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Generalized juggling patterns, quiver Grassmannians and affine flag varieties

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Abstract

The goal of this paper is to clarify the connection between certain structures from the theory of totally nonnegative Grassmannians, quiver Grassmannians for cyclic quivers and the theory of local models of Shimura varieties. More precisely, we generalize the construction from our previous paper relating the combinatorics and geometry of quiver Grassmannians to that of the totally nonnegative Grassmannians. The varieties we are interested in serve as realizations of local models of Shimura varieties. We exploit quiver representation techniques to study the quiver Grassmannians of interest and, in particular, to describe explicitly embeddings into affine flag varieties which allow us to realize our quiver Grassmannians as a union of Schubert varieties therein.

Keywords Quiver Grassmannians · Totally nonnegative Grassmannians · Affine flag varieties

1 Introduction

Quiver Grassmannians are natural generalizations of the classical Grassmannians and flag varieties. In short, given a quiver Q, a Q-representation M and a dimension vector $\mathbf{e}$ one considers the variety $\mathrm{Gr}_{\mathbf{e}}(M)$ consisting of $\mathbf{e}$-dimensional subrepresentations of M [2, 4, 39]. Quiver Grassmannians were extensively studied during the last two decades and proved to be useful in various areas of mathematics [5, 8, 35]. In this paper we consider certain quiver Grassmannians $X(k, n, \omega)$ for cyclic quivers, generalizing [12]. The varieties $X(k, n, \omega)$ naturally show up in the theory of local models of Shimura varieties [14, 16, 28] and in

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the theory of totally nonnegative Grassmannians [12, 18, 24–26, 31]. We give some details below.

Let Δ_n be a cyclic quiver on n vertices. For a positive integer ω we consider a Δ_n module $U_{n\omega}$ defined as follows. Let A_∞ be an infinite (in both directions) equioriented quiver of type A. Let us consider a natural $\mod n$ projection from A_∞ to Δ_n. Let $V(i, j)$, $i \leq j$, be the indecomposable representation of A_∞ supported on vertices from i to j. Then the Δ_n module $U_{n\omega}$ is obtained as the projection of the direct sum of A_∞ modules $V(i+1, i+n\omega)$ for $i = 0, \dots, n-1$. In particular, the dimension of $U_{n\omega}$ is equal to $(n\omega, \dots, n\omega)$.

Now let us fix $k = 1, \dots, n-1$. We define $X(k, n, \omega)$ to be the Δ_n quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(U_{n\omega})$, where $\mathbf{e} = (k\omega, \dots, k\omega)$. The $\omega = 1$ case was considered in our previous paper [12]. In particular, we showed that the topological and combinatorial properties of $X(k, n, 1)$ are closely related to that of the totally nonnegative Grassmannians. The following theorem holds.

Theorem 1 *The variety $X(k, n, \omega)$ is a projective equidimensional variety of dimension $\omega k(n-k)$. The number of irreducible components is $\binom{n}{k}$. Each component is normal, Cohen-Macaulay, has rational singularities and admits a desingularization by a certain smooth quiver Grassmannian.*

Recall that in [12] a link between the varieties $X(k, n, 1)$ and the totally nonnegative Grassmannians was described. More precisely, we proved that $X(k, n, 1)$ admits a Bialynicki-Birula decomposition [1], which is also a cellular decomposition. The poset of cells was identified with the (reversed) cell poset of the corresponding totally nonnegative Grassmannian. We prove the following generalization.

Theorem 2 *The varieties $X(k, n, \omega)$ admit a Bialynicki-Birula decomposition with each stratum being an affine cell. Each cell contains a unique fixed point under an appropriate algebraic torus action and the cell is the orbit of this point under the action of the automorphism group of the Δ_n module $U_{n\omega}$. The cells are labeled by a natural ω-generalization of the bounded affine permutations.*

The cellular decomposition is stable under the action of the above mentioned algebraic torus. We describe the moment graph resulting from the torus action on $X(k, n, \omega)$, and investigate the poset structure on the set of cells.

The last part of the paper is devoted to the realization of the quiver Grassmannians $X(k, n, \omega)$ inside the affine Grassmannian of type A. The varieties $X(k, n, \omega)$ show up in the literature as an explicit realization of the local models of Shimura varieties (see [14, 15, 27]). Certain properties of these local models have been studied. In particular, the embeddings into the affine Grassmannians were constructed. We use the techniques of quiver representations in order to prove the following.

Theorem 3 *The varieties $X(k, n, \omega)$ can be identified with the union of certain $\binom{n}{k}$-many Schubert subvarieties inside the type A affine flag variety. The Weyl group elements corresponding to these Schubert varieties are explicitly described. The action of the automorphism group $Aut_{\Delta_n}(U_{n\omega})$ is identified with the action of the Iwahori subgroup.*

The statements from the theorem above are proved in Proposition 6.10, Corollary 6.12 and Theorem 6.14. The link between the quiver Grassmannians and affine Schubert varieties is established via the formalism of semi-infinite wedge spaces and Sato Grassmannians.

Finally, let us formulate a natural question which remains open. For all $\omega \geq 1$ construct spaces $\mathrm{Gr}(k, n, \omega)_{\geq 0}$ which generalize the totally nonnegative Grassmannians $\mathrm{Gr}(k, n)_{\geq 0}$, in the sense that $\mathrm{Gr}(k,n,\omega)_{\geq 0}$ admits a decomposition into cells of the form $\mathbb{R}^M_{>0}$ such that the cell poset is dual to that of $X(k,n,\omega)$. (recall that the latter is described in terms of the generalized bounded affine permutations).

The paper is organized as follows. In Sect. 2 we study the geometric properties of the quiver Grassmannians $X(k,n,\omega)$; Theorem 1 is proved here. In Sect. 3 we describe the moment graphs of $X(k,n,\omega)$ and discuss their cohomology; in particular, we construct a cyclic group action on the equivariant cohomology. Theorem 2 is proved in Sect. 4. We introduce the ω-generalized versions of the bounded affine permutations, juggling patterns and provide a combinatorial model for the Poincaré polynomials. Theorem 3 is proved in Sect. 6 and the preliminaries on the Sato Grassmannians and affine flag varieties can be found in Sect. 5. In the appendix we provide a correction of a computation from [12].

2 Geometric properties

2.1 Quiver Grassmannians

In this section we recall the definition of quiver Grassmannians. For more detail on the representation theory of quivers see [38]. A finite quiver Q consists of a finite set of vertices Q_0, a finite set of oriented edges Q_1 between the vertices. A Q-representation M is a pair of tuples, with a tuple $(M^{(i)})_{i\in Q_0}$ of $\mathbb{C}$-vector spaces over the vertices, and a tuple $(M_a)_{a\in Q_1}$ containing linear maps between the vector spaces, along the arrows in Q_1.

A morphism ψ of Q-representations M and N is a collection of linear maps $\psi_i : M^{(i)} \to N^{(i)}$ such that $\psi_j \circ M_a = N_a \circ \psi_i$ holds for all edges $a : i \to j$. The set of all Q-morphisms from M to N is denoted by $\mathrm{Hom}_Q(M,N)$. The category of finite dimensional complex Q-representations is $\mathrm{rep}_{\mathbb{C}}(Q)$.

The entries of the dimension vector $\mathbf{dim}\, M \in \mathbb{Z}^{Q_0}$ of a quiver representation $M \in \mathrm{rep}_{\mathbb{C}}(Q)$ are given by $\dim_{\mathbb{C}} M^{(i)}$ for all $i \in Q_0$. A subrepresentation $N \subseteq M$ is parameterized by a tuple of vector subspaces $N^{(i)} \subset M^{(i)}$, such that $M_a(N^{(i)}) \subseteq N^{(j)}$ holds for all arrows $a : i \to j$ of Q.

Definition 2.1 For $\mathbf{e} \in \mathbb{Z}^{Q_0}$ and $M \in \mathrm{rep}_{\mathbb{C}}(Q)$, the **quiver Grassmannian** $\mathrm{Gr}_{\mathbf{e}}(M)$ is the variety of all $\mathbf{e}$-dimensional subrepresentations of M.

