BRAIDINGS OF POISSON GROUPS
WITH QUASITRIANGULAR DUAL

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Abstract. Let \( g \) be a quasitriangular Lie bialgebra over a field \( k \) of characteristic zero, and let \( g^* \) be its dual Lie bialgebra. We prove that the formal Poisson group \( F[[g^*]] \) is a braided Hopf algebra. More generally, we prove that if \( (U_h, R) \) is any quasitriangular QUEA, then \( (U_h', Ad(R)[|_{U_h' \otimes U_h'}) \) — where \( U_h' \) is defined by Drinfeld — is a braided QFSHA. The first result is then just a consequence of the existence of a quasitriangular quantization \( U_h(R) \) of \( U(g) \) and of the fact that \( U_h' \) is a quantization of \( F[[g^*]] \).

Introduction

Let \( g \) be a Lie Lie bialgebra over a field \( k \) of characteristic zero; let \( g^* \) be the dual Lie bialgebra associated to \( g^* \). If \( g \) is quasitriangular, endowed with the \( r \)–matrix \( r \), this gives \( g \) some additional properties. A question then rises: what new structure one obtains on the dual bialgebra \( g^* \)? In this work we shall show that the topological Poisson Hopf algebra \( F[[g^*]] \) is a braided Poisson algebra (we’ll give the definition later on). This was already proved for \( g = \mathfrak{sl}(2, k) \) by Reshetikhin (cf. [Re]), and generalised to the case where \( g \) is Kac-Moody of finite (cf. [G1]) or affine (cf. [G2]) type by the first author.

In order to prove the result, we shall use quantization of universal enveloping algebras. After Etingof-Kazhdan (cf. [EK]), each Lie bialgebra admits a quantization \( U_h(g) \), namely a topological Hopf algebra over \( k[[h]] \) whose specialisation at \( h = 0 \) is isomorphic to \( U(g) \) as a co-Poisson Hopf algebra; in addition, if \( g \) is quasitriangular and \( r \) is its \( r \)–matrix, then such a \( U_h(g) \) exists which is quasitriangular too, as a Hopf algebra, with an \( R \)–matrix \( R_h (\in U_h(g) \otimes U_h(g)) \) such that \( R_h \equiv 1 + r h^2 \mod h^2 \) (where we have identified, as vector spaces, \( U_h(g) \cong U(g)[[h]] \)).

Now, after Drinfel’d (cf. [Dr]), for any quantised universal enveloping algebra \( U \) one can define also a certain Hopf subalgebra \( U' \) such that, if the semiclassical limit of \( U \) is

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$U(\mathfrak{g})$ (with $\mathfrak{g}$ a Lie bialgebra), then the semiclassical limit of $U'$ is $F[[\mathfrak{g}^*]]$. In our case, when considering $U_h(\mathfrak{g})'$ one can observe that the $R$–matrix does not belong, a priori, to $U_h(\mathfrak{g})' \otimes U_h(\mathfrak{g})'$; nevertheless, we prove that its adjoint action $R_h := \text{Ad}(R_h) : U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}) \to U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$, $x \otimes y \mapsto R_h \cdot (x \otimes y) \cdot R_h^{-1}$, stabilises $U_h(\mathfrak{g})' \otimes U_h(\mathfrak{g})'$, hence it induces by specialisation an operator $R_h$ on $F[[\mathfrak{g}^*]] \otimes F[[\mathfrak{g}^*]]$. Finally, the properties which make $R_h$ an $R$–matrix imply that $R_h$ is a braiding operator, whence the same holds for $R_0$: thus the pair $(F[[\mathfrak{g}^*]], R_0)$ is braided Poisson algebra.

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§ 1. Recalls and definitions

1.1 The classical objects. Let $k$ be a fixed field of characteristic zero. In the following $k$ will be the ground field of all the objects — Lie algebras and bialgebras, Hopf algebras, etc. — which we’ll introduce.

Following [CP], §1.3, we call Lie bialgebra a pair $(\mathfrak{g}, \delta_\mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra and $\delta_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is an antisymmetric linear map — called Lie cobracket — such that its dual $\delta_\mathfrak{g}^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ be Lie bracket and that $\delta_\mathfrak{g}^*$ itself be a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$. Then it happens that also $\mathfrak{g}^*$, the linear dual of $\mathfrak{g}$, is a Lie bialgebra on its own. Following [CP], §2.1.B, we call quasitriangular Lie bialgebra a pair $(\mathfrak{g}, r)$ such that $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a solution of the classical Yang-Baxter equation (CYBE) $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ and $\mathfrak{g}$ be a Lie bialgebra with respect to the cobracket $\delta = \delta_\mathfrak{g}$ defined by $\delta(x) = [x, r]$; the element $r$ is then called $r$–matrix of $\mathfrak{g}$.

If $\mathfrak{g}$ is a Lie algebra, its universal enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra; if, in addition, $\mathfrak{g}$ is a Lie bialgebra, then $U(\mathfrak{g})$ is in fact a co-Poisson Hopf algebra (cf. [CP], §6.2.A).

Let $\mathfrak{g}$ be any Lie algebra: then we call function algebra on the formal group associated to $\mathfrak{g}$, or simply formal group associated to $\mathfrak{g}$, the space $F[[\mathfrak{g}]] := U(\mathfrak{g})^*$ linear dual of $U(\mathfrak{g})$. As $U(\mathfrak{g})$ is a Hopf algebra, its dual $F[[\mathfrak{g}]]$ is on its own a formal Hopf algebra (following [Di], Ch. 1). Note that, if $G$ is a connected algebraic group whose tangent Lie algebra is $\mathfrak{g}$, letting $F[G]$ be the Hopf algebra of regular functions on $G$ and letting $\mathfrak{m}_e$ be the maximal ideal of $F[G]$ of functions vanishing at the unit point $e \in G$, the formal Hopf algebra $F[[\mathfrak{g}]]$ is nothing but the $\mathfrak{m}_e$–adic completion of $F[G]$ (cf. [On], Ch. 1). When, in addition, $\mathfrak{g}$ is a Lie bialgebra, $F[[\mathfrak{g}]]$ is in fact a formal Poisson Hopf algebra (cf. [CP], §6.2.A).

