LECTURE VI

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1 The Boson Fock space

In this lecture we outline the main ideas underlying the new, white noise, approach to classical and quantum stochastic calculi. We refer to [AcLuVo00], [AcLuVo99] for more details, and to [AcSk99], [AcFrSk00], [AHK00], [Śni00] for more recent developments on stochastic calculus with higher powers of the white noise.

A standard way to construct white noises is through Fock spaces. One could give an abstract, purely algebraic, definition of Fock space but in this lecture we describe a concrete representation which is most often used in the physical literature because it is well suited for explicit calculations. Such a representation can be used whenever the 1–particle space is concretely realized as an L^2–space over some measure space (finite or σ–finite) (S, μ). In this case the n–particle space can be realized as the space $L^2_{\mathrm{sym}}(S^n, \otimes^n \mu)$ of all the symmetric, square integrable functions on the product space

$$S^n := S \times S \times \ldots \times S \qquad (n - \text{times})$$

with the measure $\otimes^n \mu$, which is the product of n copies of the measure μ. In the following we shall fix the choice

$$S = \mathbb{R}^d \qquad ; \qquad \mu = \text{ Lebesgue measure}$$

Let $\mathcal{F}_1 = L^2(\mathbb{R}^d)$ be the Hilbert space of functions on $\mathbb{R}^d$ with the inner product

$$(f, g) = \int_{\mathbb{R}^d} \overline{f}(s) g(s) ds \quad f, g \in \mathcal{F}_1 \tag{1}$$

and $\mathcal{F}_n = L^2_{\mathrm{sym}}(\mathbb{R}^{nd})$, $n = 1, 2, \ldots$ be the Hilbert space of square integrable functions of n–variables in $\mathbb{R}^d$, symmetric under the permutation of their arguments. The elements of $\mathcal{F}_n$ are called *n–particle vectors*. For an element $\psi_n \in \mathcal{F}_n$ we write $\psi_n = \psi_n(s_1, \ldots, s_n)$, $s_i \in \mathbb{R}^d$ and one has $\psi_n(s_1, \ldots, s_n) = \psi_n(s_{\pi(1)}, \ldots, s_{\pi(n)})$ for any permutation π.

Definition 1 *The symmetric representation of the scalar Boson Fock space $\mathcal{F}$ is the direct sum of the Hilbert spaces $\mathcal{F}_n$*

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} L^2_{sym}(\mathbb{R}^{dn}) = \mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n \tag{2}$$

Here we set $\mathcal{F}_0 = \mathbb{C}$*. So, an element of the Boson Fock space* $\mathcal{F}$ *is a sequence of functions*

$$\Psi = \{\psi_0, \psi_1, \psi_2, \ldots\}$$

where $\psi_0 \in \mathbb{C}$*,* $\psi_n \in \mathcal{F}_n$*,* $n = 1, 2, \ldots$ *and*

$$\|\psi\|^2 = \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2_{L^2(\mathbb{R}^{dn})} < \infty \tag{3}$$

More explicitly

$$\|\psi\|^2 = |\psi^{(0)}|^2 + \sum_{n=1}^{\infty} \int_{\mathbb{R}^{dn}} |\psi^{(n)}(s_1, \ldots, s_n)|^2 ds_1 \ldots ds_n \tag{4}$$

The inner product of elements $\Psi = \{\psi_n\}_{n=0}^{\infty}$ *and* $\Phi = \{\phi_n\}_{n=0}^{\infty}$ *from* $\mathcal{F}$ *is given by*

$$(\Psi, \Phi) = \sum_{n=0}^{\infty} (\psi_n, \phi_n) = \overline{\psi}_0 \phi_0 + \sum_{n=1}^{\infty} \int_{\mathbb{R}^{nd}} \overline{\psi_n(s_1, \ldots, s_n)} \phi_n(s_1, \ldots, s_n) ds_1 \ldots ds_n \tag{5}$$

The vector $\Psi_0 = (1, 0, 0, \ldots)$ *is called the* **vacuum vector***. It describes the state of the system in which no particle is present.*

Remark. More generally, if $\mathcal{H}$ an abitrary Hilbert space then one defines the Boson Fock space $\mathcal{F}$ over $\mathcal{H}$ as the symmetric tensor algebra on $\mathcal{H}$:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s n}$$

where $\otimes_s$ means the symmetric tensor product.

2 Creation and annihilation operators

We define now the annihilation and creation operators on $\mathcal{F}$. They play an important role because as we will see a very wide class of operators in $\mathcal{F}$ (in particular all bounded operators) can be expressed in terms of annihilation and creation operators.

Definition 2 *For any* $f \in \mathcal{F}_1$ *one defines the* **annihilation operator**

$$A(f) : \mathcal{F}_n \to \mathcal{F}_{n-1} \quad n = 1, 2, \dots$$

by the following formula: the annihilation operator annihilates the vacuum

$$A(f)\Phi_0 = 0 \tag{6}$$

and, if $\Psi = (0, \dots, 0, \psi_n, 0, \dots) \in \mathcal{F}_n$*, then* Ψ *is in the domain of* $a(f)$ *and* $a(f)\Psi = \Phi$*, where* $\Phi = (0, \dots, 0, \phi_{n-1}, 0, \dots) \in \mathcal{F}_{n-1}$ *and*

$$\phi_{n-1}(s_1, \dots, s_{n-1}) = \sqrt{n} \int f(s_n)\psi_n(s_1, \dots, s_n) ds_n \tag{7}$$

One defines the **creation operator**

$$A^+(f) : \mathcal{F}_n \to \mathcal{F}_{n+1} \quad n = 0, 1, 2, \dots$$

by the following formula: any $\Psi = (0, \dots, 0, \psi_n, 0, \dots) \in \mathcal{F}_n$ *is in the domain of* $A^+(f)$ *and* $A^+(f)\Psi = \Phi$*, where* $\Phi = (0, \dots, 0, \phi_{n+1}, 0, \dots) \in \mathcal{F}_{n+1}$ *and*

$$\phi_{n+1}(s_1, \dots, s_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \overline{f(s_i)}\psi_n(s_1, \dots, \hat{s}_i, \dots, s_{n+1}) \tag{8}$$

Here $\hat{s}_i$ *means that this argument is missing.*

Remark. Using shorter notations one can write (109) and (7) as

$$(A(f)\psi_n)_{n-1}(s_1, \dots, s_{n-1}) = \sqrt{n} \int \overline{f(s_n)}\psi_n(s_1, \dots, s_n) ds_n \tag{9}$$

$$(A^+(f)\psi_n)_{n+1}(s_1, \dots, s_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} f(s_i)\psi_n(s_1, \dots, \hat{s}_i, \dots, s_{n+1}) \tag{10}$$

In particular the action of the creation operator $A^+(f)$ on the vacuum $\Psi_0 := (1, 0, \dots)$ gives the 1–particle state

$$A^+(f)\Phi_0 = (0, \overline{f}, 0, \dots)$$

and the action of the creation operator on the 1–particle state $\Psi_1 = (0, \psi_1(s_1), 0, \dots)$ gives the 2–particle state

$$A^+(f)\Psi_1 = \left(0, 0, \frac{1}{2}\left(\overline{f}(s_1)\psi_1(s_2) + \overline{f}(s_2)\psi_1(s_1)\right), 0, \dots\right) = \left(0, 0, \frac{1}{2}\left(f \otimes \psi_1 + \psi_1 \otimes f, \dots\right)\right)$$

The exponential vectors

$$\psi(f) = \psi_f = (\psi^{(n)}(f)) \tag{11}$$

are defined by

$$\psi^{(n)}(f)(s_1, \ldots, s_n) = \frac{1}{\sqrt{n!}} \prod_{j=1}^{n} f(s_j) \tag{12}$$

Vectors in $\mathcal{F}$ for which all components, with the exception of at most a finite number are equal to zero are called *finite* particle vectors or simply number vectors. The number vectors form a dense set in the Hilbert space $\mathcal{F}$. We denote this set D.

