Semilinear equations with exponential nonlinearity
and measure data

Équations semi linéaires avec non linéarité exponentielle
et données mesures

Daniele Bartolucci a, Fabiana Leoni b, Luigi Orsina b, Augusto C. Ponce c,d,*

a Dipartimento di Matematica, Università di Roma “Tre”, Largo S. Leonardo Murialdo 1, 00146 Roma, Italy
b Dipartimento di Matematica, Università di Roma “La Sapienza”, Piazza A. Moro 2, 00185 Roma, Italy
c Laboratoire Jacques-Louis Lions, université Pierre et Marie Curie, boîte courrier 187, 75252 Paris cedex 05, France
d Rutgers University, Department of Mathematics, Hill Center, Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854, USA

Received 27 July 2004; accepted 15 December 2004
Available online 22 April 2005

Abstract
We study the existence and non-existence of solutions of the problem

\[
\begin{aligned}
&-\Delta u + e^u - 1 = \mu \quad \text{in } \Omega, \\
&u = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N, N \geq 3 \), and \( \mu \) is a Radon measure. We prove that if \( \mu \leq 4\pi H^{N-2} \), then (0.1) has a unique solution. We also show that the constant \( 4\pi \) in this condition cannot be improved.

Résumé
Nous étudions l’existence et la non existence des solutions de l’équation

\[
\begin{aligned}
&-\Delta u + e^u - 1 = \mu \quad \text{dans } \Omega, \\
&u = 0 \quad \text{sur } \partial \Omega,
\end{aligned}
\]
1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary. We consider the problem

$$
\begin{cases}
-\Delta u + e^u - 1 = \mu & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\mu \in \mathcal{M}(\Omega)$, the space of bounded Radon measures in $\Omega$. We say that a function $u$ is a solution of (1.1) if $u \in L^1(\Omega), e^u \in L^1(\Omega)$ and the following holds:

$$
- \int_{\Omega} u \partial u + \int_{\Omega} (e^u - 1) \zeta = \int_{\Omega} \zeta \, d\mu \quad \forall \zeta \in C^2_0(\overline{\Omega}).
$$

Here $C^2_0(\overline{\Omega})$ denotes the set of functions $\zeta \in C^2(\overline{\Omega})$ such that $\zeta = 0$ on $\partial \Omega$. A measure $\mu$ is a good measure for problem (1.1) if (1.1) has a solution. We shall denote by $G$ the set of good measures. Problem (1.1) has been recently studied by Brezis, Marcus and Ponce in [1], where the general case of a continuous nondecreasing nonlinearity $g(u)$, with $g(0) = 0$, is dealt with. Applying Theorem 1 of [1] to $g(u) = e^u - 1$, it follows that for every $\mu \in \mathcal{M}(\Omega)$ there exists a largest good measure $\leq \mu$ for (1.1), which we shall denote by $\mu^*$. In the case $N = 2$, the set of good measures for problem (1.1) has been characterized by Vázquez in [9]. More precisely, a measure $\mu$ is a good measure if and only if $\mu(\{x\}) \leq 4\pi$ for every $x$ in $\Omega$. Note that any $\mu \in \mathcal{M}(\Omega)$ can be decomposed as

$$
\mu = \mu_0 + \sum_{i=1}^{\infty} \alpha_i \delta_{x_i},
$$

with $\mu_0(\{x\}) = 0$ for every $x$ in $\Omega$, and $\delta_{x_i}$ is the Dirac mass concentrated at $x_i$. Using Vázquez’s result, it is not difficult to check that (see [1, Example 5])

$$
\mu^* = \mu_0 + \sum_{i=1}^{\infty} \min\{4\pi, \alpha_i\} \delta_{x_i}.
$$

This paper is devoted to the study of problem (1.1) in the case $N \geq 3$. First of all, let us recall that if $\mu$ is a good measure, then (1.1) has a unique solution $u$ (see [1, Corollary B.1]). This solution can be either obtained as the limit of the sequence $(u_n)$ of solutions of

$$
\begin{cases}
-\Delta u_n + \min\{e^{u_n} - 1, n\} = \mu & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}
$$

or as the limit of a sequence $(v_n)$ of solutions of

$$
\begin{cases}
-\Delta v_n + e^{v_n} - 1 = \mu_n & \text{in } \Omega, \\
v_n = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where
with $\mu_n = \rho_n \ast \mu$, where $(\rho_n)$ is a sequence of mollifiers. If $\mu$ is not a good measure, then both sequences $(u_n)$ and $(v_n)$ converge to the solution $u^*$ of problem (1.1) with datum $\mu^*$ (see [1]). It has also been proved in [1] that the set $\mathcal{G}$ of good measures is convex and closed with respect to the strong topology in $\mathcal{M}(\Omega)$. Moreover, it is easy to see that if $\nu \leq \mu$ and $\mu \in \mathcal{G}$, then $\nu \in \mathcal{G}$.

Before stating our results, let us briefly recall the definitions of Hausdorff measure and Hausdorff dimension of a set. Let $s \geq 0$, and let $A \subset \mathbb{R}^N$ be a Borel set. Given $\delta > 0$, let
\[
\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_i \omega_s r_i^s : K \subset \bigcup_i B_{r_i} \text{ with } r_i < \delta, \ \forall i \right\},
\]
where the infimum is taken over all coverings of $A$ with open balls $B_{r_i}$ of radius $r_i < \delta$, and $\omega_s = \pi^{s/2} / \Gamma(s/2 + 1)$. We define the (spherical) $s$-dimensional Hausdorff measure in $\mathbb{R}^N$ as
\[
\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A),
\]
and the Hausdorff dimension of $A$ as
\[
\dim_{\mathcal{H}}(A) = \inf \{ s \geq 0 : \mathcal{H}^s(A) = 0 \}.
\]

Given a measure $\mu$ in $\mathcal{M}(\Omega)$, we say that it is concentrated on a Borel set $E \subset \Omega$ if $\mu(A) = \mu(E \cap A)$ for every Borel set $A \subset \Omega$. Given a measure $\mu$ in $\mathcal{M}(\Omega)$, and a Borel set $E \subset \Omega$, the measure $\mu|_E$ is defined by $\mu|_E(A) = \mu(E \cap A)$ for every Borel set $A \subset \Omega$.

One of our main results is the following

**Theorem 1.** Let $\mu \in \mathcal{M}(\Omega)$. If $\mu \leq 4\pi \mathcal{H}^{N-2}$, that is, if $\mu(A) \leq 4\pi \mathcal{H}^{N-2}(A)$ for every Borel set $A \subset \Omega$ such that $\mathcal{H}^{N-2}(A) < \infty$, then there exists a unique solution $u$ of (1.1).

As a corollary of Theorem 1, we have

**Corollary 1.** Let $\mu \in \mathcal{M}(\Omega)$. If $\mu \leq 4\pi \mathcal{H}^{N-2}$, then $\mu^* = \mu$.

The proof of Theorem 1 relies on a decomposition lemma for Radon measures (see Section 3 below) and on the following sharp estimate concerning the exponential summability for solutions of the Laplace equation. We denote by $M^{N/2}(\Omega)$ the Morrey space with exponent $\frac{N}{2}$ equipped with the norm $\| \cdot \|_{N/2}$ (see Definition 1 below).

