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MALLIAVIN REPRESENTATION FORMULAS FOR ASIAN OPTIONS IN A JUMP-DIFFUSION MODEL

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Abstract
We first develop a unifying Malliavin calculus in a jump-diffusion context, by taking into account all the randomness involved (Brownian motion, jump times and jump amplitudes) and by stating an integration by parts formula which gives the starting point of our work. The results are then applied to study representation formulas both for sensitivities (delta) and conditional expectations (in terms of non conditional ones) for a two dimensional process $Z_t = (X_t, Y_t)$, in which X stands for a jump diffusion and $Y_t = \int_0^t X_r dr$. Therefore, the link with problems arising from Finance (price/delta of Asian options) is studied. Several examples are analyzed in details and equipped with numerical studies.

Keywords
Malliavin calculus, Monte Carlo methods, Asian options.

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Contents

Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ denote a probability space, where the following ingredients are defined:

- a standard one-dimensional Brownian motion W; we use the notation $\mathscr{W}_t = \sigma\{W_s\,;\, s \leq t\} \vee \mathcal{N}$, where $\mathcal{N}$ stands for the $\mathbb{P}$-null sets;
- a sequence $\{T_i\}_{i\geq 1}$ of positive r.v.'s such that $T_1, T_2 - T_1, T_3 - T_2, \ldots$ are i.i.d. r.v.'s exponentially distributed with parameter $\lambda > 0$; we denote by J_t the associated counting process, i.e. $J_t = \sum_{i\geq 1} \mathbf{1}_{\{T_i \leq t\}}$, and by $\mathscr{J}_t = \sigma\{J_s\,;\, s \leq t\}$;
- a sequence $\{\Delta_i\}_{i\geq 1}$ of independent and real r.v.'s; we assume that there exists the probability density function g of Δ_i.

From now on, we assume that W, $\{T_i\}_{i\geq 1}$ and $\{\Delta_i\}_{i\geq 1}$ are independent.

Take now a time horizon $T > 0$ and the vector fields $b = b(t, x)$, $\sigma = \sigma(t, x)$ and $c = c(t, a, x)$, with

$$b\,:\,[0,T]\times\mathbb{R}\to\mathbb{R},\quad \sigma\,:\,[0,T]\times\mathbb{R}\to\mathbb{R},\quad c\,:\,[0,T]\times\mathbb{R}^2\to\mathbb{R}.$$

The, we assume to be in a one-dimensional framework but the general case gives no technical difficulties and makes the notations much more complicated. Suppose for the moment only that b, σ, c are smooth enough in order to ensure the existence of the process X solution to the following jump-diffusion stochastic differential equation (s.d.e.)

$$X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r + \sum_{i=1}^{J_t} c(T_i, \Delta_i, X_{T_i^-}), \quad t \leq T. \tag{1}$$

Now, let us define the pair $Z_t = (X_t, Y_t)$, $t \in [0, T]$, in which

$$Y_t = \int_0^T X_r dr \tag{2}$$

This is our framework, and we can now describe the aim of this paper, which is twice.

The first part is devoted to an attempt to give a unifying approach to the Malliavin calculus jointly in the Gaussian direction as well as in the the jump times and the jump amplitudes one. More precisely, we aim to develop integration by parts formulas for functionals of the Brownian motion, the jump times and the jump amplitudes taking into account all the three random sources involved, even if one has to say that sometimes this cannot be done (see e.g. next Example 1.1.25). For this purposes, we inspire to the classical Gaussian Malliavin calculus (there are many references, e.g. the classical book of Nualart [25] or also the approach by Bally [2], which we have used) and the mixed jump times and amplitudes Malliavin calculus developed by Bally, Bavouzet and Messaoud [3]. But several other papers handling jump-diffusions should be cited: a Malliavin calculus in the direction of the Brownian motion has been recently developed by Davis and Johansson [13] and Forster, Lütkebohmert and Teichman [15]; in the direction of the jump-times, let us cite also the book of Bichteler, Gravereaux and Jacod [9] and the recent works by Privault and Wei [27], [28], by El-Khatib and Privault [14] and the paper of Vives, León, Utzet and Solé [29].

Once a suitable integration by parts formula is obtained, we can tackle the second goal: the study either of sensitivities and representation formulas for the conditional expectation for the pair $Z_t = (X_t, Y_t)$. This has an immediate application to Finance. In fact, if X models the underlying asset price then the payoff of an Asian option of European type with maturity T is given by $f(T, X_T, Y_T)$. For example, one can take

$$f(T,x,y) = f_{\text{put}}(T,x,y) = \begin{cases} (K - y/T)_+ & \text{standard Asian put option} \\ (x - y/T)_+ & \text{floating Asian put option} \end{cases}$$

and similarly for the call option case, in which K plays the role of the strike price. Therefore, in our framework the sensitivity analysis relies the study of the Greeks for Asian options.

In the American context, the payoff function is given by $f(t, X_t, Y_t)$ and the price is usually approximated by means of a Dynamic Programming Principle, which can be roughly summarized as follows. Set $\Delta t = T/n \in (0,1)$ and let $(\bar X_{k\Delta t}, \bar Y_{k\Delta t})_{k=0,1,\dots,n}$ be a discretization for $(X_t, Y_t)_{t\in[0,T]}$. Since the pair (X_t, Y_t) is Markovian, the American price P_0 is approximated by $\bar P_0(x, 0)$, where $\bar P_{k\Delta t}(\bar X_{k\Delta t}, \bar Y_{k\Delta t})$ solves the following backward problem:

$$\begin{aligned} &\bar P_{n\Delta t}(\bar X_{n\Delta t}, \bar Y_{n\Delta t}) = f(n\Delta t, \bar X_{n\Delta t}, \bar Y_{n\Delta t}) \text{ and for any } k = n-1, \dots, 1, 0 \\ &\bar P_{k\Delta t}(\bar X_{k\Delta t}, \bar Y_{k\Delta t}) = \max\Big(f(k\Delta t, \bar X_{k\Delta t}, \bar Y_{k\Delta t}), \\ &\qquad\qquad e^{-r\Delta t}\, \mathrm{T}[\bar P_{(k+1)\Delta t}](\bar X_{k\Delta t}, \bar Y_{k\Delta t})\Big) \end{aligned}$$

where the operator T is defined as

$$\mathrm{T}[\bar P_{(k+1)\Delta t}](\bar X_{k\Delta t}, \bar Y_{k\Delta t}) = \mathbb{E}^*\Big(\bar P_{(k+1)\Delta t}(\bar X_{(k+1)\Delta t}, \bar Y_{(k+1)\Delta t}) \,\Big|\, \bar X_{k\Delta t}, \bar Y_{k\Delta t}\Big),$$

in which r stands for the (constant, for the sake of simplicity) spot rate and $\mathbb{E}^*$ for the expectation under the risk neutral measure. Therefore, a representation formula for the conditional expectation in terms of non-conditional ones is really helpful in order to set up Monte Carlo methods for the pricing of American Asian options, as already observed and done in the literature. As for the pure diffusion case, such arguments have been developed by Fournié, Lasry, Lebuchoux and Lions [17], Lions and Regnier [23], Bouchard, Ekeland and Touzi [11], [10], Bally, Caramellino and Zanette [4].

Let us finally add that in the case of Asian options, in the already cited literature one can find formulas allowing to represent the delta as an expectation of the payoff multiplied by a suitable weight even in the presence of jumps. Nevertheless, these formulas require the payoff to be dependent on the integral coordinate only, as in the case of the standard call/put Asian option, and are not suitable to handle for example floating call/put Asian options, that is Asian options in which the payoff depends also on the underlying asset price. This last case has been studied only for pure diffusions, see e.g. Benhamou [6]. Notice that this is not a trivial complication: the pair (X_t, Y_t) solves a degenerate jump-diffusion, and some standard generalizations cannot be applied. This holds mainly for the problem of finding representation formulas for the conditional expectation in terms of non conditional ones. And as for this latter problem, as far as we know, no formulas are available in the presence of jumps also if one is interested to work with the coordinate X only.

Chapter 1

Malliavin calculus: a primer

In abstract terms, when one is interested to develop a differential calculus of Malliavin type and the aim is to find representation formulas for sensitivities (e.g. derivative of expectations w.r.t. the initial state) in terms of weighted expectations or representation formulas for the conditional expectation in terms of non conditional ones, the following steps have to be performed.

1.1 Malliavin calculus in the jump noise direction

This section is inspired to the paper by Bally, Bavouzet and Messaoud [3] (see also [5]), where all proofs and generalizations can be found. The aim is to deal with a stochastic variational calculus corresponding to Poisson processes, to be used to the context of our interest: jump-diffusions and applications to Finance. Many other references should be added (see the exhaustive citations in [3]). In particular, a special look to financial problems has been given by El-Khatib and Privault [14], Privault and Wei [27], [28] and by Vives, León, Utzet and Solé [29].

Let $(\Omega, \mathscr{F}, \mathbb{P})$ denote the underlying probability space, i.e. the space such that all the further random instruments are suppose to be defined on.

Let us consider a sequence $\{\Delta_i\}_{i\geq 1}$ of i.i.d. r.v.'s on $\mathbb{R}$ (the jump-amplitudes) such that there exists the probability density function (pdf) g of Δ_i. We assume that

Assumption 1.1.1. *There exist some points* $-\infty \leq a_1 < b_1 < a_2 \cdots < b_k \leq +\infty$ *such that the density* g *of* Δ_i *is strictly positive on*

$$I = \bigcup_{j=1}^{k} (a_j, b_j).$$

Moreover, the function $\log g$ *is continuously differentiable on* I *and both* Δ_i *and* $\log g(\Delta_i)$ *belong to* L^p *for any* p.

Take now a sequence $\{T_i\}_{i\geq 1}$ of positive r.v.'s (the jump-times) such that $T_1, T_2 - T_1, T_3 - T_2, \ldots$ are i.i.d. r.v.'s exponentially distributed with parameter $\lambda > 0$. Let us set $T_0 = 0$. We denote by J_t the associated counting process, i.e. $J_t = \sum_{i\geq 1} \mathbf{1}_{\{T_i \leq t\}}$, and $\mathscr{J}_t = \sigma\{J_s\,;\, s \leq t\}$. It is well known (see e.g. Bertoin [8]) that for a fixed $t > 0$, the conditional law of $T_1, \ldots, T_n$ given that $\{J_t = n\}$ is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^n$ and has the following probability density function:

$$p(\omega, t_1, \ldots, t_n) = \frac{n!}{t^n}\, \mathbf{1}_{\{0<t_1<\cdots<t_n<t\}}(t_1, \ldots, t_n) \quad \text{for } \omega \in \{J_t = n\}.$$

Moreover, as $i = 1, \ldots, n$, the marginal conditional law of T_i given that $\{J_t = n\}$ is uniformly distributed, with probability density function given by

$$p_i(\omega, r) = \frac{1}{T_{i+1}(\omega) - T_{i-1}(\omega)}\, \mathbf{1}_{\{T_{i-1}(\omega)<r<T_{i+1}(\omega)\}} \quad \text{for } \omega \in \{J_t = n\}, \tag{1.1}$$

in which we have set $T_0 = 0$ and for $i = n$, (1.1) has to be intended with $T_{n+1} = t$, i.e.

$$p_n(\omega, r) = \frac{1}{t - T_n(\omega)}\, \mathbf{1}_{\{T_n(\omega)<r<t\}} \quad \text{for } \omega \in \{J_t = n\}, \tag{1.2}$$

From now on, we assume the following:

Assumption 1.1.2. *$\{T_i\}_{i\geq 1}$ and $\{\Delta_i\}_{i\geq 1}$ are independent.*

The pair $\{T_i\}_{i\geq 1}$ and $\{\Delta_i\}_{i\geq 1}$ defines the compound Poisson process

$$N_t = \sum_{i=1}^{J_t} \Delta_i.$$

Our aim is here to study a Malliavin type derivative D and a Skorohod type integral δ in order to achieve a duality relationship of the type

$$\mathbb{E}(\langle DF, u\rangle_\pi \mathbf{1}_{\{J_t=n\}}) = \mathbb{E}(F\delta(u)\mathbf{1}_{\{J_t=n\}})$$

or something of similar, for a random variable F and a random process u, suitably depending on the jump times and amplitudes, and for some inner product $\langle \cdot, \cdot \rangle_\pi$.

To our purposes, let us add a standard Brownian motion W, independent of $\{T_i\}_{i\geq 1}$ and $\{\Delta_i\}_{i\geq 1}$. Set moreover $\mathscr{W}_s$ as the associated filtration augmented by the $\mathbb{P}$-null sets.

Let us first introduce the functional spaces we are going to work with.

Fix $t > 0$ and $n \geq 1$. We denote by $\mathcal{C}^k_{n,t}(\Delta)$ the set of the $\mathscr{W}_t \times \mathscr{B}(\mathbb{R}^n) \times \mathscr{B}(\mathbb{R}^n_+)$ measurable functions $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^n_+ \to \mathbb{R}$ such that for any $i = 1, \ldots, n$,

$\mathbb{T}_i(f)(\omega,\cdot) \in \mathcal{C}^k(I_i)$ a.s., where

$$\begin{aligned}
&I_i = I \text{ and} \\
&\mathbb{T}_i(f)(\omega,y) = f\Big(\omega, \Delta_1(\omega), \ldots, \Delta_{i-1}(\omega), y, \Delta_{i+1}(\omega), \ldots, \Delta_n(\omega), \\
&\qquad\qquad T_1(\omega), \ldots, T_n(\omega)\Big)
\end{aligned} \tag{1.3}$$

Recall that I is defined in Assumption 1.1.1.
Similarly, we denote by $\mathcal{C}^k_{n,t}(T)$ the set of the $\mathscr{W}_t \times \mathscr{B}(\mathbb{R}^n) \times \mathscr{B}(\mathbb{R}^n_+)$ measurable functions $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^n_+ \to \mathbb{R}$ such that for any $i = 1, \ldots, n$, $\mathbb{T}_{i+n}(f)(\omega,\cdot) \in \mathcal{C}^k(I_{i+n})$ a.s.,where

$$\begin{aligned}
&I_{i+n} = (T_{i-1}(\omega), T_{i+1}(\omega)) \quad \text{as } i = 1, ..., n \quad \text{and} \\
&\mathbb{T}_{i+n}(f)(\omega,r) = f\Big(\omega, \Delta_1(\omega), \ldots, \Delta_n(\omega), \\
&\qquad\qquad T_1(\omega), \ldots, T_{i-1}(\omega), r, T_{i+1}(\omega), \ldots, T_n(\omega)\Big)
\end{aligned} \tag{1.4}$$

Again, we have set $T_{n+1} = t$ in the above expression.
We also set

$$\mathcal{C}^k_{n,t} = \mathcal{C}^k_{n,t}(\Delta) \cap \mathcal{C}^k_{n,t}(T).$$

For a multi index $\alpha \in \{1, \ldots, 2n\}^k$, set $\partial^k_\alpha f = \partial^k_{x_{\alpha_1} \ldots x_{\alpha_k}} f$. Now we can define $\mathscr{C}^k_{n,t}(\Delta)$ and $\mathscr{C}^k_{n,t}(T)$ as the set of the function $f \in \mathcal{C}^k_{n,t}(\Delta)$ and $f \in \mathcal{C}^k_{n,t}(T)$ respectively such that $\partial^j_\alpha f(\omega, \Delta_1, \ldots, \Delta_n, T_1, \ldots, T_n)\mathbf{1}_{\{J_t=n\}} \in L^p(\Omega)$, for any p, $0 \le j \le k$ and $\alpha = (\alpha_1, \ldots, \alpha_j) \in \{1, \ldots, 2n\}^j$. We also set

$$\mathscr{C}^k_{n,t} = \mathscr{C}^k_{n,t}(\Delta) \cap \mathscr{C}^k_{n,t}(T).$$

Let us now define some further functions, whose meaning is the following: they are weights allowing to set up the proper scalar product.

Definition 1.1.3. [Weights for the scalar product]

- *As* $i = 1, \ldots, n$, *set*

  $$\mathscr{G}_i = \mathscr{W}_t \vee \sigma(\{\mathbf{1}_{\{J_t=n\}}, \Delta_1, \ldots, \Delta_n, T_1, \ldots, T_n\} \setminus \{\Delta_i\}).$$

  Let

  $$\pi_i : \Omega \times \mathbb{R} \to \mathbb{R}_+$$

  be a $\mathscr{G}_i \times \mathscr{B}(\mathbb{R})$*-measurable function such that*

  $$\pi_i(\omega, y) = 0 \text{ when } y \notin I \text{ and } I \ni y \mapsto \pi_i(\omega, y) \in \mathcal{C}^1(I).$$

- *Similarly, set*

  $$\mathscr{G}_{i+n} = \mathscr{W}_t \vee \sigma(\{\mathbf{1}_{\{J_t=n\}}, \Delta_1, \ldots, \Delta_n, T_1, \ldots, T_n\} \setminus \{T_i\}),$$

$i = 1, \dots, n$. Let

$$\pi_{i+n} : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$$

be a $\mathscr{G}_{i+n} \times \mathscr{B}(\mathbb{R}_+)$-measurable function such that

$$\pi_{i+n}(\omega, r) = 0 \text{ when } r \notin I_i \text{ and } I_i \ni r \mapsto \pi_{i+n}(\omega, r) \in \mathcal{C}^1(I_i).$$

Assume that

Assumption 1.1.4. *As $i = 1, \dots, n$, $\pi_i(\omega, \Delta_i), \pi_{i+n}(\omega, T_i) \in L^p(\Omega)$ for any p and there exists $\delta > 0$ such that $\partial_y \pi_i(\omega, \Delta_i)$, $\partial_r \pi_i(\omega, T_i) \in L^{1+\delta}(\Omega)$.*

We can now define the set of the simple functionals and of the simple processes.

Definition 1.1.5. [Simple functionals] *The set $\mathscr{S}^k_{n,t}(\Delta)$ of the simple functionals of length n up to time t w.r.t. the jump amplitudes is given by the r.v.'s on $\mathbb{R}$ of the form*

$$F = f(\omega, \Delta_1, \dots, \Delta_n, T_1, \dots, T_n) \text{ for some } f \in \mathcal{C}^k_{n,t}(\Delta).$$

Similarly, the set $\mathscr{S}^k_{n,t}(T)$ of the simple functionals of length n up to time t w.r.t. the jump times is given by the r.v.'s on $\mathbb{R}$ of the form

$$F = f(\omega, \Delta_1, \dots, \Delta_n, T_1, \dots, T_n) \text{ for some } f \in \mathcal{C}^k_{n,t}(T)$$

We set $\mathscr{S}^k_{n,t} = \mathscr{S}^k_{n,t}(\Delta) \cap \mathscr{S}^k_{n,t}(T)$ to denote the simple functionals of length n up to time t. Whenever $f \in \mathscr{C}^k_{n,t}(\{J_t = n\})$, we will write $\mathscr{S}^k_{n,t}(\{J_t = n\})$.

Definition 1.1.6. [Simple processes] *The set $\mathscr{P}^k_{n,t}(\Delta)$ of the simple processes of length n up to time t w.r.t. the jump amplitudes is given by the r.v.'s on $\mathbb{R}^n$ of the form $U = (U_1, \dots, U_n)$, with*

$$U_i = u_i(\omega, \Delta_1, \dots, \Delta_n, T_1, \dots, T_n) \text{ for some } u_i \in \mathcal{C}^k_{n,t}(\Delta),\ i = 1, \dots, n.$$

Similarly, the set $\mathscr{P}^k_{n,t}(T)$ of the simple processes of length n up to time t w.r.t. the jump times is given by the r.v.'s on $\mathbb{R}^n$ of the form $U = (U_{n+1}, \dots, U_{2n})$, with

$$U_{i+n} = u_{i+n}(\omega, \Delta_1, \dots, \Delta_n, T_1, \dots, T_n) \text{ for some } u_{i+n} \in \mathcal{C}^k_{n,t}(T),$$

as $i = 1, \dots, n$. Finally, the set $\mathscr{P}^k_{n,t}$ of the simple processes of length n up to time t is given by the r.v.'s on $\mathbb{R}^{2n}$ of the form $U = (U_1, \dots, U_{2n})$, with $U^\Delta = (U_1, \dots, U_n) \in \mathscr{P}^k_{n,t}(\Delta)$ and $U^T = (U_{n+1}, \dots, U_{2n}) \in \mathscr{P}^k_{n,t}(T)$.
The set $\mathscr{P}^k_{n,t}(\{J_t = n\})$ denotes the simple processes corresponding to functions $u_i \in \mathscr{C}^k_{n,t}(\{J_t = n\})$, $i = 1, \dots, 2n$.

As $U, V \in \mathscr{P}^k_{n,t}(\Delta)$ and $U, V \in \mathscr{P}^k_{n,t}(T)$ we define

$$\langle U, V\rangle^\Delta_\pi = \sum_{i=1}^n \pi_i U_i V_i \quad \text{and} \quad \langle U, V\rangle^T_\pi = \sum_{i=1}^n \pi_{i+n} U_{i+n} V_{i+n}.$$

respectively; if $U, V \in \mathscr{P}^k_{n,t}$, we set

$$\langle U, V\rangle_\pi = \langle U^\Delta, V^\Delta\rangle^\Delta_\pi + \langle U^T, V^T\rangle^T_\pi = \sum_{i=1}^{2n} \pi_i U_i V_i.$$

Notice that $\langle U^\Delta, V^\Delta\rangle^\Delta_\pi$, $\langle U^T, V^T\rangle^T_\pi$ and $\langle U, V\rangle_\pi$ are all (random) scalar products (on the proper space) when $\omega \in \{J_t = n\}$, whenever the involved weights π_i's are all non null.

We are now ready to define the differential operator.

Definition 1.1.7. [Malliavin derivative] *For $F \in \mathscr{S}^1_{n,t}$, $F = f(\omega, \Delta_1, \dots, \Delta_n, T_1, \dots, T_n)$, the Malliavin derivative $Df = (D_1F, \dots, D_{2n}F)$ is defined by*

$$\begin{aligned} D_iF &= \partial_{x_i} f(\omega, \Delta_1, \dots, \Delta_n, T_1, \dots, T_n)\mathbf{1}_{\Delta_i \in I} \\ D_{i+n}F &= \partial_{t_i} f(\omega, \Delta_1, \dots, \Delta_n, T_1, \dots, T_n)\mathbf{1}_{T_i \in I_i}, \end{aligned}$$

as $i = 1, \dots, n$. Therefore,

$$D : \mathscr{S}^1_{n,t} \to \mathscr{P}^0_{n,t}.$$

Remark 1.1.8. *The operator D_i has to be intended as the Malliavin derivative in the direction of the jump amplitudes if $i \le n$ and in the direction of the jump times whenever $i \ge n+1$. In fact, $D = (D_1, \dots, D_{2n})$ can be defined also separately on the jump amplitudes or times only. In fact, $D = (D^\Delta, D^T)$, with*

$$\begin{aligned} D^\Delta &\equiv (D_1, \dots, D_n) &&: \mathscr{S}^1_{n,t}(\Delta) \to \mathscr{P}^0_{n,t}(\Delta) \\ D^T &\equiv (D_{n+1}, \dots, D_{2n}) &&: \mathscr{S}^1_{n,t}(T) \to \mathscr{P}^0_{n,t}(T) \end{aligned}$$

Definition 1.1.9. [Skorohod integral] *For $U = (U_1, \dots U_{2n})$ with $U_i = u_i(\omega, \Delta_1, \dots, \Delta_n, T_1, \dots, T_n)$, $i = 1, \dots 2n$, the Skorohod integral $\delta(U)$ is defined by*

$$\delta(U) = \sum_{j=1}^{2n} \delta_j(U_j)$$

where, as i=1,...,n (recall that g is the pdf of the jump amplitudes)

$$\begin{aligned} \delta_i(U_i) &= -\Big(\partial_{x_i}(\pi_i u_i) + (\pi_i u_i)\partial_y \ln g\Big)(\omega, \Delta_1, \dots, \Delta_n, T_1, \dots, T_n) \\ \delta_{i+n}(U_{i+n}) &= -\Big(\partial_{t_i}(\pi_{i+n} u_{i+n})\Big)(\omega, \Delta_1, \dots, \Delta_n, T_1, \dots, T_n) \end{aligned}$$

Therefore,

$$\delta : \mathscr{P}^1_{n,t} \to \mathscr{S}^0_{n,t}.$$

Remark 1.1.10. *The operator δ_j has to be intended as the Skorohod integral in the direction of the jump amplitudes if $j \leq n$ and in the direction of the jump times whenever $j \geq n+1$. Also, the operator δ can be defined separately on the jump amplitudes or times only. In fact, $\delta = \delta^\Delta + \delta^T$ with*

$$\begin{aligned}\delta^\Delta &: \mathscr{P}^1_{n,t}(\Delta) \to \mathscr{S}^0_{n,t}(\Delta), \quad \delta^\Delta(U) = \sum_{i=1}^n \delta_i(U_i)\\ \delta^T &: \mathscr{P}^1_{n,t}(T) \to \mathscr{S}^0_{n,t}(T), \quad \delta^T(U) = \sum_{i=1}^n \delta_{i+n}(U_{i+n})\end{aligned}$$

Let us finally generalize the definition of Malliavin derivative and Skorohod integral.

Definition 1.1.11. *Fix $n \geq 1$ and $t > 0$.*

[Malliavin derivable r.v.'s on the set $\{J_t = n\}$]
The class $\mathscr{D}^k_{n,t}$ ($\mathscr{D}^k_{n,t}(\Delta)$, $\mathscr{D}^k_{n,t}(T)$ resp.) is given by the r.v.'s F such that $F\mathbf{1}_{\{J_t=n\}} = F_{n,t}\mathbf{1}_{\{J_t=n\}}$ for some $F_{n,t} \in \mathscr{S}^k_{n,t}$ ($\mathscr{S}^k_{n,t}(\Delta)$, $\mathscr{S}^k_{n,t}(T)$ resp.). For $F \in \mathscr{D}^k_{n,t}$ we write

$$D_i F\mathbf{1}_{\{J_t=n\}} = D_i F_{n,t}\mathbf{1}_{\{J_t=n\}}$$

[Skorohod integrable processes on the set $\{J_t = n\}$]
The class $\mathscr{I}^k_{n,t}$ ($\mathscr{I}^k_{n,t}(\Delta)$, $\mathscr{I}^k_{n,t}(T)$ resp.) is given by the r.v.'s U such that $U\mathbf{1}_{\{J_t=n\}} = U_{n,t}\mathbf{1}_{\{J_t=n\}}$ for some $U_{n,t} \in \mathscr{P}^k_{n,t}$ ($\mathscr{P}^k_{n,t}(\Delta)$, $\mathscr{P}^k_{n,t}(T)$ resp.). For $U \in \mathscr{I}^k_{n,t}$ we write

$$\delta_i(U)\mathbf{1}_{\{J_t=n\}} = \delta_i(U_{n,t})\mathbf{1}_{\{J_t=n\}}$$

The denomination "Malliavin derivative" and "Skorohod integral" is justified by a duality relationship, stated in next Proposition 1.1.15. In order to do this, let us introduce the border term operator, allowing to set up the integration by parts formula.
For our purposes, we will need only the border term operator with respect to the jump times, which is defined as follows.

Definition 1.1.12. [The border term operator] *For $F \in \mathscr{S}^0_{n,t}$ and $U \in \mathscr{P}^0_{n,t}$, the border term operator is defined as*

$$[F, U]_\pi = [F, U]^\Delta_\pi + [F, U]^T_\pi$$

where (recall that g, defined on $I = \bigcup_{j=1}^{k-1}(a_j, b_j)$ as in Assumption 1.1.1, is the pdf of the jump amplitudes Δ_i and p_i is the conditional pdf of T_i given that $\{J_t = n\}$ as in (1.1))

$$[F, U]^\Delta_\pi = \sum_{i=1}^n \sum_{j=1}^k \Big[\Big(\mathbb{T}_i(fu_i)\cdot \pi_i g\Big)(\omega, b_j^-) - \Big(\mathbb{T}_i(fu_i)\cdot \pi_i g\Big)(\omega, a_j^+)\Big] \tag{1.5}$$

$$
\begin{aligned}
[F,U]^T_\pi = \sum_{i=1}^n \Big[\Big(\mathbb{T}_{i+n}(fu_{i+n}) \cdot \pi_{i+n} p_i \Big)(\omega, T^-_{i+1}) + \\
- \Big(\mathbb{T}_{i+n}(fu_{i+n}) \cdot \pi_{i+n} p_i \Big)(\omega, T^+_{i-1}) \Big]
\end{aligned}
\tag{1.6}
$$

where f and u_i, $i = 1, \dots, 2n$, denote as usual the functions representing F and U_i, $i = 1, \dots, 2n$, respectively, and the $\mathbb{T}_i$'s being defined in (1.3) and (1.4).

Remark 1.1.13. *Obviously, the operators $[\cdot,\cdot]^\Delta_\pi$ and $[\cdot,\cdot]^T_\pi$ can be defined separately on $\mathscr{S}^0_{n,t}(\Delta) \times \mathscr{P}^0_{n,t}(\Delta)$ and $\mathscr{S}^0_{n,t}(T) \times \mathscr{P}^0_{n,t}(T)$ respectively.*
There are cases in which the border term operators are null, for example if, for any $i = 1, \dots, n$, $\pi_i(\omega, y) = 0$ on ∂I and $\pi_{i+n}(\omega, y) = 0$ on ∂I_i. The property $[\cdot,\cdot]_\pi \equiv 0$ turns out to be important, because it gives a duality formula similar to the classical one (see the following Proposition 1.1.15). Such a case is widely studied in Bally, Bavouzet and Messaoud [3] (see also [5]). For practical purposes, they assume $g \equiv 0$ on ∂I and take $\pi_i = 1$ for any $i = 1, \dots, n$, which give $[\cdot,\cdot]^\Delta_\pi \equiv 0$. Moreover, they set

$$
\pi_{i+n}(\omega, r) = (T_{i+1}(\omega) - r)^\lambda (r - T_{i-1}(\omega))^\lambda \mathbf{1}_{r \in (T_{i-1}(\omega), T_{i+1}(\omega))}, \quad i = 1, ..., n, \tag{1.7}
$$

for suitable values of λ giving $[\cdot,\cdot]^T_\pi \equiv 0$.

As it will be clear later on, we will always set $\pi_i \equiv 1$ for any $i = 1, \dots, 2n$ and take $g \equiv 0$ on ∂I. Therefore, we will have $[\cdot,\cdot]^\Delta_\pi \equiv 0$, but $[\cdot,\cdot]^T_\pi$ will not be equal to zero. In fact, let us consider the following example.

Example 1.1.14. *Let us write down the border term operators in the case we are going to study, that is: g is a lognormal type density and $\pi_i = 1$ for any i. In fact, we will take $1 + \Delta_i \overset{\mathscr{L}}{=} e^Z$, with $Z \sim \mathrm{N}(m, \varrho^2)$, so that*

$$
g(y) = \frac{1}{(1+y)\sqrt{2\pi\varrho^2}} \exp\Big(- \frac{(\log(1+y) - m)^2}{2\varrho^2} \Big) \mathbf{1}_{y > -1}
$$

which gives $I = (-1, +\infty)$, i.e. $k = 1$, $a_1 = -1$, $b_1 = +\infty$. Since $g(-1^+) = 0 = g(+\infty)$, and assuming $\pi_i = 1$, one immediately has $[F,U]^\Delta_\pi = 0$ for any F and U. Concerning the border term operator w.r.t. the jump times, its formula simplifies as follows:

$$
[F,U]^T_\pi = \sum_{i=1}^n \frac{\mathbb{T}_{i+n}(f)(\omega, T^-_{i+1}) - \mathbb{T}_{i+n}(f)(\omega, T^+_{i-1})}{T_{i+1} - T_{i-1}}.
$$

Let us add here also another useful observation. The Skorohod integral δ_i, $i = 1, \dots, n$, contains the quantity $\partial_y \ln g(\Delta_i)$. For our choice of g, we obtain

$$
\partial_y \ln g(\Delta_i) = \frac{m - \varrho^2 - \ln(1+\Delta_i)}{(1+\Delta_i)\varrho^2} \overset{\mathscr{L}}{=} \frac{m - \varrho^2 - Z}{\varrho^2} e^{-Z}.
$$

Notice that $\Delta_i, \partial_y \ln g(\Delta_i) \in L^p$ for any $p \geq 1$.

By elementary arguments one has that (recall Remarks 1.1.8, 1.1.10 and 1.1.13)

Proposition 1.1.15. *For any $F \in \mathscr{D}^1_{n,t}$ and $U \in \mathscr{I}^1_{n,t}$, such that*

$$\mathbb{E}\Big(\Big[|D_iFU_i\pi_i| + |F\delta_i(U_i)|\Big]\mathbf{1}_{\{J_t=n\}}\Big) < \infty \text{ for any } i = 1, \dots, 2n,$$

one has

$$\mathbb{E}(\langle DF, U\rangle_\pi \mathbf{1}_{\{J_t=n\}}) = \mathbb{E}(F\delta(U)\mathbf{1}_{\{J_t=n\}}) + \mathbb{E}([F,U]_\pi \mathbf{1}_{\{J_t=n\}}). \tag{1.8}$$

Moreover, (1.8) holds also if one is interested to work with the jump amplitudes or the jump times only, i.e. with $\mathscr{D}^1_{n,t}$, $\mathscr{I}^1_{n,t}$, D, δ, $[\cdot,\cdot]_\pi$ replaced by $\mathscr{D}^1_{n,t}(\Delta)$, $\mathscr{I}^1_{n,t}(\Delta)$, D^Δ, δ^Δ, $[\cdot,\cdot]^\Delta_\pi$ respectively or by $\mathscr{D}^1_{n,t}(T)$, $\mathscr{I}^1_{n,t}(T)$, D^T, δ^T, $[\cdot,\cdot]^T_\pi$ respectively.

Let us here recall some of the main properties of D and δ we are going to use.

Property 1.1.16. [Chain rule] *For any continuously differentiable $\phi : \mathbb{R}^d \to \mathbb{R}$ and $F_1, \dots, F_d \in \mathscr{D}^1_{n,t}$, $\phi(F_1, \dots, F_d) \in \mathscr{D}^1_{n,t}$ and*

$$D\phi(F_1, \dots, F_d)\mathbf{1}_{\{J_t=n\}} = \sum_{k=1}^d \partial_{x_k}\phi(F_1, \dots, F_d) DF_k \mathbf{1}_{\{J_t=n\}}.$$

Moreover, the above statement holds also with $\mathscr{D}^1_{n,t}$, D replaced by $\mathscr{D}^1_{n,t}(\Delta)$, D^Δ respectively or by $\mathscr{D}^1_{n,t}(T)$, D^T respectively.

Property 1.1.17. [Skorohod integral of a (special) product] *If $F \in \mathscr{D}^1_{n,t}$ and $U \in \mathscr{I}^1_{n,t}$ then $FU \in \mathscr{I}^1_{n,t}$ and*

$$\delta(FU)\mathbf{1}_{\{J_t=n\}} = F\delta(U)\mathbf{1}_{\{J_t=n\}} - \langle DF\mathbf{1}_{\{J_t=n\}}, U\rangle_\pi.$$

Moreover, the above statement holds also with $\mathscr{D}^1_{n,t}$, $\mathscr{I}^1_{n,t}$, D, δ replaced by $\mathscr{D}^1_{n,t}(\Delta)$, $\mathscr{I}^1_{n,t}(\Delta)$, D^Δ, δ^Δ respectively or by $\mathscr{D}^1_{n,t}(T)$, $\mathscr{I}^1_{n,t}(T)$, D^T, δ^T respectively.

In the following, we will need the existence of Malliavin derivatives of higher order. We define the associated space in the natural way: we say that $F \in \mathscr{D}^2_{n,t}$ if $F \in \mathscr{D}^1_{n,t}$ and $D_iF \in \mathscr{D}^1_{n,t}$. Similarly, we can define $\mathscr{D}^\ell_{n,t}$ for any $\ell \geq 1$.

Property 1.1.18. [Integration by parts formula] *Let $n \geq 1$ and $t > 0$ be fixed. Let $d \geq 1$, $F = (F^1, ..., F^d)$ and $v = (v^1, \dots, v^d)$, with $F^j \in \mathscr{D}^2_{n,t}$ and $v^j \in \mathscr{I}^1_{n,t}$, $j = 1, \dots, d$. Define $\gamma_{DF,v}$ as the following $d \times d$ random matrix:*

$$\gamma^{i,j}_{DF,v} = \langle DF^j \mathbf{1}_{\{J_t=n\}}, v^i\rangle_\pi, \quad i,j = 1, \dots, d. \tag{1.9}$$

Suppose that on the set $\{J_t = n\}$ there exists the inverse matrix $\hat\gamma_{DF,v}$ of $\gamma_{DF,v}$ and that each entry $\hat\gamma^{ij}_{DF,v}$ belongs to $\mathscr{D}^1_{n,t}$. Let $\phi \in \mathcal{C}^1_p(\mathbb{R}^d)$ and $G = (G^1,\dots,G^d)$, with $G^j \in \mathscr{S}^1_{n,t}$ for any j, be such that

$$\begin{aligned}&\mathbb{E}\Big(|D_i\phi(F)(G\hat\gamma_{DF,v})^i\pi_i|\mathbf{1}_{\{J_t=n\}}\Big) < \infty \quad i=1,\dots,2n\\ &\mathbb{E}\Big(|\phi(F)\delta_i((G\hat\gamma_{DF,v})^i)|\mathbf{1}_{\{J_t=n\}}\Big) < \infty \quad i=1,\dots,2n\end{aligned} \tag{1.10}$$

Then, one has

$$\begin{aligned}\mathbb{E}\Big(\sum_{i=1}^d \partial_{x_i}\phi(F)G^i\mathbf{1}_{\{J_t=n\}}\Big) = \;& \mathbb{E}\Big(\phi(F)\delta\big(\sum_{i=1}^d (G\hat\gamma_{DF,v})^i v^i\big)\mathbf{1}_{\{J_t=n\}}\Big)+\\ &+\mathbb{E}\Big([\phi(F),\sum_{i=1}^d (G\hat\gamma_{DF,v})^i v^i]_\pi\Big).\end{aligned}$$

Moreover, the above statements holds also with $\mathscr{D}^\ell_{n,t}$, $\mathscr{S}^1_{n,t}$, D, δ, $[\cdot,\cdot]_\pi$ replaced by $\mathscr{D}^\ell_{n,t}(\Delta)$, $\mathscr{S}^1_{n,t}(\Delta)$, D^Δ, δ^Δ, $[\cdot,\cdot]^\Delta_\pi$ respectively ((1.10) holding for $i=1,\dots,n$) or by $\mathscr{D}^\ell_{n,t}(T)$, $\mathscr{S}^1_{n,t}(T)$, D^T, δ^T, $[\cdot,\cdot]^T_\pi$ respectively ((1.10) holding for $i=n+1,\dots,2n$).

The (non-symmetric) matrix $\gamma_{DF,v}$ defined in (1.9) is a generalization of the well famous Malliavin covariance matrix σ_F, that is

$$\sigma_F = \gamma_{F,DF} = \Big(\langle DF^i\mathbf{1}_{\{J_t=n\}}, DF^j\mathbf{1}_{\{J_t=n\}}\rangle_\pi\Big)_{i,j=1,\dots,d}. \tag{1.11}$$

The proof of Property 1.1.18, to be seen in Appendix 1.4, is straightforward.

Remark 1.1.19. *Let us briefly discuss how the assumptions in (1.10) will work in our special context. As observed, we will take $\pi_i = 1$ for any i and g as a lognormal type density (see Example 1.1.14). Assuming that F, G and all the involved Malliavin derivatives belong to L^p for any p, and if we take similarly the v^j_i's, a sufficient condition allowing to get (1.10) is that all entries of $\hat\gamma_{DF,v}$ belong to L^p for any p. Now, this holds if*

$$|\det\gamma_{DF,v}|^{-1} \in L^p, \text{ for any } p \ge 1.$$

In the case $v = DF$, it reduces to ask for the p-integrability condition of the inverse of the determinant of the Malliavin covariance matrix, which is a quite standard request in Malliavin calculus.

Let us now consider the case of a pure jump diffusion:

$$X_t = x + \int_0^t b(r,X_r)dr + \sum_{i=1}^{J_t} c(T_i,\Delta_i,X_{T_i-}). \tag{1.12}$$

Assumption 1.1.20. *Suppose that* $(t,x) \mapsto c(t,a,x)$ *and* $x \mapsto b(t,x)$ *are twice differentiable, with bounded derivatives of first and second order, and having linear growth with respect to* x*, uniformly with respect to* t *and* a *(that is* $|c(t,a,x)| + |b(t,x)| \le K(1+|x|)$*).*

Assumption 1.1.20 guarantees the existence of the solution X to the pure-jump sde (1.12). Moreover, also its first variation process ξ is well defined ($\xi = \partial_x X$) and is given by

$$\xi_t = 1 + \int_0^t \partial_x b(r, X_r)\xi_r dr + \sum_{i=1}^{J_t} \partial_x c(T_i, \Delta_i, X_{T_i-})\xi_{T_i-}. \tag{1.13}$$

Let us recall that, by Assumption 1.1.20, both X_t and ξ_t belong to L^p for any t and the same holds for $\hat{\xi}_t = \xi_t^{-1}$, being the solution to

$$\hat{\xi}_t = 1 - \int_0^t \hat{\xi}_r \big(\partial_x b(r, X_r) - (\partial_x \sigma(r, X_r))^2\big) dr - \sum_{i=1}^{J_t} \hat{\xi}_{T_i-} \frac{\partial_x c(T_i, \Delta_i, X_{T_i-})}{1 + \partial_x c(T_i, \Delta_i, X_{T_i-})}. \tag{1.14}$$

In order to study the Malliavin derivatives, we have to add the following

Assumption 1.1.21. *There exists* $\eta > 0$ *such that, for any* r, a, x*,*

$$|\partial_a c(r,a,x)| + |1 + \partial_x c(r,a,x)| \ge \eta.$$

Assumption 1.1.22. *There exists* $\eta > 0$ *such that, for any* r, a, x*,*

$$|q(r,a,x)| + |1 + \partial_x c(r,a,x)| \ge \eta$$

where

$$q(r,a,x) = \Big(\partial_r c + b(1 + \partial_x c)\Big)(r,a,x) - b(r, x + c(r,a,x)). \tag{1.15}$$

Then, one has

Property 1.1.23. [Malliavin derivative of a pure jump diffusion] *Suppose Assumption 1.1.20 holds and let* X *and* ξ *be the solution to (1.12) and (1.13) respectively. Under Assumption 1.1.21 and 1.1.22, for any* $t > 0$ *and* $n \ge 1$*, the r.v.'s* $X_{T_p^-}, X_{T_p}, X_r \mathbf{1}_{T_p < r < T_{p+1}}$*, with* $p = 1, \dots, n$ *(with* $T_{n+1} \equiv t$*), and* $\int_0^t X_r dr$ *all belong to* $\mathscr{D}^2_{n,t}$*. Moreover, on the set* $\{J_t = n\}$ *one has, as* $i = 1, \dots, n$*,*

$$\begin{aligned} D_i X_{T_p} &= \mathbf{1}_{\{T_i \le T_p\}}\, \xi_{T_p} \xi_{T_i}^{-1} \partial_a c(T_i, \Delta_i, X_{T_i^-}) \\ D_i X_r &= \mathbf{1}_{\{T_i < r\}}\, \xi_r \xi_{T_i}^{-1} \partial_a c(T_i, \Delta_i, X_{T_i^-}) \\ D_{i+n} X_r &= \mathbf{1}_{\{T_i < r\}}\, \xi_r \xi_{T_i}^{-1} q(T_i, \Delta_i, X_{T_i^-}) \end{aligned}$$

and

$$D_{i+n}X_{T_p} = \begin{cases} 0 & \text{if } p < i \\ q(T_i, \Delta_i, X_{T_i^-}) + b(T_i, X_{T_i^-} + c(T_i, \Delta_i, X_{T_i^-})) & \text{if } p = i \\ \xi_{T_p}\xi_{T_i}^{-1} q(T_i, \Delta_i, X_{T_i^-}) & \text{if } p > i \end{cases}$$

with q defined in (1.15), the formulas for $X_{T_p^-}$ being immediate to derive from $X_{T_p^-} = X_{T_p} - c(T_p, \Delta_p, X_{T_p^-}))$. Moreover, one has

$$\begin{aligned} D_i \int_0^t X_r dr &= \mathbf{1}_{\{T_i<t\}}\, \xi_{T_i}^{-1} \partial_a c(T_i, \Delta_i, X_{T_i-}) \int_{T_i}^t \xi_r dr \\ D_{i+n} \int_0^t X_r dr &= \mathbf{1}_{\{T_i<t\}}\, \xi_{T_i}^{-1} q(T_i, \Delta_i, X_{T_i-}) \int_{T_i}^t \xi_r dr \end{aligned}$$

Finally, the above statement holds also with Assumption 1.1.21 plus Assumption 1.1.22, $\mathscr{D}^2_{n,t}$, D replaced by Assumption 1.1.21, $\mathscr{D}^2_{n,t}(\Delta)$, D^Δ respectively or by Assumption 1.1.22, $\mathscr{D}^2_{n,t}(T)$, D^T respectively.