Proposition 1 *The creation and annihilation operators are adjoint to each other. On the finite vectors one has*

$$(\Phi, A(f)\Psi) = (A^+(f)\Phi, \Psi) \tag{13}$$

Proof. It is enough to check (8) for n–particle vectors

$$(\Phi_{n-1}, A(f)\Psi_n) = (A^+(f)\Phi_{n-1}, \Psi_n) \tag{14}$$

The identity (86) follows from (81) and (75). More details are in the proof of Proposition **??**.

3 Annihilator and creator densities

In the notations of section (2) define

$$\mathcal{D}_{\mathcal{S}} := \{\psi \in \mathcal{F} | \psi^{(n)} \in \mathcal{S}(\mathbb{R}^{dn})\} \tag{15}$$

In the remaining of this chapter, unless otherwise specified, all the n–particle vectors shall belong to $\mathcal{D}_{\mathcal{S}}$. Define moreover

$$\mathcal{D}^o_{\mathcal{S}} := \{\psi \in \mathcal{D}_{\mathcal{S}} | \psi^{(n)} = 0 \ \text{ for almost all } n \in \mathbb{N} \ \} \tag{16}$$

$$\mathcal{D}(a) := \left\{ \psi \in \mathcal{D}_{\mathcal{S}} \ : \ \sum_{n=1}^{\infty} n \|\psi^{(n)}\|^2 < \infty \right\} \tag{17}$$

and notice that $\mathcal{D}(a)$ is a vector space containing both the number and the exponential vectors with test functions in $\mathcal{S}$. Define the *annihilation density*$^{(1)}$ a_s

$$(a_s\psi)^{(n)}(s_1,\ldots,s_n) = \sqrt{n+1}\psi^{(n+1)}(s,s_1,\ldots,s_n) \qquad ; \qquad s\in\mathbb{R}^d \;\; n\in\mathbb{N} \tag{18}$$

The right hand side of (7) is well defined whenever it makes sense to speak of the values $\psi^{(n)}$ on any point, for example when $\psi^{(n)}$ is in the L^2–equivalence class of a continuous function for each n, and the sequence of functions $\{(a_s\psi)^{(n)}\}$ defines an element of $\mathcal{F}$. This is surely the case if ψ is in $\mathcal{D}(a)$. Thus, for any $t\in\mathbb{R}^d$ the annihilator a_t is a densely defined operator which maps $\mathcal{D}(a)$ into $\mathcal{F}$.

From (85) it follows that the map a_s is weakly measurable and therefore, for any square integrable function g, the integral

$$A(g) = \int_{\mathbb{R}^d} ds\ \overline{g}(s)a_s \tag{19}$$

is well defined as a Bochner integral on $\mathcal{D}(a)$. That it is a pre-closed operator will follow from Proposition 107 below. The explicit action of $A(g)$ on vectors in $\mathcal{D}(a)$ is deduced from (8) to be, for $n\in\mathbb{N}$:

$$(A(g)\psi)^{(n)} = \int_{\mathbb{R}^d} ds\,\overline{g}(s)(a_s\psi)^{(n)}(s_1,\ldots,s_n) = \sqrt{n+1}\int_{\mathbb{R}^d} ds\,\overline{g}(s)\psi^{(n+1)}(s,s_1,\ldots,s_n) \tag{20}$$

For example the explicit action of $A(g)$ on the exponential vectors is also deduced from (4) and it is simpler:

$$A(g)\psi_f = \int_{\mathbb{R}^d} ds\ \overline{g}(s)a_s\psi_f = \int_{\mathbb{R}^d} ds\ \overline{g}(s)f(s)\psi_f = \langle g,f\rangle\psi_f \tag{21}$$

The *creation density* a_s^+ is defined for $\psi\in\mathcal{D}_{\mathcal{S}}$ by

$$(a_s^+\psi)^{(n)}(s_1\ldots s_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta(s-s_i)\psi^{(n-1)}(s_1\ldots\hat{s}_i\ldots s_n) \tag{22}$$

The δ–function on the right hand side of (85) shows that the creation density $a^+(t)$ is not an operator but a sesquilinear form on the number vectors. The following Proposition shows that the condition of Definition **??** below is satisfied if we choose the domaian $\mathcal{D}$ to be the number vectors.

Proposition 2 *For any square integrable function g there exists a preclosed operator $A^+(g)$, defined on the n–particle vectors, represented by continuous functions, by the relation*

$$(A^+(g)\psi)^{(n)}(s_1,\dots,s_n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g(s_i)\psi^{(n-1)}(s_1,\dots,\hat{s}_i,\dots,s_n) \tag{23}$$

Moreover, on the n–particle space, $A^+(g)$ is bounded with norm less or equal to $n^{1/2} \parallel g \parallel$ (L^2–norm of g) and, on $\mathcal{D}(a)$, $A^+(g)$ satisfies the relation

$$\langle A^+(g)\psi, \psi' \rangle = \langle \psi, A(g)\psi' \rangle \tag{24}$$

Proof. Let ψ be as in the statement. Then, using the definition (86) of $A^+(g)$:

$$\parallel (A^+(g)\psi)^{(n)} \parallel^2 = \frac{1}{n} \sum_{i,j=1}^{n} \langle g_i\psi_i^{(n-1)}, g_j\psi_j^{(n-1)} \rangle$$

where

$$g_i(s_1,\dots,s_i,\dots,s_n) := g(s_i) \quad ; \quad \psi_i^{(n-1)}(s_1,\dots,s_i,\dots,s_n) := \psi^{(n-1)}(s_1,\dots,\hat{s}_i,\dots,s_n)$$

Since $\psi^{(n-1)}$ is a symmetric function it follows that

$$|\langle g_i\psi_i^{(n-1)}, g_j\psi_j^{(n-1)} \rangle| \le \|g_i\psi_i^{(n-1)}\|\|g_j\psi_j^{(n-1)}\| = \|g\|^2\|\psi^{(n-1)}\|^2$$

and therefore

$$\parallel (A^+(g)\psi)^{(n)} \parallel^2 \le n\|g\|^2\|\psi^{(n-1)}\|^2$$

This shows that $A^+(g)$ is a well defined operator on the domain $\mathcal{D}(a)$, bounded on each n-particle space. To prove (10) we compute:

$$\langle \psi, A(g)\psi' \rangle = \sum_n \langle \psi^{(n)}, (A_g\psi')^n \rangle = \sum_n \sqrt{n+1} \int ds\overline{g}(s) \langle \psi^{(n)}, \psi'^{(n+1)}(s,\cdot) \rangle$$

$$= \sum_n \sqrt{n+1} \int ds\overline{g}'_s \int \overline{\psi}^{(n)}(s_1,\dots,s_n)\psi'^{(n+1)}(s,s_1,\dots,s_n)$$

$$= \sum_n \sqrt{n+1} \int ds\overline{g}_s \int \overline{\psi}^{(n)}(s,\dots,s_n)\psi'^{(n+1)}(s,s_1,\dots,s_n)$$

$$= \sum_n \int ds \int ds_1 \dots \int ds_n \left[\frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} g_{s_i}\psi^{(n)}(s_1,\dots,\hat{s}_i,\dots,s_{n+1}) \right]$$

$$\psi'^{(n+1)}(s_1,s_2,\dots,s_{n+1}) = \langle A^+(g)\psi, \psi' \rangle$$

Lemma 1 *The following formulae hold on $\mathcal{D}_a$:*

$$(a(t_1)a^+(t_2)\psi)^{(n)}(s_1,\ldots,s_n) = \tag{25}$$

$$= \sum_{i=1}^{n} \delta(t_2 - s_i)\psi^{(n)}(s_1,\ldots,\hat{s}_i,\ldots,s_n,t_1) + \delta(t_2 - t_1)\psi^{(n)}(s_1,\ldots,s_n)$$

$$(a^+(t_1)a(t_2)\psi)^{(n)}(s_1,\ldots,s_n) = \sum_{i=1}^{n} \delta(t_2 - s_i)\psi^{(n)}(s_1,\ldots,\hat{s}_i,\ldots,s_n,t_1) \tag{26}$$

