

Arithmetic Hecke operators as completely positive maps

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Abstract. We prove that the arithmetic Hecke operators are completely positive maps with respect to the Berezin's quantization deformation product of functions on  $\mathbb{H}/\Gamma$ . We then show that the associated subfactor defined by the Connes' correspondence associated to the completely positive map has integer index and graph  $A_\infty$ . The same construction for  $PSL(3, \mathbb{Z})$  gives a finite index subfactor of  $\mathcal{L}(PSL(3, \mathbb{Z}))$  of infinite depth.

Les opérateurs de Hecke arithmétiques et l'algèbre de déformation de Berezin

Résumé. On montre que les opérateurs de Hecke arithmétiques sont des applications complètement positives par rapport à la structure de l'algèbre définie par la déformation quantique de Berezin, des fonctions sur le quotient  $\mathbb{H}/\Gamma$  du demi-plan supérieur par un sous-groupe discret  $\Gamma$  de  $PSL(2, \mathbb{R})$ . On montre ensuite que si  $\Gamma$  est  $PSL(2, \mathbb{Z})$ , alors les sous-facteurs associés par la méthode des correspondances de Connes ont comme indice un nombre entier et le type  $A_\infty$ . La même construction pour  $PSL(3, \mathbb{Z})$  nous permet de construire un sous-facteur de  $\mathcal{L}(PSL(3, \mathbb{Z}))$  d'indice entier et de profondeur infinie.

Version française abrégée. Dans cet article nous montrons que les opérateurs de Hecke opérant sur le sous-espace des fonctions sur le demi-plan supérieur et invariantes par rapport au groupe  $PSL(2, \mathbb{Z})$ , sont des applications complètement positives pour la structure multiplicative donnée par la déformation quantique,

$\Gamma$ -invariante, de Berezin du demi plan supérieur. La structure de l'algèbre de déformation est déterminée par les calculs, (dans [GHJ], [Ra]), de la dimension de von Neumann pour les  $\mathcal{L}(PSL(2, \mathbb{Z}))$ -modules de Hilbert à gauche, obtenus par la restriction au groupe  $PSL(2, \mathbb{Z})$  des représentations unitaires de  $PSL(2, \mathbb{R})$ .

Chaque opérateur de Hecke est une somme d'applications complètement positives par rapport à la structure de  $C^*$ -algèbre de l'algèbre de déformation.

Par la méthode des correspondances de Connes, chaque application complètement positive détermine un sous-facteur. Nous calculons les invariants provenant de ces sous-facteurs par la construction de base de Jones pour les sous-facteurs. Nous montrerons que chaque sous-facteur a comme indice le carré d'un nombre entier.

Nous montrerons ainsi que l'algèbre de fusion du sous-facteur est un quotient d'une sous-algèbre de l'algèbre de Hecke d'un ensemble de classes doubles. Le bimodule de Hilbert est simplement, dans ce cas, l'espace  $l^2$  associé à une double classe de  $\Gamma$  dans  $GL(2, \mathbb{Q})$ .

Ce genre de correspondances d'indice fini peut être construit chaque fois qu'on s'est donné un sous-groupe presque normal d'un groupe discret. Par conséquent, nous pensons qu'un autre exemple intéressant pourrait être obtenu à partir des sous-groupes presque normaux considérés dans [BC]. On a ci-dessous le résultat principal de cet article.

**Théorème.** *Soit  $\Gamma = PSL(2, \mathbb{Z})$ , alors les sous-facteurs correspondants aux opérateurs de Hecke sont irréductibles et ont comme indice un nombre entier. Les commutants relatifs supérieurs sont engendrés par les projecteurs de Jones. Par conséquent le sous-facteur est de type  $A_\infty$ . La construction analogue pour  $\Gamma = PSL(3, \mathbb{Z})$  a*

*comme graphe principal ([Oc], [Po1]) un arbre infini. Par conséquent,  $\mathcal{L}(PSL(3, \mathbb{Z}))$  est un sous-facteur d'indice fini et de profondeur infinie.*

In this paper we show that the arithmetic Hecke operators acting on  $PSL(2, \mathbb{Z})$ -invariant functions on the upper halfplane, are completely positive maps for the multiplicative structure given by the  $PSL(2, \mathbb{Z})$ -invariant Berezin deformation quantization of the upper half plane. The structure of this algebra may be determined by the computations of the von Neumann dimensions in ([GHJ], [Ra2]) for the left  $\mathcal{L}(PSL(2, \mathbb{Z}))$ -Hilbert modules defined by the restriction of unitary representations of  $PSL(2, \mathbb{R})$  to  $PSL(2, \mathbb{Z})$ .

