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Benci, Vieri (I-PISA-DA); **D'Aprile, Teresa** (I-BARI-DIM)
The semiclassical limit of the nonlinear Schrödinger equation in a radial potential. (English summary)
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In this paper, the authors are concerned with the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+V(x)\psi-\gamma_\hbar|\psi|^{p-2}\psi,\quad \gamma_\hbar>0,\ x\in\mathbf{R}^2,$$

where $\hbar>0$, $2<p<6$, $\psi\colon \mathbf{R}^2\to\mathbf{C}$, and the potential V is radially symmetric. Upon denoting by (r,θ) the polar coordinates in the plane, the authors' purpose is to obtain positive solutions of the form $\psi(r,\theta,t)=e^{\hbar^{-1}(iM_\hbar\theta+iEt)}v(r)$. They assume $M_\hbar>0$, which implies that all such functions have nontrivial angular momentum. This kind of solution exhibits a "spike-layer" pattern as $\hbar\to 0+$; that is, as $\hbar\to 0+$ the solutions concentrate on a circle centered at the origin while approximating uniformly zero away from it. In order to locate the asymptotic peaks, the authors analyze the appearance of such a concentration's asymptotic behavior by means of a suitable auxiliary functional.

Reviewed by *Alberto Parmeggiani*

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