Proof. Define

$$\begin{aligned}
\phi_{t_2}^{(n+1)}(s_1,\ldots,s_{n+1}) : & \\
&= (a^+(t_2)\psi)^{(n+1)}(s_1,\ldots,s_{n+1}) = \\
&= \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \delta(t_2 - s_i)\psi^{(n)}(s_1,\ldots,\hat{s}_i,\ldots,s_{n+1})
\end{aligned}$$

$\phi^{(n+1)}{}_{t_2}$ is a distribution with values in $\mathcal{F}_{n+1}$, i.e. for any $g \in \mathcal{S}$

$$\left[\int dt_2 g(t_2)\phi_{t_2}\right]^{(n+1)} = \frac{1}{\sqrt{n+1}} \sum_{i} g(s_i)\psi^{(n)}(s_1,\ldots,\hat{s}_i,\ldots,s_{n+1})$$

Now, applying a_{t_1}, as defined by (85), we find

$$\left[a_{t_1} \int dt_2 g(t_2)\phi_{t_2}\right]^{(n)} (s_1,\ldots,s_n) =$$

$$\begin{aligned}
&= g(t_1)\psi^{(n)}(s_1,\ldots,s_n) + \sum_{i=1}^{n} g(s_i)\psi^{(n)}(t_1,\ldots,\hat{s}_i,\ldots,s_n) \\
&= \int dt_2 g(t_2)\delta(t_2 - t_1)\psi^{(n)}(s_1,\ldots,s_n) + \\
&+ \sum_{i=1}^{n} \delta(t_2 - s_i) g(t_2)\psi^{(n)}(t_1,\ldots,\hat{s}_i,\ldots,s_n) \\
&= \int dt_2 g(t_2)\sqrt{n+1}\phi_{t_2}^{(n+1)}(t_1,s_1,\ldots,s_n)
\end{aligned}$$

Therefore in the sense of distributions:

$$(a(t_1)\phi_{t_2})^{(n)}(s_1,\dots,s_n) = \sqrt{n+1}\phi_{t_2}^{(n+1)}(t_1,s_1,\dots,s_n) \tag{27}$$

and from (94) we get (88), (89) is proved in a similar way.

Remark. Comparing (88) and (89) one deduces the **Boson commutation relations** for a scalar Boson white noise

$$a(t_1)a^+(t_2) - a^+(t_2)a(t_1) = \delta(t_2 - t_1)$$

4 Stochastic integrals with respect to the Boson Fock white noises

In this chapter we shall discuss white noises and stochastic integrals in $\mathbb{R}^d$ rather than in $\mathbb{R}$ because exactly the same formulae are valid in the 1– and in the d–dimensional case.

In section (4) we have defined the operators

$$A(F) = \langle F, A\rangle = \int_{\mathbb{R}^d} dsF_s a_s \quad ; \quad A^+(F) = \langle F, A^+\rangle = \int_{\mathbb{R}^d} dsF_s a_s^+ \tag{28}$$

when F is a complex valued function on $\mathbb{R}$. The generalization of these integrals to the case when F is an operator valued function are called *right stochastic integrals* with respect to a_s (resp. a_s^+). One has also to define the *left stochastic integrals*

$$\langle A, F\rangle = \int_{\mathbb{R}^d} dsa_s F_s \quad ; \quad \langle A^+, F\rangle = \int_{\mathbb{R}^d} dsa_s^+ F_s \tag{29}$$

and the *two–sided stochastic integrals*

$$\int_{\mathbb{R}^d} dsF_s a_s^{\pm} G_s \quad ; \quad \int_{\mathbb{R}^d} dsa_s^+ F_s a_s \tag{30}$$

Let $\mathcal{D}_S^0$ be as in (**??**) and let $\mathcal{L}$ be a space of maps from $\mathbb{R}^d$ to linear operators from a dense subspace of $\mathcal{D}_{\mathcal{S}}^o$ to $\mathcal{F}$ with the property that the maps

$$s \mapsto \langle \psi, F_s\varphi\rangle \ ; \quad s \mapsto \|F_s\psi\|^2 \ ; \quad \varphi, \psi \in \mathcal{D}_{\mathcal{S}}^0$$

are locally integrable. Clearly $s \mapsto a_s$ then $a \in \mathcal{L}$, while $s \mapsto a_s^+$ is not in $\mathcal{L}$.

If P_n denotes the projection onto the n–particle space of the Fock space then for any t we can write

$$F_t = \sum_{n,k} P_n F_t P_k =: \sum_{n,k} F_t^{(n,k)}$$

Remark. By inspection from formulae (108) and (109) one can guess that, even if the *integrand* F_s is bounded, in general the stochastic integrals will not be bounded operators. So a precise definition of the notion of stochastic integral should always specify the domain of vectors where this inegral is defined. The general scheme we shall adopt to define stochastic integrals is the following. If G_s denotes any of the integrands in formulae (107) or (108) or (109), I denotes the corresponding stochastic integral and ψ an arbitrary vector, then I will be characterized by the following two properties:

(i) The n–particle component of $I\psi$ is the Bochner integral of the n–particle component of $G_s\psi$:

$$\left(\int_{\mathbb{R}^d} ds G_s \psi\right)^n := \int_{\mathbb{R}^d} ds \left(G_s\psi\right)^n$$

(ii) The n–particle component of $G_s\psi$ is explicitly computed using the rules of the previous section.

In the following sections we shall show how these general principles work in concrete applications.

5 Right annihilator integrals

Let $\mathcal{D}_S^0 =: \mathcal{D}$ and $\mathcal{L} := \mathcal{L}(\mathcal{D})$ be as in section (6).

Definition 3 *The right annihilator stochastic integral of $F \in \mathcal{L}$ is the operator:*

$$\psi = \int F_s a_s \psi ds = \langle F^*, A\rangle\psi := \int F_s A_s \psi ds \tag{31}$$

where the integral is meant as a Bochner integral in the Fock space. It is defined for each $\psi \in \mathcal{D}_S^o$ such that $a_s\psi$ is in the domain of F_s for each s and the vector valued function $s \in \mathbb{R}^d \mapsto F_s a_s \psi$ is Bochner integrable.

