COMPUTATION OF THE FORCE BETWEEN TWO PLANAR ARRAYS OF CYLINDRICAL MAGNETS

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Abstract

In this note I present an example of application of Fourier-transform techniques to the computation of the interaction force between two magnetic bodies. The first body is composed of four magnetized right cylinders of height $2d$ and circular cross section of radius R. The bases of the cylinders belong to a common plane, and their centers lie at the vertices of a square of edge $2R$. Within each cylinder $\mathcal{C}$, the magnetization is uniform, parallel to the axis, and antiparallel to the magnetizations of the two cylinders in contact with $\mathcal{C}$. The second body is obtained by placing copies of the first one at the edges of a square lattice on a plane.

Consider a magnetic body having the shape of a right cylinder with base radius R and height $2d$. The cylinder occupies the region $\mathcal{C} = \{\mathbf{r} \in \mathbb{R}^3 : r_x^2 + r_y^2 \le R, r_z^2 \le d^2\}$ where $\mathbf{r} = (r_x, r_y, r_z)$ are the coordinates in a Cartesian frame $\{O; r_x, r_y, r_z\}$. The cylinder has axis parallel to the z-axis and is uniformly magnetized. The magnetization density may be written as:

$$\mathbf{M}_1(\mathbf{r}) = M_0 D_1(\mathbf{r})\mathbf{e}_z, \qquad \mathbf{r} = (r_x, r_y, r_z),$$

where M_0 is the saturation magnetization, $\mathbf{e}_z$ is the unit vector along the z-axis, and $D_1(\cdot)$ is the characteristic function of the cylinder defined by

$$D_1(\mathbf{r}) = 1 \text{ if } \mathbf{r} \in \mathcal{C}, \qquad D_1(\mathbf{r}) = 0 \text{ otherwise}.$$

The function $\mathbf{M}_4$ defined by

$$\mathbf{M}_4(\mathbf{r}) := \sum_{\substack{\alpha\in\{-1,+1\}\\ \beta\in\{-1,+1\}}} (-1)^{\alpha+\beta}\mathbf{M}_1(\mathbf{r} - R\alpha\mathbf{e}_x - R\beta\mathbf{e}_y)$$

is the magnetization field of a group of four right cylinders whose mid sections belong to the $x-y$ plane. The center of the mid sections lie on the vertices of a square, and cylinders in mutual contact have opposite magnetization, parallel and anti-parallel to $\mathbf{e}_z$. By a periodic repetition of the function $\hat{\mathbf{M}}_4(\cdot)$ in the x and y direction, with period $4R$, we obtain the field $\hat{\mathbf{M}}_\infty(\cdot)$ defined by

$$\mathbf{M}_\infty(\mathbf{r}) := \sum_{n_x=-\infty}^{+\infty} \sum_{n_y=-\infty}^{+\infty} \mathbf{M}_4(\mathbf{r} + 4R(n_x\mathbf{e}_x + n_y\mathbf{e}_y)).$$

The field $\hat{\mathbf{M}}_\infty(\cdot)$ describes an infinite array of right cylinders with alternating magnetization.

We shall derive a simple formula for the interaction energy between this infinite array of cylinders, and a group of 4 cylinders with magnetization described by $\mathbf{M}_4$, placed above the plane at distance $z > 0$. The magnetization field of the whole system

$$\mathbf{M}_{4\infty}(\mathbf{r}) = \mathbf{M}_{\infty}(\mathbf{r}) + \mathbf{M}_4(\mathbf{r} - z\mathbf{e}_z). \tag{1}$$

The following expression for the magnetostatic energy of a magnetic system with magnetization $\mathbf{M}(\cdot)$ has been derived in [1, Equation (8)]:

$$\mathcal{E} := \frac{\mu_0}{16\pi^3} \iiint_{-\infty}^{+\infty} \frac{|\hat{\mathbf{M}}(\mathbf{k}) \cdot \mathbf{k}|^2}{|\mathbf{k}|^2} \mathrm{d}^3\mathbf{k} \tag{2}$$

where $\hat{\mathbf{M}}$ stands for the Fourier transform of $\mathbf{M}$, defined by $\hat{\mathbf{M}}(\mathbf{k}) := \iiint_{-\infty}^{\infty} \mathbf{M}(\mathbf{r}) \exp(-i\mathbf{r} \cdot \mathbf{k}) \mathrm{d}^3\mathbf{r}$ for all $\mathbf{k} \in \mathbb{R}^3$.

