d(x)\kappa_i(x)} e_i(x) \otimes e_i(x).$$

Since $\lim_{x\to x_0} g(x) = g(x_0) = x_0 - \theta e_n = z_0$ and

$$\lim_{x\to x_0} Jg(x) = I + \sum_{i=1}^{n-1} \frac{\theta\kappa_i(z_0)}{1 - d(z_0)\kappa_i(z_0)} e_i \otimes e_i,$$

we have

$$\lim_{x\to x_0} |Jg(x)| = \prod_{i=1}^{n-1} \left(1 + \frac{\theta\kappa_i(z_0)}{1 - d(z_0)\kappa_i(z_0)}\right),$$

and then we conclude

$$\begin{aligned} &\lim_{r\to 0} \frac{1}{\omega_{n-1} r^{n-1}} \int_{S_1(r)} v(x)\langle Dd(x), \nu(x)\rangle \, d\mathcal{H}^{n-1}(x) \\ &= -v(z_0) \prod_{i=1}^{n-1} \left(1 + \frac{\theta\kappa_i(z_0)}{1 - d(z_0)\kappa_i(z_0)}\right). \end{aligned} \tag{4.2.10}$$

So now it only remains to estimate $\lim_{r\to 0} \frac{1}{\omega_{n-1} r^{n-1}} \int_{D(r)} f(x)\, dx$. Exploiting the structure of the set $D(r)$, it is easy to see that we can write

$$\int_{D(r)} f(x)\, dx = \int_0^\theta dt \int_{S_t(r)} f(z)\, d\mathcal{H}^{n-1}(z) \tag{4.2.11}$$

where

$$S_t(r) := \left\{ y - \frac{t}{\langle Dd(y), e_n\rangle} Dd(y) \mid y \in S_1(r) \right\}.$$

Hence, using the previous computations and the continuity of f we finally find

$$\begin{aligned} &\lim_{r\to 0} \frac{1}{\omega_{n-1} r^{n-1}} \int_{D(r)} f(x)\, dx \\ &= \int_0^\theta f(x_0 - tDd(x_0)) \prod_{i=1}^{n-1} \left(1 + \frac{t\kappa_i(x_0)}{1 - d(x_0)\kappa_i(x_0)}\right) dt \end{aligned} \tag{4.2.12}$$

Collecting together (4.2.8), (4.2.10) and (4.2.12) and recalling identity (4.2.6), we can write

$$\begin{aligned} &\int_0^\theta f(x_0 - tDd(x_0)) \prod_{i=1}^{n-1} \left(1 + \frac{t\kappa_i(x_0)}{1 - d(x_0)\kappa_i(x_0)}\right) dt \\ &= -v(x_0) + v(z_0) \prod_{i=1}^{n-1} \left(1 + \frac{\theta\kappa_i(z_0)}{1 - d(z_0)\kappa_i(z_0)}\right). \end{aligned} \tag{4.2.13}$$

In order to represent (4.2.13) in the form (4.2.5) we only have to divide both sides of (4.2.13) by

$$\prod_{i=1}^{n-1}\left(1+\frac{\theta\kappa_i(x_0)}{1-d(x_0)\kappa_i(x_0)}\right)$$

and make a change of variable in the right–hand integral. Indeed, recalling that $x_0 = z_0 + \theta Dd(z_0)$, $Dd(x_0) = Dd(z_0)$ and $\kappa_i(x_0) = \kappa_i(z_0)$, the above computation gives

$$\begin{aligned}\int_0^\theta f(z_0+(\theta-t)Dd(z_0))\prod_{i=1}^{n-1}\left(\frac{1-(d(z_0)+\theta-t)\kappa_i(z_0)}{1-d(z_0)\kappa_i(z_0)}\right)\,dt\\ = -v(z_0+\theta Dd(z_0)) + v(z_0)\prod_{i=1}^{n-1}\left(\frac{1-(d(z_0)+\theta)\kappa_i(z_0)}{1-d(z_0)\kappa_i(z_0)}\right).\end{aligned}\tag{4.2.14}$$

Now the representation formula (4.2.5) follows as soon as we replace the variable t by $\theta - s$ in the above right–hand integral. □

Proposition 4.2.6. *If (d, v) is a solution of system* (4.0.1)–(4.0.2), *then v vanishes on $\overline{\Sigma}$.*

Proof—Since v is a continuous function, it suffices to prove that $v \equiv 0$ on Σ. So, let us fix any $x_0 \in \Sigma$ and choose $\varepsilon > 0$ sufficiently small such that $B_\varepsilon(x_0) \subset \Omega$. Then, for any $x \in B_1(0)$ set

$$\begin{cases} d_\varepsilon(x) := \dfrac{d(x_0+\varepsilon x)-d(x_0)}{\varepsilon} \\ v_\varepsilon(x) := v(x_0+\varepsilon x), \quad f_\varepsilon(x) := f(x_0+\varepsilon x). \end{cases}\tag{4.2.15}$$