Remark 1.1.24. *The conditions of Assumption 1.1.22*

$$|\partial_a c(r, a, x)| \geq \eta \quad |q(r, a, x,)| \geq \eta$$

are asked in the paper of Bally, Bavouzet and Messaoud [3] and assure that the non degeneracy condition, that allows to use the integration by parts formula, holds true, when one considers the jump amplitudes and times direction of calculus respectively. We can ask a weaker condition on $q(r, a, x)$ that is the following:

$$q_i \equiv q(T_i, \Delta_i, X_{T_i-}) \in L^p \; \forall p.$$

Notice that in the Ornstein-Uhlenbeck model, for which we will consider the jump times direction of calculus (see Section 2.5.2), we have $q(r, a, x) = -ba$. Therefore the condition $|q(r, a, x)| > \eta$ does not hold true. By the way $q_i \in L^p \; \forall p$, being Δ_i log-normal distributed.
In conclusion we will always ask the condition $|1 + \partial_x c(r, a, x)| \geq \eta$ whenever we will consider the jumps (amplitudes or times) direction, that assures the existence of the inverse of the first variation process ξ^{-1}

The writing of Property 1.1.23 is new and its proof is postponed to Appendix 1.4.
Let us now consider the case of a jump diffusion:

$$X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r + \sum_{i=1}^{J_t} c(T_i, \Delta_i, X_{T_i-}). \tag{1.16}$$

When dealing with Malliavin derivatives, no general formulas can be stated in the direction of the jump times, as it follows from next simple

Example 1.1.25. *The r.v. W_{T_1} is not derivable in the Malliavin sense in the direction of the jump times. In fact, one has $W_{T_1} = f(\omega, T_1)$, in which $f(\omega, t) = W_t(\omega)$. Since $\partial_t f(\omega, t)$ does not exist, the Malliavin derivative in the direction of the jump times cannot be done, that is $W_{T_1} \notin \mathscr{D}^1_{n,t}(T)$ for any $n \geq 1$ and $t > 0$. Nevertheless, as observed in Remark 1.2.2, the (generalized) Gaussian Malliavin derivative exists for any t.*

Therefore, we concentrate our attention on the Malliavin derivatives w.r.t. the jump amplitudes.

Assumption 1.1.26. *In addition to what required in Assumption 1.1.20, one requires that $(t, x) \mapsto \sigma(t, x)$ is continuous and $x \mapsto \sigma(t, x)$ is twice differentiable, with bounded derivatives of first and second order, and having linear growth with respect to x, uniformly with respect to t (that is $|\sigma(t, x)| \leq K(1 + |x|)$).*

Under Assumption 1.1.26 also the first variation process ξ is well defined and given by

$$\begin{aligned} \xi_t = \;& 1 + \int_0^t \partial_x b(r, X_r)\xi_r dr + \int_0^t \partial_x \sigma(r, X_r)\xi_r dW_r \\ & + \sum_{i=1}^{J_t} \partial_x c(T_i, \Delta_i, X_{T_i-})\xi_{T_i-}. \end{aligned} \tag{1.17}$$

Again, Assumption 1.1.20 ensures that both X_t and ξ_t belong to L^p for any t and the same holds for $\hat{\xi}_t = \xi_t^{-1}$, being the solution to

$$\begin{aligned} \hat{\xi}_t = \;& 1 - \int_0^t \hat{\xi}_r \big(\partial_x b(r, X_r) - (\partial_x \sigma(r, X_r))^2\big) dr - \int_0^t \hat{\xi}_r \partial_x \sigma(r, X_r) dW_r \\ & - \sum_{i=1}^{J_t} \hat{\xi}_{T_i-} \frac{\partial_x c(T_i, \Delta_i, X_{T_i-})}{1 + \partial_x c(T_i, \Delta_i, X_{T_i-})}. \end{aligned} \tag{1.18}$$

Then, one has

Property 1.1.27. [Malliavin derivative of a jump diffusion] *Suppose Assumption 1.1.26 holds and let X and ξ be the solution to (1.16) and (1.17) respectively. Under Assumption 1.1.21, for any $t > 0$ and $n \geq 1$, the r.v.'s $X_{T_p^-}, X_{T_p}, X_r \mathbf{1}_{T_p < r < T_{p+1}}$, with $p = 1, \ldots, n$ (with $T_{n+1} \equiv t$), and $\int_0^t X_r dr$ all belong to $\mathscr{D}^2_{n,t}(\Delta)$. Moreover, on the set $\{J_t = n\}$ one has, as $i = 1, \ldots, n$,*

$$\begin{aligned} D_i X_{T_p} &= \mathbf{1}_{\{T_i \leq T_p\}}\, \xi_{T_p} \xi_{T_i}^{-1} \partial_a c(T_i, \Delta_i, X_{T_i^-}) \\ D_i X_r &= \mathbf{1}_{\{T_i < r\}}\, \xi_r \xi_{T_i}^{-1} \partial_a c(T_i, \Delta_i, X_{T_i^-}) \end{aligned}$$

the formulas for $X_{T_p^-}$ being immediate from $X_{T_p^-} = X_{T_p} - c(T_p, \Delta_p, X_{T_p^-})$. Moreover, one has

$$D_i \int_0^t X_r dr = \mathbf{1}_{\{T_i < t\}}\, \xi_{T_i}^{-1} \partial_a c(T_i, \Delta_i, X_{T_i-}) \int_{T_i}^t \xi_r dr$$

Remark 1.1.28. *It is worth to observe that, due to Assumption 1.1.20 and 1.1.26, all the above Malliavin derivatives, also of the second order, belong to* L^p *for any* $p \geq 1$*.*

1.2 Malliavin calculus in the Gaussian direction

Let W_t denote a Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$ and le $\mathscr{W}_t$ be the augmentation w.r.t. $\mathbb{P}$ of the filtration generated by W. Let us briefly recall how the standard Gaussian Malliavin calculus works. The most used and well known reference for this framework is given by Nualart [25]. Here, we follow the approach of Bally as in [2]. Set T as the time horizon and for $N \in \mathbb{N}$, set

$$\Delta^N W = (\Delta_0^N W, \dots, \Delta_{\lfloor 2^N T \rfloor - 1}^N W)$$

where

$$\Delta_k^N W = W_{\frac{k+1}{2^N}} - W_{\frac{k}{2^N}}, k = 0, 1, \dots, \lfloor 2^N T \rfloor - 1.$$

Set now $\mathscr{S}_N$ the set of the simple functionals of order N: $F \in \mathscr{S}_N$ if there exists $f^F \in C_b^\infty(\mathbb{R}^{\lfloor 2^N T \rfloor}, \mathbb{R})$ such that

$$F = f^F(\Delta^N W).$$

It is easy to see that $\mathscr{S}_N \subset \mathscr{S}_{N+1}$ and we set $\mathscr{S} = \cup_{N \geq 1} \mathscr{S}_N$. Finally, notice that $\mathscr{S} \subset L^p(\Omega)$ for any p.
Now, we can define the class $\mathscr{P}_N$ of the simple processes of order N: $u \in \mathscr{P}_N$ if there exists $U_0, \dots, U_{\lfloor 2^N T \rfloor - 1} \in \mathscr{S}_N$ such that

$$u_s = \sum_{k=0}^{\lfloor 2^N T \rfloor - 1} U_k \mathbf{1}_{s \in \left[\frac{k}{\lfloor 2^N T \rfloor}, \frac{k+1}{\lfloor 2^N T \rfloor}\right)}.$$

Again, $\mathscr{P}_N \subset \mathscr{P}_{N+1}$ and we set $\mathscr{P} = \cup_{N \geq 1} \mathscr{P}_N$. Notice that a simple process $u \in \mathscr{P}$ is not necessarily adapted and if it is, one has that each U_k depends on $\Delta_i^N W$ up to $i = k - 1$.
Le us also observe that $\mathscr{P} \subset L^p(H)$ for any p, where

$$H = \{\text{progressively measurable processes } u \,:\, u \in L^2([0, T]) \text{ a.s.}\}$$

and

$$L^p(H) = \{u \in H \,:\, \mathbb{E}(\|u\|_{L^2([0,T])}^p) < \infty\}$$

Therefore, for $u \in L^p(H)$ we set

$$\|u\|_{L^p(H)}^p = \mathbb{E}\Big(\|u\|_{L^2[0,T]}^p\Big) = \mathbb{E}\Big(\Big|\int_0^T |u_s|^2 ds\Big|^{p/2}\Big)$$

which makes $L^p(H)$ a Banach space. Moreover, since $L^2(H) = L^2(\Omega \times [0,T])$, in the special case $p = 2$, one obtains that $L^2(H)$ is a Hilbert space, with inner product given by

$$\langle u, v\rangle_{L^2(H)} = \mathbb{E}(\langle u, v\rangle_{L^2[0,T]}) = \mathbb{E}\Big(\int_0^T u_s v_s ds\Big).$$

We can pass now to define the Malliavin derivative and the Skorohod integral. For $F \in \mathscr{S}$, the Malliavin derivative is defined as

$$D_s F = \sum_{k=0}^{\lfloor 2^N T\rfloor -1} \partial_{x_k} f^F(\Delta^N W)\mathbf{1}_{s\in\left[\frac{k}{\lfloor 2^N T\rfloor}, \frac{k+1}{\lfloor 2^N T\rfloor}\right)}$$

and for $u \in \mathscr{P}$, the Skorohod integral is given by

$$\delta(u) = \sum_{k=0}^{\lfloor 2^N T\rfloor -1} \Big(U_k \Delta_k^N W - \partial_{x_k} f^{U_k}(\Delta^N W)\frac{1}{2^{\lfloor 2^N T\rfloor}}\Big)$$

being N the one such that $F \in \mathscr{S}_N$ and $u \in \mathscr{P}_N$ (recall that f^{U_k} is the C_b^∞ function representing $U_k \in \mathscr{S}_N$). It easily follows that both D and δ are really independent of N, so that they are both well defined. Notice that $\delta(u) = \int_0^T u_s dW_s$ (the standard Ito integral) whenever u is adapted.
Therefore, for any p,

$$D : \mathscr{S} \subset L^p(\Omega) \to \mathscr{P} \subset L^p(H) \;\text{ and }\; \delta : \mathscr{P} \subset L^p(H) \to \mathscr{S} \subset L^p(\Omega) \quad (1.19)$$

Thinking to (1.19) in the special case $p = 2$, the following duality property is immediate to check:

$$\text{for any } F \in \mathscr{S} \text{ and } u \in \mathscr{P} \text{ then } \mathbb{E}(\langle DF, u\rangle_{L^2[0,T]}) = \mathbb{E}(F\delta(u)) \quad (1.20)$$

which allows to extend the (unbounded, linear) operators in (1.19). In fact, when the case $p = 2$ is considered, by using (1.20) one can show that both D and δ are closable: if $F_n \to 0$ in $L^2(\Omega)$ and $DF_n \to u$ in $L^2(H)$ then $u = 0$, as well as if $u_n \to 0$ in $L^2(H)$ and $\delta(u_n) \to G$ in $L^2(H)$ then $G = 0$. So, by setting

$$\|F\|_{1,2} = \|F\|_{L^2(\Omega)} + \|DF\|_{L^2(H)}, \quad F \in \mathscr{S}, \quad (1.21)$$
$$\|u\|_{\delta,2} = \|u\|_{L^2(H)} + \|\delta(u)\|_{L^2(\Omega)}, \quad u \in \mathscr{P} \quad (1.22)$$

one can define the set of the Malliavin derivable r.v.'s as

$$\mathbb{D}^{1,2} = \overline{\mathscr{S}}^{\|\cdot\|_{1,2}} \quad (1.23)$$

and the set of the Skorohod integrable processes as

$$Dom_2(\delta) = \overline{\mathscr{P}}^{\|\cdot\|_{\delta,2}} \quad (1.24)$$

Such spaces are both Hilbert ones, with inner product given by

$$\begin{aligned}\langle F, G\rangle_{1,2} &= \langle F, G\rangle_{L^2(\Omega)} + \langle DF, DG\rangle_{L^2(H)}, \quad F, G \in \mathbb{D}^{1,2}\\ \langle u, v\rangle_{\delta,2} &= \langle u, v\rangle_{L^2(H)} + \langle \delta(u), \delta(v)\rangle_{L^2(\Omega)}, \quad u, v \in Dom_2(\delta)\end{aligned}$$

respectively. Now, by construction the duality relationship continues to hold between $\mathbb{D}^{1,2}$ and $Dom_2(\delta)$:

$$\text{for any } F \in \mathbb{D}^{1,2} \text{ and } u \in Dom_2(\delta) \text{ then } \mathbb{E}(\langle DF, u\rangle) = \mathbb{E}(F\delta(u)) \tag{1.25}$$

Unfortunately, both $\mathbb{D}^{1,2}$ and $Dom_2(\delta)$ are not algebras. So, it comes natural to consider the forthcoming spaces. For a generical p, set

$$\|F\|_{1,p} = \|F\|_{L^p(\Omega)} + \|DF\|_{L^p(H)}, \quad F \in \mathscr{S}, \tag{1.26}$$

$$\|u\|_{\delta,p} = \|u\|_{L^p(H)} + \|\delta(u)\|_{L^p(\Omega)}, \quad u \in \mathscr{P} \tag{1.27}$$

and

$$\mathbb{D}^{1,p} = \overline{\mathscr{S}}^{\|\cdot\|_{1,p}} \tag{1.28}$$

$$Dom_p(\delta) = \overline{\mathscr{P}}^{\|\cdot\|_{\delta,p}} \tag{1.29}$$

By using standard arguments for L^p spaces, for $p \geq \bar{p}$, one has $\mathbb{D}^{1,p} \subset \mathbb{D}^{1,\bar{p}}$ and $Dom_p(\delta) \subset Dom_{\bar{p}}(\delta)$. Moreover, if $\frac{1}{p} + \frac{1}{q} = 1$, one has that if $F \in \mathbb{D}^{1,rp}$ and $G \in \mathbb{D}^{1,rq}$ then $FG \in \mathbb{D}^{1,r}$, as well as if $F \in \mathbb{D}^{1,rp}$ and $u \in Dom_{rq}(\delta)$ then $Fu \in Dom_r(\delta)$. So, one sets

$$\mathbb{D}^{1,\infty} = \bigcap_p \mathbb{D}^{1,p} \quad \text{and} \quad Dom_\infty(\delta) = \bigcap_p Dom_p(\delta). \tag{1.30}$$

Now the algebra properties hold. For example, $FG \in \mathbb{D}^{1,\infty}$ when $F, G \in \mathbb{D}^{1,\infty}$, as well as $Fu \in Dom_\infty(\delta)$ if $F \in \mathbb{D}^{1,\infty}$ and $u \in Dom_\infty(\delta)$. A lot of further interesting and useful facts turn out to be true, and we will soon refer to some of them specialized in our context. The forthcoming ideas to generalize the Gaussian Malliavin calculus to objects depending on other variables independent of the Brownian motion have been already used e.g. by Davis and Johansson [13] or by Forster, Lütkebohmert and Teichman [15].

Let $\{T_i\}_{i\geq 1}$ and $\{\Delta_i\}_{i\geq 1}$ denote the jump times and amplitudes respectively, as before, and suppose they are both independent from the Brownian motion. Let us also set, as $s \geq 0$,

$$\mathscr{G}_s = \sigma(J_u\,;\, u \leq s) \vee \sigma(\Delta_j \mathbf{1}_{\{j \leq J_s\}}\,;\, j \geq 1)$$

(recall that $J_s = \sum_{j\geq 1} \mathbf{1}_{\{T_j \leq s\}}$). Therefore, $\mathscr{G}_s$ gives all the information coming out from the jump times and amplitudes noise up to time s.

For $n \geq 1$ and $t > 0$ fixed, let us consider the notation

$$z_{n,t} = (y_1, \ldots, y_n, t_1, \ldots, t_n) \in \mathbb{R}^n \times [0, t]^n$$
$$\text{and } Z_{n,t} = (\Delta_1, \ldots, \Delta_n, T_1, \ldots, T_n).$$

Definition 1.2.1. [Generalized Gaussian Malliavin derivative] *Fix* $n \geq 1$ *and* $t > 0$. *Let* $f : \Omega \times \mathbb{R}^n \times [0,t]^n \to \mathbb{R}$ *be* $\mathscr{W}_t \times \mathscr{B}(\mathbb{R}^n) \times \mathscr{B}([0,t]^n)$*-measurable such that* $\omega \mapsto f(\omega, z_{n,t}) \in \mathbb{D}^{1,p}$ *for any* $z_{n,t} \in \mathbb{R}^n \times [0,t]^n$. *On the set* $\{J_t = n\}$, *set*

$$F = f(\cdot, z_{n,t})\Big|_{z_{n,t}=Z_{n,t}} \quad \text{and} \quad D_0 F = Df(\cdot, z_{n,t})\Big|_{z_{n,t}=Z_{n,t}}$$

The class $\mathbb{D}^{1,p}_{n,t}$ *is defined as the random variables* F *as above and such that*

$$F\mathbf{1}_{\{J_t=n\}} \in L^p(\Omega) \quad \text{and} \quad D_0 F \mathbf{1}_{\{J_t=n\}} \in L^p(H)$$

For $F \in \mathbb{D}^{1,p}_{n,t}$, $D_0 F \mathbf{1}_{\{J_t=n\}}$ *will stand for the Malliavin derivative in the Gaussian direction on the set* $\{J_t = n\}$.
We similarly define $\mathbb{D}^{k,p}_{n,t}$ *and set* $\mathbb{D}^{k,\infty}_{n,t} = \bigcap_p \mathbb{D}^{k,p}_{n,t}$.

Example 1.2.2. *As an example,* $W_{T_1} \in \mathbb{D}^{1,\infty}_{n,t}$ *for any* $n \geq 1$ *and* $t > 0$. *In fact,* $W_{T_1} = f(\omega, T_1)$, *in which* $f(\omega, t) = W_t(\omega)$. *The Gaussian Malliavin derivative* $D = (D_u)_u$ *of* $f(\omega, t)$ *exists for any* t: $D_u f(\omega, t) = \mathbf{1}_{\{u \leq t\}}$, *so that one has*

$$D_{0,u} W_{T_1} = D_u W_t\Big|_{t=T_1} = \mathbf{1}_{\{u<T_1\}}$$

Similarly we can extend the definition of the Skorohod integral.

Definition 1.2.3. [Generalized Gaussian Skorohod integral] *Fix* $n \geq 1$ *and* $t > 0$. *Let* $u : [0,t] \times \Omega \times \mathbb{R}^n \times [0,t]^n \to \mathbb{R}$ *be* $\mathscr{B}([0,t]) \times \mathscr{W}_t \times \mathscr{B}(\mathbb{R}^n) \times \mathscr{B}([0,t]^n)$*-measurable such that* $(r, \omega) \mapsto u_r(\omega, z_{n,t}) \in Dom_p(\delta)$ *for any* $z_{n,t} \in \mathbb{R}^n \times \mathbb{R}^n_+$. *On the set* $\{J_t = n\}$, *set*

$$u = u(\cdot, z_{n,t})\Big|_{z_{n,t}=Z_{n,t}} \quad \text{and} \quad \delta_0(u) = \delta\big(u(\cdot, z_{n,t})\big)\Big|_{z_{n,t}=Z_{n,t}}$$

The class $Dom_{n,t,p}(\delta_0)$ *is defined as the random processes* u *as above and such that*

$$u\mathbf{1}_{\{J_t=n\}} \in L^p(H) \quad \text{and} \quad \delta_0(u)\mathbf{1}_{\{J_t=n\}} \in L^p(\Omega)$$

For $u \in Dom_{n,t,p}(\delta_0)$, $\delta_0(u)\mathbf{1}_{\{J_t=n\}}$ *will stand for the Skorohod integral in the Gaussian direction on* $\{J_t = n\}$. *We set* $Dom_{n,t,\infty}(\delta_0) = \bigcap_p Dom_{n,t,p}(\delta_0)$.

The duality relationship between D_0 and δ_0 holds as well if restricted to the set $\{J_t = n\}$. In fact

Proposition 1.2.4. *Fix* $n \geq 1$ *and* $t > 0$. *For any* $F \in \mathbb{D}^{1,2}_{n,t}$ *and* $u \in Dom_{n,t,2}(\delta_0)$ *one has*

$$\mathbb{E}(\langle D_0 F, u\rangle_{L^2[0,T]} \mathbf{1}_{\{J_t=n\}}) = \mathbb{E}(F\delta_0(u)\mathbf{1}_{\{J_t=n\}}).$$

Proof. Let us use the following notation: $F^{z_{n,t}} = f(\cdot, z_{n,t})$ and $u^{z_{n,t}} = u(\cdot, z_{n,t})$. Recalling that a $\mathscr{W}_t$-measurable r.v. is independent of $\mathscr{G}_t = \sigma(J_u\,;\, u \leq t) \vee \sigma(\Delta_j \mathbf{1}_{\{j \leq J_t\}}\,;\, j \geq 1)$, we can write

$$\begin{aligned}
\mathbb{E}\Big(\langle D_0 F, u\rangle \mathbf{1}_{\{J_t=n\}}\Big) &= \mathbb{E}\Big(\mathbb{E}(\langle D_0 F, u\rangle_{L^2[0,T]} \mathbf{1}_{\{J_t=n\}} \,|\, \mathscr{G}_t\Big) \\
&= \mathbb{E}\Big(\mathbb{E}(\langle D_0 F^{z_{n,t}}, u^{z_{n,t}}\rangle_{L^2[0,T]}) \,|_{z_{n,t}=Z_{n,t}} \mathbf{1}_{\{J_t=n\}}\Big) \\
&= \mathbb{E}\Big(\mathbb{E}(\langle D F^{z_{n,t}}, u^{z_{n,t}}\rangle_{L^2[0,T]}) \,|_{z_{n,t}=Z_{n,t}} \mathbf{1}_{\{J_t=n\}}\Big) \\
&= \mathbb{E}\Big(\mathbb{E}(F^{z_{n,t}} \delta(u^{z_{n,t}})) \,|_{z_{n,t}=Z_{n,t}} \mathbf{1}_{\{J_t=n\}}\Big) \\
&= \mathbb{E}\Big(\mathbb{E}(F^{z_{n,t}} \delta_0(u^{z_{n,t}})) \,|_{z_{n,t}=Z_{n,t}} \mathbf{1}_{\{J_t=n\}}\Big) \\
&= \mathbb{E}\Big(\mathbb{E}(F \delta_0(u) \,|\, \mathscr{G}_t) \mathbf{1}_{\{J_t=n\}}\Big) \\
&= \mathbb{E}\Big(F \delta_0(u) \mathbf{1}_{\{J_t=n\}}\Big)
\end{aligned}$$

and the statement holds. □

By using similar arguments, one can extend the classical properties of the Malliavin derivative and the Skorohod integral. In fact, the following properties easily follow.

Property 1.2.5. [Chain rule] *For any $\phi \in C_b^1(\mathbb{R}^d, \mathbb{R})$ and $F_1, \ldots, F_d \in \mathbb{D}_{n,t}^{1,2}$, one has*

$$D_0 \phi(F_1, \ldots, F_n) \mathbf{1}_{\{J_t=n\}} = \sum_{i=1}^{d} \partial_{x_i} \phi(F_1, \ldots, F_n) D_0 F_i \mathbf{1}_{\{J_t=n\}}.$$

Moreover, the statement holds if $F_1, \ldots, F_d \in \mathbb{D}_{n,t}^{1,\infty}$ and $\phi \in C_p^1(\mathbb{R}^d, \mathbb{R})$.

Property 1.2.6. [Skorohod integral of a (special) product] *If $F \in \mathbb{D}_{n,t}^{1,2}$ and $u \in Dom_{n,t,2}(\delta_0)$ are such that $Fu \in Dom_{n,t,2}(\delta_0)$ then*

$$\delta_0(Fu) \mathbf{1}_{\{J_t=n\}} = F \delta_0(u) \mathbf{1}_{\{J_t=n\}} - \int_0^t D_{0,s} F \mathbf{1}_{\{J_t=n\}} u_s ds.$$

Remark 1.2.7. *As in the classical case, one has $Fu \in Dom_{n,t,r}(\delta_0)$ whenever $F \in \mathbb{D}_{n,t}^{1,rp}$ and $u \in Dom_{n,t,rq}(\delta_0)$, with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

Property 1.2.8. [Integration by parts formula] *Let $d \geq 1$, $F = (F^1, ..., F^d)$ and $v = (v^1, \ldots, v^d)$, with $F^j \in \mathbb{D}_{n,t}^{1,2}$ and $v^j \in Dom_{n,t,2}(\delta_0)$, $j = 1, \ldots, d$. On the set $\{J_t = n\}$, define $\gamma_{D_0 F, v}$ as the following $d \times d$ random matrix:*

$$\gamma_{D_0 F, v}^{i,j} = \langle D_0 F^j, v^i \rangle, \quad i, j = 1, \ldots, d. \tag{1.31}$$

Suppose there exists the inverse matrix $\hat{\gamma}_{DF,v}$ of $\gamma_{D_0 F, v}$ on the set $\{J_t = n\}$ and that each entry $\hat{\gamma}_{D_0 F, v}^{ij}$ belongs to $\mathbb{D}_{n,t}^{1,2}$. Then for any $\phi \in \mathcal{C}_b^1(\mathbb{R}^d)$, $G =$

$(G^1,\dots,G^d)$ *such that* $(G\hat\gamma_{DF,v})^i v^i \in Dom_{n,t,2}(\delta_0)$ *for any* $i=1,\dots,d$, *one has*

$$\mathbb{E}\Big(\sum_{i=1}^d \partial_{x_i}\phi(F)G^i\mathbf{1}_{\{J_t=n\}}\Big)=\mathbb{E}\Big(\phi(F)\delta_0\big(\sum_{i=1}^d (G\hat\gamma_{DF,v})^i v^i\big)\mathbf{1}_{\{J_t=n\}}\Big).$$

The case $v=D_0F$ gives the Malliavin covariance matrix σ_F on $\{J_t=n\}$, that is

$$\sigma_F=\gamma_{D_0F,D_0F}=\Big(\langle D_0F^j,D_0F^i\rangle_{L^2([0,T])}\Big)_{i,j=1,\dots,d},\quad \text{on } \{J_t=n\}. \tag{1.32}$$

The proof of Property 1.2.8 is straightforward (see Appendix 1.4).

Remark 1.2.9. *The proof of the above integration by parts formula reduces to the application of the duality formula between* D_0F *and* $\sum_i (G\hat\gamma_{D_0F,v})^i v^i$. *Therefore, one should have* $(G\hat\gamma_{D_0F,v})^i v^i \in Dom_{n,t,2}(\delta_0)$ *for any* $i=1,\dots,d$. *In view of Remark 1.2.7, it would be sufficient that* $F\in\mathbb{D}^{1,2p}_{n,t}$ *and* $(G\hat\gamma_{D_0F,v})^i v^i \in Dom_{n,t,2q}(\delta_0)$, *with* $p,q\geq 1$ *and* $\frac1p+\frac1q=1$. *Again, one needs* $(G\hat\gamma_{D_0F,v})^i\in\mathbb{D}^{1,2p\bar p}_{n,t}$ *and* $v^i\in Dom_{n,t,2q\bar q}(\delta_0)$, *with* $\bar p,\bar q\geq 1$ *and* $\frac1{\bar p}+\frac1{\bar q}=1$. *As for the first requirement, it holds if* $G^i\in\mathbb{D}^{1,2p\bar p\hat p}_{n,t}$ *and* $\hat\gamma^{ij}_{D_0F,v}\in\mathbb{D}^{1,2p\bar p\hat q}_{n,t}$, $\frac1{\hat p}+\frac1{\hat q}=1$. *And notice that* $\hat\gamma_{D_0F,v}$ *is an inverse matrix, so that in its entries also* $(\det\hat\gamma_{D_0F,v})^{-1}$ *and* D_0F *has to be taken into account. Therefore, the only reasonable general assumptions giving the integration by parts formula are the following:* $F^i\in\mathbb{D}^{2,\infty}_{n,t}$, $v^i\in Dom_{n,t,\infty}(\delta_0)$ *and* $G^i\in\mathbb{D}^{1,\infty}_{n,t}$ *for all* $i=1,\dots,i$, *and finally*

$$\mathbb{E}\big(|\det\hat\gamma_{D_0F,v}|^{-p}\big)<\infty \quad \textit{for any } p.$$

The only good news are that one can require $\phi\in C^1_p$ *instead of* $\phi\in C^1_b$ *(see Property 1.2.5).*

Let us now consider the case of a diffusion with jumps:

$$X_t=x+\int_0^t b(r,X_r)dr+\int_0^t\sigma(r,X_r)dW_r+\sum_{i=1}^{J_t}c(T_i,\Delta_i,X_{T_i-}) \tag{1.33}$$

under the following

Assumption 1.2.10. *Suppose that* $(t,x)\mapsto c(t,a,x),\sigma(t,x),b(t,x)$ *are continuous and* $x\mapsto c(t,a,x),\sigma(t,x),b(t,x)$ *are twice differentiable, with bounded derivatives of first and second order, and having linear growth with respect to* x, *uniformly with respect to* t *and* a.

Assumption 1.2.10 guarantees the existence of the solution X to (1.33). Moreover, also its first variation process ξ is well defined ($\xi=\partial_xX$) and solves the sde with jumps as in (1.17). Let us also recall that, by Assumption 1.2.10, both X_t and ξ_t belong to L^p for any t and the same holds for $\hat\xi_t=\xi_t^{-1}$, being the solution to Equation (1.18). Roughly speaking, the assumptions on the first and second order derivative of the diffusion coefficients allow to handle the first order and the second order Malliavin derivative respectively. In fact, one has (see e.g. Forster, Lütkebohmert and Teichman [15])

Property 1.2.11. [Gaussian Malliavin derivative of a jump diffusion] *If Assumption (1.2.10) holds then for any $n \geq 1$ and $t > 0$, X_r and $\int_0^r X_u du$, as $r \leq t$, and X_{T_i}, as $i \leq n$, all belong to $\mathbb{D}^{2,\infty}_{n,t}$ and on the set $\{J_t = n\}$ one has*

$$\begin{aligned} D_{0,s} X_r &= \xi_r \xi_s^{-1} \sigma(s, X_s) \mathbf{1}_{\{s \leq r\}}, \\ D_{0,s} \int_0^r X_u du &= \int_0^r D_{0,s} X_u du = \int_s^r \xi_u du \, \xi_s^{-1} \sigma(s, X_s). \\ D_{0,s} X_{T_i} &= \xi_{T_i} \xi_s^{-1} \sigma(s, X_s) \mathbf{1}_{\{s \leq T_i\}}, \end{aligned}$$

Moreover, if in addition and there exists $\eta > 0$ such that, for any r, a, x,

$$|1 + \partial_x c(r, a, x)| \geq \eta.$$

then X_{T_i-}, as $i \leq n$, belongs to $\mathbb{D}^{2,\infty}_{n,t}$ and on the set $\{J_t = n\}$ one has

$$D_{0,s} X_{T_i-} = \frac{\xi_{T_i} \xi_s^{-1} \sigma(s, X_s)}{1 + \partial_x c(T_i, \Delta_i, X_{T_i-})} \mathbf{1}_{\{s \leq T_i\}}$$

1.3 Jump diffusions: unifying Malliavin calculus

In this section we develop a unifying Malliavin calculus, which takes into account all the randomness involved. In particular, we consider Malliavin derivatives and Skorohod integrals in the direction of the Brownian motion, of the jump times and the jump amplitudes, and we get a duality relationship allowing to set up a stochastic calculus of variation in the context of jump-diffusion processes.
It has to be said that the joint Malliavin derivatives in the Brownian direction and the jump times cannot be stated in general for jump diffusions (see Example 1.1.25).
Now, in the first two sections we have seen the Malliavin derivative for processes depending on the Brownian motion and on the jump amplitudes and times, in all directions of randomness. Now, in order to unifying the notations, we consider the following definitions. Let $n \geq 1$ and $t > 0$ be fixed.

Definition 1.3.1. [Unified Malliavin derivative on the set $\{J_t = n\}$] *Given a random variable F we define a Malliavin operator on the set $\{J_t = n\}$ as*

$$\tilde{D} F = (\tilde{D}_0 F, \tilde{D}_1 F, ..., \tilde{D}_n F, \tilde{D}_{n+1} F, ..., \tilde{D}_{2n} F)$$

where:

- *if* $F \in \mathbb{D}^{1,2}_{n,t}$,
$$\tilde{D}_0 F = D_0 F,$$
being D_0 *as in Definition 1.2.1, otherwise we set* $\tilde{D}_0 F = 0$*;*

- *if* $F \in \mathscr{D}^1_{n,t}(\Delta)$,
$$\tilde{D}_i F \mathbf{1}_{\{J_t=n\}} = D_i F \mathbf{1}_{\{J_t=n\}}, \quad i = 1, ..., n,$$
being D_i *as in Definition 1.1.11, otherwise we set* $\tilde{D}_i F = 0$ *for* $i = 1, ..., n$.

- *if* $F \in \mathscr{D}^1_{n,t}(T)$,
$$\tilde{D}_{i+n} F \mathbf{1}_{\{J_t=n\}} = D_{i+n} F \mathbf{1}_{\{J_t=n\}}, \quad i = 1, ..., n,$$
being D_{i+n} *as in Definition 1.1.11, otherwise we set* $\tilde{D}_{i+n} F = 0$ *for* $i = 1, ..., n$.

Example 1.3.2. *Consider for example the r.v.* $F = W_{T_1} + \sum_{j=1}^{J_t} \Delta_j$*. As remarked in Example 1.1.25,* F *is not Malliavin differentiable in the direction of jump times but* $\tilde{D}F$ *can be written, having that on* $\{J_t = n\}$

$$\begin{aligned} \tilde{D}F &= (D_{0,s}F, D_1 F, ..., D_n F, 0, ..., 0) \\ &= (\mathbf{1}_{s \le T_1}, 1, ..., 1, 0, ..., 0) \end{aligned}$$

Definition 1.3.3. [Unified Skorohod operator on the set $\{J_t = n\}$] *Given a process* $u = (u_0, U_1, ..., U_{2n})$ *made by a random process* $u_0 = (U_0(t))_{t \ge 0}$ *and a r.v.* $(U_1, ..., U_{2n})$ *taking values on* $\mathbb{R}^{2n}$*, we define the unified Skorohod operator on the set* $\{J_t = n\}$ *as*

$$\tilde{\delta}(u) = \tilde{\delta}_0(u_0) + \sum_{j=1}^{2n} \tilde{\delta}_j(U_j)$$

where:

- *if* $u_0 \in Dom_{n,t,2}(\delta_0)$,
$$\tilde{\delta}_0(u_0) = \delta_0(u_0)$$
being δ_0 *as in Definition 1.2.3, otherwise we set* $\tilde{\delta}_0(u_0) = 0$*;*

- *if* $U^\Delta = (U_1, \ldots, U_n) \in \mathscr{I}^0_{n,t}(\Delta)$,
$$\tilde{\delta}_i(U_i) \mathbf{1}_{\{J_t=n\}} = \delta_i(U_i) \mathbf{1}_{\{J_t=n\}}, \quad i = 1, ..., n,$$
being δ_i *as in Definition 1.1.11, otherwise we set* $\tilde{\delta}_i(U_i) = 0$ *for* $i = 1, ..., n$*;*

- *if* $U^T = (U_{1+n}, \ldots, U_{2n}) \in \mathscr{I}^0_{n,t}(T)$,

$$\tilde{\delta}_{i+n}(U_{i+n})\mathbf{1}_{\{J_t=n\}} = \delta_{i+n}(U_{i+n})\mathbf{1}_{\{J_t=n\}} \quad i = 1, ..., n,$$

being δ_{i+n} *as in Definition 1.1.11, otherwise we set* $\tilde{\delta}_i(U_i) = 0$ *for* $i = 1, ..., n$.

Definition 1.3.4. [Compatibility in the Malliavin sense] *Fix* $n \geq 1$ *and* $t > 0$. *Let* F *and* u *be a random variable and a random process as in Definition 1.3.1 and 1.3.3 respectively. We say that* F *and* u *are "compatible in the Malliavin sense on* $\{J_t = n\}$*" (shortly:* (n, t)*-compatible) if*

$$\begin{aligned} F \in \mathbb{D}^{1,2}_{n,t} &\iff u_0 \in Dom_{n,t,2}(\delta_0) \\ F \in \mathscr{D}^1_{n,t}(\Delta) &\iff U^\Delta = (U_1, ..., U_n) \in \mathscr{I}^1_{n,t}(\Delta) \\ F \in \mathscr{D}^1_{n,t}(T) &\iff U^T = (U_{n+1}, ..., U_{2n}) \in \mathscr{I}^1_{n,t}(T) \end{aligned}$$

If F *and* U *are* (n, t)*-compatible, on the set* $\{J_t = n\}$ *we put*

$$\langle \tilde{D}F, U\rangle = \int_0^t \tilde{D}_{0,s}Fu_0(s)ds + \sum_{i=1}^{2n} \tilde{D}_iFU_i\pi_i.$$

The compatibility in the Malliavin sense relation means, roughly speaking, that we can perform the Malliavin calculus in the same direction for both F and U. In fact, it is immediate to check that

Proposition 1.3.5. *For any* (n, t)*-compatible* F *and* u *such that*

$$\mathbb{E}\Big(\Big[|\tilde{D}_iFU_i\pi_i| + |F\tilde{\delta}_i(U_i)|\Big]\mathbf{1}_{\{J_t=n\}}\Big) < \infty \text{ for any } i = 1, \ldots, 2n,$$

the following duality relation holds

$$\mathbb{E}(\langle \tilde{D}F, u\rangle\mathbf{1}_{\{J_t=n\}}) = \mathbb{E}(F\tilde{\delta}(u)\mathbf{1}_{\{J_t=n\}}) + \mathbb{E}([F, u]_\pi\mathbf{1}_{\{J_t=n\}}).$$

Let us give a brief discussion in order to avoid confusion. Here, on the set on $\{J_t = n\}$, the process $u = (u_0, U_1, \ldots, U_{2n})$ has $2n + 1$ coordinates, being made by a stochastic process (its 0th coordinate) and a simple process (the remaining $2n$ coordinates) in the sense of the jump noise. Now, with $[F, u]_\pi$ we intend the border term operator applied to $(U_1, \ldots, U_{2n})$, which is effectively given by formulas (1.5) and (1.6).

Remark 1.3.6. *Let us stress that if the compatibility Property does not hold, the duality can fail. For example, take* $n \geq 2$, $t > 0$, $F = W^2_{T_1}$ *and ,* $u_0 = 0$, $U_i = 0$ *for any* i *except for the case* $i = n + 1$*:* $U_{n+1} = T_1$. *One has then* $\langle \tilde{D}F, u\rangle = 0$ *and* $F\tilde{\delta}(u) = -W^2_{T_1}$ *because* $\tilde{\delta}_{1+n}(T_1) = -1$. *Therefore,* $\mathbb{E}(\langle \tilde{D}F, u\rangle\mathbf{1}_{\{J_t=n\}}) = 0$. *But, here*

$$
\begin{aligned}
\mathbb{E}\big(F\tilde{\delta}(u)\mathbf{1}_{\{J_t=1\}}\big)+\mathbb{E}\big([F,U]_\pi\mathbf{1}_{\{J_t=1\}}\big) &= -\mathbb{E}\Big(W_{T_1}^2\mathbf{1}_{\{J_t=1\}}\big)+\mathbb{E}\big(\frac{W_{T_2}^2}{T_2}\mathbf{1}_{\{J_t=1\}}\big)\\
&= \mathbb{E}\Big(-\mathbb{E}\big(W_{T_1}^2\mathbf{1}_{\{J_t=1\}}|\mathscr{J}_t\big)\Big)\\
&+ \mathbb{E}\Big(\mathbb{E}\big(\frac{W_{T_2}^2}{T_2}\mathbf{1}_{\{J_t=1\}}|\mathscr{J}_t\big)\Big)\\
&= \mathbb{E}\big((-T_1+\frac{T_2}{T_2})\mathbf{1}_{\{J_t=1\}}\big)\\
&= \mathbb{P}(J_t=1)\mathbb{E}\big(1-T_1|J_t=1\big)\\
&= \mathbb{P}(J_t=1)(1-\frac{t}{2})\neq 0
\end{aligned}
$$

where T_1 is uniformly distributed on $(0,t)$
Then, the duality fails and in fact, F and U are not (n,t)-compatible.

Property 1.3.7. [Properties of the Unified operators] *The properties of the Malliavin derivative and the Skorohod integral stated in the previous sections, can be extended to the unified differential operators. More precisely,*

- **[Chain rule]** *for every continuously differentiable function $\phi\in C_b(\mathbb{R}^d,\mathbb{R})$ and d r.v.'s $F_1,...,F_n$ one has*

$$\tilde{D}\phi(F_1,...,F_n)\mathbf{1}_{\{J_t=n\}}=\sum_{i=1}^n\partial_{x_i}\phi(F_1,...,F_n)\tilde{D}F_i\mathbf{1}_{\{J_t=n\}};$$

- **[Skorohod integral of a (special) product]** *given F and U (n,t)-compatible, and such that $Fu\in Dom_{n,t,2}(\delta_0)$ if $u\in Dom_{n,t,2}(\delta_0)$, then*

$$\tilde{\delta}(Fu)\mathbf{1}_{\{J_t=n\}}=F\tilde{\delta}(u)\mathbf{1}_{\{J_t=n\}}-\langle\tilde{D}F\mathbf{1}_{\{J_t=n\}},u\rangle;$$

Moreover,

Property 1.3.8. [Integration by parts formula] *Let $d\geq 1$, $F=(F^1,...,F^d)$ and $v=(v^1,...,v^d)$ such that F^i and v^i are (n,t)-compatible for any $i=1,\dots,d$. On the set $\{J_t=n\}$, let us define the matrix*

$$\gamma^{i,j}_{\tilde{D}F,v}=\langle\tilde{D}F^j\mathbf{1}_{\{J_t=n\}},v^i\rangle,\quad i,j=1,\dots,d$$

and suppose it is invertible and let $\hat{\gamma}_{\tilde{D}F,v}$ denote the inverse. Suppose moreover that (1.10) holds with D_i replaced by $\tilde{D}_i$, $i=1,\dots,2n$, and that the hypothesis in Remark 1.2.9 are true with D_0 replaced by $\tilde{D}_0$.

Let $G = (G^1, \dots, G^d)$ and $\phi \in C_p^1$ be such that $\sum_{i=1}^d (G\hat{\gamma}_{\tilde{D}F,v})^i v^i$ and $\phi(F)$ are (n,t)-compatible. Then,

$$\begin{aligned} \mathbb{E}\Big(\sum_{i=1}^d \partial_{x_i}\phi(F) G^i \mathbf{1}_{\{J_t=n\}}\Big) &= \mathbb{E}\Big(\phi(F)\tilde{\delta}\big(\sum_{i=1}^d (G\hat{\gamma}_{\tilde{D}F,v})^i v^i\big)\mathbf{1}_{\{J_t=n\}}\Big) \\ &+ \mathbb{E}\Big([\phi(F), \sum_{i=1}^d (G\hat{\gamma}_{\tilde{D}F,v})^i v^i]_\pi\Big) \end{aligned}$$

In conclusion, consider the s.d.e. (1.12) and (1.33)

$$X_t = x + \int_0^t b(r, Y_r)dr + \sum_{i=1}^{J_t} c(T_i, \Delta_i, X_{T_i-})$$

$$\hat{X}_t = x + \int_0^t b(r, \hat{X}_r)dr + \int_0^t \sigma(r, \hat{X}_r)dW_r + \sum_{i=1}^{J_t} c(T_i, \Delta_i, \hat{X}_{T_i-})$$

For both the solutions of the equations, we can apply the definition of the unified Malliavin and Skorohod operators on the set $\{J_t = n\}$. For X we have $\tilde{D} = (0, D_1, ..., D_{2n})$, missing the Brownian part. For the process $\hat{X}$, to avoid non differentiable problems in the jump time direction (see Example 1.1.25), we can in general consider $\tilde{D} = (\tilde{D}_0, \tilde{D}_1, ..., \tilde{D}_n, 0, ..., 0)$.