To perform calculations, it is convenient to introduce

$$D_4(\mathbf{r}) := \sum_{\substack{\alpha \in \{-1,+1\} \\ \beta \in \{-1,+1\}}} (-1)^{\alpha+\beta} D_1(\mathbf{r} - R\alpha\mathbf{e}_x - R\beta\mathbf{e}_y) \tag{3}$$

and

$$D_{\infty}(\mathbf{r}) := \sum_{n_x=-\infty}^{+\infty} \sum_{n_y=-\infty}^{+\infty} D_4(\mathbf{r} + 4R(n_x\mathbf{e}_x + n_y\mathbf{e}_y)). \tag{4}$$

Now, we can write

$$\mathbf{M}_{4\infty}(\mathbf{r}) = (D_{\infty}(\mathbf{r}) + D_4(\mathbf{r} - z\mathbf{e}_z))M_0\mathbf{e}_z. \tag{5}$$

As a rule, we shall henceforth denote Fourier transforms by a superimposed hat. We will also use the notation $\mathbf{k} = (k_x, k_y, k_z)$, and

$$k_{\perp} = \sqrt{k_x^2 + k_y^2}.$$

According to [3, eq. (10)], the Fourier transform of D_1, is

$$\hat{D}_1(\mathbf{k}) = 2V \frac{J_1(Rk_{\perp})}{Rk_{\perp}} \frac{\sin(dk_z)}{dk_z}, \tag{6}$$

where $J_1(\cdot)$ is the Bessel Function of the first kind of order 1 and $V = 2\pi dR^2$ is twice the volume of the cylinder. The Fourier transform of the *translated function* $\mathbf{r} \mapsto D_1(\mathbf{r} - \mathbf{r}_0)$ is given by $\exp(-i\mathbf{r}_0 \cdot \mathbf{k})\hat{D}_1(\mathbf{k})$. This can be used to compute

$$\hat{D}_4(\mathbf{k}) = -4\sin(Rk_x)\sin(Rk_y)\hat{D}_1(\mathbf{k}). \tag{7}$$

Using the results in the Appendix, we can compute the Fourier transform of D_{∞}, which turns out to be the a tempered distribution:

$$\hat{D}_{\infty}(\mathbf{k}) = \left(\frac{\pi}{2R}\right)^2 \sum_{n_x=-\infty}^{+\infty} \sum_{n_y=-\infty}^{+\infty} \hat{D}_4(\frac{\pi n_x}{2R}, \frac{\pi n_y}{2R}, k_z)\delta(k_x - \frac{\pi n_x}{2R}) \otimes \delta(k_x - \frac{\pi n_x}{2R}), \tag{8}$$

where $\otimes$ denotes the tensor product operation. We remark that a δ distribution in Fourier space has dimension of length, hence (8) is dimensionally consistent.

The Fourier transform of $\mathbf{M}_{4\infty}$ is thus

$$\hat{\mathbf{M}}_{4\infty}(\mathbf{k}) = (\hat{D}_\infty(\mathbf{k}) - \hat{D}_4(\mathbf{k})\exp(izk_z))M_0\mathbf{e}_z.$$

We cannot plug $\hat{\mathbf{M}}_{4\infty}(\cdot)$ in place of $\hat{\mathbf{M}}(\cdot)$ in (2) since multiplication between distributions is not defined. However, the energy can be split, at least formally, into a self-energy and an interaction energy

$$\mathcal{E} = \mathcal{E}_{self} + \mathcal{E}_i,$$

where

$$\mathcal{E}_i = \frac{\mu_0 M_0^2}{8\pi^3}\iiint_{-\infty}^{+\infty} \frac{k_z^2}{k_z^2 + k_\perp^2}\hat{D}_\infty(\mathbf{k})\hat{D}_4(\mathbf{k})\cos(zk_z)\mathrm{d}^3\mathbf{k}, \tag{9}$$

and where $\mathcal{E}_{self}$, which is independent on z, can be ignored, since it not needed to determine the interaction force. [1]

The integral on the right–hand side of (9) is to be understood as the application of the distribution $\hat{D}_\infty$ to the function

$$\mathbf{k} \mapsto \frac{k_z^2}{k_z^2 + k_\perp^2}\hat{D}_4(\mathbf{k})\cos(zk_z). \tag{10}$$