**Theorem 2.** Let $f$ be a function in $M^{N/2}(\Omega)$, and let $u$ be the solution of
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Then, for every $0 < \alpha < 2N\omega_N$, it holds
\[
\int_{\Omega} e^{((2N\omega_N - \alpha)\|f\|_{N/2})|u|} \leq \frac{(N\omega_N)^2}{\alpha} \text{diam}(\Omega)^N.
\]

This theorem is the counterpart in the case $N \geq 3$ of a result proved, for $N = 2$ and $f \in L^1(\Omega)$, by Brezis and Merle in [2]. Note that, for $N = 2$, the space $M^{N/2}(\Omega)$ coincides with $L^1(\Omega)$.

As a consequence of Theorem 1, we have that the set of good measures $\mathcal{G}$ contains all measures $\mu$ which satisfy $\mu \leq 4\pi \mathcal{H}^{N-2}$. If $N = 2$, then the result of Vázquez states that the converse is also true. In our case, that is $N \geq 3$, this is false. After this work was completed, A.C. Ponce found explicit examples of good measures which are not
The existence of such measures was conjectured by L. Véron in a personal communication.

We now present some necessary conditions a measure \( \mu \in \mathcal{G} \) has to satisfy. We start with the following

**Theorem 3.** Let \( \mu \in \mathcal{M}(\Omega) \). If \( \mu(A) > 0 \) for some Borel set \( A \subset \Omega \) such that \( \dim_H(\Omega) < N - 2 \), then (1.1) has no solution.

Observe that in the case of dimension \( N = 2 \), no measure \( \mu \) satisfies the assumptions of Theorem 3.

As a consequence of Theorem 3 we have

**Corollary 2.** Let \( \mu \in \mathcal{M}(\Omega) \). If \( \mu^+ \) is concentrated on a Borel set \( A \subset \Omega \) with \( \dim_H(A) < N - 2 \), then \( \mu^* = -\mu^- \).

The next theorem, which is one of the main results of this paper, states that there exists no solution of (1.1) if \( \mu \) is strictly larger than \( 4\pi \mathcal{H}^{N-2} \) on an \( (N - 2) \)-rectifiable set.

**Theorem 4.** Let \( \mu \in \mathcal{M}(\Omega) \). Assume there exist \( \varepsilon > 0 \) and an \( (N - 2) \)-rectifiable set \( E \subset \Omega \), with \( \mathcal{H}^{N-2}(E) > 0 \), such that \( \mu(E) \geq (4\pi + \varepsilon) \mathcal{H}^{N-2}(E) \). Then, (1.1) has no solution.

**Corollary 3.** Assume \( \mu = \alpha(x)\mathcal{H}^{N-2} \; \text{E} \), where \( E \subset \Omega \) is \( (N - 2) \)-rectifiable and \( \alpha \) is \( \mathcal{H}^{N-2} \)-integrable. Then, \( \mu^* = \min\{4\pi, \alpha(x)\}\mathcal{H}^{N-2} \; \text{E} \).

In Theorem 4 (and also in Corollary 3), the assumption that \( E \) is \( (N - 2) \)-rectifiable is important. In fact, one can find \( (N - 2) \)-unrectifiable sets \( F \subset \Omega \), with \( 0 < \mathcal{H}^{N-2}(F) \leq \infty \), such that \( v = \alpha\mathcal{H}^{N-2} \; \text{F} \) is a good measure for every \( \alpha > 0 \) (see [7]).

As a consequence of the previous results, we can derive some information on \( \mu^* \). To this extent, let \( \mu \in \mathcal{M}(\Omega) \). Since \( e^u - 1 \) is bounded for \( u < 0 \), \( \mu^- \) will play no role in the existence-nonexistence theory for (1.1). Therefore, we only have to deal with \( \mu^+ \), which we recall can be uniquely decomposed as

\[
\mu^+ = \mu_1 + \mu_2 + \mu_3,
\]

where

\[
\begin{align*}
\mu_1(A) &= 0 \quad \text{for every Borel set } A \subset \Omega \text{ such that } \mathcal{H}^{N-2}(A) < \infty, \quad (1.6) \\
\mu_2 &= \alpha(x)\mathcal{H}^{N-2} \; \text{E} \text{ for some Borel set } E \subset \Omega, \text{ and some } \mathcal{H}^{N-2} \text{-measurable } \alpha, \quad (1.7) \\
\mu_3(\Omega \setminus F) &= 0 \quad \text{for some Borel set } F \subset \Omega \text{ with } \mathcal{H}^{N-2}(F) = 0. \quad (1.8)
\end{align*}
\]

By a result of Federer (see [4] and also [6, Theorem 15.6]), the set \( E \) can be uniquely decomposed as a disjoint union \( E = E_1 \cup E_2 \), where \( E_1 \) is \( (N - 2) \)-rectifiable and \( E_2 \) is purely \( (N - 2) \)-unrectifiable. In particular,

\[
\mu_2 = \alpha(x)\mathcal{H}^{N-2} \; \text{E}_1 + \alpha(x)\mathcal{H}^{N-2} \; \text{E}_2. \quad (1.9)
\]

Combining Corollaries 1–3, we establish the following

**Theorem 5.** Given \( \mu \in \mathcal{M}(\Omega) \), decompose \( \mu^+ \) as in (1.5)–(1.9). Then,

\[
\mu^* = (\mu_1)^* + (\mu_2)^* + (\mu_3)^* - \mu^-.
\]

In addition,
\begin{align}
(\mu_1)^\ast &= \mu_1, \\
(\mu_2)^\ast &= (\alpha(x)H^{N-2} \ominus E_1)^\ast + (\alpha(x)H^{N-2} \ominus E_2)^\ast, \\
(\alpha(x)H^{N-2} \ominus E_1)^\ast &= \min\{4\pi, \alpha(x)\}H^{N-2} \ominus E_1, \\
(\alpha(x)H^{N-2} \ominus E_2)^\ast &\geq \min\{4\pi, \alpha(x)\}H^{N-2} \ominus E_2, \\
(\mu_3)^\ast (A) &= 0 \text{ for every Borel set } A \subset \Omega \text{ with } \dim_H(A) < N - 2.
\end{align}

In view of the examples presented in [7], one can find measures \( \mu \geq 0 \) for which equality in (1.14) fails and such that \((\mu_3)^\ast(F) > 0\) for some Borel set \( F \subset \Omega \), with \( H^{N-2}(F) = 0 \).

The plan of the paper is as follows. In the next section we will prove Theorem 2. In Section 3 we will present a decomposition result for Radon measures. Theorem 1 will then be proved in Section 4. Theorems 3 and 4 will be established in Section 5. The last section will be devoted to the proof of Theorem 5 and Corollaries 1–3.

2. Proof of Theorem 2

We first recall the definition of the Morrey space \( M^p(\Omega) \); see [5].

**Definition 1.** Let \( p \geq 1 \) be a real number. We say that a function \( f \in L^1(\Omega) \) belongs to the Morrey space \( M^p(\Omega) \) if
\[
\|f\|_p = \sup_{B_r} \frac{1}{r^{N(1-1/p)}} \int_{\Omega \cap B_r} |f(y)| \, dy < +\infty,
\]
where the supremum is taken over all open balls \( B_r \subset \mathbb{R}^N \).

The following theorem is well-known (for the proof, see for example [5, Section 7.9]).

**Theorem 6.** Let \( f \in M^p(\Omega) \) for some \( p \geq \frac{N}{2} \), and let \( u \) be the solution of
\[
\begin{cases}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{cases}
\]
If \( p > \frac{N}{2} \), then \( u \) belongs to \( L^\infty(\Omega) \). If \( p = \frac{N}{2} \), then \( e^{\beta|u|} \) is uniformly bounded in \( L^1(\Omega) \) norm, for every \( \beta < \beta_0 = \frac{2N\omega_N}{\|f\|_{N/2}} \).