For a point $U \in \mathrm{Gr}_{\mathbf{e}}(M)$ the isomorphism class $\mathcal{S}_U$ in the quiver Grassmannian is called stratum and is irreducible (cf. [5, Lemma 2.4]). The automorphism group $\mathrm{Aut}_Q(M) \subset \mathrm{End}_Q(M) = \mathrm{Hom}_Q(M,M)$ acts on $\mathrm{Gr}_{\mathbf{e}}(M)$ as

$$A.\big(U^{(i)}\big)_{i\in Q_0} := \Big(A_i\big(U^{(i)}\big)\Big)_{i\in Q_0} \quad \text{for } A \in \mathrm{Aut}_Q(M) \text{ and } U \in \mathrm{Gr}_{\mathbf{e}}(M).$$

2.2 Cyclic quiver Grassmannians

The equioriented cycle Δ_n is the quiver with vertex set $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ and arrows $a : i \to i+1$ for all $i \in \mathbb{Z}_n$. For a Δ_n-representation we write M_i instead of M_a for the map along the arrow $a : i \to i+1$. Now we define the Δ_n-representation U_m, for $m \geq 2$: Take the vector spaces $M^{(i)} := \mathbb{C}^m$ for all $i \in \mathbb{Z}_n$ and let $B^{(i)} := \{v^{(i)}_j : j \in [m]\}$ be the standard basis of the i-th copy of $\mathbb{C}^m$. Then each map M_i sends $v^{(i)}_j$ to $v^{(i+1)}_{j+1}$ for $j \in [m-1]$ and $v^{(i)}_m$ to zero.

Definition 2.2 Let us fix numbers $k, \omega \geq 1$ with $k \leq n$. Define the quiver Grassmannian

$$X(k, n, \omega) := \mathrm{Gr}_{(k\omega,\dots,k\omega)}(U_{\omega n}).$$

In particular, $X(k, n, 1)$ coincides with the quiver Grassmannian $X(k, n)$ studied in our paper [12].

Remark 2.3 We note that the varieties $X(k, n, \omega)$ show up in the theory of local models of Shimura varieties (see [15, 16, 27–30]). More precisely, $X(k, n, \omega)$ appear as concrete realizations of the local models of Shimura varieties for $G = GL_n$ and minuscule coweights, as discussed, for example, in [28, §7.1].

Remark 2.4 One may vary the representation $U_{\omega n}$ keeping its dimension unchanged. Then one gets a family of quiver Grassmannians in the spirit of [7, 9]. It would be interesting to study this family.

2.3 Torus actions

The torus $\mathbb{C}^*$ acts on the vector spaces of $U_{\omega n}$ with the weights $\mathrm{wt}(v_j^{(i)}) := j$ for all $i \in \mathbb{Z}_n$ and $j \in [\omega n]$. This action extends to $X(k, n, \omega)$ by [3, Lemma 1.1].

The above $\mathbb{C}^*$ action coincides with a cocharacter of an $n+1$-dimensional algebraic torus $T := (\mathbb{C}^*)^{n+1}$ which acts on the vector spaces of $U_{\omega n}$ via

$$\gamma.v_j^{(i)} = \gamma_0^{j-1}\gamma_{i-j+1}v_j^{(i)} \quad \text{for } \big(\gamma_0, (\gamma_i)_{i\in\mathbb{Z}_n}\big) \in T.$$

Remark 2.5 This coincides with the torus action on Δ_n-representations as defined in [22]. Hence it extends to $X(k, n, \omega)$ by [22, Lemma 5.12].

Lemma 2.6 *The fixed points of the $\mathbb{C}^*$ action and T action on $X(k, n, \omega)$ coincide, and the number of fixed points is finite.*

Proof The first part is a special case of [22, Theorem 5.14] and the second part follows from [3, Theorem 1]. □

Now we want to introduce an explicit parametrization of the T-fixed points of $X(k, n, \omega)$. For $k \leq n$ we denote by $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. The following definition generalizes the standard definition of juggling patterns (see [21] or Sect. 4.1).

Definition 2.7 For $k, n, \omega \in \mathbb{N}$ with $k \leq n$, the set of (k, n, ω) juggling patterns is

$$\mathcal{J}ug(k, n, \omega) := \Big\{(J_i)_{i\in\mathbb{Z}_n} \in \prod_{i\in\mathbb{Z}_n}\binom{[\omega n]}{k\omega} : \tau_1(J_i\setminus\{\omega n\}) \subset J_{i+1} \text{ for all } i \in \mathbb{Z}_n\Big\},$$

where $\tau_1(x) = x + 1$.

Lemma 2.8 *The fixed points $X(k, n, \omega)^T$ are in bijection with $\mathcal{J}ug(k, n, \omega)$.*

Proof It follows from [3, Theorem 1] that the vector spaces parameterizing the fixed points of $X(k, n, \omega)$ are spanned by subsets $P^{(i)} \subset B^{(i)}$ for $i \in \mathbb{Z}_n$, where each subset has cardinality $k\omega$. This is encoded with the index sets from $\binom{[\omega n]}{k\omega}$. Hence the condition

$$M_i\Big(\big\langle v : v \in P^{(i)}\big\rangle\Big) \subset \big\langle w : w \in P^{(i+1)}\big\rangle$$

translates to $\tau_1(J_i\setminus\{\omega n\}) \subset J_{i+1}$ where $P^{(i)} = \{v_j^{(i)} : j \in J_i\}$. □

2.4 Geometric properties

In order to apply the desired results concerning the geometry of quiver Grassmannians for Δ_n-representations, we need an alternative realization of U_m. By A_∞ we denote the infinite equioriented quiver of type A. Let $V(i, j)$ be the indecomposable A_∞-representation with vector spaces $V(i, j)^{(k)} = \mathbb{C}$ for any $k \in [i, j]$ and maps $V(i, j)_k = \mathrm{id}_\mathbb{C}$ for any $k \in [i, j-1]$. All other maps and vector spaces are zero. Let $F : A_\infty \to \Delta_n$ send k to $k \mod n$, and $(a : k \to k+1)$ to $(\overline{a} : k \mod n \to k+1 \mod n)$. This induces the Δ_n-representation $U_i(\ell)$, with vector spaces $U_i(\ell)^{(j)} := \bigoplus_{k\in F^{-1}(j)} V(i+1, i+\ell)^{(k)}$ for any $j \in \mathbb{Z}_n$ and obvious linear maps.

Proposition 2.9 *There is an isomorphism of Δ_n-representations:*

$$U_m \cong \bigoplus_{i\in\mathbb{Z}_n} U_i(m).$$

Proof The representation U_m decomposes into the direct sum of n indecomposable summands labeled by elements $i \in \mathbb{Z}_n$: the i-th summand contains the vector $v_1^{(i)}$. Now one easily sees that the i-th summand is isomorphic to $U_{i-1}(m)$. □

Proposition 2.10 *The elements of the automorphism group $Aut_{\Delta_n}(U_m)$ are exactly the matrix tuples $A = (A_i)_{i\in\mathbb{Z}_n}$ with*

$$A_i = \begin{pmatrix} a_{1,1}^{(i)} & & & & \\ a_{2,1}^{(i)} & a_{1,1}^{(i-1)} & & & \\ \vdots & \vdots & \ddots & & \\ a_{m-1,1}^{(i)} & a_{m-2,1}^{(i-1)} & \dots & a_{1,1}^{(i-m+2)} & \\ a_{m,1}^{(i)} & a_{m-1,1}^{(i-1)} & \dots & a_{2,1}^{(i-m-2)} & a_{1,1}^{(i-m+1)} \end{pmatrix}$$

where $a_{k,1}^{(i)} \in \mathbb{C}$ for all $i \in \mathbb{Z}_n$, $k \in [2, m]$ and $a_{1,1}^{(i)} \in \mathbb{C}^*$ for all $i \in \mathbb{Z}_n$. In particular, $\dim_\mathbb{C} Aut_{\Delta_n}(U_m) = mn$.