1.4 Appendix

Proof of Property 1.1.18, 1.2.8

We prove here Property (1.1.18), the proof of Property (1.2.8) being identical with the appropriate notations (mainly, the scalar product $\langle\cdot,\cdot\rangle_\pi$ has to be replaced by $\langle\cdot,\cdot\rangle_{L^2([0,T])}$).
By using the chain rule (cfr. Property 1.1.16), on the set $\{J_t = n\}$ we can write, as $j = 1, \dots, d$,

$$\langle D\phi(F), v^j\rangle_\pi = \sum_{\ell=1}^d \partial_{x_\ell}\phi(F)\langle DF^\ell, v^j\rangle_\pi = \sum_{\ell=1}^d \partial_{x_\ell}\phi(F)\gamma_{F,v}^{\ell j},$$

so that one obtains

$$\sum_{i=1}^d \partial_{x_i}\phi(F)G^i = \sum_{i=1}^d\sum_{j=1}^d \langle D\phi(F), v^j\rangle_\pi \hat{\gamma}_{F,v}^{ji} G^i = \langle D\phi(F), \sum_{j=1}^d (G\hat{\gamma}_{F,v})^j v^j\rangle_\pi$$

Notice that $\phi(F) \in \mathscr{D}_{n,t}^1$, $v \in \mathscr{I}_{n,t}^1$ and $G^i \in \mathscr{D}_{n,t}^1$ satisfy (1.10), so that we can apply the duality formula as in Proposition 1.1.15 and the statement immediately follows.

Proof of Property 1.1.23

First we fix some useful notations for the proof of the Property (1.1.23), following mainly the paper of Bally, Bavouzet and Messaoud [3]. Consider the pure jump process solution of the equation (1.12)

$$X_t = x + \int_0^t b(r, Y_r)dr + \sum_{i=1}^{J_t} c(T_i, \Delta_i, X_{T_i-}).$$

We can associate to this equation a deterministic one: fix $0 = u_0 < u_1 < ... < u_n$, $u = (u_1, ..., u_n) \in \mathbb{R}^n$ and $a = (a_1, ..., a_n) \in \mathbb{R}^n$. Therefore we have

$$x_t = x + \int_0^t b(r, x_r)dr + \sum_{i=1}^{j_t(u)} c(u_i, a_i, x_{u_i-}). \tag{1.34}$$

where $j_t(u) = k$ for $u_k \leq t < u_{k+1}$. As a consequence the solution x_t of (1.34) is such that we have

$$X_t = x_t(\Delta_1, ..., \Delta_n, T_1, ..., T_n) \tag{1.35}$$

where X_t is the solution of 1.12.
We introduce the flow $\Phi = \Phi_u(t, x)$, $0 \leq u \leq t$, $x \in \mathbb{R}$, solution of the integral equation

$$\Phi_u(t, x) = x + \int_u^t b(r, \Phi_u(r, x))dr, \quad t \geq u.$$

Let us introduce moreover

$$e_{u,t}(x) := \exp\Big(\int_u^t \partial_x b(r, \Phi_u(r, x))dr\Big)$$

for which the following relation holds

$$e_{u,t}(x) = \partial_x \Phi_u(t, x)$$

exploiting that

$$\partial_x \Phi_u(t, x) = 1 + \int_u^t \partial_x b(r, \Phi_u(r, x))\partial_x \Phi_u(r, x)dr. \tag{1.36}$$

(For the details see Bally, Bavouzet and Messaoud [3]). Finally remember the definition of $q(r, a, x)$ given by (1.15) that is

$$q(r, a, x) = \Big(\partial_r c + b(1 + \partial_x c)\Big)(r, a, x) - b(r, x + c(r, a, x)).$$

We are ready to prove the Property (1.1.23).
Consider first the relation in the jump amplitudes direction, that is

$$D_iX_r\mathbf{1}_{\{T_p<r<T_{p+1}\}} = \mathbf{1}_{\{T_i\le r\}}\,\xi_r\xi_{T_i}^{-1}\partial_a c(T_i,\Delta_i,X_{T_i-}). \tag{1.37}$$

First of all notice that, as a consequence of (1.35), the following relation holds

$$\partial_{a_j}x_t(T_1,...,T_n,\Delta_1,...,\Delta_n) = D_jX_t. \tag{1.38}$$

Moreover, for $t\le u_j$, $\partial_{a_j}x_t(u,a)=0$ and for $t\ge u_j$, $\partial_{a_j}x_t(u,a)$ satisfies the equation

$$\begin{aligned}\partial_{a_j}x_t = \; & \partial_a c(u_j,a_j,x_{u_j-}) + \sum_{i=j+1}^{j_t(u)} \partial_x c(u_i,a_i,x_{u_i-})\partial_{a_j}x_{u_i-} + \\ & + \int_{u_j}^t \partial_x b(r,x_r)\partial_{a_j}x_r dr.\end{aligned} \tag{1.39}$$

Now consider the first variation process ξ_t. We have that this process satisfies the stochastic equation

$$\xi_t = 1 + \int_0^t \partial_x b(r,X_r)\xi_r dr + \sum_{i=1}^{J_t} \partial_x c(\Delta_i,T_i,X_{T_i-})\xi_{T_i-}.$$

Therefore, remembering the relations (1.38) and (1.39), we can conclude that $\xi_r\xi_{T_i}^{-1}\partial_a c(T_i,\Delta_i,X_{T_i-})$ on $\{T_i\le r\}$ and D_iX_r on $\{T_p<r<T_{p+1}\}$ are solution of the same stochastic differential equation and so we can end the proof of ((1.37)).

Consider now the relation in the jump times direction:

$$D_{i+n}X_t = \xi_t\xi_{T_i}^{-1}q(T_i,\Delta_i,X_{t_i-}). \tag{1.40}$$

Let $T_j<t<T_{j+1}$. The first variation process can be written as

$$\xi_t = \xi_{T_j} + \int_{T_j}^t \partial_x b(r,X_r)\xi_r dr$$

because between T_j and t there are no jumps. Let us denote $\eta_t = \xi_t\xi_{T_i}^{-1}$. Therefore we have that

$$\eta_t = 1 + \int_{u_j}^t \partial_x g\cdot\eta_r dr.$$

As in jump amplitudes case, as a consequence of the property (1.35), the process x, solution of the associated deterministic equation, satisfies the following relations for $u_j\le t$

$$\partial_{u_j}x_t(T_1,...,T_n,\Delta_1,...,\Delta_n) = D_{j+n}X_t \tag{1.41}$$

and

$$\partial_{u_j} x_t = q(u_j, a_j, x_{u_j-}) e_{u_j,t}(x_{u_j}). \tag{1.42}$$

But we have that $e_{u_j,t}(x_{u_j}) = \partial_x \Phi_{u_j}(t, x_{u_j})$ and, by (1.36), η_t and $\partial_x \Phi_{u_j}(t, x_{u_j})$ are solution of the same equation. Finally, remembering the relation (1.42), we have that

$$D_{i+n} X_t = \partial_{u_i} x_t(T_1, ..., T_n, \Delta_1, ..., \Delta_n)$$

and

$$\xi_t \xi_{T_i}^{-1} q(T_i, \Delta_i, X_{t_i-})$$

satisfy the same equation and therefore the relation (1.40) holds.
The proof of the relation between the first variation process and the integral process follows trivially by the stated properties for the process X.

Chapter 2

Sensitivity analysis

The aim of this chapter is to find representation formulas for sensitivities, in particular for the Delta, i.e. the derivative of the price function with respect to the initial data. The same problem can be found in other contexts, as in the insurance ones, whenever one has to find the variation with respect to parameters of interest.
In general, looking at the option hedging problem, given the price of the considered option, expressed as $\mathbb{E}[payoff]$, the Greeks analysis is the study of sensitivity of w.r.t. some typical parameters (initial condition, diffusion coefficient, drift coefficient...). The so called "Greeks" can be expressed as

$$\frac{\partial}{\partial \lambda}(Price) = \mathbb{E}[payoff \times weight]$$

using the Malliavin integration by parts formula, previously developed and stated in Property 1.3.8.
The computation of the Delta has been developed in many papers: Fournié Lasry, Lebuchoux and Lions [17], Fournié Lasry, Lebuchoux, Lions and Touzi [18], Davis and Johansson [13], Benhamou [6], El-Khatib and Privault [14], Chen and Glasserman [12], Bally, Bavouzet-Morel and Messaoud [3], [5].
Here, we consider mainly payoff functions depending on the pair $Z_t = (X_t, Y_t)$, in which X_t stands for a jump diffusion and Y_t is integral (w.r.t. the Lebesgue measure) up to time t (see below) and not only the two components separately. In financial terms, we are going to study complex or floating Asian options. To our knowledge, in the literature there are few results for this case, mainly Benhamou [6], and only the Brownian noise is considered. Here, we extend this case by discussing several weights, turning out also by handling the jump noise.

2.1 Delta for payoffs depending on the pair

We are able to focus on the following case: consider the process

$$Z_t = (X_t, Y_t)$$

where

- X_t is a n-dimensional process. Given a d-dimensional Brownian motion $W_t = (W_t^1, ..., W_t^d)$ and an m-dimensional Poisson process $J_t = (J_t^1, ..., J_t^m)$ we have

$$
\begin{aligned}
X_t^i = \; & x^i + \int_0^T b^i(r, X_r)dr + \sum_{j=1}^{d} \int_0^T \sigma^{ij}(r, X_r)dW_r^j + \\
& + \sum_{j=1}^{m} \sum_{k=1}^{J_t^j} c^{ij}(T_k, \Delta_k, X_{T_k-})
\end{aligned}
$$

- given $f : \mathbb{R}^n \to \mathbb{R}$, Y_t is defined as

$$
Y_t = \int_0^t f(X_r)dr.
$$

In this general framework X_t can be thought as a pure diffusion or as a pure jump process, respectively of the form (1.33), if $c^{ij} = 0$ for all $i = 1, ..., n$, $j = 1, ..., m$, or like (1.12), if $\sigma^{ij} = 0$ for all $i = 1, ..., n$, $j = 1, ..., d$. In the following we consider $f(y) = y$, but the same machinery can be thought for a general smooth function.
Now, fix a time $T > 0$, standing for the maturity of the option, and let

$$
\phi(X_T, Y_T) \tag{2.1}
$$

denote the payoff function. In order to find representation formulas for the Delta, we are interested to the computation of the following derivative

$$
\frac{\partial}{\partial x}\mathbb{E}(\phi(X_T, Y_T)) \tag{2.2}
$$

that is the initial data sensitivity.
To avoid complex and misleading notations, we suppose in the following that X is one-dimensional and $Y_t = \int_0^t X_r dr$. The forthcoming results extend what developed by Fournié, Lasry, Lebuchoux and Lions [17], Davis and Johansson [13], Benhamou [6] to our general framework.
First, let us state the following

Definition 2.1.1. *We say that a jump diffusion X in satisfies the* **standard hypothesis** *if Assumption*

- *1.1.20, 1.1.21 and 1.1.22 hold in the pure jump case (i.e. $\sigma \equiv 0$)*
- *1.2.10 and 1.1.21 hold in the jump diffusion case*

are satisfied.

Roughly speaking, we will see that things are slightly different according to the case that a pure jump diffusion or a jump diffusion is taken into account (for example, we have already seen that a Malliavin calculus in the direction of the jump times cannot be performed if a jump diffusion is considered, i.e. if the Brownian motion is present). Therefore, Definition 2.1.1 ensures the hypothesis on the jump diffusion coefficients allowing to get the Malliavin derivative in the right direction.

Proposition 2.1.2. [Sensitivity w.r.t. the initial condition] *Let X satisfy the standard hypothesis and $Z_t = (X_t, Y_t)$, with $Y_t = \int_0^t X_u du$. Let the payoff function ϕ have polynomial growth and let ξ be the first variation process related to the process X. Let v a process such that $F = Z_T = (X_T, Y_T)$, v and $G = (\xi_T, \int_0^T \xi_u du)$ satisfy the hypothesis in Property 1.3.8, with $n \geq 1$ and $t = T$. Then, with the convention $\mathbf{1}_{\{J_T=n\}} = 1$ in the pure diffusion case (i.e. when $c \equiv 0$),*

$$\frac{\partial}{\partial x}\mathbb{E}(\phi(X_T, Y_T)\mathbf{1}_{\{J_T=n\}}) = \mathbb{E}(\phi(X_T, Y_T)\tilde{\delta}(w\mathbf{1}_{\{J_T=n\}})) + \mathbb{E}([\phi(X_T, Y_T), w\mathbf{1}_{\{J_T=n\}}]_\pi) \tag{2.3}$$

where

$$w = \xi_T \sum_{j=1}^{2} \hat{\gamma}^{1j}_{Z_T,v} v^j + \int_0^T \xi_r dr \sum_{j=1}^{2} \hat{\gamma}^{2j}_{Z_T,v} v^j \tag{2.4}$$

and $\hat{\gamma}_{Z_T,v} = \gamma^{-1}_{Z_T,v}$ on the set $\{J_T = n\}$, being

$$\gamma^{ij}_{Z_T,v} = \langle \tilde{D} Z^j \mathbf{1}_{\{J_t=n\}}, v^i \rangle, \quad i, j = 1, 2.$$

Proof. of the Theorem 2.1.2
The proof is straightforward, as a consequence of the Integration by parts formula. As a matter of fact, we can suppose that $\phi \in C^1_b$, because the general case can be handled by standard density arguments using mollifiers. Then, one has

$$\frac{\partial}{\partial x}\mathbb{E}(\phi(X_T, Y_T)\mathbf{1}_{J_T=n})) = \mathbb{E}\big(\partial_x \phi(X_T, Y_T))\xi_T \mathbf{1}_{J_T=n}\big) + \mathbb{E}\big(\partial_y \phi(X_T, Y_T)) \int_0^T \xi_r dr \cdot \mathbf{1}_{J_T=n}\big).$$

Therefore, taking $G = (\xi_T, \int_0^T \xi_r dr)$ in the Integration by Parts formula, the statement holds. □

Remark 2.1.3. *Let us discuss the hypothesis allowing to get the result in Proposition 2.1.2. First, let us recall that ξ_T and $\int_0^T \xi_r dr$ are both Malliavin differentiable (except in the jump times direction in the strict jump-diffusion case) and that their Malliavin derivatives belong to the desired L^p spaces for any p. Then, one can resume the conditions to require case by case in the following way: if w denotes the process as in (2.4),*

- $(w_k)_{k=1,\dots,2n} \in \mathscr{I}^1_{n,T}$ *for* $i,j = 1,2$ *in the pure jump case (i.e.* $\sigma \equiv 0$*);*
- $w_0 \in Dom_{n,T,2}(\delta)$ *and* $(w_k)_{k=1,\dots,n} \in \mathscr{I}^1_{n,T}(\Delta)$ $i,j = 1,2$ *in the jump diffusion case;*
- $w_0 \in Dom_{n,T,2}(\delta)$ *for* $i,j = 1,2$ *in the pure diffusion case (i.e.* $c \equiv 0$*).*

Moreover the border term written operator appearing in the weight given by 2.3 is non null only when we are considering the jump times direction of the differential calculus (see Definition 1.1.12 and Remark 1.1.13). Therefore in the direction of the Brownian motion and the jump amplitudes we can state the following formula:

$$\frac{\partial}{\partial x}\mathbb{E}(\phi(X_T,Y_T)\mathbf{1}_{J_T=n}) = \mathbb{E}(\phi(X_T,Y_T)\tilde{\delta}(w\mathbf{1}_{J_T=n})) \tag{2.5}$$

For practical purposes, Proposition 2.1.2 gives the Delta, since

$$\begin{aligned}\partial_x\mathbb{E}(\phi(X_T,Y_T)) &= \partial_x\mathbb{E}(\phi(X_T,Y_T)\mathbf{1}_{\{J_T=0\}}) + \partial_x\mathbb{E}(\phi(X_T,Y_T)\mathbf{1}_{\{J_T\geq 1\}}) \\ &= \partial_x\mathbb{E}(\phi(X_T,Y_T)\mathbf{1}_{\{J_T=0\}}) + \sum_{n\geq 1}\partial_x\mathbb{E}(\phi(X_T,Y_T)\mathbf{1}_{\{J_T=n\}})\end{aligned}$$

Thanks to Proposition 2.1.2, the last term of the above expression is equal to

$$\sum_{n\geq 1}\mathbb{E}(\phi(X_T,Y_T)\tilde{\delta}(w\mathbf{1}_{J_T=n})).$$

Concerning the first term, i.e. $\partial_x\mathbb{E}(\phi(X_T,Y_T)\mathbf{1}_{\{J_T=0\}})$, we have to consider separately the pure jump diffusion and the jump-diffusion case. Let us summarize all in the following remarks.

Remark 2.1.4. [The case $n = 0$ for pure jump diffusions] *In the pure jump case, on the set* $\{J_T = 0\}$*, the process* Z_t *follows a deterministic equation, having no jumps and therefore stochastic noises. We have that* $(X_t,Y_t)_{t\leq T} = (x_t,y_t)_{t\leq T}$*, where*

$$\begin{aligned}\dot{x}_t &= b(r,x_r) \quad & x_0 = x \\ \dot{y}_t &= x_t & y_0 = 0.\end{aligned}$$

Therefore

$$\partial_x\mathbb{E}(\phi(X_T,Y_T)\mathbf{1}_{\{J_T=0\}}) = \partial_x\phi(x_T,y_T)\mathbb{P}(J_T=0)$$

Remark 2.1.5. [The case $n = 0$ for jump diffusions] *In the jump-diffusion case (i.e.* σ *non null), on the set* $\{J_T = 0\}$*, the process* Z_t *is a pure diffusion and coincides with* $(\bar{X},\bar{Y})$ *over* $[0,T]$*, where*

$$\begin{aligned}d\bar{X}_t &= b(t,\bar{X}_t)dt + \sigma(t,\bar{X}_t)dW_t \quad & \bar{X}_0 = x \\ d\bar{Y}_t &= \bar{X}_t dt & \bar{Y}_0 = 0\end{aligned}$$

Let $\bar{w}$ *be the* $\mathscr{B}([0,T]) \times \mathscr{W}_T$*-measurable process given by Proposition 2.1.2 when the pure diffusion* $(\bar{X},\bar{Y})$ *is considered. Notice that* $w = \bar{w}\mathbf{1}_{\{J_T=0\}}$ *is* *measurable w.r.t.* $\mathscr{B}([0,T])\times\mathscr{W}_T\times\mathscr{J}_T$. *By recalling that the Brownian motion and the Poisson process are independent, we can write*

$$\begin{aligned}\frac{\partial}{\partial x}\mathbb{E}(\phi(X_T,Y_T)\mathbf{1}_{J_T=0}) &= \frac{\partial}{\partial x}\mathbb{E}(\phi(X_T,Y_T)|J_T=0)\cdot\mathbb{P}(J_T=0)\\ &= \frac{\partial}{\partial x}\mathbb{E}(\phi(\bar{X}_T,\bar{Y}_T))\cdot\mathbb{P}(J_T=0)\\ &= \mathbb{E}(\phi(\bar{X}_T,\bar{Y}_T)\tilde{\delta}(\bar{w}))\cdot\mathbb{P}(J_T=0)\\ &= \mathbb{E}(\phi(X_T,Y_T)\tilde{\delta}(w)|J_T=0)\cdot\mathbb{P}(J_T=0)\\ &= \mathbb{E}(\phi(X_T,Y_T)\tilde{\delta}(w)\mathbf{1}_{J_T=0})\end{aligned}$$

and a representation formula holds as well, by using the Gaussian Malliavin calculus.

Let us focus on the case $n=1$, that is on the set $\{J_T=1\}$. Here, some degenerate facts may happen, as summarized in the next

Remark 2.1.6. [The case $n=1$**]** *On the set* $\{J_T=1\}$, *the process* X *has performed only one jump in the time interval* $[0,T]$. *Suppose that one wants to find the weight for the Delta, using the Malliavin calculus only in the direction of the jump amplitude* Δ_1 *or only in the direction of the jump time* T_1 *(the two cases are similar, and so we will consider only the first one). Therefore the differential operators are the following:*

$$\tilde{D}=(0,\tilde{D}_1,0)\qquad \textit{and}\qquad \tilde{\delta}=(0,\tilde{\delta}_1,0)$$

Thus the n*-dimensional scalar product degenerates in a one-dimensional one, that is,* $\langle U,V\rangle_\pi=U_1\cdot V_1$, U,V *simple processes, and the matrix* $\gamma_{\tilde{D}Z_T,v}$ *becomes*

$$\gamma_{\tilde{D}Z_T,v}=\begin{pmatrix}\tilde{D}X_T\cdot v^1 & \tilde{D}Y_T\cdot v^1\\ \tilde{D}X_T\cdot v^2 & \tilde{D}Y_T\cdot v^2\end{pmatrix}.$$

So $\det\gamma_{\tilde{D}Z_T,v}=0$ *for every choice of the process* $v=(v_1^1,v_1^2)$ *and the inverse matrix* $\hat{\gamma}_{\tilde{D}Z_T,v}$ *does not exist. As a consequence the machinery described by the Proposition 2.1.2 cannot be applied. Therefore, in order to find a weight, when the jump amplitudes or jump times noise is considered, one is forced to add a noise in order to get a non-degenerate Malliavin covariance matrix. Or, obviously, the lonely Gaussian noise direction can be considered. By the way, it can happen that only one direction of noise must be considered (see for example the Merton model, where there is not an explicit dependence on the jump times). In this cases, the pair* (X_t,Y_t) *has to be considered as a one-dimensional process, when there is only one jump on* $[0,T]$.

The above Remarks 2.1.4, 2.1.5 and 2.1.6, giving relations when no jumps or at least one jump are observed, will be used later, for numerical applications.

The Proposition 2.1.2 gives an expression for the Delta depending on the process v. In the next sections we will suggest some possible choices for v, in order to simplify the weight given by (2.5).

Remark 2.1.7. *For a 2-dimensional process $v = (v^1, v^2)$, let $v^{\alpha,\beta}$ be a the following transformation of v:*

$$v^{\alpha,\beta} = (\alpha v^1, \beta v^2)$$

α and β being non null and possibly random constants. The associated matrix $\gamma_{\tilde{D}Z, v^{\alpha,\beta}}$ is therefore defined as

$$\gamma_{\tilde{D}Z_T, v^{\alpha,\beta}} = \begin{pmatrix} \alpha\langle \tilde{D}X, v^1\rangle & \alpha\langle \tilde{D}Y, v^1\rangle \\ \beta\langle \tilde{D}X, v^2\rangle & \beta\langle \tilde{D}Y, v^2\rangle \end{pmatrix}$$

Now, it is immediate to see that the corresponding process $w^{\alpha,\beta}$ defined in (2.4), coincides with the process w turning out from v. In conclusion, the weight $\tilde{\delta}(w\mathbf{1}_{\{J_T=n\}})$, defined by (2.5), does not change by considering the process $v^{\alpha,\beta}$.

Remark 2.1.8. Notation
We will use the notations:

$$\begin{aligned} \partial_a c_i &= \partial_a c(T_i, \Delta_i, X_{T_i-}) \\ \partial_x c_i &= \partial_x c(T_i, \Delta_i, X_{T_i-}) \\ q_i &= q(T_i, \Delta_i, X_{T_i-}) \end{aligned}$$

where q is defined by (1.15).
Moreover let us define the following process, which we need in the following

$$\beta_t = \int_0^t \xi_r dr$$

From now on, when is not specified, we refer to X as a one dimensional process, in order to simplify the notations. It is clear that the multidimensional case follows straightforwardly by the one 1-dimensional one.

2.2 Malliavin covariance matrix

The first natural choice of the 2-dimensional process v is the Malliavin derivative $\tilde{D}Z$, so that $\gamma_{\tilde{D}Z,v}$ becomes the standard Malliavin covariance matrix (see Bally, Bavouzet, Messaud [3] for more details), usually denoted through $\sigma_{\tilde{D}Z}$:

$\sigma_{\tilde{D}Z} = \gamma_{\tilde{D}Z,\tilde{D}Z}$. In this case the weight is not simple to be used for numerical purposes, due to a complicated structure of the covariance matrix. Actually, we have introduced the process v and defined the generalized covariance matrix $\gamma_{\tilde{D}Z,v}$ for the sake of generalization and mainly, to reduce the complicated structure of the covariance matrix.
For example, in the pure diffusion case, the Malliavin covariance matrix is less complicated than in other cases, although difficult to use in practice. As a matter of fact, remembering the relations between the Malliavin derivative and the first variation process, stated by Property 1.2.11, we have

$$\sigma_{\tilde{D}Z_T} = \begin{pmatrix} \int_0^T (\xi_t \xi_u^{-1} \sigma(X_u))^2 du & \int_0^T (\xi_u^{-1}\sigma(X_u))^2 \xi_t(\beta_T - \beta_u) du \\ \int_0^T (\xi_u^{-1}\sigma(X_u))^2 \xi_t(\beta_T - \beta_u) du & \int_0^T (\xi_u^{-1}\sigma(X_u)(\beta_T - \beta_u))^2 du \end{pmatrix}$$

In fact, to our knowledge, the only result appearing in the literature, due to Benhamou [7] does not use such an instrument. In that paper it is shown that the weight is given by $\delta_0(w)$, with

$$w_u = \frac{\int_0^T Y_r^2 dr - Y_u \int_0^T Y_r dr}{\sigma x \big(T \int_0^T Y_r^2 dr - (\int_0^T Y_r^2 dr)^2\big)}$$

The technique allowing to prove such a result are different from ours but we will see that such a weight can be reproduced by our approach, that is by using the generalized Malliavin covariance matrix, as it is shown in next Remark 2.3.3.

Before to consider the jump diffusion case, we stress the fact that the process $Z = (X, Y)$ does not satisfy the ellipticity condition in every point of the space (the ellepticity condition assures that the Malliavin covariance matrix is invertible and non degenerate). In fact

$$\sigma = \begin{pmatrix} \sigma(r, X_r) \\ 0 \end{pmatrix}$$

and

$$\sigma\sigma^* = \begin{pmatrix} \sigma(r, X_r) & 0 \\ 0 & 0 \end{pmatrix}$$

that is a degenerate matrix (much more than non elliptic). By the way it can be shown that the Hörmander condition (see Nualart [25] Section 2.3) holds true (with the notations of Nualart A_0 and $[A_0, A]$ span $\mathbb{R}^2$). By this condition it can be proved that the standard covariance matrix is invertible and has finite moments of every order, but there are not references in literature, also if it is a widely accepted statement. Notice moreover that for practical implementations, to check Hörmander condition is not useful.

In order to stress the fact that the use of the Malliavin covariance matrix is unfeasible in practice for our problem, let us write down it in the case of the jump diffusion process $Z_t = (X_t, Y_t)$ (the pure jump one is similar), which solves

$$
\begin{aligned}
X_t &= x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r + \sum_{i=1}^{J_t} c(T_i, \Delta_i, X_{T_i-}) \\
Y_t &= \int_0^t X_r dr
\end{aligned}
$$

As remarked, in this case we can consider (at least in the general case) the Malliavin derivative only in the Gaussian and jump amplitudes directions. So we have that $\tilde{D} = (\tilde{D}_0, \tilde{D}_1, ..., \tilde{D}_n, 0..., 0)$. Using the formulas for the Malliavin derivatives involving the first variation process ξ (w.r.t. the Gaussian and the jump amplitude noise, see (1.2.11) and (1.1.23) respectively), the standard Malliavin covariance matrix $\sigma_{\tilde{D}Z}$ is given by

$$
\begin{pmatrix}
\begin{array}{c} \Big[\int_0^T (\xi_u)^{-2}(\sigma(X_u))^2 du + \\ +\sum_{i=1}^n (\xi_{T_i})^{-2}(\partial_a c_i)^2\Big](\xi_T)^2 \end{array} & \begin{array}{c} \Big[\int_0^T (\xi_u)^{-2}(\sigma(X_u))^2(\beta_T - \beta_u)du + \\ +\sum_{i=1}^n (\xi_{T_i})^{-2}(\partial_a c_i)^2(\beta_T - \beta_{T_i})\Big]\xi_T \end{array} \\
\begin{array}{c} \Big[\int_0^T (\xi_u)^{-2}(\sigma(X_u))^2(\beta_T - \beta_u)du + \\ +\sum_{i=1}^n (\xi_{T_i})^{-2}(\partial_a c_i)^2(\beta_T - \beta_{T_i})\Big]\xi_T \end{array} & \begin{array}{c} \int_0^T (\xi_u)^{-2}(\sigma(X_u))^2(\beta_T - \beta_u)^2 du + \\ +\sum_{i=1}^n (\xi_{T_i})^{-2}(\partial_a c_i)^2(\beta_T - \beta_{T_i})^2 \end{array}
\end{pmatrix}
$$

It appears clear that the inverse matrix $\hat{\sigma}_{\tilde{D}Z}$ and, as a consequence, the weight for the Delta given by (2.5), are too complicated to be used in practice. So in the next section we will see more suitable choices for the process v, differently constructed according to the direction along which the Malliavin calculus is considered.
It is worth to be observed that in the one-dimensional case, the choice of v giving the standard Malliavin covariance matrix, does not produce the same difficulties as in the multidimensional case (see for example Bally, Bavouzet and Messaud [3] or Fournié, Lasry, Lebuchoux, Lions and Touzi [18]).

2.3 Generalized Malliavin covariance matrix

In this section we consider the different directions of the Malliavin calculus separately, in order to define the process v in each case. We stress that by using the generalized covariance matrix, the calculus of the weight for the Delta is considerably simplified and the writing of the resulting weight is really plain. From now on, suppose to have fixed the maturity time T, at which n jumps have been observed, i.e. we work on the set $\{J_T = n\}$. In the following, we will consider the jump diffusion process

$$
\begin{aligned}
X_t &= x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r + \sum_{i=1}^{J_t} c(T_i, \Delta_i, X_{T_i-}) \\
Y_t &= \int_0^t X_r dr
\end{aligned}
$$

We will speak about a "pure diffusion" if $c \equiv 0$; we will say that a "the pure jump diffusion" is considered when $\sigma \equiv 0$.
Let us define the following classes of deterministic processes, widely used later on:

$$
\mathscr{A}^W = \Big\{ a \,:\, [0,T] \to \mathbb{R} \text{ such that } \int_0^T a_u du = 0 \Big\} \tag{2.6}
$$

and for $a \in \mathscr{A}^W$, we set $A(t) = \int_0^t a_u du$;

$$
\mathscr{A}^J = \Big\{ a \in \mathbb{R}^n \text{ such that } \sum_{i=1}^n a_i = 0 \Big\} \tag{2.7}
$$

As we will see, $\mathscr{A}^W$ and $\mathscr{A}^J$ are constructed in order to simplify the structure of the matrix $\gamma_{\tilde{D}Z,v}$.
Finally we refer to U and V as simple processes, whose general form is $U = ((U_u^0)_{u \in [0,T]}, U^1, ..., U^{2n})$ and $V = ((V_u^0)_{u \in [0,T]}, V^1, ..., V^{2n})$.

2.3.1 Gaussian direction

Let us first consider the classical Gaussian direction for the Malliavin calculus. This means that

$$
\begin{aligned}
&\tilde{D} = D_0 \quad \text{and} \quad \tilde{D}_i \equiv 0 \text{ for } i = 1, ..., 2n \\
&\tilde{\delta} = \delta_0 \quad \text{and} \quad \tilde{\delta}_i \equiv 0 \text{ for } i = 1, ..., 2n.
\end{aligned}
$$

Moreover, the underlying scalar product is the standard one in $L^2([0,T])$, that is

$$
\langle U, V \rangle_\pi = \int_0^T U_u^0 V_u^0 du.
$$

We can consider the Brownian Malliavin derivative for general jump diffusions, whenever the right hypothesis on the coefficients are taken into account (see Assumption 1.2.10 and 1.1.21).
Following Property 1.2.11 giving the relation between the first variation process ξ and the process X, we have

$$
\begin{aligned}
\tilde{D}_{0,u} X_T &= \xi_T \xi_u^{-1} \sigma(X_u) \\
\tilde{D}_{0,u} Y_T &= (\beta_T - \beta_u) \xi_u^{-1} \sigma(X_u)
\end{aligned}
$$

where from now on we set

$$\beta_t = \int_0^t \xi_u \, du = \partial_x Y_t.$$

Thus we define the process $v = (v^1, v^2)$ as follows: for some $a \in \mathscr{A}^W$,

$$\begin{array}{llll} v^1 : & v^1_{0,u} = \xi_u \sigma^{-1}(X_u) a_u & v^1_i \equiv 0 & i = 1, \dots, 2n \\ v^2 : & v^2_{0,u} = \xi_u \sigma^{-1}(X_u) & v^2_i \equiv 0 & i = 1, \dots, 2n \end{array}$$

So the matrix $\gamma_{\tilde{D}Z_T,v}$ is

$$\gamma_{\tilde{D}Z_T,v} = \begin{pmatrix} 0 & -\int_0^T \beta_u a_u du \\ & \\ T\xi_T & T\beta_T - \int_0^T \beta_u du \end{pmatrix}.$$

Finally, the process w as in (2.4) is then given by

$$\begin{aligned} w_{0,u} &= \frac{\xi_u \sigma^{-1}(X_u)}{T} \Big(1 - \frac{\int_0^T \beta_r dr}{\int_0^T \beta_r a_r dr} \, a_u \Big) \\ w_i &\equiv 0 \qquad\qquad\qquad\qquad\qquad\qquad i = 1, \dots, 2n \end{aligned}$$

In order to show that the machinery runs, it remains to show that, on the set $\{J_T = n\}$, $(\det \gamma_{\tilde{D}Z_T,v})^{-1} \in L^p$ for any p. One has,

$$\mathbf{1}_{\{J_T=n\}} |\det \gamma_{\tilde{D}Z_T,v}|^{-1} = \mathbf{1}_{\{J_T=n\}} \, T\xi_T^{-1} \cdot | \int_0^T \beta_u a_u \, du|^{-1}.$$

Now, $\xi_T^{-1} = \hat{\xi}_T \in L^q$ for any q. Concerning the latter term, one has

Proposition 2.3.1. *Suppose that* $A_u = \int_0^u a_s \, ds \geq 0$. *Then,*

$$\mathbf{1}_{\{J_T=n\}} \Big| \int_0^T \beta_u a_u \, du \Big|^{-1} \in L^p \textit{ for any } p.$$

Proof. One has

$$\int_0^T \sqrt{A_u} \, du \leq \Big(\int_0^T A_u \xi_u \, du \Big)^{1/2} \Big(\int_0^T \xi_u^{-1} \, du \Big)^{1/2}$$

so that, setting $\hat{\xi} = \xi^{-1}$,

$$\Big(\int_0^T A_u \xi_u \, du \Big)^{-1} \leq c_A^{-2} \cdot \int_0^T \hat{\xi}_u \, du$$

being $c_A = \int_0^T \sqrt{A_u} \, du > 0$. Therefore,

$$\Big(\int_0^T A_u \xi_u \, du \Big)^{-p} \leq c_A^{-2p} \cdot \Big(\int_0^T \hat{\xi}_u \, du)^p \leq d_{A,p} \cdot \int_0^T \hat{\xi}^p_u \, du$$

for a suitable positive constant $d_{A,p}$. Now, since $(X,\hat{\xi})$ is a diffusion process with (very) good coefficients, $\sup_{u\in[0,T]}\hat{\xi}_u \in L^p$ for any $p\geq 1$, and the statement holds. $\square$

Remark 2.3.2. *In practice, we will take, for $s\in[0,T]$, either*

$$a_s = \frac{T}{2} - s \quad or \quad a_s = \sin\big(2\pi\frac{s}{T}\big)$$

Notice that, in each case, $A_s = \int_0^s a_u\,du \geq 0$ for any $s\in[0,T]$.

Remark 2.3.3. *In literature Benhamou has considered the floating Asian options (see [7]), giving an expression of a weight using the Gaussian direction of the Malliavin calculus, obtained using some considerations on the conditional expectation. As a process w, he has considered the following one*

$$w_u^{Ben} = \xi_u\sigma^{-1}(X_u)\Big(\frac{\beta_T^2 + 2\xi_u(\int_0^T r\xi_r dr - T\beta_T)}{\beta_T(2\int_0^T r\xi_r dr - T\beta_T)}\Big)$$

where we have used our standard notation. We can find this particular process w, choosing in our machinery the process v as

$$\begin{array}{ll} v^1: & v^1_{0,u} = \xi_u^2\sigma^{-1}(X_u) \quad v^1_i \equiv 0 \quad i=1,\dots,2n \\ v^2: & v^2_{0,u} = \xi_u\sigma^{-1}(X_u) \quad v^1_i \equiv 0 \quad i=1,\dots,2n \end{array}$$

Notice that this choice of the process v, makes more complicated the calculus of the weight, instead of an easy one given by the choice introduced in the Section 2.3.1. As a mattter of fact the matrix $\gamma_{\tilde{D}Z_T,v}$ is

$$\gamma_{\tilde{D}Z_T,v} = \begin{pmatrix} \xi_T\beta_T & \beta_T^2 - \int_0^T \beta_r dr \\ & \\ T\xi_T & T\beta_T - \int_0^T \beta_r dr \end{pmatrix}$$

and we have that $det(\gamma_{\tilde{D}Z_T,v}) = \xi_T\int_0^T \beta_r dr(T-\beta_T)$. By straightforward computations, one can verify that the following equality actually holds:

$$w^{Ben} = \xi_T\sum_{j=1}^2 \gamma^{1j}_{\tilde{D}Z_T,v} v^j + \beta_T\sum_{j=1}^2 \gamma^{2j}_{\tilde{D}Z_T,v} v^j$$

2.3.2 Jump amplitudes direction

We consider now the jump amplitudes direction, that is

$$\begin{array}{lll} \tilde{D}_0 \equiv 0, & \tilde{D}_i = \partial_{\Delta_i} \text{ for } i=1,...,n, & \tilde{D}_i \equiv 0 \text{ for } i=n+1,...,2n \\ \tilde{\delta}_0 \equiv 0, & \tilde{\delta}_i = \delta_i^{\Delta} \text{ for } i=1,...,n, & \tilde{\delta}_i \equiv 0 \text{ for } i=n+1,...,2n. \end{array}$$

The scalar product is now defined as

$$\langle U, V\rangle_\pi = \langle U^\Delta, V^\Delta\rangle_\pi^\Delta = \sum_{i=1}^n U^i V^i \pi_i.$$

Again, we can think to handle a general jump diffusion, on which the right hypothesis have to be set in order to get the differentiability (see Definition 2.1.1)

Here, we will always assume that $\pi_i = 1$ and that the jump amplitudes have a smooth density g, which is null outside $I = \cup_{j=1}^k (a_j, b_j)$ and such that $g \equiv 0$ on ∂I. This means that the border term operator is null:

$$[\cdot,\cdot]_\pi^\Delta = 0 \text{ on } \mathscr{S}_{n,t}(\Delta) \times \mathscr{P}_{n,t}(\Delta)$$

(see Remark 1.1.13 for details). We recall that for our numerical applications we will consider g as a log-normal pdf.

Following Property 1.1.23 on the first variation process ξ and the process X, we have

$$\begin{aligned}\tilde{D}_i X_T &= \xi_T \xi_{T_i}^{-1} \partial_a c_i \\ \tilde{D}_i Y_T &= (\beta_T - \beta_{T_i}) \xi_{T_i}^{-1} \partial_a c_i.\end{aligned}$$

Thus we define the process $v = (v^1, v^2)$ as follows: for some $a \in \mathscr{A}^J$,

$$\begin{array}{llll} v^1: & v_i^1 = \xi_{T_i} (\partial_a c_i)^{-1} a_i & i = 1, \ldots, n, \quad v_i^1 \equiv 0 & i = 0, n+1, \ldots, 2n \\ v^2: & v_i^2 = \xi_{T_i} (\partial_a c_i)^{-1} & i = 1, \ldots, n, \quad v_i^2 \equiv 0 & i = 0, n+1, \ldots, 2n \end{array}$$

The matrix $\gamma_{\tilde{D}Z_T, v}$ is then

$$\gamma_{\tilde{D}Z_T, v} = \begin{pmatrix} 0 & -\sum_{i=1}^n \beta_{T_i} a_i \\ \\ n\xi_T & n\beta_T - \sum_{i=1}^n \beta_{T_i} \end{pmatrix}.$$

It is easy to verify that the process v and the matrix $\hat{\gamma}_{\tilde{D}Z_T, v}$ verify the hypothesis of the Theorem 2.1.2, that is $\hat{\gamma}_{\tilde{D}Z_T, v}^{ij} v_j \in \mathscr{I}_{n,T}^1(\Delta)$ for $i, j = 1, 2$. We have supposed the smoothness of the p.d.f. g of the jump amplitudes and therefore we have that the exists

$$-(\partial_{\Delta_k} U_k + U_k \partial_{\Delta_k} \log g(\Delta_k))$$

where $U = (U_1, ..., U_n)$ and $U_k = \gamma_{\tilde{D}Z_T, v}^{ij} v_k^j$. This means that $\hat{\gamma}_{\tilde{D}Z_T, v}^{ij} v_j \in \mathscr{I}_{n,T}^1(\Delta)$.

Finally, the process w as in (2.4) is then given by

$$\begin{array}{ll} w_i = \dfrac{\xi_{T_i} (\partial_a c_i)^{-1}}{n} \Big(1 - \dfrac{\sum_{j=1}^n \beta_{T_j}}{\sum_{j=1}^n \beta_{T_j} a_j} a_i\Big) & i = 1, \ldots, n \\ w_i \equiv 0 & \text{otherwise} \end{array}$$

Let us discuss the integrability properties for $|\det \gamma_{\tilde{D}Z_T,v}|^{-p}$ on the set $\{J_T = n\}$. One has,

$$\mathbf{1}_{\{J_T=n\}} |\det \gamma_{\tilde{D}Z_T,v}|^{-1} = \mathbf{1}_{\{J_T=n\}}\, n\xi_T^{-1} \cdot |\sum_{i=1}^n \beta_{T_i} a_i|^{-1}.$$

Now, $\xi_T^{-1} = \hat{\xi}_T \in L^q$ for any q. Concerning the latter term, one has

Proposition 2.3.4. *Suppose that $A_k = \sum_{i=1}^k a_i > 0$ for any $k = 1, \dots, n-1$. Then, $\mathbf{1}_{\{J_T=n\}} |\sum_{i=1}^n \beta_{T_i} a_i|^{-1} \in L^p$ if there exists $\delta > 0$ such that $p < n-1-\delta$.*

Let us give a discussion before to prove Proposition 2.3.4.

The real important fact is that (1.10) holds in the direction of the jump amplitudes. Now, it is simple to see that it reduces to ask for the following fact: one needs that $\mathbf{1}_{\{J_T=n\}} |\det \gamma_{\tilde{D}Z_T,v}|^{-1} \in L^p$ for some $p > 2$. Now, this holds if $\mathbf{1}_{\{J_T=n\}} |\sum_{i=1}^n \beta_{T_i} a_i|^{-1} \in L^{2+\delta}$ for some $\delta > 0$. By Proposition 2.3.4, this happens whenever $n \geq 4$. However, it is worth to say that the above formulas will be implemented for any n, and we will see how they work from a practical point of view.

In order to prove Proposition 2.3.4, let us give the following simple

Lemma 2.3.5. *Let V be a r.v. Then, $|V|^{-1} \in L^p$ if and only if*

$$\int_0^1 \frac{\mathbb{P}(|V| < \xi)}{\xi^{p+1}}\, d\xi < +\infty.$$

Proof. The statement easily follows from the fact that

$$\mathbb{E}(|V|^{-p}) = \int_0^\infty \mathbb{P}(|V|^{-p} > v)dv = \int_0^\infty p\, \frac{\mathbb{P}(|V| < \xi)}{\xi^{p+1}} d\xi$$

in which we have performed the change of variable $\xi = v^{-1/p}$. Now, the above integral if finite, i.e. $|V|^{-1} \in L^p$, if and only if integral in a neighborhood of the origin is finite.