Theorem 2 in the Introduction improves the upper bound \( \beta_0 \) given in [5]. It turns out that the constant \( \frac{2N\omega_N}{\|f\|_{N/2}} \) is sharp. Indeed we have the following

**Example 1.** Let \( E = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N: x_1 = x_2 = 0\} \), and let \( \mu = 4\pi H^{N-2} \ominus E \). Define \( \mu_n = \rho_n * \mu \), where \( (\rho_n) \) is a sequence of mollifiers, and let \( u_n \) be the solution of
\[
\begin{cases}
-\Delta u_n &= \mu_n \quad \text{in } B_2(0), \\
u_n &= 0 \quad \text{on } \partial B_2(0).
\end{cases}
\]
By standard elliptic estimates, \( u_n \to u \) in \( W^{1,q}_0(B_2(0)) \), for every \( q < \frac{N}{N-1} \) and a.e., where \( u \) is the solution of
\[
\begin{cases}
-\Delta u &= 4\pi H^{N-2} \ominus E \quad \text{in } B_2(0), \\
u &= 0 \quad \text{on } \partial B_2(0).
\end{cases}
\]
Using the Green representation formula, and setting \( \rho(x) = \text{dist}(x, E) \), one can prove that \( u(x) \) behaves as \( -2\ln \rho(x) \), for any \( x \) in a suitable neighborhood of \( E \cap B_1(0) \). Moreover, it is easy to verify that

\[
\|\mu_n\|_{N/2} \to 2N\omega_N \quad \text{as} \quad n \to \infty.
\]

Then, by Fatou’s lemma

\[
\liminf_{n \to +\infty} \int_{B_2(0)} e^{(2N\omega_N/\|\mu_n\|_{N/2})u_n} \geq \int_{B_2(0)} e^u = +\infty.
\]

We now turn to the proof of Theorem 2. We start with the following well-known

**Lemma 1.** Let \( f : [0, d] \to \mathbb{R}^+ \) be a \( C^1 \)-function, and

\[
g(r) = \sup_{t \in [0, r]} f(t).
\]

Then, \( g \) is absolutely continuous on \([0, d]\), and its derivative satisfies the following inequality:

\[
0 \leq g'(r) \leq \left[ f'(r) \right]^+ \quad \text{a.e.,} \quad (2.1)
\]

where \( s^+ = \max\{s, 0\} \) is the positive part of \( s \in \mathbb{R} \).

**Proof.** First of all, observe that since \( f \) is continuous, then so is \( g \). We now prove that, for every \( x < y \) in \([0, d]\), there exist \( \tilde{x} \leq \tilde{y} \) in \([x, y]\) such that

\[
0 \leq g(y) - g(x) \leq \left[ f(\tilde{y}) - f(\tilde{x}) \right]^+.
\]

Indeed, if \( g(y) = g(x) \), then it is enough to choose \( \tilde{x} = x \) and \( \tilde{y} = y \). If \( g(y) > g(x) \), then let us define

\[
\tilde{x} = \max\{z \geq x : g(z) = g(x)\} \quad \text{and} \quad \tilde{y} = \min\{z \leq y : g(z) = g(y)\}.
\]

Clearly, since \( g \) is nondecreasing, we have \( \tilde{x} \leq \tilde{y} \). In order to prove (2.2), simply observe that \( f(\tilde{x}) = g(x) \) and \( f(\tilde{y}) = g(y) \). Indeed, if for example \( f(\tilde{x}) \neq g(x) \), then it must be \( f(\tilde{x}) < g(x) \), and this implies that \( g(z) = g(x) \) for some \( z > x \), thus contradicting the definition of \( \tilde{x} \).

Since \( f \) is absolutely continuous, (2.2) implies that \( g \) is absolutely continuous, as required, so that \( g'(r) \) exists for almost every \( r \). We now establish (2.1). Starting from (2.2), and applying the mean value problem to \( f \), we have that there exists \( \xi \in [\tilde{x}, \tilde{y}] \) such that

\[
0 \leq g(y) - g(x) \leq \left[ f(\tilde{y}) - f(\tilde{x}) \right]^+ = \left[ f'(\xi) \right]^+(\tilde{y} - \tilde{x}) \leq \left[ f'(\xi) \right]^+(y - x).
\]

Dividing by \( y - x \), and letting \( y \to x \), the result follows. \( \square \)

**Proof of Theorem 2.** We split the proof into two steps:

**Step 1.** Given \( f \in C_c^\infty(\Omega) \), \( f \geq 0 \), let

\[
v(x) = \frac{1}{N(N-2)\omega_N} \int_\Omega \left( \frac{1}{|x-y|^{N-2}} - \frac{1}{d^{N-2}} \right) f(y) \, dy \quad \forall x \in \Omega,
\]

where \( d \) is the diameter of \( \Omega \). Then, for every \( 0 < \alpha < 2N\omega_N \), it holds

\[
\int_\Omega e^{((2N\omega_N-\alpha)/\|f\|_{N/2})v(x)} \, dx \leq \frac{(N\omega_N)^2}{\alpha} d^N.
\]
Let us set
\[ \nu(x,r) = \int_{B_r(x)} f(y) \, dy \quad \forall x \in \Omega. \]

In particular,
\[ \nu(x,r) \leq \omega_N r^N \| f \|_{L^\infty} \quad \text{and} \quad \nu'(x,r) = \int_{\partial B_r(x)} f(y) \, d\sigma(y) \leq N\omega_N r^{N-1} \| f \|_{L^\infty}, \quad (2.5) \]

where ' denotes the derivative with respect to \( r \) and \( d\sigma \) is the \((N-1)\)-dimensional measure on \( \partial B_r(x) \). Then,
\[ \nu(x) = \frac{1}{N(N-2)\omega_N} \int_0^d \left( \frac{1}{r^{N-2}} - \frac{1}{d^{N-2}} \right) \left( \int_{\partial B_r(x)} f(y) \, d\sigma(y) \right) \, dr \]
\[ = \frac{1}{N(N-2)\omega_N} \int_0^d \left( \frac{1}{r^{N-2}} - \frac{1}{d^{N-2}} \right) \nu'(x,r) \, dr. \]

Integrating by parts, we have
\[ \nu(x) = \frac{1}{N(N-2)\omega_N} \left( \frac{1}{r^{N-2}} - \frac{1}{d^{N-2}} \right) \nu(x,r) \bigg|_0^d + \frac{1}{N\omega_N} \int_0^d \nu(x,r) \, r^{N-1} \, dr. \]

By (2.5),
\[ \lim_{r \to 0} \frac{\nu(x,r)}{r^{N-2}} = 0, \]
and so
\[ \nu(x) = \frac{1}{N\omega_N} \int_0^d \frac{\nu(x,r)}{r^{N-1}} \, dr. \]

Define now
\[ \psi(x,r) = \sup_{t \in [0,r]} \frac{\nu(x,t)}{t^{N-2}}. \]

It follows from Lemma 1 that \( \psi(x, \cdot) \) is absolutely continuous. Then, integrating by parts,
\[ \nu(x) \leq \frac{1}{N\omega_N} \int_0^d \frac{\psi(x,r)}{r} \, dr = -\frac{1}{N\omega_N} \int_0^d \left( \ln \left( \frac{d}{r} \right) \right) ' \psi(x,r) \, dr \]
\[ = -\frac{1}{N\omega_N} \psi(x,r) \ln \left( \frac{d}{r} \right) \bigg|_0^d + \frac{1}{N\omega_N} \int_0^d \ln \left( \frac{d}{r} \right) \psi'(x,r) \, dr. \]

