J. Differential Equations 250 (2011) 4289–4335

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

# Integro-differential equations of hyperbolic type with positive definite kernels

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ARTICLE INFO

*Article history:*
Received 11 August 2009
Revised 13 October 2010
Available online 29 March 2011

*MSC:*
45N05
45M10
93D20
35L70

*Keywords:*
Integro-differential equations
Global existence
Stability
Exponential decay
Weakly singular kernels

ABSTRACT

The main purpose of this work is to study the damping effect of memory terms associated with singular convolution kernels on the asymptotic behavior of the solutions of second order evolution equations in Hilbert spaces. For kernels that decay exponentially at infinity and possess strongly positive definite primitives, the exponential stability of weak solutions is obtained in the energy norm. It is also shown that this theory applies to several examples of kernels with possibly variable sign, and to a problem in nonlinear viscoelasticity.



## 1. Introduction

In this work, we investigate the global existence and asymptotic behavior at $\infty$ of the solutions of the semilinear integro-differential equation

$$u''(t) + Au(t) - \int_0^t g(t-s)Au(s)\,ds = \nabla F\big(u(t)\big) + f(t), \quad t \in (0,\infty), \tag{1.1}$$

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0022-0396/$ – see front matter 
doi:10.1016/j.jde.2011.03.005

where $A$ is a positive operator on a Hilbert space $X$, with dense domain $D(A)$, and $\nabla F$ denotes the gradient of a Gâteaux differentiable functional $F : D(A^{1/2}) \to \mathbb{R}$. It is well known that the above abstract model applies to a large variety of elastic systems, where $u$ represents displacement. Indeed, experience confirms that the behavior of some viscoelastic materials (polymers, suspensions, emulsions) shows memory properties. Consequently, in the constitutive assumptions, stress depends not only on the present values of the strain and/or velocity gradient, but on the entire temporal history of the motion as well. Typically, memory fades with time: disturbances which occurred in the more recent past have more influence on the present stress than those which occurred in the distant past. For these motivations, many constitutive models for viscoelastic materials lead to equations of motion which have the form of a linear hyperbolic PDE perturbed by a dissipative integral term of Volterra type, having a nonnegative, decreasing convolution kernel, see, e.g., [13,14,23,24,33].

When the problem is linear ($F \equiv 0$), Eq. (1.1) can be rewritten as an integral equation. Thus, the theory developed by Prüss in [31] provides a general framework for the existence and uniqueness of solutions, which is the initial step of the analysis even in the nonlinear case.

Stability results for the solutions of specific PDE models related to the above abstract context, were obtained by Dafermos [13,14] for smooth convolution kernels $g$. Following this, several contributions were added to the literature providing exponential and polynomial decay estimates for the energy of the solutions of nonlinear wave equations and elasticity systems, possibly with additional frictional damping. Examples of such results can be found, for instance, in [33,10–12,26–28,5,3,36]. The abstract evolution equation (1.1) has been recently studied in [1] where $g$ is supposed to be a nonnegative absolutely continuous function satisfying $g(0) > 0$ and a suitable differential inequality which—for exponentially decaying kernels—reduces to a bound of the form $g'(t) \leqslant -kg(t)$ for some $k > 0$.

On the other hand, fewer results are available when $g$ is not assumed to be absolutely continuous but just integrable, so that the decay properties of $g$ at infinity cannot be expressed by a differential relation like the one above. Alternatively, one can assume that the primitive

$$G(t) = \int_t^\infty g(s)\,ds$$

is *positive definite*, or *strongly positive definite*. We recall that $G$ is called positive definite if the convolution operator defined by $G$ is a positive operator in the $L^2$ topology, and $G$ is strongly positive definite if $G(t) - \delta e^{-t}$ is positive definite for some $\delta > 0$. Such classes of kernels, introduced in [19] (see also [29]), have proven extremely useful in the asymptotic analysis of integro-differential equations. It is also noteworthy that the positive definiteness of $G$ implies a commonly accepted thermodynamical restriction on $g$ (see (2.7) below) for the concrete models described by (1.1) (see, e.g., [16,17]).

For a one-dimensional nonlinear wave equation, with memory damping associated to the derivative of a strongly positive kernel, Kawashima [22] obtained global existence and strong stability results for sufficiently small initial data. Related results for nonlinear problems in higher space dimension can be found in [18]. Recently, abstract integro-differential equations associated to an integrable kernel $g$ satisfying

$$\int_0^\infty \big|g(t)\big| e^{\alpha_0 t}\,dt < \infty \tag{1.2}$$

for some $\alpha_0 > 0$, were studied in [9] and [30]. In [9], the energy of solutions is shown to decay exponentially at $\infty$ assuming $g$ to be the derivative of a positive definite kernel and adding a nonlinear frictional damping term to the equation. In [30], Prüss replaced positive definiteness by the stronger requirement that $g$ be a nonnegative locally absolutely continuous in $(0,\infty)$ with negative derivative, deducing exponential and polynomial[1] stability without additional frictional damping. A typical example of kernel considered in [30] is the following

$$g(t) = k_0 \frac{t^{\beta-1}}{\Gamma(\beta)} e^{-\gamma t}, \quad t > 0, \tag{1.3}$$

where $\gamma > 0$, $\beta \in (0,1)$ and $0 < k_0 < \gamma^\beta$. Further stability results for equations with boundary damping and integrable kernels under sign conditions have been also announced in [2].

In this paper, still under condition (1.2), we will prove the global existence and exponential stability of the solution to (1.1) assuming that

$$G_{\alpha_0}(t) := \int_t^\infty g(s) e^{\alpha_0 s}\, ds$$

is strongly positive definite. Compared to [9], in this paper we make it clear that the damping phenomenon is due to the effect of memory with no need of additional frictional terms. In fact, we compensate for the lack of frictional damping using the strong positive definiteness of $G_{\alpha_0}$. Moreover, such an assumption seems to be quite natural in light of the fading memory principle recalled at the beginning of this introduction. Indeed, the fact that $g$ is a nonnegative decreasing function *just near* 0 is a key property in Proposition 5.5 below, where we give sufficient conditions for the strong positive definiteness of $G$.

As we show in Section 5 of this paper, in addition to (1.3) our results apply to several other examples of kernels such as

- $g(t) = \frac{1}{\Gamma(1/2)} e^{-at} t^{-1/2} \cos(ct)$, $t > 0$, $a > 0$, $c \in \mathbb{R}$,
- $g(t) = \theta \sum_{n=1}^\infty \cos(c_n t) e^{-n^\beta t}$, $t > 0$, $\beta > 1$,
- $g(t) = -e^{-\beta t} \log t$, $t > 0$, $\beta > 0$.

This fact supports the idea that the damping effect of the convolution term in (1.1) depends on the positivity of the convolution operator associated with $G$ rather than of $g$ itself. Another difference between this paper and [30] is the technique we use for our stability analysis. In [30] the asymptotic behavior of solutions is investigated by frequency domain methods. Our approach, on the contrary, takes advantage of the coercivity properties of strongly positive definite kernels (see Lemmas 2.8 and 2.9 below) to adapt the so-called multiplier method to Eq. (1.1).

The outline of this paper is the following. In Section 2, we recall basic properties of positive definite kernels as well as the definition of the resolvent for the linear equation associated with (1.1). Section 3 is devoted to the well-posedness of (1.1) and Section 4 to exponential decay. In Section 5, we compare our results with those obtained in [1] and [30] and discuss several examples of kernels, possibly of variable sign. Section 6 contains an application to a nonlinear wave equation with memory. The paper ends with Appendix A intended to assist the reader with some technical properties for linear systems.

## 2. Preliminaries

Let $X$ be a real Hilbert space with scalar product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$.

For any $T \in (0,\infty]$ and $p \in [1,\infty]$ we denote by $L^p(0,T;X)$ the usual spaces of measurable functions $v : (0,T) \to X$ such that one has

[1] For polynomial stability the exponential in assumption (1.2) is replaced by a power of $t$.

$$\|v\|_{p,T}^p := \int_0^T \|v(t)\|^p \, dt < \infty, \quad 1 \leqslant p < \infty,$$

$$\|v\|_{\infty,T} := \operatorname*{ess\,sup}_{0 \leqslant t \leqslant T} \|v(t)\| < \infty,$$

respectively. We shall use the shorter notation $\|v\|_p$ for $\|v\|_{p,\infty}$, $1 \leqslant p \leqslant \infty$. We denote by $L^p_{loc}(0,\infty;X)$ the space of functions belonging to $L^p(0,T;X)$ for any $T \in (0,\infty)$. In the case of $X = \mathbb{R}$, we will use the abbreviations $L^p(0,T)$ and $L^p_{loc}(0,\infty)$ to denote the spaces $L^p(0,T;\mathbb{R})$ and $L^p_{loc}(0,\infty;\mathbb{R})$, respectively.

$C^k([0,T];X)$, $k = 0,1,2$, stands for the space of continuous functions from $[0,T]$ to $X$ having continuous derivatives up to the order $k$ in $[0,T]$. In particular, we write $C([0,T];X)$ for $C^0([0,T];X)$.

For any $h \in L^1_{loc}(0,\infty)$ and any $v \in L^1_{loc}(0,\infty;X)$ we define

$$h * v(t) = \int_0^t h(t-s)v(s)\, ds, \quad t \geqslant 0.$$

Throughout the paper, for $v \in L^1_{loc}(0,\infty;X)$ we denote by $\widehat{v}$ the Laplace transform of $v$, that is

$$\widehat{v}(z) := \int_0^\infty e^{-zt} v(t)\, dt, \quad z \in \mathbb{C}.$$

Given $h \in L^1_{loc}(0,\infty)$, recall that $h$ is a *positive definite kernel* if

$$\int_0^t \langle h * y(s), y(s) \rangle \, ds \geqslant 0, \quad t \geqslant 0, \tag{2.1}$$

for any $y \in L^2_{loc}(0,\infty;X)$. Also, $h$ is said to be a *strongly positive definite kernel* if there exists a constant $\delta > 0$ such that $h(t) - \delta e^{-t}$ is positive definite, namely

$$\int_0^t \langle h * y(s), y(s) \rangle \, ds \geqslant \delta \int_0^t \langle e * y(s), y(s) \rangle \, ds, \quad t \geqslant 0, \tag{2.2}$$

for any $y \in L^2_{loc}(0,\infty;X)$, where $e(t) = e^{-t}$.

If $h \in L^\infty(0,\infty)$, $h$ is positive definite if and only if

$$\operatorname{Re} \widehat{h}(z) \geqslant 0 \quad \text{for any } z \in \mathbb{C},\ \operatorname{Re} z > 0 \tag{2.3}$$

(see, e.g., [31, p. 38]). For any $h \in L^1(0,\infty)$ the following characterization is well known (see [29, Theorem 2]): $h$ is a positive definite kernel if and only if $\operatorname{Re} \widehat{h}(i\omega) \geqslant 0$ for any $\omega \in \mathbb{R}$. We note that this result can be given as follows: $h \in L^1(0,\infty)$ is positive definite if and only if

$$\operatorname{Re} \widehat{h}(i\omega) \geqslant 0 \quad \text{for any } \omega > 0. \tag{2.4}$$

From this it follows that $h \in L^1(0,\infty)$ is a strongly positive definite kernel if and only if there is a constant $\delta > 0$ such that

$$\operatorname{Re}\widehat{h}(i\omega) \geqslant \delta/\big(1+\omega^2\big) \quad \text{for any } \omega > 0, \tag{2.5}$$

where we have used $\operatorname{Re}\widehat{e}(i\omega) = 1/(1+\omega^2)$.

In order to check that a kernel is strongly positive definite, a handy result can be found in the literature, see [29, Corollary 2.2]. For the reader's convenience we recall it in the following.

**Theorem 2.1.** *A twice differentiable function $h(t)$ with $h' \not\equiv 0$ satisfying*

$$(-1)^n h^{(n)}(t) \geqslant 0 \quad \forall t > 0,\ n = 0, 1, 2 \tag{2.6}$$

*is a strongly positive definite kernel.*

From now on, let $g \in L^1(0,\infty)$ be a function and $G(t) := \int_t^\infty g(s)\,ds$.

The following results are useful to study exponential decay. The first proposition was stated and proved in [9].

**Proposition 2.2.**

(a) *If $G$ is a positive definite kernel, then*

$$\int_0^\infty \sin(\omega t) g(t)\,dt \geqslant 0 \quad \text{for any } \omega > 0. \tag{2.7}$$

(b) *If $G \in L^1(0,\infty)$ and $g$ verifies* (2.7)*, then $G$ is a positive definite kernel.*

Now, we will prove an analogous result for strongly positive definite kernels.

**Corollary 2.3.**

(a) *If $G$ is a strongly positive definite kernel, then there exists $\delta > 0$ such that*

$$\int_0^\infty \sin(\omega t) g(t)\,dt \geqslant \delta \frac{\omega}{1+\omega^2} \quad \text{for any } \omega > 0. \tag{2.8}$$

(b) *If $G \in L^1(0,\infty)$ and there exists $\delta > 0$ such that $g$ verifies* (2.8)*, then $G$ is a strongly positive definite.*

**Proof.** (a) Since $G(t) = \int_t^\infty g(s)\,ds$ is strongly positive definite, there exists $\delta > 0$ such that

$$G(t) - \delta e^{-t} = \int_t^\infty \big(g(s) - \delta e^{-s}\big)\,ds$$

is positive definite. Therefore, applying Proposition 2.2(a) to $g(t) - \delta e^{-t}$ and using the identity

$$\int_0^\infty \sin(\omega t)e^{-t}\,dt = \frac{\omega}{1+\omega^2}, \tag{2.9}$$

the conclusion follows.

(b) If $G \in L^1(0,\infty)$ and there exists $\delta > 0$ such that $g$ verifies (2.8), then $G(t) - \delta e^{-t}$ verifies (2.7), taking into account (2.9). By Proposition 2.2(b) we obtain that $G(t) - \delta e^{-t}$ is positive definite, so the statement holds true. □

We now recall a known result (see [7, Lemma 3.4]) that will be used next.

**Lemma 2.4.** *If $g \in L^1(0,\infty)$ is a function satisfying* (2.7)*, then the perturbed function $e^{-\sigma t}g(t)$, $\sigma > 0$, satisfies* (2.7) *as well.*

**Proposition 2.5.** *Let $g \in L^1(0,\infty)$ be a function such that $t \to \int_t^\infty g(s)\,ds$ is strongly positive definite. Then $t \to \int_t^\infty e^{-\sigma s}g(s)\,ds$, $\sigma > 0$, is strongly positive definite as well.*

*In addition, if $\delta$ is the constant in* (2.2) *corresponding to $t \to \int_t^\infty g(s)\,ds$, then the analogous constant $\delta_\sigma$ for $t \to \int_t^\infty e^{-\sigma s}g(s)\,ds$ is given by*

$$\delta_\sigma = \frac{\delta}{(\sigma+1)^2}. \tag{2.10}$$

**Proof.** By Corollary 2.3(a) there exists $\delta > 0$ such that $g$ verifies (2.8), and hence, taking into account that $\frac{\omega}{1+\omega^2} = \int_0^\infty \sin(\omega t)e^{-t}\,dt$, we have

$$\int_0^\infty \sin(\omega t)\big(g(t) - \delta e^{-t}\big)\,dt \geqslant 0 \quad \text{for any } \omega > 0.$$

In view of Lemma 2.4 we get for any $\sigma > 0$

$$\int_0^\infty \sin(\omega t)e^{-\sigma t}\big(g(t) - \delta e^{-t}\big)\,dt \geqslant 0 \quad \text{for any } \omega > 0,$$

whence, since $\int_0^\infty \sin(\omega t)e^{-(\sigma+1)t}\,dt = \frac{\omega}{(\sigma+1)^2+\omega^2}$, it follows

$$\int_0^\infty \sin(\omega t)e^{-\sigma t}g(t)\,dt \geqslant \delta\frac{\omega}{(\sigma+1)^2+\omega^2} \geqslant \frac{\delta}{(\sigma+1)^2}\frac{\omega}{1+\omega^2},$$

for any $\omega > 0$. Now, we observe that, for any $\sigma > 0$, the function $t \mapsto te^{-\sigma t}g(t)$ is in $L^1(0,\infty)$, so also the function $t \to \int_t^\infty e^{-\sigma s}g(s)\,ds$ is in $L^1(0,\infty)$. Therefore, we can apply Corollary 2.3(b) to $e^{-\sigma t}g(t)$ to obtain the conclusion. □

Now, we recall some estimates for positive definite kernels, which can be found in the previous literature.