□

We can now pass to the

Proof of Proposition 2.3.4. First, notice that

$$\sum_{i=1}^n \beta_{T_i} a_i = \sum_{i=1}^n \int_0^{T_i} \xi_s ds a_i = \int_0^T \xi_s \sum_{i=1}^n a_i \mathbf{1}_{s<T_i} ds$$
$$= \int_0^T \xi_s \sum_{i=J_s+1}^n a_i ds = -\int_0^T \xi_s A_{J_s} ds$$

where $A_{J_s} = \sum_{i=1}^{J_s} a_i \geq 0$. Therefore,

$$\int_0^T \sqrt{A_{J_s}} ds = \int_0^T \sqrt{A_{J_s}\xi_s}\sqrt{\xi_s^{-1}}\, ds \leq \Big(\int_0^T A_{J_s}\xi_s ds\Big)^{1/2} \Big(\int_0^T \hat{\xi}_s ds\Big)^{1/2}$$

in which we have set again $\hat{\xi} = \xi^{-1}$. Then,

$$|\sum_{i=1}^{n} \beta_{T_i} a_i|^{-1} = \Big| \int_0^T A_{J_s} \xi_s ds \Big|^{-1} \leq \int_0^T \hat{\xi}_s ds \cdot \Big(\int_0^T \sqrt{A_{J_s}} ds \Big)^{-2}$$

Then, since $\mathbf{1}_{\{J_T=n\}} \int_0^T \hat{\xi}_s ds \in L^q$ for any q, one has that, on the set $\{J_T = n\}$, $|\sum_{i=1}^n \beta_{T_i} a_i|^{-1} \in L^p$ if $\Big(\int_0^T \sqrt{A_{J_s}} ds \Big)^{-1} \in L^{p+\delta}$, for some $\delta > 0$. Let us then discuss whenever this happens.
First, notice that

$$\begin{aligned}
\int_0^T \sqrt{A_{J_s}} ds &= \int_0^T \sqrt{A_{J_s}} \sum_{i=0}^{n} \mathbf{1}_{\{J_s=i\}} ds = \sum_{i=1}^{n-1} \int_{T_i}^{T_{i+1}} \sqrt{A_i}\, ds \\
&= \sum_{i=1}^{n-1} \sqrt{A_i} (T_{i+1} - T_i) \geq c_A \sum_{i=1}^{n-1} (T_{i+1} - T_i) = c_A (T_n - T_1)
\end{aligned}$$

where $c_A = \inf\{A_i\,;\, i = 1, \dots, n-1\} > 0$. Therefore, the problem reduces to study when $\mathbf{1}_{\{J_T=n\}} (T_n - T_1)^{-1} \in L^{p+\delta}$. To this purpose, we use Lemma 2.3.5. Recall that, conditional to $\{J_T = n\}$, the probability density function of $(T_1, \dots, T_n)$ is given by

$$p_n(\omega, t_1, \dots, t_n) = \frac{n!}{T^n} \mathbf{1}_{0<t_1<\dots<t_n<T}$$

as $\omega \in \{J_T = n\}$, so that the marginal law of (T_1, T_n) is

$$p_{T_1,T_n}(\omega, t_1, t_n) = \frac{n(n-1)}{T^n} (t_n - t_1)^n \mathbf{1}_{0<t_1<t_n<T}$$

Simple but straightforward computations allow to give that

$$\mathbb{P}(T_n - T_1 \leq \xi \mid J_T = n) = \frac{\xi^{n-1}}{T^n} (nT - \xi)$$

for any $\xi \in (0, T)$. By using Lemma 2.3.5, $(T_n - T_1)^{-(p+\delta)}$ is integrable conditional to $\{J_T = n\}$ if and only if $n - 1 - p - \delta > 0$. Now, by recalling that $\mathbb{E}(V \mathbf{1}_A) = \mathbb{E}(V \mid A) \mathbb{P}(A)$, the statement follows.
□

Remark 2.3.6. *In practice, we will do the following choice: $a_i = 1$ for any $i < n$ and $a_n = 1 - n$, so that $A_k = k > 0$ for any $k = 1, \dots n-1$.*

2.3.3 Jump times direction

We consider now the jump times direction, that is

$$\begin{aligned}
&\tilde{D}_i \equiv 0 \text{ for } i = 0, 1, ..., n \quad \text{and} \quad \tilde{D}_i = \partial_{T_i} \text{ for } i = n+1, ..., 2n \\
&\tilde{\delta}_i \equiv 0 \text{ for } i = 0, 1, ..., n \quad \text{and} \quad \tilde{\delta}_i = \delta_i \text{ for } i = n+1, ..., 2n.
\end{aligned}$$

Recall that in this case we can perform a Malliavin differential calculus only for pure jump processes (see Example 1.1.25). Obviously, the right hypothesis on the coefficients have to be set (see Definition 2.1.1).
The scalar product reduces to

$$\langle U, V\rangle_\pi = \langle U^T, V^T\rangle_\pi^T = \sum_{i=1}^n U_{i+n}V_{i+n}\pi_{i+n}.$$

Again, we always consider the case $\pi_{i+n} \equiv 1, i = n+1, ..., 2n$. But, as observed in Remark 1.1.13, the border term operator appearing in the integration by parts formula is now non null, as already seen in Formula (2.3). Let us write down how the border term operator applies on the elements in $\mathscr{S}_{n,t}(T)$ of the form

$$F = f(X_T, Y_T) \equiv \bar{f}(T_1, \ldots, T_n, \omega) \text{ on the set } \{J_T = n\}$$

in which f denotes a quite general function, in which ω denotes generically the dependence on the jump amplitudes. By recalling Formula 1.1 for the p_i's, for $U \in \mathscr{P}_{n,t}(T)$ by (1.6) one has

$$[f(X_T, Y_T), U]_\pi = \sum_{i=1}^n \frac{f_i^+(X_T, Y_T)\cdot U_i^+ - f_i^-(X_T, Y_T)U_i^-}{T_{i+1} - T_{i-1}} \tag{2.8}$$

in which we have set

$$\begin{aligned} f_i^\pm(X_T, Y_T) &= \lim_{t\to T_{i\pm1}} \bar{f}(T_1, ..., T_{i-1}, t, T_{i+1}, ..., T_n, \omega) \\ U_i^\pm &= \lim_{t\to T_{i\pm1}} g_i^U(T_1, ..., T_{i-1}, t, T_{i+1}, ..., T_n, \omega) \end{aligned}$$

(recall that $T_0 = 0$ and $T_{n+1} = T$), where we have denoted through g_i^U the function such that $U_i = g_i^U(T_1, ..., T_n, T, \omega)$, again ω gives generically the dependence on the jump amplitudes on the set $\{J_T = n\}$.

Remark 2.3.7. *Let us observe that we are performing a choice for the π_i's differently from Bally, Bavouzet and Messaud [3], whose π_i's are set in order to keep a null border term operator. Our main purpose is to handle a (generalized) Malliavin covariance matrix $\gamma_{\tilde{Z},v}$ as simpler as possible (recall that we are in a multidimensional framework and $\gamma_{\tilde{Z},v}$ has to be inverted), and in fact, we obtain a triangular matrix $\gamma_{\tilde{Z},v}$. In other words, our choice $\pi_i \equiv 1$ turns out to be fundamental to simplify in practice the calculus of the Greeks.*

Recalling Property 1.1.23 on the first variation process ξ and the process X, we have

$$\begin{aligned} \tilde{D}_i X_T &= \xi_T \xi_{T_i}^{-1} q_i \\ \tilde{D}_i Y_T &= (\beta_T - \beta_{T_i})\xi_{T_i}^{-1} q_i. \end{aligned}$$

Thus we define the process $v = (v^1, v^2)$ as follows: for some $a \in \mathscr{A}^J$,

$$\begin{array}{llll} v^1: & v_i^1 = \xi_{T_i}(q_i)^{-1} a_i \quad i = n+1, \ldots, n, & v_i^1 \equiv 0 & i = 0, 1, \ldots, n \\ v^2: & v_i^2 = \xi_{T_i}(q_i)^{-1} \quad\;\; i = n+1, \ldots, 2n, & v_i^2 \equiv 0 & i = 0, 1, \ldots, n \end{array}$$

Therefore the matrix $\gamma_{\tilde{D}Z_T,v}$ is

$$\gamma_{\tilde{D}Z_T,v} = \left(\begin{array}{cc} 0 & -\sum_{i=1}^n \beta_{T_i} a_i \\ & \\ n\xi_T & n\beta_T - \sum_{i=1}^n \beta_{T_i} \end{array} \right).$$

Notice that in this way we have the same generalized covariance matrix that in the jump amplitudes case.
In order to verify the hypothesis of the Theorem 2.1.2, we have to verify that $\hat{\gamma}^{ij}_{\tilde{D}Z_T,v} v_j \in \mathscr{I} I^1_{n,T}(T)$. By the definition of the Skorohod integral in the jump times direction 1.1.10 it easy to check that

$$-(\partial_{T_k} U_{k+n})$$

exists. We have denoted $U_{k+n} = \hat{\gamma}^{ij}_{\tilde{D}Z_T,v} v^j_k$.
Finally, the process w as in (2.4) is then given by

$$\begin{array}{ll} w_i = \dfrac{\xi_{T_i}(q_i)^{-1}}{n} \Big(1 - \dfrac{\sum_{j=1}^n \beta_{T_j}}{\sum_{j=1}^n \beta_{T_j} a_j} a_i \Big) & i = n+1, \ldots, 2n \\ w_i \equiv 0 & \text{otherwise} \end{array}$$

Concerning the p-integrability properties for $|\det \gamma_{\tilde{D}Z_T,v}|^{-1}$ on the set $\{J_T = n\}$, the same arguments as in Section 2.3.2 (see page 42) can be performed. Again, The real important fact is that (1.10) holds in the direction of the jump times, which reduces to require that $\mathbf{1}_{\{J_T=n\}} |\det \gamma_{\tilde{D}Z_T,v}|^{-1} \in L^p$ for some $p > 2$. Now, this holds if

$$\mathbf{1}_{\{J_T=n\}} |\sum_{i=1}^n \beta_{T_i} a_i|^{-1} \in L^{2+\delta} \text{ for some } \delta > 0.$$

By Proposition 2.3.4, if $A_k = \sum_{i=1}^k > 0$ for any $k = 1, \ldots, n-1$, this happens whenever $n \geq 4$.

Remark 2.3.8. *For our numerical purposes, we will take $a_i = 1$ for $i < n$ and $a_n = 1 - n$. Therefore, $A_k > 0$ for any $k = 1, \ldots, n-1$.*

2.3.4 Joint Gaussian and jump amplitudes direction

Once considered separately the three directions of noises, we can take into account the joint differential calculus with respect to the Brownian motion and the amplitudes of the jumps. Then, differential operators are

$$\begin{array}{lll} \tilde{D}_0 = D_0, & \tilde{D}_i = \partial_{\Delta_i} \text{ for } i = 0, 1, ..., n, & \tilde{D}_i \equiv 0 \text{ for } i = n+1, ..., 2n \\ \tilde{\delta}_0 = \delta_0, & \tilde{\delta}_i = \delta_i \text{ for } i = 0, 1, ..., n, & \tilde{\delta}_i \equiv 0 \text{ for } i = n+1, ..., 2n \end{array}$$

and the underlying scalar product is the defined as

$$\langle U,V\rangle_\pi=\int_0^T U_u^0V_u^0du+\sum_{i=1}^n U_iV_i\pi_i.$$

As remarked in the previous section, we will define $\pi_i\equiv 1$ for $i=1,...,n$.
We define the process $v=(v^1,v^2)$ as a mixture, in some sense, of the v's seen in Section 2.3.1 and 2.3.2: for some $a\in\mathscr{A}^W$

$$\begin{array}{llll} v^1: & v^1_{0,u}=\xi_u\sigma^{-1}(X_u)a_u & v^1_i\equiv 0 & i=1,\dots,2n\\ v^2: & v^2_i=\xi_{T_i}(\partial_a c_i)^{-1} i=1,\dots,n, & v^2_i\equiv 0 & i=0,n+1,\dots,n \end{array}$$

Therefore the matrix $\gamma_{\tilde{D}Z_T,v}$ is

$$\gamma_{\tilde{D}Z_T,v}=\begin{pmatrix} 0 & -\int_0^T\beta_u a_u du\\ n\xi_T & n\beta_T-\sum_{i=1}^n\beta_{T_i}\end{pmatrix}.$$

Finally, the process w as in (2.4) is then given by

$$\begin{aligned} w_{0,u}&=-\frac{-\sum_{i=1}^n\beta_{T_i}}{n\int_0^T\beta_r a_r dr}\cdot\xi_u\sigma^{-1}(X_u)a_u\\ w_i&=\frac{1}{n}\cdot\xi_{T_i}(\partial_a c_i)^{-1} && i=1,\dots,n\\ w_i&\equiv 0 && i=n+1,\dots,2n \end{aligned}$$

Obviously, the functions v^1 and v^2 can be chosen also by a different mixtue involving a process a in $\mathcal{A}^{\mathcal{J}}$. In fact, we can also take, for some $a\in\mathscr{A}^J$,

$$\begin{array}{llll} v^1: & v^1_{0,u}=\xi_u\sigma^{-1}(X_u) & v^1_i\equiv 0 & i=1,\dots,2n\\ v^2: & v^2_i=\xi_{T_i}(\partial_a c_i)^{-1}a_i \quad i=1,\dots,n, & v^2_i\equiv 0 & i=0,n+1,\dots,n \end{array}$$

Then, in this case we have

$$\gamma_{\tilde{D}Z_T,v}=\begin{pmatrix} T\xi_T & T\beta_T-\int_0^T\beta_u du\\ 0 & -\sum_{i=1}^n\beta_{T_i}a_i\end{pmatrix}.$$

and the process w as in (2.4) is given by

$$\begin{aligned} w_{0,u}&=\frac{1}{T}\xi_u\sigma^{-1}(X_u)\\ w_i&=-\frac{\int_0^T\beta_u du}{T\sum_{j=1}^n\beta_{T_j}a_j}\cdot\xi_{T_i}(\partial_a c_i)^{-1}a_i && i=1,\dots,n\\ w_i&\equiv 0 && i=n+1,\dots,2n \end{aligned}$$

As for the p-integrability properties, we refer to Section 2.3.1 (page 39) and Section 2.3.2 (page 42).

2.3.5 Joint jump amplitudes and times direction

We conclude this analysis with the joint differential calculus with respect to the jump amplitudes and times. The differential operators are in this case defined as

$$\begin{array}{lll} \tilde{D}_0 \equiv 0, & \tilde{D}_i = \partial_{\Delta_i} \text{ for } i = 1, ..., n, & \tilde{D}_i = \partial_{T_i} \text{ for } i = n+1, ..., 2n \\ \tilde{\delta}_0 \equiv 0, & \tilde{\delta}_i = \delta_i^{\Delta} \text{ for } i = 1, ..., n, & \tilde{\delta}_i = \delta_i^{T} \text{ for } i = n+1, ..., 2n \end{array}$$

The scalar product is

$$\langle U, V \rangle_\pi = \sum_{i=1}^{2n} U_i V_i \pi_i$$

and again we put $\pi_i \equiv 1$ for $i = 1, ..., 2n$.
The process v is defined as a mixture of the v's seen in Section 2.3.2 and 2.3.3: for some $a \in \mathscr{A}^J$,

$$\begin{array}{lll} v^1: \; v_i^1 = \xi_{T_i} (\partial_a c_i)^{-1} a_i & i = 1, \ldots, n, & v_i^1 \equiv 0 \quad i = 0, n+1, \ldots, 2n \\ v^2: \; v_i^2 = \xi_{T_i} q_i^{-1} & i = n+1, \ldots, 2n, & v_i^2 \equiv 0 \quad i = 0, 1, \ldots, n \end{array}$$

so that the matrix $\gamma_{\tilde{D}Z_T, v}$ is

$$\gamma_{\tilde{D}Z_T, v} = \begin{pmatrix} 0 & -\sum_{i=1}^n \beta_{T_i} a_i \\ n\xi_T & n\beta_T - \sum_{i=1}^n \beta_{T_i} \end{pmatrix}.$$

Therefore, the process w as in (2.4) is given by

$$\begin{aligned} w_0 &\equiv 0 \\ w_i &= -\frac{-\sum_{j=1}^n \beta_{T_j}}{n \sum_{j=1}^n \beta_{T_j} a_j} \cdot \xi_{T_i} (\partial_a c_i)^{-1} a_i \quad i = 1, \ldots, n \\ w_i &= \frac{1}{n} \cdot \xi_{T_i} (q_i)^{-1} \qquad\qquad i = n+1, \ldots, 2n \end{aligned}$$

Again, we can also choose v^1 and v^2 in order to have a super-triangular generalized covariance matrix. In fact, for some $a \in \mathscr{A}^J$, set

$$\begin{array}{lll} v^1: \; v_i^1 = \xi_{T_i} (\partial_a c_i)^{-1} & i = 1, \ldots, n, & v_i^1 \equiv 0 \quad i = 0, n+1, \ldots, 2n \\ v^2: \; v_i^2 = \xi_{T_i} q_i^{-1} a_i & i = n+1, \ldots, 2n, & v_i^2 \equiv 0 \quad i = 0, 1, \ldots, n \end{array}$$

Then, $\gamma_{\tilde{D}Z_T, v}$ is

$$\gamma_{\tilde{D}Z_T, v} = \begin{pmatrix} n\xi_T & n\beta_T - \sum_{i=1}^n \beta_{T_i} \\ 0 & -\sum_{i=1}^n \beta_{T_i} a_i \end{pmatrix}.$$

and the process w as in (2.4) is given by

$$\begin{aligned} w_0 &\equiv 0 \\ w_i &= \frac{1}{n} \cdot \xi_{T_i} (\partial_a c_i)^{-1} && i = 1, \ldots, n \\ w_i &= -\frac{\sum_{j=1}^n \beta_{T_j}}{n \sum_{j=1}^n \beta_{T_j} a_j} \cdot \xi_{T_i} (q_i)^{-1} a_i && i = n+1, \ldots, 2n \end{aligned}$$

Recall that in this case also the border term operator has to be considered, and the formula we will use is given by (2.8).
Finally, concerning the p-integrability properties of the inverse of the determinant of the Malliavin covariance matrix, we refer to Section 2.3.2 (page 42) and Section 2.3.3 (page 45).

2.4 Payoffs depending on each single component

The arguments previously developed for the 2-dimensional framework, can be applied in the one-dimensional case. Obviously, the generalized covariance matrix is now simply a scalar, the structure of the weight becoming significantly simpler. By the way, the introduction of the process v gives better results than considering the standard Malliavin derivative.
In financial terms, we are now going to consider a payoff function ϕ of the type $\phi(X_T)$ or $\phi(Y_T)$, i.e. depending on each single component: underlying asset price at maturity time T or the mean of the underlying asset price over $[0, T]$. The result given in next Proposition 2.4.1, is simply a particular case of Proposition 2.1.2 and for this reason we omit the proof.

Proposition 2.4.1. . *Suppose that X satisfies the standard hypothesis. Let ϕ be the payoff function with polynomial growth and let ξ denote the first variation process related to the process X. Let F and G denote either X_T and ξ_T or Y_T and $\beta_T = \int_0^T \xi_u \, du$.*
Let v^F be a process such that F, v^F, and G satisfy the hypothesis in Property 1.3.8, with $n \geq 1$ and $t = T$. Then,

$$\frac{\partial}{\partial x} \mathbb{E}(\phi(F) \mathbf{1}_{J_T = n}) = \mathbb{E}(\phi(F) \tilde{\delta}(w^F \mathbf{1}_{J_T = n})) + \mathbb{E}([\phi(F), w^F \mathbf{1}_{J_T = n}]_\pi) \tag{2.9}$$

with

$$\text{if } F = X_T: \quad w^F = w^X = \xi_T (\gamma_{DX_T, v})^{-1} \cdot v^X \tag{2.10}$$
$$\text{if } F = Y_T: \quad w^F = w^Y = \beta_T (\gamma_{DY_T, v})^{-1} \cdot v^Y \tag{2.11}$$

The above result holds in the pure jump diffusion case (i.e. $c \equiv 0$) with the convention $\mathbf{1}_{\{J_T = n\}} \equiv 1$.

For the sake of clearness, we can write the hypothesis on w^F, with $F = X_T$ and $F = Y_T$, case by case as follows:

- $(w_k^F)_{k=1,...,2n} \in \mathscr{I}^1_{n,T}$ for $i,j = 1,2$ in the pure jump case (i.e. $\sigma \equiv 0$);
- $w_0^F \in Dom_{n,T,2}(\delta)$ and $(w_k^F)_{k=1,...,n} \in \mathscr{I}^1_{n,T}(\Delta)$ $i,j = 1,2$ in the jump diffusion case;
- $w_0^F \in Dom_{n,T,2}(\delta)$ for $i,j = 1,2$ in the pure diffusion case (i.e. $c \equiv 0$).

Moreover we recall that in our choice $\pi_i = 1$, in the direction of the Brownian motion and of the jump amplitudes we have not a border term (see Definition 1.1.12 and Remark 1.1.13) and therefore we can write Formula 2.9 as

$$\frac{\partial}{\partial x}\mathbb{E}(\phi(X_T)\mathbf{1}_{J_T=n}) = \mathbb{E}(\phi(X_T)\tilde{\delta}(w^X\mathbf{1}_{J_T=n}))$$

$$\frac{\partial}{\partial x}\mathbb{E}(\phi(Y_T)\mathbf{1}_{J_T=n}) = \mathbb{E}(\phi(Y_T)\tilde{\delta}(w^Y\mathbf{1}_{J_T=n})) \tag{2.12}$$

In the following, without entering in details but referring to the section previously developed, we consider different directions of the Malliavin calculus. We follow the notation of Proposition 2.4.1 and we write explicitely only the process w whose Skorohod integral gives the final weight.

- **The Gaussian Direction** [Section 2.3.1]

  We define $v_u^X = v_u^Y = \xi_u \sigma(X_u)^{-1}$. For simplicity we do not consider the components of v related to the jump noises, to be thought all equal to zero. Thus we have

  $$w_u^X = \frac{1}{T}\xi_u\sigma(X_u)^{-1} \quad \text{and} \quad w_u^Y = \frac{\beta_T\xi_u\sigma(X_u)^{-1}}{T\beta_T - \int_0^T \beta_u du}$$

- **The jump amplitudes direction** [Section 2.3.2]

  We define $v_i^X = v_i^Y = \xi_{T_i}(\partial_{\Delta_i}c_i)^{-1}$, for $i = 1,...,n$ and $v_i^X = v_i^Y = 0$ for $i = 0, n+1,...,2n$. Therefore $w_i^X = v_i^Y = 0$ for $i = 0, n+1,...,2n$ and for $i = 1,...,n$ we have

  $$w_i^X = \frac{1}{n}\xi_{T_i}(\partial_{\Delta_i}c_i)^{-1} \quad \text{and} \quad w_i^Y = \frac{\beta_T\xi_{T_i}(\partial_{\Delta_i}c_i)^{-1}}{n\beta_T - \sum_{i=1}^n \beta_{T_i} du}$$

- **The jump times direction** [Section 2.3.3]

  Let $v_i^X = v_i^Y = \xi_{T_i}q_i^{-1}$, for $i = n+1,...,2n$ and $v_i^X = v_i^Y = 0$ for $i = 0,1,...,n$. Therefore $w_i^X = v_i^Y = 0$ for $i = 0,1,...,n$ and for $i = n+1,...,2n$ we have

$$w_i^X = \frac{1}{n}\xi_{T_i} q_i^{-1} \quad \text{and} \quad w_i^Y = \frac{\beta_T \xi_{T_i} q_i^{-1}}{n\beta_T - \sum_{i=1}^n \beta_{T_i}}$$

Notice that we have defined $\pi_i = 1$ and therefore the border term operator has to be taken into account. The formula is given again by (2.3).

- **The joint Gaussian and jump amplitudes direction** [Section 2.3.4]

  In this case we define

$$\begin{aligned} v^X:\ \ v_{0,u}^X &= \frac{\xi_u \sigma(X_u)^{-1}}{T} \quad v_i^X = \frac{\xi_{T_i}(\partial_{\Delta_i} c_i)^{-1}}{n} \quad i = 1, \ldots, n \\ v^Y:\ \ v_{0,u}^Y &= \frac{\xi_u \sigma(X_u)^{-1}}{T} \quad v_i^Y = -\frac{\xi_{T_i}\partial_{\Delta_i} c^{-1}}{n} \quad i = 1, \ldots, n \end{aligned}$$

  and $v_i^X = v_i^Y = 0$ for $i = n+1, \ldots, 2n$. Thus the processes w^X and w^Y become: $w_i^X = w_i^Y = 0$ for $i = n+1, \ldots, 2n$ and

$$w^X:\ \ w_{0,u}^X = \frac{1}{2T}\xi_u \sigma(X_u)^{-1} \quad w_i^X = \frac{1}{2n}\xi_{T_i}\partial_{\Delta_i} c^{-1} \quad i = 1, \ldots, n$$

  and

$$\begin{aligned} w^Y:\ \ w_{0,u}^Y &= \frac{\beta_T \xi_u \sigma(X_u)^{-1}}{\frac{1}{nT}\sum_{i=1}^n \beta_{T_i} - \int_0^T \beta_u du} \\ w_i^Y &= -\frac{\beta_T \xi_{T_i}\partial_{\Delta_i} c^{-1}}{\sum_{i=1}^n \beta_{T_i} - \frac{1}{nT}\int_0^T \beta_u du} \quad i = 1, \ldots, n \end{aligned}$$

- **The joint jump times and amplitudes direction** [Section 2.3.5]

  We define in this case $v^X = v^Y = v$, that is in the same way for payoffs depending on X or on Y only. We have

$$v_i = \xi_{T_i}(\partial_{\Delta_i} c_i)^{-1} \quad i = 1, \ldots, n, \quad v_i = \xi_{T_i} q_i^{-1} \quad i = n+1, \ldots, 2n$$

  and $v_0 \equiv 0$. This gives

$$\begin{aligned} w^X:\ \ w_i^X &= \frac{1}{2n}\xi_{T_i}(\partial_{\Delta_i} c_i)^{-1} && i = 1, \ldots, n \\ w_i^X &= \frac{1}{2n}\xi_{T_i} q_i^{-1} && i = n+1, \ldots, 2n \\ w^Y:\ \ w_i^Y &= \frac{\beta_T \xi_{T_i}(\partial_{\Delta_i} c_i)^{-1}}{2(n\beta_T - \sum_{i=1}^n \beta_{T_i})} && i = 1, \ldots, n \\ w_i^Y &= \frac{\beta_T \xi_{T_i} q_i^{-1}}{2(n\beta_T - \sum_{i=1}^n \beta_{T_i})} && i = n+1, \ldots, 2n \end{aligned}$$

  and $w_0^X = w_0^Y \equiv 0$.

  Again, since $\pi_i = 1$, the border term operator as in (2.3) has to be taken into account.

2.5 Examples

Here, we consider some examples of financial interest, seeing in details what developed up to now. In some examples, we will prefer to use some specifical choice of the process v, according to the considered model, instead of the general definition previously given. In this way we can further simplify the Malliavin Delta weight.
The models are singularly discussed, analyzing the different directions for the differential calculus.

2.5.1 Black-Scholes-Merton model

We consider the process $Z_t = (X_t, Y_t)$ under the dynamics

$$\begin{aligned} X_t &= x + \int_0^t bX_r dr + \int_0^t \sigma X_r dW_r + \sum_{i=1}^{J_t} \Delta_i X_{T_i-} \\ Y_t &= \int_0^t X_r dr \end{aligned}$$

where $b \in \mathbb{R}$ and $\sigma > 0$.
When considering only the diffusion part of the process X, one has the classic Black-Scholes model; keeping the jump part, one obtains the Merton model. Notice moreover that Z_t satisfies all the standard hypothesis. We can write the solution process X_t explicitly:

$$X_t = x \exp\Big((b - \frac{1}{2}\sigma^2)t + \sigma W_t\Big) \prod_{i=1}^{J_t}(1 + \Delta_i).$$

As a consequence, the first variation process ξ_t and $\beta_t = \int_0^t \xi_r dr$ are given by

$$\xi_t = \frac{X_t}{x} \qquad \beta_t = \frac{Y_t}{x}.$$

Now, fix the maturity time T and the set $\{J_T = n\}$. Suppose moreover that the jump times are as usual (that is, T_1, $T_2 - T_1$, ... are i.i.d exponentially distributed r.v.'s) and that the density g associated to the jump amplitudes is smooth, non null only on $I = \cup_k(a_k, b_k)$ and $g \equiv 0$ on ∂I. In the numerical simulations we will take the jump amplitudes as log-normal distributed.

Remark 2.5.1. *Notice that X (and as a consequence Y) does not depend on the jump times T_i explicitly, the dependence being on J_t only. Therefore we cannot apply the differentia calculus in the jump times direction (see Bally, Bavouzet, Messaoud, [3])*

Gaussian direction

We apply here the results in Section 2.3.1. Then, the process w whose Skorohod integral gives the final weight has only the first component non null and given by

$$w_{0,u} = \frac{1}{\sigma x T}\Big(1 - \frac{\int_0^T Y_r dr}{\int_0^T Y_r a_r dr} \cdot a_u\Big), \qquad u \in [0,T]$$

where $a \in \mathscr{A}^W$ (i.e. a is a deterministic function such that $\int_0^T a_u du = 0$). So, by recalling Property 1.2.6 on the Skorohod integral of a special product, we can write

$$\begin{aligned}\tilde{\delta}_0(w_0) &= \frac{1}{\sigma x T}\Big[W_T - \frac{\int_0^T Y_r dr}{\int_0^T Y_r a_r dr}\int_0^T a_u dW_u + \\ &\qquad + \int_0^T \tilde{D}_{0,u}\Big(\frac{\int_0^T Y_r dr}{\int_0^T Y_r a_r dr}\Big) a_u du\Big]\end{aligned}$$

Now,

$$\begin{aligned}\tilde{D}_{0,u}\Big(\frac{\int_0^T Y_r dr}{\int_0^T Y_r a_r dr}\Big) &= \frac{\tilde{D}_{0,u}(\int_0^T Y_r dr)}{\int_0^T Y_r a_r dr} - \int_0^T Y_r dr \cdot \frac{\tilde{D}_{0,u}(\int_0^T Y_r a_r dr)}{(\int_0^T Y_r a_r dr)^2} \\ &= \frac{\sigma \int_u^T (Y_r - Y_u) dr}{\int_0^T Y_r a_r dr} - \int_0^T Y_r dr \cdot \frac{\sigma \int_u^T (Y_r - Y_u) a_r dr}{(\int_0^T Y_r a_r dr)^2}\end{aligned}$$

in which we have used that, here, $\tilde{D}_{0,u} Y_r = \sigma(Y_r - Y_u)\mathbf{1}_{u \le r}$. Therefore, the weight has the final form

$$\begin{aligned}\tilde{\delta}(w) &= \frac{1}{\sigma x T}\Big[W_T - \frac{\int_0^T Y_r dr}{\int_0^T Y_r a_r dr}\int_0^T a_u dW_u + \\ &\qquad + \frac{\sigma}{\int_0^T Y_r a_r dr}\int_0^T \Big(\int_u^T (Y_r - Y_u) dr\Big) a_u du \\ &\qquad - \frac{\sigma \int_0^T Y_r dr}{(\int_0^T Y_r a_r dr)^2}\int_0^T \Big(\int_u^T (Y_r - Y_u) a_r dr\Big) a_u du\Big]\end{aligned} \tag{2.13}$$

Remark 2.5.2. *Consider for the Black-Scholes model the weight of Benhamou considered in Remark 2.3.3. The weight of Benhamou can be written as $\tilde{\delta}(w_u^{Ben})$ where*

$$w_u^{Ben} = \frac{1}{\sigma x}\Big(\frac{Y_T^2 + 2(\int_0^T r X_r dr - T Y_T) X_u}{Y_T(2\int_0^T r X_r dr - T Y_T)}\Big)$$

Therefore in order to handle the Skorohod integral $\tilde{\delta}(w_u^{Ben})$, we apply Property

1.2.6 and we obtain

$$\tilde\delta(w_u^{Ben}) = \frac{1}{\sigma x}\Big\{\frac{Y_T W_T}{2\int_0^T rX_r dr - TY_T} - \int_0^T \frac{\sigma(Y_T - Y_u)}{2\int_0^T rX_r dr - TY_T} du$$

$$- \int_0^T \frac{\sigma Y_T\big(2\int_u^T rX_r dr - T(Y_T - Y_u)\big)}{(2\int_0^T rX_r dr - TY_T)^2} du$$

$$+ \frac{2(\int_0^T rX_r dr - TY_T)}{Y_T(2\int_0^T rX_r dr - TY_T)} \int_0^T X_u dW_u - 2\sigma \int_0^T \frac{\int_u^T rX_r dr - T(Y_T - Y_u)}{Y_T(2\int_0^T rX_r dr - TY_T)} X_u du$$

$$+2\sigma(\int_0^T rX_r dr - TY_T)\times$$

$$\times \int_0^T \Big(\frac{(Y_T - Y_u)(2\int_0^T rX_r dr - TY_T)) + Y_T(2\int_u^T - T(Y_T - Y_u))}{(2\int_0^T rX_r dr - TY_T))^2}\Big) X_u du\Big\} \tag{2.14}$$

A comparison with Formula (2.13) shows that (2.14) has a much more complicated structure, which in turn suggests the importance of the introduction of suitable processes-directions v's for practical purposes.

Jump amplitudes direction

Following Section 2.3.2, the proces w is non null only along the direction of the jump amplitudes. Here, by using the fact that

$$\xi_{T_i} = \frac{X_{T_i}}{x}, \quad \partial_a c_i = X_{T_i} \quad \text{and} \quad \frac{X_{T_i}}{X_{T_i^-}} = 1 + \Delta_i.$$

it follows that

$$w_i = \frac{1}{nx}\Big(1 - \frac{\sum_{j=1}^n Y_{T_j}}{\sum_{j=1}^n Y_{T_j} a_j} a_i\Big)(1 + \Delta_i), \quad i = 1, \dots, n$$

where $a \in \mathscr{A}^J$ (i.e. $a \in \mathbb{R}^n$ with $\sum_{i=1}^n a_i = 0$). By recalling Property 1.1.17 on the Skorohod integral of a special product, we can write for $i = 1, \dots, n$,

$$\tilde\delta_i(w_i) = \frac{1}{nx}\Big[\Big(1 - \frac{\sum_{j=1}^n Y_{T_j}}{\sum_{j=1}^n Y_{T_j} a_j} a_i\Big)\tilde\delta_i(1 + \Delta_i) + \tilde D_i\Big(\frac{\sum_{j=1}^n Y_{T_j}}{\sum_{j=1}^n Y_{T_j} a_j} a_i\Big)(1 + \Delta_i)\Big]$$

Now, it is easy to see that

$$\tilde D_i X_u = \frac{X_u}{1 + \Delta_i} \mathbf{1}_{T_i \le u}$$

so that

$$\tilde{D}_i Y_{T_j} = \frac{Y_{T_j} - Y_{T_i}}{1+\Delta_i} \mathbf{1}_{T_i \le T_j}$$

By using the chain rule, the term in which the derivative $\tilde{D}_i$ appears can be written, and the final formula for the weight is

$$\begin{aligned}\tilde{\delta}(w) &= \frac{1}{n\,x}\Big[\Big(1 - \frac{\sum_{j=1}^n Y_{T_j}}{\sum_{j=1}^n Y_{T_j} a_j}\, a_i\Big)\sum_{i=1}^n \tilde{\delta}_i(1+\Delta_i) + \\ &\quad + \frac{1}{\sum_{j=1}^n Y_{T_j} a_j}\sum_{i=1}^n \sum_{j=i+1}^n (Y_{T_j} - Y_{T_i}) + \\ &\quad - \frac{\sum_{j=1}^n Y_{T_j}}{(\sum_{j=1}^n Y_{T_j} a_j)^2}\sum_{i=1}^n a_i \sum_{j=i+1}^n (Y_{T_j} - Y_{T_i}) a_j\Big]\end{aligned} \tag{2.15}$$

where, recalling Definition 1.1.9 for the Skorohod integral,

$$\tilde{\delta}_i(1+\Delta_i) = -\sum_{i=1}^n (1 + (1+\Delta_i)\partial \log g(\Delta_i)).$$

For practical purposes, we will consider $1+\Delta_1$ as a log-normal r.v., that is $1+\Delta_1 = \exp\Gamma$, with $\Gamma \sim \mathrm{N}(m, \varrho^2)$. Then, $I = (-1, +\infty)$ and

$$g(x) = \phi\Big(\frac{\log(1+x) - m}{\varrho}\Big)\frac{1}{\varrho(1+x)}\,\mathbf{1}_{\{x>-1\}}$$

where ϕ denotes the standard Gaussian probability density function and notice that $g \equiv 0$ on ∂I. In this case, we obtain for $x > -1$

$$\partial \log g(x) = \frac{m - \log(1+x) - \varrho^2}{\varrho^2(1+x)}.$$

so that

$$\tilde{\delta}_i(1+\Delta_i) = -\sum_{i=1}^n \frac{m - \log(1+\Delta_i)}{\varrho^2}, \quad i = 1, \dots, n. \tag{2.16}$$

Joint Gaussian and jump amplitudes direction

In Section 2.3.4, we suggested two possible choices for the process $v = (v^1, v^2)$, giving two different processes w. Let us write down the associated weights in this context. In both cases, w is non null along the directions 0 and $1, \dots, n$. In the first case, w is is given by

$$w_{0,u} = -\frac{1}{n\sigma x}\cdot\frac{\sum_{j=1}^n Y_{T_j}}{\int_0^T Y_r a_r dr}\cdot a_u \text{ and } w_i = \frac{1}{nx}\cdot(1+\Delta_i), \quad 1 \le i \le n$$

where $a \in \mathscr{A}^W$. The associated weight is then $\tilde{\delta}(w) = \tilde{\delta}_0(w_0) + \sum_{i=1}^n \tilde{\delta}_i(w_i)$. Let us write more explicitely $\tilde{\delta}(w)$. First, the i-th Skorohod integral in the direction of the jump amplitudes is simple to write: using directly Definition 1.1.10,

$$\tilde{\delta}_i(w_i) = \frac{1}{nx}(1 + (1+\Delta_i)\partial \log g(\Delta_i)) \tag{2.17}$$

where g is the density of Δ_i. Moreover, by applying Property 1.2.6 for handling the Skorohod integral $\tilde{\delta}_0(w_0)$, we obtain

$$\begin{aligned} \tilde{\delta}_0(w_0) &= \frac{1}{n\sigma x}\Big[- \frac{\sum_{j=1}^n Y_{T_j}}{\int_0^T Y_r a_r dr} \int_0^T a_u dW_u + \\ &\quad + \frac{\sigma}{\int_0^T Y_r a_r dr} \int_0^T a_u \sum_{j=1}^n (Y_{T_j} - Y_u)\, \mathbf{1}_{T_j > u}\, du + \\ &\quad - \frac{\sigma \sum_{j=1}^n Y_{T_j}}{\left(\int_0^T Y_r a_r dr\right)^2} \cdot \int_0^T a_u \int_u^T (Y_r - Y_u) dr\, du \Big] \end{aligned} \tag{2.18}$$

By resuming, the Delta weight is given by

$$\tilde{\delta}(w) = \tilde{\delta}_0(w_0) + \sum_{i=1}^n \tilde{\delta}_i(w_i) \tag{2.19}$$

with $\tilde{\delta}_0(w_0)$ given by (2.18) and for $i \geq 1$, $\tilde{\delta}_i(w_i)$ given by (2.17).

Let us now consider the second possible weight, as discussed in Section 2.3.4. In this case, again w is null along the direction of the jump times and

$$w_{0,u} = \frac{1}{\sigma T x} \text{ and } w_i = -\frac{\int_0^T Y_u du}{Tx \sum_{j=1}^n Y_{T_j} a_j} a_i (1+\Delta_i), \quad 1 \leq i \leq n$$

so that one immediately obtains that

$$\tilde{\delta}_0(w_0) = \frac{W_T}{\sigma T x} \tag{2.20}$$

Moreover, by using the chain rule and Property 1.1.17, we have

$$\begin{aligned} \tilde{\delta}_i(w_i) &= -\frac{1}{Tx}\Big[\frac{\int_0^T Y_u du}{\sum_{j=1}^n Y_{T_j} a_j}(1 + (1+\Delta_i)\partial \log g(\Delta_i)) + \\ &\quad - \frac{\int_{T_i}^T (Y_u - Y_{T_i}) du}{\sum_{j=1}^n Y_{T_j} a_j} + \\ &\quad + \frac{\sum_{j=i+1}^n (Y_{T_j} - Y_{T_i}) a_j}{(\sum_{j=1}^n Y_{T_j} a_j)^2} \int_0^T Y_u du \Big]\, a_i \end{aligned} \tag{2.21}$$

By resuming, the second possible Delta weight is given by

$$\tilde{\delta}(w) = \tilde{\delta}_0(w_0) + \sum_{i=1}^{n} \tilde{\delta}_i(w_i) \tag{2.22}$$

with $\tilde{\delta}_0(w_0)$ given by (2.20) and for $i \geq 1$, $\tilde{\delta}_i(w_i)$ given by (2.21).

2.5.2 Ornstein-Uhlenbeck model

Consider now the jump version of the Ornstein-Uhlenbeck process. Our process of interest $Z_t = (X_t, Y_t)$ is then the solution to the following equation

$$\begin{aligned} X_t &= x - \int_0^t b \cdot (X_r - \theta) dr + \int_0^t \sigma dW_r + \sum_{i=1}^{J_t} \alpha \Delta_i \\ Y_t &= \int_0^t X_r dr \end{aligned} \tag{2.23}$$

In other words, one has $b(r, x) = b \cdot (x - \theta)$, $\sigma(r, x) = \sigma$ and $c(r, a, x) = \alpha \cdot a$, with $b, \theta, \alpha \in \mathbb{R}$ and $\sigma > 0$. Notice that X satisfies the standard hypothesis. The process X_t has a closed form solution:

$$X_t = xe^{-bt} + \theta(1 - e^{-bt}) + \sigma e^{-bt} \int_0^t e^{br} dW_r + \alpha \sum_{i=1}^{J_t} e^{-b(t-T_i)} \Delta_i. \tag{2.24}$$

Therefore, the first variation processes ξ_t and $\beta_t = \int_0^t \xi_r dr$ are simply deterministic and given by

$$\xi_t = e^{-bt} \text{ and } \beta_t = \frac{1 - e^{-bt}}{b}$$

respectively.
Let a maturity $T > 0$ and a number n of jumps be fixed, so that we are going to implicitly work on the set $\{J_t = n\}$.