Proof We begin by computing the endomorphism algebra $\mathrm{End}_{\Delta_n}(U_m)$. By definition we have that $(E_i)_{i\in\mathbb{Z}_n} \in \mathrm{End}_{\Delta_n}(U_m)$ if and only if

$$E_{i+1}\tau_1 = \tau_1 E_i \quad \text{for all } i \in \mathbb{Z}_n.$$

This is the same as

$$E_{i+1}\tau_1(v_l^{(i)}) = \tau_1 E_i(v_l^{(i)}) \quad \text{for all } i \in \mathbb{Z}_n, l \in [m]. \tag{2.1}$$

We write $e_{k,l}^{(i)} := (E_i)_{k,l}$ for the matrix entries, so that

$$E_i(v_l^{(i)}) = \sum_{k=1}^{m} e_{k,l}^{(i)} v_k^{(i)}.$$

Then the equations (2.1) are equivalent to

$$e_{k,l}^{(i)} = e_{k+1,l+1}^{(i+1)}, \quad e_{k,m}^{(i)} = 0, \quad e_{m,l}^{(i)} = 0, \qquad \text{for all } k, l \in [m-1].$$

From the above equations, it follows by induction on $m-l$ that $e_{k,l}^{(i)}=0$ for any $l>k$, and by induction on l that $e_{k,l}^{(i)}=e_{k+1,l+1}^{(i+1)}$. This implies that the E_i's are of the claimed lower triangle form. Now we obtain the automorphism group by imposing the additional condition that all diagonal entries are invertible. □

The $\mathbb{C}^*$ action on $X := X(k,n,\omega)$ induces the decomposition:

$$X=\bigcup_{p\in X^{\mathbb{C}^*}} W_p, \qquad \text{with} \qquad W_p := \left\{x\in X : \lim_{z\to 0} z.x = p\right\}.$$

We call this a BB-decomposition since decompositions of this type were first studied by Bialynicki-Birula in [1].

Theorem 2.11 *For $\omega \geq 1$ and $k\leq n$ the variety $X(k,n,\omega)$ satisfies the following:*

(i) *it is a projective variety of dimension $\omega k(n-k)$;*
(ii) *its irreducible components are equidimensional;*
(iii) *the BB-decomposition is a cellular decomposition;*
(iv) *the irreducible components are normal, Cohen-Macaulay and have rational singularities;*
(v) *the irreducible components $X_I(k,n,\omega)$ are labeled by the k-element subsets $I\subset[n]$.*

Proof Part (i) is a special case of [33, Lemma 4.9]. Part (ii) and the labeling of the irreducible components as in part (v), are obtained from [33, Lemma 4.10]. The defining weight function of the $\mathbb{C}^*$ action introduced above coincides with the one in [33, Section 4.4]. Hence [33, Theorem 4.13] implies that the parts of the BB-decomposition are cells and thus item (iii) holds. [33, Lemma 4.12] implies item (iv). □

Remark 2.12 Using the methods from [33], it is also possible to study $\mathrm{Gr}_{(q,\dots,q)}(U_m)$, where the Δ_n-representation U_m is defined as above. But in this setting it is only possible to prove part (iii) and (iv) of the above theorem. For part (i), (ii) and (v) it is crucial that n divides m and that m/n divides q. Otherwise, the irreducible components are not of the same dimension, their parametrization is unknown and hence there is no dimension formula.

Theorem 2.13 *Each cell $C\subset X(k,n,\omega)$ is T-stable and contains exactly one T-fixed point P_C. The $\mathrm{Aut}_{\Delta_n}(U_{\omega n})$-orbit and stratum of P_C coincide with C.*

Proof The first part follows from [22, Theorem 5.7] in combination with [22, Theorem 5.14]. The bijection between strata and $\mathrm{Aut}_{\Delta_n}(U_{\omega n})$-orbits follows from [32, Lemma 2.28]. It follows from the explicit description of the T-fixed points in Lemma 2.8 that they are pairwise not isomorphic. Hence they belong to different strata. □

Remark 2.14 Observe that in general the $\mathrm{Aut}_Q(M)$-orbits in a quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ are not cells and contain more than one torus fixed point. This is very special to the $X(k,n,\omega)$'s. Otherwise it already fails in small examples:

Consider the Δ_2-representation M with $M^{(1)}=M^{(2)}=\mathbb{C}^2$, $M_1=\mathrm{id}$ and $M_2=0$. For $\mathbf{e}=(1,1)$, let $\mathbb{C}^*$ act on the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ induced by the weight function $\mathrm{wt}(e_i)=i$, where $\{e_1,e_2\}$ is the standard basis of $\mathbb{C}^2$. It follows from [3, Theorem 1] that $\mathrm{Gr}_{\mathbf{e}}(M)$ has two isomorphic $\mathbb{C}^*$-fixed points. Hence they live in the same $\mathrm{Aut}_{\Delta_2}(M)$-orbit.

Remark 2.15 Theorem 2.13 implies that every cell closure in $X(k,n,\omega)$ is the union of smaller cells. The bijection between cells and generalized juggling patterns allows to make this description explicit (Corollary 4.7). But first we have to describe the moment graph for the T action on $X(k,n,\omega)$ (Lemma 3.4).

2.5 Desingularization

In this subsection we give a desingularization of $X(k, n, \omega)$ following the general construction in [34] (see also [6, 10, 17, 37]).

For $I \in \binom{[n]}{k}$, define the Δ_n-representation

$$U_I := \bigoplus_{i \in I} U_i(\omega n).$$

Proposition 2.16 *For all $I \in \binom{[n]}{k}$, the closed stratum $\overline{\mathcal{S}_{U_I}}$ is an irreducible component of $X(k, n, \omega)$ and all irreducible components are of this form.*

Proof The parametrization for the representatives of the top dimensional strata is obtained in the proof of [33, Lemma 4.10]. The second part follows from Theorem 2.11.(v). □

Let $\hat{\Delta}_n$ be the quiver with vertex set

$$\big\{(i, k) \;\; i \in \mathbb{Z}_n \text{ and } k \in [\omega n]\big\}$$

and arrows

$$\big\{a_{i,k} : (i, k) \to (i, k+1) \; : \; i \in \mathbb{Z}_n \text{ and } k \in [\omega n - 1]\big\} \cup$$
$$\big\{b_{i,k} : (i, k) \to (i+1, k-1) \; : \; i \in \mathbb{Z}_n \text{ and } k \in [\omega n] \setminus \{1\}\big\}.$$

Let $M \in \mathrm{rep}_{\mathbb{C}}(\Delta_n)$ be a nilpotent representation, i.e. there exists $N > 0$ such that $M_{i+N-2} \circ M_{i+k-3} \circ \cdots \circ M_{i+1} \circ M_i = 0$. We define the $\hat{\Delta}_n$-representation

$$\hat{M} := \big((\hat{M}^{(i,k)})_{i \in \mathbb{Z}_n, k \in [N]}, (\hat{M}_{a_{i,k}}, \hat{M}_{b_{i,k+1}})_{i \in \mathbb{Z}_n, k \in [N-1]}\big)$$

with

$$\begin{aligned}
\hat{M}^{(i,1)} &:= M^{(i)} && \text{for } k = 1\\
\hat{M}^{(i,k)} &:= M_{i+k-2} \circ M_{i+k-3} \circ \cdots \circ M_{i+1} \circ M_i(M^{(i)}) && \text{for } k \geq 2\\
\hat{M}_{a_{i,k}} &:= M_{i+k-1} && \text{for } k \geq 1\\
\hat{M}_{b_{i,k}} &:= \iota : \hat{M}^{(i,k)} \hookrightarrow \hat{M}^{(i+1,k-1)} && \text{for } k \geq 2
\end{aligned}$$

Here the inclusion maps along $b_{i,k}$ arise naturally from the definition of the vector spaces of $\hat{M}$.

Every $W \in \mathrm{rep}_{\mathbb{C}}(\hat{\Delta}_n)$ restricts to a Δ_n-representation

$$\mathrm{res}\, W := \Big(\big(W^{(i,1)}\big)_{i \in \mathbb{Z}_n}, \big(W_{b_{i,2}} \circ W_{a_{i,1}}\big)_{i \in \mathbb{Z}_n}\Big).$$

For $I \in \binom{[n]}{k}$ set $\hat{X}_I(k, n, \omega) := \mathrm{Gr}_{\mathbf{dim}\, \hat{U}_I}\big(\hat{U}_{\omega n}\big)$ define the map

$$\pi_I : \hat{X}_I(k, n, \omega) \longrightarrow X(k, n, \omega)$$

by $\pi_I(V) := \mathrm{res} V$ for all $V \in \hat{X}_I(k, n, \omega)$.