**Lemma 2.6.** *(See [34,22].) Let $h \in C([0,\infty))$ be a positive definite kernel. Then $h(0) \geqslant 0$ and*

$$\|h * y(t)\|^2 \leqslant 2h(0)\int_0^t \langle h * y(\tau), y(\tau)\rangle\, d\tau, \quad t \geqslant 0, \tag{2.11}$$

*for any $y \in L^1_{loc}(0,\infty;X)$.*

**Remark 2.7.** Note that if $h$ is a strongly positive definite kernel and $\delta$ is the constant in (2.2), then by Lemma 2.6 we have $h(0) - \delta \geqslant 0$, that is

$$h(0) \geqslant \delta. \tag{2.12}$$

**Lemma 2.8.** *(See [35,22].) Let $h$ be a strongly positive definite kernel satisfying $h, h' \in L^1(0,\infty)$. Then*

$$\int_0^t \|h * y(\tau)\|^2\, d\tau \leqslant \frac{1}{\delta}\left(\|h\|_1^2 + 4\|h'\|_1^2\right)\int_0^t \langle h * y(\tau), y(\tau)\rangle\, d\tau, \quad t \geqslant 0, \tag{2.13}$$

*for any $y \in L^1_{loc}(0,\infty;X)$, where $\delta$ is the constant in (2.2).*

**Lemma 2.9.** *(See [21,22].) Let $h \in L^1_{loc}(0,\infty)$ be a strongly positive definite kernel. Then*

$$\int_0^t \|y(\tau)\|^2\, d\tau \leqslant \|y(0)\|^2 + \frac{2}{\delta}\left(\int_0^t \langle h * y(\tau), y(\tau)\rangle\, d\tau + \int_0^t \langle h * y'(\tau), y'(\tau)\rangle\, d\tau\right), \tag{2.14}$$

*for any $t \geqslant 0$ and $y \in L^2_{loc}(0,\infty;X)$ with $y' \in L^1_{loc}(0,\infty;X)$, where $\delta$ is the constant in (2.2).*

Classical results for integral equations (see, e.g., [20, Theorem 2.3.5]) ensure that, for any kernel $h \in L^1_{loc}(0,\infty)$ and any $v \in L^1_{loc}(0,\infty;X)$, the problem

$$y(t) - h * y(t) = v(t), \quad t \geqslant 0, \tag{2.15}$$

admits a unique solution $y \in L^1_{loc}(0,\infty;X)$. In particular, there is a unique solution $r \in L^1_{loc}(0,\infty)$ of

$$r(t) - h * r(t) = h(t), \quad t \geqslant 0. \tag{2.16}$$

Such a solution is called the *resolvent kernel* of $h$. Furthermore, the solution $y$ of (2.15) is given by the variation of constants formula

$$y(t) = v(t) + r * v(t), \quad t \geqslant 0, \tag{2.17}$$

where $r$ is the resolvent kernel of $h$.

Now, we recall the classical Paley–Wiener Theorem (see, e.g., [20, Theorem 2.4.5]), which gives a necessary and sufficient condition for the resolvent of a kernel $h \in L^1(0,\infty)$ to belong to $L^1(0,\infty)$.

**Theorem 2.10.** *Let $h \in L^1(0,\infty)$. Then, the resolvent kernel of $h$ belongs to $L^1(0,\infty)$ if and only if $\widehat{h}(z) \neq 1$ for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geqslant 0$.*

**Proposition 2.11.** *Let $g \in L^1(0,\infty)$ be such that $\int_0^\infty g(s)\,ds < 1$ and suppose that $G(t) = \int_t^\infty g(s)\,ds$ is a strongly positive definite kernel. Then $\widehat{g}(z) \neq 1$, for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geqslant 0$.*

**Proof.** Since $G(t) = \int_0^\infty g(s)\,ds - \int_0^t g(s)\,ds$, we have that

$$\widehat{G}(z) = \frac{1}{z}\int_0^\infty g(s)\,ds - \frac{\widehat{g}(z)}{z}. \tag{2.18}$$

Now, for $\operatorname{Re} z > 0$,

$$\frac{1}{z}\int_0^\infty g(s)\,ds - \frac{\widehat{g}(z)}{z} \neq \frac{1}{z}\left(\int_0^\infty g(s)\,ds - 1\right)$$

because the real part of the left-hand side is nonnegative on account of (2.3), whereas the real part of the right-hand side is negative. Therefore, $\widehat{g}(z) \neq 1$ for $\operatorname{Re} z > 0$.

Moreover, since $G$ is strongly positive definite, by Corollary 2.3(a) there exists $\delta > 0$ such that for any $\omega > 0$

$$\int_0^\infty \sin(\omega t) g(t)\,dt \geqslant \delta \frac{\omega}{1+\omega^2},$$

whence for any $\omega \neq 0$ we have $\operatorname{Im}\widehat{g}(i\omega) \neq 0$. So, taking into account that $\int_0^\infty g(s)\,ds < 1$, we get $\widehat{g}(z) \neq 1$ also for $\operatorname{Re} z = 0$. $\square$

The following corollary of Proposition 2.11 and Theorem 2.10 provides uniform estimates for solutions of integral equations.

**Corollary 2.12.** *Let $g \in L^1(0,\infty)$ be such that $\int_0^\infty g(s)\,ds < 1$ and suppose that $G(t) = \int_t^\infty g(s)\,ds$ is a strongly positive definite kernel. Then,*

(a) *the resolvent kernel $r$ of $g$ belongs to $L^1(0,\infty)$;*
(b) *for any $v \in L^p(0,\infty;X)$, $1 \leqslant p \leqslant \infty$, the solution $y$ of equation*

$$y(t) - g * y(t) = v(t), \quad t \geqslant 0,$$

*belongs to $L^p(0,\infty;X)$ and*

$$\|y\|_p \leqslant \big(1 + \|r\|_1\big)\|v\|_p. \tag{2.19}$$

For the reader's convenience, we recall the notion of resolvent for the equation

$$u''(t) + Au(t) - \int_0^t g(t-s)Au(s)\,ds = 0, \tag{2.20}$$

where $A$ is a self-adjoint linear operator on $X$ with dense domain $D(A)$ and $g \in L^1_{loc}(0,\infty)$ (see [9,8] for smooth kernels).

**Definition 2.13.** A family $\{S(t)\}_{t\geqslant 0}$ of bounded linear operators in $X$ is called a resolvent for Eq. (2.20) if the following conditions are satisfied:

(S1) $S(0)=I$ and $S(t)$ is strongly continuous on $[0,\infty)$, that is, for all $x\in X$, $S(\cdot)x$ is continuous;
(S2) $S(t)$ commutes with $A$, which means that $S(t)D(A)\subset D(A)$ and

$$AS(t)x=S(t)Ax,\quad x\in D(A),\ t\geqslant 0;$$

(S3) for any $x\in D(A)$, $S(\cdot)x$ is twice continuously differentiable in $X$ on $[0,\infty)$ and $S'(0)x=0$;
(S4) for any $x\in D(A)$ and any $t\geqslant 0$,

$$S''(t)x+AS(t)x-\int_0^t g(t-\tau)AS(\tau)x\,d\tau=0.$$

Finally, we give a lemma establishing a sort of integration by parts, which will be useful in Section 5.

**Lemma 2.14.** *Let $f\in W^{1,1}_{loc}(0,\infty)\cap L^1(0,\infty)$ and $h\in C^1([0,\infty))$ with $h(0)=0$ be such that $f'(t)h(t)\geqslant 0$ for a.e. $t>0$, $fh$ and $fh'$ belong to $L^1(0,\infty)$. Then*

$$\int_0^\infty f'(t)h(t)\,dt=-\int_0^\infty f(t)h'(t)\,dt, \tag{2.21}$$

*in the sense of generalized integrals.*

**Proof.** First, we observe that there exists a sequence $\{a_n\}$ of positive numbers such that $\lim_{n\to\infty}a_n=0$ and $\lim_{n\to\infty}a_n f(a_n)=0$, because otherwise $|f(t)|\geqslant c_0/t$ near 0, for some $c_0>0$, which contradicts $f\in L^1(0,\infty)$. Since $h\in C^1([0,\infty))$ and $h(0)=0$, it follows that $\lim_{n\to\infty}f(a_n)h(a_n)=0$. In addition, since $fh\in L^1(0,\infty)$, there exists a sequence $\{b_n\}$ such that $\lim_{n\to\infty}b_n=\infty$ and $\lim_{n\to\infty}f(b_n)h(b_n)=0$.

Integrating by parts, we get for any $n\in\mathbb{N}$

$$\int_{a_n}^{b_n} f'(t)h(t)\,dt=\big[f(t)h(t)\big]_{a_n}^{b_n}-\int_{a_n}^{b_n} f(t)h'(t)\,dt. \tag{2.22}$$

Now, for any $0<a<b$ we have $a_n\leqslant a<b\leqslant b_n$ for $n\in\mathbb{N}$ sufficiently large, so

$$\int_a^b f'(t)h(t)\,dt\leqslant\int_{a_n}^{b_n} f'(t)h(t)\,dt,$$

whence, taking into account (2.22) and passing to the limit as $n\to\infty$, $a\to 0^+$ and $b\to\infty$, we obtain

$$\int_0^\infty f'(t)h(t)\,dt\leqslant-\int_0^\infty f(t)h'(t)\,dt. \tag{2.23}$$

On the other hand, again by (2.22) we see that

$$\big[f(t)h(t)\big]_{a_n}^{b_n} - \int_{a_n}^{b_n} f(t)h'(t)\,dt = \int_{a_n}^{b_n} f'(t)h(t)\,dt \leqslant \int_0^\infty f'(t)h(t)\,dt,$$

and hence

$$-\int_0^\infty f(t)h'(t)\,dt \leqslant \int_0^\infty f'(t)h(t)\,dt. \tag{2.24}$$

Finally, (2.21) follows from (2.23) and (2.24). □

**Remark 2.15.** We observe that, if $f \in L^1(0,T) \cap W^{1,1}(\varepsilon,T)$, for every $\varepsilon > 0$, and $h \in C^1([0,T])$ is such that $h(0) = 0$ and $f'(t)h(t) \geqslant 0$ for a.e. $t \in (0,T)$, then, in the sense of generalized integrals,

$$\int_0^T f'(t)h(t)\,dt = f(T)h(T) - \int_0^T f(t)h'(t)\,dt. \tag{2.25}$$

The proof is similar to that of Lemma 2.14.

## 3. Existence and uniqueness

### 3.1. Local existence of mild and strong solutions

Let us consider the semilinear equation

$$u''(t) + Au(t) - \int_0^t g(t-s)Au(s)\,ds = \nabla F\big(u(t)\big) + f(t), \quad t \geqslant 0. \tag{3.1}$$

Throughout this section we will assume that the following conditions are satisfied:

**Assumptions (H1).**

1. $A$ is a self-adjoint linear operator on $X$ with dense domain $D(A)$ such that

$$\langle Ax, x\rangle \geqslant M\|x\|^2 \quad \forall x \in D(A) \tag{3.2}$$

   for some $M > 0$.
2. $g \in L^1(0,\infty)$ such that

$$\int_0^\infty g(t)\,dt < 1, \tag{3.3}$$

$$t \mapsto \int_t^\infty g(s)\,ds \quad \text{is positive definite.} \tag{3.4}$$

3. $F : D(A^{1/2}) \to \mathbb{R}$ is a functional such that
   (a) $F$ is Gâteaux differentiable at any point $x \in D(A^{1/2})$;
   (b) for any $x \in D(A^{1/2})$ there exists a constant $c(x) > 0$ such that

$$\big|DF(x)(y)\big| \leqslant c(x)\|y\|, \quad \text{for any } y \in D\big(A^{1/2}\big), \tag{3.5}$$

   where $DF(x)$ denotes the Gâteaux derivative of $F$ in $x$; consequently, $DF(x)$ can be extended to the whole space $X$ (and we will denote by $\nabla F(x)$ the unique vector representing $DF(x)$ in the Riesz isomorphism, that is, $\langle \nabla F(x), y\rangle = DF(x)(y)$, for any $y \in X$);
   (c) for any $R > 0$ there exists a constant $C_R > 0$ such that

$$\big\|\nabla F(x) - \nabla F(y)\big\| \leqslant C_R \big\|A^{1/2}x - A^{1/2}y\big\| \tag{3.6}$$

   for all $x, y \in D(A^{1/2})$ satisfying $\|A^{1/2}x\|, \|A^{1/2}y\| \leqslant R$.

Let $0 < T \leqslant \infty$ be given and $f \in L^1(0, T; X)$. To begin with, we recall some notions of solution.

**Definition 3.1.** We say that $u$ is a *strong solution* of (3.1) on $[0, T]$ if

$$u \in C^2\big([0, T]; X\big) \cap C\big([0, T]; D(A)\big)$$

and $u$ satisfies (3.1) for every $t \in [0, T]$.

Let $u_0, u_1 \in X$. A function $u \in C^1([0, T]; X) \cap C([0, T]; D(A^{1/2}))$ is a *mild solution* of (3.1) on $[0, T]$ with initial conditions

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{3.7}$$

if

$$u(t) = S(t)u_0 + \int_0^t S(\tau)u_1\,d\tau + \int_0^t 1 * S(t-\tau)\big(\nabla F\big(u(\tau)\big) + f(\tau)\big)\,d\tau, \tag{3.8}$$

where $\{S(t)\}$ is the resolvent for (2.20) (see Definition 2.13).

Notice that the convolution term in (3.8) is well defined, thanks to (3.6). A strong solution is also a mild one.

Another useful notion of generalized solution of (3.1) is the so-called *weak solution*, that is a function $u \in C^1([0, T]; X) \cap C([0, T]; D(A^{1/2}))$ such that, for any $v \in D(A^{1/2})$, $\langle u'(t), v\rangle \in C^1([0, T])$ and for any $t \in [0, T]$ one has

$$\frac{d}{dt}\big\langle u'(t), v\big\rangle + \big\langle A^{1/2}u(t), A^{1/2}v\big\rangle - \left\langle \int_0^t g(t-s)A^{1/2}u(s)\,ds, A^{1/2}v\right\rangle$$
$$= \big\langle \nabla F\big(u(t)\big), v\big\rangle + \big\langle f(t), v\big\rangle. \tag{3.9}$$

Adapting a classical argument due to Ball [4], one can show that any mild solution of (3.1) is also a weak solution, and the two notions of solution are equivalent when $F \equiv 0$ (see also [31]).

The next proposition ensures the local existence and uniqueness of mild solutions. The proof relies on suitable regularity estimates for the resolvent $\{S(t)\}$ (see, e.g., [9, Section 3]) and a standard fixed point argument (see [6] for an analogous proof).

**Proposition 3.2.** *Let $u_0 \in D(A^{1/2})$, $u_1 \in X$ and $f \in L^1(0,T;X)$. Then, a positive number $T_0 = T_0^u \leqslant T$ exists so that Eq. (3.1) with initial conditions $u(0) = u_0$ and $u'(0) = u_1$ admits a unique mild solution on $[0,T_0]$.*

*In addition, for any $u_0, v_0 \in D(A^{1/2})$, $u_1, v_1 \in X$ and $f^u \in L^1(0,T;X)$, $f^v \in L^1(0,T;X)$, there exists a constant $C_T > 0$ (depending on $T$) such that, called $u$ and $v$ the mild solutions with data $u_0$, $u_1$, $f^u$ and $v_0$, $v_1$, $f^v$ respectively, we have*

$$\begin{aligned}&\left\|A^{1/2}u(t) - A^{1/2}v(t)\right\| + \left\|u'(t) - v'(t)\right\| \\ &\quad \leqslant C_T\left(\left\|A^{1/2}u_0 - A^{1/2}v_0\right\| + \|u_1 - v_1\| + \left\|f^u - f^v\right\|_{1,T}\right),\end{aligned} \tag{3.10}$$

*for any $t \in [0, T_0^u \wedge T_0^v]$.*

Assuming more regular data and using standard argumentations, one can show that the mild solution is a strong one.

**Proposition 3.3.** *Let $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in W^{1,1}(0,T;X)$. Then, the mild solution of the Cauchy problem (3.1)–(3.7) in $[0,T_0]$, $T_0 \in (0,T]$, is a strong solution.*

*In addition, $u$ belongs to $C^1([0,T_0];D(A^{1/2}))$.*

*3.2. Global existence of mild and strong solutions*

In this section we will investigate the existence in the large of the solution to the Cauchy problem

$$\begin{cases} u''(t) + Au(t) - \displaystyle\int_0^t g(t-s)Au(s)\,ds = \nabla F\big(u(t)\big) + f(t), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{3.11}$$

We define the energy of a mild solution $u$ of (3.11) on a given interval $[0,T]$, as

$$E_u(t) := \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\left(1 - \int_0^\infty g(s)\,ds\right)\left\|A^{1/2}u(t)\right\|^2 - F\big(u(t)\big). \tag{3.12}$$

Throughout this section let Assumptions (H1) be satisfied and we set

$$G(t) := \int_t^\infty g(s)\,ds, \quad t \geqslant 0.$$

First, we will show two preliminary results, which will be used later.

**Lemma 3.4.** *For any $v \in C^1([0,T];X)$, $T > 0$, the following holds true for any $t \in [0,T]$*

$$\begin{aligned}\int_0^t \big\langle g * v(s), v'(s)\big\rangle\,ds = &-\int_0^t \big\langle G * v'(s), v'(s)\big\rangle\,ds + \frac{G(0)}{2}\left(\left\|v(t)\right\|^2 + \left\|v(0)\right\|^2\right) \\ &- G(t)\big\langle v(0), v(t)\big\rangle - \int_0^t g(s)\big\langle v(0), v(s)\big\rangle\,ds,\end{aligned} \tag{3.13}$$

$$\langle v(t), v(t) - g * v(t)\rangle = \big(1 - G(0)\big)\|v(t)\|^2 + G(t)\langle v(0), v(t)\rangle + \langle v(t), G * v'(t)\rangle. \tag{3.14}$$

*In addition, if $G \in L^2(0,\infty)$, then we have for any $t \in [0,T]$*

$$\int_0^t \|v(s) - g * v(s)\|^2\,ds \leqslant 2\big(1 - G(0)\big)\int_0^t \langle v(s), v(s) - g * v(s)\rangle\,ds \\ + 2\|G\|_2^2\|v(0)\|^2 + 2\int_0^t \|G * v'(s)\|^2\,ds. \tag{3.15}$$

**Proof.** As $g = -G'$, the convolution term $g * v(t)$ can be estimated integrating by parts as follows:

$$g * v(t) = -\int_0^t G'(t-r)v(r)\,dr \\ = G(0)v(t) - G(t)v(0) - \int_0^t G(t-r)v'(r)\,dr. \tag{3.16}$$

Therefore, we have

$$\int_0^t \langle g * v(s), v'(s)\rangle\,ds \\ = \frac{G(0)}{2}\big(\|v(t)\|^2 - \|v(0)\|^2\big) - \left\langle v(0), \int_0^t G(s)v'(s)\,ds\right\rangle - \int_0^t \langle G * v'(s), v'(s)\rangle\,ds.$$

Another integration by parts yields

$$\int_0^t G(s)v'(s)\,ds = G(t)v(t) - G(0)v(0) + \int_0^t g(s)v(s)\,ds,$$

so, putting the above identity into the previous one, we have (3.13).