By (2.5),
\[ \lim_{r \to 0} \psi(x,r) \ln \left( \frac{d}{r} \right) = 0, \]
and then, observing that \( \psi(x,d) \geq v(x,d)/d^{N-2} = \|f\|_{L^1}/d^{N-2} > 0 \),

\[
v(x) \leq \frac{1}{N\omega_N} \int_0^d \ln \left( \frac{d}{r} \right) \psi'(x,r) \, dr = \int_0^d \frac{\psi(x,d)}{N\omega_N} \ln \left( \frac{d}{r} \right) \frac{\psi'(x,r)}{\psi(x,d)} \, dr.
\]

Therefore, for any \( 0 < \alpha < 2N\omega_N \),

\[
e^{(2N\omega_N - \alpha)/\|f\|_{N/2}}v(x) \leq \exp \left( \int_0^d \frac{2N\omega_N - \alpha}{\|f\|_{N/2}} \ln \left( \frac{d}{r} \right) \frac{\psi'(x,r)}{\psi(x,d)} \, dr \right).
\]

Since \( \frac{\psi'(x,r)}{\psi(x,d)} \, dr \) is a probability measure on \((0, d)\), Jensen’s inequality implies

\[
e^{(2N\omega_N - \alpha)/\|f\|_{N/2}}v(x) \leq \int_0^d \left( \frac{2N\omega_N - \alpha}{\|f\|_{N/2}} \right)^\psi(x,d) N\omega_N \ln \left( \frac{d}{r} \right) \psi''(x,r) \psi(x,d) \, dr.
\]

Clearly,

\[
\psi(x,d) \leq \sup_{y \in \Omega} \psi(y,d) = \|f\|_{N/2} \quad \text{and} \quad \psi(x,d) \geq \frac{\|f\|_{L^1}}{d^{N-2}}.
\]

Thus,

\[
e^{(2N\omega_N - \alpha)/\|f\|_{N/2}}v(x) \leq \left( \frac{d^{N-\alpha/\omega_N}}{\|f\|_{L^1}} \right) \int_0^d \frac{\psi'(x,r)}{r^{2-\alpha/\omega_N}} \, dr.
\] (2.6)

Now, by (2.1) we have

\[
\psi'(x,r) \leq \left[ \left( \frac{\psi(x,r)}{r^{N-2}} \right) \right]^{+} \leq \frac{\psi'(x,r)}{r^{N-2}},
\]

so that

\[
\int_\Omega \psi'(x,r) \, dx \leq \frac{1}{r^{N-2}} \int_\Omega \left( \int_{\partial B_r(x)} f(y) \, d\sigma(y) \right) \, dx = \frac{1}{r^{N-2}} \int_\Omega \left( \int_{\partial B_r(0)} f(y + x) \, d\sigma(y) \right) \, dx
\]

\[
= \frac{1}{r^{N-2}} \int_{\partial B_r(0)} \left( \int_\Omega f(y + x) \, dx \right) d\sigma(y) \leq N\omega_N \|f\|_{L^1}.
\]

Hence, from (2.6),

\[
\int_\Omega e^{(2N\omega_N - \alpha)/\|f\|_{N/2}}v(x) \, dx \leq N\omega_N d^{N-\alpha/\omega_N} \int_0^d \frac{dr}{r^{1-\alpha/\omega_N}} = \frac{(N\omega_N)^2}{\alpha} d^N
\]

which is (2.4). This concludes the proof of Step 1.

**Step 2.** Proof of Theorem 2 completed.

Let \( f \in M^{N/2}(\Omega) \). Clearly, it suffices to prove the theorem for \( f \geq 0 \). By extending \( f \) to be identically zero outside \( \Omega \), we have

\[
\int_{B_r} f(y) \, dy \leq \|f\|_{N/2} r^{N-2} \quad \text{for every ball } B_r \subset \mathbb{R}^N.
\] (2.7)
Let \((\rho_n) \subset C_\infty^c(B_1), \rho_n \geq 0\), be a sequence of mollifiers. Take \((\zeta_n) \subset C_\infty^c(\Omega)\) to be such that \(0 \leq \zeta_n \leq 1\) in \(\Omega\), and \(\zeta_n(x) = 1\) if \(d(x, \partial \Omega) \geq \frac{1}{n}\). Set \(f_n = \zeta_n(\rho_n * f)\). We claim that
\[ \|f_n\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \quad \forall n \geq 1. \] (2.8)

In fact, given any ball \(B_r(z) \subset \mathbb{R}^N\), we have
\[ \int_{B_r(z)} f_n(x) \, dx \leq \int_{B_r(z)} (\rho_n * f)(x) \, dx = \int_{\mathbb{R}^N} \rho_n(x - y) f(y) \, dy \int_{\mathbb{R}^N} f(y) \, dy \rho_n(t) \, dt. \]

Since (2.7) holds, we get
\[ \int_{B_r(z)} f_n(x) \, dx \leq \|f\|_{L^2(\Omega)} r^{N-2} \int_{\mathbb{R}^N} \rho_n(t) \, dt = \|f\|_{L^2(\Omega)} r^{N-2}, \]

which is precisely (2.8).

Let \(u_n\) be the unique solution of
\[ \begin{cases} -\Delta u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial \Omega. \end{cases} \]

We shall denote by \(v_n\) the function given by (2.3), with \(f\) replaced by \(f_n\). Note that, by the standard maximum principle, \(0 \leq u_n \leq v_n\) in \(\Omega\), \(\forall n \geq 1\). Given \(0 < \alpha < 2N\omega_N\), it follows from (2.8) and the previous step that
\[ \int_{\Omega} e^{((2N\omega_N - \alpha)\|f\|_{L^2(\Omega)/2})u_n(x)} \, dx \leq \int_{\Omega} e^{((2N\omega_N - \alpha)\|f\|_{L^2(\Omega)/2})v_n(x)} \, dx \leq \frac{(N\omega_N)^2}{\alpha} d^N \quad \forall n \geq 1. \] (2.9)

Since \(f_n \to f\) in \(L^1(\Omega)\), standard elliptic estimates imply that \(u_n \to u\) in \(L^1(\Omega)\) a.e. Thus, as \(n \to \infty\) in (2.9), it follows from Fatou’s lemma that \(\int_{\Omega} e^{((2N\omega_N - \alpha)\|f\|_{L^2(\Omega)/2})u(x)} \, dx \leq \frac{(N\omega_N)^2}{\alpha} d^N\).

This concludes the proof of the theorem.

\[ \square \]

3. A useful decomposition result

Our goal in this section is to establish the following:

**Lemma 2.** Let \(\mu \in \mathcal{M}(\mathbb{R}^N), \mu \geq 0\). Given \(\delta > 0\), there exists an open set \(A \subset \mathbb{R}^N\) such that

(a) \(\mu(B_r \setminus A) \leq 2N\omega_N r^{N-2}\) for every ball \(B_r \subset \mathbb{R}^N\) with \(0 < r < \delta\);
(b) for every compact set \(K \subset A,\)
\[ \mu(N_{2\delta}(K)) \geq 4\pi \mathcal{H}_{N-2}^\delta(K), \]

where \(N_{2\delta}(K)\) denotes the neighborhood of \(K\) of radius \(2\delta\).