Remark 2.17 The vector spaces of $\hat{U}_{\omega n}$ are spanned by subsets of the bases for the vector spaces of $U_{\omega n}$. Hence the T action on $U_{\omega n}$ extends to the quiver Grassmannians $\hat{X}_I(k, n, \omega)$ for $I \in \binom{[n]}{k}$ in the obvious way. The same holds for the $\mathbb{C}^*$ action.

The following result is a special case of [34, Theorem 3.18, Lemma 5.3].

Theorem 2.18 *The map*

$$\pi := \bigsqcup_{I \in \binom{[n]}{k}} \pi_I \quad \bigsqcup_{I \in \binom{[n]}{k}} \hat{X}_I(k, n, \omega) \longrightarrow X(k, n, \omega)$$

is a T-equivariant desingularization of $X(k, n, \omega)$.

Theorem 2.19 *For each $I \in \binom{[n]}{k}$ the quiver Grassmannian $\hat{X}_I(k, n, \omega)$ is isomorphic to a tower of fibrations*

$$\hat{X}_I(k, n, \omega) = X_1 \to X_2 \to \cdots \to X_{\omega n} = pt$$

where each map $X_k \to X_{k+1}$ for $k \in [\omega n - 1]$ is a fibration with fiber isomorphic to a product of Grassmannians of subspaces.

This result is a special case of [34, Theorem 3.21] and generalizes [12, Theorem 7.10].

2.6 Properties of the desingularization

Lemma 2.20 (c.f. [34, Theorem 5.5]) *For $I \in \binom{[n]}{k}$ the T-fixed points of $\hat{X}_I(k, n, \omega)$ are exactly the preimages of the T-fixed points of $X_I(k, n, \omega) \subset X(k, n, \omega)$ under π_I (where $X_I(k, n, \omega) := \overline{\mathcal{S}_{U_I}}$). The $\mathbb{C}^*$-attracting sets of these points provide a cellular decomposition of $\hat{X}_I(k, n, \omega)$.*

Proposition 2.21 *The automorphism group of $\hat{U}_{\omega n}$ satisfies*

$$Aut_{\hat{\Delta}_n}\big(\hat{U}_{\omega n}\big) \cong Aut_{\Delta_n}\big(U_{\omega n}\big).$$

Proof From the compositions $\beta_{i+1,2} \circ \alpha_{i,1}$ for all $i \in \mathbb{Z}_n$ we obtain the same relations on each matrix $A^{(i,1)}$ of $A \in \mathrm{Aut}_{\hat{\Delta}_n}(\hat{U}_{\omega n})$ as for the matrix $B^{(i)}$ of $B \in \mathrm{Aut}_{\Delta_n}(U_{\omega n})$ (see Proposition 2.10). Now it follows from the construction of $\hat{U}_{\omega n}$ that all other components $A^{(i,r)}$ are the lower diagonal blocks of size $\omega n - r + 1$ in the matrices $A^{(i,1)}$. This implies the desired isomorphism. □

Lemma 2.22 *The strata in the quiver Grassmannian $\hat{X}_I(k, n, \omega)$ are exactly the $Aut_{\hat{\Delta}_n}\big(\hat{U}_{\omega n}\big)$ orbits of the T-fixed points and coincide with their $\mathbb{C}^*$-attracting sets.*

Proof The representation $\hat{U}_{\omega n}$ is an injective bounded $\hat{\Delta}_n$ representation. Hence we can apply [32, Lemma 2.28], to conclude that all strata are $\mathrm{Aut}_{\hat{\Delta}_n}\big(\hat{U}_{\omega n}\big)$ orbits and vice versa. It follows from the explicit description of the T-fixed points of $\hat{X}_I(k, n, \omega)$ from Lemma 2.20 that they are pairwise isomorphic. This implies that each $\mathrm{Aut}_{\hat{\Delta}_n}\big(\hat{U}_{\omega n}\big)$ orbit contains exactly one T-fixed point. Hence it has to coincide with the $\mathbb{C}^*$-attracting set of that fixed point. □

3 Moment graph and cohomology

3.1 Moment graph

There is a combinatorial object called moment graph which captures the structure of fixed points and one-dimensional orbits for suitable torus actions on complex projective varieties.

The structure of this graph helps to understand the equivariant geometry of the variety. Before we describe it for $X(k, n, \omega)$ we recall the definition and some required terminology.

Let X be complex projective algebraic variety X acted upon by a torus T with finitely many fixed points and finitely many one-dimensional T orbits (i.e. the action is skeletal). The definition below is specialized to the setting that X admits a T-stable cellular decomposition (as in our case).

Definition 3.1 Let T be an algebraic torus and let X be a complex projective algebraic T-variety. Assume that X admits a T-stable cellular decomposition where every cell has exactly one fixed point. If the action of T on X is skeletal, then the corresponding moment graph $\mathcal{G}(X, T)$ is given by

- the vertex set is the fixed point set: $\mathcal{V} = X^T$;
- there is an edge $x \to y$ if and only if x and y belong to the same one dimensional T orbit closure $\overline{\mathcal{O}_{x\to y}}$ and y belongs to the closure of the cell containing x;
- the label of the edge $x \to y$ is the character $\alpha \in \mathrm{Hom}(T, \mathbb{C}^*)$ the torus acts by on $\mathcal{O}_{x\to y}$.

The edge labels are only well defined up to a sign, but since this does not play any role in the applications (e.g. computation of equivariant cohomology), we assume the labels to be fixed once and for all, and forget about this ambiguity.

The following parametrization of the T-fixed points of $X(k, n, \omega)$ helps to describe the structure of the one-dimensional T-orbits.

Proposition 3.2 *For $k, n, \omega \in \mathbb{N}$ with $k \le n$, there is a bijection between $\mathcal{J}ug(k, n, \omega)$ and*

$$\mathcal{C}_{k,n,\omega} := \Big\{(\ell_j)_{j\in\mathbb{Z}_n} \in [0, \omega n]^{\mathbb{Z}_n} : \mathbf{dim} \bigoplus_{j\in\mathbb{Z}_n} U_j(\ell_j) = (k\omega, \dots, k\omega) \in \mathbb{N}^{\mathbb{Z}_n}\Big\}.$$

Proof Let $\varphi : \mathcal{J}ug(k, n, \omega) \to \mathcal{C}_{k,n,\omega}$ send $\mathcal{J}_\bullet = (J_i)_{i\in\mathbb{Z}_n}$ to $\ell_\bullet = (\ell_j)_{j\in\mathbb{Z}_n}$ with

$$\ell_j := \max\big(\{r \in [\omega n] : \omega n - r + 1 \in J_{j-r+1}\} \cup \{0\}\big).$$

It follows immediately from the definition of $U_{\ell_\bullet} := \bigoplus_{j\in\mathbb{Z}_n} U_j(\ell_j)$ that $\mathbf{dim}\, U_{\ell_\bullet} = (k\omega, \dots, k\omega)$ since each J_i contains $k\omega$-many elements.

The inverse map $\varphi^{-1} : \mathcal{C}_{k,n,\omega} \to \mathcal{J}ug(k, n, \omega)$ sends $\ell_\bullet$ to $\mathcal{J}_\bullet$ where for each $\ell_j \neq 0$ and $s \in [\ell_j]$ the set J_{j-s+1} contains the element $\omega n - s + 1$. Clearly, the ℓ_j's contribute to each J_i exactly $k\omega$-many times since $\mathbf{dim}\, U_{\ell_\bullet} = (k\omega, \dots, k\omega)$. □

Example 3.3 For $k = 1, n = \omega = 2$, $X(k, n, \omega)$ has five $T = (\mathbb{C}^*)^{2+1}$-fixed points labeled by the tuples $(2, 2), (3, 1), (1, 3), (4, 0),$ and $(0, 4)$.

Now we describe certain cut and paste moves on the segments of the elements in $\mathcal{C}_{k,n,\omega}$. For every element $\ell_\bullet \in \mathcal{C}_{k,n,\omega}$ there are maps of the form $f_{i,j,r} : \mathcal{C}_{k,n,\omega} \to \mathcal{C}_{k,n,\omega}$ with

$$\big(f_{i,j,r}(\ell_\bullet)\big)_s := \begin{cases} \ell_s & s \notin \{i, j\} \\ \ell_i - r & s = i \\ \ell_j + r & s = j \end{cases}.$$

whenever $r \in [0, \min\{\ell_i, \omega n - \ell_j\}]$ and $i - \ell_i = j - \ell_j - r \mod n$. It is straightforward to check that $f_{i,j,r}(\ell_\bullet)$ is again an element of $\mathcal{C}_{k,n,\omega}$. These cut and paste moves describe all one-dimensional T-orbits.