By (3.16) we get

$$v(t) - g * v(t) = \big(1 - G(0)\big)v(t) + G(t)v(0) + G * v'(t), \tag{3.17}$$

whence, multiplying by $v(t)$, (3.14) follows.

Now, we multiply both sides of (3.17) by $v(t) - g * v(t)$ and evaluate the second and the third terms on the right-hand side by applying the elementary inequality $|ab| \leqslant a^2/4 + b^2$. This yields

$$\|v(t) - g * v(t)\|^2 \\ \leqslant 2\big(1 - G(0)\big)\langle v(t), v(t) - g * v(t)\rangle + 2|G(t)|^2\|v(0)\|^2 + 2\|G * v'(t)\|^2.$$

Integrating the above estimate over $[0,t]$ gives (3.15). □

**Lemma 3.5.**

(i) *If* $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ *and* $f \in W^{1,1}(0,T;X)$, *then the strong solution* $u$ *of problem* (3.11) *on* $[0,T_0]$, $T_0 \leqslant T$, *satisfies the identity*

$$\begin{aligned} &E_u(t) + \int_0^t \langle G * A^{1/2}u'(s), A^{1/2}u'(s)\rangle\, ds \\ &\quad = E_u(0) + G(0)\|A^{1/2}u_0\|^2 - \langle A^{1/2}u_0, G(t)A^{1/2}u(t)\rangle \\ &\qquad - \int_0^t g(s)\langle A^{1/2}u_0, A^{1/2}u(s)\rangle\, ds + \int_0^t \langle f(s), u'(s)\rangle\, ds, \end{aligned} \tag{3.18}$$

*for any* $t \in [0,T_0]$.

(ii) *If* $u_0 \in D(A^{1/2})$, $u_1 \in X$ *and* $f \in L^1(0,T;X)$, *then the mild solution u of problem* (3.11) *on* $[0,T_0]$ *verifies*

$$\begin{aligned} E_u(t) &\leqslant E_u(0) + G(0)\|A^{1/2}u_0\|^2 - \langle A^{1/2}u_0, G(t)A^{1/2}u(t)\rangle \\ &\quad - \int_0^t g(s)\langle A^{1/2}u_0, A^{1/2}u(s)\rangle\, ds + \int_0^t \langle f(s), u'(s)\rangle\, ds, \end{aligned} \tag{3.19}$$

*for any* $t \in [0,T_0]$.

**Proof.** (i) Let us suppose, first, that $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in W^{1,1}(0,T;X)$; in particular, the strong solution $u$ belongs to $C^1([0,T_0];D(A^{1/2}))$, see Proposition 3.3.

If we multiply the equation in (3.11) by $u'(t)$, then we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u'(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|A^{1/2}u(t)\|^2 - \langle g * A^{1/2}u(t), A^{1/2}u'(t)\rangle \\ &\quad = \langle \nabla F(u(t)), u'(t)\rangle + \langle f(t), u'(t)\rangle. \end{aligned}$$

Integrating from 0 to $t$ yields

$$\begin{aligned} &\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|A^{1/2}u(t)\|^2 - \int_0^t \langle g * A^{1/2}u(\tau), A^{1/2}u'(\tau)\rangle\, d\tau - F(u(t)) \\ &\quad = \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|A^{1/2}u_0\|^2 - F(u_0) + \int_0^t \langle f(\tau), u'(\tau)\rangle\, d\tau. \end{aligned}$$

To estimate the integral on the left-hand side of the above estimate, we use (3.13) with $v = A^{1/2}u$, so we have

$$\frac{1}{2}\|u'(t)\|^2 + \frac{1-G(0)}{2}\|A^{1/2}u(t)\|^2 + \int_0^t \langle G * A^{1/2}u'(\tau), A^{1/2}u'(\tau)\rangle\, d\tau - F(u(t))$$

$$= \frac{1}{2}\|u_1\|^2 + \frac{1+G(0)}{2}\left\|A^{1/2}u_0\right\|^2 - F(u_0)$$

$$- \left\langle A^{1/2}u_0, G(t)A^{1/2}u(t)\right\rangle - \int_0^t g(\tau)\left\langle A^{1/2}u_0, A^{1/2}u(\tau)\right\rangle d\tau + \int_0^t \left\langle f(\tau), u'(\tau)\right\rangle d\tau,$$

whence (3.18) follows.

(ii) Since $G$ is a positive definite kernel, we have

$$\int_0^t \left\langle G * A^{1/2}u'(s), A^{1/2}u'(s)\right\rangle ds \geqslant 0;$$

so, an approximation argument based on (3.10) suffices to prove (3.19) for mild solutions. □

Assuming an extra condition on functional $F$, global existence will follow with sufficiently small data.

**Theorem 3.6.** *Suppose that there exists an upper semicontinuous function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that*

$$\left|F(x)\right| \leqslant \psi\left(\left\|A^{1/2}x\right\|\right)\left\|A^{1/2}x\right\|^2 \quad \forall x \in D\left(A^{1/2}\right). \tag{3.20}$$

*Then a number $\rho_0 > 0$ exists such that for any $(u_0, u_1) \in D(A^{1/2}) \times X$ and any $f \in L^1(0,\infty; X)$, satisfying*

$$\left\|A^{1/2}u_0\right\| + \|u_1\| + \|f\|_1 < \rho_0,$$

*problem (3.11) admits a unique mild solution $u$ on $[0,\infty)$.*

*Moreover, $E_u(t)$ is positive and for $\varepsilon \in (0, \frac{1-G(0)}{2})$ we have*

$$E_u(t) \geqslant \frac{1}{2}\left\|u'(t)\right\|^2 + \varepsilon\left\|A^{1/2}u(t)\right\|^2, \tag{3.21}$$

$$\psi\left(\left\|A^{1/2}u(t)\right\|\right) \leqslant \frac{1-G(0)}{2} - \varepsilon, \tag{3.22}$$

$$E_u(t) \leqslant C\left(\|u_1\| + \left\|A^{1/2}u_0\right\| + \|f\|_1\right), \tag{3.23}$$

$$\left\|u'(t)\right\|^2 + \left\|A^{1/2}u(t)\right\|^2 \leqslant C\left(\|u_1\| + \left\|A^{1/2}u_0\right\| + \|f\|_1\right) \tag{3.24}$$

*for any $t \geqslant 0$, where $C(R)$ is a positive, increasing, upper semicontinuous function such that $C(0) = 0$.*

*Furthermore, if $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in W^{1,1}_{loc}(0,\infty; X) \cap L^1(0,\infty; X)$, then $u$ is a strong solution of the equation in (3.11) on $[0,\infty)$, $u \in C^1([0,\infty); D(A^{1/2}))$ and for any $t \geqslant 0$*

$$E_u(t) + \int_0^t \left\langle G * A^{1/2}u'(s), A^{1/2}u'(s)\right\rangle ds \leqslant C\left(\|u_1\| + \left\|A^{1/2}u_0\right\| + \|f\|_1\right). \tag{3.25}$$

**Proof.** Let $[0,T)$ be the maximal domain of the mild solution $u$ of (3.11). To prove $T=\infty$, we will show that the energy of $u$ is nonnegative and bounded from above by a constant independent of $T$.

First, we claim that for $\varepsilon \in (0, \frac{1-G(0)}{2})$, if the initial data are suitably small, then

$$\psi\big(\big\|A^{1/2}u(t)\big\|\big) \leqslant \frac{1-G(0)}{2} - \varepsilon \quad \forall t \in [0,T). \tag{3.26}$$

To prove this, we observe that there exists $\overline{\rho} > 0$ such that

$$\psi(\rho) \leqslant \frac{1-G(0)}{2} - \varepsilon, \quad \forall 0 < \rho \leqslant \overline{\rho}. \tag{3.27}$$

We assume that

$$\big\|A^{1/2}u_0\big\| \leqslant \overline{\rho}/2, \tag{3.28}$$

so the set $\Lambda$ of all $s \in [0,T)$ such that $\|A^{1/2}u(t)\| \leqslant \overline{\rho}$ for any $t \in [0,s]$ is not empty. Let $\tau$ be the supremum of $\Lambda$. We will show that $\tau = T$. Reasoning by contradiction, we suppose $\tau < T$. By continuity,

$$\big\|A^{1/2}u(t)\big\| \leqslant \overline{\rho} \quad \forall t \in [0,\tau],$$

whence, by (3.27), we have

$$\psi\big(\big\|A^{1/2}u(t)\big\|\big) \leqslant \frac{1-G(0)}{2} - \varepsilon \quad \forall t \in [0,\tau]. \tag{3.29}$$

Assumption (3.20) and estimate (3.29) yield for any $t \in [0,\tau]$

$$\begin{aligned} E_u(t) &\geqslant \frac{1}{2}\big\|u'(t)\big\|^2 + \frac{1-G(0)}{2}\big\|A^{1/2}u(t)\big\|^2 - \psi\big(\big\|A^{1/2}u(t)\big\|\big)\big\|A^{1/2}u(t)\big\|^2 \\ &\geqslant \frac{1}{2}\big\|u'(t)\big\|^2 + \varepsilon\big\|A^{1/2}u(t)\big\|^2. \end{aligned} \tag{3.30}$$

Now, we observe that by (3.19), (3.20) and (3.28) we get

$$\begin{aligned} E_u(t) &\leqslant \frac{1}{2}\|u_1\|^2 + \big\|A^{1/2}u_0\big\|^2 + \frac{1}{2\varepsilon}\|g\|_1\big(\|g\|_1+1\big)\big\|A^{1/2}u_0\big\|^2 + \frac{1}{2}\|f\|_1 \\ &\quad + \frac{\varepsilon}{2}\big\|A^{1/2}u(t)\big\|^2 + \frac{\varepsilon}{2}\int_0^t \big|g(\sigma)\big|\big\|A^{1/2}u(\sigma)\big\|^2\,d\sigma + \frac{1}{2}\int_0^t \big|f(\sigma)\big|\big\|u'(\sigma)\big\|^2\,d\sigma \\ &\leqslant \left(2 + \frac{1}{2\varepsilon}\|g\|_1\big(\|g\|_1+1\big)\right)\big(\|u_1\|^2 + \big\|A^{1/2}u_0\big\|^2 + \|f\|_1\big) \\ &\quad + \frac{\varepsilon}{2}\big\|A^{1/2}u(t)\big\|^2 + \frac{\varepsilon}{2}\int_0^t \big|g(\sigma)\big|\big\|A^{1/2}u(\sigma)\big\|^2\,d\sigma + \frac{1}{2}\int_0^t \big|f(\sigma)\big|\big\|u'(\sigma)\big\|^2\,d\sigma. \end{aligned} \tag{3.31}$$

Putting together (3.31) and (3.30), we get

$$\|u'(t)\|^2+\varepsilon\|A^{1/2}u(t)\|^2 \leqslant \left(4+\frac{1}{\varepsilon}\|g\|_1\big(\|g\|_1+1\big)\right)\big(\|u_1\|^2+\|A^{1/2}u_0\|^2+\|f\|_1\big)$$
$$+\int_0^t \big(|g(\sigma)|+|f(\sigma)|\big)\big(\|u'(\sigma)\|^2+\varepsilon\|A^{1/2}u(\sigma)\|^2\big)\,d\sigma. \quad (3.32)$$

Applying Gronwall Lemma, we have for any $t\in[0,\tau]$

$$\|u'(t)\|^2+\varepsilon\|A^{1/2}u(t)\|^2$$
$$\leqslant e^{\|g\|_1}\left(4+\frac{1}{\varepsilon}\|g\|_1\big(\|g\|_1+1\big)\right)e^{\|f\|_1}\big(\|u_1\|^2+\|A^{1/2}u_0\|^2+\|f\|_1\big). \quad (3.33)$$

Therefore, set

$$C:=\frac{e^{\|g\|_1}}{\varepsilon}\left(4+\frac{1}{\varepsilon}\|g\|_1\big(\|g\|_1+1\big)\right)$$

if we take $\delta>0$ such that $\delta e^{\delta}<\frac{\overline{\rho}^2}{4C}$, then for $\|f\|_1<\delta$ and $\|u_1\|^2+\|A^{1/2}u_0\|^2<e^{-\delta}\frac{\overline{\rho}^2}{4C}-\delta$ we have

$$e^{\|f\|_1}\big(\|u_1\|^2+\|A^{1/2}u_0\|^2+\|f\|_1\big)\leqslant\frac{\overline{\rho}^2}{4C},$$

so, by (3.33) we get

$$\|A^{1/2}u(\tau)\|\leqslant\frac{\overline{\rho}}{2}<\overline{\rho}.$$

This contradicts the maximality of $\tau$. So, $\tau=T$.

To sum up, there exists $\rho_0>0$ such that for any $(u_0,u_1)\in D(A^{1/2})\times X$ and any $f\in L^1(0,\infty;X)$ with $\|A^{1/2}u_0\|+\|u_1\|+\|f\|_1\leqslant\rho_0$, we have

$$\|A^{1/2}u(t)\|\leqslant\overline{\rho}\quad\forall t\in[0,T),$$

so

$$\psi\big(\|A^{1/2}u(t)\|\big)\leqslant\frac{1-G(0)}{2}-\varepsilon\quad\forall t\in[0,T).$$

Thus, for any $t\in[0,T)$

$$E_u(t)\geqslant\frac{1}{2}\|u'(t)\|^2+\varepsilon\|A^{1/2}u(t)\|^2,$$

and

$$\|u'(t)\|^2+\varepsilon\|A^{1/2}u(t)\|^2$$
$$\leqslant e^{\|g\|_1+\|f\|_1}\left(4+\frac{1}{\varepsilon}\|g\|_1\big(\|g\|_1+1\big)\right)\big(\|u_1\|^2+\|A^{1/2}u_0\|^2+\|f\|_1\big).$$

Moreover, putting the above estimate into (3.31) we obtain

$$E_u(t) \leqslant C\big(\|u_1\| + \big\|A^{1/2}u_0\big\| + \|f\|_1\big) \quad \forall t \in [0,T),$$

where $C(R)$ is a positive, increasing, upper semicontinuous function such that $C(0) = 0$ and independent of $T$. Therefore, $u$ is global and estimates (3.21)–(3.23) hold true. Finally, (3.24) follows from (3.21) and (3.23), while (3.25) holds for strong solutions in view of (3.18). □

**Corollary 3.7.** *Under the assumptions of Theorem* 3.6*, for any* $R > 0$ *there exists* $\rho_R > 0$ *such that for any* $(u_0, u_1) \in D(A^{1/2}) \times X$ *and any* $f \in L^1(0,\infty; X)$*, satisfying*

$$\big\|A^{1/2}u_0\big\| + \|u_1\| + \|f\|_1 < \rho_R,$$

*the mild solution* $u$ *of problem* (3.11) *verifies*

$$\psi\big(\big\|A^{1/2}u(t)\big\|\big) \leqslant R \quad \forall t \geqslant 0.$$

**Proof.** The statement follows from the upper semicontinuity of $\psi$ and (3.24). □

## 4. Exponential decay of the energy

This section is devoted to the study of the asymptotic behavior of the solution to the Cauchy problem

$$\begin{cases} u''(t) + Au(t) - \displaystyle\int_0^t g(t-s)Au(s)\,ds = \nabla F\big(u(t)\big) + f(t), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{4.1}$$

For any measurable function $h : (0,\infty) \to X$ and $\alpha \in \mathbb{R}$ let us set

$$h_\alpha(t) := e^{\alpha t}h(t), \quad t > 0.$$

We will provide conditions to ensure that the energy of the solution decays exponentially at $\infty$. Such conditions are obtained by strengthening Assumptions (H1) as follows.

**Assumptions (H2).**

1. There exists $\alpha_0 \geqslant 0$ such that $g_{\alpha_0} \in L^1(0,\infty)$ and $t \mapsto \int_t^\infty g_{\alpha_0}(s)\,ds$ is a strongly positive definite kernel.
2. $F(0) = 0$ and there exists an upper semicontinuous function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that

$$\big|\big\langle \nabla F(x), y\big\rangle\big| \leqslant \psi\big(\big\|A^{1/2}x\big\|\big)\big\|A^{1/2}x\big\|\|y\|, \tag{4.2}$$

for any $x \in D(A^{1/2})$ and $y \in X$.

Notice that (H2)-1 is satisfied (with $\alpha_0 = 0$) if $G$ is a strongly positive definite kernel. For our next result, however, we need that (H2)-1 holds true with $\alpha_0 > 0$.

**Theorem 4.1.** *Assume* (H1) *and* (H2) *with* $\alpha_0 > 0$*. Then, there exist positive numbers* $\rho_1 \leqslant {\rho_0}^2$ [2] *and* $C$ *such that, for any* $(u_0, u_1) \in D(A^{1/2}) \times X$ *and any* $f_{\eta_0} \in L^1(0, \infty; X)$, $\eta_0 > 0$*, satisfying*

$$\left\| A^{1/2} u_0 \right\| + \|u_1\| + \|f_{\eta_0}\|_1 < \rho_1,$$

*the energy* $E_u(t)$ *of the mild solution* $u$ *of* (4.1) *decays as*

$$E_u(t) \leqslant C\left(\left\| A^{1/2} u_0 \right\| + \|u_1\| + \|f_{\eta_0}\|_1\right) e^{-2\alpha t} \quad \forall t \geqslant 0, \tag{4.3}$$

*and*

$$\int_0^\infty e^{2\alpha t} E_u(t)\, dt \leqslant C\left(\left\| A^{1/2} u_0 \right\| + \|u_1\| + \|f_{\eta_0}\|_1\right) \tag{4.4}$$

*for any* $\alpha \in [0, \alpha^*]$*, where* $\alpha^* \in (0, \alpha_0 \wedge \eta_0]$ *and* $C(R)$ *is a positive, increasing, upper semicontinuous function such that* $C(0) = 0$.

*Moreover, one can take* $\rho_1 = \infty$ *if* $F \equiv 0$.