The explicit form of the right annihilator stochastic integral on the n–particle vectors can be obtained easily by using the same technique as in section (4). In fact one has that

$$(a_s\psi)^{(n)} = \sqrt{n+1}\psi^{(n+1)}(s,\cdot) \tag{32}$$

where $\psi^{(n+1)}(s,\cdot)$ is the function

$$(s_1,\dots,s_n) \in \mathbb{R}^{dn} \mapsto \psi^{(n+1)}(s,s_1,\dots,s_n) \tag{33}$$

Therefore (107) is equivalent to

$$\int F_s a_s \psi ds = \sum_{n\geq 0} \sqrt{n+1} \int ds F_s \psi^{(n+1)}(s,\cdot) \tag{34}$$

In particular on the exponential vectors this explicit form is

$$\int F_t a_t dt \psi_f = \int dt F_t f(t) \psi_f \tag{35}$$

where the right hand side of (107) is a usual Bochner integral. The right hand side of (81) is defined on the set of the exponential vectors ψ_f with test function in $\mathcal{H}_1$ such that the vector valued function $s \mapsto f(s)F(s)\psi_f$ is Bochner integrable. From Definition 107 we have that,

$$\langle F^*, A\rangle := \int F a_s ds \tag{36}$$

In case $S = \mathbb{R}$ and $F = \chi_I F$ with $I = [0,t]$ we shall simply write

$$\langle F, A_t\rangle := \int_0^t F_s a_s ds \tag{37}$$

Thus the right annihilator integral maps functions $F : \mathbb{R}^d \to \mathcal{L}(\mathcal{D})$ into elements of $\mathcal{L}(\mathcal{D})$.

From (7) and (81) we deduce the estimate

$$\left\| \int F_s a_s \psi ds \right\| \leq \sum_{n\geq 0} \sqrt{n+1} \int ds \|F_s \psi^{(n+1)}(s,\cdot)\| = \int ds \|F_s (N+1)^{1/2} \psi^{(n+1)}_{(s,\cdot)} \|(s,\cdot)\| \tag{38}$$

The definition of exponential vector implies that

$$(N+1)^{1/2}\psi_f^{(n+1)}(s,\cdot) = f(s)\psi_f^{(n)} \tag{39}$$

therefore (81) implies that for any exponential vector ψ_f one has

$$\left\| \int F_s a_s ds \psi_f \right\| \leq \int |f(s)| \cdot \|F_s\psi_f\| ds \tag{40}$$

A sufficient condition for the finiteness of the right hand side of (87) is that vector valued function $s \mapsto f(s)F(s)\psi(s)$ is Bochner integrable.

6 The left creator stochastic integral

Definition 4 *The definition of left creator stochastic integrals is the natural extension of formula (??) for the scalar case:*

$$\left(\int a_t^+ F_t \psi dt\right)^{(n)} (s_1,\dots,s_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (F(s_i)\psi)^{(n-1)}(s_1,\dots,\hat{s}_i,\dots,s_n) \tag{41}$$

Definition 107 has a meaning for any measurable function F_s and, given such an F, the natural domain of its left creator stochastic integral is

$$\mathcal{D}\left(\int a_t^+ F_t dt\right) = \left\{\psi \middle| \sum_{n=1}^{\infty} \|(\left(\int a_t^+ F_t \psi dt\right)^{(n)} \|^2 < \infty\right\} \tag{42}$$

or, more explicitly, a vector ψ is in $\mathcal{D}\left(\int a_t^+ F_t dt\right)$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{\mathbb{R}^{dn}} \left| \sum_{i=1}^{n} (F(s_i)\psi)^{n-1}(s_1,\dots,\hat{s}_i,\dots,s_n) \right|^2 ds_1 \dots ds_n < \infty \tag{43}$$

We want now to obtain an estimate on the norm of $\int a_s^+ F_s ds \psi$ which guarantees that the stochastic integral exists. This is given by the following theorem:

Lemma 2 *Let $\psi^{(n-1)}$ belong to $\mathcal{D}(F_s)$ for all $s \in \mathbb{R}^d$. Then one has, for each $n \in \mathbb{N}$*

$$\left\| P_n \left(\int ds a_s^+ F_s \psi\right) \right\|^2 \leq n \int ds ||P_{n-1}(F_s\psi)\|^2 = \int ds \| \sqrt{(N+1)}(F_s\psi)^{(n-1)}\|^2 \tag{44}$$

In particular

$$\left\| \int dsa_s^+ F_s \psi \right\|^2 \leq \int ds \|(\sqrt{N+1}) F_s \psi\|^2 \tag{45}$$

Proof. The norm square of (107) is

$$\int \dots \int ds_1 \dots ds_n \frac{1}{n} \sum_{i,j} \langle (F_{s_i}\psi)^{(n-1)}(s_1, \dots, \hat{s}_i, \dots, s_n), (F_{s_j}\psi)^{(n-1)}(s_1, \dots, \hat{s}_j, \dots, s_n) \rangle$$

$$\leq \frac{1}{n} \sum_{ij=1}^{n} \int ds_1 \dots ds_n \left\| (F_{s_i}\psi)^{(n-1)}(s_1, \dots, \hat{s}_i, \dots, s_n) \right\| \|(F_{s_j}\psi)^{(n-1)}(s_1, \dots, \hat{s}_j, \dots, s_n)\| =$$

$$= \frac{n^2}{n} \int_{\mathbb{R}^d} ds \int_{\mathbb{R}^{d(n-1)}} ds_2 \dots ds_n \|(F_s\psi)^{(n-1)}(s_2, \dots, s_{n-1})\|^2 = n \int_{\mathbb{R}^d} ds \|(F_s\psi)^{(n-1)}\|^2$$

Corollary (2). Let L_s be a function with values in $\mathcal{B}(\mathcal{F})$ such that (i) For any $0 < T < +\infty$ $\sup_{s \in [0,T]} \|L_s\|_\infty < \sqrt{C_T}$ (ii) L_s and L_s^+ commute with every a_t, a_t^+, P_k $t \in \mathbb{R}$, $k \in \mathbb{N}$.

Then

$$\left\| P_n \int dsa_s^+ L_s F_s \psi \right\|^2 \leq C_T n \int ds \|P_{n-1} F_s \psi\|^2 \tag{46}$$

Proof. From Lemma 107 the left hand side of (75) is less or equal than

$$C_T n \int ds \|P_{n-1} L_s F_s \psi\|^2$$

and, using (ii) and (i) the thesis follows.

7 The normally ordered two–sided integral

Definition 5 *The 2–sided (normally ordered) integral*

$$\int dsb_s^+ F_s b_s$$

is defined, weakly on the exponential or number vectors by:

$$\langle \xi, \int dsb_s^+ F_s b_s \eta \rangle = \int ds \langle b_s \xi, F_s b_s \eta \rangle \tag{47}$$

In particular, on exponential vectors one has

$$\langle\psi_f, \int dsb_s^+ F_s b_s \psi_g\rangle = \int ds\overline{f}(s)g(s)\langle\psi_f, F_s\psi_g\rangle \tag{48}$$

Lemma 3 *For any $n \in \mathbb{N}$ and for any exponential vector ψ_f one has the estimate*

$$\left\|\left(\int dsb_s^+ F_s b_s \psi_f\right)^{(n)}\right\|^2 \leq n \int ds|f(s)|^2 \|(F_s\psi_f)\|^2 \tag{49}$$

In particular

$$\left\|\int dsb_s^+ F_s b_s \psi_f\right\|^2 \leq \int ds|f(s)|^2 \|(N+1)^{1/2}\psi_f\|^2 \tag{50}$$

Proof. Using $b_s\psi_f = f(s)\psi_f$, the left hand side of (109) becomes

$$\left\|\left(\int dsb_s^+ f(s)F_s\psi_f\right)^{(n)}\right\|$$

and, because of (**??**) this is

$$\leq n \int ds\|(F_s\psi_f)^{(n)}\|^2|f(s)|^2$$

i.e. (107). (108) is obtained from (107) by summing over all n.

