FOR HAUSDORFF SPACES, H-CLOSED = D-PSEUDOCOMPACT FOR ALL ULTRAFILTERS D

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ABSTRACT. We prove that, for an arbitrary topological space X, the following two conditions are equivalent: (a) Every open cover of X has a finite subset with dense union (b) X is D-pseudocompact, for every ultrafilter D.

Locally, our result asserts that if X is weakly initially λ-compact, and $2^\mu \leq \lambda$, then X is D-pseudocompact, for every ultrafilter D over any set of cardinality $\leq \mu$. As a consequence, if $2^\mu \leq \lambda$, then the product of any family of weakly initially λ-compact spaces is weakly initially μ-compact.

Throughout this note λ and μ are infinite cardinals. No separation axiom is assumed, if not otherwise specified. By a product of topological spaces we shall always mean the Tychonoff product.

The notion of weak initial λ-compactness has been introduced by Z. Frolík [F] under a different name and subsequently studied by various authors. See, e. g., Stephenson and Vaughan [SV]. See [L, Remark 3] for further references about this and related notions.

For Tychonoff spaces, and for D an ultrafilter over ω, the notion of D-pseudocompactness has been introduced by Ginsburg and Saks [GS]. Their paper contains also significant applications. The notion has been extensively studied by many authors in the setting of Tychonoff spaces, especially in connection with various orders on ω^*. See, e. g., [GF1, HST, ST] and further references there for results and related notions. In the case of an ultrafilter over an arbitrary cardinal, the notion of D-pseudocompactness has been introduced and studied in García-Ferreira [GF2].

In this note we show that weak initial λ-compactness and D-pseudocompactness are tightly connected. In fact, D-pseudocompactness for

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every ultrafilter D is equivalent to weak initial λ-compactness for every infinite cardinal λ. No separation axiom is needed to prove the equivalence. As mentioned in the abstract, our result has a local version (Theorem 1 below).

The situation described in this note has some resemblance with the connections between initial λ-compactness and D-compactness. See, e. g., the survey by R. Stephenson [S] for definitions and results, in particular, Section 3 therein. However, Remark 7 here points out a significant difference.

We now recall the relevant definitions. A topological space is said to be *weakly initially λ-compact* if and only if every open cover of cardinality at most λ has a finite subset with dense union. Notice that, for Tychonoff spaces, weak initial ω-compactness is well known to be equivalent to pseudocompactness.

If D is an ultrafilter over some set I, a topological space X is said to be *D-pseudocompact* if and only if every I-indexed sequence of nonempty open sets of X has some D-limit point, where x is called a *D-limit point* of the sequence $(O_i)_{i\in I}$ if and only if, for every neighborhood U of x in X, $\{i \in I \mid U \cap O_i \neq \emptyset\} \in D$.

Theorem 1. *If X is a weakly initially λ-compact topological space, and $2^\mu \leq \lambda$, then X is D-pseudocompact, for every ultrafilter D over any set of cardinality $\leq \mu$.*

Proof. Suppose by contradiction that X is weakly initially λ-compact, D is an ultrafilter over I, $2^{|I|} \leq \lambda$, and X is not D-pseudocompact. Thus, there is a sequence $(O_i)_{i\in I}$ of nonempty open sets of X which has no D-limit point in X. This means that, for every $x \in X$, there is an open neighborhood U_x of x such that $\{i \in I \mid U_x \cap O_i \neq \emptyset\} \notin D$, that is, $\{i \in I \mid U_x \cap O_i = \emptyset\} \in D$, since D is an ultrafilter. For each $x \in X$, choose some U_x as above, and let $Z_x = \{i \in I \mid U_x \cap O_i = \emptyset\}$. Thus, $Z_x \in D$.

For each $Z \in D$, let $V_Z = \bigcup\{U_x \mid x \text{ is such that } Z_x = Z\}$. Notice that if $i \in Z \in D$, then $V_Z \cap O_i = \emptyset$. Notice also that $(V_Z)_{Z\in D}$ is an open cover of X. Since $|D| \leq 2^{|I|} \leq \lambda$, then, by weak initial λ-compactness, there is a finite number $Z_1, \dots, Z_n$ of elements of D such that $V_{Z_1} \cup \dots \cup V_{Z_n}$ is dense in X. Since D is a filter, $Z = Z_1 \cap \dots \cap Z_n \in D$, hence $Z_1 \cap \dots \cap Z_n \neq \emptyset$. Choose $i \in Z_1 \cap \dots \cap Z_n$. Then $O_i \cap V_{Z_1} = \emptyset$, $\dots$, $O_i \cap V_{Z_n} = \emptyset$, hence $O_i \cap (V_{Z_1} \cup \dots \cup V_{Z_n}) = \emptyset$, contradicting the conclusion that $V_{Z_1} \cup \dots \cup V_{Z_n}$ is dense in X, since, by assumption, O_i is nonempty. $\square$

Theorem 1 shows that weak initial λ-compactness implies D-pseudocompactness, for ultrafilters over sets of sufficiently small cardinality. The next proposition presents an easy result in the other direction.