Lemma 3.4 *The vertices of the moment graph for the action of the torus T on $X(k, n, \omega)$ are labelled by the elements of $\mathcal{C}_{k,n,\omega}$. There is an oriented edge in the moment graph from $\ell_\bullet$ to $f_{i,j,r}(\ell_\bullet)$ if and only if $\ell_i > \ell_j + r$. The label of the edge $\ell_\bullet \to f_{i,j,r}(\ell_\bullet)$ is $\epsilon_j - \epsilon_i + \delta \cdot (\ell_i - \ell_j - r)$, where $\delta(\gamma) := \gamma_0$ and $\epsilon_i(\gamma) := \gamma_i$ for any $i \in [n]$ and $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n) \in T$.*

Proof This is a special case of the description of the edges in the moment graph and their labels as given in [22, Theorem 6.15], translated to the description of the T-fixed points from Proposition 3.2. □

Example 3.5 The one-dimensional T-orbits between the T-fixed points from Example 3.3 are captured in the following graph:

Observe that there is no edge between (4, 0) and (0, 4) because $1 \neq 2 \mod 2$. For the same reason there can't be an edge between (3, 1) and (1, 3), and (4, 0) or (0, 4) and (2, 2). The label of the edge $(4, 0) \to (1, 3)$ is $\epsilon_2 - \epsilon_1 + \delta$, since $(1, 3) = f_{1,2,3}(4, 0)$. All other labels are computed in the same way.

3.2 *T*-equivariant cohomology

By [22, Theorem 6.6], we can use the structure of the moment graph $\mathcal{G}$ as described in Lemma 3.4 to compute the (T-equivariant) cohomology ring of $X := X(k, n, \omega)$. Let $R := \mathbb{Q}[\epsilon_1, \dots, \epsilon_n, \delta]$ and consider it as a $\mathbb{Z}$-graded ring with grading induced by $\deg(\epsilon_i) = \deg(\delta) = 2$ for all $i \in [n]$. By $\alpha(\ell_\bullet, \ell'_\bullet)$ we denote the label of the edge $\ell_\bullet \to \ell'_\bullet$. [13, Theorem 1.2.2] gives the following result.

Corollary 3.6 *There is an isomorphism of ($\mathbb{Z}$-graded) rings*

$$H_T^\bullet(X, \mathbb{Q}) \simeq \left\{ (z_{\ell_\bullet})_{\ell_\bullet \in \mathcal{C}(k,n,\omega)} \in \bigoplus_{\ell_\bullet \in \mathcal{C}(k,n,\omega)} R \;\middle|\; \begin{array}{l} z_{\ell_\bullet} \equiv z_{\ell'_\bullet} \mod \alpha(\ell_\bullet, \ell'_\bullet) \\ \text{for every edge } \ell_\bullet \to \ell'_\bullet \end{array} \right\}.$$

Remark 3.7 By [23, Theorem 3.22], $H_T^\bullet(X(k, n, \omega), \mathbb{Q})$ admits a very nice basis as a free module over R, namely a so-called Knutson–Tao (KT) basis. We briefly recall the definition (see [40, Definition 2.12]). A Knutson–Tao class for a torus fixed point $x \in X^T$ is an equivariant class $p^x = (p^x_y)_{y \in X^T} \in H_T^\bullet(X)$ such that: first, $p^x_x = \prod \alpha(x, y)$ for all edges $x \to y$ from the moment graph; second, each p^y_x is a homogeneous polynomial in $\mathbb{Q}[T]$ such that the degrees of p^x_x and of p^y_x coincide; third, $p^y_x = 0$ for $y \in X^T$ such that x cannot be reached from y via an oriented path on the graph.

Example 3.8 For instance, in the case the moment graph is the one from Example 3.5, if we denote by $\alpha = \epsilon_1 - \epsilon_2$, the KT classes are the following:

$$\xi_{(0,4)} = \begin{array}{ccc} (\alpha+\delta)(\alpha+3\delta) & & 0 \\ \downarrow & \swarrow\!\!\!\searrow & \downarrow \\ 0 & & 0 \\ & \searrow \quad \swarrow & \\ & 0 & \end{array} \qquad \xi_{(4,0)} = \begin{array}{ccc} 0 & & (-\alpha+\delta)(-\alpha+3\delta) \\ \downarrow & \swarrow\!\!\!\searrow & \downarrow \\ 0 & & 0 \\ & \searrow \quad \swarrow & \\ & 0 & \end{array}$$

$$\xi_{(1,3)} = \begin{array}{ccc} \alpha+\delta & & -\alpha+3\delta \\ \downarrow & \swarrow\!\!\!\searrow & \downarrow \\ \alpha+\delta & & 0 \\ & \searrow \quad \swarrow & \\ & 0 & \end{array} \qquad \xi_{(3,1)} = \begin{array}{ccc} \alpha+3\delta & & -\alpha+\delta \\ \downarrow & \swarrow\!\!\!\searrow & \downarrow \\ 0 & & -\alpha+\delta \\ & \searrow \quad \swarrow & \\ & 0 & \end{array}$$

$$\xi_{(2,2)} = \begin{array}{ccc} 1 & & 1 \\ \downarrow & \swarrow\!\!\!\searrow & \downarrow \\ 1 & & 1 \\ & \searrow \quad \swarrow & \\ & 1 & \end{array}$$

Above the equivariant cohomology class $(z_{\ell_\bullet})_{\ell_\bullet \in \mathcal{C}(k,n,\omega)}$ is represented by a collection of polynomials arranged on the corresponding vertices of the moment graph (with $z_{\ell_\bullet}$ on the vertex $\ell_\bullet$).

3.3 Cyclic group action on equivariant cohomology

Consider the $\mathbb{Z}_n$ action on T given by

$$m \cdot (\gamma_0, \gamma_1, \ldots, \gamma_n) = (\gamma_0, \gamma_{1+m}, \ldots, \gamma_{n+m}), \quad (m \in \mathbb{Z}_n, \gamma \in T).$$

Such an action induces a $\mathbb{Z}_n$ action on the character lattice $\mathfrak{X}^*(T)$ via $m \cdot \alpha(\gamma) = \alpha((-m)\cdot\gamma)$ for any $m \in \mathbb{Z}_n,\ \alpha \in \mathfrak{X}^*(T),\ \gamma \in T$. In this way we also get a $\mathbb{Z}_n$ action on R, uniquely determined by $m(\epsilon_i) = \epsilon_{i-m}$ and $m(\delta) = \delta$ for any $m, i \in \mathbb{Z}_n$.

Clearly, the set $\mathcal{C}_{k,n,\omega}$ is also equipped with a $\mathbb{Z}_n$ action given by $m \cdot (\ell_j)_{j\in\mathbb{Z}_n} = (\ell'_j)_{j\in\mathbb{Z}_n}$, where $\ell'_j = \ell_{j-m}$.

Proposition 3.9 *There is an action of $\mathbb{Z}_n$ on $H_T^\bullet(X(k,n,\omega),\mathbb{Q})$ given by*

$$m \cdot \left(z_{\ell_\bullet}\right)_{\ell_\bullet\in\mathcal{C}(k,n,\omega)} = \left(z'_{\ell_\bullet}\right)_{\ell_\bullet\in\mathcal{C}(k,n,\omega)},$$

where $m \in \mathbb{Z}_n$, $(z_{\ell_\bullet}) \in H_T^\bullet(X(k,n,\omega),\mathbb{Q})$, and $z'_{\ell_\bullet} = m(z_{m\cdot\ell_\bullet})$.