**Remark 4.2.**

1. In the proof of Theorem 4.1 we give an estimate of the decay rate $\alpha^*$. Precisely, it can be taken as in the following formula

$$2\alpha^* := \frac{\delta_{\alpha_0}}{4(1+\alpha_0)^2} \left( \left\| A^{-1/2} \right\|^2 + \frac{(1+\alpha_0)^2}{\delta_{\alpha_0}} + \frac{8\alpha_1^2 \|A^{-1/2}\|^2}{1 - G(0)} \right)^{-1} \wedge \frac{\alpha_1}{2} \wedge \eta_0,$$

   with

$$\alpha_1 := \frac{\alpha_0^2 \delta_{\alpha_0}^2}{8(1+\alpha_0^2)(1+\alpha_0)^4 \|g_{\alpha_0}\|_1^2} \wedge \frac{\alpha_0}{2},$$

   see (4.15) and (4.25) below.
2. As we shall see in Section 5, the above set-up can be used to treat kernels with variable sign. For instance, our assumptions for convolution kernels, namely (H1)-2 and (H2)-1, hold true for

$$g_{a,b,c}(t) := \frac{1}{\Gamma(1-b)} e^{-at} t^{-b} \cos(ct)$$

   when $a, b, c$ satisfy suitable constraints.
3. For any $\alpha \in [0, \alpha_0]$, $g_\alpha \in L^1(0, \infty)$ and

$$G_\alpha(t) := \int_t^\infty g_\alpha(s)\, ds$$

   is a strongly positive definite kernel with

[2] $\rho_0$ is given by Theorem 3.6.

$$\delta_\alpha := \frac{\delta_{\alpha_0}}{(1+\alpha_0-\alpha)^2} \geqslant \frac{\delta_{\alpha_0}}{(1+\alpha_0)^2}, \tag{4.5}$$

as one can see noting that $G_\alpha(t) = \int_t^\infty e^{-(\alpha_0-\alpha)s} g_{\alpha_0}(s)\,ds$ and applying Proposition 2.5 to $g_{\alpha_0}$. Moreover, thanks to (2.12) and (4.5), we have

$$G_\alpha(0) \geqslant \delta_\alpha \geqslant \frac{\delta_{\alpha_0}}{(1+\alpha_0)^2}. \tag{4.6}$$

4. Thanks to $g_{\alpha_0} \in L^1(0,\infty)$, the dominated convergence theorem and $G(0) < 1$, we have for any $\alpha \in [0,\alpha']$

$$1 - G_\alpha(0) \geqslant \frac{1-G(0)}{2} > 0, \tag{4.7}$$

for some $\alpha' \in (0,\alpha_0]$. For simplicity, in the following we assume $\alpha' = \alpha_0$, so (4.7) is satisfied for any $\alpha \in [0,\alpha_0]$.
5. For any $\alpha \in (0,\alpha_0-\varepsilon_0]$, $0 < \varepsilon_0 < \alpha_0$, $G_\alpha \in L^1(0,\infty)$ and

$$\|G_\alpha\|_1 \leqslant \frac{1}{\varepsilon_0}\|g_{\alpha_0}\|_1, \tag{4.8}$$

because it suffices to note that

$$\begin{aligned}\int_0^\infty \big|G_\alpha(t)\big|\,dt &\leqslant \int_0^\infty s e^{\alpha s}\big|g(s)\big|\,ds \\ &= \int_0^\infty s e^{-(\alpha_0-\alpha)s} e^{\alpha_0 s}\big|g(s)\big|\,ds \leqslant \frac{1}{\varepsilon_0}\int_0^\infty e^{\alpha_0 s}\big|g(s)\big|\,ds. \end{aligned} \tag{4.9}$$

6. Observe that for any $\alpha \in (0,\alpha_0-\varepsilon_0]$, $0 < \varepsilon_0 < \alpha_0$, $G_\alpha \in L^2(0,\infty)$ and also by (4.8) we have

$$\|G_\alpha\|_2^2 \leqslant \|G_\alpha\|_1\|g_\alpha\|_1 \leqslant \frac{1}{\varepsilon_0}\|g_{\alpha_0}\|_1^2. \tag{4.10}$$

7. We note that $\nabla F(0) = 0$, as follows from (4.2).
8. Observe that assumption (H2)-2 ensures that hypothesis (3.20) is satisfied. Indeed, since $F(0) = 0$ for any $x \in D(A^{1/2})$ we have

$$\begin{aligned}\big|F(x)\big| &\leqslant \int_0^1 \big|\big\langle \nabla F(tx), x\big\rangle\big|\,dt \\ &\leqslant \big\|A^{1/2}x\big\|\,\|x\| \int_0^1 \psi\big(t\big\|A^{1/2}x\big\|\big)t\,dt \leqslant \frac{1}{2\sqrt{M}}\psi\big(\big\|A^{1/2}x\big\|\big)\big\|A^{1/2}x\big\|^2,\end{aligned}$$

also in view of (3.2).

**Proof of Theorem 4.1.** Due to some tricky technical features of the proof, we prefer to treat, at first, the linear case $F = 0$.

Let us suppose that $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in C^1([0,\infty); X)$, so that the mild solution $u$ is a strong one. It is easy to see that, for any $\alpha \geqslant 0$, the function $u_\alpha(t) = e^{\alpha t}u(t)$ solves the problem

$$\begin{cases} u_\alpha''(t) - 2\alpha u_\alpha'(t) + \alpha^2 u_\alpha(t) + Au_\alpha(t) - g_\alpha * Au_\alpha(t) = f_\alpha(t), \\ u_\alpha(0) = u_0; \quad u_\alpha'(0) = u_0 + \alpha u_1. \end{cases} \tag{4.11}$$

As in the nonintegral case, the more difficult term to estimate is the kinetic energy. Indeed, to evaluate $\int_0^t \|u_\alpha'(s)\|^2\,ds$, we apply inequality (2.14) with $h = G_\alpha$ and $y = u_\alpha'$, obtaining for any $t \geqslant 0$

$$\int_0^t \|u_\alpha'(s)\|^2\,ds \leqslant \|u_\alpha'(0)\|^2 + \frac{2}{\delta_\alpha}\left(\int_0^t \langle G_\alpha * u_\alpha'(s), u_\alpha'(s)\rangle\,ds + \int_0^t \langle G_\alpha * u_\alpha''(s), u_\alpha''(s)\rangle\,ds\right). \tag{4.12}$$

Therefore, to proceed with the proof, we need to give suitable estimates for the integrals on the right-hand side. That is our goal in the following lemmas.

From now on, we denote by the symbol $C_i$, $i \in \mathbb{N}$, positive constants depending only on data and $\|A^{1/2}u_0\|$, $\|u_1\|$, $\|f_{\eta_0}\|_1$.

**Lemma 1.** *For any $\alpha \in [0, \alpha_0]$ and $t \geqslant 0$ we have*

$$\int_0^t \langle G_\alpha * u_\alpha'(s), u_\alpha'(s)\rangle\,ds \leqslant C_1 + H_{\alpha,1}(t) + 2\alpha \|A^{-1/2}\|^2 \int_0^t \|u_\alpha'(s)\|^2\,ds, \tag{4.13}$$

*where $C_1 > 0$ and*

$$H_{\alpha,1}(t) := -G_\alpha(t)\langle u_0, u_\alpha(t)\rangle - \int_0^t g_\alpha(s)\langle u_0, u_\alpha(s)\rangle\,ds + \int_0^t \langle f_\alpha(s), A^{-1}u_\alpha'(s)\rangle\,ds. \tag{4.14}$$

**Proof.** If we multiply the equation in (4.11) by $A^{-1}u_\alpha'(t)$ and integrate from 0 to $t$, then we obtain

$$\begin{aligned} &\frac{1}{2}\|A^{-1/2}u_\alpha'(t)\|^2 + \frac{\alpha^2}{2}\|A^{-1/2}u_\alpha(t)\|^2 + \frac{1}{2}\|u_\alpha(t)\|^2 - \int_0^t \langle g_\alpha * u_\alpha(s), u_\alpha'(s)\rangle\,ds \\ &= \frac{1}{2}\|A^{-1/2}u_\alpha'(0)\|^2 + \frac{\alpha^2}{2}\|A^{-1/2}u_0\|^2 + \frac{1}{2}\|u_0\|^2 \\ &\quad + \int_0^t \langle f_\alpha(s), A^{-1}u_\alpha'(s)\rangle\,ds + 2\alpha \int_0^t \|A^{-1/2}u_\alpha'(s)\|^2\,ds. \end{aligned}$$

Therefore, applying (3.13) with $v = u_\alpha$ to the integral on the left-hand side, one gets (4.13)–(4.14). □

**Lemma 2.** *For any $\alpha \in [0, \alpha_1 - \varepsilon_0]$, $0 < \varepsilon_0 < \alpha_1$, with*

$$\alpha_1 := \frac{\alpha_0^2 \delta_{\alpha_0}^2}{8(1+\alpha_0^2)(1+\alpha_0)^4 \|g_{\alpha_0}\|_1^2} \wedge \frac{\alpha_0}{2}, \tag{4.15}$$

*and $t \geqslant 0$ the following two estimates hold true*

$$\int_0^t \langle G_\alpha * u_\alpha''(s), u_\alpha''(s)\rangle\, ds \leqslant \frac{\alpha_1}{\varepsilon_0}\big(C_2 + H_{\alpha,2}(t)\big) + \alpha \frac{\alpha_1}{\varepsilon_0}\left(\frac{(1+\alpha_0)^2}{\delta_{\alpha_0}} + \frac{8\alpha_1^2}{1-G(0)}\|A^{-1/2}\|^2\right)\int_0^t \|u_\alpha'(s)\|^2\, ds, \tag{4.16}$$

$$\frac{1-G_\alpha(0)}{2}\|u_\alpha'(t)\|^2 + \frac{1}{2}\|A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(t)\|^2 + \alpha^2 \frac{1-G_\alpha(0)}{4}\|u_\alpha(t)\|^2 \leqslant C_2 + H_{\alpha,2}(t) + \alpha\left(\frac{(1+\alpha_0)^2}{\delta_{\alpha_0}} + \frac{8\alpha_1^2}{1-G(0)}\|A^{-1/2}\|^2\right)\int_0^t \|u_\alpha'(s)\|^2\, ds, \tag{4.17}$$

*where $C_2 > 0$ and*

$$H_{\alpha,2}(t) := \frac{2\alpha^2}{1-G_\alpha(0)} H_{\alpha,1}(t) - G_\alpha(t)\langle u_\alpha'(0), u_\alpha'(t)\rangle - \int_0^t g_\alpha(s)\langle u_\alpha'(0), u_\alpha'(s)\rangle\, ds - \int_0^t g_\alpha(s)\langle A^{1/2}u_0, A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(s)\rangle\, ds + \int_0^t \langle f_\alpha(s), u_\alpha'(s) - g_\alpha * u_\alpha'(s)\rangle\, ds. \tag{4.18}$$

**Proof.** First, we multiply the equation in (4.11) by

$$u_\alpha'(t) - g_\alpha * u_\alpha'(t) = \frac{d}{dt}(u_\alpha - g_\alpha * u_\alpha)(t) + g_\alpha(t)u_0 \tag{4.19}$$

and, since $u \in C^1([0,\infty); D(A^{1/2}))$ (see Theorem 3.6), we obtain

$$\begin{aligned}&\langle u_\alpha''(t), u_\alpha'(t) - g_\alpha * u_\alpha'(t)\rangle - 2\alpha\langle u_\alpha'(t), u_\alpha'(t) - g_\alpha * u_\alpha'(t)\rangle \\ &\quad + \alpha^2\langle u_\alpha(t), u_\alpha'(t) - g_\alpha * u_\alpha'(t)\rangle + \frac{1}{2}\frac{d}{dt}\|A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(t)\|^2 \\ &\quad + g_\alpha(t)\langle A^{1/2}u_0, A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(t)\rangle = \langle f_\alpha(t), u_\alpha'(t) - g_\alpha * u_\alpha'(t)\rangle.\end{aligned}$$

Integrating from 0 to $t$ yields

$$\frac{1}{2}\left\|u_\alpha'(t)\right\|^2-\int_0^t\left\langle u_\alpha''(s),g_\alpha*u_\alpha'(s)\right\rangle ds+\frac{1}{2}\left\|A^{1/2}(u_\alpha-g_\alpha*u_\alpha)(t)\right\|^2$$

$$=\frac{1}{2}\left\|u_\alpha'(0)\right\|^2+\frac{1}{2}\left\|A^{1/2}u_0\right\|^2-\int_0^t g_\alpha(s)\left\langle A^{1/2}u_0,A^{1/2}(u_\alpha-g_\alpha*u_\alpha)(s)\right\rangle ds$$

$$+\int_0^t\left\langle f_\alpha(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\right\rangle ds+2\alpha\int_0^t\left\langle u_\alpha'(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\right\rangle ds$$

$$-\alpha^2\int_0^t\left\langle u_\alpha(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\right\rangle ds.$$

To estimate the integral on the left-hand side, we use (3.13) with $v=u_\alpha'$, so we have

$$\frac{1-G_\alpha(0)}{2}\left\|u_\alpha'(t)\right\|^2+\int_0^t\left\langle G_\alpha*u_\alpha''(s),u_\alpha''(s)\right\rangle ds+\frac{1}{2}\left\|A^{1/2}(u_\alpha-g_\alpha*u_\alpha)(t)\right\|^2$$

$$=\frac{1+G_\alpha(0)}{2}\left\|u_\alpha'(0)\right\|^2+\frac{1}{2}\left\|A^{1/2}u_0\right\|^2-G_\alpha(t)\left\langle u_\alpha'(0),u_\alpha'(t)\right\rangle$$

$$-\int_0^t g_\alpha(s)\left\langle u_\alpha'(0),u_\alpha'(s)\right\rangle ds-\int_0^t g_\alpha(s)\left\langle A^{1/2}u_0,A^{1/2}(u_\alpha-g_\alpha*u_\alpha)(s)\right\rangle ds$$

$$+\int_0^t\left\langle f_\alpha(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\right\rangle ds+2\alpha\int_0^t\left\langle u_\alpha'(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\right\rangle ds$$

$$-\alpha^2\int_0^t\left\langle u_\alpha(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\right\rangle ds. \tag{4.20}$$

We must evaluate the last two terms on the right-hand side. By means of (3.15) with $v=u_\alpha'$ we obtain

$$2\int_0^t\left\langle u_\alpha'(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\right\rangle ds$$

$$\leqslant\int_0^t\left\|u_\alpha'(s)\right\|^2 ds+\int_0^t\left\|u_\alpha'(s)-g_\alpha*u_\alpha'(s)\right\|^2 ds$$

$$\leqslant\int_0^t\left\|u_\alpha'(s)\right\|^2 ds+2\big(1-G_\alpha(0)\big)\int_0^t\left\langle u_\alpha'(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\right\rangle ds$$

$$+2\|G_\alpha\|_2^2\big\|u_\alpha'(0)\big\|^2+2\int_0^t\big\|G_\alpha*u_\alpha''(s)\big\|^2\,ds,$$

whence, thanks also to (2.13),

$$\begin{aligned}
&2G_\alpha(0)\int_0^t\big\langle u_\alpha'(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\big\rangle\,ds\\
&\quad\leqslant 2\|G_\alpha\|_2^2\big\|u_\alpha'(0)\big\|^2+2\int_0^t\big\|G_\alpha*u_\alpha''(s)\big\|^2\,ds+\int_0^t\big\|u_\alpha'(s)\big\|^2\,ds\\
&\quad\leqslant 2\|G_\alpha\|_2^2\big\|u_\alpha'(0)\big\|^2+\frac{2}{\delta_\alpha}\big(\|G_\alpha\|_1^2+4\|g_\alpha\|_1^2\big)\int_0^t\big\langle G_\alpha*u_\alpha''(s),u_\alpha''(s)\big\rangle\,ds+\int_0^t\big\|u_\alpha'(s)\big\|^2\,ds.
\end{aligned}\tag{4.21}$$

Next, if we use (4.19), then we get

$$\begin{aligned}
&\int_0^t\big\langle u_\alpha(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\big\rangle\,ds\\
&\quad=\langle u_\alpha,u_\alpha-g_\alpha*u_\alpha\rangle\big|_0^t+\int_0^t g_\alpha(s)\big\langle u_0,u_\alpha(s)\big\rangle\,ds\\
&\qquad-\int_0^t\big\langle u_\alpha'(s),u_\alpha(s)-g_\alpha*u_\alpha(s)\big\rangle\,ds\\
&\quad=\langle u_\alpha,u_\alpha-g_\alpha*u_\alpha\rangle\big|_0^t+\int_0^t g_\alpha(s)\big\langle u_0,u_\alpha(s)\big\rangle\,ds+\frac{1}{2}\|u_0\|^2\\
&\qquad-\frac{1}{2}\big\|u_\alpha(t)\big\|^2+\int_0^t\big\langle u_\alpha'(s),g_\alpha*u_\alpha(s)\big\rangle\,ds.
\end{aligned}$$

If now we evaluate the last term on the right-hand side by means of (3.13) with $v=u_\alpha$, we have

$$\begin{aligned}
&\int_0^t\big\langle u_\alpha(s),u_\alpha'(s)-g_\alpha*u_\alpha'(s)\big\rangle\,ds\\
&\quad=\langle u_\alpha,u_\alpha-g_\alpha*u_\alpha\rangle\big|_0^t+\frac{1+G_\alpha(0)}{2}\|u_0\|^2-G_\alpha(t)\big\langle u_0,u_\alpha(t)\big\rangle\\
&\qquad-\frac{1-G_\alpha(0)}{2}\big\|u_\alpha(t)\big\|^2-\int_0^t\big\langle G_\alpha*u_\alpha'(s),u_\alpha'(s)\big\rangle\,ds.
\end{aligned}\tag{4.22}$$