8 Operator valued distributions: scalar test functions

Since a white noise is an operator valued distribution, we begin by recalling in some generality the definition of this notion[(1)].

For $\varepsilon = 0, 1$ denote

$$\mathcal{S}(\mathbb{R}^d)^\varepsilon = \begin{cases} \overline{\mathcal{S}}(\mathbb{R}^d), & \text{if } \varepsilon = 1 \\ \mathcal{S}(\mathbb{R}^d), & \text{if } \varepsilon = 0 \end{cases} \tag{51}$$

where $\overline{\mathcal{S}}(\mathbb{R}^d)$ coincides with $\mathcal{S}(\mathbb{R}^d)$ as a set but is a complex vector space for the action of the complex numbers defined by

$$(\lambda, f) \in \mathbb{C} \times \mathcal{S}(\mathbb{R}^d) \mapsto \overline{\lambda} f \in \mathcal{S}(\mathbb{R}^d)$$

Then, for any $\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}$ the topological space

$$\mathcal{S}(\mathbb{R}^d)^{\varepsilon_1} \otimes \ldots \otimes \mathcal{S}(\mathbb{R}^d)^{\varepsilon_n} \tag{52}$$

(algebraic tensor product) is naturally embedded in $\mathcal{S}'(\mathbb{R}^{nd})$ hence it inherits the topology of the duality $\langle \mathcal{S}'(\mathbb{R}^{nd}), \mathcal{S}(\mathbb{R}^{nd})\rangle$. The completion of the space (75) for this topology will be denoted

$$\Big(\mathcal{S}(\mathbb{R}^d)^{\varepsilon_1} \otimes \ldots \otimes \mathcal{S}(\mathbb{R}^d)^{\varepsilon_n}\Big)' \tag{53}$$

An n–linear functional on the space (75), continuous for this topology corresponds to a distribution which is linear on the test functions in the spaces $\mathcal{S}(\mathbb{R}^d)^{\varepsilon}$ with $\varepsilon = 1$ and anti–linear on the test functions in the spaces with $\varepsilon = 0, 1$.

Definition 6 *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{D} \subseteq \mathcal{H}$ be a dense subspace. An operator valued distributions on $\mathbb{R}^d$ with domain $\mathcal{D} \subseteq \mathcal{H}$ is a sesquilinear form A^+ on that domain with values in the tempered distributions $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ and with the following additional properties:*
(i) for each function g in the Schwartz space $\mathcal{S}$ there exists two linear operators $A(g)$, $A^+(g)$, having $\mathcal{D}$ as an invariant domain i.e.

$$A(g)\mathcal{D} \subseteq \mathcal{D} \qquad ; \qquad A^+(g)\mathcal{D} \subseteq \mathcal{D} \tag{54}$$

and such that, $A(g)$ and $A^+(g)$ are adjoint of each other on $\mathcal{D}$, i.e. for any $\xi, \eta \in \mathcal{D}$

$$\langle A^+(\xi,\eta), g\rangle := \langle A(g)\xi, \eta\rangle = \langle \xi, A^+(g)\eta\rangle \tag{55}$$

where the brackets $\langle\ \rangle$ in the left hand side of (81) refer to the duality $\langle \mathcal{S}', \mathcal{S}\rangle$ and in the remaining two terms denote the scalar product in $\mathcal{H}$.
(ii) for any $\xi, \eta \in \mathcal{D}$, $n \in \mathbb{N}$, $\varepsilon_1 g_1 \ldots\ldots \varepsilon_n \in \{0,1\}$, and with the notation

$$X^{\varepsilon} = \begin{cases} X, & \text{if } \varepsilon = 1 \\ X^+, & \text{if } \varepsilon = 0 \end{cases} \tag{56}$$

the map

$$(g_1 \otimes \ldots \otimes g_n) \in \mathcal{S}(\mathbb{R}^d)^{\varepsilon_1} \otimes \ldots \otimes \mathcal{S}(\mathbb{R}^d)^{\varepsilon_n} \mapsto \langle \xi, A^{\varepsilon_1}(g_1)\ldots A^{\varepsilon_n}(g_n)\eta\rangle$$

(which is well defined because of (7)) defines an element of $\Big(\mathcal{S}(\mathbb{R}^d)^{\varepsilon_1} \otimes \ldots \otimes \mathcal{S}(\mathbb{R}^d)^{\varepsilon_n}\Big)'$.

Remark. From the above definition it follows that the map $g \mapsto A^+(g)$ is linear and the map $g \mapsto A(g)$ is antilinear. In the following we shall use the notation

$$A(g) = \int_{\mathbb{R}^d} ds\overline{g}(s)a_s \qquad ; \qquad A^+(g) = \int_{\mathbb{R}^d} dsg(s)a_s^+ \tag{57}$$

9 $*$–algebras of distributions

Usual distributions can be multiplied only at the cost of a rather elaborate formalism, but operator valued distributions have a natural structure of $*$–algebra defined as follows.

The algebra generated (algebraically) by the identity in $\mathcal{H}$ and the operators $A(f)$, $A^+(g)$, with $f, g \in \mathcal{S}(\mathbb{R}^d)$ is a $*$–algebra of operators on the common invariant domain $\mathcal{D}$, called the *polynomial algebra* in A, A^+.

Definition 7 *For any $\xi, \eta \in \mathcal{D}$, $n \in \mathbb{N}$, $\varepsilon_1 \ldots\ldots \varepsilon_n \in \{0,1\}$, the unique element of $\Big(\mathcal{S}(\mathbb{R}^d)^{\varepsilon_1} \otimes \ldots \otimes \mathcal{S}(\mathbb{R}^d)^{\varepsilon_n}\Big)'$ defined by*

$$(g_1 \otimes \ldots \otimes g_n) \in \mathcal{S}(\mathbb{R}^{nd}) \mapsto \langle \xi, A^{\varepsilon_1}(g_1) \ldots A^{\varepsilon_n}(g_n)\eta \rangle \tag{58}$$

will be denoted

$$a_{s_1}^{\varepsilon_1} \ldots a_{s_n}^{\varepsilon_n} \tag{59}$$

*This notation amounts to extending to arbitrary products of the operators $A(f)$, $A^+(g)$ the notations (**??**) as follows:*

$$A^{\varepsilon_1}(g_1) \ldots A^{\varepsilon_n}(g_n) =: \int_{\mathbb{R}^d} ds_1 \ldots \int_{\mathbb{R}^d} ds_n g_1^{\varepsilon_1}(s_1) \ldots g_n^{\varepsilon_n}(s_n) a_{s_1}^{\varepsilon_1} \ldots a_{s_n}^{\varepsilon_n} \tag{60}$$

where

$$g^\varepsilon = \begin{cases} \overline{g}, & \textit{if } \varepsilon = 1 \\ g, & \textit{if } \varepsilon = 0 \end{cases} \tag{61}$$

Lemma 4 *Let $\mathcal{P}(\{a, a^+\})$ denote the free unital complex $*$–algebra generated by the symbols*

$$\{a, a^+\} \tag{62}$$

the map

$$a^{\varepsilon_1} \ldots a^{\varepsilon_n} \mapsto a_{s_1}^{\varepsilon_1} \ldots a_{s_n}^{\varepsilon_n} \tag{63}$$

is a vector space homomorphism.

Proof. Clear

We use the homomorphism defined in Lemma 108 to define a structure of $*$–algebra on the distributions (108). This homomorphism will not be, in general an isomorphism because among the $A(f)$, $A^+(g)$ there will be some algebraic relations. The best known examples of these relations are the *commutation relations*.

