More Appendices for paper
“Robust trajectory tracking for a class of hybrid systems: an internal model principle approach” by
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Abstract
This report contains the proofs of Lemmas 3-7.
C. Proof of Lemma 3: the tube $X_{n,\varepsilon_0}$ around $\bar{x}_{ik}(\cdot)$.

By periodicity, it is enough to consider the case $h = 0$ so that $k = \kappa$. Since the only switching surface hit by the motion $\bar{x}_{ik}(\cdot)$ in the interval $[\bar{t}_k, \bar{t}_{k+1}]$ is $C_{\bar{t}_{k+1}}$, denoting by dist($\bar{x}_{ik}(t), C_{\bar{t}_{k+1}}$) the distance between $\bar{x}_{ik}(t)$ and $C_{\bar{t}_{k+1}}$, it follows that $\varepsilon_\kappa := \inf_{t \in [\bar{t}_k, \bar{t}_{k+1}]} \text{dist}(\bar{x}_{ik}(t), C_{\bar{t}_{k+1}})$ is strictly positive. Defining $\varepsilon^*_0 = \min_{\kappa \in \cal{X}} \{\varepsilon_\kappa\}$, it follows that for any $\varepsilon_0 \in (0, \varepsilon^*_0)$ all points contained in the set $X_{n,\varepsilon_0}$ in (33) can belong at most to the switching surface $C_{\bar{t}_{k+1}}$.

D. Proof of Lemma 4: ensuring that reference and actual switching times are pairwise close.

Since $\varepsilon_1 \leq \varepsilon_0$, by Lemma 3 the switching event at time $t_{k+1}$ can only happen when the switching surface $C_{\bar{t}_{k+1}}$ is hit. In the case $t_{k+1} = \bar{t}_{k+1}$, there is nothing to prove. Hence, the following computations consider the two cases $t_{k+1} > \bar{t}_{k+1}$ and $t_{k+1} < \bar{t}_{k+1}$. In both cases, the proof requires to compute the solutions between $t_{k+1}$ and $t_{k+1} \pm \omega$, and to use the constants $\omega$ and $M\omega$ computed in the procedure in Subsection VI-A to show that if a switching event does not occur in the considered interval then a contradiction arise.

Consider the case $t_{k+1} > \bar{t}_{k+1}$ first. Since $t_{k+1} \leq \bar{t}_{k+1}$, it follows that $J_{\bar{t}_{k+1}} \bar{x}_{ik}(t) - b_{\bar{t}_{k+1}} < 0$, $\forall t \in [\bar{t}_k, \bar{t}_{k+1}]$. By contradiction, assume that there is no switching time $t_{k+1} \in ([\bar{t}_{k+1}, \bar{t}_{k+1} + \omega]$; this implies that the scalar function $J_{\bar{t}_{k+1}} \bar{x}_{ik}(t) - b_{\bar{t}_{k+1}}$ remains strictly negative also for all $t \in (\bar{t}_{k+1}, \bar{t}_{k+1} + \omega$. Recalling (20) (where $u_{b,k+1} = \bar{u}(\bar{t}_{k+1})$) and (17c), one has for $t > \bar{t}_{k+1}$:

$$J_{\bar{t}_{k+1}} \bar{x}_{ik}(t) - b_{\bar{t}_{k+1}} = J_{\bar{t}_{k+1}} e^{A_{\bar{t}_{k+1}}(t-\bar{t}_{k+1})} \left(\bar{x}_{ik}(\bar{t}_{k+1}) + \bar{x}_{ik}(\bar{t}_{k+1}) + \int_{\bar{t}_{k+1}}^{t} e^{A_{\bar{t}_{k+1}}(t-t')} B_{ik} u_{b,k+1} d\tau\right) - b_{\bar{t}_{k+1}}$$

$$= (J_{\bar{t}_{k+1}} \bar{x}_{ik}(t) - b_{\bar{t}_{k+1}}) + J_{\bar{t}_{k+1}} e^{A_{\bar{t}_{k+1}}(t-\bar{t}_{k+1})} \bar{x}_{ik}(\bar{t}_{k+1})$$