Recall that an ultrafilter over μ is *regular* if and only if there is a family of μ elements of D such that the intersection of any infinite subset of the family is empty. As a consequence of the Axiom of Choice (actually, the Prime Ideal Theorem suffices), for every infinite cardinal μ there is a regular ultrafilter over μ.

Proposition 2. *If the topological space X is D-pseudocompact, for some regular ultrafilter D over μ, then X is weakly initially μ-compact. Actually, every power of X is weakly initially μ-compact.*

Proof. E. g., by [L, Corollary 15]. □

Corollary 3. *If $2^\mu \leq \lambda$, then the product of any family of weakly initially λ-compact spaces is weakly initially μ-compact.*

Proof. Choose some regular ultrafilter D over μ. Given any family of weakly initially λ-compact spaces, then, by Theorem 1, each member of the family is D-pseudocompact. Since D-pseudocompactness is productive [GS], the product is D-pseudocompact, hence weakly initially μ-compact, because of the choice of D, and by Proposition 2. □

Let us say that a topological space is *weakly initially $< \nu$-compact* if and only if every open cover of cardinality $< \nu$ has a finite subset with dense union. That is, weak initial $< \nu$-compactness means weak initially λ-compactness for all $\lambda < \nu$. Recall that a topological space is said to be *initially λ-compact* if and only if every open cover of cardinality at most λ has a finite subcover.

Corollary 4. *Suppose that ν is a strong limit cardinal.*

(1) *Any product of a family of weakly initially $< \nu$-compact topological spaces is weakly initially $< \nu$-compact.*
(2) *If ν is singular, then a product of a family of topological spaces is weakly initially ν-compact, provided that each factor is both weakly initially ν-compact and initially $2^{\mathrm{cf}\,\nu}$-compact.*

Proof. (1) is immediate from Corollary 3, and the assumption that ν is a strong limit cardinal.

(2) Suppose that we have a product as in the assumption. By (1), the product is weakly initially $< \nu$-compact. By known results, or by a variation on the proof of Theorem 1 (see Remark 7 or Theorem 8), any product of initially $2^{\mathrm{cf}\,\nu}$-compact spaces is initially $\mathrm{cf}\,\nu$-compact. (2) now follows from the easy fact that a weakly initially $< \nu$-compact

and initially cf ν-compact space is weakly initially ν-compact (actually, a weakly initially $< \nu$-compact and $[\mathrm{cf}\,\nu, \mathrm{cf}\,\nu]$-compact space is weakly initially ν-compact.) □

We now give the characterization of Hausdorff-closed spaces announced in the title. Recall that a topological space X is said to be $H(i)$ if and only if every open filter base on X has nonvoid adherence. Equivalently, a topological space is $H(i)$ if and only if every open cover has a finite subset with dense union. A Hausdorff space is *H-closed* (or *Hausdorff-closed*, or *absolutely closed*) if and only if it is closed in every Hausdorff space in which it is embedded. It is well known that a Hausdorff topological space is H-closed if and only if it is $H(i)$. A regular Hausdorff space is H-closed if and only if it is compact. See, e. g., [SS] for references.

Theorem 5. *For every topological space X, the following conditions are equivalent.*

1. *X is $H(i)$.*
2. *X is weakly initially λ-compact, for every infinite cardinal λ.*
3. *X is D-pseudocompact, for every ultrafilter D.*
4. *For every infinite cardinal λ, there exists some regular ultrafilter D over λ such that X is D-pseudocompact.*

If X is Hausdorff (respectively, Hausdorff and regular) then the preceding conditions are also equivalent to, respectively:

5. *X is H-closed.*
6. *X is compact.*

Proof. (1) and (2) are equivalent, because of the above mentioned characterization of $H(i)$ spaces.

(2) $\Rightarrow$ (3) is immediate from Theorem 1.

(3) $\Rightarrow$ (4) follows from the fact that, as we mentioned right before Proposition 2, for every infinite cardinal λ, there does exist some regular ultrafilter over λ.

(4) $\Rightarrow$ (2) follows from Proposition 2.

The equivalences of (1) and (5), and of (1) and (6), under the respective assumptions, follow from the remarks before the statement of the theorem. □

As a consequence of Theorem 5, we get another proof of some classical results.

Corollary 6. *Any product of a family of $H(i)$ spaces is an $H(i)$ space.*

Any product of a family of H-closed Hausdorff spaces is H-closed.