Definition 8 *Let q be a complex number. The operator valued distribution A, A^+ is said to satisfy the q–commutation relations if the identity*

$$A(f)A^+(g) - qA^+(g)A(f) = \langle f, g\rangle \ ; \quad \forall\, f, g \in \mathcal{S} \tag{64}$$

holds for any $f, g \in \mathcal{S}$, where $\langle f, g\rangle$ denotes the standard scalar product in $L^2(\mathbb{R}^d)$, i.e.:

$$\langle f, g\rangle = \int_{\mathbb{R}^d} \overline{f}(x)g(x)dx \tag{65}$$

The corresponding distribution identity is

$$a_s a_t^+ - q a_t^+ a_s = \delta(t - s) \tag{66}$$

The commutation relations (85) with $q = +1$ and with the additional relations

$$a_s a_t - a_t a_s = a_s^+ a_t^+ - a_t^+ a_s^+ = 0 \tag{67}$$

define the Boson distribution algebra.

The commutation relations (85) with $q = -1$ and with the additional relations

$$a_s a_t + a_t a_s = a_s^+ a_t^+ + a_t^+ a_s^+ = 0 \tag{68}$$

define the *Fermion distribution algebra*.

The commutation relations (85) with

$$q = 0 \tag{69}$$

and without additional relations, define the *Boltzmannian* (or *free*) *distribution algebra*

10 States on distribution algebras

Let $A(g)$, $A^+(g)$ be an operator valued distribution in the sense of section (8).

Definition 9 *A state on the $*$–algebra with unit generated by the operators $A(g)$, $A^+(g)$ is a normalized positive linear functional φ on this algebra with the additional property that for any $n \in \mathbb{N}$ and for any $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}$, the map*

$$g_1 \otimes \dots \otimes g_n \in \otimes^n \mathcal{S} \mapsto \varphi(A^{\varepsilon_1}(g_1) \dots A^{\varepsilon_n}(g_n)) \tag{70}$$

is a Schwarz distribution on $\otimes^n \mathcal{S}$.

For the distribution defined by (1) we will use the notations

$$\varphi(a^{\varepsilon_1}_{s_1} \dots a^{\varepsilon_n}_{s_n}) = \langle a^{\varepsilon_1}_{s_1} \dots a^{\varepsilon_n}_{s_n} \rangle =: \psi(\varepsilon_1, \dots, \varepsilon_n; s_1, \dots, s_n) \tag{71}$$

Thus, by definition

$$\langle A^{\varepsilon_1}(g_1) \dots A^{\varepsilon_n}(g_n) \rangle = \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} ds_1 \dots ds_n g_1^{\varepsilon_1}(s_1) \dots g_n^{\varepsilon_n}(s_n) \langle a^{\varepsilon_1}_{s_1} \dots a^{\varepsilon_n}_{s_n} \rangle \tag{72}$$

Proposition 3 *Any state on the polynomial algebra of $A(g)$, $A^+(g)$ ($g \in \mathcal{S}$) determines a family of distributions $\psi^{(n)}(\varepsilon_1, \dots, \varepsilon_n) \otimes^n \mathcal{S}$*

$$\psi^{(n)}(\varepsilon_1, \dots, \varepsilon_n;\ s_1, \dots, s_n) \tag{73}$$

The distribution (7) satisfies the following positivity condition

$$g^{1-\varepsilon_{h1}}(s_{h1}) \dots g^{1-\varepsilon_h n_h}(s_{hn_h}) g^{\varepsilon_{k1}}(s_{k1}) \dots g^{\varepsilon_{kn_k}}(s_{nk})$$

$$\psi^{(n)}(1 - \varepsilon_{h1}, \dots, 1 - \varepsilon_{hn_h}, \varepsilon_{k1}, \dots, \varepsilon_{kn_k}, s_{h1}, \dots, s_{hn_h}, s_{k1}, \dots, s_{kn_k}) \geq 0 \tag{74}$$

(summation and integration over all variables being understood.

Conversely any family of distributions with the above properties defines a unique state on the polynomial algebra.

Proof. Condition (81) expresses the positivity of

$$\langle \left| \sum_{k=1}^{N} A^{\varepsilon_{k1}}(g_{k1}) \dots A^{\varepsilon_k n_k}(g_{kn_k}) \right|^2 \rangle \tag{75}$$

Conversely, given the family $(\psi^{(n)})$ one can use (109) to define a linear functional on the polynomial algebra and (81) expresses the positivity of this functional.

So, for example, when we say that the *mean value* $\langle a_s \rangle$ of the distribution a_s is equal to a constant, say c, we mean that, for every test function g, one has

$$\int_{\mathbb{R}^d} ds\overline{g}(s)\langle a_s\rangle = \langle A(g)\rangle = c\int_{\mathbb{R}^d} ds\overline{g}(s) \tag{76}$$

11 Normally ordered white noise equations in R^d

Given a notion of stochastic integral, one can study the problem of the meaning, existence, uniqueness and unitarity of the corresponding integral equations. We will study integral equations of the form

$$Y_t = Y_0 + \int_{\mathbb{R}^d} L_{01}(s,t)Y_s a_s ds + \tag{77}$$

$$+ \int_{\mathbb{R}^d} L_{10}(s,t)a_s^+ Y_s ds + \int_{\mathbb{R}^d} L_{11}(s,t)a_s^+ Y_s a_s ds + \int_{\mathbb{R}^d} L_{00}(s,t)Y_s ds$$

where the coefficients $L_{\varepsilon,\varepsilon'}(s,t)$ $(\varepsilon,\varepsilon' = 0,1)$ are linear operators acting on $\mathcal{H}_S$ such that,

i) for any $(\varepsilon,\varepsilon' = 0,1)$ and $s,t \in \mathbb{R}^d$, the operator $L_{\varepsilon,\varepsilon'}(s,t)$ is bounded;

ii) defining

$$\max_{\varepsilon,\varepsilon'=0,1} \|L_{\varepsilon,\varepsilon'}(s,t)\| =: l(s,t) \tag{78}$$

then for any bounded set $B \subseteq \mathbb{R}^d$, the functions

$$s \in \mathbb{R}^d \mapsto l(s,t) \tag{79}$$

are integrable for each $t \in B$ and the set of integrals, as a function of $t \in B$, is bounded.

iii) for any bounded set $B \subseteq \mathbb{R}^d$ there exists a constant $L \geq 0$ such that, for any natural integer k one has:

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} ds_1 \dots ds_k l(s_1, t) l(s_2, s_1) \dots l(s_k, s_{k-1}) \leq \frac{L^k}{k!} \tag{80}$$

uniformly in $t \in B$.

In section (12) we shall can give examples of coefficients $L_{\varepsilon,\varepsilon'}(s,t)$ which satisfy condition iii).

We shall write equation (107) in the notation

$$Y_t = Y_0 + \int_{\mathbb{R}^d} L_{\varepsilon,\varepsilon'}(s,t) d\Lambda_\varepsilon^{\varepsilon'}(s) Y_s \tag{81}$$

where summation is understood in the indices ε, $\varepsilon' \in \{0, 1\}$.