Remark 2.5.3. *By the closed form solution 2.24, we can observe that there is not dependence between the Gaussian and the jump times parts. Therefore we can consider the jump time direction of the calculus also in the presence of the Ito integrals. Thus the condition $\sigma = 0$ does not hold anymore. Moreover we can consider a joint Gaussian-jump times direction and even a joint Gaussian-jump amplitudes and times direction.*

Gaussian direction

We follow now what developed in Section 2.3.1. First, notice that the following useful equalities hold:

$$\tilde{D}_{0,u} X_r = \sigma e^{-b(r-u)} \mathbf{1}_{u \leq r} \qquad \tilde{D}_{0,u} Y_r = \frac{\sigma}{b}(1 - e^{-b(r-u)}) \mathbf{1}_{u \leq r}.$$

The process w is sensible only in the 0-th direction and

$$w_{0,u} = \frac{e^{-bu}}{\sigma T}\Big(1 - \frac{1 - Tb - e^{-bT}}{b\int_0^T e^{-br} a_r dr}\, a_u\Big)$$

for some $a \in \mathscr{A}^W$. The associated Skorohod integral agrees with the Ito one and the final weight is

$$\tilde{\delta}(w) \equiv \tilde{\delta}_0(w_0) = \frac{1}{\sigma T}\Big(\int_0^T e^{-bu} dW_u - \frac{1 - Tb - e^{-bT}}{b\int_0^T e^{-br} a_r dr}\int_0^T e^{-bu}\, a_u dW_u\Big) \quad (2.25)$$

Jump amplitudes direction

Let us now consider the differential calculus with respect the jump amplitudes, as in Section 2.3.2. First notice that for $i = 1, .., n$,

$$\tilde{D}_i X_T = \alpha\, e^{-b(T-T_i)} \mathbf{1}_{T_i \le T} \quad \text{and} \quad \tilde{D}_i Y_T = \frac{\alpha}{b}\,(1 - e^{-b(T-T_i)})\mathbf{1}_{T_i \le T}.$$

Since here $\partial_a c_i = \alpha$, we have, for some $a \in \mathscr{A}^J$,

$$w_i = \frac{e^{-bT_i}}{\alpha n}\Big(1 - \frac{n - \sum_{j=1}^n e^{-bT_j}}{\sum_{j=1}^n e^{-bT_j} a_j} a_i\Big)$$

for $i = 1, \dots, n$, and $w_i = 0$ otherwise. Notice that in this case w depends on the jump times but not on the jump amplitudes. By recalling that, for any constant c and $i = 1, \dots, n$, $\tilde{\delta}_i(c) = c\tilde{\delta}_i(1) = -c\partial \log g(\Delta_i)$, the final weight is

$$\tilde{\delta}(w) = -\frac{1}{\alpha n}\sum_{i=1}^n e^{-bT_i}\Big(\Big(1 - \frac{n - \sum_{j=1}^n e^{-bT_j}}{\sum_{j=1}^n e^{-bT_j} a_j} a_i\Big)\partial \log g(\Delta_i) \quad (2.26)$$

Jump times direction

In general, when considering the jump times direction, one has to deal with a pure jump diffusion. In this particular case, we do not have to set $\sigma = 0$ in (2.23) (see Remark 2.5.3). Moreover, following Section 2.3.3, in the final weight also the border term operator has to be taken into account.
First, notice that we have the following formulas for the Malliavin derivatives hold for $i = n+1, ..., 2n$:

$$\tilde{D}_i X_T = \alpha b\, e^{-b(T-T_i)} \Delta_i \mathbf{1}_{T_i \le T} \quad \text{and} \quad \tilde{D}_i Y_T = \alpha \Delta_i\,(1 - e^{-b(T-T_i)})\mathbf{1}_{T_i \le T}.$$

Moreover, here we have $q_i = \alpha b \Delta_i$, because $\partial_x c_i = \partial_t c_i = 0$.
Here, form Section 2.3.3, we obtain for $a \in \mathscr{A}^J$,

$$w_i = \frac{e^{-bT_i}}{b\alpha n \Delta_i}\Big(1 - \frac{\sum_{j=1}^n e^{-bT_j} - n}{\sum_{j=1}^n e^{-bT_j} a_j}\, a_i\Big)$$

for $i = n+1, \ldots, 2n$, and $w_i \equiv 0$ otherwise. By recalling that, for $i = n+1, \ldots, 2n$, $\tilde{\delta}_i(ce^{-bT_i}) = -cbe^{-bT_i}$ for any possibly random but not dependent on T_i (see Definition 1.1.9) and by using Property 1.1.17, we can write

$$\begin{aligned}\tilde{\delta}(w) &= \frac{1}{\alpha n}\sum_{i=1}^{n}\frac{e^{-bT_i}}{\Delta_i}\Big(1 - \frac{\sum_{j=1}^{n} e^{-bT_j}}{\sum_{j=1}^{n} e^{-bT_j} a_j}\, a_i + \\ &\qquad -\frac{1}{\sum_{j=1}^{n} e^{-bT_j} a_j}\, a_i e^{-bT_i} + \\ &\qquad + \frac{\sum_{j=1}^{n} e^{-bT_j}}{(\sum_{j=1}^{n} e^{-bT_j} a_j)^2}\, a_i^2 e^{-bT_i}\Big)\end{aligned} \tag{2.27}$$

Now, in this special case the above $\tilde{\delta}(w)$ does give the weight but the final formula for the delta involves also the border term operator, given in Formula (2.8). In oder to use it, first let us use the notation $X_T^{T_1,\ldots,T_n}(\omega)$ and $Y_T^{T_1,\ldots,T_n}(\omega)$ to denote the functions giving the dependence of X_T and Y_T respectively on the jump times (the other noises being contempled in the "generical" ω). Now, for any $i = 1, \ldots, n$, notice that

$$\begin{aligned}\lim_{t\to T_i\pm 1} X_T^{T_1,\ldots,T_{i-1},t,T_{i+1},\ldots,T_n} &= X_T + \alpha e^{-bT}\ (e^{bT_{i\pm 1}} - e^{bT_i})\Delta_i \text{ and} \\ \lim_{t\to T_i\pm 1} Y_T^{T_1,\ldots,T_{i-1},t,T_{i+1},\ldots,T_n} &= Y_T - \frac{\alpha}{b}(e^{-bT} - e^{-bT_i})\ (e^{bT_{i\pm 1}} - e^{bT_i})\Delta_i\end{aligned}$$

(recall that $T_0 = 0$ and $T_{n+1} = T$).

Let us briefly see how the border term operator will work to handle an option with payoff $f(X_T, Y_T)$. Using the notations of (2.8), we have, for a continuous function f,

$$f_i^{\pm} = f\Big(X_T + \alpha e^{-bT}\ (e^{bT_{i\pm 1}} - e^{bT_i})\Delta_i, Y_T - \frac{\alpha}{b}(e^{-bT} - e^{-bT_i})\ (e^{bT_{i\pm 1}} - e^{bT_i})\Delta_i\Big) \tag{2.28}$$

Moreover,

$$\begin{aligned} w_i^{\pm} &= \lim_{t\to T_i\pm 1} w_i(T_1, ..., T_{i-1}, t, T_{i+1}, ..., T_n, T, \omega) = \\ &= \frac{e^{-bT_{i\pm 1}} a_i}{\alpha\, b\, n\, \Delta_i}\Big(1 - \frac{\sum_{j=1, j\neq i}^{n} e^{-bT_j} + e^{-bT_{i\pm 1}} - n}{\sum_{j=1, j\neq i}^{n} e^{-bT_j} a_j + e^{-bT_{i\pm 1}} a_i}\Big).\end{aligned} \tag{2.29}$$

We are now ready to write the border term operator:

$$[f(X_T, Y_T), w]_\pi = \sum_{i=1}^{n}\frac{f_i^+ w_i^+ - f_i^- w_i^-}{T_{i+1} - T_{i-1}} \tag{2.30}$$

with $f_i^{\pm}$ and $w_i^{\pm}$ given in (2.28) and (2.29) respectively

Joint Gaussian and jump amplitudes direction

Following Section 2.3.4 (recall that we are going here to write two different weights), the first process w here becomes

$$w_{0,u} = \frac{1}{n\sigma} \cdot \frac{\sum_{j=1}^{n} e^{-bT_j} - n}{\int_0^T e^{-br} a_r dr} \cdot e^{-bu} a_u \text{ and } w_i = \frac{1}{n\alpha} \cdot e^{-bT_i}, \quad 1 \le i \le n$$

for some $a \in \mathscr{A}^W$, and $w_i = 0$ otherwise. The finaly weight is immediate to write down:

$$\tilde{\delta}(w) = \frac{\sum_{j=1}^{n} e^{-bT_j} - n}{n\sigma \int_0^T e^{-br} a_r dr} \cdot \int_0^T e^{-bu} a_u dW_u - \frac{1}{n\alpha} \cdot \sum_{i=1}^{n} e^{-bT_i} \partial \log g(\Delta_i). \quad (2.31)$$

As for the second case, we have

$$w_{0,u} = \frac{1}{T\sigma} e^{-bu} \text{ and } w_i = \frac{1}{T\sigma} \cdot \frac{T - \int_0^T e^{-br} dr}{\sum_{j=1}^{n} e^{-bT_j} a_j} \cdot e^{-bT_i}, \quad 1 \le i \le n$$

for some $a \in \mathscr{A}^J$, with $w_i = 0$ otherwise, and we obtain

$$\tilde{\delta}(w) = \frac{1}{T\sigma} \int_0^T e^{-bu} dW_u - \frac{1}{T\sigma} \sum_{i=1}^{n} \frac{T - \int_0^T e^{-br} dr}{\sum_{j=1}^{n} e^{-bT_j} a_j} \cdot e^{-bT_i} \partial \log g(\Delta_i). \quad (2.32)$$

Joint jump amplitudes and times direction

We have set two choice of the process v for this direction of Calculus (see 2.3.5. In the first one, for a fixed $a \in \mathscr{A}^J$, we have $w_0 \equiv 0$ and

$$\begin{aligned} w_i &= \frac{\sum_{j=1}^{n} e^{-bT_j} - n}{n\alpha \sum_{j=1}^{n} e^{-bT_j} a_j} \cdot e^{-bT_i} a_i & 1 \le i \le n \\ w_i &= \frac{e^{-bT_i}}{n\alpha b \, \Delta_i} & n+1 \le i \le 2n \end{aligned}$$

Therefore the Skorohod integral is given by

$$\tilde{\delta}(w) = \frac{n - \sum_{j=1}^{n} e^{-bT_j}}{n\alpha \sum_{j=1}^{n} e^{-bT_j} a_j} \sum_{i=1}^{n} e^{-bT_i} a_i \partial \log g(\Delta_i) + \frac{1}{n\alpha b} \sum_{i=1}^{n} \frac{b e^{-bT_i}}{\Delta_i}. \quad (2.33)$$

As for the border term in the integration by parts, in the jump times direction (see (2.3)), in this case we have that

$$w_{i+n}^{\pm} = \frac{1}{\alpha b n} \frac{e^{-bT_{i\pm1}}}{\Delta_i} \quad (2.34)$$

and for $f_i^\pm$ we can look at Formula (2.28). We can summarize the border term as follows:

$$[f(X_T, Y_T), w]_\pi = \sum_{i=1}^{n} \frac{f_i^+ w_i^+ - f_i^- w_i^-}{T_{i+1} - T_{i-1}} \tag{2.35}$$

with $f_i^\pm$ and $w_i^\pm$ given in (2.28) and (2.34) respectively

Let us final discuss the second w introduced in Section 2.3.5: here, one has, for a fixed $a \in \mathscr{A}^J$, $w_0 \equiv 0$ and

$$\begin{aligned} w_i &= \frac{e^{-bT_i}}{n\alpha} && 1 \le i \le n \\ w_{i+n} &= \frac{n - \sum_{j=1}^n e^{-bT_j}}{n\alpha b \sum_{j=1}^n e^{-bT_j} a_j} \cdot \frac{e^{-bT_i} a_i}{\Delta_i} && 1 \le i \le n \end{aligned}$$

and the Skorohod integral is given by

$$\begin{aligned} \tilde{\delta}(w) = & -\frac{e^{-bT_i}}{n\alpha} \sum_{i=1}^{n} \partial \log g(\Delta_i) + \\ & + \sum_{i=1}^{n} \frac{e^{-bT_i} a_i}{n\alpha \Delta_i} \Big(- \frac{n - \sum_{j=1}^n e^{-bT_j}}{\sum_{j=1}^n e^{-bT_j} a_j} + \\ & \qquad + \frac{e^{-bT_i}}{\sum_{j=1}^n e^{-bT_j} a_j} + \\ & \qquad + \frac{n - \sum_{j=1}^n e^{-bT_j}}{(\sum_{j=1}^n e^{-bT_j} a_j)^2} e^{-bT_i} a_i \Big) \end{aligned} \tag{2.36}$$

As for the border term in the integration by parts, in the jump times direction (see (2.3)), in this case we have that

$$w_{i+n}^\pm = \frac{n - \sum_{j=1, j\neq i}^n e^{-bT_j} - e^{-bT_{i\pm1}}}{n\alpha b \sum_{j=1, j\neq i}^n e^{-bT_j} a_j + e^{-bT_{i\pm1}} a_i} \cdot \frac{e^{-bT_{i\pm1}} a_i}{\Delta_i} \tag{2.37}$$

We can summarize the border term as follows:

$$[f(X_T, Y_T), w]_\pi = \sum_{i=1}^{n} \frac{f_i^+ w_i^+ - f_i^- w_i^-}{T_{i+1} - T_{i-1}} \tag{2.38}$$

with $f_i^\pm$ and $w_i^\pm$ given in (2.28) and (2.37) respectively

Joint Gaussian and jump times direction

We consider in this Section the Joint Gaussian and jump times direction of the Malliavin calculus, that can be taken in account in this particular case (see Remark 2.5.3. Let introduce the process v as

$$\begin{aligned} & v^1 : \ v^1_{0,u} = \xi_u \sigma^{-1}(X_u) a_u = \tfrac{e^{-bu} a_u}{\sigma} && v_i^1 \equiv 0 \quad i = 1, \dots, 2n \\ & v^2 : \ v_i^2 = \xi_{T_i}(q_i)^{-1} = \tfrac{e^{-bT_i}}{\alpha b \Delta_i} \quad i = n+1, \dots, 2n, && v_i^2 \equiv 0 \quad i = 0, 1, \dots, n \end{aligned}$$

We follow the same machinery as in previous joint direction of calculus. Therefore the process $w = (w_{0,u}, w_1, ..., w_{2n})$ is defined as

$$\begin{array}{lll} w_{0,u} = & \left(\frac{n-\sum_{j=1}^n e^{-bT_j}}{n\sigma\int_0^T e^{-br}a_r dr}\right)e^{-bu}a_u & \\ w_i = & 0 & i = 1, .., n \\ w_i = & \frac{e^{-bT_i}}{\alpha bn\Delta_i} & i = n+1, .., 2n \end{array}$$

Thus the integral Skorohod of w is given by

$$\begin{array}{ll} \tilde{\delta}(w) & = \delta_0(w_0) + \sum_{i=1}^n \delta_{n+i}(w_{n+i}) \\ & = \left(\frac{n-\sum_{j=1}^n e^{-bT_j}}{n\sigma\int_0^T e^{-br}a_r dr}\right)\int_0^T e^{-bu}a_u dW_u - \frac{1}{\alpha n}\sum_{i=1}^n \frac{e^{-bT_i}}{\Delta_i} \end{array}$$

Notice that the process w has the same $n+1, .., 2n$ components than in the joint jump amplitudes and times direction of calculus case. Than we have that the border term, that has to be considered, is the same one (see (2.37) and (2.38)).

Joint Gaussian and jump amplitudes and times direction

All the the direction of noises are considered in this Section. Thus we have for this special case that the process involved in the calculus have all the components different to zero.
Thus let define in a suitable way the process v.

$$\begin{array}{llll} v^1: & v^1_{0,u} = \xi_u\sigma^{-1}(X_u)a_u = \frac{e^{-bu}a_u}{\sigma} & v^1_i \equiv 0 & i = 1, \ldots, 2n \\ v^2: & v^2_i = \frac{1}{2}\xi_{T_i}(\partial_a c_i)^{-1} = \frac{e^{-bT_i}}{2\alpha} & & i = 1, \ldots, n \\ & v^2_i = \frac{1}{2}\xi_{T_i}(q_i)^{-1} = \frac{e^{-bT_i}}{2\alpha b\Delta_i} & & i = n+1, \ldots, 2n. \end{array}$$

Then the process w is

$$\begin{array}{lll} w_{0,u} = & \left(\frac{n-\sum_{j=1}^n e^{-bT_j}}{n\sigma\int_0^T e^{-br}a_r dr}\right)e^{-bu}a_u & \\ w_i = & \frac{e^{-bT_i}}{2\alpha n} & i = 1, .., n \\ w_i = & \frac{e^{-bT_i}}{2\alpha bn\Delta_i} & i = n+1, .., 2n \end{array}$$

We remark that the $w_{0,u}$ is the same as in the joint Gaussian and jump times direction, w_i, $i = 1, ..., n$ is the same (multiplied for one half) as in the joint Gaussian and jump amplitudes direction and, w_i, $i = n+1, ..., 2n$ is the same (multiplied for one half) as in the joint Gaussian and jump times direction. So the Skorohod integral $\tilde{\delta}(w)$ can be easily deduced. The same can be done for the border term operator.

2.5.3 CIR Model

We study in this Section the following model

$$
\begin{aligned}
X_t &= x + \int_0^t (\nu - \eta X_r)dr + \int_0^t \sigma \sqrt{X_r} dW_r + \alpha \sum_{i=1}^{J_t} \Delta_i \\
Y_t &= \int_0^t X_r dr
\end{aligned}
\tag{2.39}
$$

where X_t is a C.I.R. model with a jump part.
Let us resume some important facts concerning the pure diffusion version of this process.

Proposition 2.5.4. *Suppose here that X is a pure diffusion:*

$$X_t = x + \int_0^t (\nu - \eta X_r)dr + \int_0^t \sigma \sqrt{X_r} dW_r.$$

Then the following holds.

- (i) *Set τ the first time at which the process X hits 0. If $2\nu \geq \sigma^2 > 0$ then $\mathbb{P}(\tau = +\infty) = 1$ and $X_t \in L^p$ for any p.*
- (ii) *If $2\nu > p\sigma^2$ then $X_t^{-1} \in L^p$ and moreover*

$$\mathbb{E}\Big(\Big(\frac{1}{X_t}\Big)^p\Big) \leq \frac{C_p}{L(t)} \Big(\frac{e^{\eta t}}{\sqrt{v_0}}\Big)^{p-1}$$

where C_p denotes a suitable positive constant and $L(t) = \frac{\sigma^2}{4\eta}(1 - e^{-\eta t})$.

- (iii) *If $2\nu > \sigma^2 > 0$ then $X_t \in \mathbb{D}^{1,\infty}$.*
- (iv) *If $4\nu > 3p\sigma^2 > 0$ then $X_t \in \mathbb{D}^{2,p}$*

Proof. (i) is a well known result (see e.g. Lamberton and Lapeyre [22] and the references quoted therein). (ii) is proved in Lemma 4.1 in Alos and Ewald in [1]. As for (ii) and (iii), they can be easily proved as an immediate consequence of what developed by Alos and Ewald in [1]. □

As for the jump diffusion case, we always assume that

$$2\nu \geq \sigma^2 > 0 \quad \text{and} \quad \Delta_i > 0$$

(in particular, Δ_i lognormal distributed), which imply that the solution to (2.39) never hits 0.
Here, we have not a closed form solution for the process X_t and therefore we can refer to the first variation process ξ_t as the solution process of the stochastic differential equation

$$\xi_t = 1 - \int_0^t \eta \xi_r dr + \int_0^t \frac{\sigma \xi_r}{2\sqrt{X_r}} dW_r \tag{2.40}$$

in which the dependence on the jumps is implicit in X_t. As for the existence and uniqueness of the solution to above Equation 2.40 see again Alos and Ewald in [1].
Now we can consider the different directions of the differential calculus, having fixed the set $\{J_T = n\}$.

Gaussian direction

Following the definition in Section 2.3.1 we have that

$$w_{0,u} = \frac{\xi_u}{\sigma T \sqrt{X_u}} \Big(1 - \frac{\int_0^T \beta_r dr}{\int_0^T \beta_r a_r dr} a_u \Big)$$

where $a \in \mathscr{A}^W$. As remarked we have not an explicit expression for the processes ξ_t and β_t. For simplicity of notation, we write

$$\Upsilon = \frac{\int_0^T \beta_r dr}{\int_0^T \beta_r a_r dr}.$$

Thus using Property 1.2.6 we can handle the Skorohod integral having

$$\begin{aligned} \tilde{\delta}_0(w_{0,u}) &= \frac{1}{\sigma T} \Big\{ \int_0^T \frac{\xi_u}{\sqrt{X_u}} dW_u - \Upsilon \int_0^T \frac{\xi_u a_u}{\sqrt{X_u}} dW_u \\ &+ \int_0^T \frac{\xi_u a_u}{\sqrt{X_u}} \cdot (\tilde{D}_{0,u} \Upsilon) du. \Big\} \end{aligned}$$

Consider now the term $\tilde{D}_{0,u}\Upsilon$. In view of numerical applications, it is sufficient to think of the Malliavin derivative of the first variation process ξ_t, being $\beta_t = \int_0^t \xi_r dr$. We have that $\tilde{D}_{0,u}\xi_t := \chi_t^u$ is the solution process of the following equation

$$\begin{aligned} \chi_t^u &= \frac{\sigma \xi_u}{2\sqrt{X_u}} - \int_0^t \eta \chi_r^u dr \\ &+ \frac{1}{2} \int_0^t \sigma \Big(\frac{\chi_r^u}{\sqrt{X_r}} - \frac{\xi_r (\tilde{D}_{0,u} X_r)}{2 (X_r)^{3/2}} \Big) dW_r \end{aligned}$$

Jump amplitudes direction

As developed in Section 2.3.2, we have that for $i = 1, ..., n$

$$w_i = \frac{\xi_{T_i}}{n}\Big(1 - \frac{\sum_{j=1}^n \beta_{T_j}}{\sum_{j=1}^n \beta_{T_j} a_j}\Big)$$

where $a = (a_1, ..., a_n) \in \mathscr{A}^\Delta$. For simplicity of notation, we write

$$\Upsilon = \frac{\sum_{j=1}^n \beta_{T_j}}{\sum_{j=1}^n \beta_{T_j} a_j}.$$

Using Property (1.1.17) we have

$$\begin{aligned} \tilde{\delta}_i(w_i) &= \frac{1}{n}\Big\{\sum_{i=1}^n \tilde{\delta}_i(\xi_{T_i}) - \Upsilon \sum_{i=1}^n \tilde{\delta}_i(\xi_{T_i} a_i) \\ &+ \sum_{i=1}^n \xi_{T_i} a_i \cdot \tilde{D}_i \Upsilon\Big\} \end{aligned}$$

We can handle the Malliavin derivative of Υ as in the previous Section, using the same Properties of the calculus.

Joint Gaussian and jump amplitudes direction

In this last case, we consider the joint direction of differential calculus, remembering the definitions of Section 2.3.4.
We have

$$\begin{aligned} w_{0,u} &= -\frac{\sum_{i=1}^n \beta_{T_i}}{n\int_0^T \beta_r a_r dr} \frac{\xi_u a_u}{\sigma\sqrt{X_u}} \\ w_i &= \frac{\xi_{T_i}}{n}, \qquad \text{for } i = 1, ..., n \end{aligned}$$

Also in this case, for simplicity let us denote

$$\Upsilon = \sum_{i=1}^n \frac{\beta_{T_i}}{n\int_0^T \beta_r a_r dr}.$$

In conclusion we have

$$\begin{aligned} \tilde{\delta}_0(w_{0,u}) &= -\frac{1}{n}\Big\{\Upsilon \int_0^T \frac{\xi_u a_u}{\sigma\sqrt{X_u}} dW_u - \int_0^T \frac{\xi_u a_u}{\sigma\sqrt{X_u}} \cdot (\tilde{D}_{0,u}\Upsilon) du.\Big\} \\ \sum_{i=1}^n \tilde{\delta}_i(w_i) &= \frac{1}{n}\sum_{i=1}^n \tilde{\delta}_i(\xi_{T_i}) \end{aligned}$$

2.5.4 A jump-diffusion stochastic volatility model

We consider in this section a stochastic volatility model, in which the underlying process X_t is a Merton model and the volatility is an Ornstein-Uhlenbeck one, written on a correlated Brownian motion. As usual Y_t denotes the integral of X over $[0,T]$.

Without loss of generality, for a fixed $\rho \in (-1,1)$ (giving the correlation process for the two correlated Brownian motions), let us consider the following 3-dimensional process

$$\begin{aligned} X_t &= x + \int_0^t \mu X_r dr + \int_0^t \sigma_r X_r \Big(\sqrt{1-\rho^2} dW_r^1 + \rho\, dW_r^2\Big) + \sum_{i=1}^{J_t} \Delta_i X_{T_i-} \\ \sigma_t &= y + \int_0^t k(\theta - \sigma_r) dr + \int_0^t \beta dW_r^2 \\ Y_t &= \int_0^t X_r dr \end{aligned} \tag{2.41}$$

where $\mu, k, \theta, \beta \in \mathbb{R}$ and $W = (W^1, W^2)$ is a standard 2-dimensional Brownian motion.

We are interested in computing the sensitivity of an option price with respect to x, the initial value of the process X. In other words, we have to deal with

$$\partial_x \mathbb{E}\Big(f(X_T, Y_T)\Big)$$

and we consider also in this case the 2-dimensional process $Z_t = (X_t, Y_t)$. Notice that the dependence on the volatility process σ is inside the process X_t. First, notice that both X and σ can be written more explicitely as

$$\begin{aligned} X_t &= x e^{\int_0^t (\mu - \frac{1}{2}\sigma_r^2) dr + \int_0^t \sigma_r (\sqrt{1-\rho^2}\, dW_r^1 + \rho dW_r^2)} \prod_{i=1}^{J_t} (1 + \Delta_i) \\ \sigma_t &= y + k\theta \int_0^t e^{-k(t-r)} dr + \beta \int_0^t e^{-k(t-r)} dW_r^2 \end{aligned}$$

Moreover, the first variation process of X (w.r.t. x) and its integral are

$$\xi_t = \frac{X_t}{x} \qquad \beta_t = \frac{Y_t}{x}.$$

Notice that if one is interested in $\partial_x \mathbb{E}(\phi(X_t, Y_t))$, then the first variation process of the volatility process does not need to be taken into account. Moreover, σ satisfies an autonomous stochastic differential equation, in which X and then its starting point does not appear.

Moreover, as in the Black-Scholes and Merton models, there is not an explicit dependence on the jump times T_i and therefore we can consider only the Gaussian or/and the jump amplitudes direction for the Malliavin calculus.

The Malliavin derivatives of the process X are given by

$$\begin{aligned}\tilde{D}_{0,u}X_T &= \sqrt{1-\rho^2}\sigma_u X_T \mathbf{1}_{u\leq T}\\ \tilde{D}_i X_T &= \frac{X_T}{1+\Delta_i}\mathbf{1}_{T_i\leq T} \qquad \text{for } i=1,...,n.\end{aligned}$$

Now fix the maturity time T and the set $\{J_T = n\}$. We briefly consider the Malliavin calculus with respect to the Brownian motion W^1, the amplitudes of the jumps and joint direction, remarking that this case can be easily deduced by the Merton model Section 2.5.1.

Gaussian direction

We apply here the results in Section 2.3.1, considering the Malliavin calculus with respect to W^1. We take into account that in this case σ is not constant, but is a process governed by an independent stochastic differential equation (independent of W^1). Then, the process w whose Skorohod integral gives the final weight has only the first component non null and given by

$$w_{0,u} = \frac{1}{\sigma_u x T\sqrt{1-\rho^2}}\Big(1 - \frac{\int_0^T Y_r dr}{\int_0^T Y_r a_r dr}\cdot a_u\Big), \quad u\in[0,T]$$

where $a \in \mathscr{A}^W$ (i.e. a is a deterministic function such that $\int_0^T a_u du = 0$). Notice that $w_{0,u}$ is the same as in the Merton model, and σ_u, not depending on W^1, is not involved in the differential calculus. Thus we can follow the same steps of Section 2.5.1 in order to compute the Skorohod integral, that is

$$\begin{aligned}\tilde{\delta}(w) &= \frac{1}{xT\sqrt{1-\rho^2}}\Big[\int_0^T \frac{1}{\sigma_u} dW_u^1 - \frac{\int_0^T Y_r dr}{\int_0^T Y_r a_r dr}\int_0^T \frac{a_u}{\sigma_u}\, dW_u^1 + \\ &\quad + \frac{1}{\int_0^T Y_r a_r dr}\int_0^T \Big(\int_u^T (Y_r - Y_u)dr\Big)\frac{a_u}{\sigma_u}\, du \\ &\quad - \frac{\int_0^T Y_r dr}{(\int_0^T Y_r a_r dr)^2}\int_0^T \Big(\int_u^T (Y_r - Y_u)a_r dr\Big)\frac{a_u}{\sigma_u}\, du\Big]\end{aligned} \tag{2.42}$$

Jump amplitudes direction

The process X does not depend on σ_t in the jump part. Then the Malliavin derivative of X with respect to the jump amplitudes is the same as in the Merton case.
As a matter of fact, the i-th coordinate of the process w is given by

$$w_i = \frac{1}{nx}\Big(1 - \frac{\sum_{j=1}^n Y_{T_j}}{\sum_{j=1}^n Y_{T_j} a_j} a_i\Big)(1+\Delta_i), \quad i = 1, \dots, n$$

where $a \in \mathscr{A}^J$ (i.e. $a \in \mathbb{R}^n$ with $\sum_{i=1}^n a_i = 0$).
Finally we have

$$\begin{aligned} \tilde{\delta}(w) &= \frac{1}{n\,x}\Big[\Big(1 - \frac{\sum_{j=1}^n Y_{T_j}}{\sum_{j=1}^n Y_{T_j} a_j}\, a_i\Big)\sum_{i=1}^n \tilde{\delta}_i(1+\Delta_i) + \\ &\quad + \frac{1}{\sum_{j=1}^n Y_{T_j} a_j}\sum_{i=1}^n \sum_{j=i+1}^n (Y_{T_j} - Y_{T_i}) + \\ &\quad - \frac{\sum_{j=1}^n Y_{T_j}}{(\sum_{j=1}^n Y_{T_j} a_j)^2}\sum_{i=1}^n a_i \sum_{j=i+1}^n (Y_{T_j} - Y_{T_i}) a_j\Big] \end{aligned} \tag{2.43}$$

where, recalling Definition 1.1.9 for the Skorohod integral,

$$\tilde{\delta}_i(1+\Delta_i) = -\sum_{i=1}^n (1 + (1+\Delta_i)\partial \log g(\Delta_i)).$$

As for the Merton model, we will consider $1+\Delta_1$ as a log-normal r.v., that is $1+\Delta_1 = \exp\Gamma$, with $\Gamma \sim \mathrm{N}(m, \varrho^2)$. Then, as seen in (2.16), one has

$$\tilde{\delta}_i(1+\Delta_i) = -\sum_{i=1}^n \frac{m - \log(1+\Delta_i)}{\varrho^2}, \quad i = 1, \dots, n.$$

Joint Gaussian and jump amplitudes direction

We conclude this Section with the joint direction. We can follow exactly what done in Section 2.5.1 having two possible choices for the process v that lead to the following two processes w. In the first one we have

$$w_{0,u} = -\frac{1}{nx\sqrt{1-\rho^2}} \cdot \frac{\sum_{j=1}^n Y_{T_j}}{\int_0^T Y_r a_r dr} \cdot \frac{a_u}{\sigma_u} \text{ and } w_i = \frac{1}{nx} \cdot (1+\Delta_i), \quad 1 \le i \le n$$

where $a \in \mathscr{A}^W$. The associated weight is then $\tilde{\delta}(w) = \tilde{\delta}_0(w_0) + \sum_{i=1}^n \tilde{\delta}_i(w_i)$.
In the second case we have

$$w_{0,u} = \frac{1}{Tx\sqrt{1-\rho^2}\sigma_u} \text{ and } w_i = -\frac{\int_0^T Y_u du}{Tx\sum_{j=1}^n Y_{T_j} a_j} a_i(1+\Delta_i), \quad 1 \le i \le n.$$

In Section 2.5.1 an explicit expression of the Skorohod integrals of the process w is given in both cases.

Joint Gaussian direction

In this case we use the Malliavin calculus considering both the directions of the Brownian motions W^1 and W^2. Then the definitions of the differential operators and, in particular, of the process v are slightly different with respect to the definitions of Section 2.3.1, having one more component. By the way, we will follow the same idea to have a triangular Malliavin generalized covariance matrix.
So we have to consider the derivative as $\tilde{D} = (\tilde{D}^1_0, \tilde{D}^2_0, ..., 0, ..., 0)$ where $\tilde{D}^i_0$ is the derivative in the W^i direction, $i = 1, 2$. For simplicity let write $\tilde{D}_0 = (\tilde{D}^1_0, \tilde{D}^2_0)$, leaving out the components related on the jump amplitudes and times. We have that

$$\tilde{D}^1_{0,u} X_t = \sqrt{1-\rho^2}\sigma_u X_t \mathbf{1}_{u\leq t}$$

$$\begin{aligned} D^2_{0,u} X_t &= X_t \mathbf{1}_{u\leq t}\Big(-\int_u^t \sigma_r \tilde{D}^2_{0,u}\sigma_r dr + \sqrt{1-\rho^2}\int_u^t D^2_{0,u}\sigma_r dW^1_r \\ &+ \rho\sigma_u + \rho\int_u^t D^2_{0,u}\sigma_r dW^2_r\Big) \end{aligned}$$

$$\tilde{D}^1_{0,u}\sigma_t = 0 \qquad \tilde{D}^2_{0,u}\sigma_t = \beta e^{-k(t-u)}\mathbf{1}_{u\leq t}$$

$$\tilde{D}^1_{0,u} Y_t = \sqrt{1-\rho^2}\sigma_u (Y_t - Y_u)\mathbf{1}_{u\leq t} \qquad \tilde{D}^2_{0,u} Y_t = \rho\sigma_u (Y_t - Y_u)\mathbf{1}_{u\leq t}$$

Now let us define the process v, that have $2n+2$ components, the first two related to the Brownian motions W^1 and W^2. Also in this case we consider only the pair of the non null components. We set

$$\begin{aligned} v^1 &= \Big(\frac{a_u}{\sigma_u\sqrt{1-\rho^2}}, \frac{a_u}{\sigma_u\rho}\Big) \\ v^2 &= \Big(\frac{1}{\sigma_u\sqrt{1-\rho^2}}, \frac{1}{\sigma_u\rho}\Big) \end{aligned} \tag{2.44}$$

Therefore the matrix $\gamma_{\tilde{D}Z_T,v}$ is

$$\gamma_{\tilde{D}Z_T,v} = \begin{pmatrix} 0 & -2\int_0^T Y_r a_r dr \\ 2TX_T & 2(TY_T - \int_0^T Y_r dr) \end{pmatrix}$$

Thus the process w is such that the first component related to the Brownian motion W^1, is

$$w^1_{0,u} = \frac{1}{2xT\sqrt{1-\rho^2}}\Big(\frac{1}{\sigma_u} - \frac{\int_0^T Y_r dr}{\int_0^T Y_r a_r dr}\frac{a_u}{\sigma_u}\Big)$$

and the second one, related to the Brownian motion W^2, is given by

$$w^2_{0,u} = \frac{1}{2xT\rho}\Big(\frac{1}{\sigma_u} - \frac{\int_0^T Y_r dr}{\int_0^T Y_r a_r dr}\frac{a_u}{\sigma_u}\Big).$$

Notice that $w^1_{0,u}$ and $w^2_{0,u}$ are, unless the constant of correlation ρ, identical. Moreover σ_u is adapted and therefore we do not have to use Property 1.2.6 of the Skorohod integral. Thus, the weight for the Delta is given by

$$\begin{aligned}
\tilde{\delta}(w) &= \sum_{i=1}^{2} \frac{1}{2xT\chi_i}\Big\{ -\frac{\int_0^T Y_r dr}{\int_0^T Y_r a_r dr}\int_0^T \frac{a_u}{\sigma_u} dW^i_u + \int_0^T \frac{1}{\sigma_u} dW^i_u \\
&+ \frac{\chi_i}{\int_0^T Y_r a_r dr}\int_0^T \Big(\int_u^T (Y_r - Y_u) dr\Big) a_u du \\
&- \frac{\chi_i \int_0^T Y_r dr}{\big(\int_0^T Y_r a_r dr\big)^2}\int_0^T \Big(\int_u^T (Y_r - Y_u) a_r dr\Big) a_u du \Big\}
\end{aligned}$$

where we have denoted δ^i_0 the Skorohod integral with respect to the Brownian motion W^i and with $\chi_1 = \sqrt{1-\rho^2}$ and $\chi_2 = \rho$.

2.6 Numerical experiments

We present here several numerical experiments in order to compare the different Malliavin approaches (in the direction of the Brownian motion or the jump amplitudes and times) and the finite difference method for the Delta hedging. We consider the models analyzed in Section 2.5 and we look at different types of options: floating Asian call/put options, standard and Asian call/put options, standard and Asian digital options. For the sake of clearness, let us write down the above mentioned payoffs $f(X_T, Y_T)$:

- floating Asian call and put option:

$$f(X_T, Y_T) = (X_T - Y_T/T)_+ \text{ and } f(X_T, Y_T) = (Y_T/T - X_T)_+$$

- standard call and put option:

$$f(X_T, Y_T) = (X_T - K)_+ \text{ and } f(X_T, Y_T) = (K - X_T)_+$$

- Asian call and put option:

$$f(X_T, Y_T) = (Y_T/T - K)_+ \text{ and } f(X_T, Y_T) = (K - Y_T/T)_+$$

- standard digital option:

$$f(X_T, Y_T) = \mathbf{1}_{X_T > K}$$

- Asian digital option:

$$f(X_T, Y_T) = \mathbf{1}_{Y_T/T > K}$$

Let us summarize some facts concerning the forthcoming numerical experiments.

- In our simulations, we have to deal with $n = 0$ and $n = 1$. We have already observed how to handle these cases in Remark 2.1.5, 2.1.4 and 2.1.6.
- In the following experiments, we will always consider a benchmark value, obtained using the finite difference method with $250,000$ simulations.
- When a discretization of the time interval is needed (e.g. for the simulation of the process Y, as well as for the simulation of the CIR process), we consider the time interval split in 100 subintervals.
- Some parameters are set once for all. More precisely,
  - as for the jump amplitudes, se always set $\Delta_i \overset{\mathscr{L}}{=} e^{m+\varrho\mathcal{N}}$, with $m = 0$, $\varrho = 0.05$ and $\mathcal{N}$ denoting a standard Gaussian r.v.;
  - as for the intensity λ, we generally set $\lambda = 5$, unless specified (somewhere we test the weights also for a small value of λ);
  - as for the maturity time, we set $T = 5$.
  - as for the starting underlying asset price value, we set $x = 100$.

  The parameters strictly connected to the chosen model will be declared case by case.
- The functions $a \in \mathscr{A}^W$ and $a \in \mathscr{A}^J$ are set as in Remark 2.3.2, 2.3.6 and 2.3.8.

We consider separately the different models and for each model, we look at some of the above options.
But, let us first open a brief discussion on a localization method we are going to use in practice.

2.6.1 Localization methods

Here, we present a variance reduction technique based on a localization method. For European and Asian call/put options, we follow the localization function used by Bavouzet-Morel and Messaud [5]. For Floating Asian call/put options, we introduce here a natural generalization.

Call options

Roughly speaking, we are interested in the evaluation of the derivative

$$\partial_x[(X_T - Y_T/T)_+]$$

where x denotes the initial value of the process X. For a fixed $\varepsilon > 0$, let us define the following functions:

$$B^1_\varepsilon(s,t) = \begin{cases} 0 & \text{for } s \le t-\varepsilon \\ \dfrac{s-(t-\varepsilon)}{2\varepsilon} & \text{for } s \in [t-\varepsilon, t+\varepsilon] \\ 1 & \text{for } s \ge t+\varepsilon \end{cases}$$

$$B^2_\varepsilon(s,t) = \begin{cases} 0 & \text{for } s \le t-\varepsilon \\ -\dfrac{s-(t-\varepsilon)}{2\varepsilon T} & \text{for } s \in [t-\varepsilon, t+\varepsilon] \\ -\frac{1}{T} & \text{for } s \ge t+\varepsilon \end{cases}$$

$$G_\varepsilon(s,t) = \begin{cases} 0 & \text{for } s \le t-\varepsilon \\ \dfrac{(s-(t-\varepsilon))^2}{4\varepsilon} & \text{for } s \in [t-\varepsilon, t+\varepsilon] \\ s-t & \text{for } s \ge t+\varepsilon \end{cases}$$

$$F_\varepsilon(s,t) = \begin{cases} 0 & \text{for } s \le t-\varepsilon \\ \dfrac{(s-(t-\varepsilon))^2}{4\varepsilon} & \text{for } s \in [t-\varepsilon, t] \\ \dfrac{s-t-(s-(t-\varepsilon))^2}{4\varepsilon} & \text{for } s \in [t, t+\varepsilon] \\ 0 & \text{for } s \ge t+\varepsilon \end{cases}$$

Notice that the following relations hold:

$$B^1_\varepsilon(s,t) = \partial_s G_\varepsilon(s,t), \quad B^2_\varepsilon(s,t) = \partial_t G_\varepsilon(s,t) \text{ and}$$
$$F_\varepsilon(s,t) = (s-t)_+ - G_\varepsilon(s,t)$$

so that, on the set $\{J_t = n\}$, we have

$$\begin{aligned} \partial_x \mathbb{E}\Big(\big(X_T - \tfrac{Y_T}{T}\big)_+\Big) &= \partial_x \mathbb{E}\Big(G_\varepsilon\big(X_T, \tfrac{Y_T}{T}\big)\Big) + \partial_x \mathbb{E}\Big(F_\varepsilon\big(X_T, \tfrac{Y_T}{T}\big)\Big) \\ &= \mathbb{E}\Big(\partial_x G_\varepsilon\big(X_T, \tfrac{Y_T}{T}\big)\Big) + \mathbb{E}\Big(F_\varepsilon\big(X_T, \tfrac{Y_T}{T}\big)\tilde{\delta}(w)\Big) \end{aligned}$$

where we have used Proposition 2.1.2 and in particular (2.5). Looking at the first term we have

$$\begin{aligned} \mathbb{E}\Big(\partial_x G_\varepsilon\Big(X_T, \tfrac{Y_T}{T}\Big)\Big) &= \mathbb{E}\Big(\partial_s G_\varepsilon\Big(X_T, \tfrac{Y_T}{T}\Big)\partial_x X_T + \partial_t G_\varepsilon\Big(X_T, \tfrac{Y_T}{T}\Big)\partial_x \tfrac{Y_T}{T}\Big) \\ &= \mathbb{E}\Big(B^1_\varepsilon\Big(X_T, \tfrac{Y_T}{T}\Big)\cdot \xi_T + B^2_\varepsilon\Big(X_T, \tfrac{Y_T}{T}\Big)\cdot \beta_T\Big). \end{aligned}$$

Digital options

We can use the same idea of the previous section with digital options with payoff

$$\mathbf{1}_{X_T>K} \qquad \text{or} \qquad \mathbf{1}_{Y_T/T>K}.$$

Thus, given $\varepsilon > 0$, we can introduce the following functions:

$$B_\varepsilon(s) = \begin{cases} 0 & \text{for } s \le K-\varepsilon \\ \dfrac{3}{2\varepsilon}\Big(\dfrac{s-K+\varepsilon}{\varepsilon}\Big)^2 & \text{for } s \in [K-\varepsilon, K] \\ \dfrac{3}{2\varepsilon}\Big(\dfrac{s-K-\varepsilon}{\varepsilon}\Big)^2 & \text{for } s \in [K, K+\varepsilon] \\ 0 & \text{for } s \ge K+\varepsilon \end{cases}$$

$$G_\varepsilon(s) = \begin{cases} 0 & \text{for } s \le K-\varepsilon \\ \dfrac{1}{2}\Big(\dfrac{s-K+\varepsilon}{\varepsilon}\Big)^3 & \text{for } s \le K-\varepsilon \\ \dfrac{1}{2\varepsilon}\Big(\dfrac{s-K-\varepsilon}{\varepsilon}\Big)^3 + 1 & \text{for } s \in [K, K+\varepsilon] \\ 1 & \text{for } s \ge K+\varepsilon \end{cases}$$

$$F_\varepsilon(s) = \begin{cases} 0 & \text{for } s \le K-\varepsilon \\ -\dfrac{1}{2}\Big(\dfrac{s-K+\varepsilon}{\varepsilon}\Big)^3 & \text{for } s \in [K-\varepsilon, K] \\ \dfrac{1}{2\varepsilon}\Big(\dfrac{s-K-\varepsilon}{\varepsilon}\Big)^3 & \text{for } s \in [K, K+\varepsilon] \\ 0 & \text{for } s \ge K+\varepsilon \end{cases}$$

Notice that

$$G'_\varepsilon(s) = B_\varepsilon(s) \quad \text{and} \quad F_\varepsilon(s) = \mathbf{1}_{s>K} - G_\varepsilon(s).$$

Fix the set $\{J_T = n\}$. Therefore we have:

$$\begin{aligned} \partial_x \mathbb{E}[\mathbf{1}_{X_T>K}] &= \partial_x \mathbb{E}[G_\varepsilon(X_T)] + \partial_x \mathbb{E}[F_\varepsilon(X_T)] \\ &= \mathbb{E}[\partial_x G_\varepsilon(X_T)] + \mathbb{E}[F_\varepsilon(X_T)\tilde{\delta}(w)] \end{aligned}$$

where we have used Proposition 2.1.2 and in particular (2.5). Finally the first term is

$$\mathbb{E}[\partial_x G_\varepsilon(X_T)] = \mathbb{E}[B_\varepsilon(X_T)\xi_T].$$

When we will consider the payoff $\mathbf{1}_{Y_T/T>K}$, we will use the same localization, replacing X_T with Y_T/T and ξ_T with β_T.