By construction, for any $\varepsilon > 0$ as above $d_\varepsilon(0) = 0$ and $|Dd_\varepsilon(x)| = |Dd(x_0 + \varepsilon x)| = 1$ almost everywhere in $B_1(0)$. Hence, there exist a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$, $\varepsilon_j \to 0^+$ and a 1-Lipschitz function $d_0 : B_1(0) \to \mathbb{R}$ such that $d_{\varepsilon_j} \to d_0$ in the uniform topology in $B_1(0)$. Moreover, being $|Dd_{\varepsilon_j}(x)| = 1$ in the viscosity sense in $B_1(0)$, by [8, Proposition 2.2] also $|Dd_0(x)| = 1$ in the viscosity sense in $B_1(0)$, which gives $|Dd_0(x)| = 1$ almost everywhere. Thus,

$$\lim_{j\to\infty}\int_{B_1(0)} |Dd_{\varepsilon_j}(x)|^2\,dx = \omega_n = \int_{B_1(0)} |Dd_0(x)|^2\,dx,$$

which implies, together with the uniform convergence of d_{ε_j} to d_0, the convergence of Dd_{ε_j} to Dd_0 in the L^2 topology.

Also, an easy computation shows that d_{ε_j} is a semiconcave function in $B_1(0)$ with constant $C\varepsilon_j$, where $C > 0$ is the semiconcavity constant of d in $B_\varepsilon(x_0) \subset \Omega$. Therefore, d_0 is a concave function. Finally, the functions v_{ε_j} and f_{ε_j} defined above uniformly converge to $v(x_0)$ and $f(x_0)$ respectively and the pair $(d_{\varepsilon_j}, v_{\varepsilon_j})$ solves

$$-\mathrm{div}\,(v_{\varepsilon_j} Dd_{\varepsilon_j}) = \varepsilon_j f_{\varepsilon_j} \qquad \text{in } B_1(0)$$

in the weak sense, because (d, v) solves (4.0.1)–(4.0.2). Passing to the limit as $j \to \infty$ we then obtain that d_0 is a weak solution of

$$-\mathrm{div}\,(v(x_0) Dd_0(x)) = 0 \qquad x \in B_1(0).$$

Now, if $v(x_0) \neq 0$, the previous equation turns out to be the classical Laplace equation

$$\triangle d_0 = 0 \qquad \text{in } B_1(0)$$

and it is well-known that any weak Lipschitz solution in the ball of this equation is actually analytic. On the other hand, d_0 cannot be differentiable in $x = 0$, because d_0 is the 'blow up' of the distance function around a singular point x_0. Hence $v(x_0) = 0$ and the proof is complete. □

The last two propositions allow us to prove Theorem 4.2.1 as a simple corollary. Indeed, we already know by Proposition 4.2.4 that if (u, v) is a solution of system (4.0.1)–(4.0.2), then $u \equiv d$ on the set $\Omega_{v_f} = \{v_f > 0\}$. So it only remains to prove that $v \equiv v_f$ in Ω, where v_f is given by (3.1.22). But Proposition 4.2.6 guarantees that $v \equiv 0$ in $\overline{\Sigma}$, while Proposition 4.2.5 tells us that for any $z_0 \in \Omega \setminus \overline{\Sigma}$ and $\theta \in (0, \tau(z_0))$

$$\begin{aligned} & v(z_0) - \prod_{i=1}^{n-1} \frac{1 - (d(z_0) + \theta)\kappa_i(z_0)}{1 - d(z_0)\kappa_i(z_0)} v(z_0 + \theta Dd(z_0)) \\ &= \int_0^\theta f(z_0 + tDd(z_0)) \prod_{i=1}^{n-1} \frac{1 - (d(z_0) + t)\kappa_i(z_0)}{1 - d(z_0)\kappa_i(z_0)}\, dt. \end{aligned}$$

Hence, letting $\theta \to \tau(z_0)^-$ and using the continuity of v we obtain the coincidence of v and v_f at the point z_0. □

Bibliography

[1] Albano P., Cannarsa P., (1999) Structural properties of singularities of semiconcave functions. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 28, No.4, 719-740.

[2] Alberti G., Ambrosio L., Cannarsa P., (1992) On the singularities of convex functions. Manuscripta Math. 76, 421–435.

[3] Ambrosio L., Lecture Notes on Optimal Transport Problems. Mathematical aspects of evolving interfaces (Funchal, 2000), Lecture Notes in Math. 1812, 1–52, Springer, Berlin, (2003).

[4] Ambrosio L., Cannarsa P., Soner H.M., (1993) On the propagation of singularities of semi-convex functions. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 20, no. 4, 597–616.

[5] Aronsson G., (1972) A mathematical model in sand mechanics: Presentation and analysis. SIAM J. Appl. Math. 22, 437–458.

[6] Aronsson G., Evans L. C., (2002) An asymptotic model for compression molding. Indiana Univ. Math. J. 51 , no. 1, 1–36.

[7] Aronsson G., Evans L. C., Wu Y., (1996) Fast/slow diffusion and growing sandpiles. J. Differential Equations 131, no. 2, 304–335.