Now, thanks to (3.14), we obtain

$$\begin{aligned}&\big\langle u_\alpha(t), u_\alpha(t) - g_\alpha * u_\alpha(t)\big\rangle - G_\alpha(t)\big\langle u_0, u_\alpha(t)\big\rangle \\ &\quad = \big(1 - G_\alpha(0)\big)\big\|u_\alpha(t)\big\|^2 + \big\langle u_\alpha(t), G_\alpha * u'_\alpha(t)\big\rangle,\end{aligned}$$

so (4.22) can be written in the form

$$\begin{aligned}\int_0^t \big\langle u_\alpha(s), u'_\alpha(s) - g_\alpha * u'_\alpha(s)\big\rangle\,ds &= \frac{G_\alpha(0)-1}{2}\|u_0\|^2 + \frac{1-G_\alpha(0)}{2}\big\|u_\alpha(t)\big\|^2 \\ &\quad + \big\langle u_\alpha(t), G_\alpha * u'_\alpha(t)\big\rangle - \int_0^t \big\langle G_\alpha * u'_\alpha(s), u'_\alpha(s)\big\rangle\,ds. \qquad (4.23)\end{aligned}$$

Put (4.21) and (4.23) into (4.20), to get

$$\begin{aligned}&\frac{1-G_\alpha(0)}{2}\big\|u'_\alpha(t)\big\|^2 + \int_0^t \big\langle G_\alpha * u''_\alpha(s), u''_\alpha(s)\big\rangle\,ds + \frac{1}{2}\big\|A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(t)\big\|^2 \\ &\quad + \alpha^2 \frac{1-G_\alpha(0)}{2}\big\|u_\alpha(t)\big\|^2 \\ &\leqslant \frac{1+G_\alpha(0)}{2}\big\|u'_\alpha(0)\big\|^2 + \frac{1}{2}\big\|A^{1/2}u_0\big\|^2 + \alpha\frac{2\|G_\alpha\|_2^2}{G_\alpha(0)}\big\|u'_\alpha(0)\big\|^2 \\ &\quad + \alpha^2\frac{1-G_\alpha(0)}{2}\|u_0\|^2 - G_\alpha(t)\big\langle u'_\alpha(0), u'_\alpha(t)\big\rangle - \int_0^t g_\alpha(s)\big\langle u'_\alpha(0), u'_\alpha(s)\big\rangle\,ds \\ &\quad - \int_0^t g_\alpha(s)\big\langle A^{1/2}u_0, A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(s)\big\rangle\,ds + \int_0^t \big\langle f_\alpha(s), \big(u'_\alpha - g_\alpha * u'_\alpha\big)(s)\big\rangle\,ds \\ &\quad + 2\alpha\frac{\|G_\alpha\|_1^2 + 4\|g_\alpha\|_1^2}{G_\alpha(0)\delta_\alpha}\int_0^t \big\langle G_\alpha * u''_\alpha(s), u''_\alpha(s)\big\rangle\,ds + \frac{\alpha}{G_\alpha(0)}\int_0^t \big\|u'_\alpha(s)\big\|^2\,ds \\ &\quad - \alpha^2\big\langle u_\alpha(t), G_\alpha * u'_\alpha(t)\big\rangle + \alpha^2\int_0^t \big\langle G_\alpha * u'_\alpha(s), u'_\alpha(s)\big\rangle\,ds. \qquad (4.24)\end{aligned}$$

Thanks to (4.8), (4.6), and (4.5), for $\alpha \leqslant \alpha_0/2$ we have

$$\frac{\|G_\alpha\|_1^2 + 4\|g_\alpha\|_1^2}{G_\alpha(0)\delta_\alpha} \leqslant \frac{4(1+\alpha_0^2)(1+\alpha_0)^4}{\alpha_0^2\delta_{\alpha_0}^2}\|g_{\alpha_0}\|_1^2 \leqslant \frac{1}{2\alpha_1},$$

and by (2.11)

$$-\langle u_\alpha(t), G_\alpha * u'_\alpha(t)\rangle$$
$$\leqslant \frac{1-G_\alpha(0)}{4}\|u_\alpha(t)\|^2 + \frac{2G_\alpha(0)}{1-G_\alpha(0)}\int_0^t \langle G_\alpha * u'_\alpha(s), u'_\alpha(s)\rangle\,ds,$$

so, in view also of (4.13), from (4.24) we get

$$\frac{1-G_\alpha(0)}{2}\|u'_\alpha(t)\|^2 + \int_0^t \langle G_\alpha * u''_\alpha(s), u''_\alpha(s)\rangle\,ds + \frac{1}{2}\|A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(t)\|^2$$
$$+\alpha^2\frac{1-G_\alpha(0)}{4}\|u_\alpha(t)\|^2$$
$$\leqslant C_2 + \frac{2\alpha^2}{1-G_\alpha(0)}H_{\alpha,1}(t) - G_\alpha(t)\langle u'_\alpha(0), u'_\alpha(t)\rangle$$
$$-\int_0^t g_\alpha(s)\langle u'_\alpha(0), u'_\alpha(s)\rangle\,ds - \int_0^t g_\alpha(s)\langle A^{1/2}u_0, A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(s)\rangle\,ds$$
$$+\int_0^t \langle f_\alpha(s), (u'_\alpha - g_\alpha * u'_\alpha)(s)\rangle\,ds + \frac{\alpha}{\alpha_1}\int_0^t \langle G_\alpha * u''_\alpha(s), u''_\alpha(s)\rangle\,ds$$
$$+\alpha\left(\frac{1}{G_\alpha(0)} + \frac{4\alpha^2}{1-G_\alpha(0)}\|A^{-1/2}\|^2\right)\int_0^t \|u'_\alpha(s)\|^2\,ds,$$

for some $C_2 > 0$. Finally, the term $\frac{\alpha}{\alpha_1}\int_0^t \langle G_\alpha * u''_\alpha(s), u''_\alpha(s)\rangle\,ds$ can be absorbed by the similar term on the left-hand side, to deduce (4.16) and (4.17) for $\alpha \leqslant \alpha_1 - \varepsilon_0$. □

**Lemma 3.** *Set*

$$\alpha_2 := \frac{\delta_{\alpha_0}}{4(1+\alpha_0)^2}\left(\|A^{-1/2}\|^2 + \frac{(1+\alpha_0)^2}{\delta_{\alpha_0}} + \frac{8\alpha_1^2\|A^{-1/2}\|^2}{1-G(0)}\right)^{-1} \wedge \frac{\alpha_1}{2} \wedge \eta_0, \tag{4.25}$$

*then for any $\alpha \in [0, \alpha_2 - \varepsilon_0]$, $0 < \varepsilon_0 < \alpha_2$, and $t \geqslant 0$ we have*

$$\frac{1-G_\alpha(0)}{2}\|u'_\alpha(t)\|^2 + \frac{1}{2}\|A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(t)\|^2 + \alpha^2\frac{1-G_\alpha(0)}{2}\|u_\alpha(t)\|^2 \leqslant C_3, \tag{4.26}$$
$$\int_0^t \|u'_\alpha(\tau)\|^2\,d\tau \leqslant C_4, \tag{4.27}$$

*where $C_3, C_4 > 0$.*

**Proof.** First, we estimate the term $\int_0^t \|u'_\alpha(\tau)\|^2\,d\tau$ by applying inequality (4.12). Indeed, in view of (4.5), (4.13) and (4.16), we obtain, for $\alpha \leqslant \alpha_1/2$ and $t \geqslant 0$,

$$\int_0^t \|u'_\alpha(\tau)\|^2\,d\tau \leqslant \|u'_\alpha(0)\|^2 + \frac{2(1+\alpha_0)^2}{\delta_{\alpha_0}}(C_1+2C_2) + \frac{2(1+\alpha_0)^2}{\delta_{\alpha_0}}\big(H_{\alpha,1}(t)+2H_{\alpha,2}(t)\big) + \frac{\alpha}{\alpha_2}\int_0^t \|u'_\alpha(\tau)\|^2\,d\tau,$$

whence for $\alpha \leqslant \alpha_2 - \varepsilon_0$ the inequality

$$\int_0^t \|u'_\alpha(\tau)\|^2\,d\tau \leqslant \frac{\alpha_2}{\epsilon_0}\|u'_\alpha(0)\|^2 + \frac{2\alpha_2(1+\alpha_0)^2}{\delta_{\alpha_0}\epsilon_0}(C_1+2C_2) + \frac{2\alpha_2(1+\alpha_0)^2}{\delta_{\alpha_0}\epsilon_0}\big(H_{\alpha,1}(t)+2H_{\alpha,2}(t)\big) \tag{4.28}$$

holds true. Moreover, if we put (4.28) into (4.17), we have, taking into account (4.14) and (4.18),

$$\begin{aligned}
&\frac{1-G_\alpha(0)}{2}\|u'_\alpha(t)\|^2 + \frac{1}{2}\|A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(t)\|^2 + \alpha^2\frac{1-G_\alpha(0)}{2}\|u_\alpha(t)\|^2 \\
&\leqslant \varepsilon\big(\|u'_\alpha(t)\|^2 + \alpha^2\|u_\alpha(t)\|^2\big) + C_\varepsilon + C\int_0^t \big(\|f_{\eta_0}(s)\| + |g_{\alpha_0}(s)|\big)\|u'_\alpha(s)\|^2\,ds \\
&\quad + C\int_0^t |g_{\alpha_0}(s)|\,\|A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(s)\|^2\,ds + C\int_0^t |g_{\alpha_0}(s)|\,\|u_\alpha(s)\|^2\,ds \\
&\quad + C\int_0^t \|f_{\eta_0}(s)\|\,\|u'_\alpha(s) - g_\alpha * u'_\alpha(s)\|\,ds,
\end{aligned}$$

for any $\varepsilon > 0$ and some $C_\varepsilon, C > 0$. By using Gronwall Lemma we get

$$\begin{aligned}
&\frac{1-G_\alpha(0)}{2}\|u'_\alpha(t)\|^2 + \frac{1}{2}\|A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(t)\|^2 \\
&\quad + \alpha^2\frac{1-G_\alpha(0)}{2}\|u_\alpha(t)\|^2 \leqslant C + C\int_0^t \|f_{\eta_0}(s)\|\,\|u'_\alpha(s) - g_\alpha * u'_\alpha(s)\|\,ds,
\end{aligned}$$

for some $C > 0$. Since for any $T > 0$

$$\|u'_\alpha - g_\alpha * u'_\alpha\|_{\infty,T} \leqslant \big(1 + \|g_{\alpha_0}\|_1\big)\|u'_\alpha\|_{\infty,T},$$

we have

$$\begin{aligned}
&\frac{1-G_\alpha(0)}{2}\|u'_\alpha(t)\|^2 + \frac{1}{2}\|A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(t)\|^2 + \alpha^2\frac{1-G_\alpha(0)}{2}\|u_\alpha(t)\|^2 \\
&\leqslant \varepsilon\|u'_\alpha\|^2_{\infty,T} + C_\varepsilon,
\end{aligned}$$

for any $t \in [0,T]$, so we are able to control the term $\|u'_\alpha\|^2_{\infty,T}$ by some positive constant independent of $T$, and hence (4.26) follows.

Finally, functions $H_{\alpha,1}(t)$ and $H_{\alpha,2}(t)$ in (4.28) can be bounded thanks to (4.26), so one obtains (4.27). □

**Lemma 4.** *For any $\alpha \in [0, \alpha_2 - \varepsilon_0]$, $0 < \varepsilon_0 < \alpha_2$, and $t \geqslant 0$ we have*

$$\int_0^t \left\| A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(s) \right\|^2 ds \leqslant C_5, \tag{4.29}$$

*where $C_5 > 0$.*

**Proof.** We will follow the same method as in the proofs of Lemmas 1 and 2, now by using $u_\alpha(t) - g_\alpha * u_\alpha(t)$ as a multiplier. If we take the scalar product of both sides of the equation in (4.11) with $u_\alpha(t) - g_\alpha * u_\alpha(t)$ and integrate from 0 to $t$, we obtain

$$\begin{aligned}
&\int_0^t \left\| A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(s) \right\|^2 ds \\
&\quad = \int_0^t \big\langle f_\alpha(s), u_\alpha(s) - g_\alpha * u_\alpha(s) \big\rangle ds - \big\langle u'_\alpha, u_\alpha - g_\alpha * u_\alpha \big\rangle \Big|_0^t \\
&\qquad + \int_0^t \big\langle u'_\alpha(s), u'_\alpha(s) - g_\alpha * u'_\alpha(s) \big\rangle ds - \int_0^t g_\alpha(s) \big\langle u'_\alpha(s), u_0 \big\rangle ds \\
&\qquad + 2\alpha \int_0^t \big\langle u'_\alpha(s), u_\alpha(s) - g_\alpha * u_\alpha(s) \big\rangle ds - \alpha^2 \int_0^t \big\langle u_\alpha(s), u_\alpha(s) - g_\alpha * u_\alpha(s) \big\rangle ds.
\end{aligned} \tag{4.30}$$

To estimate the last term on the right-hand side we use (3.15) and get

$$\begin{aligned}
&-\big(1 - G_\alpha(0)\big) \int_0^t \big\langle u_\alpha(s), u_\alpha(s) - g_\alpha * u_\alpha(s) \big\rangle ds \\
&\quad \leqslant -\frac{1}{2} \int_0^t \left\| u_\alpha(s) - g_\alpha * u_\alpha(s) \right\|^2 ds + \|G_\alpha\|_2^2 \|u_0\|^2 + \int_0^t \left\| G_\alpha * u'_\alpha(s) \right\|^2 ds.
\end{aligned}$$

If we put the above inequality into (4.30), we have

$$\begin{aligned}
&\int_0^t \left\| A^{1/2}(u_\alpha - g_\alpha * u_\alpha)(s) \right\|^2 ds + \frac{\alpha^2}{2(1 - G_\alpha(0))} \int_0^t \left\| (u_\alpha - g_\alpha * u_\alpha)(s) \right\|^2 ds \\
&\quad \leqslant \int_0^t \left\| f_{\eta_0}(s) \right\| \left\| u_\alpha(s) - g_\alpha * u_\alpha(s) \right\| ds - \big\langle u'_\alpha, u_\alpha - g_\alpha * u_\alpha \big\rangle \Big|_0^t
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \langle u_\alpha'(s), u_\alpha'(s) - g_\alpha * u_\alpha'(s)\rangle\,ds - \int_0^t g_\alpha(s)\langle u_\alpha'(s), u_0\rangle\,ds\\
&+\frac{\alpha^2\|G_\alpha\|_2^2}{1-G_\alpha(0)}\|u_0\|^2 + \frac{\alpha^2}{1-G_\alpha(0)}\int_0^t \|G_\alpha * u_\alpha'(s)\|^2\,ds\\
&+4\big(1-G_\alpha(0)\big)\int_0^t \|u_\alpha'(s)\|^2\,ds + \frac{\alpha^2}{4(1-G_\alpha(0))}\int_0^t \|u_\alpha(s) - g_\alpha * u_\alpha(s)\|^2\,ds.
\end{aligned}
$$

To conclude, in view of (4.26) and (4.27), the desired estimate (4.29) follows. □

**Proof of Theorem 4.1** (*continued*). First, by Remark 4.2-3,4 we can apply Corollary 2.12 to give a uniform estimate for $\|A^{1/2}u_\alpha(t)\|$. Indeed, since the resolvent kernel of $g_\alpha$ is given by $r_\alpha(t) = e^{\alpha t}r(t)$ ($r$ is the resolvent kernel of $g$) thanks to (2.19) we have

$$
\|A^{1/2}u_\alpha\|_\infty \leqslant \big(1 + \|r_{\alpha_0}\|_1\big)\|A^{1/2}(u_\alpha - g_\alpha * u_\alpha)\|_\infty, \tag{4.31}
$$

whence, by using (4.26) and (4.7) we get, for any $t \geqslant 0$,

$$
\|A^{1/2}u_\alpha(t)\|^2 + \|u_\alpha'(t)\|^2 \leqslant C\big(\|A^{1/2}u_0\| + \|u_1\| + \|f_{\eta_0}\|_1\big), \tag{4.32}
$$

where $C(R)$ is a positive, increasing, upper semicontinuous function such that $C(0) = 0$. Reasoning as in (4.31), by (2.19) and (4.29) we can show a uniform estimate for $\int_0^\infty \|A^{1/2}u_\alpha(s)\|^2\,ds$ as well. Therefore, thanks also to (4.27), we get

$$
\int_0^\infty \|A^{1/2}u_\alpha(s)\|^2\,ds + \int_0^\infty \|u_\alpha'(s)\|^2\,ds \leqslant C\big(\|A^{1/2}u_0\| + \|u_1\| + \|f_{\eta_0}\|_1\big). \tag{4.33}
$$

In the nonlinear case $F \neq 0$ the function $u_\alpha(t) = e^{\alpha t}u(t)$ fulfills the equation

$$
u_\alpha''(t) - 2\alpha u_\alpha'(t) + \alpha^2 u_\alpha(t) + Au_\alpha(t) - g_\alpha * Au_\alpha(t) = e^{\alpha t}\nabla F\big(u(t)\big) + f_\alpha(t). \tag{4.34}
$$

Now, we can repeat the same argumentations used in the linear case: as in Lemmas 1, 2 and 4 we multiply Eq. (4.34) by the three multipliers $A^{-1}u_\alpha'(t)$, $u_\alpha'(t) - g_\alpha * u_\alpha'(t)$ and $u_\alpha(t) - g_\alpha * u_\alpha(t)$ and then integrate from 0 to $t$. So, to complete our analysis, we must only evaluate the integrals related to the nonlinear term $e^{\alpha t}\nabla F(u(t))$.