1.2 Braiding and quasitriangularity. Let $H$ be a Hopf algebra in a tensor category $(A, \otimes)$ (cf. [CP], §5): $H$ is called braided (cf. [Re], Définition 2) if there exists an algebra automorphism $\mathcal{R}$ of $H \otimes H$, called braiding operator of $H$, different from the flip $\sigma : H^{\otimes 2} \to H^{\otimes 2}$, $a \otimes b \mapsto b \otimes a$, and such that

$$\mathcal{R} \circ \Delta = \Delta^{\text{op}}$$

$$(\Delta \otimes \text{id}) \circ \mathcal{R} = \mathcal{R}_{13} \circ \mathcal{R}_{23} \circ (\Delta \otimes \text{id})$$

$$(\text{id} \otimes \Delta) \circ \mathcal{R} = \mathcal{R}_{13} \circ \mathcal{R}_{12} \circ (\text{id} \otimes \Delta)$$

where $\Delta^{\text{op}}$ is the opposite comultiplication, i.e. $\Delta^{\text{op}}(a) = \sigma \circ \Delta(a)$, and $\mathcal{R}_{12}$, $\mathcal{R}_{13}$, and $\mathcal{R}_{23}$ are the automorphisms of $H \otimes H \otimes H$ defined by $\mathcal{R}_{12} = \mathcal{R} \otimes \text{id}$, $\mathcal{R}_{23} = \text{id} \otimes \mathcal{R}$, $\mathcal{R}_{13} = (\sigma \otimes \text{id}) \circ (\text{id} \otimes \mathcal{R}) \circ (\sigma \otimes \text{id})$. 
Finally, when $H$ is, in addition, a Poisson Hopf algebra, we’ll say that it is braided — as a Poisson Hopf algebra — if it is braided — as a Hopf algebra — by a braiding which is also an automorphism of Poisson algebra.

If the pair $(H, \mathcal{R})$ is a braided algebra, it follows from the definition that $\mathcal{R}$ satisfies the quantum Yang-Baxter equation — QYBE in the sequel — in $\text{End}(H^{\otimes 3})$, that is

$$\mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} = \mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12}$$

which implies that, for all $n \in \mathbb{N}$ the braid group $B_n$ acts on $H^{\otimes n}$, from which one can also obtain some knot invariants, according to the recipe given in [CP], §15.12.

A Hopf algebra $H$ (in a tensor category) is said to be quasitriangular (cf. [Dr], [CP]) if there exists an invertible element $R \in H \otimes H$, called the $R$–matrix of $H$, such that

$$R \cdot \Delta(a) \cdot R^{-1} = \text{Ad}(R)(\Delta(a)) = \Delta^\text{op}(a)$$

$$\left(\Delta \otimes \text{id}\right)(R) = R_{13}R_{23}, \quad \left(\text{id} \otimes \Delta\right)(R) = R_{13}R_{12}$$

where $R_{12}, R_{13}, R_{23} \in H^{\otimes 3}$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, $R_{13} = (\sigma \otimes \text{id})(R_{23}) = (\text{id} \otimes \sigma)(R_{12})$. Then it follows from the identities above that $R$ satisfies the QYBE in $H^{\otimes 3}$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$ 

Thus, the tensor products of $H$–modules are endowed with an action of the braid group. Moreover, it is clear that if $(H, R)$ is quasitriangular, then $(H, \text{Ad}(R))$ is braided.

### 1.3 The quantum objects

Let $\mathcal{A}$ be the category whose objects are the $k[[h]]$–modules which are topologically frees and complete in $h$–adic sense, and the morphisms are the $k[[h]]$–linear continuous maps. For all $V, W$ in $\mathcal{A}$, we define $V \otimes W$ to be the projective limit of the $k[[h]]/(h^n)$–modules $(V/h^nV) \otimes k[[h]]/(h^n) (W/h^nW)$: this makes $\mathcal{A}$ into a tensor category (see [CP] for further details). After Drinfel’d (cf. [Dr]), we call quantised universal enveloping algebra — QUEA in the sequel — any Hopf algebra in the category $\mathcal{A}$ whose semiclassical limit (= specialisation at $h = 0$ ) is the universal enveloping algebra of a Lie bialgebra. Similarly, we call quantised formal series Hopf algebra — QFSHA in the sequel — any Hopf algebra in the category $\mathcal{A}$ whose semiclassical limit is the function algebra of a formal group.

In the sequel, we shall need the following result:

**Theorem 1.4.** (cf. [EK]) Let $\mathfrak{g}$ be a Lie bialgebra. Then there exists a QUEA $U_h(\mathfrak{g})$ whose semiclassical limit is isomorphic to $U(\mathfrak{g})$; furthermore, there exists an isomorphism of $k[[h]]$–modules $U_h(\mathfrak{g}) \cong U(\mathfrak{g})[[h]]$.

In addition, if $\mathfrak{g}$ is quasitriangular, with $r$–matrix $r$, then there exists a QUEA $U_h(\mathfrak{g})$ as above and an element $R_h \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ such that $(U_h(\mathfrak{g}), R_h)$ be a quasitriangular Hopf algebra and $R_h = 1 + r h + O(h^2)$ (with $O(h^2) \in h^2 \cdot H \otimes H$). □

### 1.5 The Drinfel’d’s functor

Let $H$ be a Hopf algebra over $k[[h]]$. For all $n \in \mathbb{N}$, define $\Delta^n: H \rightarrow H^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := id_H$, and $\Delta^n := (\Delta \otimes id_H^{(n-2)}) \circ \Delta^{n-1}$ if $n > 2$. For all ordered subset $\Sigma = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ with $i_1 < \cdots < i_k$, define the homomorphism $j_{\Sigma}: H^{\otimes k} \rightarrow H^{\otimes n}$ by $j_{\Sigma}(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := 1$ if $i \notin \Sigma$ and $b_{im} := a_m$ for $1 \leq m \leq k$; then set $\Delta_{\Sigma} := j_{\Sigma} \circ \Delta^k$. Finally, define
\[ \delta_n : H \rightarrow H^{\otimes n} \text{ by } \delta_n := \sum_{\Sigma \subseteq \{1, \ldots, n\}} (\Delta_{\Sigma}) ; \text{ for all } n \in \mathbb{N}_+. \]