(42)

The scalar, continuously differentiable function $J_{\bar{t}_{k+1}} \bar{x}_{ik}(t) - b_{\bar{t}_{k+1}}$ is positive (since $J_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1}) - b_{\bar{t}_{k+1}} = 0$ and $J_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1}) > 0$) and by (22) it is lower bounded by $\frac{1}{2} J_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1}) \omega$, in particular, $J_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1} + \omega) - b_{\bar{t}_{k+1}} > \frac{1}{2} J_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1}) \omega$. On the other hand, in order for the right hand side of (42) to be negative the term $J_{\bar{t}_{k+1}} e^{A_{\bar{t}_{k+1}}(t-\bar{t}_{k+1})} \bar{x}_{ik}(\bar{t}_{k+1})$ must be negative, and by (23) it satisfies $|J_{\bar{t}_{k+1}} e^{A_{\bar{t}_{k+1}}(t-\bar{t}_{k+1})} \bar{x}_{ik}(\bar{t}_{k+1})| < M\omega \frac{1}{2} J_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1}) \omega$, $\forall t \in ([\bar{t}_{k+1}, \bar{t}_{k+1} + \omega]$, contradicting $J_{\bar{t}_{k+1}} \bar{x}_{ik}(t) - b_{\bar{t}_{k+1}} < 0$, $\forall t \in ([\bar{t}_{k+1}, \bar{t}_{k+1} + \omega]$. Since $J_{\bar{t}_{k+1}} e^{A_{\bar{t}_{k+1}}(t_{k+1} - \bar{t}_{k+1} - \bar{t}_{k+1})} \bar{x}_{ik}(\bar{t}_{k+1}) = |J_{\bar{t}_{k+1}} \bar{x}_{ik}(t_{k+1}) - b_{\bar{t}_{k+1}}|$, at $t_{k+1}$, then $|\bar{t}_{k+1}| < M\omega \left|\bar{x}_{ik}(\bar{t}_{k+1})\right|$. Finally, consider the case $t_{k+1} < \bar{t}_{k+1}$. By hypothesis,

$$J_{\bar{t}_{k+1}} \bar{x}_{ik}(t_{k+1}) - b_{\bar{t}_{k+1}} = (J_{\bar{t}_{k+1}} \bar{x}_{ik}(t_{k+1}) - b_{\bar{t}_{k+1}}) + J_{\bar{t}_{k+1}} \bar{x}_{ik}(t_{k+1}) = 0.$$  

(43)

Since $(J_{\bar{t}_{k+1}} \bar{x}_{ik}(t_{k+1}) - b_{\bar{t}_{k+1}}) < -\frac{1}{2} J_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1}) \|\bar{x}_{ik}(t_{k+1})\| < 0$, it follows that $J_{\bar{t}_{k+1}} \bar{x}_{ik}(t_{k+1}) > 0$ and $J_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1}) < \|\bar{x}_{ik}(t_{k+1})\| < \frac{1}{2} J_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1}) \|\bar{x}_{ik}(t_{k+1})\|$, $\frac{1}{2} J_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1}) \|\bar{x}_{ik}(t_{k+1})\|$, $\|\bar{x}_{ik}(t_{k+1})\|$, $\|\bar{x}_{ik}(t_{k+1})\|$, $\|\bar{x}_{ik}(t_{k+1})\|$, $\|\bar{x}_{ik}(t_{k+1})\|$, $\|\bar{x}_{ik}(t_{k+1})\|$, $\|\bar{x}_{ik}(t_{k+1})\|$, and then $t_{k+1} \in ([\bar{t}_{k+1} - \omega, \bar{t}_{k+1})$. The bound $|\bar{t}_{k+1}| < M\omega \left|\bar{x}_{ik}(\bar{t}_{k+1})\right|$ follows as in the case $t_{k+1} > \bar{t}_{k+1}$.