The Hecke operators are a sum over a set of cosets and each of these summands will be a completely positive map. Each of these completely positive maps is consequently a matrix coefficient for some embedding of the above mentioned Berezin's quantization algebra into a tensor product of the algebra itself with  $M_n(\mathbb{C})$  for some  $n$ , depending on the coset.

By Connes' work on correspondences, each completely positive map determines a subfactor. We compute the invariants coming from the Jones' basic construction for the subfactors obtained by the above method and show that each subfactor has as index the square of an integer, depending on the coset. We will show that the fusion algebra is a quotient of a subalgebra of the Hecke algebra of double cosets. The Hilbert bimodule is simply in this case the  $l^2$  space  $l^2(\Gamma\alpha\Gamma)$  associated with a double coset  $\Gamma\alpha\Gamma$  of  $\Gamma$ .

This type of finite index correspondences may be in fact constructed whenever we are given an almost normal subgroup of a discrete group. Consequently, an-

other interesting example could possibly be derived from the almost normal groups considered in [BC].

For  $\Gamma = PSL(2, \mathbb{Z})$  the corresponding subfactors are irreducible and the higher relative commutants are generated by the Jones' projections. Hence the associated graph is of type  $A_\infty$ . A similar construction for  $\Gamma = PSL(3, \mathbb{Z})$  will have as principal graph ([Oc], [Po1]) an infinite tree. Consequently,  $\mathcal{L}(PSL(3, \mathbb{Z}))$  has a finite index subfactor of infinite depth (a fact that can not be derived directly from the construction of D. Bisch and U. Haagerup ([BH])).

First we describe an extension of the Hecke operators to the linear space of intertwiners between representations of  $\Gamma = PSL(2, \mathbb{Z})$  that are obtained from the restriction to  $\Gamma$  of the representations in the analytic, discrete series, of unitary representations of  $PSL(2, \mathbb{R})$ .

Let  $\pi_n, n \geq 2$  be the discrete series of unitary representation of  $PSL(2, \mathbb{R})$  on the Hilbert spaces  $H_n = H^2(\mathbb{H}, y^{n-2} dx dy)$  where  $\mathbb{H}$  is the upper part of the complex plane. By restricting this representations to  $\Gamma$  one obtains that  $H_n$  are left Hilbert modules over the von Neumann algebra  $\mathcal{L}(\Gamma)$  (see [AS], [Co3], [GHJ]). The corresponding von Neumann dimension ([GHJ]) of this left Hilbert modules over  $\mathcal{L}(PSL(2, \mathbb{Z}))$  is  $(n - 1)/12$ .

Let  $\text{Int}_\Gamma(H_n, H_m)$  be the set of all linear operators  $A : H_n \rightarrow H_m$  that are intertwiners for the two representations  $\pi_n|_\Gamma$  and  $\pi_m|_\Gamma$  (i. e.  $A\pi_n(\gamma) = \pi_m(\gamma)A$  for all  $\gamma$  in  $\Gamma$ ). Let  $f$  be an automorphic form of weight  $k$ . As it was proved in [GHJ] the linear operator of multiplication with  $f$  is a bounded operator from  $H_n$  into  $H_{n+k}$  which intertwines the representations  $\pi_n|_\Gamma$  and  $\pi_{n+k}|_\Gamma$ .

The arithmetic Hecke operators act linearly on the space of automorphic forms and thus on a subspace of the set of intertwiners. In the next definition we will

show that the action of the Hecke operators can be canonically extended to the linear space of all intertwiners in  $\text{Int}_\Gamma(H_n, H_{n+k})$ .