More in general, for all \( \Sigma = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} \), with \( i_1 < \cdots < i_k \), define

\[ \delta_{\Sigma} := \sum_{\Sigma' \subseteq \Sigma} (\Delta_{\Sigma'}) ; \quad (1.1) \]

(in particular, \( \delta_{\{1, \ldots, n\}} = \delta_n \)). Thanks to the inclusion-exclusion principle, this is equivalent to

\[ \Delta_{\Sigma} = \sum_{\Sigma' \subseteq \Sigma} \delta_{\Sigma'} ; \quad (1.2) \]

for all \( \Sigma = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} \) with \( i_1 < \cdots < i_k \). Finally, define

\[ H' := \{ a \in H \mid \delta_n(a) \in h^n H^{\otimes n} \} , \]

a subspace of \( H \) which we consider endowed with the induced topology. Then we have

**Theorem 1.6.** (cf. [Dr], §7, or [G3]) Let \( H \) be a Hopf algebra in the category \( \mathcal{A} \). Then \( H' \) is a QFSHA. Moreover, if \( H = U_h(\mathfrak{g}) \) is a QUEA, with \( U(\mathfrak{g}) \) as semiclassical limit, then the semiclassical limit of \( U_h(\mathfrak{g})' \) is \( F[[\mathfrak{g}^*]] \). □

**§ 2. The main results**

From the technical point of view, the main result of this paper concerns the general framework of quasitriangular Hopf algebras:

**Theorem 2.1.** Let \( H \) be a quasitriangular Hopf algebra in the category \( \mathcal{A} \), and let \( R \) be its \( R \)-matrix. Then, the inner automorphism \( \text{Ad}(R) : H \otimes H \rightarrow H \otimes H \) restricts to an automorphism of \( H' \otimes H' \), and the pair \( (H', \text{Ad}(R)|_{H' \otimes H'}) \) is a braided Hopf algebra in the category \( \mathcal{A} \). □

The proof of this theorem will be given in section 3. Nevertheless, we can already get out of it as a consequence the main result announced by the title and in the introduction, which gives us a geometrical interpretation of the classical \( r \)-matrix:

**Theorem 2.2.** Let \( \mathfrak{g} \) be a quasitriangular Lie bialgebra. Then the topological Poisson Hopf algebra \( F[[\mathfrak{g}^*]] \) is braided. Moreover, there exists a quantisation of \( F[[\mathfrak{g}^*]] \) which is a braided Hopf algebra whose braiding operator specialises into that of \( F[[\mathfrak{g}^*]] \).

**Proof.** Let \( r \) be the \( r \)-matrix of \( \mathfrak{g} \). By Theorem 1.4, there exists a quasitriangular QUEA \( (U_h(\mathfrak{g}), R_h) \) whose semiclassical limit is exactly \( (U(\mathfrak{g}), r) \) : that is, \( U_h(\mathfrak{g})/h U_h(\mathfrak{g}) \cong U(\mathfrak{g}) \) and \( (R - 1)/h \equiv r \mod h U_h(\mathfrak{g})^{\otimes 2} \); and by Theorem 1.6, the semiclassical limit of \( U_h(\mathfrak{g})' \) is \( F[[\mathfrak{g}^*]] \). Let \( \mathcal{R}_h := \text{Ad}(R_h) \); then Theorem 2.1 ensures that \( \left( U_h(\mathfrak{g})', \mathcal{R}_h|_{U_h(\mathfrak{g})' \otimes U_h(\mathfrak{g})'} \right) \) is a braided Hopf algebra, hence its semiclassical limit \( \left( F[[\mathfrak{g}^*]]; \mathcal{R}_h|_{U_h(\mathfrak{g})' \otimes U_h(\mathfrak{g})'} \right|_{h=0} \)
is braided as well. Furthermore, as $\mathcal{R}_h$ is an algebra automorphism and the Poisson bracket of $F[[\mathfrak{g}^*]]$ is given by $\{a, b\} = (\alpha, \beta)/h$ for all $a, b \in F[[\mathfrak{g}^*]]$ and $\alpha, \beta \in U_h(\mathfrak{g})'$ such that $\alpha|_{h=0} = a$, $\beta|_{h=0} = b$, we have that $\left(\mathcal{R}_h|_{U_h(\mathfrak{g})' \otimes U_h(\mathfrak{g})'}\right)|_{h=0}$ is also an automorphism of Poisson algebra. □

The theorem above gives a geometrical interpretation of the $r$–matrix of a quasitriangular Lie bialgebra. This very result had been proved for $\mathfrak{g} = \mathfrak{sl}(2, k)$ by Reshetikhin (cf. [Re]), and generalised to the case when $\mathfrak{g}$ is Kac-Moody of finite type (cf. [G1], where a more precise analysis is carried on) or affine type (cf. [G2]) by the first author.

Theorem 2.2 has also an important consequence. Let $\mathfrak{g}$ and $\mathfrak{g}^*$ be as above, let $\mathcal{R}$ be the braiding of $F[[\mathfrak{g}^*]]$, and let $\epsilon$ be the (unique) maximal ideal of $F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]] = F[[\mathfrak{g}^*]] \otimes F[[\mathfrak{g}^*]]$ (topological tensor product, following [Di], Ch. 1). Now, $\mathcal{R}$ is an algebra automorphism, hence $\mathcal{R}(\epsilon) = \epsilon$, and $\mathcal{R}$ induces an automorphism of vector space $\mathbb{R}: \epsilon/\epsilon^2 \rightarrow \epsilon/\epsilon^2$; in addition, $\epsilon/\epsilon^2 \cong \mathfrak{g} \oplus \mathfrak{g}$, and since $\mathcal{R}$ is also an automorphism of Poisson algebra, one has that $\mathbb{R}$ is a Lie algebra automorphism of $\mathfrak{g} \oplus \mathfrak{g} = \epsilon/\epsilon^2$; the other properties of the braiding $\mathcal{R}$ make so that $\mathbb{R}$ have other corresponding properties. Finally, the dual $\mathbb{R}^*: \mathfrak{g}^* \oplus \mathfrak{g}^* \rightarrow \mathfrak{g}^* \oplus \mathfrak{g}^*$ is a Lie coalgebra automorphism of $\mathfrak{g}^* \oplus \mathfrak{g}^*$, enjoying many other properties dual of those of $\mathcal{R}$. In particular, $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}^*$ are solutions of the QYBE, whence there is an action of the braid group $B_n$ on $F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]]^{\otimes n}$, on $(\mathfrak{g} \oplus \mathfrak{g})^{\otimes n}$, and on $(\mathfrak{g}^* \oplus \mathfrak{g}^*)^{\otimes n}$ ($n \in \mathbb{N}$), and from that one can obtain knot invariants (following [CP], §§9, related to the so-called “global $R$–matrix”), which also yields a geometrical interpretation of the classical $r$–matrix: comparing our results with those of [WX], as well as the functoriality properties of our construction, will be the matter of a forthcoming article.