Proof Let $(z_{\ell_\bullet}) \in H_T^\bullet(X(k,n,\omega),\mathbb{Q})$ and $m \in \mathbb{Z}_n$. We have to check that for any $\ell_\bullet \in \mathcal{C}(k,n,\omega)$ and any triple i, j, r such that $f_{i,j,r}(\ell_\bullet)$ is well defined, the following holds

$$mz_{m\cdot\ell_\bullet} \equiv mz_{m\cdot f_{i,j,r}\ell_\bullet} \mod \alpha_{\ell_\bullet, f_{i,j,r}\ell_\bullet} = \epsilon_j - \epsilon_i + (\ell_i - \ell_j - r)\delta.$$

Observe that

$$m \cdot f_{i,j,r}(\ell_\bullet)_s = \begin{cases} \ell_{s-m} & s \notin \{i+m, j+m\} \\ \ell_{s-m} - r & s = i+m \\ \ell_{s-m} + r & s = j+m \end{cases} = f_{i+m,j+m,r}(m\cdot\ell_\bullet)_s.$$

Thus, since $(z_{\ell_\bullet}) \in H_T^\bullet(X(k,n,\omega),\mathbb{Q})$, we have

$$z_{m\cdot\ell_\bullet} \equiv z_{m\cdot f_{i,j,r}\ell_\bullet} \mod \epsilon_{j+m} - \epsilon_{i+m} + (\ell_i - \ell_j - r)\delta.$$

We conclude that

$$mz_{m\cdot\ell_\bullet} \equiv mz_{m\cdot f_{i,j,r}\ell_\bullet} \mod m(\epsilon_{j+m}-\epsilon_{i+m}+(\ell_i-\ell_j-r)\delta)=\epsilon_j-\epsilon_i+(\ell_i-\ell_j-r)\delta.$$

□

Example 3.10 In terms of the KT classes from Example 3.8, the $\mathbb{Z}_2$ action is uniquely determined by

$$\sigma\cdot\xi_{(2,2)}=\xi_{(2,2)},\ \sigma\cdot\xi_{(3,1)}=\xi_{(1,3)},\ \sigma\cdot\xi_{(4,0)}=\xi_{(0,4)},$$

where σ is the generator of $\mathbb{Z}_2$.

Let P denote $\mathbb{Z}_2$ representation given by the space R equipped with the action introduced above. As a $\mathbb{Z}_2$ representation, the equivariant cohomology decomposes as

$$\begin{aligned} H_T^\bullet(X(1,2,2)) &\simeq P\xi_{(2,2)}\oplus P(\xi_{(3,1)}+\xi_{(1,3)})\oplus P(\xi_{(4,0)}+\xi_{(0,4)}) \\ &\quad \oplus P(\xi_{(3,1)}-\xi_{(1,3)})\oplus P(\xi_{(4,0)}-\xi_{(0,4)}) \\ &\simeq \mathbf{1}^0\otimes P\oplus\mathbf{1}^2\otimes P\oplus\mathbf{1}^4\otimes P\oplus\epsilon^2\otimes P\oplus\boldsymbol{\epsilon}^4\otimes P, \end{aligned}$$

where $\mathbf{1}^j$ and $\boldsymbol{\epsilon}^j$ represent the one dimensional representation concentrated in degree j, on which $1\in\mathbb{Z}_2$ acts via multiplication by 1 and -1, respectively (we recall that the cohomological degree of a cocharacter is 2).

Remark 3.11 It would be interesting to investigate the structure of $\mathbb{Z}_n$ representation on $H_T^\bullet(X(k,n,\omega),\mathbb{Q})$ in general.

4 Poset structures on the set of fixed points

4.1 Affine permutations, Grassmann necklaces and juggling patterns

In this subsection we briefly recall several combinatorial objects playing an important role in the theory of tnn Grassmannians. The details can be found in [19, 21, 36, 41].

Let $\mathcal{S}_{k,n}$ be the set of (k,n) affine permutations, i.e. $\mathcal{S}_{k,n}$ consists of bijections $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(i+n)=f(i)+n$ for all $i\in\mathbb{Z}$ and $\sum_{i=1}^n(f(i)-i)=kn$. We denote by $\mathrm{id}_k\in\mathcal{S}_{k,n}$ the permutation given by $\mathrm{id}_k(j)=j+k$ for all $j\in\mathbb{Z}$. In particular, $(0,n)$ affine permutations form a group W_n isomorphic to the Weyl group of the affine type $A_{n-1}^{(1)}$. The group W_n acts on $\mathcal{S}_{k,n}$ by left multiplication. In particular, the map $w\mapsto w\mathrm{id}_k$ gives a bijection $W_n\to\mathcal{S}_{k,n}$. Hence the Bruhat order on W_n induces an order on the set of affine permutations.

A (k,n) affine permutation is called bounded if $i\le f(i)\le i+n$ for all $i\in\mathbb{Z}$. The set of bounded (k,n) affine permutations is denoted by $\mathcal{B}_{k,n}$. The following fact will be important for us:

$$\mathcal{B}_{k,n} \text{ is a lower order ideal in } \mathcal{S}_{k,n}\simeq W_n. \tag{4.1}$$

A collection $\mathcal{I}=(I_a)_{a\in[n]}$ of subsets of the set $[n]$ is called a (k,n) Grassmann necklace if $|I_a|=k$ for all a and $I_a\subset I_{a+1}\cup\{a\}$ for all $a\in[n]$ (for $a=n$ we put $I_{a+1}=I_1$). We denote the set of (k,n) Grassmann necklaces by $\mathcal{GN}_{k,n}$. The bijection between $\mathcal{GN}_{k,n}$ and $\mathcal{B}_{k,n}$ is given by the following rule. For $\mathcal{I}\in\mathcal{GN}_{k,n}$ the corresponding $f\in\mathcal{B}_{k,n}$ fixes all a such that $a\notin I_a$. If $a\in I_a$, then $I_{a+1}=I_a\backslash\{a\}\cup\{b\}$; we put $f(a)=c$, where $a<c\le a+n$ and $b=c$ modulo n.

Example 4.1 For a subset $I \subset [n]$, $|I| = k$ we have a Grassmann necklace $\mathcal{I} = (I, \dots, I)$. The bounded (k, n) affine permutation f_I corresponding to $\mathcal{I}$ is determined by $f_I(a) = \begin{cases} a, & a \notin I, \\ a + n, & a \in I \end{cases}$.

We denote by $w_I \in W_n$ the Weyl group element corresponding to f_I under the identification $W_n \simeq \mathcal{S}_{k,n}$.

Juggling patterns are close cousins of the Grassmann necklaces. By definition, a collection $\mathcal{J} = (J_1, \dots, J_n)$ of k-element subsets of $[n]$ is a (k, n) juggling pattern if $\tau_1(J_a \backslash \{n\}) \subset J_{a+1}$ for all $a \in [n]$, where $\tau_1(x) = x + 1$. Let $\mathcal{J}ug(k, n)$ be the set of (k, n) juggling patterns. Then there is a bijection between $\mathcal{J}ug(k, n)$ and $\mathcal{B}_{k,n}$ given by the following rule. For a juggling pattern $\mathcal{J} = (J_a)_{a=1}^n$ the corresponding element $f_{\mathcal{J}} \in \mathcal{B}(k, n)$ is defined by

$$f_{\mathcal{J}}(a) = \begin{cases} a, & n \notin J_a, \\ n + a + 1 - x, & n \in J_a,\ J_{a+1} = \tau_1(J_a \backslash \{n\}) \cup \{x\}. \end{cases} \tag{4.2}$$

4.2 Generalized bounded affine permutations

As shown in [12, Section 1], there is a bijection between cells of $X(k, n, 1)$, juggling patterns and bounded affine permutations. Now, we introduce generalized bounded affine permutations which are in bijection with generalized juggling patterns. This gives rise to an alternative way to parameterize the T-fixed points of $X(k, n, \omega)$.

Definition 4.2 For $k, n, \omega \in \mathbb{N}$ with $k \leq n$, a (k, n, ω) bounded affine permutation is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the following properties:

(1) $f(i + n) = f(i) + n$ for all $i \in \mathbb{Z}$,
(2) $\sum_{i=1}^n (f(i) - i) = kn\omega$,
(3) $i \leq f(i) \leq i + \omega n$ for all $i \in \mathbb{Z}$.

The set of (k, n, ω) bounded affine permutations is denoted by $\mathcal{B}_{k,n,\omega}$.