**Definition.** Fix  $\alpha$  in  $GL(2, \mathbb{Q})/\{\pm 1\}$  and let  $(\gamma_i)_{i=1}^k$  be a system of representatives for the cosets of  $\Gamma/(\Gamma \cap \alpha \Gamma \alpha^{-1})$ . This system is always finite as  $\Gamma$  is an almost normal subgroup of  $GL(2, \mathbb{Q})/\{\pm 1\}$ . By [Ko] we have  $\Gamma \alpha \Gamma = \cup \Gamma \alpha \gamma_i$ . The arguments in [Ko], for the case of automorphic forms, prove that the following linear map  $\Phi_\alpha$  on  $\text{Int}_\Gamma(H_n, H_m)$  is actually an endomorphism. The formula for  $\Phi_\alpha$  is

$$\Phi_\alpha(A) = \frac{1}{k} \sum_{i=1}^k \pi_m((\alpha \gamma_i)^{-1}) A \pi_n(\alpha \gamma_i).$$

In particular, if  $n = m$ , then  $\Phi_\alpha$  defines a completely positive map on  $\text{Int}_\Gamma(H_n, H_n)$ , i. e. on the commutant  $\{\pi_n(\Gamma)\}'$  (which is a type  $II_1$  factor). Note that the Hecke operators are a sum of the  $\Phi_\alpha$ 's over a set of representatives of double  $\Gamma$ -cosets.

**Remark.** For any bounded  $\Gamma$ -invariant function  $f$  on  $\mathbb{H}$  let  $T_f^n$  be the Toeplitz operator acting on the space of analytic functions  $H_n$  with symbol  $f$ . In [Ra2] it was proved that  $\{\pi_n(\Gamma)\}'$  is the weak closure of the Toeplitz operators with  $\Gamma$ -invariant symbol. With the above identification, the map  $\Phi_\alpha$  will be given by the formula:

$$\Phi_\alpha f(z) = \frac{1}{n} \sum_i f((\alpha \gamma_i)^{-1} z), z \in \mathbb{H}.$$

In particular  $\sum_{\det \alpha = k} \Phi_\alpha$  is the usual Hecke operator (commuting with the invariant Laplacian) acting on  $\Gamma$ -invariant functions (see e.g. [Za1] for this definition of the Hecke operators).

The main result of this paper is the identification of the subfactor associated with  $\Phi_\alpha$  via Connes' correspondence construction. Recall from [GHJ] that in this

case  $\dim_{\Gamma} H_{13} = 1$  so that in this case  $\mathcal{A}_{13} = \{\pi_{13}(\Gamma)\}'$  is isomorphic to the group von Neumann algebra  $\mathcal{L}(\Gamma)$ .

**Theorem.** For  $r \geq 13, r \in \mathbb{N}$ , let  $\mathcal{A}_r = \{\pi_r(\Gamma)\}'' \subseteq B(H_r)$ . Fix  $\alpha$  in  $GL(2, \mathbb{Q})/\{\pm 1\}$  and let  $(\gamma_i)_{i=1}^n$  be a system of representatives for the cosets of  $\Gamma/(\Gamma \cap \alpha\Gamma\alpha^{-1})$ . For  $A$  in  $\mathcal{A}_r$ , let  $\Phi_{\alpha}(A) = (1/n) \sum_{i=1}^n \pi_r((\alpha\gamma_i)^{-1})A\pi_r(\alpha\gamma_i)$ .

Let  $H_{\alpha}$  be the completion of the tensor product  $\mathcal{A}_r \otimes \mathcal{A}_r$  with respect to the norm given by the scalar product  $\langle x \otimes y, x \otimes y \rangle = \tau_{\mathcal{A}_r}(y^* \Phi_{\alpha}(x^*x)y)$ , for  $x, y \in \mathcal{A}_r$ . We let  $\mathcal{A}_r$  act on the left and on the right on  $H_{\alpha}$  by multiplication ([Co1]). Then  $H_{\alpha}$  is the Connes' correspondence for the completely positive map  $\Phi_{\alpha}$ .

If  $r=13$ , then by identifying  $\{\pi_r(\Gamma)\}'$  with  $\mathcal{L}(\Gamma)$  (which is possible by [GHJ]), the correspondence  $H_{\alpha}$  is isomorphic with the  $\mathcal{L}(\Gamma)$  bimodule  $l^2(\Gamma\alpha\Gamma)$ , with the canonical action on the left and on the right of  $\Gamma$  by multiplication. If  $r > 13$  then  $H_{\alpha}$  is obtained from the correspondence  $l^2(\Gamma\alpha\Gamma)$  by reduction by a projection.