§ 3. Proof of theorem 2.1

In this section $(H, R)$ will be a quasitriangular Hopf algebra as in the statement of Theorem 2.1. We want to study the adjoint action of $R$ on $H \otimes H$, where the latter is endowed with its natural structure of Hopf algebra; we denote by $\tilde{\Delta}$ its coproduct, defined by $\tilde{\Delta} := s_{23} \circ (\Delta \otimes id_H \otimes id_H) \circ (id_H \otimes \Delta)$ where $s_{23}$ denotes the flip in the positions 2 and 3. We’ll denote also $I := 1 \otimes 1$ the unit in $H \otimes H$. After our definition of tensor product in $\mathcal{A}$, we have $(H \otimes H)' = H' \otimes H'$. Our goal is to show that, although $R$ do not necessarily belong to $(H \otimes H)'$, its adjoint action $a \mapsto R \cdot a \cdot R^{-1}$ leaves stable $(H \otimes H)' = H' \otimes H'$.

First of all set, for $\Sigma = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, always with $i_1 < \cdots < i_k$, $R_{\Sigma} := R_{21i-1, 2i-1} \cdots R_{2i-1, 2i} R_{2i-1, 2i+1} \cdots R_{2i+1, 2i} R_{2i+1, 2i+2} \cdots R_{2i+2, 2i+1} \cdots R_{2i+1, 2i} \cdots R_{2i, 2i+1} \cdots R_{2i+1, 2i} \cdots R_{2i, 2i+1} \cdots R_{2i+1, 2i}$ (product of $k^2$ terms) where $R_{i,j} := j_{(i,j)}(R)$, defining $j_{(r,s)} : H \otimes H \rightarrow H^{\otimes 2n}$ as before. We shall always write $|\Sigma|$ for the cardinality of $\Sigma$ (here $|\Sigma| = k$).

Lemma 3.1. In $(H \otimes H)^{\otimes n}$, for all $\Sigma \subseteq \{1, \ldots, n\}$, we have: $\tilde{\Delta}_\Sigma(R) = R_{\Sigma}$.

Proof. With no loss of generality, we’ll prove the result for $\Sigma = \{1, \ldots, n\}$, i.e.

$\tilde{\Delta}_{\{1, \ldots, n\}}(R) = R_{\{1, \ldots, n\}} = R_{1, 2n} \cdot R_{1, 2n-2} \cdots R_{1, 2} \cdot R_{3, 2n} \cdots R_{2n-3, 2} \cdot R_{2n-1, 2n} \cdots R_{2n-1, 2}$.
The result is evident at rank $n = 1$. Assume it be true at rank $n$, and prove it at rank $n + 1$; by definition of $\tilde{\Delta}$ and by the properties of the $R$-matrix we have
\[
\tilde{\Delta}_{\{1, \ldots, n+1\}}(R) = (\tilde{\Delta} \otimes id_{H@H}^{\otimes n-1})(\tilde{\Delta}_{\{1, \ldots, n\}}(R)) = (\tilde{\Delta} \otimes id_{H@H}^{\otimes n-1})(R_{\{1, \ldots, n\}}) \\
= s_{23}(\Delta \otimes id_{H}^{\otimes 2n})(id_{H} \otimes \Delta \otimes id_{H}^{\otimes (2n-2)})(R_{1,2n} \cdots R_{1,2} \cdots R_{3,2} \cdots R_{2n-1,2} ) \\
= s_{23}(\Delta \otimes id_{H}^{\otimes 2n})(R_{1,2n+1} \cdots R_{1,3} R_{1,2} \cdots R_{4,3} R_{4,2} \cdots R_{2n,3} R_{2n,2} ) \\
= s_{23}(R_{1,2n+2} R_{2,2n+2} \cdots R_{1,4} R_{2,4} R_{1,3} R_{2,3} \cdots R_{5,4} R_{5,3} \cdots R_{2n+1,4} R_{2n+1,3} ) \\
= R_{1,2n+2} R_{3,2n+2} \cdots R_{1,4} R_{3,4} \cdots R_{5,4} \cdots R_{5,2} \cdots R_{2n+1,4} R_{2n+1,2} \\
= R_{1,2n+2} \cdots R_{1,4} R_{1,2} R_{3,2n+2} \cdots R_{3,4} R_{3,2} \cdots R_{5,4} R_{5,2} \cdots R_{2n+1,4} R_{2n+1,2} \\
= R_{\{1, \ldots, n+1\}}, \text{ q.e.d. } \Box
\]

From now on we shall use the notation $C^a_b := (\binom{b}{a})$ for all $a, b \in \mathbb{N}$.