**Proof.** Given a sequence of open sets \((A_k)_{k \geq 0}\), for each \(k \geq 1\) we let
\[ R_k = \sup\{r \in [0, \delta): \mu(B_r \setminus A_{k-1}) \geq 2N\omega_N r^{N-2}\} \quad \text{for some ball } B_r \subset \mathbb{R}^N \}. \] (3.1)
We now construct the sequence \((A_k)\) inductively as follows. Let \(A_0 = \phi\). We have two possibilities. If \(R_1 = 0\), then we take \(A_k = \phi\) for every \(k \geq 1\). Otherwise, \(R_1 > 0\) and there exists \(r_1 \in (\frac{R_1}{2}, R_1]\) and \(x_1 \in \mathbb{R}^N\) such that
\[
\mu(B_{r_1}(x_1)) \geq 2N\omega_N r_1^{N-2}.
\]
Let \(A_1 = B_{r_1}(x_1)\). If \(R_2 = 0\), then we let \(A_k = \phi\) for every \(k \geq 2\). Assume \(R_2 > 0\). In this case, we may find \(r_2 \in \left(\frac{R_2}{2}, R_2\right]\) and \(x_2 \in \mathbb{R}^N\) such that
\[
\mu(B_{r_2}(x_2) \setminus A_1) \geq 2N\omega_N r_2^{N-2}.
\]
Proceeding by induction, we obtain a sequence of balls \(B_{r_1}(x_1), B_{r_2}(x_2), \ldots\) and open sets
\[
A_k = B_{r_1}(x_1) \cup \cdots \cup B_{r_k}(x_k)
\]
such that
\[
\frac{R_k}{2} < r_k \leq R_k
\]
and
\[
\mu(B_{r_k}(x_k) \setminus A_{k-1}) \geq 2N\omega_N r_k^{N-2} \quad \forall k \geq 1.
\]
Note that \(R_k \to 0\) as \(k \to \infty\). In fact, by (3.3) and (3.4) we have
\[
\frac{N\omega_N}{2^{N-3}} \sum_{k=1}^{\infty} R_k^{N-2} \leq 2N\omega_N \sum_{k=1}^{\infty} r_k^{N-2} \leq \sum_{k=1}^{\infty} \mu(B_{r_k}(x_k) \setminus A_{k-1}) = \mu\left(\bigcup_{k} B_{r_k}(x_k)\right) \leq \|\mu\|_{\mathcal{M}}.
\]
In particular, \(\sum_k R_k^{N-2} < \infty\), which implies the desired result.

Let
\[
A = \bigcup_{j=1}^{\infty} A_j = \bigcup_{k=1}^{\infty} B_{r_k}(x_k).
\]
We claim that \(A\) satisfies (a) and (b).

**Proof of (a).** Given \(B_r \subset \mathbb{R}^N\) such that \(0 < r < \delta\), let \(k \geq 1\) be sufficiently large so that \(R_k < r\). By the definition of \(R_k\), we have \(\mu(B_{r}(x_k) \setminus A_k) \leq 2N\omega_N r_k^{N-2}\). Since \(A_k \subset A\), we have \(B_{r} \setminus A \subset B_r \setminus A_k\) and the result follows.

**Proof of (b).** Given a compact set \(K \subset A\), let
\[
J = \{ j \geq 1: B_{r_j}(x_j) \cap K \neq \phi \}.
\]
In particular,
\[
K \subset \bigcup_{j \in J} B_{r_j}(x_j).
\]
Moreover, since \(r_j < \delta\), we have \(B_{r_j}(x_j) \subset N_{2\delta}(K)\) for every \(j \in J\). Thus,
\[
\mu(N_{2\delta}(K)) \geq \mu\left(\bigcup_{j \in J} B_{r_j}(x_j)\right) \geq \mu\left(\bigcup_{j \in J} [B_{r_j}(x_j) \setminus A_{j-1}]\right)
\]
\[
= \sum_{j \in J} \mu(B_{r_j}(x_j) \setminus A_{j-1}) \geq 2N\omega_N \sum_{j \in J} r_j^{N-2} \geq \frac{2N\omega_N}{\omega_{N-2}} H_{2\delta}^{N-2}(K).
\]
Since \(2N\omega_N/\omega_{N-2} = 4\pi\), we get
\[
\mu(N_{2\delta}(K)) \geq 4\pi H_{2\delta}^{N-2}(K).
\]
This concludes the proof of Lemma 2. \(\square\)
4. Proof of Theorem 1

We first observe that, as a consequence of Theorem 2, we have the following

**Proposition 1.** Let \( \mu \in \mathcal{M}(\Omega) \) be such that
\[
\mu^+(\Omega \cap B_r) \leq 2N\omega_Nr^{N-2} \quad \text{for every ball } B_r \subset \mathbb{R}^N.
\]

Then, \( \mu \) is a good measure for (1.1).

**Proof.** Since \( \mu \leq \mu^+ \), it is enough to show that \( \mu^+ \) is a good measure. Thus, without loss of generality, we may assume that \( \mu \geq 0 \). Moreover, extending \( \mu \) to be identically zero outside \( \Omega \), we may also assume that \( \mu \in \mathcal{M}(\mathbb{R}^N) \) and
\[
\mu(B_r) \leq 2N\omega_Nr^{N-2} \quad \text{for every ball } B_r \subset \mathbb{R}^N.
\]

We shall split the proof of Proposition 1 into two steps:

**Step 1.** Assume there exists \( \varepsilon > 0 \) such that
\[
\mu(B_r) \leq 2N\omega_N(1-\varepsilon)r^{N-2} \quad \text{for every ball } B_r \subset \mathbb{R}^N.
\]

Then, \( \mu \) is a good measure.

Let \( (\rho_n) \subset C_c^\infty(B_1) \), \( \rho_n \geq 0 \), be a sequence of mollifiers. Set \( \mu_n = \rho_n * \mu \). Proceeding as in the proof of Theorem 2, Step 2, we have
\[
\|\mu_n\|_{N/2} \leq 2N\omega_N(1-\varepsilon) \quad \forall n \geq 1.
\]

Let \( v_n \) be the unique solution of
\[
\begin{cases}
-\Delta v = \mu_n & \text{in } \Omega, \\
v_n = 0 & \text{on } \partial\Omega.
\end{cases}
\]

Applying Theorem 2 to \( \alpha = 2N\omega_N - \|\mu_n\|_{N/2} \geq 2N\omega_N\varepsilon > 0 \), we conclude that
\[
\int_{\Omega} e^{v_n} \leq C \quad \forall n \geq 1, \quad (4.1)
\]

for some constant \( C > 0 \) independent of \( n \). By standard elliptic estimates \( v_n \to v \) a.e., where \( v \) is a solution for
\[
\begin{cases}
-\Delta v = \mu & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega.
\end{cases}
\]

Hence, by Fatou’s lemma and (4.1), it follows that \( e^v \in L^1(\Omega) \). Since
\[
-\Delta v + e^v - 1 = \mu + e^v - 1 \quad \text{in } \Omega,
\]
\( \mu + e^v - 1 \) is a good measure. In particular, \( \mu \leq \mu + e^v - 1 \) and \( v \geq 0 \), imply that \( \mu \) is a good measure as well.