2.6.2 Merton Model

Let Z_t the Merton model considered in the Section 2.5.1. In this case we compare the three different Malliavin direction of calculus of the Section 2.5.1. Moreover we compare these results with the localized methods and the finite difference method.

We take the following values for the constants of the model: $\sigma = 0.2$, $b = 0.1$.

Floating Asian options

Figure 2.1 and 2.2 show a comparison among different Malliavin weights, using the different directions of calculus, and the finite difference method, for the Delta of a Floating Asian Call and Put options. In general, as shown in the figures, to consider the weight obtained using the joint direction of Malliavin calculus gives better results than to consider the weight obtained using separately the Gaussian and jump amplitudes calculus, perhaps because in this case both noises give a similar contribution in the Malliavin weight.

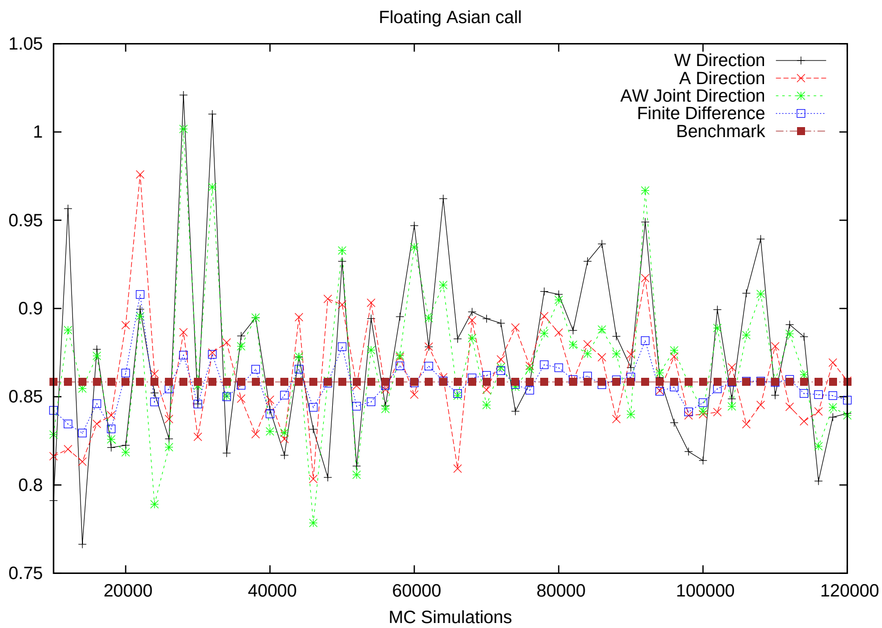

Figure 2.1: Merton model

Using the localization method introduced in Section 2.6.1 we obtain sensible better results. We have results very similar to the exact Monte Carlo Delta, as shown in the Table 2.1. Let us explain better what we are speaking about. In the Merton model we have an explicit expression for the Delta, that is

$$delta = \frac{price}{x}.$$

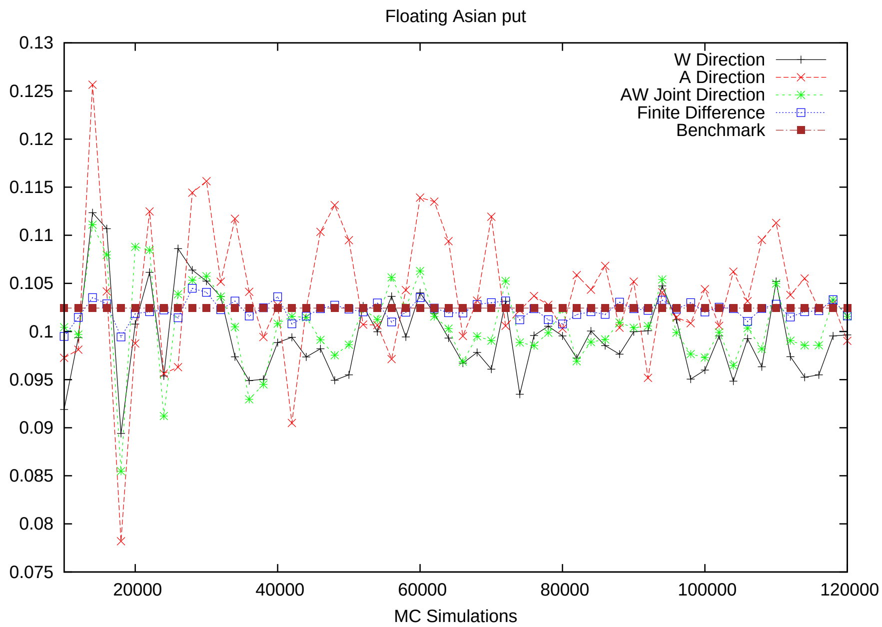

Figure 2.2: Merton model

This is due to the special form of the payoff. In fact,

$$\partial_x \mathbb{E}\Big(\Big(X_T - \frac{1}{T}Y_T\Big)_+\Big) = \mathbb{E}\Big(\partial_x\Big(X_T - \frac{1}{T}Y_T\Big)_+\Big)$$

$$= \mathbb{E}\Big(\mathbf{1}_{X_T - \frac{1}{T}Y_T > 0}\frac{1}{x}\Big(X_T - \frac{1}{T}Y_T\Big)\Big) = \frac{1}{x}\mathbb{E}\Big(\Big(X_T - \frac{1}{T}Y_T\Big)_+\Big)$$

Therefore with "exact Monte Carlo Delta value" we mean the Monte Carlo price divided by the initial value x.

MC sim.	exact MC	DF	JA	W	JA and W
20000	0.862895	0.863427	0.862905	0.863654	0.864142
40000	0.839891	0.840280	0.840981	0.840961	0.840279
60000	0.858012	0.857856	0.857685	0.856851	0.856760
80000	0.866198	0.866429	0.866416	0.866496	0.866236
100000	0.846873	0.846675	0.845960	0.847124	0.847096
120000	0.848036	0.848036	0.847720	0.848024	0.848097

Table 2.1: Comparison of the Localized Malliavin Delta with the finite difference method and the exact Monte Carlo.

Figure 2.3 shows the result for a Floating Call Option. In order to look at the improved results using the localization methods, in Figure 2.4 we compare the localized and the non localized Malliavin delta in the joint direction Gaussian and jump amplitudes, for the Floating Asian call.

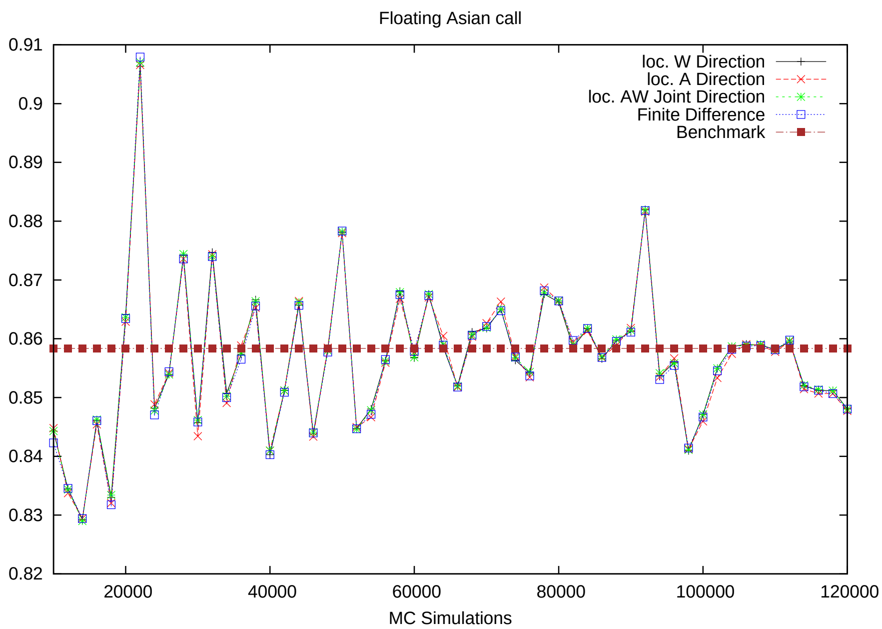

Figure 2.3: Merton model with localization

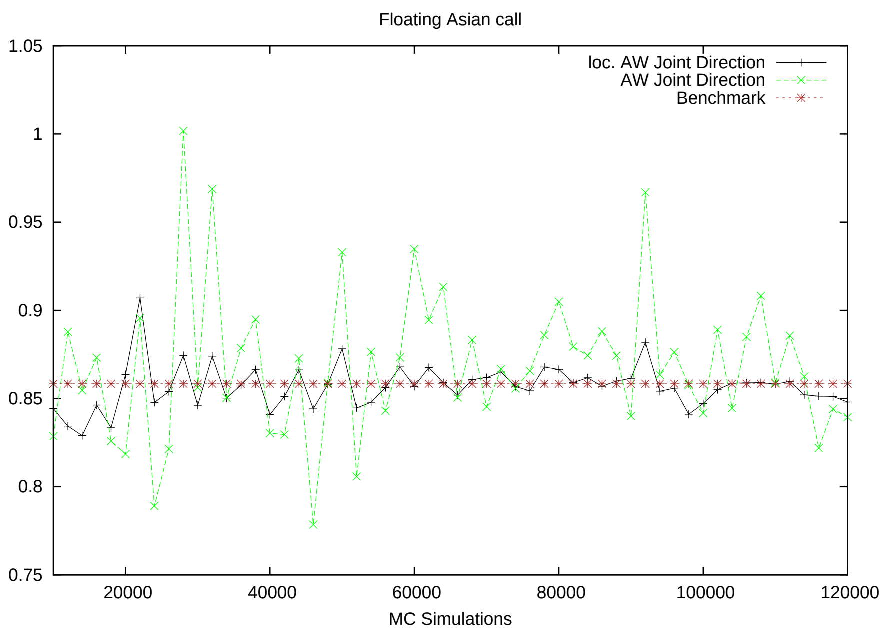

Figure 2.4: Merton model. Comparison between localized and non localized weights

The choice $\lambda = 5$ and $T = 5$ assures that we have a (quite) large number of the jumps (in mean 25). The numerical experiments shows that for a too small number of the jumps the weight obtained considering the jump amplitudes direction of the calculus does not give good results (see also Bavouzet and Messaoud [5]). As a matter of fact, we expect that a small number of jumps does not assure sufficient noise to perform Malliavin methods in the direction of jumps.
In the Merton model, we tested also a small value for λ. When $\lambda = 0.5$, the jump amplitudes contribution is too small and the Malliavin weight coming from taking into account the jump amplitudes direction (both lonely and jointly with the Gaussian one) lose accuracy, as it follows from Figure 2.5. It is worth to say that in such a case one presumably has not the desired L^p properties for the inverse of the determinant of the generalized Malliavin covariance matrix (see the discussion in Section 2.3.2, at page 42).
In other models, the lack of contribution given by a direction of calculus is due to the process structure and does not depend on the parameters choice, as we will see in the Ornstein-Ulhenbeck and Cox-Ingersoll-Ross models.

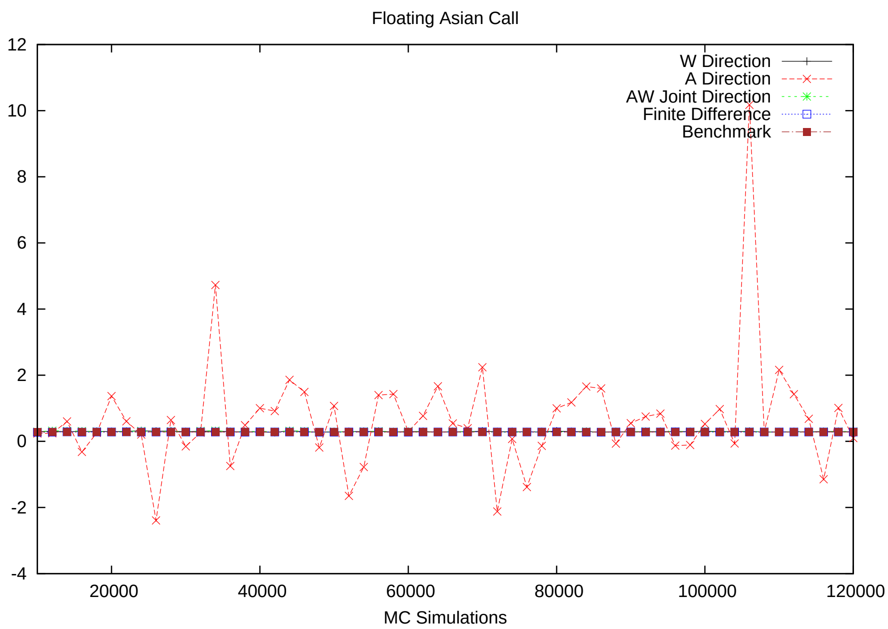

Figure 2.5: Merton model with a small value for λ ($\lambda = 0.5$).

Figure 2.5 shows that the results of the Gaussian direction and of the jumps amplitudes directions are not comparable: all the graphs practically overlap except for the case of the single jump amplitudes (by the way, the joint direction give a good result, due to the presence of the Brownian noise).
Also considering the localization method for the Floating Asian Call option introduced in the Section 2.6.1 we do not have nice results using only the jump amplitudes direction of the calculus. In Figure 2.6 we compare the localized

and the non localized Delta using only the jump amplitudes direction of the Malliavin calculus. In Figure 2.7 we show the result using the localized method in the other directions of the calculus, not considering the jump amplitudes direction.

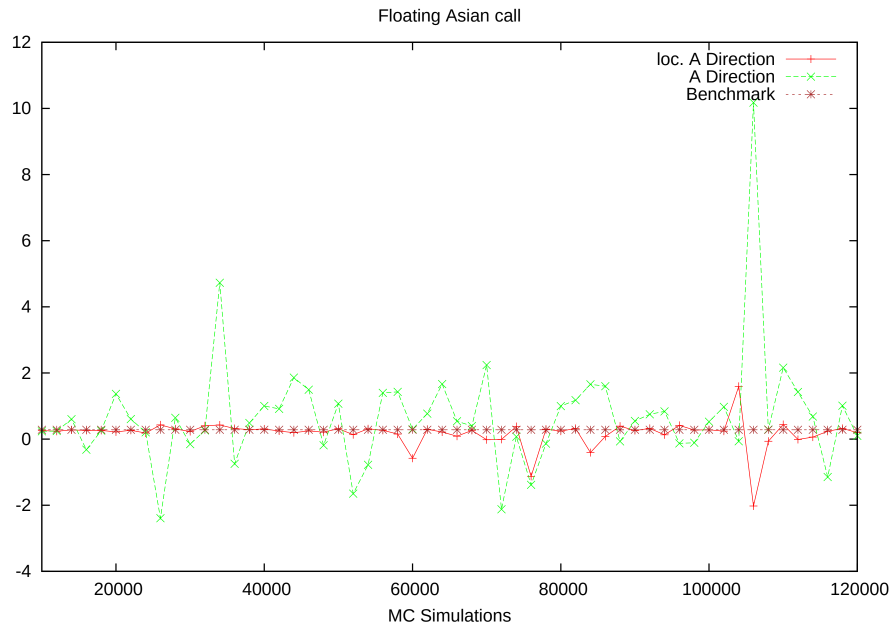

Figure 2.6: Merton model with localization and a small value for λ ($\lambda = 0.5$).

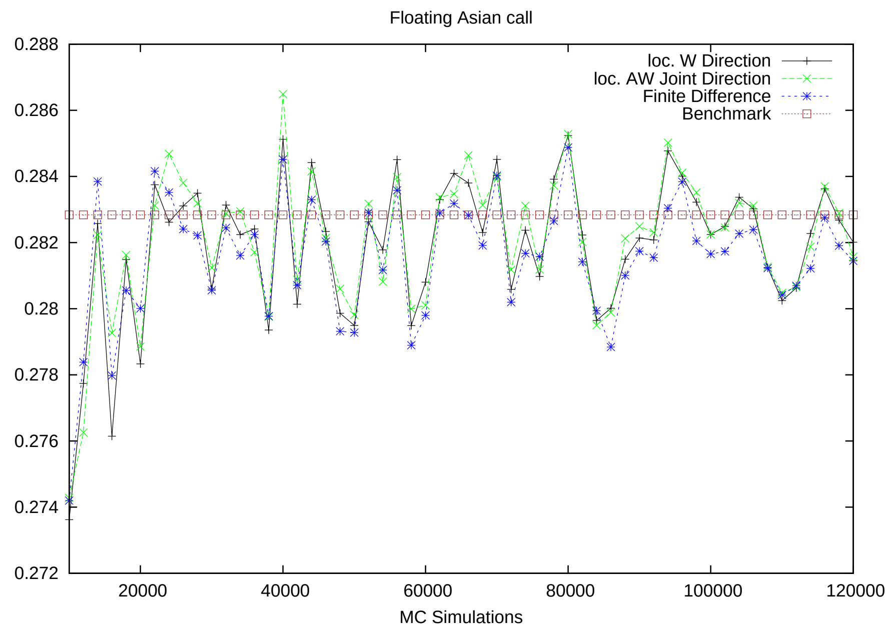

Figure 2.7: Merton model with localization and a small value for λ ($\lambda = 0.5$). All weights except for the lonely jump amplitudes one.

Floating Asian Call for the Black-Scholes Model. Comparison with the Benhamou weight

Fix $\alpha = 0$, that means that we consider the pure diffusion Black-Scholes model. We compare now our Malliavin weight with the Benhamou one (see Benhamou [7]). We have already remarked that from a theoretical point of view, our approach is less complicated, due to a simpler process w (see Remark 2.3.3). The comparison is shown in Figure 2.8.

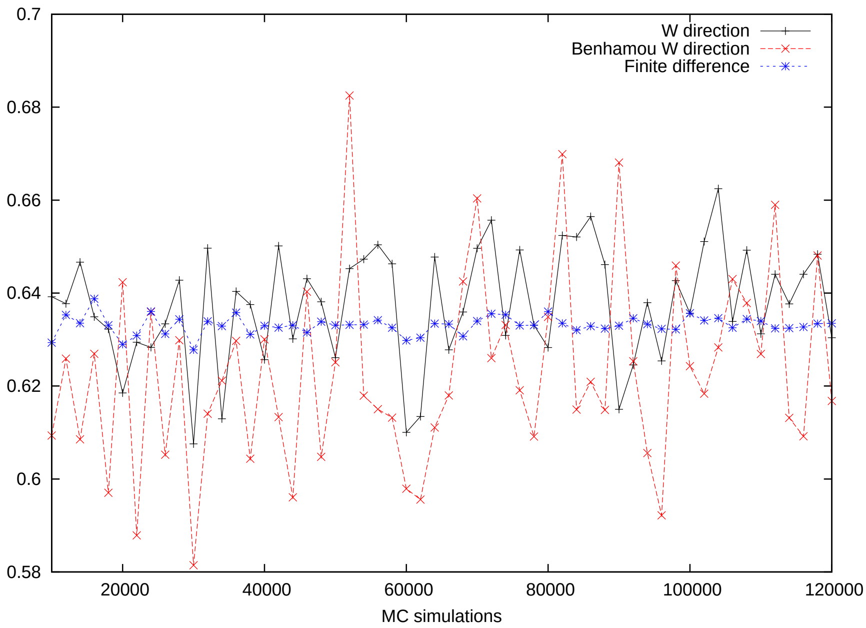

Figure 2.8: Black-Scholes model. Floating option.

To consider the localization method gives very nice results, for both weights. In the next Figure 2.9 we compare both the localized Delta with the finite difference and the Benchmark value.

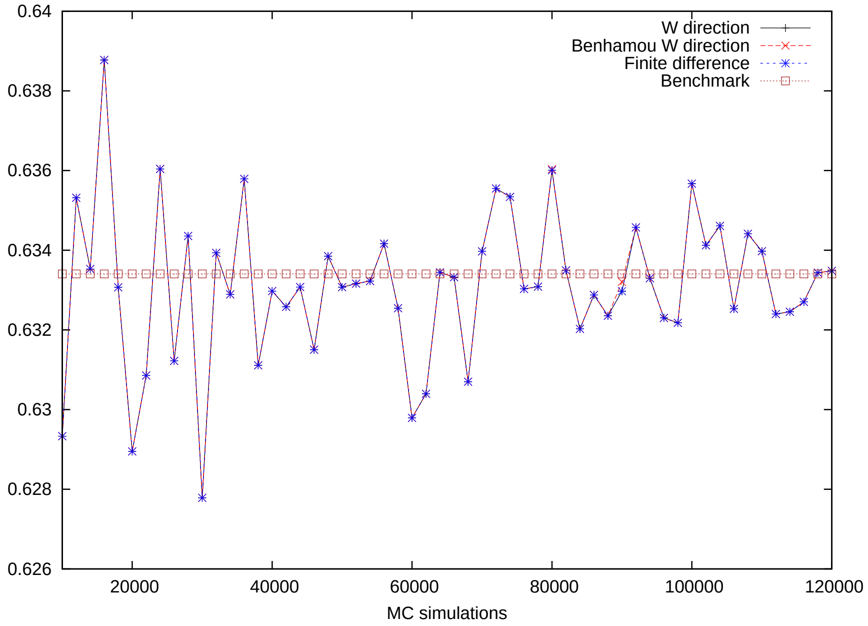

Figure 2.9: Black-Scholes model. Floating option.

Standard call option

Consider now a standard call option, with payoff

$$f(X_T, Y_T) = (X_T - K)_+$$

with $K = 100$. In this case we consider only the joint direction of the Malliavin calculus in order to compare the finite difference method with the one dimensional weight of Proposition 2.4.1, the two dimensional weight of Proposition 2.1.2 and the weight coming from the standard Malliavin covariance matrx (that is, choosing v as the Malliavin derivative of X_T). This last case is the one considered by Bally, Bavouzet and Messaud [3]. Figure 2.10 shows that also by a numerical point of view, to choose v in a suitable way and then consider the generalized covariance matrix, give better results.

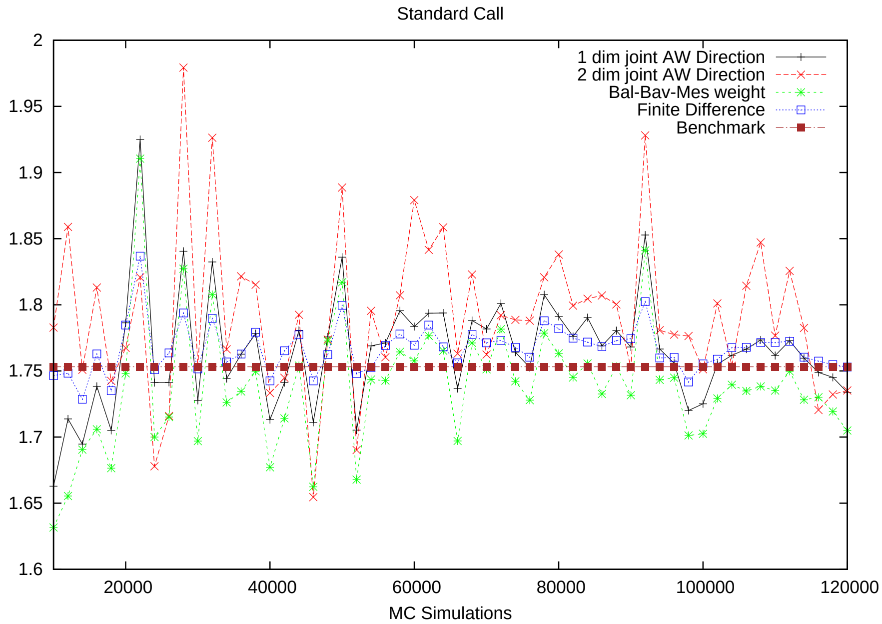

Figure 2.10: Merton model.

Digital options

In the case of digital options we have the following payoffs:

$$f(X_T, Y_T) = \mathbf{1}_{Y_T/T>K} \quad \text{and} \quad f(X_T, Y_T) = \mathbf{1}_{X_T>K}$$

for the Asian digital option and the Standard digital one, again with $K = 100$. Being the payoff a non smooth function, we have that the finite difference does not give nice results. In this case, the Malliavin methods give better ones. In Figure 2.11 and 2.12 we consider respectively a Standard digital and an Asian digital option. These graphics show that the Malliavin methods, in all directions, give sensible better results than the finite difference method.

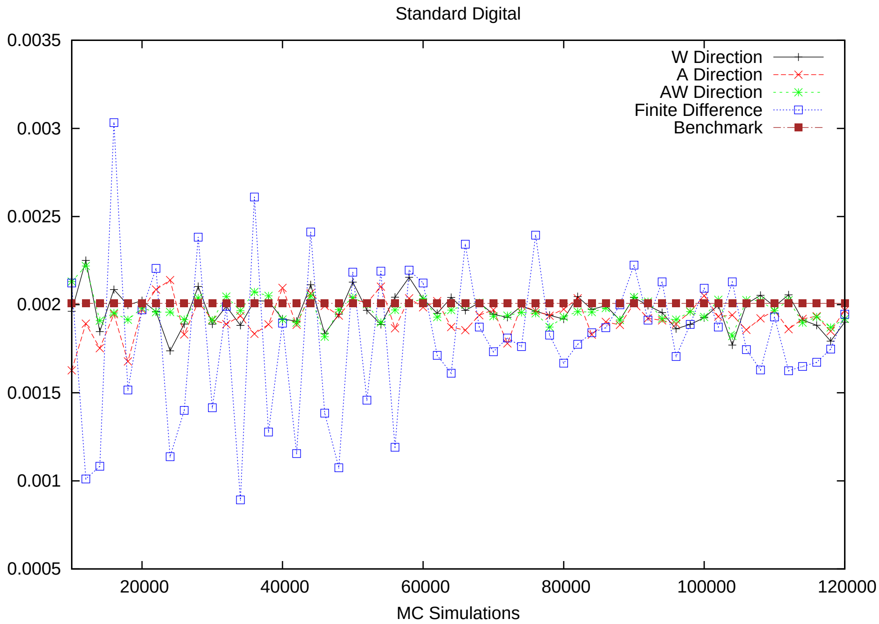

Figure 2.11: Merton model.

Moreover we consider for the Digital options the localized methods introduced in Section 2.6.1. Figure 2.13 and 2.14 show a Standard digital option using the localization. In the latter we have not included in the graphic the Finite Difference. The first figure shows that the other methods give comparable results.

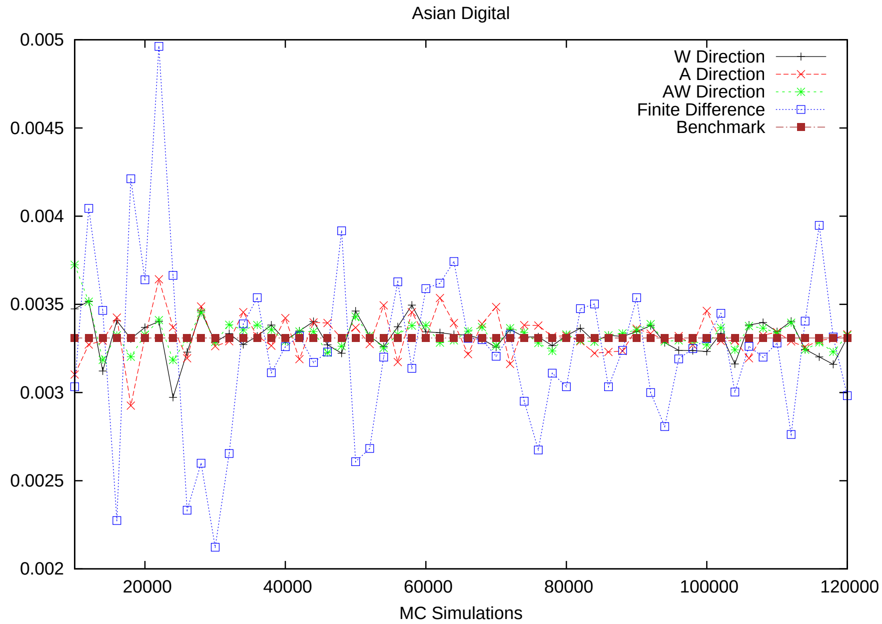

Figure 2.12: Merton model.

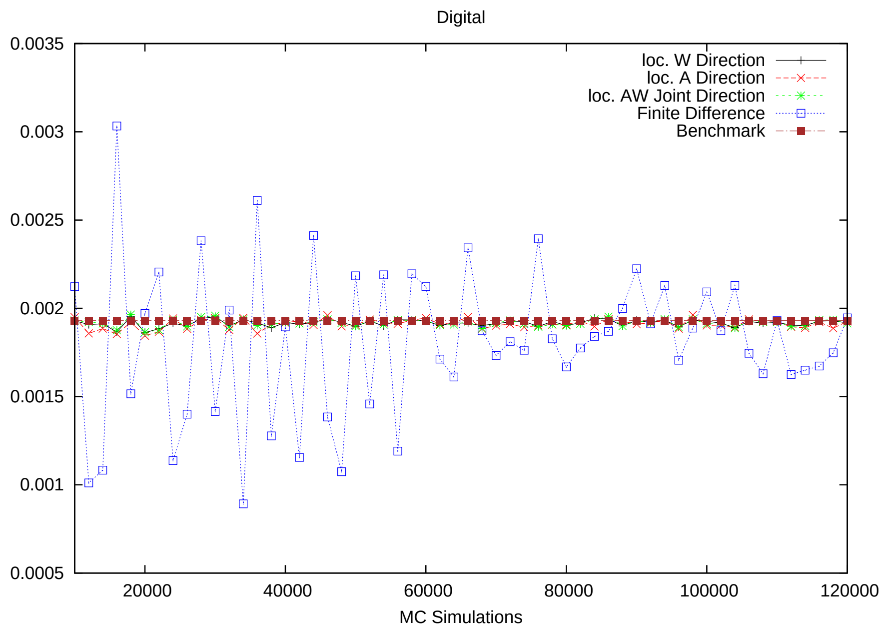

Figure 2.13: Merton model with localization.

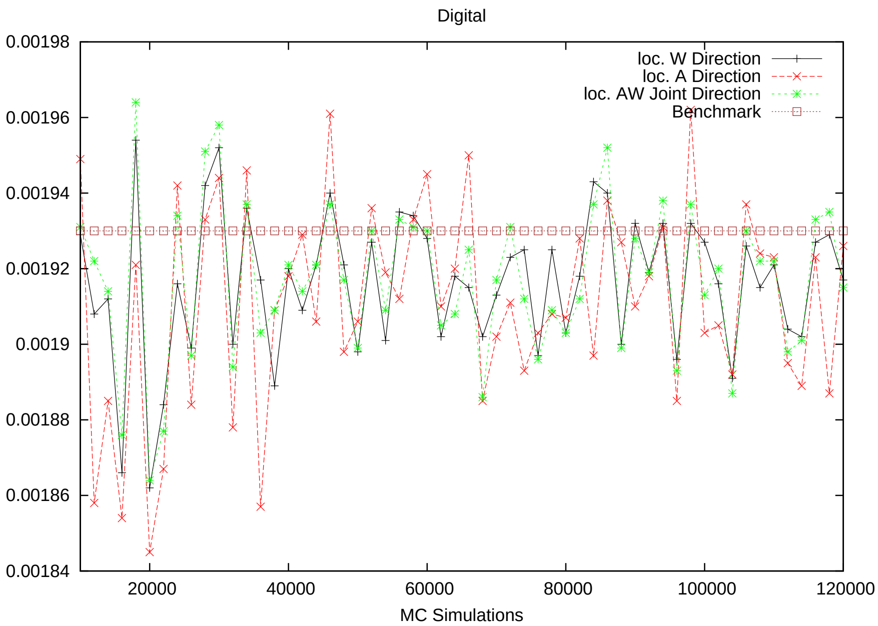

Figure 2.14: Merton model with localization, without the finite difference method.

2.6.3 Ornstein-Ulhenbeck Model

We consider here a jump-diffusion Ornstein-Ulhenbeck process. As remarked, for this process we can consider all directions of noises, the Gaussian one and both related to the jumps amplitudes and times (see Section 2.5.2). For this model to use only the jump times direction of the differential calculus give the best results.

We choose the following parameters: $r = 0.4$, $\theta = 5$, $\alpha = 10$, $\sigma = 5$.

Floating Asian Options

In this Section, as remarked, we consider a Floating Asian Option, noticing that we obtain the best results using the jump times direction of the calculus. Figure 2.15 shows the outcome of the numerical simulations, considering the three different directions of the Malliavin calculus.

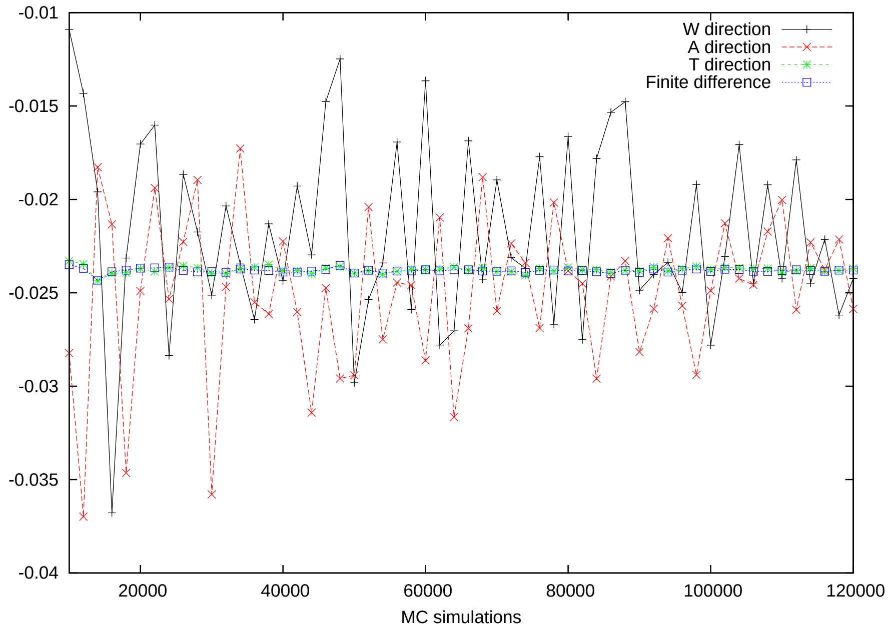

Figure 2.15: OU model. Floating Asian option.

In Figure (2.16) we have considered the joint direction of calculus. Remember that the Ornstein-Ulhenbeck model allows to consider also the joint Gaussian and jump times, and the Joint Gaussian, jump amplitudes and times directions (see Section 2.5.2)

To avoid confusion we have not considered the Benchmark value in the previous two Figures.

We stress that the jump times direction of calculus gives the best result (see Figure (2.15)). Figure (2.17) compares the jump times direction with localized joint Gaussian and jump amplitudes. The results are very similar, proving the high accuracy of the jump times direction.

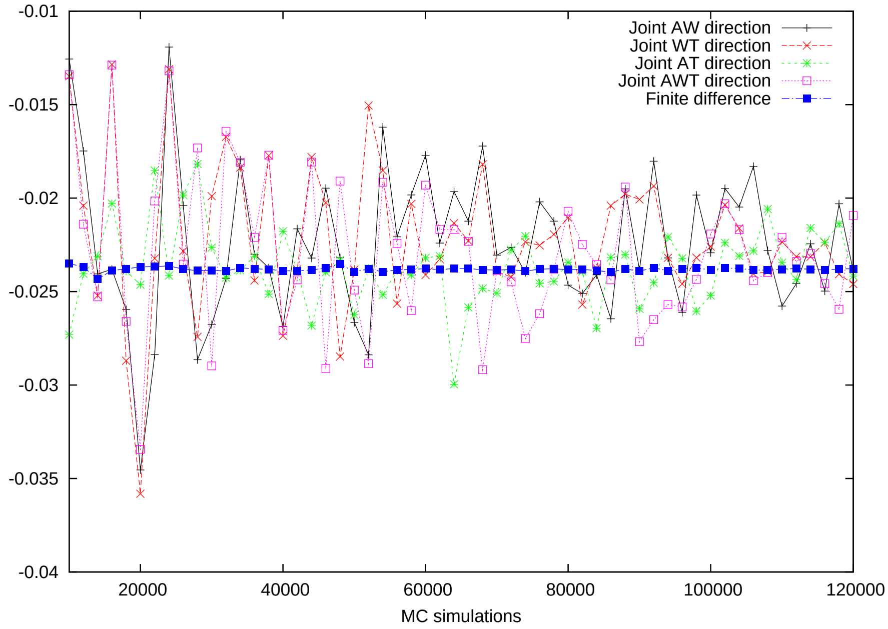

Figure 2.16: OU model. Floating Asian option.

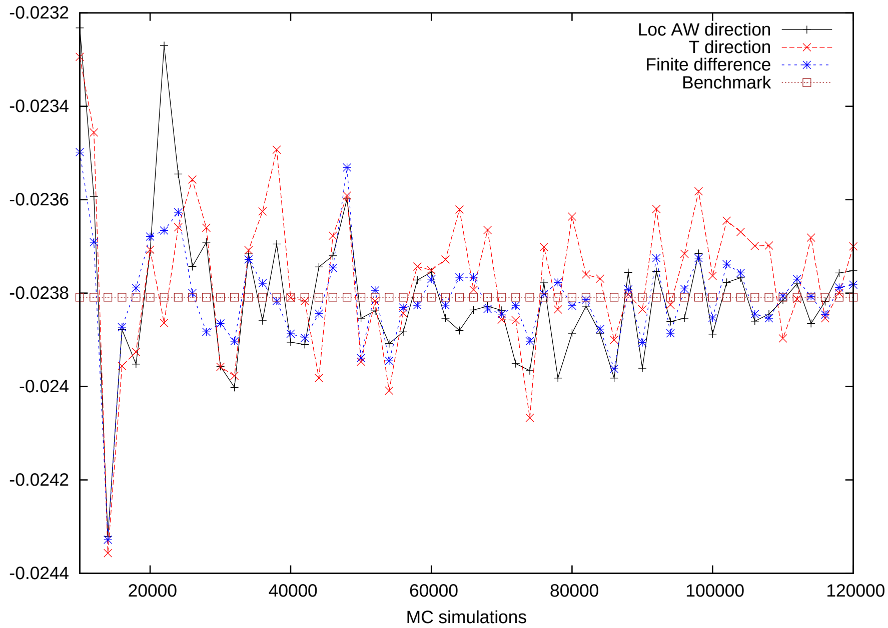

Figure 2.17: OU model. Floating Asian option.

Introducing the localization method, we obtain nice results, as shown in Figure (2.18)
In order to give a general comparison between the different direction of calculus, we show the numerical results in the Table 2.2

Remark 2.6.1. *The first variation process of the Ornstein-Ulhenbeck process*

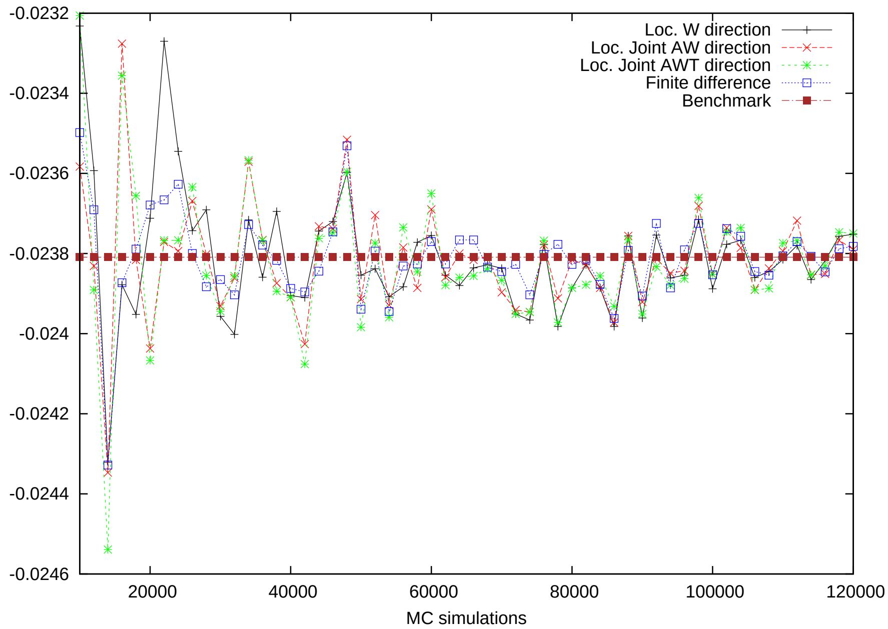

Figure 2.18: OU model. Floating Asian option.

MC sim.	DF	W	A	T	Wloc
10000	-0.023498	-0.010911	-0.028231	-0.023294	-0.023232
20000	-0.023679	-0.017034	-0.024909	-0.023707	-0.023712
30000	-0.023865	-0.025125	-0.035790	-0.023958	-0.023957
40000	-0.023887	-0.024354	-0.022254	-0.023810	-0.023905
50000	-0.023939	-0.029813	-0.029413	-0.023947	-0.023854
60000	-0.023770	-0.013654	-0.028609	-0.023751	-0.023755
70000	-0.023845	-0.018955	-0.025972	-0.023857	-0.023837
80000	-0.023827	-0.016633	-0.023818	-0.023636	-0.023886
90000	-0.023906	-0.024866	-0.028164	-0.023835	-0.023961
100000	-0.023853	-0.027804	-0.024888	-0.023763	-0.023888
110000	-0.023806	-0.024226	-0.020025	-0.023897	-0.023815
120000	-0.023782	-0.024233	-0.025865	-0.023700	-0.023752

Table 2.2: Comparison of the Malliavin Delta with the finite difference Method

is $\xi_t = e^{-bt}$ (see 2.5.2) and does not depend on the jump amplitudes or Gaussian noise. In particular ξ_{T_i} gives instead a dependence on the jump times. Therefore in the Malliavin weight for the Delta there is a little dependence on the Gaussian noise and jump amplitudes, while there is a strong dependence on the jump times(see (2.26), (2.36)). This can explain why the numerical results using only the jump times direction of calculus are more performing than the other, independently by the parameters choice.

Digital Options

We consider in this Section a standard Digital option with payoff $\mathbf{1}_{X_T > K}$.
It is a non smooth payoff, and therefore the Malliavin methods give better

results than the finite difference one. We remark that the jump times direction gives the best result.
Figure (2.19) shows the comparison between the different direction of Malliavin calculus and the finite difference.

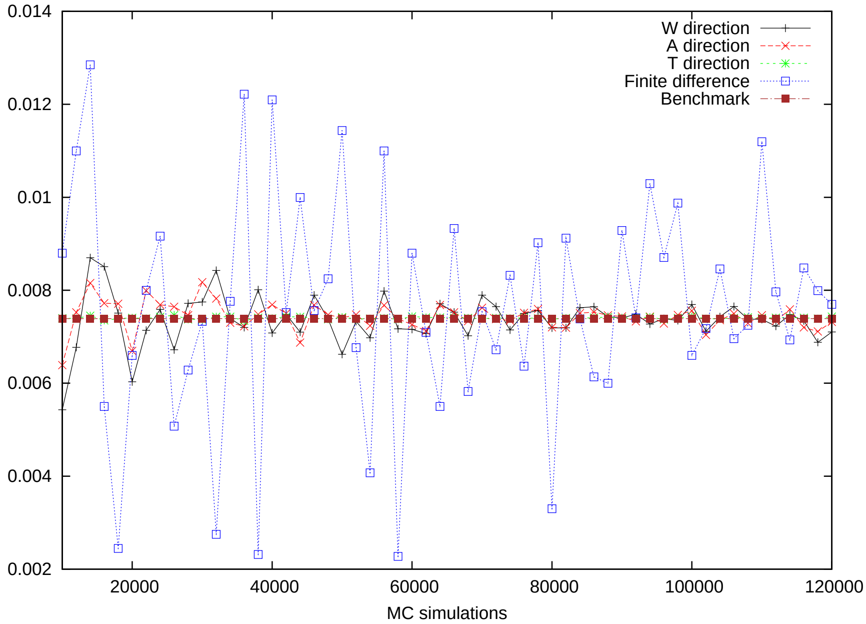

Figure 2.19: OU model. Digital option.

The high variance of the finite difference, does not allows to compare the different direction of the Malliavin calculus. Thus in the next Figure (2.20) we have taken off the finite difference.
In Figure (2.21) we compare the jump times direction of the calculus with the localized joint Gaussian and jump amplitudes as in the Floating Call Section, showing the high accuracy of the T-direction.
Figures (2.22) and (2.23) show the joint directions of calculus and the localized ones.