Here, Condition (1) is the same as for the (k, n) bounded affine permutations from Sect. 4.1. For $\omega = 1$ Condition (2) and (3) are the same as in Sect. 4.1. Without condition (3) we say f is a $(k \cdot \omega, n)$ affine permutation. This definition is valid for all $q \in \mathbb{Z}$, not only $q = \omega k$. Recall the notation $\mathcal{S}_{q,n}$ for the set of all (q, n) affine permutations.

There is a special (q, n) affine permutation id_q given by $\mathrm{id}_k(i) = i + q$. In the setting $q = k\omega$ this is a (k, n, ω) bounded affine permutation. Without dependence on ω, the length of an affine permutation is defined as

$$l(f) = |\{(i, j) \in [n] \times \mathbb{Z} :\ i < j \text{ and } f(i) > f(j)\}|.$$

We note that the set of $(0, n)$ affine permutations is a group isomorphic to the affine Weyl group W_n of type $A_{n-1}^{(1)}$. For general q the group W_n acts freely and transitively on $\mathcal{S}_{q,n}$, because we can write $q = rn + k$ with $0 \leq k < n$ and use the same arguments as for $\mathcal{S}_{k,n}$. The action of the permutation $s_i = (i, i+1) \in W_n$, for $i = 0, \dots, n-1$ permutes the values $f(i + rn)$ and $f(i + rn + 1)$ for all $r \in \mathbb{Z}$. This allows to identify the set of $\mathcal{S}_{q,n}$ with W_n by sending $w \in W_n$ to $w.\mathrm{id}_q$. Hence we obtain an induced order $\leq$ on the set $\mathcal{S}_{q,n}$ coming from the Bruhat order on W_n. Thus, the unique minimal element is id_q.

It is shown in [19, Lemma 3.6] that $\mathcal{B}_{k,n,1}$ is a lower order ideal in $\mathcal{S}_{k,n}$. With the same arguments it follows that $\mathcal{B}_{k,n,\omega}$ is a lower order ideal in $\mathcal{S}_{k\cdot\omega,n} \cong W_n$. For $f, g \in \mathcal{B}_{k,n,\omega}$ we write $f \leq_{\mathcal{B}} g$ for the order induced by the Bruhat order on W_n.

By [21, Theorem 6.2], there is an order preserving bijection between the set $\mathcal{B}_{k,n,1}$ and the set $\mathcal{J}ug(k, n, 1)$. Before generalizing this to arbitrary ω, we introduce an alternative parametrization of the T-fixed points of $X(k, n, \omega)$ which is closer to the definition of bounded affine permutations.

Proposition 4.3 *For $k, n, \omega \in \mathbb{N}$ with $k \leq n$, there is a bijection between $\mathcal{J}ug(k, n, \omega)$ and*

$$\Big\{(\ell_j)_{j\in\mathbb{Z}_n} \in [0, \omega n]^{\mathbb{Z}_n} \quad \sum_{j\in\mathbb{Z}_n} \ell_j = kn\omega, \quad \big\{j + \ell_j \, mod \, n : j \in \mathbb{Z}_n\big\} = \mathbb{Z}_n\Big\}.$$

Proof By Proposition 3.2 we have to show that for $\ell_\bullet \in [0, \omega n]^{\mathbb{Z}_n}$ the following are equivalent:

(1) $\mathbf{dim}\, U_{\ell_\bullet} = (k\omega, \dots, k\omega) \in \mathbb{N}^{\mathbb{Z}_n}$,
(2)

$$\sum_{j\in\mathbb{Z}_n} \ell_j = kn\omega \quad \text{and} \quad \big\{j + \ell_j \mod n : j \in \mathbb{Z}_n\big\} = \mathbb{Z}_n.$$

The length tuple $z_\bullet$ with $z_j = k\omega$ for all $j \in \mathbb{Z}_n$ satisfies (1) and (2). It represents the unique zero-dimensional cell C_0 of $X(k, n, \omega)$ (since the corresponding point is the span of the last $k\omega$ basis vectors at each vertex) and C_0 is contained in the closure of every other cell of $X(k, n, \omega)$. Hence, by [22, Theorem 6.15] every other $\ell_\bullet \in \mathcal{C}_{k,n,\omega}$ is obtained from $z_\bullet$ by a sequence of cut and paste moves $f_{i,j,r} : \mathcal{C}_{k,n,\omega} \to \mathcal{C}_{k,n,\omega}$. These moves preserve the properties (1) and (2). Hence (1) implies (2).

The second part of (2) implies that $\mathbf{dim}\, U_{\ell_\bullet} = (L/n, \dots, L/n)$ where $L = \sum_{j\in\mathbb{Z}_n} \ell_j$. Together with the first part of (2) this implies (1). □

Lemma 4.4 *For $k, n, \omega \in \mathbb{N}$ with $k \leq n$, there are bijections between $\mathcal{J}ug(k, n, \omega)$, $\mathcal{C}(k, n, \omega)$ and $\mathcal{B}_{k,n,\omega}$.*

Proof Proposition 3.2 gives the first bijection. We define the map $\psi : \mathcal{C}(k, n, \omega) \to \mathcal{B}_{k,n,\omega}$ sending $\ell_\bullet$ to the map $f : \mathbb{Z} \to \mathbb{Z}$ with $f(j') := j' + \ell_j$ for all $j' \in \mathbb{Z}$ with $j' = j \mod n$. It follows from the second parametrization of cells from Proposition 4.3 that this f is bijective and satisfies part (2) of Definition 4.2. $\ell_j \in [0, \omega n]$ implies part (3) of that definition. The inverse map ψ^{-1} sends f to $(f(j) - j)_{j\in\mathbb{Z}_n}$. □

4.3 Partial orders on the set of cells

In this section we introduce partial orders on the sets $\mathcal{J}ug(k, n, \omega)$, $\mathcal{C}(k, n, \omega)$ and $\mathcal{B}_{k,n,\omega}$ and examine how they are related under the bijections from Lemma 4.4.

For $\mathcal{J}_\bullet, \mathcal{J}'_\bullet \in \mathcal{J}ug(k, n, \omega)$ we write $\mathcal{J}_\bullet \geq_{\mathcal{J}} \mathcal{J}'_\bullet$ iff $j_r^{(i)} \leq j_r^{\prime(i)}$ for all $i \in \mathbb{Z}_n$ and $r \in [k\omega]$ where we order each $J_i \in \binom{[n\omega]}{k\omega}$ as

$$\big(j_1^{(i)} < j_2^{(i)} < \dots < j_{k\omega}^{(i)}\big).$$

Given two elements $\ell_\bullet, \ell'_\bullet \in \mathcal{C}(k, n, \omega)$ we write $\ell_\bullet \geq_{\mathcal{C}} \ell'_\bullet$ if there exists an oriented path from $\ell_\bullet$ to $\ell'_\bullet$ in the moment graph for the T-action on $X(k, n, \omega)$. For $p, p' \in X(k, n, \omega)^T$ we write $p' \preceq p$ if $\overline{C_p}$ contains p'. By Theorem 2.13 we obtain the same partial order $\preceq$ if we consider closures of $\mathrm{Aut}_{\Delta_n}(U_{\omega n})$-orbits or strata. Recall that the partial order $\geq_{\mathcal{B}}$ on $\mathcal{B}_{k,n,\omega}$ is induced by the Bruhat order on W_n.

Example 4.5 For the T-fixed points from Example 3.3, the bijections from Lemma 4.4 are as follows:

$$\begin{aligned}
(2,2) &\longleftrightarrow \big(\{3,4\},\{3,4\}\big) \longleftrightarrow f \ \text{ with } f(1) := 1+2,\ f(2) := 2+2,\\
(3,1) &\longleftrightarrow \big(\{2,4\},\{3,4\}\big) \longleftrightarrow f \ \text{ with } f(1) := 1+3,\ f(2) := 2+1,\\
(1,3) &\longleftrightarrow \big(\{3,4\},\{2,4\}\big) \longleftrightarrow f \ \text{ with } f(1) := 1+1,\ f(2) := 2+3,\\
(4,0) &\longleftrightarrow \big(\{2,4\},\{1,3\}\big) \longleftrightarrow f \ \text{ with } f(1) := 1+4,\ f(2) := 2+0,\\
(0,4) &\longleftrightarrow \big(\{1,3\},\{2,4\}\big) \longleftrightarrow f \ \text{ with } f(1) := 1+0,\ f(2) := 2+4.
\end{aligned}$$

and all poset structures are the same as the one induced by Example 3.5.