Proof. Since  $\dim_{\Gamma}(H_r) \geq 1$ , (by [GHJ]), there exists a vector  $\zeta$  in  $H_r$  which is a separating trace vector in  $H_r$ . Hence, for any  $x, y \in \mathcal{A}_r$ , we have

$$\tau_{\mathcal{A}_r}(y^* \Phi_{\alpha}(x^*x)y) = \langle y^* \Phi_{\alpha}(x^*x)y\zeta, \zeta \rangle = \sum_i \langle U_i^* x^* x U_i y \zeta, y \zeta \rangle = \sum_i \|x U_i y \zeta\|_{\tau}^2.$$

Hence, as a  $\mathcal{A}_r$ -bimodule,  $H_{\alpha}$  is identified with the weak closure

$$(1) \quad K = \overline{\text{Span}}^w \{ \oplus_i x U_i y \zeta \mid x, y \in \mathcal{A}_r \} \subseteq \oplus_i H_r.$$

The left action of  $\mathcal{A}_r$  is simply left diagonal multiplication by the elements in  $\mathcal{A}_r$ .

The right action of  $z$  in  $\mathcal{A}_r$  is described by

$$\oplus_i x U_i y \zeta \rightarrow \oplus_i x U_i y z \zeta.$$

We will show later that in fact the subset relation in (1) is an equality relation.

The algebra of right multiplication operators with elements in  $\mathcal{A}_r$  is equal to the weak closure of the linear span  $\{R_\gamma | \gamma \in \Gamma\} \subseteq B(H_r^n)$ , where  $R_\gamma$  for  $\gamma \in \Gamma$  is defined by

$$\oplus_i x U_i y \zeta \rightarrow \oplus_i x U_i \pi_r(\gamma) y \zeta.$$

The commutant of the left multiplication operators is clearly identified with  $M_n(\mathbb{C}) \otimes \{\pi_r(\Gamma)\}''$ . Here, the Hilbert space on which  $M_n(\mathbb{C})$  acts is identified with  $l^2(\Gamma/(\Gamma \cap \alpha\Gamma\alpha^{-1}))$ .

The operators  $R_\gamma$  are identified with elements in  $M_n(\mathbb{C}) \otimes \{\pi_r(\Gamma)\}''$  in the following way. Let  $\phi(\gamma)$  be the operator of right multiplication by  $\gamma$  on  $l^2(\Gamma/\Gamma \cap \alpha\Gamma\alpha^{-1})$ . Then  $R_\gamma$  is identified with the restriction to  $K$  of  $(\oplus U_i)(\pi_r(\gamma) \otimes \phi(\gamma))(\oplus U_i)^*$ .

If  $n=13$  and  $K = H_{13}^n$ , then it is clear that the subfactor realized by considering the left multiplication operators as a subset of the commutant of the right multiplication operators is, because  $U_i = \pi_{13}(\alpha\gamma_i)$ , the same as the subfactor coming by embedding the left multiplication operators into the commutant of the right multiplication operators in the  $\mathcal{L}(\Gamma)$ -bimodule  $l^2(\Gamma\alpha\Gamma)$ . If we show that the subfactor corresponding to the  $\mathcal{L}(\Gamma)$ -bimodule  $l^2(\Gamma\alpha\Gamma)$  is irreducible, then it will follow automatically that  $K = H_{13}^n$ . This will be proved below. Otherwise, if  $r > 13$ , then our argument shows that the subfactor associated with the correspondence is the weak closure

$$\overline{\text{Span}}^w \{(\oplus U_i)(\pi_r(\gamma) \otimes \phi(\gamma))(\oplus U_i)^* | \gamma \in \Gamma\} \subseteq M_n(\mathbb{C}) \otimes \{\pi_r(\Gamma)\}''.$$

This last subfactor is obtained from the one corresponding to  $l^2(\Gamma\alpha\Gamma)$ , after suitable identifications, by applying  $\pi_r$ .