For the first multiplier $A^{-1}u_\alpha'(t)$, in view of (4.2) and Corollary 3.7, we have for any $\varepsilon > 0$

$$
\begin{aligned}
\int_0^t e^{\alpha s}\langle \nabla F\big(u(s)\big), A^{-1}u_\alpha'(s)\rangle\,ds &\leqslant \int_0^t \psi\big(\|A^{1/2}u(s)\|\big)\|A^{1/2}u_\alpha(s)\|\|A^{-1}u_\alpha'(s)\|\,ds\\
&\leqslant \varepsilon\int_0^t \big(\|A^{1/2}u_\alpha(s)\|^2 + \|u_\alpha'(s)\|^2\big)\,ds,
\end{aligned}
$$

by taking $\|A^{1/2}u_0\| + \|u_1\| + \|f_{\eta_0}\|_1 < \rho_\varepsilon$, with $\rho_\varepsilon \leqslant \rho_0$. In a similar way, we obtain

$$\int_0^t e^{\alpha s}\big\langle \nabla F\big(u(s)\big), u'_\alpha(s) - g_\alpha * u'_\alpha(s)\big\rangle ds \leqslant \varepsilon \int_0^t \big(\big\|A^{1/2}u_\alpha(s)\big\|^2 + \big\|u'_\alpha(s)\big\|^2\big)\, ds,$$

$$\int_0^t e^{\alpha s}\big\langle \nabla F\big(u(s)\big), u_\alpha(s) - g_\alpha * u_\alpha(s)\big\rangle ds \leqslant \varepsilon \int_0^t \big\|A^{1/2}u_\alpha(s)\big\|^2\, ds.$$

Therefore, thanks to the above three estimates, we can show that there exists $\rho_1 \leqslant \rho_0$ such that for $\|A^{1/2}u_0\| + \|u_1\| + \|f_{\eta_0}\|_1 < \rho_1$ (4.32) and (4.33) also hold in the nonlinear case. So, by Remark 4.2-8 and recalling that $u(t) = e^{-\alpha t}u_\alpha(t)$ and $u'(t) = e^{-\alpha t}u'_\alpha(t) - \alpha e^{-\alpha t}u_\alpha(t)$, we obtain (4.3) and (4.4) for strong solutions.

Finally, an approximation argument suffices to extend such a conclusion to mild solutions. □

In conclusion, we are able to recover the asymptotic behavior of the energy, thanks also to some results for the linear case (see Appendix A).

**Theorem 4.3.** *Assume* (H1) *and* (H2)*. Then, there exist positive numbers* $\rho_1 \leqslant \rho_0$[3] *and* $C$ *such that, for any* $(u_0, u_1) \in D(A^{1/2}) \times X$ *and any* $f \in L^1(0,\infty; X)$*, satisfying*

$$\big\|A^{1/2}u_0\big\| + \|u_1\| + \|f\|_1 < \rho_1,$$

*the energy* $E_u(t)$ *of the mild solution* $u$ *of* (4.1) *belongs to* $L^1(0,\infty)$ *and converges to* 0 *as* $t \to \infty$.

*Moreover, one can take* $\rho_1 = \infty$ *if* $F \equiv 0$.

**Proof.** First, the energy $E_u(t)$ belongs to $L^1(0,\infty)$ by means of Lemmas 1–4 with $\alpha = 0$. Moreover, since $\nabla F(u) \in L^2(0,\infty; X)$, in view of Theorem A.3 $E_u(t)$ is uniformly continuous on $[0,\infty)$, so our claim follows. □

**Remark 4.4.** Notice that, assuming that $t \mapsto \int_t^\infty g(s)\,ds$ is strongly positive definite and $f \in L^2(0,\infty; X)$, we can prove a global existence result similar to Theorem 3.6, by using the same argumentations as in the proof of Theorem A.2. So, the previous theorems also hold in the case $f \in L^2(0,\infty; X)$.

## 5. Positive definite kernels

As we have just seen, our main stability result holds true for a certain class of strongly positive definite kernels. Such a class is described, however, in a rather abstract way by Assumptions (H2). In this section, we shall give easy-to-check sufficient conditions for a given kernel $g$ to satisfy the positivity hypothesis (H2)-1. Moreover, the conditions we discuss below will allow us to compare Theorem 4.1 with other related results in the literature, namely [1,30]. Finally, we shall discuss a few concrete examples of strongly positive kernels.

We begin by showing that any exponentially decaying absolutely continuous kernel satisfies Assumption (H2)-1. Therefore, in the case of exponential decay, we find a stability result like in [1] under weaker regularity assumptions for $g$.

**Proposition 5.1.** *Let* $g : [0,\infty) \to [0,\infty)$ *be a locally absolutely continuous function with* $g(0) > 0$ *and for some* $k > 0$

[3] $\rho_0$ is given by Theorem 3.6.

$$g'(t) \leqslant -kg(t), \quad \textit{for a.e. } t \geqslant 0. \tag{5.1}$$

*Then, for all* $0 < \alpha_0 < k$, $g_{\alpha_0} \in L^1(0,\infty)$ *and the kernel* $t \mapsto \int_t^\infty g_{\alpha_0}(s)\,ds$ *is strongly positive.*

**Proof.** First of all, we observe that in virtue of (5.1) we have

$$g(t) \leqslant g(0)e^{-kt}, \quad t \geqslant 0,$$

so, taking $\alpha_0 < k$, we get $g_{\alpha_0} = e^{\alpha_0 \cdot} g \in L^1(0,\infty)$. To prove our claim, we apply Theorem 2.1 to the function $G_{\alpha_0}(t) = \int_t^\infty g_{\alpha_0}(s)\,ds$ with $\alpha_0 < k$. Indeed, $G_{\alpha_0}(t) \geqslant 0$ and $G'_{\alpha_0}(t) = -e^{\alpha_0 t} g(t) \leqslant 0$, with $G'_{\alpha_0}(0) = -g(0) < 0$. Moreover, if we use again (5.1), we obtain

$$G''_{\alpha_0}(t) = -\alpha_0 e^{\alpha_0 t} g(t) - e^{\alpha_0 t} g'(t) \geqslant -e^{\alpha_0 t}(\alpha_0 - k) g(t) \geqslant 0.$$

Finally, by Theorem 2.1 we can conclude that $G_{\alpha_0}(t)$ is strongly positive. □

Now, we give sufficient conditions for a given kernel $g$ ensuring that the positivity hypothesis (H2)-1 is satisfied.

**Proposition 5.2.** *Let $g$ be a locally absolutely continuous function on* $(0,\infty)$ *satisfying the following*:

(1) *there exists* $\alpha_0 > 0$ *such that* $\int_0^\infty e^{\alpha_0 t}(|g(t)| + t|g'(t)|)\,dt < \infty$;
(2) *the kernel* $G(t) = \int_t^\infty g(s)\,ds$ *is strongly positive.*

*Then, there exists* $\alpha \in (0,\alpha_0)$ *such that the kernel* $G_\alpha(t) = \int_t^\infty g_\alpha(s)\,ds$ *is strongly positive.*

**Proof.** To begin with, thanks to Remark 4.2-5, we have $G_\alpha \in L^1(0,\infty)$ for any $\alpha \in (0,\alpha_0)$. So, in view of Corollary 2.3(b) it is enough to prove that there exist $\alpha \in (0,\alpha_0)$ and $\delta > 0$ such that

$$H_\alpha(\omega) := \frac{1+\omega^2}{\omega} \int_0^\infty e^{\alpha s} g(s) \sin(\omega s)\,ds \geqslant \delta \quad \text{for any } \omega > 0. \tag{5.2}$$

First, we note that by Corollary 2.3(a) there exists $\delta_0 > 0$ such that

$$H_0(\omega) := \frac{1+\omega^2}{\omega} \int_0^\infty g(s) \sin(\omega s)\,ds \geqslant \delta_0 \quad \text{for any } \omega > 0. \tag{5.3}$$

For any $\omega \in (0,1]$ we have

$$\begin{aligned}
\big|H_\alpha(\omega) - H_0(\omega)\big| &\leqslant \frac{1+\omega^2}{\omega} \int_0^\infty \big(e^{\alpha s} - 1\big)\big|g(s)\big|\big|\sin(\omega s)\big|\,ds \\
&\leqslant 2\int_0^\infty \big(e^{\alpha s} - 1\big)s\big|g(s)\big| \frac{|\sin(\omega s)|}{\omega s}\,ds \\
&\leqslant 2\int_0^\infty \big(e^{\alpha s} - 1\big)s\big|g(s)\big|\,ds.
\end{aligned}$$

Now, we observe that for any $\alpha \in (0, \alpha_0/2]$ we have

$$\int_0^\infty e^{\alpha s} s\big|g(s)\big|\,ds \leqslant \frac{2}{\alpha_0}\int_0^\infty e^{\alpha_0 s}\big|g(s)\big|\,ds, \tag{5.4}$$

so by the dominated convergence theorem we get

$$\int_0^\infty \big(e^{\alpha s}-1\big)s\big|g(s)\big|\,ds \underset{\alpha\to 0^+}{\longrightarrow} 0.$$

Therefore, in view of (5.3) there exists $\overline{\alpha} \in (0, \alpha_0/2]$ such that for any $\alpha \in (0, \overline{\alpha}]$

$$H_\alpha(\omega) \geqslant \delta_0 - 2\int_0^\infty \big(e^{\alpha s}-1\big)s\big|g(s)\big|\,ds \geqslant \frac{\delta_0}{2} \quad \text{for any } \omega \in (0,1]. \tag{5.5}$$

To study the case $\omega > 1$, for any $\alpha \in [0, \alpha_0/2]$ we introduce the function

$$h(\omega,\alpha) := \omega\int_0^\infty e^{\alpha s} g(s)\sin(\omega s)\,ds,$$

whence

$$H_\alpha(\omega) - H_0(\omega) = \frac{1+\omega^2}{\omega^2}\big[h(\omega,\alpha) - h(\omega,0)\big]. \tag{5.6}$$

First,

$$h_\alpha(\omega,\alpha) = \omega\int_0^\infty s e^{\alpha s} g(s)\sin(\omega s)\,ds.$$

We note that the function $s \mapsto s e^{\alpha s} g(s)$ belongs to $W^{1,1}(0,\infty)$, and hence $\lim_{s\to 0^+} s e^{\alpha s} g(s) = 0$ and $\lim_{s\to\infty} s e^{\alpha s} g(s) = 0$. Therefore,

$$\begin{aligned}
h_\alpha(\omega,\alpha) &= \big\{s e^{\alpha s} g(s)\big[1-\cos(\omega s)\big]\big\}_0^\infty - \int_0^\infty \big(s e^{\alpha s} g(s)\big)'\big[1-\cos(\omega s)\big]\,ds \\
&= -\int_0^\infty e^{\alpha s} g(s)\big[1-\cos(\omega s)\big]\,ds - \alpha\int_0^\infty s e^{\alpha s} g(s)\big[1-\cos(\omega s)\big]\,ds \\
&\quad - \int_0^\infty s e^{\alpha s} g'(s)\big[1-\cos(\omega s)\big]\,ds,
\end{aligned}$$

whence, using also (5.4), for any $\alpha \in (0, \alpha_0/2]$ we get

$$\left|h_\alpha(\omega,\alpha)\right| \leqslant 4\int_0^\infty e^{\alpha_0 s}\left|g(s)\right| ds + 2\int_0^\infty s e^{\alpha_0 s}\left|g'(s)\right| ds =: C.$$

Then,

$$\left|h(\omega,\alpha) - h(\omega,0)\right| \leqslant \alpha\int_0^1 \left|h_\alpha(\omega,\alpha t)\right| dt \leqslant C\alpha, \quad \forall\omega > 0.$$

From (5.6), if we use the above estimate and (5.3), we obtain for any $\alpha \leqslant (\delta_0/4C) \wedge \alpha_0/2$

$$H_\alpha(\omega) \geqslant H_0(\omega) - \frac{1+\omega^2}{\omega^2}C\alpha \geqslant \delta_0 - 2C\alpha \geqslant \frac{\delta_0}{2} \quad \text{for any } \omega > 1. \tag{5.7}$$

Finally, putting together (5.5) and (5.7), our claim (5.2) follows. □

Next, it is useful to single out specific situations where the assumptions of Proposition 5.2 are satisfied. This is the purpose of our next result.

**Lemma 5.3.** *Let $g : (0,\infty) \to (0,\infty)$ be a locally absolutely continuous function such that $g \not\equiv 0$, $g'(t) \leqslant 0$ for a.e. $t \geqslant 0$, and*

$$\int_0^\infty g(t)e^{\alpha_0 t}\, dt < \infty \tag{5.8}$$

*for some $\alpha_0 > 0$. Then,*

(1) *for any $\alpha \in [0,\alpha_0)$ we have $\int_0^\infty t|g'(t)|e^{\alpha t}\, dt < \infty$;*
(2) *the kernel $t \mapsto \int_t^\infty g(s)\, ds$ is strongly positive.*

**Proof.** First of all, we note that (5.8) yields $\int_0^\infty tg(t)e^{\alpha t}\, dt < \infty$, for any $\alpha \in [0,\alpha_0)$. Therefore by Lemma 2.14 we have

$$\int_0^\infty t\left|g'(t)\right|e^{\alpha t}\, dt = -\int_0^\infty tg'(t)e^{\alpha t}\, dt = \int_0^\infty (1+\alpha t)g(t)e^{\alpha t}\, dt,$$

so (1) holds true.

Moreover, by means of Theorem 2.1 we have that $t \mapsto \int_t^\infty g(s)\, ds$ is strongly positive. □

**Remark 5.4.** In [30] an exponential stability result is obtained for an integro-differential equation of the same type as (3.1), under exactly the same hypotheses as in Lemma 5.3 for $g$. So, in view of Lemma 5.3 and Proposition 5.2, we see that one can derive the result of [30] from Theorem 4.1.

Our next proposition is useful to study the concrete examples we will discuss at the end of this section.

**Proposition 5.5.** *Assume that*

(1) $g \in L^1(0,\infty)$ *such that* $\int_0^\infty t|g(t)|\, dt < \infty$;
(2) $\int_0^\infty g(t)\sin(\omega t)\, dt > 0\ \forall\omega > 0$;

(3) $\int_0^\infty tg(t)\,dt > 0$;
(4) $g \in \bigcap_{\varepsilon>0} W^{1,1}(\varepsilon,\infty)$ *and there exists* $\varepsilon_0 > 0$ *such that* $g(t) > 0$, $g'(t) \leqslant 0$ *for a.e.* $t \in (0,\varepsilon_0]$.

*Then, the kernel* $G(t) = \int_t^\infty g(s)\,ds$ *is strongly positive.*

**Proof.** Let us observe that, being the function $t \mapsto tg(t)$ in $L^1(0,\infty)$, so also the function $G$ is in $L^1(0,\infty)$. Therefore, to apply Corollary 2.3(b), it suffices to verify (2.8), that is, for some $\delta > 0$,

$$H(\omega) := \frac{1+\omega^2}{\omega} \int_0^\infty g(t)\sin(\omega t)\,dt \geqslant \delta, \quad \text{for any } \omega > 0. \tag{5.9}$$

Since the function $H$ is continuous and strictly positive on $(0,\infty)$, it is enough to verify that

$$\lim_{\omega\to 0^+} H(\omega) > 0, \tag{5.10}$$

$$\liminf_{\omega\to\infty} H(\omega) > 0. \tag{5.11}$$

First, (5.10) holds true, because if we use the dominated convergence theorem we have

$$\lim_{\omega\to 0^+} H(\omega) = \int_0^\infty tg(t)\,dt > 0.$$

Moreover, to prove (5.11) we note that

$$\omega\int_0^\infty g(t)\sin(\omega t)\,dt = \omega\int_0^{\varepsilon_0} g(t)\sin(\omega t)\,dt + \omega\int_{\varepsilon_0}^\infty g(t)\sin(\omega t)\,dt.$$

By (2.25) we have

$$\omega\int_0^{\varepsilon_0} g(t)\sin(\omega t)\,dt = g(\varepsilon_0)\big(1-\cos(\omega\varepsilon_0)\big) - \int_0^{\varepsilon_0} g'(t)\big(1-\cos(\omega t)\big)\,dt. \tag{5.12}$$

Since $g \in \bigcap_{\varepsilon>0} W^{1,1}(\varepsilon,\infty)$, we have $\lim_{t\to\infty} g(t) = 0$, so an integration by parts yields

$$\omega\int_{\varepsilon_0}^\infty g(t)\sin(\omega t)\,dt = -g(\varepsilon_0)\big(1-\cos(\omega\varepsilon_0)\big) - \int_{\varepsilon_0}^\infty g'(t)\big(1-\cos(\omega t)\big)\,dt. \tag{5.13}$$

Putting together (5.12) and (5.13), we obtain

$$\begin{aligned}\omega\int_0^\infty g(t)\sin(\omega t)\,dt &= -\int_0^{\varepsilon_0} g'(t)\big(1-\cos(\omega t)\big)\,dt - \int_{\varepsilon_0}^\infty g'(t)\big(1-\cos(\omega t)\big)\,dt \\ &\geqslant -\int_{\varepsilon_0}^\infty g'(t)\,dt + \int_{\varepsilon_0}^\infty g'(t)\cos(\omega t)\,dt = g(\varepsilon_0) + \int_{\varepsilon_0}^\infty g'(t)\cos(\omega t)\,dt,\end{aligned}$$

whence, in virtue of the Riemann–Lebesgue lemma, we get

$$\liminf_{\omega\to\infty} H(\omega) \geqslant g(\varepsilon_0) > 0.$$

So, the proof is complete. □

**Remark 5.6.** (i) If a kernel $g$ verifies $\int_0^\infty t|g(t)|\,dt < \infty$, for any $\omega > 0$ $\int_0^\infty g(t)\sin(\omega t)\,dt > 0$ and $\int_0^\infty tg(t)\,dt = 0$, by Proposition 2.2(b) $G$ is positive definite, but not strongly positive, since

$$\inf_{\omega>0} \frac{1+\omega^2}{\omega} \int_0^\infty g(t)\sin(\omega t)\,dt = 0.$$

(ii) We note that $G$ is not positive definite when $g$ verifies $\int_0^\infty t|g(t)|\,dt < \infty$ and $\int_0^\infty tg(t)\,dt < 0$. Indeed,

$$\lim_{\omega\to 0^+} \frac{1}{\omega} \int_0^\infty g(t)\sin(\omega t)\,dt = \int_0^\infty tg(t)\,dt < 0$$

and we conclude owing to Proposition 2.2(a).