**Lemma 3.2.** For all $a \in (H \otimes H)'$, and for all set $\Sigma$ such that $|\Sigma| > i$, we have
\[
\tilde{\Delta}_\Sigma(a) = \sum_{\Sigma' \subseteq \Sigma, |\Sigma'| \leq i} (-1)^{|\Sigma| - |\Sigma'|} C^{i - |\Sigma'|}_{|\Sigma| - 1 - |\Sigma'|} \tilde{\Delta}_{\Sigma'}(a) + O(h^{i+1}).
\]

**Proof.** It is enough to prove the claim for $\Sigma = \{1, \ldots, n\}$, with $n > i$. Due to (1.2), we have
\[
\tilde{\Delta}_{\{1, \ldots, n\}}(a) = \sum_{\Sigma \subseteq \{1, \ldots, n\}} \delta_{\Sigma}(a) = \sum_{\Sigma \subseteq \{1, \ldots, n\}, |\Sigma| \leq i} \delta_{\Sigma}(a) + O(h^{i+1}) \\
= \sum_{\Sigma \subseteq \{1, \ldots, n\}, |\Sigma| \leq i} \sum_{\Sigma' \subseteq \Sigma} (-1)^{|\Sigma| - |\Sigma'|} \tilde{\Delta}_{\Sigma'}(a) + O(h^{i+1}) \\
= \sum_{\Sigma' \subseteq \{1, \ldots, n\}, |\Sigma'| \leq i} \tilde{\Delta}_{\Sigma'}(a) \sum_{\Sigma \subseteq \Sigma, |\Sigma| \leq i} (-1)^{|\Sigma| - |\Sigma'|} + O(h^{i+1}) \\
= \sum_{\Sigma' \subseteq \{1, \ldots, n\}, |\Sigma'| \leq i} \tilde{\Delta}_{\Sigma'}(a)(-1)^{i - |\Sigma'|} C^{i - |\Sigma'|}_{n - 1 - |\Sigma'|} + O(h^{i+1}), \text{ q.e.d. } \Box
\]

Before going on with the main result, we need still another minor technical fact about the binomial coefficients: one can easily prove it using the formal series expansion of $(1 - X)^{-(r+1)}$, namely $(1 - X)^{-(r+1)} = \sum_{k=0}^{\infty} C^r_{k+r} X^k$.

**Lemma 3.3.** Let $r, s, t \in \mathbb{N}$ be such that $r < t$. Then we have the following relations (where we set $C^{r}_{r} := 0$ if $v > u$):
\[
(a) \sum_{d=0}^{t} (-1)^{d} C^{r}_{d-1} C^{d}_{t} = -(-1)^r, \quad (b) \sum_{d=0}^{t} (-1)^{d} C^{r}_{d+s} C^{d}_{t} = 0. \Box
\]

Finally, here is the main result of this section:
Proposition 3.4. For all \(a \in (H \otimes H)'\), we have \(R a R^{-1} \in (H \otimes H)'\).

Proof. As we have to show that \(R a R^{-1}\) belongs to \((H \otimes H)'\), we have to consider the terms \(\delta_n(R a R^{-1})\), \(n \in \mathbb{N}\). For this we go and re-write \(\delta_{\{1,\ldots,n\}}(R a R^{-1})\) by using Lemma 3.1 and the fact that \(\hat{\Delta}\) and more in general \(\hat{\Delta}_{\{i_1,\ldots,i_k\}}\), for \(k \leq n\), are algebra morphisms; then \(\delta_{\{1,\ldots,n\}}(R a R^{-1}) = \sum_{\Sigma \subseteq \{1,\ldots,n\}} (-1)^{n-|\Sigma|} R_{\Sigma} \hat{\Delta}_{\Sigma}(a) R_{\Sigma}^{-1}^{-1}\).

We shall prove by induction on \(i\) that

\[
\delta_{\{1,\ldots,n\}}(R a R^{-1}) = O(h^{i+1}) \quad \text{for all } 0 \leq i \leq n - 1.
\]

In other words, we’ll see that all the terms of the expansion truncated at the order \(n - 1\) are zero, hence \(\delta_n(R a R^{-1}) = O(h^n)\), whence our claim.

For \(i = 0\), we have, for each \(\Sigma\): \(\hat{\Delta}_{\Sigma}(a) = \epsilon(a) I^\otimes_{\Sigma} + O(h)\) and \(R_{\Sigma} = I^\otimes_{\Sigma} + O(h)\), and similarly \(R_{\Sigma}^{-1} = I^\otimes_{\Sigma} + O(h)\), whence \(\delta_{\{1,\ldots,n\}}(R a R^{-1}) = \sum_{m=1}^{n} C_n^m (-1)^{n-k} \epsilon(a) I^\otimes_{\Sigma} + O(h)\), thus the result \((*)\) is true for \(i = 0\).

Let’s assume the result \((*)\) proved for all \(i'<i\). Write the \(h\)-adic expansions of \(R_{\Sigma}\) and \(R_{\Sigma}^{-1}\) in the form \(R_{\Sigma} = \sum_{\ell=0}^\infty R_{\Sigma}^{(\ell)} h^\ell\) and \(R_{\Sigma}^{-1} = \sum_{m=0}^\infty R_{\Sigma}^{-1}^{(m)} h^m\). By the previous proposition, we have an approximation of \(\hat{\Delta}_{\Sigma}(a)\) at the order \(j\)

\[
\hat{\Delta}_{\Sigma}(a) = \sum_{\Sigma' \subseteq \Sigma, |\Sigma'| \leq j} (-1)^{j-|\Sigma'|} C_{|\Sigma|-|\Sigma'|}^{j-|\Sigma'|} \hat{\Delta}_{\Sigma'}(a) + O(h^{j+1}).
\]

Then we have the following approximation of \(\delta_{\{1,\ldots,n\}}(R a R^{-1})\):

\[
\delta_{\{1,\ldots,n\}}(R a R^{-1}) = \sum_{\Sigma \subseteq \{1,\ldots,n\}} \sum_{\ell, m \leq i} (-1)^{n-|\Sigma|} R_{\Sigma}^{(\ell)} \hat{\Delta}_{\Sigma}(a) R_{\Sigma}^{-1}^{(m)} h^{\ell+m} + O(h^{i+1}) =
\]

\[
= \sum_{j=0}^{i} \sum_{\ell, m = i-j} (-1)^{n-|\Sigma|} R_{\Sigma}^{(\ell)} \hat{\Delta}_{\Sigma}(a) R_{\Sigma}^{-1}^{(m)} h^{\ell+m} + O(h^{i+1}) =
\]

\[
= \sum_{j=0}^{i} \sum_{\ell, m = j} (-1)^{n-|\Sigma|} (-1)^{j-|\Sigma'|} C_{|\Sigma|-|\Sigma'|}^{j-|\Sigma'|} R_{\Sigma}^{(\ell)} \hat{\Delta}_{\Sigma'}(a) R_{\Sigma}^{-1}^{(m)} h^{\ell+m} + O(h^{i+1}).
\]

We denote (E) the last expression in brackets, and we’ll show that this expression is zero, whence \(\delta_n(R a R^{-1}) = O(h^{i+1})\).