Proof. By Theorem 5, and the mentioned result by Ginsburg and Saks [GS] that D-pseudocompactness is productive. □

Remark 7. In conclusion, a few remarks are in order. The situation described in this note is almost entirely similar to the case dealing with initial λ-compactness and D-compactness. Indeed, the proof of Theorem 1 can be easily modified in order to show directly that if $2^\mu \leq \lambda$, then every initially λ-compact topological space is D-compact, for every ultrafilter D over any cardinal $\leq \mu$ (see also Theorem 8 and the remark thereafter). This result, however, is already an immediate consequence of implications (8) and (5) in [S, Diagram 3.6]. Since D-compactness, too, is productive, we get that if $2^\mu \leq \lambda$, then any product of initially λ-compact spaces is initially μ-compact, the result analogue to Corollary 3. The above arguments furnish also a proof of the well known result that a space is compact if and only if it is D-compact, for every ultrafilter D, a theorem which, in turn, has the Tychonoff theorem that every product of compact spaces is compact as an immediate consequence. This is entirely parallel to Theorem 5 and Corollary 6.

However, a subtle difference exists between the two cases. A sufficient condition for a topological space X to be initially λ-compact is that, for every λ' with $\omega \leq \lambda' \leq \lambda$, there exists some ultrafilter D uniform over λ' such that X is D-compact (see [S, Theorem 5.13] or, again, [S, Diagram 3.6]). The parallel statement fails, in general, for weak initial λ-compactness and D-pseudocompactness. Indeed, under some set theoretical hypothesis, [GF2, Example 1.9] constructed a space X which is D-pseudocompact, for some ultrafilter uniform D over ω_1, hence necessarily D'-pseudocompact, for some ultrafilter D' uniform over ω, but X is not weakly initially ω_1-compact, actually, not even ω_1-pseudocompact. Cf. also [L, Remark 30].

The above counterexample shows that, in our arguments, and, in particular, in Proposition 2, we do need the notion of a regular ultrafilter; on the contrary, in the corresponding theory for initial compactness, (a sufficient number of) uniform ultrafilters are enough.

Theorem 1 can be generalized to the abstract framework of [L, Section 5]. We recall here only the definitions, and refer to [L] for motivations and further references.

Suppose that X is a topological space, $\mathcal{F}$ is a family of subsets of X, and λ is an infinite cardinals. We say that X is $\mathcal{F}$-$[\omega, \lambda]$-*compact* if and only if, for every open cover $(O_\alpha)_{\alpha \in \lambda}$ of X, there exists some finite $W \subseteq \lambda$ such that $F \cap \bigcup_{\alpha \in W} O_\alpha \neq \emptyset$, for every $F \in \mathcal{F}$. If D is an ultrafilter over some set I, we say that X is $\mathcal{F}$-D-*compact* if and only if every sequence $(F_i)_{i\in I}$ of members of $\mathcal{F}$ has some D-limit point in X.

Theorem 8. *If X is an $\mathcal{F}$-$[\omega, \lambda]$-compact topological space, and $2^\mu \leq \lambda$, then X is $\mathcal{F}$-D-compact, for every ultrafilter D over any set of cardinality $\leq \mu$.*

Theorem 8 is proved in a way similar to Theorem 1, by replacing everywhere the family $(O_i)_{i\in I}$ by an appropriate family $(F_i)_{i\in I}$ of members of $\mathcal{F}$.

Notice that Theorem 1 is the particular case of Theorem 8 when $\mathcal{F}$ is the family of all nonempty open sets of X. By considering the particular case of Theorem 8 in which $\mathcal{F}$ is the family of all singletons of X we obtain the parallel result mentioned in Remark 7, asserting that if $2^\mu \leq \lambda$, then initial λ-compactness implies D-compactness, for every ultrafilter over a set of cardinality $\leq \mu$.

Corollary 9. *Suppose that X is a topological space, and $\mathcal{F}$ is a family of subsets of X. Then the following conditions are equivalent.*

(1) *X is $\mathcal{F}$-$[\omega, \lambda]$-compact, for every infinite cardinal λ.*
(2) *X is $\mathcal{F}$-D-compact, for every ultrafilter D.*
(3) *For every infinite cardinal λ, there exists some regular ultrafilter D over λ such that X is $\mathcal{F}$-D-compact.*

Proof. Same as the proof of Theorem 5. The implication (3) $\Rightarrow$ (1) follows from [L, Theorem 35(2) $\Rightarrow$ (4)] with $|T| = 1$. □

As a concluding observation, we expect that Corollary 3 gives an optimal result, but we have not checked it.

Problem 10. Characterize those pairs of cardinals λ and μ such that the product of any family of (weakly) initially λ-compact spaces is (weakly) initially μ-compact.

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