In this notation we define the k–th iterated approximation solution of equation (81) by

$$Y_t^{(0)} = Y_0 \tag{82}$$

$$Y_t^{(k+1)} = \int_{\mathbb{R}^d} L_{\varepsilon\varepsilon'}(s,t) d\Lambda_\varepsilon^{\varepsilon'}(s) Y_s^{(k)} \tag{83}$$

The iterated series, associated to equation (107) is

$$\sum_{k=0}^{\infty} Y_t^{(k)} \tag{84}$$

In this section we shall fix the set

$$\mathcal{S}_0 = \{f \in L^2(\mathbb{R}^d) : \max\{\|f\|_\infty, \|f\|_2\} \leq 1\} \tag{85}$$

and we shall denote $\mathcal{E}(\mathcal{S}_0)$ the corresponding set of exponential vectors. It is known that $\mathcal{S}_0$ is a total set in $\mathcal{F}$.

Theorem 1 *Suppose that the coefficients of equation (107) satisfy conditions i), ii), iii) and moreover*

$$\|L\| < \frac{1}{16e} \tag{86}$$

*the iterated series (**??**) converges, strongly in norm on $\mathcal{E}(\mathcal{S}_0)$ to a solution of this equation uniformly in bounded subsets of $\mathbb{R}^d$.*

For the proof of Theorem 107 we shall use several lemmata.

Lemma 5 *label2 Let $L_{\varepsilon,\varepsilon'}(s,t)$ $(\varepsilon,\varepsilon' = 0,1)$ be linear operators on $\mathcal{H}_S$ satisfying the conditions i), ii) and iii), then for any $t \in B \subseteq \mathbb{R}^d$ being a bounded set, $n \in \mathbb{N}$ and $f \in \mathcal{S}_0$ one has:*

$$\left\| P_n \int L_{\varepsilon\varepsilon'}(s,t) Y_s d\Lambda_\varepsilon^{\varepsilon'}(s)\psi_f \right\|^2 \leq 8n \int l(s,t)ds(\|P_{n-1}Y_s\psi_f\|^2 + \|P_nY_s\psi_f\|^2) \tag{87}$$

In particular, $Y_t^{(k)}$ defined by (75) verifies that

$$\|P_nY_t^{(k+1)}\psi_f\|^2 \leq 8n \int ds\ l(s,t)(\|P_{n-1}Y_s^{(k)}\psi_f\|^2 + \|P_nY_s^{(k)}\psi_f\|^2) \tag{88}$$

Proof. First of all,

$$\left\| P_n \int L_{\varepsilon,\varepsilon'}(s,t) Y_s d\Lambda_\varepsilon^{\varepsilon'}(s)\psi_f \right\|^2 \leq \left(\sum_{\varepsilon,\varepsilon'=0,1} \left\| P_n \int_{\mathbb{R}^d} L_{\varepsilon,\varepsilon'}(s,t) d\Lambda_\varepsilon^{\varepsilon'}(s) Y_s\psi_f \right\| \right)^2$$

The Schwarz inequality

$$\left(\sum_{j=1}^M a_j\right)^2 \leq M \sum_{j=1}^M a_j^2$$

(M an integer M and a_j real numbers), implies that for any $n \in \mathbb{N}$

$$\left\| P_n \int L_{\varepsilon,\varepsilon'}(s,t) Y_s d\Lambda_\varepsilon^{\varepsilon'}(s)\psi_f \right\|^2 \leq 4 \sum_{\varepsilon,\varepsilon'=0,1} \left\| P_n \int_{\mathbb{R}^d} L_{\varepsilon,\varepsilon'} Y_s d\Lambda_\varepsilon^{\varepsilon'}(s)\psi_f \right\|^2 \tag{89}$$

Now we investigate the quantity in the right hand side of (89) term by term according to varies $\varepsilon, \varepsilon'$.

By acting a_s to the exponetial vector, we deduce

$$\left\| P_n \int L_{01}(s,t) Y_s a_s \psi_f ds \right\|^2 \leq \|f\|_2^2 \int \|P_n L_{01}(s,t) Y_s\psi_f\|^2 ds \leq$$

$$\leq \|f\|_2^2 \int ds\ l(s,t) \|P_nY_s\psi_f\|^2 ds \tag{90}$$

From (17.9.4) one has

$$\left\| P_n \int ds\ L_{10}(s,t) a_s^+ Y_s \psi_f \right\|^2 \leq n \int ds\ l(s,t) \|P_{n-1} Y_s \psi_f\|^2 \tag{91}$$

and from (17.9.6)

$$\leq \left\| P_n \int ds\ L_{11}(s,t) a_s^+ Y_s a_s \psi_f \right\|^2 \leq \|f\|_\infty^2 n \int ds\ l(s,t) \|P_{n-1} Y_s \psi_f\|^2 \tag{92}$$

Finally the usual properties of Bochner's integral imply that

$$\left\| P_n \int ds L_{00}(s,t) Y_s \psi_f \right\|^2 \leq \int ds\ l(s,t) \|P_n Y_s \psi_f\|^2 \tag{93}$$

Because of our assumption (85) on f, the sum of the left hand sides of (90), (91), (92), (93) is less or equal than

$$2 \int ds\ l(s,t) \|P_n Y_s \psi_f\|^2 + 2n \int ds\ l(s,t) \|P_{n-1} Y_s \psi_f\|^2 \leq$$

$$\leq 2n \int ds\ l(s,t) (\|P_{n-1} Y_s \psi_f\|^2 + \|P_n Y_s \psi_f\|^2)$$

and this is (87). To deduce (88) one simply applies (87) to the definition of $Y_t^{(k+1)}$.

Lemma 6 *If the series*

$$\sum_{k=0}^{\infty} \|Y_t^{(k)} \psi_f\| \tag{94}$$

converges uniformly on a bounded set B in $\mathbb{R}^d$, then for each $t \in B$ there exists a unique operator Y_t on $\mathcal{H}_S \otimes \mathcal{E}(\mathcal{S}_0)$ such that

$$\sum_{k=0}^{\infty} Y_t^{(k)} = Y_t \tag{95}$$

and the series on the left hand side of (95) converges strongly in norm on $\mathcal{E}(\mathcal{S}_0)$, uniformly for $t \in B$. Moreover the function $\mapsto Y_t$ is a solution of equation (107).

Proof. We know that there exists an operator Y_t on $\mathcal{H}_S \otimes \mathcal{E}(\mathcal{S}_0)$ such that (95) holds. And the estimates of Lemma 108 also imply that the stochastic integrals of Y_t for the basic integrators exist. To prove that Y_t satisfies equation (107) it will be sufficient to prove that, for each $n \in \mathbb{N}$, $P_n Y_t$ satisfies equation (107). To show this, we use the estimate of Lemma 108 to deduce that

$$\left\| P_n \int_{\mathbb{R}^d} L_{\varepsilon\varepsilon'}(s,t) Y_s d\Lambda_\varepsilon^{\varepsilon'}(s)\psi_f - P_n \int_{\mathbb{R}^d} \sum_{k=1}^{N} L_{\varepsilon,\varepsilon'}(s,t) Y_s^{(k)} d\Lambda_\varepsilon^{\varepsilon'}(s)\psi_f \right\| \leq$$

$$\leq 8n \int_{\mathbb{R}^d} \sum_{k=N+1}^{\infty} \|P_n Y_s^{(k)} \psi_f\| l(s,t) ds \tag{96}$$

By assumption, for each $t \in B$, the function $s \mapsto l(s,t)$ is integrable. Therefore the right hand side of (96) tends to zero by dominated convergence as $N \to \infty$.

Letting $N \to \infty$ in (96) we see that Y satisfies equation (107) and this ends the proof.