Let’s look first at the terms corresponding to \(\ell + m = 0\), that is \(j = i\). Then we find back \(\delta_{\{1,\ldots,n\}}(a)\), which is in \(O(h^{i+1})\) by assumption. Therefore, by now on in the sequel of the computation we assume \(\ell + m > 0\).
Consider first how the terms $R^{(\ell)}_\Sigma$ and $R^{(-m)}_\Sigma$ act on $(H \otimes H)^{\otimes n}$ (respectively on the left and on the right) for $\ell + m$ fixed (and positive), say $\ell + m = S$.

Taking the truncated expansion of each $R_{i,j}$ which occurs in $R_\Sigma$, we see that $R^{(\ell)}_\Sigma$ and $R^{(-m)}_\Sigma$ are sums of products of at most $\ell$ and $m$ terms respectively, each one acting on at most two tensor of $(H \otimes H)^{\otimes n}$. We re-write $\sum_{\ell + m = S} R^{(\ell)}_\Sigma \bar{\Delta}^{(s)}_{\Sigma'}(a) R^{(-m)}_\Sigma$ by gathering together the terms of the sum which act on the same factors of $(H \otimes H)^{\otimes n}$: we’ll denote the set of positions of this factors by $\Sigma''$.

Now consider $\bar{\Sigma} \supseteq \Sigma$. From the very definition we have $R^{(s)}_\Sigma = R^{(s)}_\Sigma + A$, where $A$ is a sum of terms which contain factors $R_{i,j}$ with $\{i, j\} \not\subset \Sigma$: to see this, it is enough to expand every factor $R_{a,b}$ in $R_\Sigma$ as $R_{a,b} = 1^{\otimes 2n} + O(h)$. Similarly, we have also $R^{(s)}_\Sigma = R^{(s)}_\Sigma + A'$, and similarly $R^{(-m)}_\Sigma = R^{(-m)}_\Sigma + A''$. This implies that $A^{(s)}_{\Sigma',\Sigma_\Sigma}(a) = A^{(s)}_{\Sigma',\Sigma_\Sigma}(a)$, and so the $A^{(s)}_{\Sigma',\Sigma_\Sigma}(a)$ do not depend on $\Sigma$; then we write

$$\sum_{\ell + m = S} R^{(\ell)}_\Sigma \bar{\Delta}^{(s)}_{\Sigma'}(a) R^{(-m)}_\Sigma = \sum_{\Sigma'' \subseteq \Sigma} A^{(s)}_{\Sigma',\Sigma_\Sigma}(a).$$

In the sequel we re-write (E) using the $A^{(s)}_{\Sigma',\Sigma_\Sigma}(a)$. In the following we’ll denote by $\delta_{\Sigma'' \subseteq \Sigma'}$ the function whose value is 1 if $\Sigma'' \subseteq \Sigma'$ and 0 if not.

Then we obtain a new expression for $\delta_{\{1, \ldots, n\}}(RaR^{-1})$, namely

$$\delta_{\{1, \ldots, n\}}(RaR^{-1}) = \sum_{j=0}^{i-1} \sum_{\Sigma' \subseteq \{1, \ldots, n\}} \left( \sum_{\Sigma \subseteq \{1, \ldots, n\}, \Sigma \not\subseteq \{1, \ldots, n\}} (-1)^{n - |\Sigma'|} (-1)^{j - |\Sigma'|} C_{|\Sigma| - 1 - |\Sigma'|} \times \right.$$

$$\left. \times \sum_{\Sigma'' \subseteq \Sigma} A^{(i-j)}_{\Sigma''}(a) + (-1)^{n - |\Sigma'|} \sum_{\Sigma'' \subseteq \Sigma} A^{(i-j)}_{\Sigma''}(a) \right) h^{i-j} + O(h^{i+1}) =$$

$$= \sum_{j=0}^{i-1} \sum_{\Sigma' \subseteq \{1, \ldots, n\}} \sum_{|\Sigma'| \leq j} h^{i-j} \sum_{\Sigma'' \subseteq \{1, \ldots, n\}} A^{(i-j)}_{\Sigma''}(a) \times$$

$$\times \left( \sum_{\Sigma \subseteq \{1, \ldots, n\}} (-1)^{n - |\Sigma'|} (-1)^{j - |\Sigma'|} C_{|\Sigma| - 1 - |\Sigma'|} + (-1)^{n - |\Sigma'|} \delta_{\Sigma'' \subseteq \Sigma'} \right) + O(h^{i+1}).$$
We denote \((E')_{\Sigma',\Sigma''}\) the new expression in brackets; in other words, for fixed \(\Sigma'\) and \(\Sigma''\), with \(|\Sigma'| \leq j\), we set
\[
(E')_{\Sigma',\Sigma''} := \sum_{\substack{\Sigma \subseteq \{1,\ldots,n\} \\Sigma \subseteq \Sigma, \; \Sigma' \subseteq \Sigma, \; |\Sigma| > j}} (-1)^{n-|\Sigma|} (-1)^{j-|\Sigma'|} C^j_{|\Sigma|-1-|\Sigma'|} + (-1)^{n-|\Sigma'|} \delta_{\Sigma'' \subseteq \Sigma'}
\]
(by the way, we remark that this is a purely combinatorial expression); we shall show that this expression is zero when \(\Sigma'\) and \(\Sigma''\) are such that \(|\Sigma' \cup \Sigma''| \leq j - i + |\Sigma'|\) and \(|\Sigma'| \leq j\).

In force of the following lemma, this will be enough to prove Proposition 3.4.