A combination of (9) with (7) and (8) yields

$$\mathcal{E}_i = \mu_0 M_0^2 \frac{2}{\pi}\left(\frac{V}{R}\right)^2 \sum_{n_x=-\infty}^{+\infty}\sum_{n_y=-\infty}^{+\infty} \frac{J_1^2(\frac{\pi}{2}\sqrt{n_x^2+n_y^2})}{(\frac{\pi}{2})^2(n_x^2+n_y^2)}\sin^2(\frac{\pi n_x}{2})\sin^2(\frac{\pi n_y}{2})\int_{-\infty}^{+\infty}\frac{\sin^2(k_z d)\cos(k_z z)}{(k_z d)^2 + (\frac{\pi}{2})^2(n_x^2+n_y^2)(\frac{d}{R})^2}\mathrm{d}k_z.$$

It can be shown that

$$\int_{-\infty}^{+\infty}\frac{\sin^2(k_z d)\cos(k_z z)}{(k_z d)^2 + (\frac{\pi}{2})^2(n_x^2+n_y^2)(\frac{d}{R})^2}\mathrm{d}k_z = \frac{1}{d}\frac{\pi R}{4d} f\left(\frac{z}{R}, \frac{d}{R}, \frac{\pi}{2}\sqrt{n_x^2+n_y^2}\right),$$

where

$$f(\zeta,\delta,\nu) = \frac{1}{\nu}\left(2e^{-\zeta\nu} - e^{-|2\delta+\zeta|\nu} - e^{-|2\delta-\zeta|\nu}\right).$$

Thus, defining

$$F(n_x,n_y) := \frac{J_1^2(\frac{\pi}{2}\sqrt{n_x^2+n_y^2})}{(\frac{\pi}{2})^2(n_x^2+n_y^2)}\sin^2(\frac{\pi n_x}{2})\sin^2(\frac{\pi n_y}{2}),$$

the energy can be written as

$$\mathcal{E}_i = 2\pi^2\mu_0 M_0^2 R^3 \sum_{n_x=-\infty}^{+\infty}\sum_{n_y=-\infty}^{+\infty} F(n_x,n_y) f\left(\frac{z}{R}, \frac{d}{R}, \frac{\pi}{2}\sqrt{n_x^2+n_y^2}\right).$$

Now, a straightforward calculation yields

$$\frac{\partial}{\partial z} f\left(\frac{z}{R}, \frac{d}{R}, \nu\right) = -R^{-1}\nu f\left(\frac{z}{R}, \frac{d}{R}, \nu\right) = -R^{-1}\left(2e^{-\zeta\nu} - e^{-|2\delta+\zeta|\nu} - e^{-|2\delta-\zeta|\nu}\right).$$

[1] Every temperate distribution can be approximated by a weakly convergent series of smooth functions. when such an approximation process is performed, the self energy of the infinite array blows up. This may be interpreted as the self energy of the infinite array being infinite.

Hence
$$\frac{\partial}{\partial z}\mathcal{E}_i = -2\pi^2\mu_0 M_0^2 R^2 \sum_{n_x=-\infty}^{+\infty}\sum_{n_y=-\infty}^{+\infty} F(n_x,n_y)\left(2e^{-\zeta\nu} - e^{-|2\delta+\zeta|\nu} - e^{-|2\delta-\zeta|\nu}\right).$$
This can be rewritten as:
$$\frac{\partial}{\partial z}\mathcal{E}_i = -8\mu_0 M_0^2 R^2 \sum_{n_x=-\infty}^{+\infty}\sum_{n_y=-\infty}^{+\infty} \frac{J_1^2(\frac{\pi}{2}\sqrt{n_x^2+n_y^2})}{n_x^2+n_y^2}\sin^2(\frac{\pi n_x}{2})\sin^2(\frac{\pi n_y}{2})\left(2e^{-\zeta\nu} - e^{-|2\delta+\zeta|\nu} - e^{-|2\delta-\zeta|\nu}\right)$$