**Corollary.** *Let  $\alpha$  be an element in  $GL(2, \mathbb{Q})/\{\pm 1\}$ . Let  $\Gamma_0 = \Gamma \cap \alpha\Gamma\alpha^{-1}$ . Let  $(\Gamma_0\gamma_i)_{i=1, \dots, n}$  be a system of representatives for  $\Gamma/\Gamma_0$ . Let  $\Phi_\alpha$  be the (arithmetic) linear operator defined on  $\Gamma$ -invariant functions by*

$$\Phi_\alpha(f)(z) = \sum_i f((\alpha\gamma_i)^{-1}z), \text{ for } z \in \mathbb{H}.$$

*Let  $*_r$  be the associative (Berezin) product on the  $\Gamma$ -invariant functions on  $\mathbb{H}$  that is obtained by using the composition rule for Toeplitz operators on  $H_r = H^2(\mathbb{H}, y^{r-2} dx dy)$  having symbols  $\Gamma$ -invariant functions. Let  $\mathcal{A}_r$  be the von Neumann algebra which is obtained via the G. N. S. construction by using the above product  $*_r$ . The trace on this algebra of functions is the integral over a fundamental domain for  $\Gamma$  in  $\mathbb{H}$ .*

*Assume  $r \geq 13$ . Then there exist an embedding  $\Psi_\alpha = (\Psi_a)_{i,j=1, \dots, n} : \mathcal{A}_r \rightarrow \mathcal{A}_r \otimes M_n(\mathbb{C})$  whose 1-1 entry  $(\Psi_a)_{1,1}$  is equal to  $\Phi_\alpha$ . Moreover the subfactor corresponding to the embedding  $\Psi_a$  has index  $n^2$ , is irreducible and of infinite depth.*

Note that the usual arithmetic Hecke operators are a sum of  $\Phi_\alpha$  when  $\alpha$  runs in a system of representatives of 2 by 2 matrices of determinant  $n$ . The properties of the associated subfactor will be proved below.

**Proposition.** *Let  $\alpha$  be an element in  $GL(2, \mathbb{Q})/\{\pm 1\}$  and let  $l^2(\Gamma\alpha\Gamma)$  be the  $\mathcal{L}(\Gamma)$ -bimodule associated with  $\alpha$ . Let  $n$  be the index of the subgroup  $\Gamma \cap \alpha\Gamma\alpha^{-1}$  in  $\Gamma$ . Then the subfactor realized as the inclusion of the left multiplication operators into the commutant of the right action of  $\Gamma$  on  $l^2(\Gamma\alpha\Gamma)$  is irreducible, of index  $n^2$ . The  $n$ -th step in the iterated basic construction of the subfactor corresponds to the bimodule  $l^2(\Gamma\alpha\Gamma\alpha^{-1}\Gamma \dots \alpha^{\pm 1}\Gamma)$  with the left and right multiplication action of  $\Gamma$ .*

The product involves  $n$  successive occurrences of either  $\alpha$  or  $\alpha^{-1}$ . In particular the fusion algebra of this subfactor is the subalgebra of the Hecke algebra of double cosets which is generated by  $\Gamma\alpha\Gamma$  and  $\Gamma\alpha^{-1}\Gamma$ , modulo scalar multiples of the identity.

Proof. Let  $\Gamma_0 = \Gamma \cap \alpha\Gamma\alpha^{-1}$ . To compute the index of the associated subfactor we let  $(\gamma_i\Gamma_0)_{i=1,\dots,n}$  be a system of coset representatives for  $\Gamma/\Gamma_0$ . Then (see e.g. [Ko],[Se]) we have  $\Gamma\alpha\Gamma = \cup_i \Gamma\alpha\gamma_i$ . Hence, as left  $\mathcal{L}(\Gamma)$ -modules, we may identify  $l^2(\Gamma\alpha\Gamma) = \oplus_i l^2(\Gamma\alpha\gamma_i)$  with the direct sum of  $n$  copies of  $l^2(\Gamma)$ .

Let  $\Gamma_1$  be the kernel of the left multiplication map on  $l^2(\Gamma/\Gamma_0)$ . Since the right action of  $\Gamma_1$  is unitary equivalent to a diagonal action on the direct sum of the  $n$ -th copies of  $l^2(\Gamma)$ , it follows that the dimension of  $l^2(\Gamma\alpha\Gamma)$  as a right module over  $\Gamma$  is equal to  $n$ , which is also the dimension for the left action.