**Lemma 3.5.**
(a) We have \(j < i\) and \(i \leq n - 1\), hence \(j \leq n - 2\).
(b) For all \(S > 0\), in the expression
\[
\sum_{\ell+m=S} R^{(\ell)}_{\Sigma} \tilde{\Delta}_{\Sigma'}(a) R^{-m}_{\Sigma} = \sum_{\Sigma'' \subseteq \Sigma} A^{(S)}_{\Sigma',\Sigma''}(a)
\]
we have that \(A^{(S)}_{\Sigma',\Sigma''}(a) = 0\) for all \(\Sigma', \Sigma''\) such that \(|\Sigma' \cup \Sigma''| > S + |\Sigma'|\).

**Proof.** The first part of the statement is trivial; to prove the second, we study the adjoint action of \(R_{\Sigma}\) on \((H \otimes H)^{\otimes n}\).

First of all, on \(k \cdot T^{\otimes n}\) the action of these elements gives a zero term because one gets the term at the order \(S\) of the \(h\)-adic expansion of \(R_{\Sigma} \cdot R_{\Sigma}^{-1} = 1\) (for \(S > 0\)).

Second, let us consider \(\Sigma \subseteq \{1, \ldots, n\}\), and let us study the action on \((H \otimes H)^{\otimes n}\). We know that \(R_{\Sigma}\) is a product of \(|\Sigma|^2\) terms of type \(R_{a,b}\), with \(a, b \in \{2i-1, 2j\mid i, j \in \Sigma\} \); so let’s analyse what happens when one computes the product \(P := R_{\Sigma} \cdot x \cdot R_{\Sigma}^{-1}\) if \(x \in (H \otimes H)^{\otimes n}\).

Consider the rightmost factor \(R_{a,b}\): if \(a, b \not\in \{2j-1, 2j\mid j \in \Sigma'\}\), then when computing \(P\) one gets \(P := R_{\Sigma} x R_{\Sigma}^{-1} = R_{a,b} x R_{a,b}^{-1} x R_{a,b}^{-1} = R_{a,b} x R_{a,b}^{-1}\) (where \(R_{*} := R_{\Sigma} R_{a,b}^{-1}\)). Similarly, moving further on from right to left along \(R_{a,b}\) one can discard all factors \(R_{c,d}\) of this type, namely those such that \(c, d \not\in \{2j-1, 2j\mid j \in \Sigma'\}\). Thus the first factor whose adjoint action is non-trivial will be necessarily of type \(R_{a,b}\) with one of the two indices belonging to \(\{2j-1, 2j\mid j \in \Sigma'\}\), say for instance \(\tilde{a}\). Notice that the new index \(\tilde{a} \in \{1, 2, \ldots, 2n-1, 2n\}\) — which “marks” a tensor factor in \(H^{\otimes 2n}\) — corresponds to a new index \(j_{\tilde{a}} \in \{1, \ldots, n\}\) — marking a tensor factor of \((H \otimes H)^{\otimes n}\). So for the following factors — i.e. on the left of \(R_{a,b}\) one has to repeat the same analysis, but with the set \(\{2j-1, 2j\mid j \in \Sigma' \cup \{j_{\tilde{a}}\}\}\) instead of \(\{2j-1, 2j\mid j \in \Sigma'\}\); therefore, as \(R_{a,b}\) might act in non-trivial way on at most \(|\Sigma'|\) factors of \((H \otimes H)^{\otimes n}\), similarly the factor which is the closest on its left may act in a non-trivial way on at most \(|\Sigma'| + 1\) factors. The upset is that the adjoint action of \(R_{\Sigma}\) is non-trivial on at most \(|\Sigma'| + 1\) factors of \((H \otimes H)^{\otimes n}\).

Now consider the different terms \(R^{(\ell)}_{\Sigma} \cdot R^{-m}_{\Sigma}\), with \(\ell + m = S\), and study the products \(R^{(\ell)}_{\Sigma} \cdot x \cdot R^{-m}_{\Sigma}\), with \(x \in (H \otimes H)^{\otimes n}\). We already know that \(R^{(\ell)}_{\Sigma}\) and \(R^{-m}_{\Sigma}\) are sums of products, denoted \(P_{+}\) and \(P_{-}\), of at most \(\ell\) and \(m\) terms respectively, of type \(R_{i,j}^{(\pm k)}\); the terms \(A^{(S)}_{\Sigma',\Sigma''}(a)\) then are nothing but sums of terms of type \(P_{+} \tilde{\Delta}_{\Sigma'}(a) P_{-}\),
where in addition the products \( P_+ \) and \( P_- \) have their "positions" in \( \Sigma'' \). Now, since each \( P_+ \) and each \( P_- \) is a product of at most \( \ell \) and \( m \) factors \( R_{s,j}^{(k)} \), one can refine the previous argument. Consider only the term at the order \( S \) of the \( h \)-adic expansion of \( P := R_{\Sigma} \times R_{\Sigma}^{-1} = R_{*} R_{a,b} x R_{a,b}^{-1} R_{*}^{-1} = R_{*} x R_{*}^{-1} \): whenever there are factors of type \( R_{a,b}^{(k)} \) or \( R_{a,b}^{(t)} \), for fixed \( a, b \) — not belonging to \( \{ 2j - 1, 2j \mid j \in \Sigma \} \) — which appear in \( R_{\Sigma}^{(\ell)} \) or \( R_{\Sigma}^{(m)} \), for some \( \ell \) or \( m \), the total contribution of all these terms in the sum \( \sum_{\ell + m = S} R_{\Sigma}^{(\ell)} x R_{\Sigma}^{(m)} \) will be zero (this follows from the fact that \( R_{*} R_{a,b} x R_{a,b}^{-1} R_{*}^{-1} = R_{*} x R_{*}^{-1} \)). In addition, since now we are dealing only with \( S \) factors in total, we conclude that \( A^{(S)}_{\Sigma', \Sigma''}(a) = 0 \) if \( |\Sigma' \cup \Sigma''| > S + |\Sigma'| \). \( \square \)