1 Appendix

For each $N \in \mathbb{N}$, let the function $s^{(N)}$ be defined by:
$$s^{(N)}(k) = \sum_{n=-N}^{N} \exp(ink).$$
The functions $s^{(N)}$ do not converge pointwise as $N \to \infty$. However, by a slight adaptation of the proof of [2, Theorem 9.6], it can be shown that
$$\lim_{N\to\infty}\int_{-\infty}^{\infty} s^{(N)}(k)\phi(k)\mathrm{d}\,k = 2\pi\sum_{n=-\infty}^{\infty}\phi(2\pi n) \quad \forall\phi\in\mathcal{S}(\mathbb{R}),$$
where $\mathcal{S}(\mathbb{R})$ is the Schwartz class (see [2, p. 334]). In other words, the finite sums $s^{(N)}$, when regarded as distributions, converge (in the sense of temperate convergence [2, p. 334]) to the distribution $2\pi\delta^{[2\pi n]}$, where
$$\langle\delta^{[2\pi n]},\phi\rangle = \sum_{n=-\infty}^{\infty}\phi(2\pi n).$$
This result is usually stated by writing (see [2, eqs. (9.13) and (9.14)]
$$\sum_{n=-\infty}^{\infty}\exp(ink) = 2\pi\sum_{n=-\infty}^{+\infty}\delta(k-2\pi n). \tag{11}$$
Given $A \in \mathbb{R}$, a generalization of (11) is [2]
$$\sum_{n=-\infty}^{\infty}\exp(iAnk) = \frac{2\pi}{A}\sum_{n=-\infty}^{+\infty}\delta(k-\frac{2\pi n}{A}). \tag{12}$$

[2]By performing a change of variables, we can write
$$\int_{-\infty}^{\infty}\phi(k)\sum_{n=-N}^{N}\exp(iAnk)\mathrm{d}\,k = \frac{2\pi}{A}\int_{-\infty}^{\infty}\phi(\bar{k}/A)\sum_{n=-N}^{N}\exp(in\bar{k})\mathrm{d}\,\bar{k},$$
hence letting $N \to \infty$
$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\phi(k)\sum_{n=-N}^{N}\exp(iAnk)\mathrm{d}\,k = \frac{2\pi}{A}\sum_{n=-\infty}^{+\infty}\phi(\frac{2\pi n}{A}) = \frac{2\pi}{A}\delta^{[\frac{2\pi n}{A}]}[\phi].$$

Let us consider the partial sums $D_\infty^{(N)}(\cdot)$ defined by

$$D_\infty^{(N)}(\mathbf{r}) = \sum_{n_x=-N}^{N} \sum_{n_y=-N}^{N} D_4(\mathbf{r} + 4R(n_x\mathbf{e}_x + n_y\mathbf{e}_y)), \tag{13}$$

where D_4 is given by (3). If we regard both $D_\infty(\cdot)$ and $D_\infty^{(N)}(\cdot)$ as temperate distributions, we have

$$D_\infty(\cdot) = \lim_{N\to\infty} D_\infty^{(N)}(\cdot)$$

in the sense of temperate convergence. The Fourier transform of $D_\infty^{(N)}(\cdot)$ is

$$\hat{D}_\infty^{(N)}(\mathbf{k}) = \hat{D}_4(\mathbf{k}) \sum_{n_x=-N}^{+N} \sum_{n_y=-N}^{+N} \exp(i4R(n_xk_x + n_yk_y)). \tag{14}$$

Since the Fourier transform is continuous with respect to temperate convergence [2, Theorem 9.8 p. 334], we have

$$\hat{D}_\infty(\cdot) = lim_{N\to\infty}\hat{D}_\infty^{(N)}(\cdot) \tag{15}$$

in the sense of temperate convergence. Now,

$$\sum_{n_x=-N}^{+N} \sum_{n_y=-N}^{N} \exp(i4R(n_xk_x + n_yk_y)) = \sum_{n_x=-N}^{+N} \exp(i4Rn_xk_x) \otimes \sum_{n_y=-N}^{+N} \exp(i4Rn_yk_y)$$

Hence using (12) we obtain from (15) and (14)

$$\hat{D}_\infty(\mathbf{k}) = \left(\frac{2\pi}{4R}\right)^2 \hat{D}_4(\mathbf{k}) \sum_{n_x=-\infty}^{+\infty} \delta(k - \frac{\pi n_x}{2R}) \otimes \sum_{n_y=-\infty}^{+\infty} \delta(k - \frac{\pi n_y}{2R}).$$

The multiplication on the right-hand side in the above equation is legal, because $\hat{D}_4(\cdot)$ is in the Schwartz class, which is equivalent to (8).

References

[1] M. Beleggia, S. Tandon, Y. Zhu, and M. De Graef. On the magnetostatic interactions between nanoparticles of arbitrary shape. *Journal of Magnetism and Magnetic Materials*, 278(1-2):270–284, July 2004.

[2] Gerald B. Folland. *Fourier Analysis and its applications.* Brooks/Cole Publishing Company, 1992.

[3] S. Tandon, M. Beleggia, Y. Zhu, and M. De Graef. On the computation of the demagnetization tensor for uniformly magnetized particles of arbitrary shape. part i: Analytical approach. *Journal of Magnetism and Magnetic Materials*, 271(1):9–26, April 2004.