E. Proof of Lemma 5

The proof requires to compute the motions between $t^{m}_{k+1}$ and $t^{M}_{k+1}$, in order to evaluate the errors at the two instants. The two cases $t_{k+1} \geq \bar{t}_{k+1}$ and $t_{k+1} \leq \bar{t}_{k+1}$ must be considered separately due to the different definition of $u_b(t)$ and $u(t)$ in the two cases according to (17b) and (17c).

Consider the case $t_{k+1} \geq \bar{t}_{k+1}$ first, so that $\bar{t}_{k+1} \in [0, \omega]$. According to (20) and (17c),

$$x_{ik}(t_{k+1}) = \Gamma_{\bar{t}_{k+1}} \left[e^{A_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1})} + \int_{0}^{\bar{t}_{k+1}} e^{A_{\bar{t}_{k+1}}(\bar{t}_{k+1} - \tau)} B_{ik} u_{b,k+1} d\tau\right] + \gamma_{\bar{t}_{k+1}},$$  

$$\bar{x}_{ik}(t_{k+1}) = e^{A_{\bar{t}_{k+1}} \bar{x}_{ik}(\bar{t}_{k+1})} + \int_{0}^{\bar{t}_{k+1}} e^{A_{\bar{t}_{k+1}}(\bar{t}_{k+1} - \tau)} B_{ij} \bar{u}(\bar{t}_{k+1} + \tau) d\tau,$$  

(43)
with \( \tilde{u}(\bar{t}_{k+1} + \tau) = \tilde{u}(\bar{t}_{k+1}) = u_{b,k+1} \) due to (20), whereas by (17a) and (17b), \( x_a(t_{k+1}) = e^{A_{tk} \bar{t}_{k+1}} x_a(\bar{t}_{k+1}) + \tilde{A}_{k+1}(h-1) \),\( x_a(t_{k+1}) = e^{A_{tk} \bar{t}_{k+1}} x_a(\bar{t}_{k+1}) + f_{1,k}(\bar{t}_{k+1}) \), \( x_a(t_{k+1}) = e^{A_{tk} \bar{t}_{k+1}} x_a(\bar{t}_{k+1}) + f_{1,k}(\bar{t}_{k+1}) \). Recalling (21), it follows that \( \tilde{x}_j(t_{k+1}) = \Gamma_{j,k} e^{A_{tk} \bar{t}_{k+1}} \tilde{x}_j(\bar{t}_{k+1}) + f_{2,k}(\bar{t}_{k+1}) \), from Lemma 1, the choice of the gains \( K \) guarantees that the solution is inside \( X_{\kappa-\varepsilon} \subset X_{\kappa,\varepsilon} \) (so that Lemma 3 and Lemma 4 can be applied and that \( \| \tilde{x}_j^e(t_{k+1}) \| < \gamma \| \tilde{x}_j^e(t_{k+1}) \| \)). Applying Lemma 5 and Lemma 4 yields (34a) and (34b). Finally, (34c) can be obtained noting that \( \gamma(t) = \gamma(t), \gamma(t) = \gamma(t) \subset X_{\kappa,\varepsilon} \), \( \forall t \in [t_{k+1}, t_{k+1}] \). Using (32c), the bound (34c) follows by choosing \( M_1 = \delta M_0 = \gamma e^{\gamma t_{k+1}} M_0 \) where \( b = \gamma e^{\gamma t_{k+1}} M_0 = \max_{k \in N}\{\| \rho \| \} \).

**G. Proof of Lemma 7.**

Proceeding by induction on \( \kappa \), it will now be shown that \( \| \tilde{x}_h \| < \delta_0, \| t_{1+h,N} \| < \delta_0 \) imply

\[
\| \chi_{\kappa+h,N} \| < \| \tilde{t}_{1+h,N} \|, \quad \kappa \in N, \tag{44a}
\]

\[
\| \tilde{t}_{1+(h+1)} \| < \| \tilde{t}_{1+h,N} \|, \quad \tilde{\xi}_{h+1} \| \| \tilde{\xi}_{h+1} \|, \quad \kappa = 2, \ldots, N. \tag{44b}
\]