The result now follows from the following lemmas

**Lemma.** *Let  $K_1, K_2$  be two  $\Gamma$  double cosets in  $GL(2, \mathbb{Q})/\{\pm 1\}$ . The Connes' tensor product of the two  $\mathcal{L}(\Gamma)$  correspondences over  $\mathcal{L}(\Gamma)$  is isomorphic to the Hilbert module  $l^2(K_1K_2)$  with the corresponding left and right action of  $\Gamma$ .*

**Lemma.** *Let  $\alpha, \beta$  be two elements in  $GL(2, \mathbb{Q})/\{\pm 1\}$  and assume that  $l^2(\Gamma\alpha\Gamma)$  and  $l^2(\Gamma\beta\Gamma)$  are isomorphic as  $\mathcal{L}(\Gamma)$ -bimodules (correspondences). Then the two cosets  $\Gamma\alpha\Gamma$  and  $\Gamma\beta\Gamma$  differ only by a rational multiple of the identity.*

Proof. Assume that the two bimodules are isomorphic. Let  $\zeta$  in  $l^2(\Gamma\alpha\Gamma)$  be the image of  $\alpha$  under this isomorphism. Then we must have  $\gamma\zeta = \zeta\alpha^{-1}\gamma\alpha$  for all  $\gamma \in \Gamma \cap \alpha\Gamma\alpha^{-1}$ , and hence  $\eta = \zeta\alpha^{-1}$  must simultaneously belong to  $l^2(\Gamma\beta\Gamma\alpha^{-1})$  and to the commutant of  $\Gamma \cap \alpha\Gamma\alpha^{-1}$ . Hence  $\eta$  is a scalar.

**Lemma.** *The conjugate correspondence of  $l^2(\Gamma\alpha\Gamma)$  is  $l^2(\Gamma\alpha^{-1}\Gamma)$ .*

The theorem now follows because the iterated term in the Jones' basic construction of a subfactor given by a correspondence  $H$  over a type  $II_1$  factor  $M$  is obtained by recursively taking the tensor product of the initial correspondence with its opposite. This also completes the proof of our first theorem, because it shows that the factor associated to  $\Phi_\alpha$  is irreducible (since we have proven that equality holds in the relation (1)).

**Corollary.** *Let  $\alpha$  be given by the matrix  $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$  in  $GL(2, \mathbb{Q})/\pm 1$ . Let  $\Gamma = PSL(2, \mathbb{Z})$ . Then the subfactor associated with the correspondence  $l^2(\Gamma\alpha\Gamma)$  is irreducible and has index  $n^2$  where  $n$  is the group index  $[\Gamma : \Gamma \cap \alpha\Gamma\alpha^{-1}]$ . Moreover the higher relative commutants are generated only by the corresponding Jones' projections. In particular the graph of the subfactor is  $A_\infty$ . The similar construction for  $PSL(3, \mathbb{Z})$  instead of  $PSL(2, \mathbb{Z})$  will yield a subfactor of integer index and whose graph is an infinite tree.*

*Proof.* We use the double coset computations for the Hecke algebra as explained in the book of Andrianov (see also [Shi]). Because our computations are modulo the scalar multiples of the identity, the bimodule for the coset  $\Gamma\alpha^{-1}\Gamma$  is the same as for the coset  $\Gamma\alpha\Gamma$ . The algebra of cosets for diagonal elements in  $GL(2, \mathbb{Q})/\{\pm 1\}$  having only powers of  $p$  on the diagonal is isomorphic with the algebra of symmetric polynomials in two variables  $x_1, x_2$  ([An]). Set  $X_k = l^2(\Gamma\alpha^k\Gamma)$  as a  $\mathcal{L}(\Gamma)$  bimodule. Then, via the above isomorphism  $X_k$  is identified with the polynomial  $x_1^k + x_2^k$ . Hence  $X_k \otimes_{\mathcal{L}(\Gamma)} X_1$  is identified with  $(x_1^{k+1} + x_2^{k+1}) + (x_1^k x_2 + x_2^k x_1)$ . The second summand corresponds to  $\Gamma \begin{pmatrix} p^k & 0 \\ 0 & p \end{pmatrix} \Gamma$ . Since we work modulo the scalars the

last bimodule will give the same correspondence as  $\Gamma\alpha^{k-1}\Gamma$ . By what we have just proved and by [An] it follows that  $X_k$  is an irreducible bimodule and that  $X_k \otimes_{\mathcal{L}(\Gamma)} X_1$  is isomorphic (as a bimodule) to  $X_{k+1} \oplus X_{k-1}$ .

The computations for  $PSL(3, \mathbb{Z})$  are similar.

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