Lemma 7 *Let $I_{n,k}(n, k \in \mathbb{N})$ be positive numbers satisfying the inequality*

$$I_{n,k+1} \leq cn(I_{n,k} + I_{n-1,k}) \tag{97}$$

where $c > 0$ is a constant, then

$$I_{n,k+1} \leq (2cn)^k \sum_{m=n-k}^{n} I_{m,0} \tag{98}$$

Proof. By iterating the inequality (97) we see that the right hand side is equal to

$$cn(cnI_{n,k-1} + cnI_{n-1,k-1} + c(n-1)I_{n-1,k-1} + c(n-1)I_{n-2,k-1})$$

$$\leq \; (cn)^2(I_{n,k-1} + 2I_{n-1,k-1} + I$$
$$\leq \; (cn)^3(I_{n,k-2} + 3I_{n-1,k-2} + 3$$
$$\leq \; (cn)^k(I_{n,0} + h_1 I_{n-1,0} + h_2 I_n$$

where the coefficients h_α satisfy

$$h_\alpha \leq 2^k$$

and from this (98) immediately follows.

Proof of Theorem 107. Introducing the notation

$$I_{n,k+1}(s) := \|P_n Y_t^{(k+1)} \psi_f\|^2$$

we have from Lemma 108

$$I_{n,k+1}(t) \leq \int ds\ l(s,t) 8n(I_{n,k}(s) + I_{n-1,k}(s))$$

therefore, arguing as in Lemma 7

$$I_{n,k+1}(t) \leq 16^k n^k \sum_{m=n-k}^{n} I_{m,0}(s_k) \int \dots \int ds_1 \dots ds_k l(s_1,t) l(s_2,s_1) \dots l(s_k,s_{k-1}) \tag{99}$$

But for any $s_k \in \mathbb{R}^d$

$$I_{m,0}(s_k) = \|P_m Y_0 \psi_f\|^2 = \|Y_0\|^2 \frac{\|f\|^{2m}}{m!}$$

and without loss of generality we can assume that

$$\|Y_0\| = 1 \tag{100}$$

Moreover, according to assumption iii), the multiple integral in (99) is dominated by $L^k/k!$. In conclusion

$$\|P_n Y_t^{(k+1)} \psi_f\|^2 \leq \frac{(16L)^k}{k!} n^k \sum_{m=n-k}^{n} \frac{\|f\|^{2m}}{m!} \tag{101}$$

Since for large m the sequence $\|f\|^{2m}/m!$ is decreasing the sum in (24) is majorized by

$$k \frac{\|f\|^{2(n-k)}}{(n-k)!}$$

Therefore

$$\|P_n Y_t^{(k+1)} \psi_f\|^2 \leq \frac{(16L)^k}{(k-1)!} \frac{n^k \|f\|^{2(n-k)}}{(n-k)!}$$

So in order to estimate

$$\|Y_t^{(k+1)} \psi_f\|^2$$

we are lead to estimate the series

$$\sum_{n\geq k} \frac{n^k \|f\|^{2(n-k)}}{(n-k)!} = \frac{d^k}{dt^k}\Big|_{t=0} \sum_{n\geq k} e^{tn} \frac{\|f\|^{2(n-k)}}{(n-k)!} = \frac{d^k}{dt^k}\Big|_{t=0} e^{tk} e^{\|f\|^2 e^t} \tag{102}$$

moreover, because of our assumption (85) on the test functions f, we can restrict our attention to the case in which $\|f\| = 1$ in (102). (We could have put $\|f\| = 1$ directly in (101), but it is convenient to leave it to show the opportunity of introducing *Bell numbers depending on a parameter*). In this case by Leibnitz rule the expression (102) is

$$\sum_{h=0}^{k} \binom{k}{h} k^h B_2(k-h) \tag{103}$$

where $B_2(k-h)$ are the Bell numbers of order 2 as defined in [CKS 98].

Under this assumption denoting

$$c := 16L \tag{104}$$

we have

$$\|P_n Y_t^{(k+1)} \psi_f\|^2 \leq \frac{c^k}{(k-1)!} \sum_{h=0}^{k} \binom{k}{h} k^h B_2(k-h) = (kc^k) \sum_{h=0}^{k} \frac{k^h}{h!} \frac{B_2(k-h)}{(k-h)!} \tag{105}$$

Now, since all the terms involved are positive, clearly

$$\sum_{h=0}^{k} \frac{k^h}{h!} \frac{B_2(k-h)}{(k-h)!} \leq \left(\sum_{h=0}^{k} \frac{k^h}{h!} \right) \left(\sum_{h'=0}^{k} \frac{B_2(k-h)}{(k-h)!} \right)$$

and, from [CKS 98] we know that this is

$$\leq e^k G_2(1)/2$$

where G_2 is an analytic function. Therefore

$$\|Y_t^{(k+1)} \psi_f\|^2 \leq G_2(1) k (ce)^k / 2 \tag{106}$$

But if $ce < 1$ or, equivalently, if

$$L < \frac{1}{16e}$$

the series on the right hand side of (106) is convergent.

12 An example

In this section we produce an example of coefficients which satisfy condition (**??**). Let, for $s, t \in \mathbb{R}^d$

$$L_{\varepsilon,\varepsilon'}(s,t) = L_{\varepsilon,\varepsilon'}\psi(|s|)\chi_{[0,|t|)}(|s|)\varphi(\hat{s},\hat{t}) \tag{107}$$

where $L_{\varepsilon,\varepsilon'} \in \mathcal{B}(\mathcal{H}_S)$ $(\varepsilon, \varepsilon' = 0, 1)$,

$$\chi_I(x) = \begin{cases} 0 & \text{if} \quad x \notin I \subseteq \mathbb{R} \\ 1 & \text{if} \quad x \in I \end{cases} \tag{108}$$

$\psi : \mathbb{R}_+ \to \mathbb{C}$ and $\varphi : S^{(d)} \times S^{(d)} \to \mathbb{C}$ are continuous functions ($S^{(d)}$ is the unit sphere in $\mathbb{R}^d$) and

$$t = |t|\hat{t} \in \mathbb{R}^d \ ; \quad |t| \in \mathbb{R}_+; \ \hat{t} \in S^{(d)} \tag{109}$$

is the polar decomposition of $t \in \mathbb{R}^d$. Then, denoting by σ_d the volume of the d–dimensional unit ball:

$$\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} ds_1 \dots ds_k l(s_1,t) l(s_1,s_2) \dots l(s_k,s_{k-1}) =$$

$$\int \dots \int \rho_1^{d-1} d\rho_1 d\hat{s}_1 \dots \rho_k^{d-1} d\rho_k d\hat{s}_k \chi_{[0,t]}(\rho_1)\chi_{[0,\rho_1)}(\rho_2) \dots \chi_{[0,\rho_{n-1})}(\rho_n) \cdot$$

$$\cdot \varphi(\hat{s}_1,\hat{t})\varphi(\hat{s}_2,\hat{s}_1) \dots \varphi(s_k,s_{k-1})\psi(\rho_1)\dots\psi(\rho_k) \leq$$

$$\leq (|t|^{d-1})^k \|\varphi\|_\infty^k \sigma_d^k \cdot \cdot \int_0^{|t|} d\rho_1 \int_0^{\rho_1} d\rho_2 \dots \int_0^{\rho_{k-1}} d\rho_k \psi(\rho_1)\dots\psi(\rho_k) =$$

$$= (|t|^{d-1})^k \|\varphi\|_\infty^k \sigma_d^k \frac{\left(\int_0^{|t|} \psi(s) d\rho\right)^k}{k!}$$

Therefore, if $B \subseteq \mathbb{R}^d$ is a bounded set and $t \in B$, condition iii) is satisfied.

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