In order to show (44a), it is enough to show that

\[
\| \tilde{t}_{1+h,N} \| < \| \tilde{t}_{1+h,N} \|, \quad \| \tilde{\xi}_{h+1} \| < \| \tilde{\xi}_{h+1} \|, \quad \kappa = 2, \ldots, N. \tag{44b}
\]
since \( \tilde{x}_{i_{1+hN}}(t_{1+hN}^M) \), \( \tilde{\Lambda}_\kappa(h-1) \), \( \kappa \in \mathcal{N} \), are subvectors of \( \tilde{\xi}_h \) and by hypothesis \( |\tilde{t}_{1+hN}| < \delta_0 \). Similarly, in order to show (44b), it is enough to show that

\[
\left\| \frac{\tilde{t}_{1+(h+1)N}}{\tilde{x}_{i_{1+hN}}(t_{1+(h+1)N}^M)} \right\| < \alpha \left\| \tilde{t}_{1+hN} \right\| , \quad \left\| \tilde{\Lambda}_\kappa(h) \right\| < \alpha \left\| \tilde{t}_{1+hN} \right\|, \quad \kappa \in \mathcal{N}.
\]

Case \( \kappa = 1 \). Since \( \tilde{x}_{i_{1+hN}}(t_{1+hN}^M), \tilde{\Lambda}_1(h-1), \tilde{\Lambda}_2(h-1) \), are subvectors of \( \tilde{\xi}_h \), it holds that \( \| \chi_{1+hN} \| \leq \left\| \tilde{t}_{1+hN} \right\| \); moreover \( \left\| \tilde{\Lambda}_1(h) \right\| = \left\| \tilde{\Lambda}_{N+1}(h-1) \right\| < \alpha \| \chi_{1+hN} \| < \left\| \tilde{t}_{1+hN} \right\| \) by (34b).

Case \( \kappa = 2, \ldots, N \). Assume that \( \| \chi_{m+hN} \| \leq \left\| \tilde{t}_{1+hN} \right\| / \| \tilde{\xi}_h \| \) and \( \left\| \tilde{\Lambda}_m(h) \right\| < \alpha \left\| \chi_{1+hN} \right\| \) have been proven for \( m = 1, \ldots, \kappa - 1 \), and recall that \( \left\| \tilde{\Lambda}_i(h-1) \right\| < \left\| \tilde{\xi}_h \right\|, \; i = 1, \ldots, N+1 \) (for \( i = 1, \ldots, N \) because \( \tilde{\Lambda}_i(h-1) \) is a subvector of \( \tilde{\xi}_h \), and for \( i = N+1 \) because it was proven in the case \( \kappa = 1 \)). The application of (34a) leads to

\[
\left\| \frac{\tilde{t}_{\kappa+hN}}{\tilde{x}_{i_{\kappa+hN}}(t_{\kappa+hN}^M)} \right\| < \left\| \tilde{t}_{1+hN} \right\|, \quad \kappa = 2, \ldots, N,
\]

which implies the required inequality for \( \| \chi_{\kappa+hN} \| \). Then, the application of (34b) implies that \( \left\| \tilde{\Lambda}_\kappa(h) \right\| < \alpha \| \chi_{1+hN} \| \), which yields the required inequality for \( \left\| \tilde{\Lambda}_\kappa(h) \right\| \). Finally, for \( \kappa = N \) the bound

\[
\left\| \tilde{t}_{1+(h+1)N} \right\| < \alpha \| \chi_{N+hN} \| \]

follows by (34a), thus proving (44).

Now, note that \( \left\| \tilde{\xi}_0 \right\| < \bar{\delta} < \delta_0 \) by hypothesis and \( |\tilde{t}_1| = 0 \) by the definition in Problem 1. By induction on \( h \), using (44b) it is then immediate to show that

\[
\left\| \frac{\tilde{t}_{1+hN}}{\tilde{\xi}_h} \right\| < \alpha^h \left\| \frac{\tilde{t}_1}{\tilde{\xi}_0} \right\| = \alpha^h \left\| \tilde{\xi}_0 \right\| < \alpha^h \bar{\delta}, \; \forall h \in \mathbb{Z}^+, \forall \kappa \in \mathcal{N}.
\]