Now we shall compute \( (E')_{\Sigma', \Sigma''} \). Thanks to the previous remark, we can limit ourselves to consider the pairs \( (\Sigma', \Sigma'') \) such that \( |\Sigma' \cup \Sigma''| \leq i - j + m + |\Sigma'| \leq i - j + j = i \leq n - 1 \). Then one can always find at least two \( \Sigma \subseteq \{1, \ldots, n\} \) such that \( |\Sigma| > j \) and \( \Sigma' \cup \Sigma'' \subseteq \Sigma \), which make us sure that there will always be at least two terms in the calculation which is to follow (such a condition will guarantee the vanishing of the expression \( (E')_{\Sigma', \Sigma''} \)). We distinguish three cases:

(I) If \( \Sigma'' \subseteq \Sigma' \), then the expression \( (E')_{\Sigma', \Sigma''} \) becomes

\[
(E' : 1)_{\Sigma', \Sigma''} = \sum_{\Sigma \subseteq \{1, \ldots, n\}, \Sigma' \subseteq \Sigma, |\Sigma| > j} (-1)^{n - |\Sigma|} (-1)^{j - |\Sigma'|} C_{|\Sigma| - 1 - |\Sigma'|}^{j - |\Sigma'|} + (-1)^{n - |\Sigma'|} .
\]

Gathering together the \( \Sigma \)'s which share the same cardinality \( d \), a simple computation gives

\[
(E' : 1)_{\Sigma', \Sigma''} = \sum_{d=j+1}^{n} (-1)^{n - d} (-1)^{j - |\Sigma'|} C_{d - 1 - |\Sigma'|}^{j - |\Sigma'|} C_{n - |\Sigma'|}^{d - |\Sigma'|} + (-1)^{n - |\Sigma'|} .
\]

Now, this last expression is zero by Lemma 3.3, for it corresponds to a sum of type \( \sum_{k=r+1}^{t} (-1)^{t+r-k} C_{k-1}^{r} C_{k}^{t} = \sum_{k=0}^{t} (-1)^{t+r-k} C_{k-1}^{r} C_{k}^{t} + (-1)^{t} \) (where \( C_{u}^{v} := 0 \) if \( v > u \)) with \( r, t \in \mathbb{N}_+ \) and \( r < t \); in our case we set \( t = n - |\Sigma'| \), \( r = j - |\Sigma'| \) and \( k = d - |\Sigma'| \); one verifies that one has just \( j - |\Sigma'| < n - |\Sigma'| \) because \( j < n \).

(II) If \( \Sigma'' \not\subseteq \Sigma' \) and \( |\Sigma' \cup \Sigma''| > j \), then the expression \( (E')_{\Sigma', \Sigma''} \) becomes

\[
(E' : 2)_{\Sigma', \Sigma''} = \sum_{\Sigma \subseteq \{1, \ldots, n\}, \Sigma' \subseteq \Sigma \cup \Sigma''} (-1)^{n - |\Sigma|} (-1)^{j - |\Sigma'|} C_{|\Sigma| - 1 - |\Sigma'|}^{j - |\Sigma'|} .
\]

Gathering together the \( \Sigma \)'s which share the same cardinality \( d \), a simple computation gives

\[
(E' : 2)_{\Sigma', \Sigma''} = \sum_{d=|\Sigma' \cup \Sigma''|}^{n} (-1)^{n - d} (-1)^{j - |\Sigma'|} C_{d - 1 - |\Sigma'|}^{j - |\Sigma'|} C_{n - |\Sigma'|}^{d - |\Sigma' \cup \Sigma''|} .
\]

Again, the last expression is zero thanks to Lemma 3.3, for it corresponds to a sum of type \( \sum_{k=0}^{t} (-1)^{t+r-k} C_{k+s}^{r} C_{k}^{t} \) with \( r, t, s \in \mathbb{N}_+ \) and \( r < t \); in our case we set
\( t = n - |\Sigma' \cup \Sigma''|, \quad r = j - |\Sigma'|, \quad s = |\Sigma' \cup \Sigma''| - |\Sigma'| - 1 \) and \( k = d - |\Sigma' \cup \Sigma''| \); then one verifies that \( j - |\Sigma'| < n - |\Sigma'| \) for \( j < n \) and \( |\Sigma' \cup \Sigma''| - |\Sigma'| - 1 \geq 0 \) since \( \Sigma'' \not\subseteq \Sigma' \).

(III) If \( \Sigma'' \not\subseteq \Sigma' \) and \( |\Sigma' \cup \Sigma''| \leq j \), then the expression \((E')_{\Sigma', \Sigma''}\) becomes

\[
(E' : 3)_{\Sigma', \Sigma''} = \sum_{\substack{\Sigma \subseteq \{1, \ldots, n\} \\ \Sigma' \cup \Sigma'' \subseteq \Sigma, \quad \Sigma \geq j}} (-1)^{n-|\Sigma|} (-1)^{j-|\Sigma'|} C^j_{|\Sigma'|} C^d_{n-|\Sigma' \cup \Sigma''|}.
\]

Gathering together the \( \Sigma \)'s which share the same cardinality \( d \), a simple computation gives

\[
(E' : 3)_{\Sigma', \Sigma''} = \sum_{d=j+1}^{n} (-1)^{n-d} (-1)^{j-|\Sigma'|} C^j_{d-1-|\Sigma'|} C^d_{n-|\Sigma' \cup \Sigma''|}.
\]

But again the last expression is zero because of Lemma 3.3, for it corresponds to a sum of type \( \sum_{k=j+1-|\Sigma' \cup \Sigma''|}^{t} (-1)^{t+r-k} C^r_{k+s} C^k_{t} = \sum_{k=0}^{t} (-1)^{1+r-k} C^r_{k+s} C^k_{t} \) (where \( C^r_{u} := 0 \) if \( v > u \)) with \( r, t, s \in \mathbb{N}_+ \) and \( r < t \); here again we set \( t = n - |\Sigma' \cup \Sigma''|, \quad r = j - |\Sigma'|, \quad s = |\Sigma' \cup \Sigma''| - |\Sigma'| - 1 \) and \( k = d - |\Sigma' \cup \Sigma''| \); one has, always for the same reasons, \( j - |\Sigma'| < n - |\Sigma'| \) and \( |\Sigma' \cup \Sigma''| - |\Sigma'| - 1 \geq 0 \).

Therefore, one has always \( (E')_{\Sigma', \Sigma''} = 0 \), whence \( (E) = 0 \), which ends the proof. \( \square \)

References


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