We want to give examples of kernels $g \in L^1(0,\infty)$ fit to the above theory, that is:

$$\begin{cases} \text{(i)} \quad t \mapsto \int_t^\infty g(s)\,ds \quad \text{is strongly positive,} \\ \text{(ii)} \quad \int_0^\infty g(s)\,ds < 1. \end{cases}$$

**Example 5.7.** Let $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt$ $(s > 0)$ be the Euler gamma function. Set

$$g_{a,b,0}(t) := \frac{1}{\Gamma(1-b)} e^{-at} t^{-b}, \quad t > 0,\ a > 0,\ 0 \leqslant b < 1.$$

Noting that

$$t \mapsto \int_t^\infty g_{a,b,0}(s)\,ds$$

verifies conditions (2.6), by Theorem 2.1 we conclude that property (i) above holds true for $g_{a,b,0}$. As for property (ii), we have

$$\int_0^\infty g_{a,b,0}(s)\,ds = \frac{1}{a^{1-b}} < 1 \quad \text{for all } a > 1.$$

Now, we give some examples of functions with variable sign.

**Example 5.8.** Let us consider

$$g_{a,1/2,c}(t) := \frac{1}{\Gamma(1/2)} e^{-at} t^{-1/2} \cos(ct), \quad t > 0,\ a > 0,\ c \in \mathbb{R}.$$

Then, $G_{a,1/2,c}(t) := \int_t^\infty g_{a,1/2,c}(s)\,ds$ is:

- strongly positive definite if $|c| < \sqrt{3}a$;
- positive definite (but not strongly positive) if $|c| = \sqrt{3}a$;
- not positive definite if $|c| > \sqrt{3}a$.

From the above assertions, that will be proved later in this section, it follows that $g_{a,1/2,c}$ has property (i) if and only if $|c| < \sqrt{3}a$.

As for property (ii), it is easy to observe that, for any $a \geqslant 1$ and $c \neq 0$,

$$\frac{1}{\Gamma(1/2)} \int_0^\infty e^{-at} t^{-1/2} \cos(ct)\,dt < \frac{1}{\Gamma(1/2)} \int_0^\infty e^{-t} t^{-1/2}\,dt = 1.$$

A more complete analysis, however, can be achieved recalling the Laplace transform formula

$$\begin{aligned} \widehat{g}_{0,1/2,0}(z) &:= \frac{1}{\Gamma(1/2)} \int_0^\infty e^{-zt} t^{-1/2}\,dt = \frac{1}{z^{1/2}} \\ &= \frac{1}{\sqrt{2}|z|} \Big[ \big(|z| + \operatorname{Re} z\big)^{1/2} - i \operatorname{sign}(\operatorname{Im} z) \big(|z| - \operatorname{Re} z\big)^{1/2} \Big], \quad z \in \mathbb{C},\ \operatorname{Re} z > 0. \end{aligned} \tag{5.14}$$

Indeed

$$\int_0^\infty g_{a,1/2,c}(t)\,dt = \operatorname{Re} \widehat{g}_{0,1/2,0}(a + ic) = \left( \frac{\sqrt{a^2 + c^2} + a}{2(a^2 + c^2)} \right)^{1/2} < 1$$

if and only if

$$c^2 > \frac{1 + 4a - 8a^2 + \sqrt{1 + 8a}}{8}.$$

To summarize, we conclude that $g_{a,1/2,c}$ satisfies both (i) and (ii) if and only if

$$a > 0 \quad \text{and} \quad \frac{1 + 4a - 8a^2 + \sqrt{1 + 8a}}{8} < c^2 < 3a^2.$$

(See Fig. 1.)

To fully justify the analysis of Example 5.8, let us prove the assertions related to property (i). Notice that, hereafter, we just need to consider the case of $c > 0$. Indeed, the case of $c = 0$ has already been studied in Example 5.7, while, for $c < 0$, the analysis is obtained by a symmetry argument. Let us begin with a preliminary result.

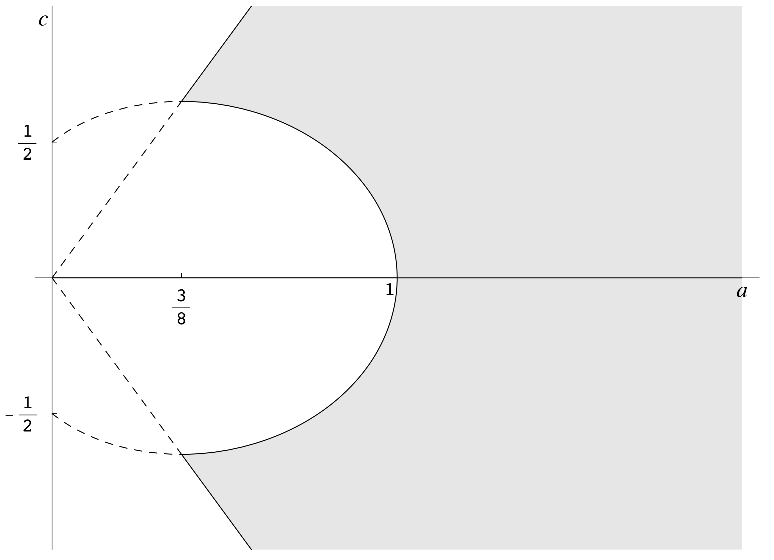


**Fig. 1.** The parameter region where (i) and (ii) hold true.

**Lemma 5.9.** *For any* $0 < c \leqslant \sqrt{3}a$ *we have that*

$$\int_0^\infty g_{a,1/2,c}(t)\sin(\omega t)\,dt > 0 \quad \forall \omega > 0. \tag{5.15}$$

**Proof.** Since

$$\int_0^\infty g_{a,1/2,c}(t)\sin(\omega t)\,dt = -\frac{1}{2}\operatorname{Im}\widehat{g}_{0,1/2,0}\big(a+i(\omega+c)\big) - \frac{1}{2}\operatorname{Im}\widehat{g}_{0,1/2,0}\big(a+i(\omega-c)\big),$$

by (5.14) and standard computations (see, for instance, [9]), one can prove that, for any $\omega > 0$,

$$\begin{aligned}
&\int_0^\infty g_{a,1/2,c}(t)\sin(\omega t)\,dt \\
&\quad = \frac{1}{2\sqrt{2a}}\bigg(\bigg(1+\frac{(\omega+c)^2}{a^2}\bigg)^{-1/2} - \bigg(1+\frac{(\omega+c)^2}{a^2}\bigg)^{-1}\bigg)^{1/2} \\
&\qquad + \frac{1}{2\sqrt{2a}}\operatorname{sign}(\omega-c)\bigg(\bigg(1+\frac{(\omega-c)^2}{a^2}\bigg)^{-1/2} - \bigg(1+\frac{(\omega-c)^2}{a^2}\bigg)^{-1}\bigg)^{1/2}.
\end{aligned} \tag{5.16}$$

Now, set $\overline{\omega} = \omega/a$, $\bar{c} = c/a$ and define

$$j_{\pm}(\overline{\omega}) := \frac{1}{1+(\overline{\omega}\pm\bar{c})^2}.$$

We have

$$\int_0^\infty g_{a,1/2,c}(t)\sin(\omega t)\,dt = \frac{1}{2\sqrt{2a}}\big(j_+(\overline{\omega})^{1/2} - j_+(\overline{\omega})\big)^{1/2}$$
$$+ \frac{1}{2\sqrt{2a}}\operatorname{sign}(\omega - c)\big(j_-(\overline{\omega})^{1/2} - j_-(\overline{\omega})\big)^{1/2}. \tag{5.17}$$

It is clear that (5.15) holds true for any $\omega \geqslant c$. So, suppose $\omega < c$. We want to check that

$$j_+(\overline{\omega})^{1/2} - j_+(\overline{\omega}) > j_-(\overline{\omega})^{1/2} - j_-(\overline{\omega}).$$

The previous inequality is equivalent to

$$\frac{4\overline{\omega}\bar{c}}{\sqrt{1+(\overline{\omega}-\bar{c})^2}\sqrt{1+(\overline{\omega}+\bar{c})^2}} > \frac{4\overline{\omega}\bar{c}}{\sqrt{1+(\overline{\omega}+\bar{c})^2}+\sqrt{1+(\overline{\omega}-\bar{c})^2}},$$

or, after simple computations,

$$\big(\overline{\omega}^2 - \bar{c}^2\big)^2 < 1 + 2\sqrt{1+(\overline{\omega}+\bar{c})^2}\sqrt{1+(\overline{\omega}-\bar{c})^2}. \tag{5.18}$$

If $c \leqslant a$, then (5.18) holds true since, in view of the fact that $\bar{c} = c/a \leqslant 1$,

$$\big|\overline{\omega}^2 - \bar{c}^2\big| = \bar{c}^2 - \overline{\omega}^2 \leqslant 1.$$

Next, suppose $a < c \leqslant \sqrt{3}a$. Then (5.18) trivially holds true for any $\overline{\omega} \geqslant (\bar{c}^2 - 1)^{1/2}$. Therefore, we only need to prove (5.18) for $\overline{\omega} < (\bar{c}^2 - 1)^{1/2}$. For this we note that, if $\overline{\omega}^2 < \bar{c}^2 - 1$, then we can rewrite (5.18) in the form

$$\big(\big(\overline{\omega}^2 - \bar{c}^2\big)^2 - 1\big)^2 < 4\big(1 + 2\big(\overline{\omega}^2 + \bar{c}^2\big) + \big(\overline{\omega}^2 - \bar{c}^2\big)^2\big). \tag{5.19}$$

Let $t = \bar{c}^2 - \overline{\omega}^2$. Then, our claim is equivalent to prove that

$$F(t) := t^4 - 6t^2 + 8t - 16\bar{c}^2 - 3 < 0, \quad \forall 1 < t < \bar{c}^2.$$

Since $F$ is strictly increasing for $t > 1$ and

$$F\big(\bar{c}^2\big) = \bar{c}^8 - 6\bar{c}^4 - 8\bar{c}^2 - 3 \leqslant 0, \quad \forall 1 \leqslant \bar{c} \leqslant \sqrt{3},$$

$F$ must be negative. We have thus proved that (5.15) holds true for all $0 < c \leqslant \sqrt{3}a$. □

**Proposition 5.10.** *For any $|c| < \sqrt{3}a$ $G_{a,1/2,c}$ is strongly positive definite and for $|c| = \sqrt{3}a$ $G_{a,1/2,c}$ is positive definite, but not strongly positive. Moreover, $G_{a,1/2,c}$ is not even positive definite for $|c| > \sqrt{3}a$.*

**Proof.** Thanks to Proposition 5.5 and Lemma 5.9, we only need to evaluate

$$\int_0^\infty t g_{a,1/2,c}(t)\,dt = \frac{1}{\Gamma(1/2)}\operatorname{Re}\int_0^\infty e^{-(a+ic)t}t^{1/2}\,dt$$
$$= \frac{1}{2}\operatorname{Re}\frac{1}{(a+ic)^{3/2}} = \frac{1}{2\sqrt{2}}\frac{c(2a - \sqrt{a^2+c^2})}{(a^2+c^2)^{3/2}(\sqrt{a^2+c^2} - a)^{1/2}}.$$

Therefore, for any $|c| < \sqrt{3}a$ we see that $\int_0^\infty t g_{a,1/2,c}(t)\,dt > 0$, whence $G_{a,1/2,c}$ is strongly positive. The last part of the claim follows by Remark 5.6. □

**Example 5.11.** Let us examine the kernels

$$g(t) := \theta \sum_{n=1}^{\infty} \cos(c_n t) e^{-n^\beta t}, \quad t > 0,\ \beta > 1, \tag{5.20}$$

where the sequence $\{c_n\}$ of real numbers and $\theta > 0$ satisfy:

(i) there exists $C \geqslant 0$ such that $|c_n| \leqslant C \wedge n^\beta$ for any $n \in \mathbb{N}$,
(ii) $\theta \sum_{n=1}^{\infty} \frac{n^\beta}{n^{2\beta} + c_n^2} < 1$.

The special case $c_n = 0$ for any $n \in \mathbb{N}$ has been introduced in polymer dynamics (see, e.g., [15,32]).

Since

$$\left|g(t)\right| \leqslant \theta \sum_{n=1}^{\infty} e^{-n^\beta t} \leqslant \theta \int_0^\infty e^{-s^\beta t}\,ds = t^{-1/\beta}\theta \int_0^\infty e^{-\tau^\beta}\,d\tau, \quad t > 0,$$

we have $g(t) = O(t^{-1/\beta})$ as $t \to 0^+$. Moreover, the kernel $g$ does not have, in general, constant sign. Indeed, taking $c_1 = 1$ and $c_n = 0$ for any $n \geqslant 2$ we have

$$g(t) = \theta\left(\cos t e^{-t} + \sum_{n=2}^{\infty} e^{-n^\beta t}\right), \quad t > 0.$$

Therefore,

$$\begin{aligned} g(\pi) &= \theta\left(-e^{-\pi} + \sum_{n=2}^{\infty} e^{-n^\beta \pi}\right) \leqslant \theta\left(-e^{-\pi} + \sum_{n=2}^{\infty} e^{-n\pi}\right) \\ &= \theta\left(-1 - 2e^{-\pi} + \frac{1}{1 - e^{-\pi}}\right) = \theta \frac{2 - e^\pi}{e^\pi (e^\pi - 1)} < 0. \end{aligned}$$

In the case $c_n = 0$ for any $n \in \mathbb{N}$, by Theorem 2.1 the kernel $t \to \int_t^\infty \sum_{n=1}^\infty e^{-n^\beta \tau}\,d\tau$ is strongly positive.

In the general case, we will prove that the kernel $t \to \int_t^\infty g(t)\,dt$ is strongly positive, applying Proposition 5.5.

We start our analysis observing that $g \in L^1(0,\infty)$, since

$$\int_0^\infty \left|g(t)\right| dt \leqslant \theta \sum_{n=1}^{\infty} \int_0^\infty e^{-n^\beta t}\,dt = \theta \sum_{n=1}^{\infty} \frac{1}{n^\beta}.$$

In addition,

$$\int_0^\infty t\left|g(t)\right| dt \leqslant \theta \sum_{n=1}^{\infty} \int_0^\infty t e^{-n^\beta t}\,dt = \theta \sum_{n=1}^{\infty} \frac{1}{n^{2\beta}}$$

and

$$\int_0^\infty tg(t)\,dt = \theta \sum_{n=1}^\infty \int_0^\infty t\cos(c_n t)e^{-n^\beta t}\,dt = \theta \sum_{n=1}^\infty \mathrm{Re} \int_0^\infty te^{-(n^\beta + ic_n)t}\,dt$$

$$= \theta\,\mathrm{Re} \sum_{n=1}^\infty \int_0^\infty \frac{e^{-(n^\beta + ic_n)t}}{n^\beta + ic_n}\,dt = \theta\,\mathrm{Re} \sum_{n=1}^\infty \frac{1}{(n^\beta + ic_n)^2} = \theta \sum_{n=1}^\infty \frac{n^{2\beta} - c_n^2}{(n^{2\beta} + c_n^2)^2} > 0,$$

in virtue of (i). In a similar way, we get

$$\int_0^\infty \sin(\omega t)g(t)\,dt = \theta \sum_{n=1}^\infty \int_0^\infty \sin(\omega t)\cos(c_n t)e^{-n^\beta t}\,dt$$

$$= \frac{\theta}{2} \sum_{n=1}^\infty \int_0^\infty \sin\big((\omega + c_n)t\big)e^{-n^\beta t}\,dt + \frac{\theta}{2} \sum_{n=1}^\infty \int_0^\infty \sin\big((\omega - c_n)t\big)e^{-n^\beta t}\,dt$$

$$= \frac{\theta}{2} \sum_{n=1}^\infty \mathrm{Im} \int_0^\infty e^{(i(\omega + c_n) - n^\beta)t}\,dt + \frac{\theta}{2} \sum_{n=1}^\infty \mathrm{Im} \int_0^\infty e^{(i(\omega - c_n) - n^\beta)t}\,dt$$

$$= \frac{\theta}{2} \sum_{n=1}^\infty \frac{\omega + c_n}{n^{2\beta} + (\omega + c_n)^2} + \frac{\theta}{2} \sum_{n=1}^\infty \frac{\omega - c_n}{n^{2\beta} + (\omega - c_n)^2}$$

$$= \theta\omega \sum_{n=1}^\infty \frac{n^{2\beta} - c_n^2 + \omega^2}{[n^{2\beta} + (\omega + c_n)^2][n^{2\beta} + (\omega - c_n)^2]} > 0 \quad \forall \omega > 0.$$

Last, also the assumption (4) of Proposition 5.5 holds. Indeed, being

$$g'(t) = -\theta \sum_{n=1}^\infty \big[c_n \sin(c_n t) + n^\beta \cos(c_n t)\big]e^{-n^\beta t}, \quad t > 0,$$

by (i) we have, for any $\varepsilon > 0$,

$$\int_\varepsilon^\infty \big|g'(t)\big|\,dt \leqslant \theta \sum_{n=1}^\infty \big[|c_n| + n^\beta\big] \int_\varepsilon^\infty e^{-n^\beta t}\,dt$$

$$\leqslant 2\theta \sum_{n=1}^\infty n^\beta \int_\varepsilon^\infty e^{-n^\beta t}\,dt = 2\theta \sum_{n=1}^\infty e^{-n^\beta \varepsilon}$$

and there exists $\varepsilon_0 > 0$ such that for any $t \in (0, \varepsilon_0]$ and $n \geqslant 1$ we have

$$c_n \sin(c_n t) + n^\beta \cos(c_n t) \geqslant -\frac{1}{4} + \frac{n^\beta}{2} > 0.$$

Therefore, thanks to Proposition 5.5, we can conclude that the kernel $t \to \int_t^\infty g(t)\,dt$ is strongly positive.