[8] Bardi, M., Capuzzo-Dolcetta, I., (1997) Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Systems and Control: Foundations and Applications. Boston, Birkhäuser.

[9] Bouchitté G., Buttazzo G., Seppechere P., (1997) Shape optimization solutions via Monge-Kantorovich equation. C. R. Acad. Sci. Paris Sr. I Math. 324, no. 10, 1185–1191.

[10] Bouchitté, G., Buttazzo, G., (2001) Characterization of optimal shapes and masses through Monge-Kantorovich equation. J. Eur. Math. Soc. 3, No.2, 139-168.

[11] Boutreux T., de Gennes P.-G., (1996) Surface flows of granular mixtures, I. General principles and minimal model. J. Phys. I France 6, 1295-1304.

[12] Bhattacharya T., Di Benedetto E., Manfredi J., (1991) Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems. Some topics in nonlinear PDEs (Turin, 1989). Rend. Sem. Mat. Univ. Politec. Torino 1989, Special Issue, 15–68.

[13] Cannarsa P., Cardaliaguet P., (2004) Representation of equilibrium solutions to the table problem for growing sandpile. J. Eur. Math. Soc. 6, 1–30.

[14] Cannarsa, P., Pignotti, C., Sinestrari, C., (2000) Semiconcavity for optimal control problems with exit time. Discrete Contin. Dynam. Systems 6, no. 4, 975–997.

[15] Cannarsa P., Sinestrari C., Semiconcave Functions, Hamilton–Jacobi Equations, and Optimal control. Progress in Nonlinear Differential Equations and Theyr Applications. Boston, Birkhäuser, (2004).

[16] Cannarsa, P. , Soner, H.M. (1989) Generalized one-sided estimates for solutions of Hamilton-Jacobi equations and applications. Nonlinear Anal., Theory Methods Appl. 13, No.3, 305-323.

[17] Choi H.I., Choi S.W., Moon H.P., (1997) Mathematical theory of medial axis transform. Pac. J. Math. 181, no. 1, 57–88.

[18] Clarke F. H., Ledyaev Yu. S., Stern R.J. & Wolenski P. R. (1998), NONSMOOTH ANALYSIS AND CONTROL THEORY. New York: Springer.

[19] Erdös P., (1945) Some remarks on the measurability of certain sets, *Bull. Amer. Math. Soc.* 51, 728–731.

[20] Evans L. C., Feldman M., Gariepy R., (1997) Fast/slow diffusion and collapsing sandpiles. J. Differential Equations 137, no. 1, 166–209.

[21] Evans L.C., Gangbo W., (1999) Differential equations methods for the Monge-Kantorovich mass transfer problem. Mem. Amer. Math. Soc. 137, no. 653,

[22] Evans, L. C., Gariepy, R. F. MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[23] Evans W. D., Harris D. J., (1987) Sobolev embeddings for generalized ridged domains. Proc. London Math. Soc. 54, 141–175.

[24] Federer H., (1959), Curvature measures. Trans. Amer. Math. Soc. 93, 418-491.

[25] Feldman M., (1999) Variational evolution problems and nonlocal geometric motion. Arch. Rational Mech. Anal. 146, 221-274.

[26] Feldman M., McCann R.J., (2002) Uniqueness and transport density in Monge's mass transportation problem. Calc. Var. 15, 81-113.

[27] Gilbarg, D., Trudinger, N.S. (1983), Elliptic Partial Differential Equations of Second Order. Grundlehren der Mathematischen Wissenschaften, 224. Berlin, Springer-Verlag.

[28] Hadeler K.P., Kuttler C., (1999) Dynamical models for granular matter. Granular Matter 2 (1999) 1, 9-18

[29] Itoh, J. , Tanaka, M., (2001) The Lipschitz continuity of the distance function to the cut locus. Trans. Am. Math. Soc. 353, No.1, 21-40.

[30] Janfalk , (1996) Behaviour in the limit, as $p \to \infty$, of minimizers of functionals involving p-Dirichlet integrals. SIAM J. Math. Anal. 27, no. 2, 341–360.

[31] Li Y.Y., Nirenberg L., (2005) The distance function to the boundary, Finsler geometry and the singular set of viscosity solutions of some Hamilton–Jacobi equations. Commun. Pure Appl. Math. 58, 85–146.

[32] Lions, P.L. Generalized solutions of Hamilton-Jacobi equations. Research Notes in Mathematics, 69. Boston - London - Melbourne: Pitman (1982).

[33] Mantegazza C., Mennucci A., (2003) Hamilton–Jacobi equations and distance functions on Riemannian manifolds. Appl. Math. Opt. 47, no. 1, 1–25.

[34] Motzkin Th., (1935) Sur quelques propriétés caractéristiques des ensembles convexes. Atti. Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 21, pp. 562-567.

[35] Prigozhin L., (1996) Variational model of sandpile growth. European J. Appl. Math. 7, no. 3, 225–235.

[36] Rockafellar R.T., Wets, R. J.-B., Variational analysis. Grundlehren der Mathematischen Wissenschaften. 317. Berlin: Springer (1998).