**Step 2.** Proof of the proposition completed.

Let \( \alpha_n \uparrow 1 \). For every \( n \geq 1 \), the measure \( \alpha_n\mu \) satisfies the assumptions of Step 1. Thus, \( \alpha_n\mu \in \mathcal{G}, \forall n \geq 1 \). Since \( \alpha_n\mu \to \mu \) strongly in \( \mathcal{M}(\Omega) \) and \( \mathcal{G} \) is closed in \( \mathcal{M}(\Omega) \), we have \( \mu \in \mathcal{G} \). \( \square \)

We recall the following result:

**Lemma 3.** If \( \mu_1, \ldots, \mu_k \in \mathcal{M}(\Omega) \) are good measures for (1.1), then so is \( \sup_i \mu_i \).
Proof. If $k = 2$, this is precisely [1, Corollary 4]. The general case easily follows by induction on $k$. □

We then have a slightly improved version of Proposition 1:

**Proposition 2.** Let $\mu \in \mathcal{M}(\Omega)$. Assume there exists $\delta > 0$ such that

$$\mu^+(\Omega \cap B_r) \leq 2N\omega_Nr^{N-2} \text{ for every ball } B_r \subset \mathbb{R}^N \text{ with } r \in (0, \delta).$$

Then, $\mu$ is a good measure for (1.1).

**Proof.** Let $B_k(x_1), \ldots, B_k(x_k)$ be a finite covering of $\Omega$. For each $i = 1, \ldots, k$, let $\mu_i = \mu \mathcal{L} B_k(x_i) \in \mathcal{M}(\Omega)$. It is easy to see that $\mu_i$ satisfies the assumptions of Proposition 1, so that each $\mu_i$ is a good measure for (1.1). Thus, by the previous lemma, $\sup_i \mu_i \in \mathcal{G}$. Since $\mu \leq \sup_i \mu_i$, we conclude that $\mu$ is also a good measure for (1.1). □

We can now present the

**Proof of Theorem 1.** As above, since $\mu \leq \mu_\ast$, it suffices to show that $\mu_\ast$ is a good measure. In particular, we may assume that $\mu \geq 0$. Moreover, it suffices to establish the theorem for a measure $\mu$ such that $\mu \leq (4\pi - \varepsilon)\mathcal{H}^{N-2}$ for some $\varepsilon > 0$. The general case follows as in Step 2 of Proposition 1.

We first extend $\mu$ to be identically zero outside $\Omega$. By Lemma 2, there exists an open set $\hat{A}_1 \subset \mathbb{R}^N$ such that (a) and (b) hold with $\delta = 1$ and $A = \hat{A}_1$. By induction, given an open set $\hat{A}_{k-1} \subset \mathbb{R}^N$, we apply Lemma 2 to $\mu \mathcal{L} \hat{A}_{k-1}$ and $\delta_k = \frac{1}{k}$ to obtain an open set $\hat{A}_k \subset \hat{A}_{k-1}$ such that

(a) $\mu \mathcal{L} \hat{A}_{k-1}(B_r \setminus \hat{A}_k) \leq 2N\omega_Nr^{N-2}$ for every ball $B_r \subset \mathbb{R}^N$ with $0 < r < \frac{1}{k}$;

(b) for every compact set $K \subset \hat{A}_k$,

$$\mu(N_{2/k}(K)) \geq \mu \mathcal{L} \hat{A}_{k-1}(N_{2/k}(K)) \geq 4\pi\mathcal{H}^{N-2}_{1/k}(K).$$

By Proposition 2, each measure $\mu \mathcal{L} \Omega \setminus \hat{A}_1, \mu \mathcal{L} \hat{A}_1 \setminus \hat{A}_2, \ldots, \mu \mathcal{L} \hat{A}_{k-1} \setminus \hat{A}_k$ is good. We now invoke Lemma 3 to conclude that

$$\mu \mathcal{L} \Omega \setminus \hat{A}_k = \sup\{\mu \mathcal{L} \Omega \setminus \hat{A}_1, \mu \mathcal{L} \hat{A}_1 \setminus \hat{A}_2, \ldots, \mu \mathcal{L} \hat{A}_{k-1} \setminus \hat{A}_k\}$$

is a good measure for every $k \geq 1$. Let $\hat{A} = \bigcap_k \hat{A}_k$. Since $\mu \mathcal{L} \Omega \setminus \hat{A}_k \rightarrow \mu \mathcal{L} \Omega \setminus \hat{A}$ strongly in $\mathcal{M}(\Omega)$ and the set $\mathcal{G}$ of good measures is closed with respect to the strong topology, we conclude that $\mu \mathcal{L} \Omega \setminus \hat{A}$ is also a good measure for (1.1).

We now claim that $\mu(\hat{A}) = 0$. In fact, let $K \subset \hat{A}$ be a compact set. In particular, $K \subset \hat{A}_k$. By (b),$ we have

$$\mu(N_{2/k}(K)) \geq 4\pi\mathcal{H}^{N-2}_{1/k}(K) \quad \forall k \geq 1.$$  \hspace{1cm} (4.2)

As $k \rightarrow \infty$, we conclude that

$$\mu(K) \geq 4\pi\mathcal{H}^{N-2}(K).$$

(4.2)

In particular, $\mathcal{H}^{N-2}(K) < \infty$. Recall that, by assumption,

$$\mu(K) \leq 4\pi(1 - \varepsilon)\mathcal{H}^{N-2}(K).$$ \hspace{1cm} (4.3)

Combining (4.2) and (4.3), we get $\mu(K) = 0$. Since $K \subset \hat{A}$ is arbitrary, we conclude that $\mu(\hat{A}) = 0$. Therefore, $\mu = \mu \mathcal{L} \Omega \setminus \hat{A}$ and so $\mu$ is a good measure. This concludes the proof of Theorem 1. □
5. Proofs of Theorems 3 and 4

In this section we derive some necessary conditions for a measure to be good for problem (1.1). Let us start with a regularity property for solutions of elliptic equations with measure data.

Lemma 4. Let \( \nu \in \mathcal{M}(\Omega) \) and let \( u \) be the solution of the Dirichlet problem
\[
\begin{cases}
-\Delta u = \nu & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]
(5.1)

If \( e^u \in L^1(\Omega) \), then \( u^+ \) belongs to \( W^{1,p}_0(\Omega) \) for every \( p < 2 \), and
\[
\|u^+\|_{W^{1,p}_0(\Omega)} \leq C \left( p, \text{meas } \Omega, \|\nu\|_{\mathcal{M}}, \|e^u\|_{L^1} \right) \quad \forall p < 2.
\]
(5.2)

Proof. Let \( \nu_n = \rho_n * \nu \), where \( (\rho_n) \) is a sequence of mollifiers, and let \( u_n \) be the solution of
\[
\begin{cases}
-\Delta u_n = \nu_n & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial\Omega.
\end{cases}
\]
(5.3)

Then it is well-known that the sequence \( (u_n) \) converges to \( u \) in \( W^{1,q}_0(\Omega) \), for every \( q < \frac{N}{N-1} \) (see [8]).