Moreover, for any $\alpha_0 \in (0,1)$ we have

$$\int_0^\infty e^{\alpha_0 t}\big|g(t)\big|\,dt \leqslant \theta \sum_{n=1}^\infty \int_0^\infty e^{(\alpha_0 - n^\beta)t}\,dt = \theta \sum_{n=1}^\infty \frac{1}{n^\beta - \alpha_0}$$

and

$$\int_0^\infty t e^{\alpha_0 t}\big|g'(t)\big|\,dt \leqslant 2\theta \sum_{n=1}^\infty n^\beta \int_0^\infty t e^{(\alpha_0 - n^\beta)t}\,dt = 2\theta \sum_{n=1}^\infty \frac{n^\beta}{(n^\beta - \alpha_0)^2},$$

so by Proposition 5.2 for some $\alpha \in (0,\alpha_0)$ the kernel $G_\alpha(t) = \int_t^\infty g_\alpha(s)\,ds$ is strongly positive.

Finally, by (ii) we have

$$\int_0^\infty g(t)\,dt = \theta \sum_{n=1}^\infty \operatorname{Re} \int_0^\infty e^{-(n^\beta + ic_n)t}\,dt = \theta \sum_{n=1}^\infty \frac{n^\beta}{n^{2\beta} + c_n^2} < 1.$$

**Example 5.12.** We consider the kernels

$$g(t) := -e^{-\beta t}\log t, \quad t > 0,\ \beta > 0. \tag{5.21}$$

This type of kernel has been also considered in [18] for the special case $\beta = e$.

Set $\gamma = 0.5772\ldots$ the Euler–Mascheroni constant, we will prove that for $\beta > e^{1-\gamma}$ the kernel $G(t) = \int_t^\infty g(t)\,dt$ is strongly positive and for $\beta = e^{1-\gamma}$ is positive definite, but not strongly positive. Moreover, $G$ is not even positive definite for $\beta < e^{1-\gamma}$.

First, we note that $G \in L^1(0,\infty)$, since the function $t \mapsto tg(t)$ belongs to $L^1(0,\infty)$. In addition, recalling that

$$\gamma = -\int_0^\infty e^{-s}\log s\,ds = -\Gamma'(1), \tag{5.22}$$

we have for any $\omega \in \mathbb{R}$

$$\int_0^\infty e^{-(\beta + i\omega)t}\log t\,dt = -\frac{\frac{1}{2}\log(\beta^2+\omega^2) + i\arctan(\omega/\beta) + \gamma}{\beta + i\omega}, \tag{5.23}$$

whence

$$\begin{aligned}\int_0^\infty \sin(\omega t) g(t)\,dt &= \operatorname{Im}\int_0^\infty e^{-(\beta+i\omega)t}\log t\,dt \\ &= \left(\frac{1}{2}\log\big(\beta^2+\omega^2\big) + \gamma - \frac{\beta}{\omega}\arctan\left(\frac{\omega}{\beta}\right)\right)\frac{\omega}{\beta^2+\omega^2}\end{aligned}$$

$$\geqslant (\log\beta+\gamma-1)\frac{1+\omega^2}{\beta^2+\omega^2}\frac{\omega}{1+\omega^2} \quad \forall\omega>0. \tag{5.24}$$

If $\beta > e^{1-\gamma}$, by (5.24) we obtain

$$\int_0^\infty \sin(\omega t)g(t)\,dt \geqslant \frac{\log\beta+\gamma-1}{\beta^2}\frac{\omega}{1+\omega^2} \quad \text{for any } \omega>0,$$

that is (2.8) is satisfied. Therefore, in virtue of Corollary 2.3(b) $G$ is strongly positive.

In addition, if $\beta = e^{1-\gamma}$, we have $\log\beta+\gamma-1=0$, so again by (5.24) and Proposition 2.2(b) $G$ is positive definite. On the other hand, since

$$\lim_{\omega\to 0}\left(\frac{1}{2}\log\bigl(\beta^2+\omega^2\bigr)+\gamma-\frac{\beta}{\omega}\arctan\left(\frac{\omega}{\beta}\right)\right) = \log\beta+\gamma-1, \tag{5.25}$$

we get $\inf_{\omega>0}\frac{1+\omega^2}{\omega}\int_0^\infty \sin(\omega t)g(t)\,dt = 0$ and hence $G$ is not strongly positive.

In the last case $\beta < e^{1-\gamma}$, we have $\log\beta+\gamma-1<0$, so by (5.25) for some $\omega>0$

$$\int_0^\infty \sin(\omega t)g(t)\,dt = \left(\frac{1}{2}\log\bigl(\beta^2+\omega^2\bigr)+\gamma-\frac{\beta}{\omega}\arctan\left(\frac{\omega}{\beta}\right)\right)\frac{\omega}{\beta^2+\omega^2} < 0,$$

whence, in view of Proposition 2.2(a), $G$ is not positive definite.

Finally, by (5.23) for $\omega=0$ we have

$$\int_0^\infty g(t)\,dt = \frac{\log\beta+\gamma}{\beta} \leqslant \frac{1}{e^{1-\gamma}} < 1, \quad \forall\beta>0. \tag{5.26}$$

## 6. Application to PDEs

We shall now give an application of our stability result to a concrete model for partial differential operators. In this section, $\Omega$ will denote a bounded open domain in $\mathbb{R}^N$, $N \geqslant 3$, with sufficiently smooth boundary $\partial\Omega$. Points in $\Omega$ will be denoted by the Greek letter $\xi$. Moreover, the lower-dimensional cases $N=1,2$ can be treated by the same method in an even easier way.

Our example concerns a semilinear wave equation with memory. In [25] was analyzed a similar problem in one-dimensional case.

Let us consider the semilinear problem

$$\begin{cases} \partial_t^2 u(t,\xi) - \Delta u(t,\xi) + \displaystyle\int_0^t g(t-s)\Delta u(s,\xi)\,ds & \\ \quad = \bigl|u(t,\xi)\bigr|^\lambda u(t,\xi) + f(t,\xi), & t\geqslant 0,\ \xi\in\Omega, \\ u(t,\xi) = 0, & t\geqslant 0,\ \xi\in\partial\Omega, \\ u(0,\xi) = u_0(\xi), & \xi\in\Omega, \\ \partial_t u(0,\xi) = u_1(\xi), & \xi\in\Omega. \end{cases} \tag{6.1}$$

Here, $u(t,\xi)$ is real-valued, and we have denoted by $\partial_t u$ the time derivative of $u$ and by $\Delta u$ the Laplacian of $u$ with respect to space variable $\xi$. As for the convolution kernel $g$, we shall assume that

$\int_0^\infty g(t)\,dt < 1$ and there exists $\alpha_0 > 0$ such that $e^{\alpha_0 t} g \in L^1(0,\infty)$ and $t \mapsto \int_t^\infty e^{\alpha_0 s} g(s)\,ds$ is strongly positive definite. Also, $\lambda > 0$ satisfies a suitable restriction to be specified later.

We can rewrite (6.1) as an abstract problem of the type (4.1). Indeed, let $X = L^2(\Omega)$ be endowed with the usual inner product and norm

$$\|x\| := \left(\int_\Omega \left|x(\xi)\right|^2 d\xi\right)^{1/2}, \quad x \in L^2(\Omega).$$

We consider the operator $A : D(A) \subset X \to X$ defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$
$$Ax(\xi) = -\Delta x(\xi), \quad x \in D(A),\ \xi \in \Omega \text{ a.e.}$$

It is well known that $A$ verifies Assumption (H1)-1. Moreover, the fractional power $A^{1/2}$ of $A$ is well defined and $D(A^{1/2}) = H_0^1(\Omega)$. Next, consider the functional

$$F(x) := \frac{1}{\lambda+2} \int_\Omega \left|x(\xi)\right|^{\lambda+2} d\xi, \quad x \in H_0^1(\Omega),$$

which, if $0 < \lambda \leqslant 4/(N-2)$, is well defined in view of Sobolev's embedding theorem. By assuming the more restrictive condition $0 < \lambda \leqslant 2/(N-2)$, $F$ satisfies Assumptions (H1) and

$$\nabla F(x)(\xi) = \left|x(\xi)\right|^\lambda x(\xi), \quad x \in H_0^1(\Omega),\ \xi \in \Omega \text{ a.e.},$$

see [1]. In addition, since $2(\lambda+1) \leqslant 2N/(N-2) = 2^*$, again by Sobolev's theorem we have for any $x \in H_0^1(\Omega)$ and $y \in L^2(\Omega)$

$$\begin{aligned}
\left|\langle \nabla F(x), y\rangle\right| &\leqslant \int_\Omega \left|x(\xi)\right|^{\lambda+1} \left|y(\xi)\right| d\xi \\
&\leqslant \left(\int_\Omega \left|x(\xi)\right|^{2(\lambda+1)} d\xi\right)^{1/2} \|y\| \\
&\leqslant C\left(\int_\Omega \left|\nabla x(\xi)\right|^2 d\xi\right)^{\lambda/2} \left(\int_\Omega \left|\nabla x(\xi)\right|^2 d\xi\right)^{1/2} \|y\|,
\end{aligned}$$

so, Assumption (H2)-2 is also satisfied with $\psi(s) = Cs^\lambda$. Therefore, by Theorem 3.6 we conclude that, if the data $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^1(0,\infty; L^2(\Omega))$ are sufficiently small, that is,

$$\int_\Omega \left(\left|\nabla u_0(\xi)\right|^2 + \left|u_1(\xi)\right|^2\right) d\xi + \int_0^\infty \left(\int_\Omega \left|f(t,\xi)\right|^2 d\xi\right)^{1/2} dt < \rho$$

for some $\rho > 0$, then problem (6.1) admits a unique mild solution $u$ on $[0,\infty)$. Moreover, as recalled in Section 3.1, $u$ is a weak solution of the equation in (6.1), that is,

$$u \in C^1\big([0,\infty); L^2(\Omega)\big) \cap C\big([0,\infty); H_0^1(\Omega)\big),$$

for all $v \in H_0^1(\Omega)$, $t \mapsto \int_\Omega \partial_t u(t,\xi)v(\xi)\,d\xi$ is of class $C^1$ and

$$\frac{d}{dt}\int_\Omega \partial_t u(t,\xi)v(\xi)\,d\xi + \int_\Omega \nabla u(t,\xi)\cdot\nabla v(\xi)\,d\xi - \int_0^t g(t-s)\int_\Omega \nabla u(s,\xi)\cdot\nabla v(\xi)\,d\xi\,ds = \int_\Omega \big|u(t,\xi)\big|^\lambda u(t,\xi)v(\xi)\,d\xi + \int_\Omega f(t,\xi)v(\xi)\,d\xi, \quad \forall t \geqslant 0.$$

Defining the energy of $u$ by

$$E_u(t) := \frac{1}{2}\int_\Omega \big|\partial_t u(t,\xi)\big|^2\,d\xi + \frac{1}{2}\left(1 - \int_0^\infty g(s)\,ds\right)\int_\Omega \big|\nabla u(t,\xi)\big|^2\,d\xi - \frac{1}{\lambda+2}\int_\Omega \big|u(t,\xi)\big|^{\lambda+2}\,d\xi,$$

we can invoke Theorem 4.1 to obtain, for any $(u_0,u_1) \in H_0^1(\Omega)\times L^2(\Omega)$ and any $e^{\eta_0 t}f \in L^1(0,\infty; L^2(\Omega))$ sufficiently small, the following decay estimates:

$$E_u(t) \leqslant C\big(\|\nabla u_0\| + \|u_1\| + \|f_{\eta_0}\|_1\big)e^{-2\alpha^* t} \quad \forall t \geqslant 0,$$
$$\int_0^\infty e^{2\alpha^* t}E_u(t)\,dt \leqslant C\big(\|\nabla u_0\| + \|u_1\| + \|f_{\eta_0}\|_1\big),$$

for some $\alpha^* \in (0, \alpha_0 \wedge \eta_0]$, where $C(R)$ is a positive, increasing, upper semicontinuous function such that $C(0) = 0$.

## Appendix A

For the reader's convenience, in this section we will list and prove some results concerning the linear equation

$$u''(t) + Au(t) - \int_0^t g(t-s)Au(s)\,ds = f(t), \quad t \geqslant 0. \tag{A.1}$$

Throughout the section, we suppose that Assumptions (H1)-1,2 and (H2)-1 are verified. To begin with, we recollect some known estimates for the resolvent (see Definition 2.13) of equation

$$u''(t) + Au(t) - \int_0^t g(t-s)Au(s)\,ds = 0.$$

**Proposition A.1.**

(i) *For any $x \in X$ and any $t > 0$, $1 * S(t)x \in D(A^{1/2})$ and*

$$\left\|S(t)x\right\|^2 + \left(1 - \int_0^\infty g(t)\,dt\right)\left\|A^{1/2}\int_0^t S(\tau)x\,d\tau\right\|^2 \leqslant \|x\|^2. \tag{A.2}$$

(ii) *For any $x \in D(A^{1/2})$, $S(\cdot)x$ is continuously differentiable on $[0,\infty)$ and*

$$\left\|S'(t)x\right\| \leqslant \left(1 + \|g\|_1\right)\left(1 - \int_0^\infty g(t)\,dt\right)^{-1/2}\left\|A^{1/2}x\right\| \quad \forall t \geqslant 0. \tag{A.3}$$

(iii) *For any $x \in X$ and any $t > 0$, $1 * 1 * S(t)x \in D(A)$ and*

$$\left\|A1 * 1 * S(t)x\right\| \leqslant C\|x\|, \tag{A.4}$$

*for some constant $C > 0$.*

**Proof.** As for (i) and (ii) see [9, Proposition 3.4], while (iii) follows by [9, Proposition 3.5] and Proposition 2.11. □

We define the energy of a mild solution $u$ of (A.1) as

$$E_u(t) := \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\left(1 - \int_0^\infty g(s)\,ds\right)\left\|A^{1/2}u(t)\right\|^2, \quad t \geqslant 0.$$

**Theorem A.2.** *For any $u_0 \in D(A^{1/2})$, $u_1 \in X$ and $f \in L^2(0,\infty;X)$ the mild solution $u$ of linear equation* (A.1) *with initial conditions*

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{A.5}$$

*has the following continuous dependence on data*

$$E_u(t) \leqslant C\left(\|u_1\|^2 + \left\|A^{1/2}u_0\right\|^2 + \|f\|_2^2\right), \tag{A.6}$$

*for any $t \geqslant 0$ and some constant $C > 0$.*

**Proof.** Assuming $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in L^2(0,\infty;X) \cap W^{1,1}_{loc}(0,\infty;X)$, $u$ is the strong solution of (A.1)–(A.5) that, in view of (3.18) (with $F = 0$), verifies

$$\begin{aligned} E_u(t) \leqslant{} & E_u(0) + G(0)\left\|A^{1/2}u_0\right\|^2 - \left\langle A^{1/2}u_0, G(t)A^{1/2}u(t)\right\rangle \\ & - \int_0^t g(s)\left\langle A^{1/2}u_0, A^{1/2}u(s)\right\rangle ds + \frac{1}{2}\int_0^t \left\|f(s)\right\|^2 ds + \frac{1}{2}\int_0^t \left\|u'(s)\right\|^2 ds. \end{aligned} \tag{A.7}$$

Next, arguing as in Lemmas 1–3 with $\alpha = 0$, we can prove that there exists a constant $C > 0$ such that for any $t \geqslant 0$

$$\int_0^t \|u'(s)\|^2 \, ds \leqslant C\left(\|u_1\|^2 + \|A^{1/2}u_0\|^2 + \|f\|_2^2\right).$$

By putting the previous estimate into (A.7), we easily obtain (A.6) for more regular data. Finally, our claim follows by using a standard approximation argument. □

**Theorem A.3.** *For any $u_0 \in D(A^{1/2})$, $u_1 \in X$ and $f \in L^1(0,\infty;X)$ or $f \in L^2(0,\infty;X)$ the energy $E_u(t)$ of the mild solution $u$ for Eq.* (A.1) *with initial conditions* (A.5) *is uniformly continuous on* $[0,\infty[$.

**Proof.** First, we verify that for $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in W^{1,1}(0,\infty;X)$, $u''(t)$ and $A^{1/2}u'(t)$ are bounded. Indeed, by using formula (3.8) (with $F = 0$) we have

$$\begin{aligned} Au(t) = S(t)Au_0 + A^{1/2}\int_0^t S(\tau)A^{1/2}u_1 \, d\tau \\ + A\int_0^t 1 * S(\tau)f(0)\, d\tau + \int_0^t A\int_0^{t-\sigma} 1 * S(\tau)f'(\sigma)\, d\tau \, d\sigma, \end{aligned}$$

whence in view of (A.2) and (A.4) we have $Au \in L^\infty(0,\infty;X)$. Since $f \in L^\infty(0,\infty;X)$, by Eq. (A.1) we deduce that $u'' \in L^\infty(0,\infty;X)$ as well. In a similar way we have

$$\begin{aligned} A^{1/2}u'(t) = S'(t)A^{1/2}u_0 + S(t)A^{1/2}u_1 \\ + A^{1/2}\int_0^t S(\tau)f(0)\, d\tau + \int_0^t A^{1/2}\int_0^{t-\sigma} S(\tau)f'(\sigma)\, d\tau \, d\sigma, \end{aligned}$$

whence by (A.2) and (A.3) we obtain $A^{1/2}u' \in L^\infty(0,\infty;X)$. Therefore, for more regular data $E_u(t)$ is uniformly continuous on $[0,\infty[$.

Finally, thanks to (3.23) (with $F = 0$) if $f \in L^1(0,\infty;X)$ or to (A.6) if $f \in L^2(0,\infty;X)$, we are able to apply an approximation argument and conclude the proof. □

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