Standard Call Options

We consider now the pure jump Ornstein-Ulhenbeck Model, taking $\sigma = 0$, in order to compare our results with Bally, Bavouzet and Messaoud (see [3]) ones. Next Figure 2.24 confirms that the use the jump Times direction gives nice results, similar to the Finite Difference, and in the case of the Call Options the result are really very precise.
The Variance of our Malliavin weight is much less than the one coming from the weight by Bally, Bavouzet and Messaoud in [3]. Table 2.3 gives such a comparison, as σ varies.

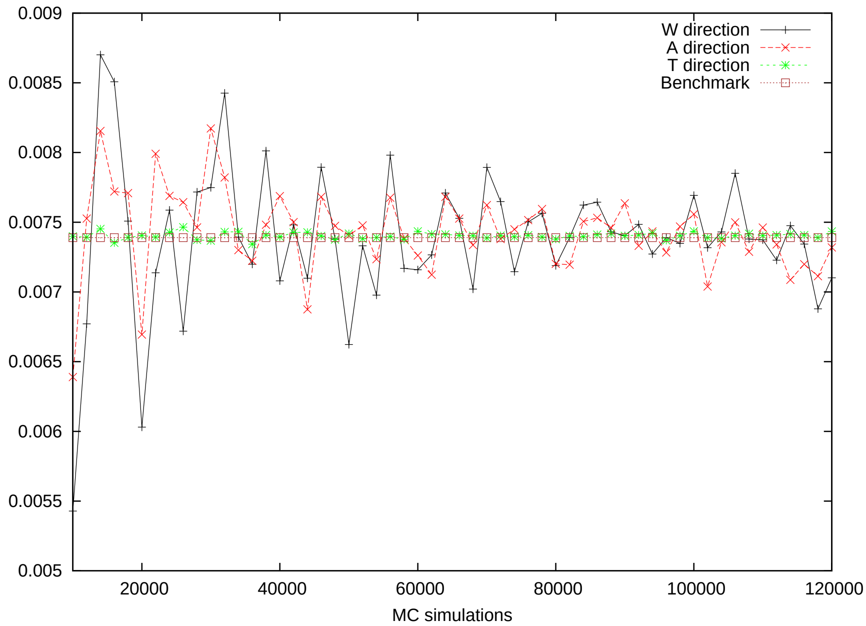

Figure 2.20: OU model. Digital option.

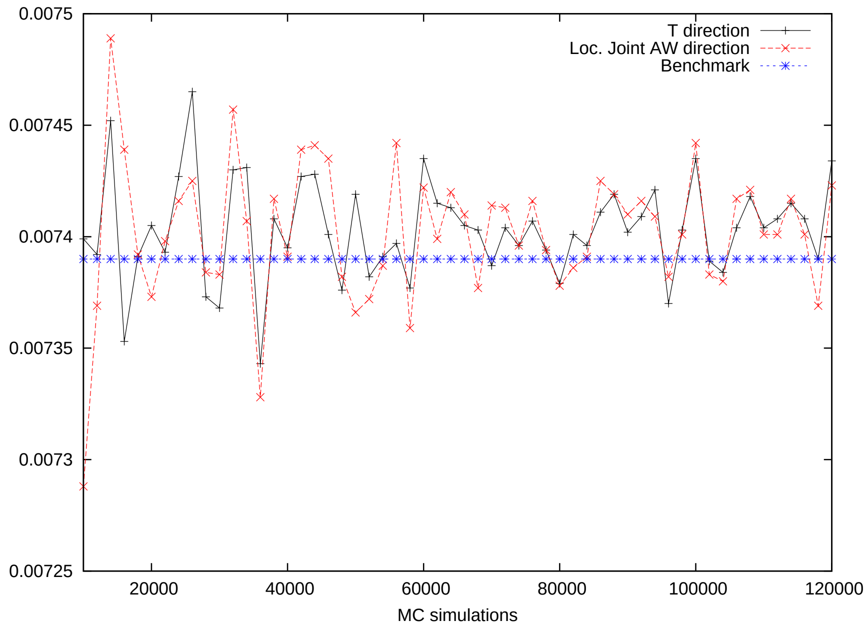

Figure 2.21: OU model. Digital option.

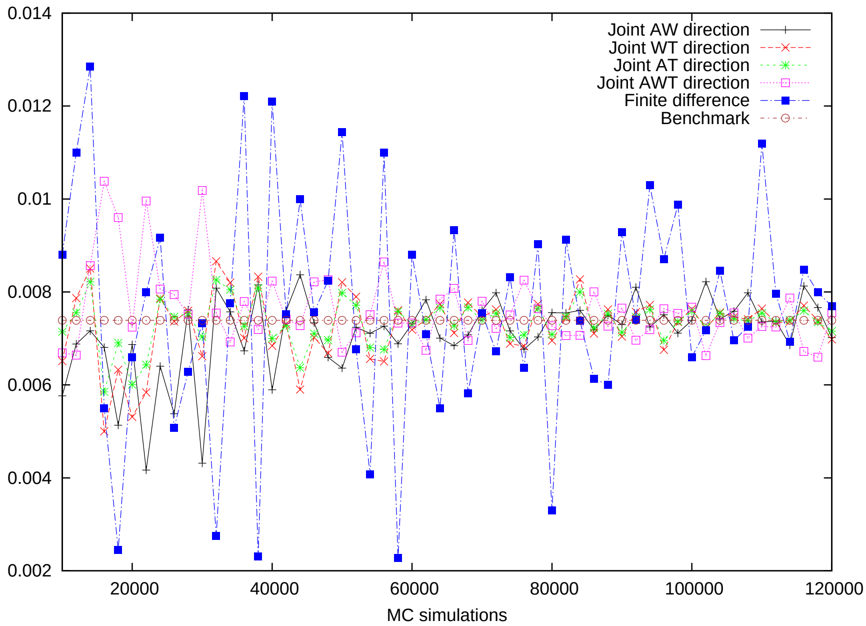

Figure 2.22: OU model. Digital option.

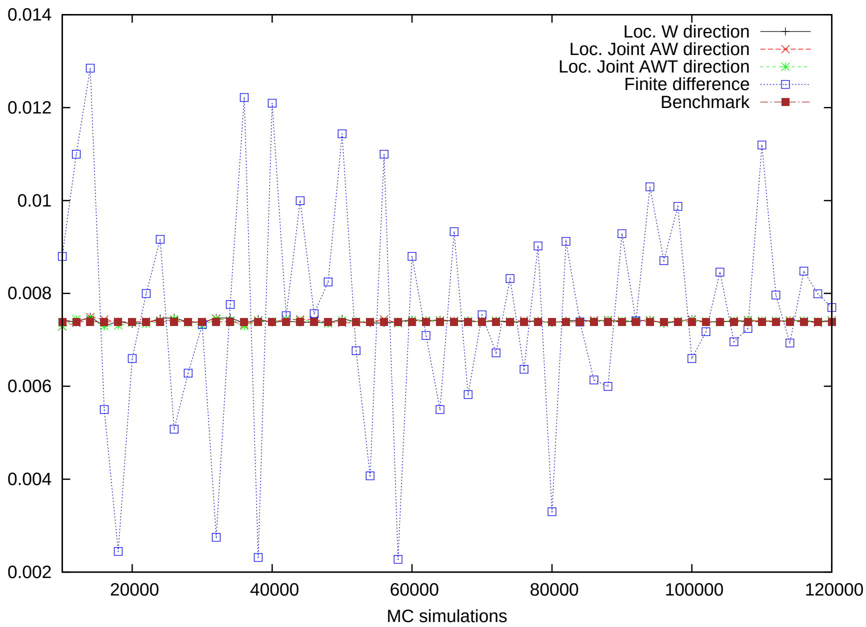

Figure 2.23: OU model. Digital option.

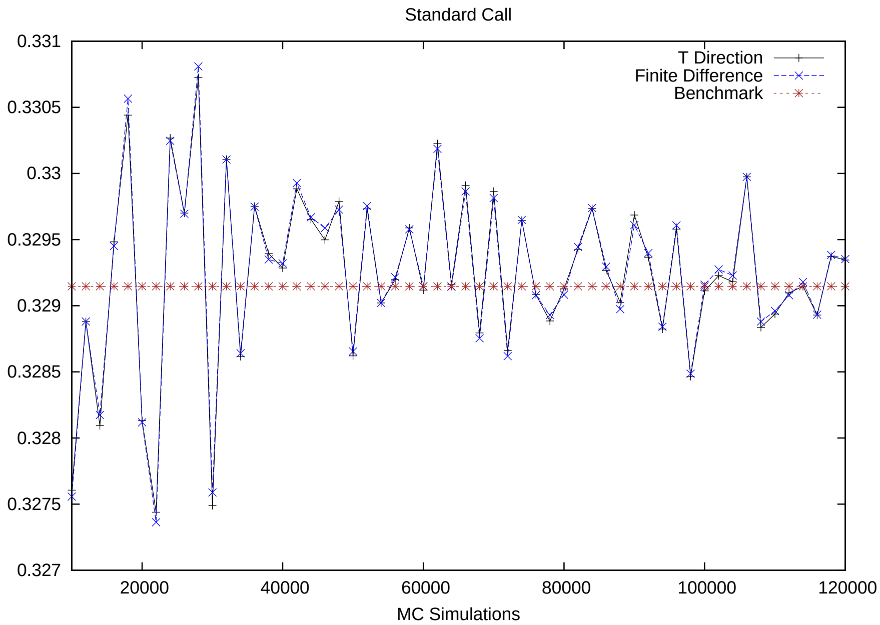

Figure 2.24: OU model. Standard call option

σ	Var weight	BBM Var weight
15.8114	0.000180	0.028512
16.6667	0.000222	0.041721
17.6777	0.000277	0.040069
18.8982	0.000341	0.041013
20.4124	0.000422	0.043306
22.3607	0.000535	0.040048
25.0000	0.000685	0.040713
28.8675	0.000902	0.036272
35.3553	0.001224	0.034315
50.0000	0.001749	0.033329

Table 2.3: Comparison between the variances of our weight and the weight by Bally, Bavouzet and Messaoud.

2.6.4 Cox Ingersoll Ross Model

We study now the C.I.R. process, as it has been considered in Section 2.5.3. The parameters of the model are: $\sigma = 0.2$, $\nu = 0.6$, $\eta = 0.6$, $\alpha = 0.6$.

Floating Asian Call Option

As remarked in Section 2.5.3, we cannot write the C.I.R. process in a closed form solution. From a numerical point of view, we have to simulate all the processes, included X_t and Y_t. Moreover, being a square root process, the C.I.R. process has an high variance that makes the numerical results less performing than the other models.
In the next figures we consider a Floating Asian Call Option. We have not included the benchmark value, because we obtained a value overlapping the finite difference ones.
Figure 2.25 shows the non localized results. As remarked, the values are not so close to the finite difference ones.

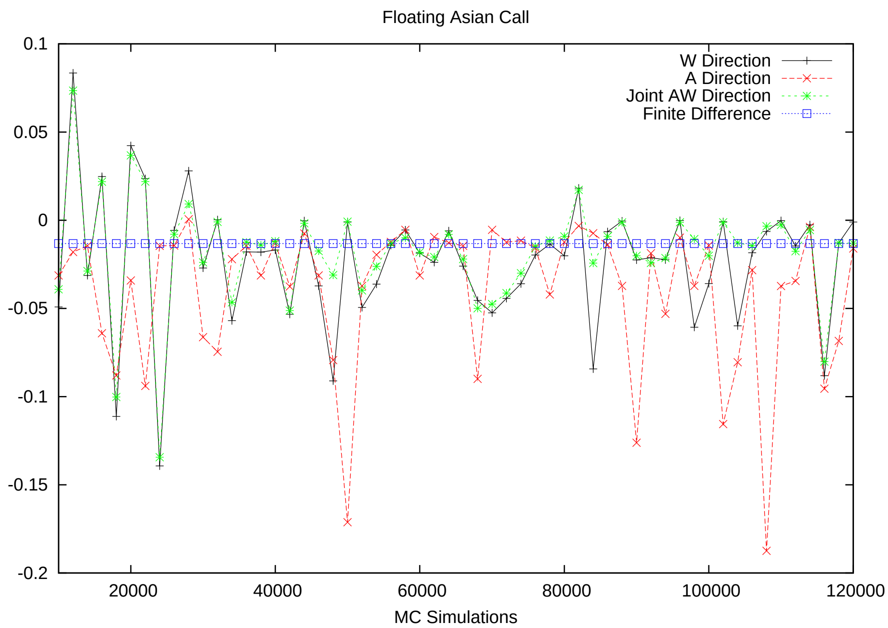

Figure 2.25: CIR model.

Following the ideas in Remark 2.6.1, we can notice that we have obtained the best numerical results using the Gaussian direction of the Malliavin calculus. In this case, the Malliavin weights do not depends explicitly on the jump amplitudes (see Section 2.5.3) and therefore, we expect a less performing weight when the jump amplitudes are taken into account.
The introduction of the localization considerably improves the results, as shown in Figure2.26.

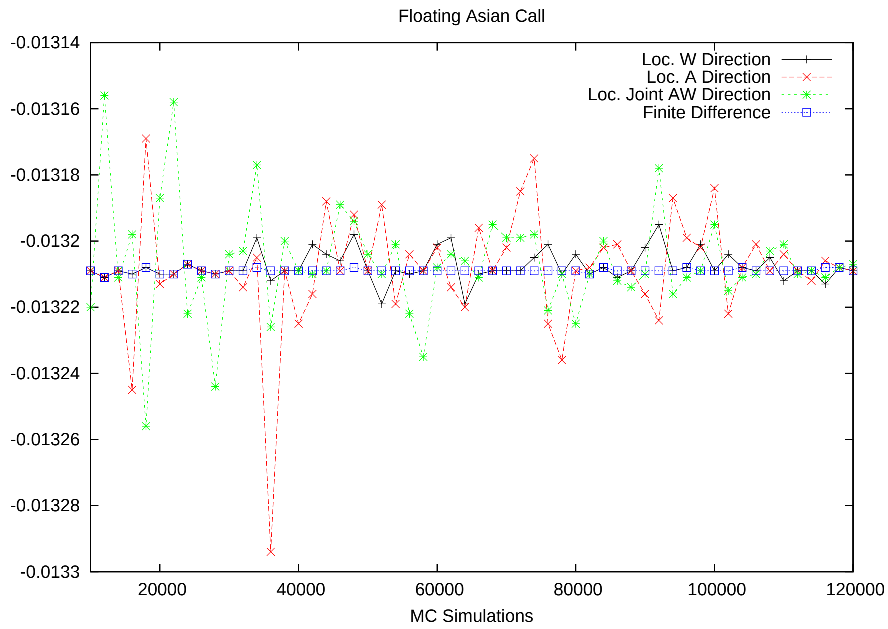

Figure 2.26: CIR model.

Last Figure 2.27 shows a comparison between the localized and the non localized weight found using the Gaussian direction of the Malliavin calculus.

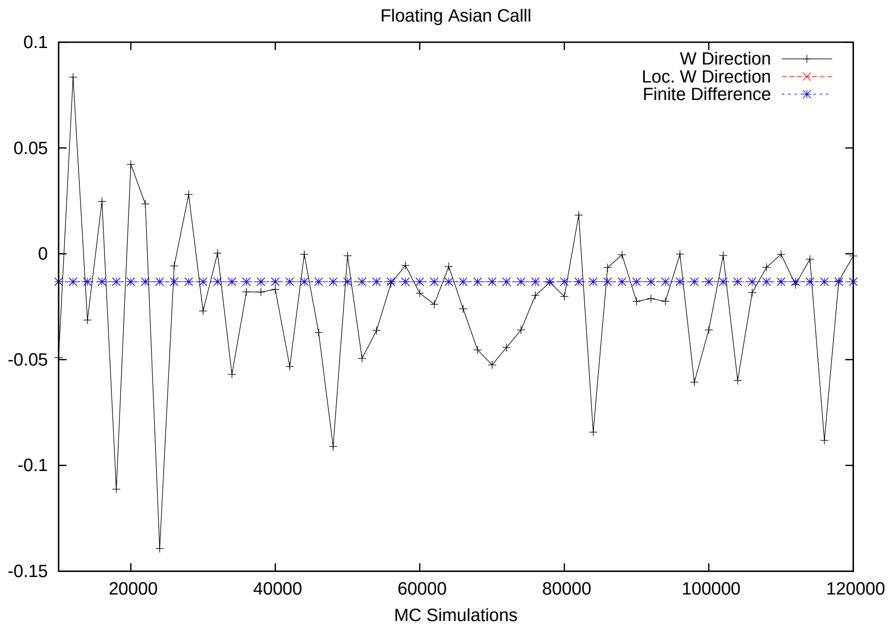

Figure 2.27: CIR model. Floating Asian call option

2.6.5 Stochastic Volatility Model

Consider now the Stochastic Volatility Model as in Section 2.5.4. We have chosen the parameters as follows: $\sigma_0 = 0.5$, $r = 0.1$, $k = 0.6$, $\theta = 0.6$, $\beta = 0.05$. The correlation between the two Brownian motions is $\rho = 0.5$.

Floating Asian Call Option

Figure 2.28 shows that, for a Floating Asian Call Option, the Malliavin methods, in all directions (a for the Gaussian one, w.r.t. the Brownian motion W^1) give result comparable with the Finite Difference.

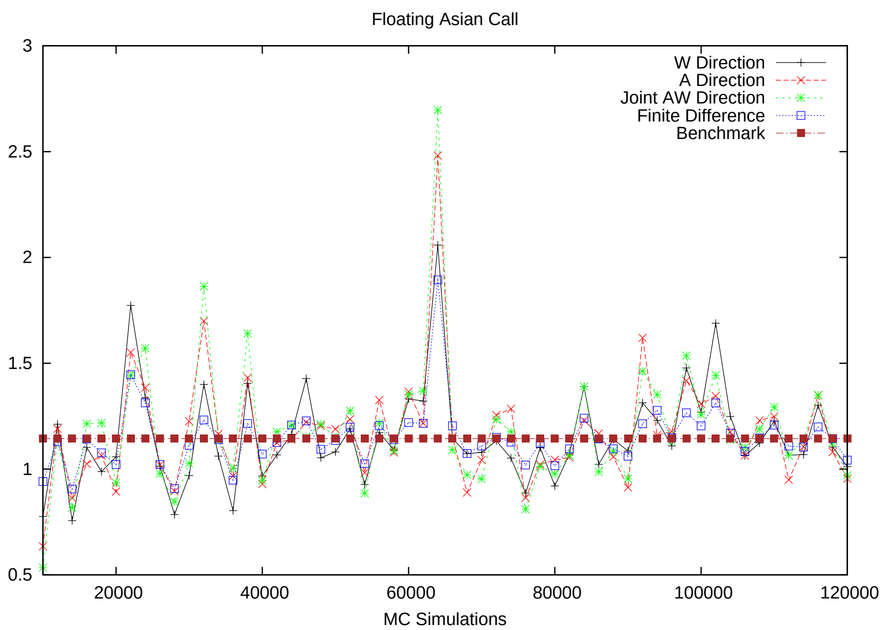

Figure 2.28: SVM model. The "W" direction is w.r.t. W^1.

Digital options

As noticed in the Merton model, the Finite Difference methods do not give good results, due to the non smoothness of the payoff. Figure 2.29 and 2.30 are related to a Standard Digital Option and an Asian Digital one respectively.

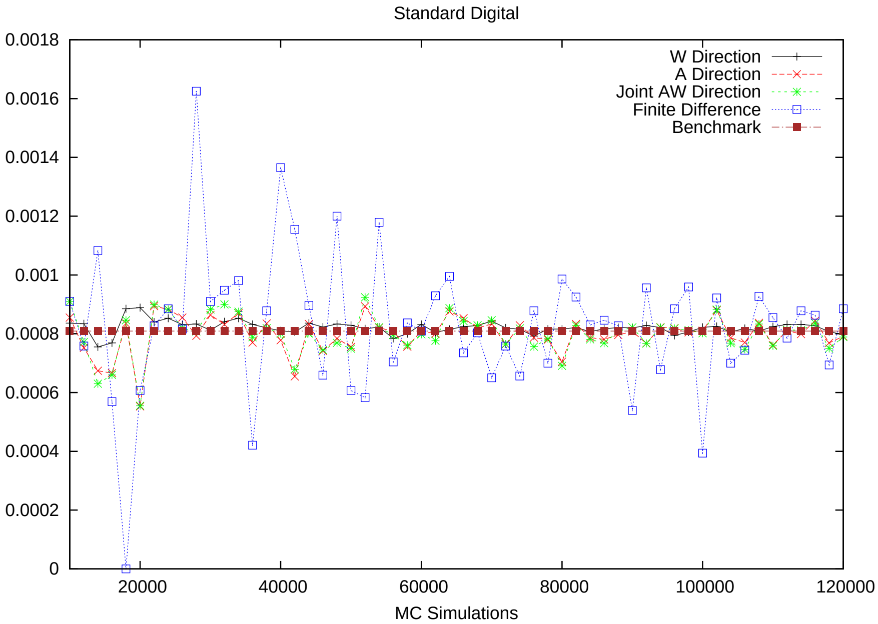

Figure 2.29: SVM model.

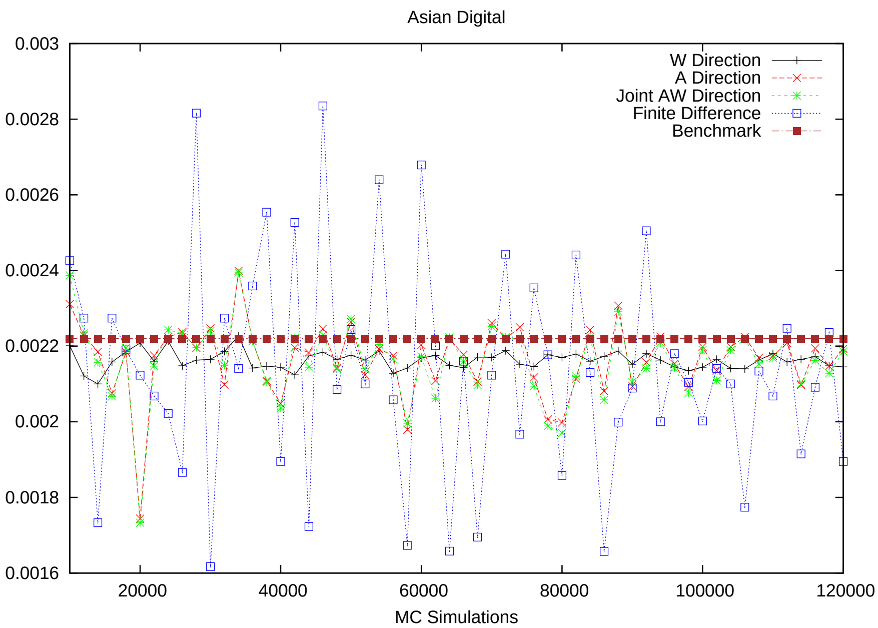

Figure 2.30: SVM model.

Chapter 3

Representation formulas for the conditional expectation

We consider here the problem of the representation of the conditional expectation in terms of non conditional ones. This problem appears in the recent literature, e.g. in Lions and Regnier [23], Bally, Caramellino and Zanette [4], Bouchard, Ekeland and Touzi [11].
More precisely, our aim is to give a representation formula for the conditional expectation

$$\mathbb{E}(F(X_t, Y_t) \,|\, X_s = \alpha_1, Y_s = \alpha_2) \tag{3.1}$$

in terms of non conditional ones, as $t > s > 0$, where (X_t, Y_t) is the two-dimensional stochastic process already considered, i.e.

$$\begin{aligned} X_t &= x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r + \sum_{i=1}^{J_t} c(T_i, \Delta_i, X_{T_i-}) \\ Y_t &= \int_0^t X_r dr \end{aligned}$$

This problem has an important relevance in finance, being related with the pricing of American Asian options. In fact, if $\Phi(t, X_t, Y_t)$ denote the payoff function, the associated American price is usually approximated by means of a Dynamic Programming Principle, which can be roughly summarized as follows. Set $\Delta t = T/n \in (0, 1)$ and let $(\bar{X}_{k\Delta t}, \bar{Y}_{k\Delta t})_{k=0,1,\ldots,n}$ be a discretization for $(X_t, Y_t)_{t\in[0,T]}$. Since the pair (X_t, Y_t) is Markovian, the American price P_0 is approximated by $\bar{P}_0(x, 0)$, where $\bar{P}_{k\Delta t}(\bar{X}_{k\Delta t}, \bar{Y}_{k\Delta t})$ solves the following backward problem:

$$\begin{aligned} &\bar{P}_{n\Delta t}(\bar{X}_{n\Delta t}, \bar{Y}_{n\Delta t}) = f(n\Delta t, \bar{X}_{n\Delta t}, \bar{Y}_{n\Delta t}) \text{ and for any } k = n-1, \ldots, 1, 0 \\ &\bar{P}_{k\Delta t}(\bar{X}_{k\Delta t}, \bar{Y}_{k\Delta t}) = \max\Big(f(k\Delta t, \bar{X}_{k\Delta t}, \bar{Y}_{k\Delta t}) \,,\, e^{-r\Delta t}\, \mathrm{T}[\bar{P}_{(k+1)\Delta t}](\bar{X}_{k\Delta t}, \bar{Y}_{k\Delta t})\Big) \end{aligned}$$

where the operator T is defined as

$$\mathrm{T}[\bar{P}_{(k+1)\Delta t}](\bar{X}_{k\Delta t}, \bar{Y}_{k\Delta t}) = \mathbb{E}^* \Big(\bar{P}_{(k+1)\Delta t}(\bar{X}_{(k+1)\Delta t}, \bar{Y}_{(k+1)\Delta t}) \,\Big|\, \bar{X}_{k\Delta t}, \bar{Y}_{k\Delta t} \Big),$$

in which r stands for the (constant, for the sake of simplicity) spot rate and $\mathbb{E}^*$ for the expectation under the risk neutral measure. Therefore, a representation formula for the conditional expectation in terms of non conditional ones is really helpful in order to set up Monte Carlo methods for the pricing of American options, as already observed and done in the literature. Moreover, in the case of American Asian options, it is worth to stress that even if the option is a non floating one, that is written on only on Y_t (for example, a standard Asian put option), the Markov property, holding for the pair and not for Y_t only, obliges to work with conditional expectation written on the pair (X, Y). In the pure diffusion case (i.e. $c \equiv 0$), representation formulas have been developed by Bouchard, Ekeland and Touzi in [11] for general d-dimensional diffusion processes, but they cannot be used in our context. In fact, one of the starting point of their work is that the diffusion coefficient in non degenerate, an hypotheses which seems to be impossible to relax and, obviously, fails in our case.
Therefore, we try to study what can be done in our degenerate case. We theoretically solve the problem using the approximation of the Dirac Delta as the derivative of the Heavyside function, by using all the different directions of the Malliavin Calculus. However, we will see that this approach lead to weights which are difficult to implement in practice, and this is due mainly to the fact that the integration by parts formula has to be applied twice. As an example, we discuss the Black-Scholes-Merton model, to give an idea of the practical application of the Theory.
Then, we briefly recall an alternative approach introduced by Malliavin and Thalmaier in [24], based on the use of the Poisson kernel.This is important because, roughly speaking, one has to use the integration by parts formula only once and in practice, one obtains simpler weights. The problem is that, at the present time, it is not clear how the resulting weights work in practice, because there are problems of high variance, as it results from some private numerical experiments of ours and recently studied in an interesting work on this subject by Kohatsu-Higa and Yasuda [19].

3.1 Classical Heavyside approach

The idea of this section is to write the conditional expectation (3.1) as a ratio, using the Dirac delta, which we consider as the derivative of the Heavyside function, in some sense. Therefore we will use similar arguments as in the chapter on Sensitivity Analysis, exploiting the integration by parts Formula in order to have suitable representation formulas.
Consider the process $Z_t = (X_t, Y_t)$, fix $0 < s < t < T$ and the number of the jumps by fixing the sets $A = \{J_s = n\} \bigcap \{J_t = m\}$, with $m \geq n$.

Notice that we can write the conditional expectation (3.1) as

$$\frac{\mathbb{E}(f(X_t,Y_t)\Lambda^0(X_s-\alpha_1)\Lambda^0(Y_s-\alpha_2))}{\mathbb{E}(\Lambda^0(X_s-\alpha_1)\Lambda^0(Y_s-\alpha_2))} \tag{3.2}$$

where Λ^0 is the 1-dimensional Dirac delta in zero. Λ^0 can be thought as the derivative of the 1-dimensional Heavyside function, that is, setting $\tilde{H}(\xi) = \mathbf{1}_{\{\xi>0\}}$,

$$\prod_{i=1}^{2}\Lambda^0(x_i) = \partial^2_{x_1,x_2}\tilde{H}_2(x) \equiv \partial^2_{x_1,x_2}\prod_{i=1}^{2}\tilde{H}(x_i).$$

Now the ratio (3.2) cannot be handled directly. So we first consider the following problem.
Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ denote infinitely differentiable functions with bounded derivatives. The problem is to find a suitable weight, that we denote with $H(Z;s,t)$, independent of ϕ and f, such that for any $0 < s < t < T$, for $i,j = 1,2$, $i \neq j$,

$$\mathbb{E}\big(\partial^2_{x_i,x_j}\phi(Z_s)f(Z_t)\mathbf{1}_A\big) = \mathbb{E}\big(\phi(Z_s)f(Z_t)\cdot H(Z;s,t)\mathbf{1}_A\big). \tag{3.3}$$

In Formula (3.3), looking at (3.2), we think $\phi(Z_s) = \tilde{H}_2(Z_s - \alpha)$. We asked smoothness properties for ϕ that of course are not satisfied by the Heavyside function. However, by using density arguments, allowing to approximate $\tilde{H}_2$ through smooth functions, one can deal with the problem in (3.3). This is a standard way to proceed, for details see Bally, Caramellino and Zanette [4] or Bouchard, Ekeland and Touzi [11].

3.1.1 Notations

Now, let us start by fixing the notations and definitions. The main difference between this problem and the problem of sensitivities is that in the latter there is only one time of interest, the maturity T. Therefore we have used one time the integration by parts formula, and we have defined the process v and the matrix $\gamma_{\tilde{D}Z_T,v}$ only on the time interval $[0,T]$, fixing moreover the set $\{J_T = n\}$. Here, we have to deal with two time intervals $[0,s]$ and $[s,t]$, using twice the properties of the differential calculus. We will use in particular the following notations.
From now on we consider $0 < s < t < T$ and fix the sets $\{J_s = n\}$ and $\{J_t = m\}$, with $m \geq n \geq 1$. Let us denote

$$A = \{J_s = n\}\bigcap\{J_t = m\}.$$

We split our calculus over the time intervals $[0,s]$ and $[s,t]$.
Up to time t, on the set A we have always ordered the random sources related to the jump noise as

$$\Delta_1,\dots,\Delta_n,\Delta_{n+1},\dots,\Delta_m,T_1,\dots,T_n,T_{n+1},\dots,T_m$$

and we will continue to do that. Moreover, on A, in each time interval of interest, the noises from the jump part are given by by:

- $\Delta_1, \dots, \Delta_n, T_1, \dots, T_n$ on the time interval $[0, s]$;
- $\Delta_{n+1}, \dots, \Delta_m, T_{n+1}, \dots, T_m$ on the time interval $[s, t]$.

We can now pass to the Malliavin derivative operator and Skorohod integral one over the two time interval $[0, s]$ and $[s, t]$. Roughly speaking, we consider only the random sources related to each interval and, in some sense, we freeze the others.

Malliavin derivatives on $[0, s]$ and $[s, t]$

On the set $\{J_s = n\} \bigcap \{J_t = m\}$, the r.v.'s of interest depend on the evolution of the Brownian and the jump one up to t. As for the latter, one has $2m$ noise sources, which we have written as:

- components $1, \dots, m$: $\Delta_1, \dots, \Delta_n, \Delta_{n+1}, \dots, \Delta_m$;
- components $m + 1, \dots, 2m$: $T_1, \dots, T_n, T_{n+1}, \dots, T_m$.

Notice that components $1, \dots, n, m + 1, \dots, n + m$ refer to what happens in the time interval $[0, s]$, while components $n+1, \dots, m, n+m+1, \dots, 2m$ refer to what happens in the time interval $[s, t]$.
We then define the following generalized Malliavin derivatives:

$$\begin{aligned} \tilde{D}_{s,n} &= (\tilde{D}_0^{0,s}, \tilde{D}_1^n, ..., \tilde{D}_n^n, 0, ...0, \tilde{D}_{m+1}^n, ..., \tilde{D}_{m+n}^n, 0, ..., 0) \\ \tilde{D}_{t,m} &= (\tilde{D}_0^{s,t}, 0, ..., 0, \tilde{D}_{n+1}^m, ..., \tilde{D}_m^m, 0, ..., 0, \tilde{D}_{m+n+1}^m, ..., \tilde{D}_{2m}^m) \end{aligned}$$

in which

- $\tilde{D}_0^{0,s}$ and $\tilde{D}_0^{s,t}$ denote the Gaussian Malliavin derivative over the time interval $[0, s]$ and $[s, t]$ respectively;
- the superscripts n and m means that we are considering the first n jumps and the remaining $m - n$ jumps in the generalized Malliavin derivative w.r.t. the jump noise over the time interval $[0, s]$ and $[s, t]$ respectively.

Therefore, we will use $\tilde{D}_0^{0,s}$ and $\tilde{D}_0^{s,t}$ to handle the Malliavin calculus over $[0, s]$ and over $[s, t]$ respectively.

Skorohod integrals on $[0, s]$ and $[s, t]$

On the set $\{J_s = n\} \bigcap \{J_t = m\}$, the processes of interest depend on the evolution of the Brownian and the jump noise up to t: $u = (u_0, U_1, \dots, U_{2m})$. In particular, they are made by $2m + 1$ components. The Skorohod integral acts on the first component, identified by the subscript 0, as the Gaussian one. As for the the remaining $2m$ components, the Skorohod integral acts

- w.r.t. Δ_i on component i, as $i = 1, \dots, m$;
- w.r.t. T_i on component $i + m$, as $i = 1, \dots, m$;

Let us recall again that components $1, \dots, n, m+1, \dots, n+m$ refer to what happens in the time interval $[0, s]$, while components $n+1, \dots, m, n+m+1, \dots, 2m$ refer to what happens in the time interval $[s, t]$.
We then define the following generalized Skorohod integrals:

$$\begin{aligned}\tilde{\delta}_{s,n}(u) &= \tilde{\delta}_0^{0,s}(u_0) + \sum_{i=1}^{n} \tilde{\delta}_i^n(U_i) + \sum_{i=m+1}^{m+n} \tilde{\delta}_i^n(U_i) \\ \tilde{\delta}_{t,m}(u) &= \tilde{\delta}_0^{s,t}(u_0) + \sum_{i=n+1}^{m} \tilde{\delta}_i^m(U_i) + \sum_{i=n+m+1}^{2m} \tilde{\delta}_i^m(U_i)\end{aligned}$$

in which

- $\tilde{\delta}_0^{0,s}$ and $\tilde{\delta}_0^{s,t}$ denote the Gaussian Skorohod integral over the time interval $[0, s]$ and $[s, t]$ respectively;
- the superscripts n and m means that we are considering the first n jumps and the remaining $m-n$ jumps in the generalized Skorohod integral w.r.t. the jump noise over the time interval $[0, s]$ and $[s, t]$ respectively.

Therefore, we will use $\tilde{\delta}_{s,n}$ and $\tilde{\delta}_{t,m}$ to handle the Malliavin calculus over $[0, s]$ and over $[s, t]$ respectively.

Generalized Malliavin covariance matrixes on $[0, s]$ and $[s, t]$

Fix a 2-dimensional process $v = (v^1, v^2)$. On the set A, we set the 2×2 generalized Malliavin covariance matrixes on $[0, s]$ and $[s, t]$ as $\gamma_{\tilde{D}_{s,n} Z_s, v}$ and $\gamma_{\tilde{D}_{t,m} Z_t, v}$ respectively. This means that, as $i, j = 1, 2$

$$\big(\gamma_{\tilde{D}_{s,n} Z_s, v}\big)^{ij} = \langle \tilde{D}_{s,n} Z_s^j, v^i \rangle \quad \text{and} \quad \big(\gamma_{\tilde{D}_{t,m} Z_t, v}\big)^{ij} = \langle \tilde{D}_{t,m} Z_t^j, v^i \rangle$$

in which, as usual, $Z^1 = X$ and $Z^2 = Y$.

3.1.2 Representation of the conditional expectation

We start by finding a formula of the type

$$\mathbb{E}(\partial_{x_i} \phi(Z_s) G \mathbf{1}_A) = \mathbb{E}(\phi(Z_s) \cdot H_i(Z_s; G) \mathbf{1}_A).$$

We will see soon that

$$H_i(Z_s; G) = \tilde{\delta}_{s,n}\Big(\sum_{j=1}^{2} \hat{\gamma}^{i,j}_{\tilde{D} Z_s, v} v^j G \Big) \tag{3.4}$$

holding when $\pi_i = 1$ for any i and considering the case σ non null (giving a null border term operator). Now, for our purposes $G = f(X_t)$, so that (3.4) gives

$$\mathbb{E}(\partial_{x_i}\phi(Z_s)f(Z_t)) = \mathbb{E}\Big(\phi(Z_s)\tilde{\delta}_{s,n}(\sum_{j=1}^{2}\hat{\gamma}^{ij}_{\tilde{D}Z_s,v}\, v^j f(Z_t))\Big).$$

It is clear that the problem is the presence of $f(Z_t)$ in the argument of the Skorohod integral. Now, $f(Z_t)$ can be put outside by using the Skorohod integral of a special product property but this will make the derivatives of f appearing. So, in order to handle them, the idea is to perform a Malliavin calculus over the time interval $[s,t]$.
As already remarked, we use here essentially the same arguments previously used. So, we will choose $\pi_i = 1$ for $i = 1, ..., 2m$.

Remark 3.1.1. *We have fixed the sets $\{J_s = n\}$ and $\{J_t = m\}$ with $m \geq n$. If $m = n$, that means that the process X has no jumps on the time interval $(s,t]$, the Malliavin calculus in the jump direction cannot be applied. Also if $n = 1$ or $m = n+1$, that means that X has only one jump on $[0,s]$ or $(s,t]$ respectively, the generalized Malliavin covariance matrix method cannot be applied because it has null determinant. We refer to Remark 2.1.4, 2.1.5 and 2.1.6, where we have discussed this topic.*

We can state now a first important result.

Lemma 3.1.2. *Let $0 < s < t < T$ be fixed and let $\phi, f : \mathbb{R}^2 \to \mathbb{R}$ differentiable functions with bounded derivatives. Suppose that X satisfies the standard hypothesis. Let v a process such that the generalized Malliavin covariance matrixes $\gamma_{\tilde{D}_{s,n}Z_s,v}$ and $\gamma_{\tilde{D}_{t,m}Z_t,v}$ are invertible on A, and let $\hat{\gamma}_{\tilde{D}_{s,n}Z_s,v}$ and $\hat{\gamma}_{\tilde{D}_{t,m}Z_t,v}$ denote the inverses. Let F denote a r.v. Fix now $i = 1,2$ and assume that:*

- *$\phi(Z_s)$ and $f(Z_t)\sum_{j=1}^{2}\hat{\gamma}^{ij}_{\tilde{D}_{s,n}Z_s,v}v^j F$ are (n,s)-compatible on $[0,s]$;*
- *$f(Z_t)$ and $\sum_{j=1}^{2}\hat{\gamma}^{ij}_{\tilde{D}_{s,n}Z_s,v}v^j$ are (n,s)-compatible and moreover, suppose that*

$$\text{if } \sum_{j=1}^{2}\hat{\gamma}^{ij}_{\tilde{D}_{s,n}Z_s,v}v^j_0(u)\mathbf{1}_{u\in[0,s]}F \in Dom_{n,s,2}(\delta_0)$$

$$\text{then } f(Z_t)\sum_{j=1}^{2}\hat{\gamma}^{ij}_{\tilde{D}_{s,n}Z_s,v}v^j_0(u)\mathbf{1}_{u\in[0,s]}F \in Dom_{n,s,2}(\delta_0);$$

- *$f(Z_t)$ and $\sum_{j,k,l=1}^{2}B^{kj}_i v^l$ are (m,t)-compatible over $[s,t]$, where*

$$B^{k,j}_i = \langle \tilde{D}_{s,n}Z^k_t, \hat{\gamma}^{i,j}_{\tilde{D}_{s,n}Z_s,v}v^j F\rangle$$

.

Then,

$$\mathbb{E}\Big(\partial_{x_i}\phi(Z_s)f(Z_t)F\mathbf{1}_A\Big) = \mathbb{E}\Big(\phi(Z_s)f(Z_t)\hat{H}_i^F(Z;s,t)\mathbf{1}_A\Big) + \mathbb{E}\Big([\phi(Z_s);f(Z_t);\hat{H}_i^F(Z;s,t)\mathbf{1}_A]_\pi\Big) \quad (3.5)$$

where

$$\hat{H}_i^F(Z;s,t) = \sum_{j=1}^{2}\tilde{\delta}_{s,n}(\hat{\gamma}^{i,j}_{\tilde{D}_{s,n}Z_s,v}v^jF) - \sum_{j,k,l=1}^{2}\tilde{\delta}_{t,m}(B_i^{k,j}\hat{\gamma}^{k,l}_{\tilde{D}_{t,m}Z_t,v}\ v^l) \quad (3.6)$$

and the border term is given by

$$[\phi(Z_s);f(Z_t);\hat{H}_i^F(Z;s,t)\mathbf{1}_A]_\pi = $$
$$= [\phi(Z_s),\sum_{j=1}^{2}\hat{\gamma}^{i,j}_{\tilde{D}_{s,n}Z_s,v}v^jF]_\pi - [f(Z_t),\phi(Z_s)\sum_{j,k,l=1}^{2}B_i^{k,j}\ \gamma^{k,l}_{\tilde{D}_{t,m}Z_t,v}\ v^l]_\pi.$$

We recall that since $\pi_i = 1$, the border term is null considering the Gaussian or jump amplitudes directions of the Malliavin calculus.

Proof of Lemma 3.1.2

We prove the Lemma assuming that $\sigma \neq 0$, that is, in the direction of the Brownian motion or jump amplitudes. The proof in the jump times direction follows straightforwardly by considering also the border term.