Using \( T_k(u^+ +) = \min\{k, \max\{u^+, 0\}\} \) as a test function in (5.3), we have
\[
\int_{\Omega} |\nabla T_k(u^+ +)|^2 \, dx = \int_{\Omega} T_k(u^+ +) \nu_n \, dx \leq k \|\nu_n\|_{L^1} \leq k \|\nu\|_{\mathcal{M}}.
\]
(5.4)

Letting \( n \to \infty \), by weak lower semicontinuity we obtain
\[
\int_{\Omega} |\nabla T_k(u^+ +)|^2 \, dx \leq k \|\nu\|_{\mathcal{M}}.
\]
(5.4)

On the other hand, assumption \( e^u \in L^1(\Omega) \) implies, for every \( k > 0 \),
\[
e^k \text{meas}\{u > k\} \leq \int e^u \, dx \leq \|e^u\|_{L^1},
\]
and so
\[
\text{meas}\{u > k\} \leq e^{-k} \|e^u\|_{L^1}.
\]
(5.5)

For every \( \eta > 1 \) we have
\[
\{ |\nabla u^+| > \eta \} = \{ |\nabla u| > \eta, u > k \} \cup \{ |\nabla u| > \eta, 0 \leq u \leq k \},
\]
so that, by (5.4) and (5.5),
\[
\text{meas}\{ |\nabla u^+| > \eta \} \leq \text{meas}\{u > k\} + \text{meas}\{ |\nabla u| > \eta, 0 \leq u \leq k \}
\leq e^{-k} \|e^u\|_{L^1} + \frac{1}{\eta^2} \int_{\Omega} |\nabla T_k(u^+ +)|^2 \, dx \leq C \left( e^{-k} + \frac{k}{\eta^2} \right),
\]
where \( C = \max\{\|e^u\|_{L^1}, \|\nu\|_{\mathcal{M}}\} \). Minimizing on \( k \), we find
\[
\text{meas}\{ |\nabla u^+| > \eta \} \leq C \frac{1 + 2 \ln \eta}{\eta^2}.
\]
Therefore, $|\nabla u^+|$ belongs to the Marcinkiewicz space of exponent $p$, for every $p < 2$. Since $\Omega$ is bounded, it follows that $|\nabla u^+| \in L^p(\Omega)$, for every $p < 2$, and that (5.2) holds. \qed

Theorem 3 can now be obtained as a consequence of the above results.

**Proof of Theorem 3.** By inner regularity, it is enough to prove that if $\mu \in \mathcal{M}(\Omega)$ is a good measure for problem (1.1), then $\mu(K) \leq 0$ for every compact set $K \subset \Omega$ with $\text{dim}_H(K) < N - 2$.

By Lemma 3, if $\mu$ is a good measure, then so is $\mu^+ := \sup\{\mu, 0\}$. Let $v \geq 0$ be the solution of problem (1.1) with datum $\mu^+$. In particular, $v$ satisfies

$$\int_\Omega \nabla v \nabla \zeta + \int_\Omega (e^v - 1) \zeta = \int_\Omega \zeta \, d\mu^+ \quad \forall \zeta \in C_c^\infty(\Omega).$$

(5.6)

Take now a compact set $K \subset \Omega$ with $\text{dim}_H(K) < N - 2$, and let $q$ be such that $2 < q < N - \text{dim}_H(K)$. Then the $q$-capacity of $K$ is zero (see e.g. [3]), and there exists a sequence of smooth functions $\zeta_n \in C_c^\infty(\Omega)$ such that

$$0 \leq \zeta_n \leq 1 \quad \text{in } \Omega, \quad \zeta_n = 1 \quad \text{in } K, \quad \zeta_n \to 0 \quad \text{in } W^{1,q}_0(\Omega) \text{ and a.e.}$$

(5.7)

Using $\zeta_n$ as test function in (5.6) yields

$$0 \leq \mu^+(K) \leq \int_\Omega \zeta_n \, d\mu^+ = \int_\Omega \nabla v \nabla \zeta_n + \int_\Omega (e^v - 1) \zeta_n.$$

Since, by Lemma 4, $v \in W^{1,q}_0(\Omega)$, the right-hand side tends to 0 as $n \to \infty$. Hence, $\mu^+(K) = 0$, which implies $\mu(K) \leq 0$, as desired. \qed

Before presenting the proof of Theorem 4, we need some preliminary lemmas. The first one is well-known (see e.g. [3]).

**Lemma 5.** If $f \in L^1(\mathbb{R}^N)$, then, for every $0 \leq s < N$,

$$\lim_{r \to 0} \frac{1}{r^s} \int_{B_r(x)} |f(y)| \, dy = 0 \quad \forall \xi^s \text{-a.e. in } \mathbb{R}^N.$$

In the following, we will denote the angular mean of a function $w \in L^1(\mathbb{R}^N)$ on the sphere centered at $x \in \mathbb{R}^N$ with radius $r > 0$ by

$$\overline{w}(x, r) = \frac{1}{N \omega_N r^{N-1}} \int_{\partial B_r(x)} w \, d\sigma.$$  

(5.8)

The next result provides an estimate of the asymptotic behavior, as $r \to 0$, of the angular mean of a function in terms of its Laplacian.

**Lemma 6.** Let $w \in L^1(\mathbb{R}^N)$ be such that $\Delta w \in \mathcal{M}(\mathbb{R}^N)$. Set $\mu = -\Delta w$. Then,

$$\frac{1}{N \omega_N} \liminf_{r \to 0} \frac{\mu(B_r(x))}{r^{N-2}} \leq \liminf_{r \to 0} \frac{\overline{w}(x, r)}{\ln(1/r)} \leq \limsup_{r \to 0} \frac{\overline{w}(x, r)}{\ln(1/r)} \leq \frac{1}{N \omega_N} \limsup_{r \to 0} \frac{\mu(B_r(x))}{r^{N-2}}.$$

**Proof.** We claim that, for every $0 < r < s < 1$, we have

$$\overline{w}(x, r) - \overline{w}(x, s) = \frac{1}{N \omega_N} \int_r^s \frac{\mu(B_\rho(x))}{\rho^{N-1}} \, d\rho.$$

(5.9)
Indeed, if $\mu \in L^1(\mathbb{R}^N)$, then, integrating by parts, we have
\[
\int_{B_\rho(x)} \mu(y) \, dy = -N \omega_N \rho^{N-1} \nabla \omega(\rho),
\] (5.10)
where $'$ denotes the derivative with respect to $\rho$. Integrating (5.10) from $r$ to $s$ we have
\[
\bar{w}(x, r) - \bar{w}(x, s) = \frac{1}{N \omega_N} \int_r^s \frac{1}{\rho^{N-1}} \left( \int_{B_\rho(x)} \mu(y) \, dy \right) d\rho,
\]
which is precisely (5.9) if $\mu \in L^1(\mathbb{R}^N)$. The general case then follows by regularizing via convolution and taking the limit. Thus, from (5.9) we have
\[
\frac{1}{N \omega_N} \inf_{0 < \rho < s} \left( \frac{\mu(B_\rho(x))}{\rho^{N-2}} \right) \ln \left( \frac{s}{r} \right) \leq \bar{w}(x, r) - \bar{w}(x, s) \leq \frac{1}{N \omega_N} \sup_{0 < \rho < s} \left( \frac{\mu(B_\rho(x))}{\rho^{N-2}} \right) \ln \left( \frac{s}{r} \right).
\]
Dividing by $\ln (1/r)$ and letting $r \to 0$ yields
\[
\frac{1}{N \omega_N} \inf_{0 < \rho < s} \left( \frac{\mu(B_\rho(x))}{\rho^{N-2}} \right) \leq \liminf_{r \to 0} \frac{\bar{w}(x, r)}{\ln(1/r)} \leq \limsup_{r \to 0} \frac{\bar{w}(x, r)}{\ln(1/r)} \leq \frac{1}{N \omega_N} \sup_{0 < \rho < s} \left( \frac{\mu(B_\rho(x))}{\rho^{N-2}} \right),
\]
and the conclusion follows by letting $s \to 0$. □

An immediate consequence of Lemmas 5 and 6 is the following

**Corollary 4.** Let $w \in L^1(\mathbb{R}^N)$ be such that $\Delta w \in L^1(\mathbb{R}^N)$. Then,
\[
\lim_{r \to 0} \frac{\bar{w}(x, r)}{\ln(1/r)} = 0 \quad \text{for } \mathcal{H}^{N-2} \text{-a.e. } x \in \mathbb{R}^N.
\]

We can now prove Theorem 4.