We first consider the Malliavin calculus over $[0,s]$. By the integration by parts formula as in Property 1.3.8, one has

$$\mathbb{E}\Big(\partial_{x_i}\phi(Z_s)f(Z_t)F\mathbf{1}_A\Big) = \mathbb{E}\Big(\phi(Z_s)H_i(Z_s;f(Z_t)F)\mathbf{1}_A\Big),$$

where

$$H_i(Z_s;f(Z_t)F) = \tilde{\delta}_{s,n}\Big(\sum_{j=1}^{2}\hat{\gamma}^{i,j}_{\tilde{D}Z_s,v}v^jf(Z_t)F\Big).$$

Exploiting the Skorohod integral of a (special) product Property 1.3.7, we can write:

$$\mathbb{E}\Big[\phi(Z_s)\sum_{j=1}^{2}\tilde{\delta}_{s,n}\Big(\hat{\gamma}^{i,j}_{\tilde{D}_{s,n}Z_s,v}v^jf(Z_t)F\Big)\mathbf{1}_A\Big]$$
$$= \mathbb{E}\Big[\phi(Z_s)f(Z_t)\sum_{j=1}^{2}\tilde{\delta}_{s,n}\Big(\hat{\gamma}^{i,j}_{\tilde{D}Z_s,v}v^jF\Big)\mathbf{1}_A\Big]$$
$$-\mathbb{E}\Big[\phi(Z_s)\sum_{j=1}^{2}\langle\tilde{D}_{s,n}f(Z_t),\hat{\gamma}^{i,j}_{\tilde{D}_{s,n}Z_s,v}v^jF\rangle\mathbf{1}_A\Big]$$

The first term of the r.h.s. contains exactly the first term of the weight $\hat{H}_i^F(Z;s,t)$ given by (3.6). So consider the second one. By the chain rule, stated in Properties 1.3.7,

$$\tilde{D}_{s,n} f(Z_t) = \sum_{k=1}^{2} \partial_{x_k} f(Z_t) D_{s,n} Z_t.$$

Therefore we have

$$\mathbb{E}\Big[\phi(Z_s)\sum_{j=1}^{2}\langle \tilde{D}_{s,n} f(Z_t), \hat{\gamma}^{i,j}_{\tilde{D}_{s,n}Z_s,v} v^j F\rangle \mathbf{1}_A\Big] = \mathbb{E}\Big[\sum_{j,k=1}^{2} \phi(Z_s)\partial_{x_k} f(Z_t) B_i^{k,j}\mathbf{1}_A\Big].$$

Now, by performing the Malliavin calculus over $[s,t]$ and by using the chain rule, one has

$$\partial_{x_k} f(Z_t) = \sum_{l=1}^{2} \hat{\gamma}^{k,l}_{\tilde{D}_{t,m}Z_t,v}\langle \tilde{D}_{t,m} f(Z_t), v^l\rangle$$

Thus, again by the integration by parts formula as in Property 1.3.8 we obtain

$$\mathbb{E}\Big[\sum_{j,k=1}^{2} \phi(Z_s)\partial_{x_k} f(Z_t) B_i^{k,j}\mathbf{1}_A\Big] =$$

$$= \mathbb{E}\Big[f(Z_t)\sum_{j,k=1}^{2}\sum_{l=1}^{2} \tilde{\delta}_{t,m}(\phi(Z_s) B_i^{k,j}\hat{\gamma}^{k,l}_{\tilde{D}_{t,m}Z_t,v} v^n)\mathbf{1}_A\Big].$$

Now

$$\tilde{\delta}_{t,m}(\phi(Z_s) B_i^{k,j}\hat{\gamma}^{k,l}_{\tilde{D}_{t,m}Z_t,v} v^n) = \phi(Z_s)\tilde{\delta}_{t,m}(B_i^{k,j}\hat{\gamma}^{k,l}_{\tilde{D}_{t,m}Z_t,v} v^n)$$

because $\phi(Z_s)$ does not depend on W_r, as $r \in [s,t]$, and on $(T_{n+1},...,T_m,\Delta_{n+1},...,\Delta_m)$ (or, in an equivalent way: $\tilde{D}_{t,m}\phi(Z_s) = 0$ in the Skorohod integral of a (special) product Property 1.3.7). Finally we have

$$\mathbb{E}\Big[\phi(Z_s)\sum_{j=1}^{2}\langle \tilde{D}_{s,n} f(Z_t), \hat{\gamma}^{i,j}_{\tilde{D}_{s,n}Z_s,v} v^j F\rangle \mathbf{1}_A\Big]$$

$$= \mathbb{E}\Big[\phi(Z_s) f(Z_t)\sum_{j,k,l=1}^{2} \tilde{\delta}_{t,m}(B_i^{k,j}\ \gamma^{k,l}_{\tilde{D}_{t,m}Z_t,v}\ v^l)\mathbf{1}_A\Big]$$

that ends the proof. □

As a consequence of the Lemma 3.1.2 we can solve the problem to finding the weight $H(Z;s,t)$ stated by (3.3) that is the core of this Chapter. Thus we have the following

Theorem 3.1.3. *Suppose the hypothesis of Lemma 3.1.2 holds under one of the two following cases:*

(i) $i = 1$ *and* $F = \hat{H}_2^1(Z; s, t)$*;*

(ii) $i = 2$ *and* $F = \hat{H}_1^1(Z; s, t)$*.*

Then,

$$\begin{aligned}\mathbb{E}\big(\partial^2_{x_i x_j}\phi(Z_s)f(Z_t)\mathbf{1}_A\big) &= \mathbb{E}\big(\phi(Z_s)f(Z_t)\cdot H(Z;s,t)\mathbf{1}_A\big)\\ &+ \mathbb{E}\Big([\phi(Z_s); f(Z_t); H(Z;s,t)]_\pi \mathbf{1}_A\Big)\\ &+ \mathbb{E}\Big([\partial_{x_j}\phi(Z_s); f(Z_t); \hat{H}_i^1(Z;s,t)]_\pi \mathbf{1}_A\Big). \quad (3.7)\end{aligned}$$

where

$$H(Z;s,t) = \begin{cases} \hat{H}_1^{\hat{H}_2^1(Z;s,t)}(Z;s,t) & \text{in case } (i)\\ \hat{H}_2^{\hat{H}_1^1(Z;s,t)}(Z;s,t) & \text{in case } (ii)\end{cases}$$

and the border terms are defined in Lemma 3.1.2.

Proof.
Using Lemma 3.1.2, one has

$$\begin{aligned}\mathbb{E}\big(\partial^2_{x_i x_j}\phi(Z_s)f(Z_t)\mathbf{1}_A\big) &= \mathbb{E}\big(\partial_{x_i}(\partial_{x_j}\phi(Z_s))f(Z_t)\mathbf{1}_A\big)\\ &= \mathbb{E}\big(\partial_{x_j}\phi(Z_s)f(Z_t)H_i^1(Z;s,t)\mathbf{1}_A\big)\\ &+ \mathbb{E}\big([\partial_{x_j}\phi(Z_s); f(Z_t); H_i^1(Z;s,t)]_\pi \mathbf{1}_A\big) \quad (3.8)\end{aligned}$$

Again by Lemma 3.1.2, setting $H(Z;s,t) = H_j^{H_i^1(Z;s,t)}(Z;s,t)$, one has

$$\begin{aligned}\mathbb{E}\big(\partial_{x_j}\phi(Z_s)f(Z_t)H_i^1(Z;s,t)\mathbf{1}_A\big) &= \mathbb{E}\big(\phi(Z_s)f(Z_t)H(Z;s,t)\mathbf{1}_A\big)\\ &+\mathbb{E}\big([\phi(Z_s); f(Z_t); H(Z;s,t)]_\pi \mathbf{1}_A\big) \quad (3.9)\end{aligned}$$

and the statement holds. □

3.1.3 Poisson kernel approach

We discuss here an alternative approximation of the Dirac delta, using the Poisson kernel, which allows to represent the solution of the heat equation. In the last years the first approach based on the Heavyside function was widely applied, studying also variance reduction techniques, localization and optimization methods (see again example Bally, Caramellino and Zanette [4]or Bouchard, Ekeland and Touzi [11]). On the other side, recently also a second approach has been introduced, by Malliavin and Thalmaier in [24], which allows to simplify the writing of the weights.

Let us come back to the relation (3.2), i.e.

$$\frac{\mathbb{E}(f(Z_t)\Lambda_2^0(Z_s-\alpha))}{\mathbb{E}(\Lambda_2^0(Z_s-\alpha)}$$

where, in general, $\Lambda_d^0(x)$, $x\in\mathbb{R}^d$, denotes the Dirac mass in the origin. Then, one can write

$$\Lambda_d^0=\Delta Q_d$$

Δ denotes the Laplace operator on $\mathbb{R}^d$ and Q_d stands for the Poisson kernel, that Q_d is such that the convolution $Q_d * f$ gives the solution to $\Delta u = f$ in $\mathbb{R}^d$, for any $f\in C_0^2$. Q_d has an explicit form:

$$Q_2(x) = \frac{1}{2\pi}\ln|x| \tag{3.10}$$

$$Q_d(x) = -c_d^{-1}|x|^{-(d-2)} \tag{3.11}$$

in which c_d denotes the area of the unit sphere in $\mathbb{R}^d$. Then,

$$\partial_i Q_d(x) = C_d\frac{x_i}{|x|^2} \tag{3.12}$$

with $C_2 = 1/(2\pi)$ and $C_d = c_d^{-1}(d-2)$ for $d\geq 3$.
Then, the idea is the following. Assuming $\sigma\neq 0$ (and then, a null border term operator), the result in Lemma 3.1.2 would give

$$\begin{aligned}\mathbb{E}(f(Z_t)\Lambda_2^0(Z_s-\alpha)) &= \sum_{i=1}^2\mathbb{E}(f(Z_t)\partial^2_{x_ix_i}Q_2(Z_s-\alpha))\\ &= \mathbb{E}\Big(f(Z_t)\sum_{i=1}^2\partial_{x_i}Q_2(Z_s-\alpha)H_i^1(Z;s,t)\Big)\end{aligned}$$

By using such an equality also for $f\equiv 1$, one would obtain

$$\mathbb{E}(f(Z_t)|Z_s=\alpha) = \frac{\sum_{i=1}^2\mathbb{E}(f(Z_t)\partial_{x_i}Q_2(Z_s-\alpha)H_i^1(Z;s,t))}{\sum_{i=1}^2\mathbb{E}(\partial_{x_i}Q_2(Z_s-\alpha)H_i^1(Z;s,t))} \tag{3.13}$$

where $\partial_{x_i}Q_2$ is given in (3.12).
Obviously, the above mentioned ideas should be formalized with the right hypothesis, and this could be actually done. Let us give here a discussion comparing with the previous results.
The main difference between formula (3.13) and the formula stated by Theorem 3.1.3 is that in the first we need only one integration by parts, instead of two. Therefore this approach brings to less complicated weights. Nevertheless, there are numerical problems in using (3.13). In fact, private numerical experiments of ours show that the variance of the Monte Carlo estimators is too high. But recent studies by Kohatsu-Higa and Yasuda [19]. could be helpful to go deep in the development of such an approach.

3.2 Examples

Propose here two examples, to give an idea of the structure of the weights defined by (3.6). The same machinery can be applied to the other models considered in the Sensitivity Analysis Chapter. But the calculus of the weights is very complicated.

3.2.1 The Black-Scholes-Merton model

Let $Z_t = (X_t, Y_t)$ defined by

$$\begin{aligned} X_t &= x + \int_0^t bX_r dr + \int_0^t \sigma X_r dW_r + \sum_{i=1}^{J_t} \Delta_i X_{T_i-} \\ Y_t &= \int_0^t X_r dr. \end{aligned}$$

with $\sigma \neq 0$. For the properties of this process and of its first variation one, see Section 2.5.1. We analyze the different directions of calculus, namely the Gaussian, jump amplitudes and the joint Gaussian and jump amplitudes direction.

Gaussian direction

We define the process $v = (v^1, v^2)$. Both v^1 and v^2 has only the first component non null and equal to

$$v^1_{0,u} = \frac{a_u}{\sigma x} \quad \text{and} \quad v^2_{0,u} = \frac{1}{\sigma x}$$

respectively, in which a is deterministic and such that

$$\int_0^s a_u \, du = \int_s^t a_u \, du = 0$$

Straightforward computations allow to write

$$\begin{aligned} \hat{H}_1^F(Z; s, t) &= \frac{1}{\sigma s} \int_0^s \frac{F}{X_s} \Big(\frac{sY_s - \int_0^s Y_r dr}{\int_0^s Y_r a_r dr} \, a_u + 1 \Big) dW_u \\ &- \frac{1}{\sigma(t-s)} \int_s^t \frac{F}{X_s} \Big(\frac{(t-s)Y_t - \int_s^t Y_r dr}{\int_s^t Y_r a_r dr} \, a_u + 1 \Big) dW_u \\ &+ \frac{1}{\sigma} \int_s^t \frac{F}{X_s} \Big(\frac{Y_t - Y_s}{\int_s^t Y_r a_r dr} \Big) a_u dW_u \end{aligned}$$

and

$$\hat{H}_2^F(Z;s,t) = - \frac{1}{\sigma}\int_0^s \frac{F}{\int_0^s Y_r a_r dr} a_u dW_u + \frac{1}{\sigma}\int_s^t \frac{F}{\int_s^t Y_r a_r dr} a_u dW_u$$

Jump Amplitudes direction

We choose here $v = (v^1, v^2)$ as follows: $v_0^1 = v_0^2 \equiv 0$ and

$$v^1 : \quad v_i^1 = \tfrac{1+\Delta_i}{x} a_i \quad i = 1, \dots, m, \quad v_i^1 \equiv 0 \quad \text{otherwise}$$
$$v^2 : \quad v_i^2 = \tfrac{1+\Delta_i}{x} \quad i = 1, \dots, m, \quad v_i^2 \equiv 0 \quad \text{otherwise}$$

where $a_1, \dots, a_n$ are deterministic and such that

$$\sum_{i=1}^n a_i = \sum_{i=n+1}^m a_i = 0.$$

Straightforward computations give

$$\begin{aligned}\hat{H}_1^F(Z;s,t) &= \sum_{i=1}^n \tilde{\delta}_i^n \Big(\frac{F(1+\Delta_i)}{nX_s} \big(\frac{nY_s - \sum_{j=1}^n Y_{T_j}}{\sum_{j=1}^n Y_{T_j} a_j} a_i + 1 \big) \Big) \\ &- \sum_{i=n+1}^m \tilde{\delta}_i^m \Big(\frac{F(1+\Delta_i)}{(m-n)X_s} \big(\frac{(m-n)Y_t - \sum_{j=n+1}^m Y_{T_j}}{\sum_{j=n+1}^m Y_{T_j} a_j} a_i + 1 \big) \Big) \\ &+ \sum_{i=n+1}^m \tilde{\delta}_i^m \Big(\frac{F(1+\Delta_i)}{X_s \sum_{j=n+1}^m Y_{T_j} a_j} \big(Y_t - Y_s \big) a_i \Big)\end{aligned}$$

and

$$\begin{aligned}\hat{H}_2^F(Z;s,t) = &- \sum_{i=1}^n \tilde{\delta}_i^n \Big(\frac{F(1+\Delta_i)}{\sum_{j=1}^n Y_{T_j} a_j} a_i \Big) \\ &+ \sum_{i=1}^n \tilde{\delta}_i^m \Big(\frac{F(1+\Delta_i)}{\sum_{j=n+1}^m Y_{T_j} a_j} a_i \Big)\end{aligned}$$

Joint Gaussian and Jump Amplitudes direction

We take here the process $v = (v^1, v^2)$ as

$$v^1 : \quad v_0^1 = \tfrac{1}{\sigma x} a_u \qquad\qquad v_i^1 \equiv 0 \quad \text{otherwise}$$
$$v^2 : \quad v_i^2 = \tfrac{1+\Delta_i}{x} \quad i = 1, \dots, m, \quad v_i^2 \equiv 0 \quad \text{otherwise}$$

in which again we ask that

$$\int_0^s a_u\, du = \int_s^t a_u\, du = 0.$$

Straightforward computations give

$$\begin{aligned}
\hat{H}_1^F(Z;s,t) &= \int_0^s \Big(\frac{F(nY_s - \sum_{j=1}^n Y_{T_j}}{n\sigma X_s \int_0^s Y_r a_r dr} a_u\Big) dW_u \\
&+ \sum_{i=1}^n \tilde{\delta}_i^n \Big(\frac{F(1+\Delta_i)}{nX_s}\Big) \\
&- \int_s^t \Big(\frac{F((m-n)Y_t - \sum_{j=n+1}^m Y_{T_j}}{(m-n)\sigma X_s \int_s^t Y_r a_r dr} a_u\Big) dW_u \\
&- \sum_{i=n+1}^m \tilde{\delta}_i^m \Big(\frac{F(1+\Delta_i)}{(m-n)X_s}\Big) \\
&+ \int_s^t \Big(\frac{F(Y_t - Y_s)a_u}{\sigma X_s \int_s^t Y_r a_r dr}\Big) dW_u
\end{aligned}$$

and

$$\begin{aligned}
\hat{H}_2^F(Z;s,t) = &- \int_0^s \Big(\frac{F}{\sigma \int_0^s Y_r a_r dr}\Big) a_u dW_u \\
&+ \int_s^t \Big(\frac{F}{\sigma \int_s^t Y_r a_r dr}\Big) a_u dW_u
\end{aligned}$$

3.2.2 Brownian motion with jumps

Consider now the following easy model

$$\begin{aligned}
X_t &= x + W_t + \sum_{i=1}^{J_t} \Delta_i \\
Y_t &= \int_0^t X_r dr. \qquad (3.14)
\end{aligned}$$

where W_t is a standard Brownian Motion. Such a model is studied with the purpose of writing explicitly the weight $H(Z;s,t)$ and really testing it in practice.

Obviously, here the first variation process is constant: $\xi_t = 1$. As a consequence, $\beta_t = \int_0^t \xi_r dr = t$. Moreover, one has $c(t,a,x) = a$, so that $\partial_i c_i = 1$.

Gaussian direction

Let $v = (v^1, v^2)$ be such that v_i^1 and v_i^2 are all null except for the component $i = 0$, and we set

$$v_{0,u}^1 = a_u \quad \text{and} \quad v_{0,u}^2 = 1$$

with

$$\int_0^s a_u\, du = \int_s^t a_u\, du = 0$$

In such a case, one has

$$\begin{aligned} \hat{H}_1^F(Z; s, t) &= \frac{1}{s}\Big(\frac{s^2}{2\int_0^s r a_r dr}\int_0^s F a_u dW_u + \int_0^s F dW_u\Big) \\ &- \frac{1}{(t-s)}\Big(\frac{(t-s)^2}{2\int_s^t r a_r dr}\int_s^t F a_u dW_u + \int_s^t F dW_u\Big) \end{aligned}$$

and

$$\begin{aligned} \hat{H}_2^F(Z; s, t) = &- \frac{1}{\int_0^s a_r dr}\int_0^s F a_u dW_u \\ &+ \frac{1}{\int_s^t a_r dr}\int_s^t F a_u dW_u. \end{aligned}$$

By Theorem 3.1.3, the final weight for the conditional expectation is given by

$$H(Z; s, t) = \hat{H}_1^{\hat{H}_2^1(Z_s)}(Z_s).$$

Let us now compute it.

First of all, notice that in the weight $\hat{H}_2^1(Z_s)$, defined by (3.6) with $F = 1$, the Skorohod integrals are Ito ones. As a matter of fact the arguments of these integrals are or a_u or 1. Therefore the only difficult is to handle the Skorohod integrals in the weight $\hat{H}_1^{\hat{H}_2^1(Z_s)}(Z_s)$. Thus, we can conclude giving an explicit expression, as follows

$$\begin{aligned}
\hat{H}(Z;s,t) &= \frac{s}{2\int_0^s ra_r dr}\Big(\frac{\int_0^s a_u^2 du - \big(\int_0^s a_r dW_r\big)^2}{\int_0^s ra_r dr} + \frac{\int_0^s a_r dW_r \int_s^t a_r dW_r}{\int_s^t ra_r dr}\Big) \\
&- \frac{W_s \int_0^s a_r dW_r}{s\int_0^s ra_r dr} + \frac{W_s \int_s^t a_r dW_r}{s\int_0^s ra_r dr} \\
&- \frac{t-s}{2\int_s^t ra_r dr}\Big(\frac{\int_s^t a_u^2 du - \big(\int_s^t a_r dW_r\big)^2}{\int_s^t ra_r dr} + \frac{\int_0^s a_r dW_r \int_s^t a_r dW_r}{\int_0^s ra_r dr}\Big) \\
&+ \frac{1}{t-s}\Big(\frac{(W_t - W_s)\int_0^s a_u dw_u}{\int_0^s ra_r dr} - \frac{(W_t - W_s)\int_s^t a_u dw_u}{\int_s^r a_r dr}\Big)
\end{aligned} \tag{3.15}$$

Jump amplitudes direction

We consider now the jump amplitudes direction of the Malliavin calculus. Firstly, we define the process $v = (v^1, v^2)$ as

$$\begin{aligned}
v^1 : \quad & v_i^1 = a_i \quad i = 1, \dots, m, \quad v_i^1 \equiv 0 \quad \text{otherwise} \\
v^2 : \quad & v_i^2 = 1 \quad i = 1, \dots, m, \quad v_i^2 \equiv 0 \quad \text{otherwise}
\end{aligned}$$

with

$$\sum_{i=1}^n a_i = \sum_{i=n+1}^m a_i = 0.$$

The final version for the weights $H_1^F(Z;s,t)$ and $H_2^F(Z;s,t)$ is the following:

$$\begin{aligned}
\hat{H}_1^F(Z;s,t) &= \frac{ns - \sum_{j=1}^n T_j}{n\sum_{j=1}^n T_j a_j} \sum_{i=1}^n \tilde{\delta}_i^n\big(Fa_i\big) + \frac{1}{n}\sum_{i=1}^n \tilde{\delta}_i^n\big(F\big) \\
&- \frac{(m-n)t - \sum_{j=n+1}^m T_j}{n\sum_{j=1} T_j a_j} \sum_{i=n+1}^m \tilde{\delta}_i^m\big(Fa_i\big) + \frac{1}{n}\sum_{i=n+1}^m \tilde{\delta}_i^m\big(F\big) \\
&+ \frac{t-s}{\sum_{j=1} T_j a_j}\tilde{\delta}_i^m\big(Fa_i\big)
\end{aligned}$$

and

$$\hat{H}_2^F(Z;s,t) = -\frac{\sum_{i=1}^n \tilde{\delta}_i^n\big(Fa_i\big)}{\sum_{j=1}^n T_j a_j} + \frac{\sum_{n+i=1}^m \tilde{\delta}_i^m\big(Fa_i\big)}{\sum_{j=n+1}^m T_j a_j}$$

By considering $F = 1$, and using the definition of the Skorohod integral (see 1.1.9), in the weight $H_2^1(Z;s,t)$ we quantities of the type

$$\sum_{i=1}^n \tilde{\delta}_i^n\big(a_i\big) = -\sum_{i=1}^n a_i \partial_{\Delta_i} \log g(\Delta)$$

where $g(\Delta_i)$ is the density of Δ_i (see Definition 1.1.1). The same holds in the time interval $[s,t]$, considering the jumps between $n+1$ and m. Therefore, proceeding similarly as in the Gaussian direction, we obtain the following explicit expression

$$\hat{H}(Z;s,t) = \hat{H}_1^{\hat{H}_2^1(Z;s,t)}(Z;s,t)$$

$$= \frac{C_n}{n}\Big(\frac{-\big(\mathcal{S}_n^{a\chi}\big)^2 - \sum_{k=1}^n \tilde{D}_k^n\big(\mathcal{S}_n^{a\chi}\big)a_k}{\sum_{k=1}^n T_k a_k} + \frac{\big(\mathcal{S}_m^{a\chi} - \mathcal{S}_n^{a\chi}\big)\mathcal{S}_n^{a\chi}}{\sum_{k=n+1}^m T_k a_k}\Big)$$

$$+\frac{1}{n}\Big(\frac{-\mathcal{S}_n^{a\chi}\mathcal{S}_n^{\chi} - \sum_{k=1}^n \tilde{D}_k^n\big(\mathcal{S}_n^{a\chi}\big)}{\sum_{k=1}^n T_k a_k} + \frac{\big(\mathcal{S}_m^{a\chi} - \mathcal{S}_n^{a\chi}\big)\mathcal{S}_n^{\chi}}{\sum_{k=n+1}^m T_k a_k}\Big)$$

$$+\frac{C_t}{n}\Big(\frac{-\mathcal{S}_n^{a\chi}\big(\mathcal{S}_m^{a\chi} - \mathcal{S}_n^{a\chi}\big)}{\sum_{k=1}^n T_k a_k} + \frac{\big(\mathcal{S}_m^{a\chi} - \mathcal{S}_n^{a\chi}\big)^2 + \sum_{k=n+1}^m \tilde{D}_k^n\big(\mathcal{S}_m^{a\chi} - \mathcal{S}_n^{a\chi}\big)a_k}{\sum_{k=n+1}^m T_k a_k}\Big)$$

$$+\frac{1}{n}\Big(\frac{-\mathcal{S}_n^{a\chi}\big(\mathcal{S}_m^{\chi} - \mathcal{S}_n^{\chi}\big)}{\sum_{k=1}^n T_k a_k} + \frac{\big(\mathcal{S}_m^{a\chi} - \mathcal{S}_n^{a\chi}\big)\big(\mathcal{S}_m^{\chi} - \mathcal{S}_n^{\chi}\big) + \sum_{k=n+1}^m \tilde{D}_k^n\big(\mathcal{S}_m^{a\chi} - \mathcal{S}_n^{a\chi}\big)}{\sum_{k=n+1}^m T_k a_k}\Big)$$

where, for simplicity, we have used the following notations:

$$\begin{aligned} C_s &= \frac{ns - \sum_{i=1}^n T_i}{\sum_{i=1}^n T_i a_i} \\ C_t &= \frac{(2n-m)t - ns - \sum_{i=n+1}^m T_i}{\sum_{i=1}^n T_i a_i} \\ \chi_i &= \partial_{\Delta_i} \log g(\Delta) \quad \text{for } i = 1, ..., m \\ \mathcal{S}_n^{\chi} &= \sum_{k=1}^n \chi_k \\ \mathcal{S}_n^{a\chi} &= \sum_{k=1}^n a_k \chi_k \end{aligned} \tag{3.16}$$

Joint Gaussian and jump amplitudes direction

We define here the process v as

$$\begin{aligned} &v^1: \ v_0^1 = a_u && v_i^1 \equiv 0 \quad \text{otherwise} \\ &v^2: \ v_i^2 = 1 \quad i = 1, \ldots, m, && v_i^2 \equiv 0 \quad \text{otherwise} \end{aligned}$$

again with

$$\int_0^s a_u\, du = \int_s^T a_u\, du = 0.$$

The weights $H_1^F(Z;s,t)$ and $H_2^F(Z;s,t)$ are as follows

$$\begin{aligned}\hat{H}_1^F(Z;s,t) &= \frac{ns-\sum_{j=1}^{n}T_j}{n\int_0^s ra_r dr}\int_0^s Fa_u dW_u + \frac{1}{n}\sum_{i=1}^{n}\tilde{\delta}_i^n(F)\\ &- \frac{(m-n)s-\sum_{k=n+1}^{m}T_j}{(m-n)\int_s^t ra_r dr}\int_s^t Fa_u dW_u + \frac{1}{m-n}\sum_{i=n+1}^{m}\tilde{\delta}_i^m(F)\end{aligned}$$

and

$$\hat{H}_2^F(Z;s,t) = -\frac{1}{\int_0^s ra_r dr}\int_0^s Fa_u dW_u + \frac{1}{\int_s^t ra_r dr}\int_s^t Fa_u dW_u.$$

Notice that in this particular case, $\hat{H}_2^F(Z;s,t)$ does not depend on Δ_i as $i = 1,..,m$, simplifying the differential calculus in the jump amplitudes direction. Thus, we have that

$$\begin{aligned}\hat{H}(Z;s,t) &= \frac{\sum_{k=1}^{n}\chi_k}{n}\Big(\frac{\int_0^s a_r dW_r}{\int_0^s ra_r dr} - \frac{\int_s^t a_r dW_r}{\int_s^t ra_r dr}\Big)\\ &+ \frac{ns-\sum_{j=1}^{n}T_j}{n\int_0^s ra_r dr}\Big(\frac{\int_0^s a_r^2 dr - \big(\int_0^s a_r dW_r\big)^2}{\int_0^s ra_r dr} + \frac{\int_0^s a_r dW_r \int_s^t a_r dW_r}{\int_s^t ra_r dr}\Big)\\ &+ \frac{\sum_{k=n+1}^{m}\chi_k}{m-n}\Big(\frac{\int_0^s a_r dW_r}{\int_0^s ra_r dr} - \frac{\int_s^t a_r dW_r}{\int_s^t ra_r dr}\Big)\\ &- \frac{(m-n)s-\sum_{j=n+1}^{m}T_j}{(m-n)\int_s^t ra_r dr}\Big(\frac{\big(\int_s^t a_r dW_r\big)^2 - \int_s^t a_r^2 dr}{\int_s^t ra_r dr} - \frac{\int_0^s a_r dW_r \int_s^t a_r dW_r}{\int_0^s ra_r dr}\Big)\end{aligned} \tag{3.17}$$

where $\chi_i = \partial_{\Delta_i}\log g(\Delta)$ for $i = 1,...,m$.

3.3 Numerical experiments

We present in this Section some numerical experiments related to the Theory developed in the Representation formulas of the Conditional expectation Chapter. We consider the Brownian motion with jumps, that is the simple model (3.14), which has been introduced actually with the purpose of numerical applications.

This Chapter is divided into two parts. In the first one we present a Malliavin localization method, based on the idea to reduce the variance of the Heavyside function with an exponential function (see Bally, Caramellino and Zanette [4]). In the second, we show the numeric results, comparing the Malliavin method with the localized one and the explicit simulation of the trajectory of the process Z.

3.3.1 Localization methods

We consider a variance reduction method introduced in Bally, Caramellino and Zanette [4] in a one-dimensional framework for the pricing of the American option.
Let $\Lambda^0(y_1, y_2) = \mathbf{1}_{y_1>0}\mathbf{1}_{y_2>0}$ be the Heavyside function on $\mathbb{R}^2$. Remembering Formula 3.2, we have to deal with he following expectation

$$\mathbb{E}\big(\varphi(Z_t)\Lambda^0(Z_s - \alpha)\big) \tag{3.18}$$

where $\varphi(Z_t) = f(Z_t)$ or $\varphi(Z_t) = 1$. Consider the following function:

$$\psi(z) = \psi(x, y) = \frac{\mu^2}{4} e^{-\mu(|x|+|y|)}$$

which a probability density function on $\mathbb{R}^2$, and we denote with Ψ the associated distribution function, i.e.

$$\Psi(z) = \Psi(x, y) = \frac{\lambda^2}{4} \int_{-\infty}^{x} e^{-\mu|u|} du \cdot \int_{-\infty}^{y} e^{-\mu|v|} dv.$$

(3.18) can be then written as

$$\mathbb{E}\Big(\varphi(Z_t)\psi(Z_s - \alpha)\Big) + \mathbb{E}\Big(\varphi(Z_t)\big(\tilde{H} - \Psi\big)(Z_s - \alpha)H(Z; s, t)\Big) \tag{3.19}$$

where $\tilde{H}$ is the Heavyside function and $H(Z; s, t)$ is the weight given by the Theorem 3.1.3. Therefore in the first term of (3.19) the Heavyside function does not appear, whilst in the second, the presence of Ψ allows to get a considerable reduction of variance of the Heavyside function, as we will show in the next numerical applications.
In the numerical experiments we have chosen $\mu = t - s$, how suggested by Bally, Caramellino and Zanette [4], which we refer for more details.

3.3.2 Brownian motion with jumps

We consider now the Brownian motion with jumps model as in (3.14), with the following choice of parameters: $s = 1$, $t = 2$ and we have supposed the jump amplitudes to be normal distributed, that is, $\Delta_i \overset{=}{\mathscr{L}} \mathcal{N}$, with $\mathcal{N}$ standard normal r.v. The intensity parameter is set as $\lambda = 25$, meaning that we have in mean 25 jumps both on $[0, s]$ and $[s, t]$. Moreover we have set $X_0 = 100$.
We have considered a grid for some possible values of X_s and Y_s, built from numerical simulation of the process in the time interval $[0, s]$. In particular the grid considers values for X_s and Y_s in $[99, 101]$, with step of 0.1.
We have performed $10,000$ simulations for each point in the grid, dividing the time intervals $[0, s]$ and $[s, t]$ in 100 steps in order to simulate the integral

component Y. In fact, we have tested the conditional expectation in (3.1) with the function

$$F(Z_t) = (X_t - \frac{1}{t}Y_t)_+$$

In order to have an exact result with which comparison with the Malliavin method can be done, we consider first the model with no jumps, that is

$$\begin{aligned} X_t &= x + W_t \\ Y_t &= \int_0^t X_r dr. \end{aligned} \tag{3.20}$$

Here, we can write the following exact formula:

$$\mathbb{E}\Big((X_t - \frac{Y_t}{t})_+ | X_s = \alpha_1, Y_s = \alpha_2\Big) = m\Phi(\frac{m}{v}) + \frac{1}{2\pi}e^{-\frac{m^2}{2v^2}}$$

being Φ the distribution function of a standard normal distribution, and

$$m = \frac{s\alpha_1 - \alpha_2}{t}, \qquad v^2 = \frac{t-s}{3t}\Big(3s + (t-s)^2\Big).$$

Figure 3.1 shows a comparison of localized Malliavin method in the Gaussian direction with the exact value and the "trajectory simulated" method on the grid, that is, the standard computation of the conditional expectation by Monte Carlo through the simulation of the process starting at time s.
For the sake of clearness, we fix $X_s = 100$ and $Y_s = 100$ to obtain the 2-dimensional sections of the surfaces in Figure 3.1, as shown in Figure 3.2 and 3.3. There, we can see how the localized Malliavin method gives results much better than the trajectory simulated.
Notice that without localization, the Malliavin technique are really not satisfactory, as well known in the literature on this topic. As an example, we propose Figure 3.4, which clearly shows this fact.
Let us consider now also the jump part, i.e. the model (3.14). Here, there is not a closed form solution, because the presence of the jumps makes impossible to calculate explicitly the conditional expectation. Thus, we can compare the (localized) Malliavin method with the trajectory simulated. We do not consider the simple Malliavin method, for the reason already explained.
Next Figure 3.5 summarize the results in all the directions of the Malliavin calculus (i.e. with respect to the Brownian motion, to the jump amplitudes and the joint one).
As in the previous case, the numerical experiments show that all the directions of the differential calculus give good results. In order to point out the numerical result for each direction of the calculus, we consider them separately. In Figure 3.3.2 and 3.7 we present the sections of the surfaces given in Figure 3.5, fixing $X_s = 100$ and $Y_s = 100$ respectively. As a consequence, it has to be stressed that the Malliavin approach with localization gives a sensible stability to the results.

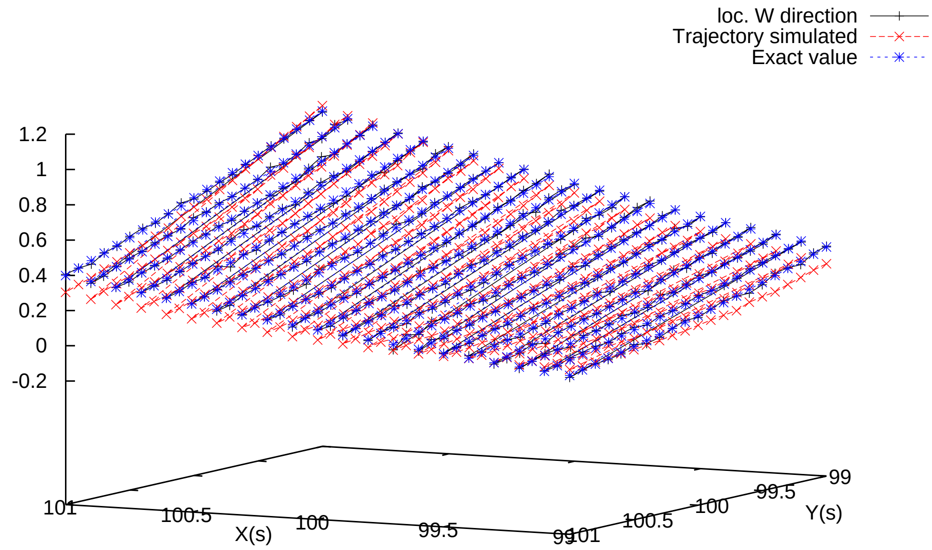

Figure 3.1: Conditional expectation for a Brownian motion without jumps.

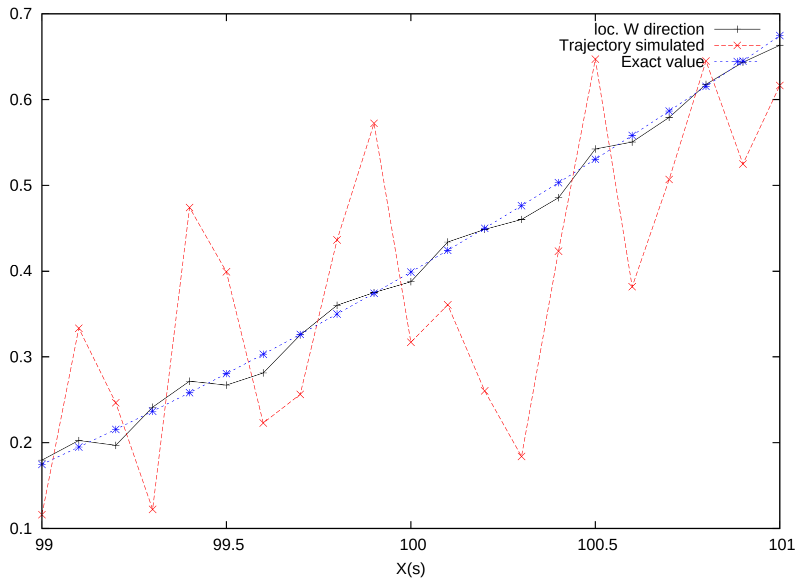

Figure 3.2: Brownian motion without jumps. Conditional expectation as a function of X_s [with $Y_s = 100$].

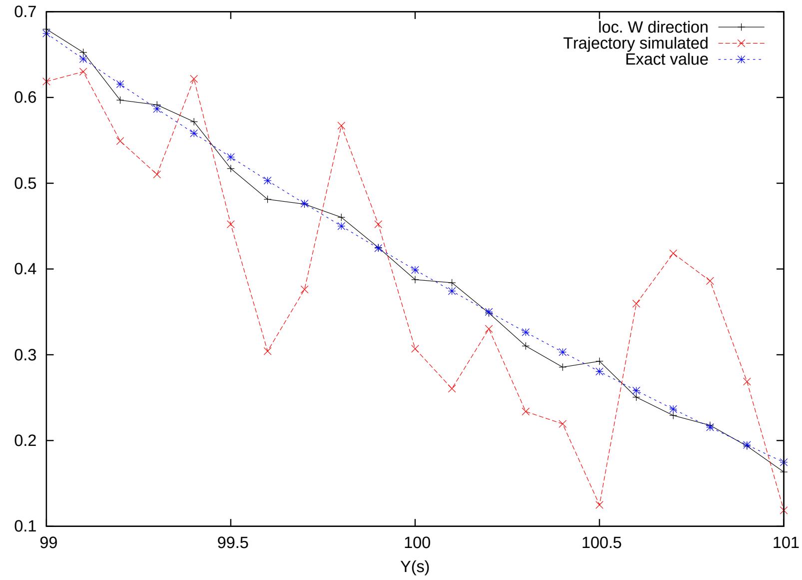

Figure 3.3: Brownian motion without jumps. Conditional expectation as a function of Y_s [with $X_s = 100$].

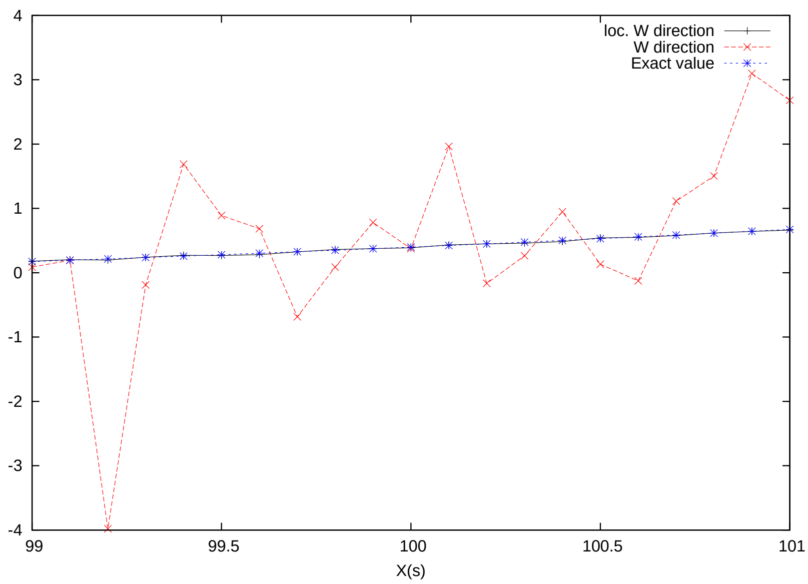

Figure 3.4: Brownian motion without jumps. Conditional expectation as a function of X_s [with $Y_s = 100$]. Comparison between the localized and the non localized formulas.

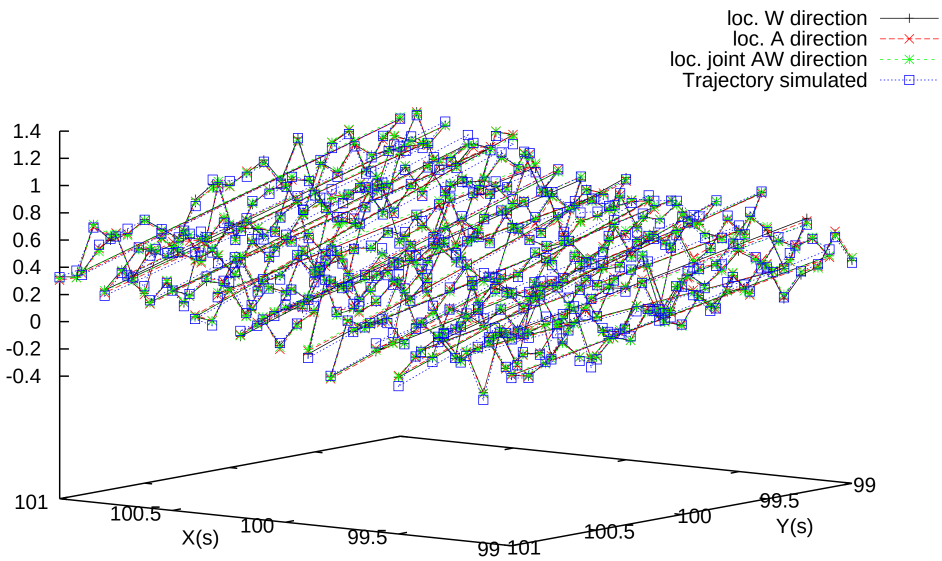

Figure 3.5: Conditional expectation for a Brownian motion with jumps. All directions.

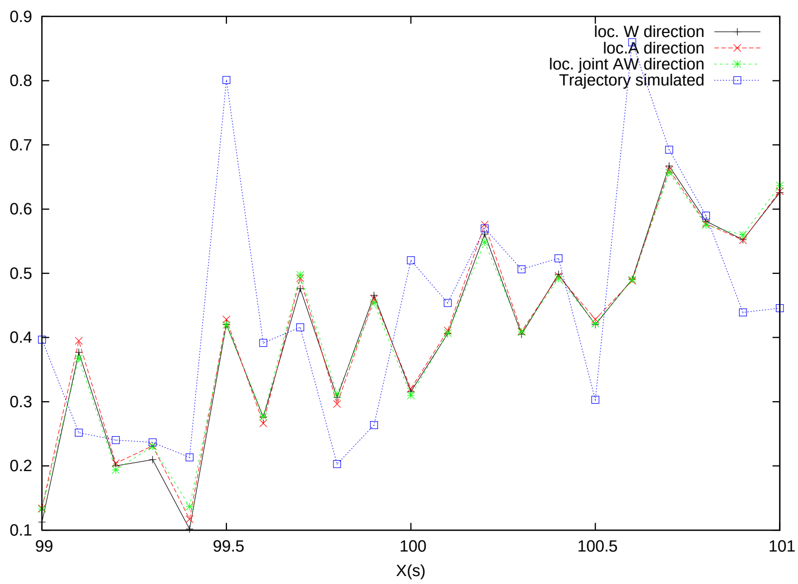

Figure 3.6: Conditional expectation for a Brownian motion with jumps as a function of X_s [with $Y_s = 100$].

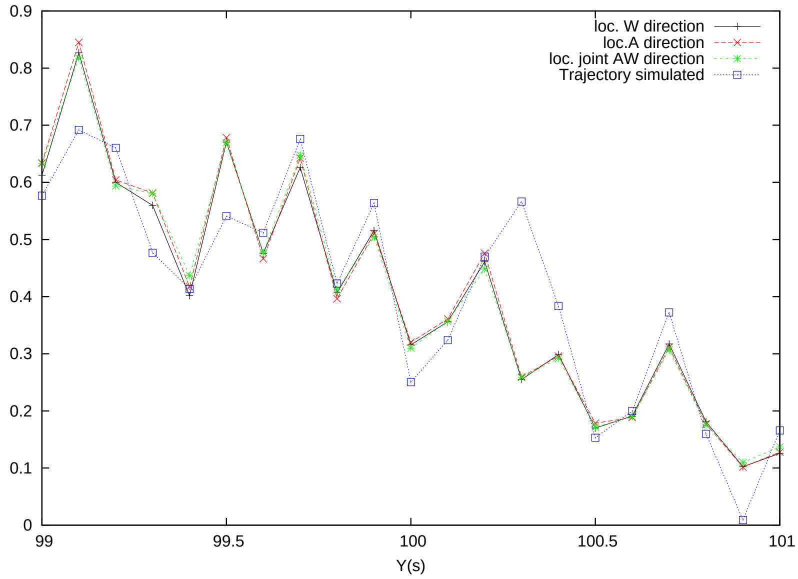

Figure 3.7: Conditional expectation for a Brownian motion with jumps as a function of Y_s [with $X_s = 100$].

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