**Proof of Theorem 4.** By contradiction, assume that $\mu$ is a good measure for problem (1.1), so that $(4\pi + \varepsilon)\mathcal{H}^{N-2} \subset E$ is also a good measure. Let $u$ be the solution of (1.1) with datum $(4\pi + \varepsilon)\mathcal{H}^{N-2} \subset E$ and let $v$ the solution of
\[
\begin{cases}
-\Delta v = (4\pi + \varepsilon)\mathcal{H}^{N-2} \subset E & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Since $E$ is $(N - 2)$-rectifiable, then (see [6])
\[
\lim_{r \to 0} \frac{\mathcal{H}^{N-2}(E \cap B_r(x))}{r^{N-2}} = \omega_{N-2} \quad \text{for } \mathcal{H}^{N-2} \text{-a.e. } x \in E.
\]
Thus, from Lemma 6 we obtain
\[
\lim_{r \to 0} \frac{\bar{v}(x, r)}{\ln(1/r)} = \frac{(4\pi + \varepsilon) \omega_{N-2}}{N \omega_N} = \frac{4\pi + \varepsilon}{2\pi} \quad \text{for } \mathcal{H}^{N-2} \text{-a.e. } x \in E. \tag{5.11}
\]
On the other hand, the function $w = v - u$ satisfies $-\Delta w = e^u - 1 \in L^1(\Omega)$, so that, by Corollary 4,
\[
\lim_{r \to 0} \frac{\bar{w}(x, r)}{\ln(1/r)} = \lim_{r \to 0} \frac{\bar{v}(x, r) - \bar{u}(x, r)}{\ln(1/r)} = 0 \quad \text{for } \mathcal{H}^{N-2} \text{-a.e. } x \in \Omega. \tag{5.12}
\]
Combining (5.11) and (5.12) we deduce
\[
\lim_{r \to 0} \frac{\overline{u}(x, r)}{\ln(1/r)} = \frac{4\pi + \varepsilon}{2\pi} > 2 \quad \text{for } \mathcal{H}^{N-2} \text{-a.e. } x \in E.
\]

Thus, for \( \mathcal{H}^{N-2} \text{-a.e. } x \in E \), there exists \( \delta = \delta(x) > 0 \) such that
\[
\frac{\overline{u}(x, r)}{\ln(1/r)} > 2 \quad \forall r \in (0, \delta).
\]  

(5.13)

Since
\[
\int_{B_\delta(x)} e^{\alpha(y)} \, dy = \int_0^\delta \left( \int_{\partial B_r(x)} e^{\alpha} \, d\sigma \right) \, dr = N \omega_N \int_0^\delta r^{N-1} \left( \int_{\partial B_r(x)} e^{\alpha} \, d\sigma \right) \, dr,
\]
by Jensen’s inequality and (5.13), it follows that
\[
\int_{B_\delta(x)} e^{\alpha(y)} \, dy \geq N \omega_N \int_0^\delta r^{N-1} e^{\overline{u}(x, r)} \, dr \geq N \omega_N \int_0^\delta r^{N-3} \, dr = \frac{N \omega_N}{N-2} \delta^{N-2}.
\]

Consequently, as \( \delta \to 0 \), we obtain
\[
\liminf_{\delta \to 0} \frac{1}{\delta^{N-2}} \int_{B_\delta(x)} e^{\alpha(y)} \, dy > 0 \quad \text{for } \mathcal{H}^{N-2} \text{-a.e. } x \in E,
\]
which contradicts Lemma 5 being \( \mathcal{H}^{N-2}(E) > 0 \). \( \square \)

6. Proof of Theorem 5

We first establish Corollaries 1–3.

**Proof of Corollary 1.** Let \( \mu \in \mathcal{M}(\Omega) \) be such that \( \mu \leq 4\pi \mathcal{H}^{N-2} \). It follows from Theorem 1 that \( \mu \) is a good measure. Since \( \mu^* \) is the largest good measure \( \leq \mu \), we must have \( \mu = \mu^* \). \( \square \)

**Proof of Corollary 2.** By Corollary 10 in [1], for every \( \mu \in \mathcal{M}(\Omega) \) we have
\[
\mu^* = (\mu^+)^* + (\mu^-)^* = (\mu^+)^* - \mu^-.
\]

(6.1)

Assume that there exists a Borel set \( A \subset \Omega \), with \( \dim_H(A) < N - 2 \), such that \( \mu^+ = \mu^+ \ll A \). We claim that \( (\mu^+)^* = 0 \).

By contradiction, suppose that \( (\mu^+)^* \neq 0 \). Since \( 0 \leq (\mu^+)^* \leq \mu^+ \), the measure \( (\mu^+)^* \) is also concentrated on \( A \). In addition, \( (\mu^+)^* \neq 0 \) implies \( (\mu^+)^*(A) > 0 \). Applying Theorem 3, we conclude that \( (\mu^+)^* \) is not a good measure, which is a contradiction. Thus, \( (\mu^+)^* = 0 \). It then follows from (6.1) that \( \mu^* = -\mu^- \). \( \square \)

**Proof of Corollary 3.** Without loss of generality we can assume that \( \alpha(x) \geq 0 \) for \( \mathcal{H}^{N-2} \text{-a.e. } x \in E \). Let
\[
v = \min\{4\pi, \alpha(x)\} \mathcal{H}^{N-2} \ll E.
\]
Since \( v \leq 4\pi \mathcal{H}^{N-2} \), Theorem 1 implies that \( v \) is a good measure. Clearly, \( v \leq \mu \); thus, \( v \leq \mu^* \). Since \( \mu^* \leq \mu = \alpha(x) \mathcal{H}^{N-2} \ll E \), there exists an \( \mathcal{H}^{N-2} \)-measurable function \( \beta \), such that \( \mu^* = \beta(x) \mathcal{H}^{N-2} \ll E \). Assume by contradiction that \( \beta \neq \min\{4\pi, \alpha\} \). Since
\[
\min\{4\pi, \alpha\} \leq \beta \leq \alpha,
\]
we conclude that there exists $\varepsilon > 0$ and a Borel set $F \subset E$, with $\mathcal{H}^{N-2}(F) > 0$, such that $$ (4\pi + \varepsilon) \leq \beta \mathcal{H}^{N-2} \text{-a.e. on } F. $$

Since $E$ is $(N - 2)$-rectifiable and $F \subset E$, then $F$ is also $(N - 2)$-rectifiable (see e.g. [6, Lemma 15.5]). Moreover, $$ (4\pi + \varepsilon)\mathcal{H}^{N-2}F \leq \beta\mathcal{H}^{N-2}F \leq \mu^*. $$

Thus, $(4\pi + \varepsilon)\mathcal{H}^{N-2}F$ is a good measure. But this contradicts Theorem 4. Therefore, $\beta = \min\{4\pi, \alpha\}$ and so $\mu^* = \nu$. \hfill \Box

We now present the

**Proof of Theorem 5.** Clearly, the measures $\mu_1$, $\mu_2$, $\mu_3$ and $-\mu^-$ are singular with respect to each other; (1.10) then follows from Theorem 8 in [1]. For the same reason, (1.12) holds. Next, Corollaries 1–3 imply (1.11), (1.13) and (1.15). Finally, since $\min\{4\pi, \alpha\}\mathcal{H}^{N-2}E_2$ is a good measure by Theorem 1, we have (1.14). \hfill \Box

**Acknowledgements**

The fourth author (A.C.P.) was partially supported by CAPES, Brazil, under grant no. BEX1187/99-6. A.C.P. gratefully acknowledges the invitation and the warm hospitality of the Math. Dept. at the University of Rome 1, where part of this work was carried out.

**References**