The next theorem shows that this identification of poset structures was no coincidence.

Theorem 4.6 *For $k, n, \omega \in \mathbb{N}$ with $k \le n$, there are order preserving poset isomorphisms between $\mathcal{J}ug(k,n,\omega)$, $\mathcal{C}(k,n,\omega)$, $X(k,n,\omega)^T$ and $\mathcal{B}_{k,n,\omega}$.*

Corollary 4.7 *The closure of every cell in $X(k,n,\omega)$ is obtained as*

$$\overline{C_{\mathcal{J}}} = \bigcup_{\mathcal{J}' \in \mathcal{J}ug(k,n,\omega) \text{ s.t.: } \mathcal{J}' \le_{\mathcal{J}} \mathcal{J}} C_{\mathcal{J}'}.$$

Moreover the moment graph of $\overline{C_{\mathcal{J}}}$ is the full subgraph of the graph described in Lemma 3.4 on the vertices corresponding to $\mathcal{J}' \le_{\mathcal{J}} \mathcal{J}$. The dimension of $C_{\mathcal{J}}$ is the number of edges in the moment graph starting at $\mathcal{J}$. This equals the length of the corresponding bounded affine permutation.

Proof The description of the closure is obtained from Theorem 4.6 in combination with Remark 2.15. Hence the moment graph of the cell closure is the full subgraph on the vertices which are smaller with respect to any of the partial orders. Finally the dimension formula is obtained from the embedding into the affine flag variety as described in Sect. 6. □

Below, we relate the poset structures of the cells and generalized juggling patterns. The relation to affine permutations is examined in Sect. 6.

Proof of Theorem 4.6 ($\le_{\mathcal{J}} \iff \le_{\mathcal{C}} \iff \preceq$) The isomorphisms on the level of sets were introduced in Lemma 4.4 and Lemma 2.8. It remains to show that they preserve the poset structures. The partial order $\le_{\mathcal{C}}$ on $\mathcal{C}(k,n,\omega)$ is obtained from the edges in the moment graph as described in Lemma 3.4. The property $\ell_i > \ell_j + r$ of the cut and paste moves implies that $\varphi^{-1}(\ell_\bullet) \ge_{\mathcal{J}} \varphi^{-1}(f_{i,j,k}(\ell_\bullet))$ where φ is the map from Proposition 3.2. Hence $\ell_\bullet \ge_{\mathcal{C}} \ell'_\bullet$ implies $\varphi^{-1}(\ell_\bullet) \ge_{\mathcal{J}} \varphi^{-1}(\ell'_\bullet)$ for all $\ell_\bullet, \ell'_\bullet \in \mathcal{C}(k,n,\omega)$.

Starting with $\mathcal{J}_\bullet, \mathcal{J}'_\bullet \in \mathcal{J}ug(k,n,\omega)$ such that $\mathcal{J}_\bullet \ge_{\mathcal{J}} \mathcal{J}'_\bullet$ we construct a path from $\ell_\bullet := \varphi(\mathcal{J}_\bullet)$ to $\ell'_\bullet := \varphi(\mathcal{J}'_\bullet)$ in the moment graph inductively: Let $d := \#\{i \in \mathbb{Z}_n : J_i \ne J'_i\}$. For $d = 0$ both juggling patterns are equal and there is nothing to show. If $d > 0$ there exists an $s \in \mathbb{Z}_n$ such that $J_s \ne J'_s$ and an $r \in [\omega k]$ such that $j_p^{(s)} = j_p'^{(s)}$ for all $p \in [r+1, \omega k]$ and $j_r^{(s)} < j_r'^{(s)}$. Here we assume that the sets belonging to the juggling patterns are ordered increasingly as introduced in the beginning of Sect. 4.3. Now, we determine a path $\ell_\bullet \to \ell''_\bullet$ in the moment graph with $\varphi^{-1}(\ell''_\bullet) =: \mathcal{J}''_\bullet \ge_{\mathcal{J}} \mathcal{J}'_\bullet$ and $d > \#\{i \in \mathbb{Z}_n : J''_i \ne J'_i\}$.

The points $j_r^{(s)}$ and $j_r'^{(s)}$ live on two different indecomposable summands of the representation $U_{\omega n}$ indexed by $a, b \in \mathbb{Z}_n$ which are obtained from $\mathcal{J}_\bullet, \mathcal{J}'_\bullet$ by the map φ as described in the proof of Proposition 3.2. This gives rise to the cut and paste map $f_{a,b,r}$

where $r := b - a + \ell_a - \ell_b$. By construction it follows that $\ell_\bullet >_{\mathcal{C}} f_{a,b,r}(\ell_\bullet)$ and $\varphi^{-1}(f_{a,b,r}(\ell_\bullet)) \geq_{\mathcal{J}} \mathcal{J}'_\bullet$. We can apply this construction of an edge in the moment graph recursively until we reach a point $\ell''_\bullet$ with $\varphi^{-1}(\ell''_\bullet) =: \mathcal{J}''_\bullet \geq_{\mathcal{J}} \mathcal{J}'_\bullet$ and $J''_s = J'_s$. This implies $d > \#\{i \in \mathbb{Z}_n : J''_i \neq J'_i\}$ and finishes the inductive step. Hence the partial orders $\leq_{\mathcal{J}}$ and $\leq_{\mathcal{C}}$ are equivalent.

With the explicit description of the cells as attracting sets of the fixed points, it is straightforward to check that for $p, p' \in X(k,n,\omega)^T$ the moment graph contains a path from p to p' if and only if $\overline{C_p}$ contains p'. This implies the equivalence of the partial orders $\leq_{\mathcal{C}}$ and $\preceq$. □

4.4 Poincaré polynomials

The Poincaré polynomial of the quiver Grassmannian $X(k,n,\omega)$ is obtained as

$$P_{k,n,\omega}(q) = \sum_{p \in X(k,n,\omega)^T} q^{\dim_{\mathbb{C}} C_p}.$$

From the computations in the previous section we obtain the following formula

Lemma 4.8 *For $k, n, \omega \in \mathbb{N}$ with $k \leq n$, the Poincaré polynomial of $X(k,n,\omega)$ is*

$$P_{k,n,\omega}(q) = \sum_{f \in \mathcal{B}_{k,n,\omega}} q^{l(f)}.$$

Here $l(f)$ denotes the length of the bounded affine permutation f as defined in Sect. 4.2.

Proof This follows immediately from Corollary 4.7. □

5 Affine flag varieties

In this section we recall some basics from the theory of affine flag varieties in type A and Sato Grassmannians [11, 20, 33]. We use this material in the next section for the explicit construction of the embeddings of the quiver Grassmannians $X(k,n,\omega)$ into the affine flag varieties.

5.1 Notation

Let $\widehat{\mathfrak{sl}_n}$ denote the affine Kac–Moody Lie algebra of type $A^{(1)}_{n-1}$. Explicitly, $\widehat{\mathfrak{sl}_n} = \mathfrak{sl}_n \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$, where K is central and d is the derivation. Let us fix a Cartan decomposition $\mathfrak{sl}_n = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ and the Borel subalgebra $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. Then the Iwahori subalgebra of $\widehat{\mathfrak{sl}_n}$ is given by $\mathfrak{b} \otimes 1 \oplus \mathfrak{sl}_n \otimes t\mathbb{C}[t]$.

We denote the Weyl group of $\widehat{\mathfrak{sl}_n}$ by W_n. For $n = 2$ the Weyl group is generated by s_0 and s_1 subject to the relations $s_0^2 = s_1^2 = e$. For $n > 2$ the group W_n is generated by reflections $s_0, s_1, \ldots, s_{n-1}$ subject to the defining relations

$$s_i^2 = e,\ i = 0, \ldots, n-1, \quad s_i s_j = s_j s_i,\ |i-j| > 1,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ i = 0, \ldots, n-1$$

(here and below we set $s_n = s_0$). The group W_n can be also realized as the group of $(0, n)$ affine permutations, i.e. bijections $f : \mathbb{Z} \to \mathbb{Z}$ subject to the conditions $f(i+n) = f(i)+n$ for all i and $\sum_{i=1}^{n}(f(i) - i) = 0$.

Let $\widehat{SL_n}$ be the affine group with the Lie algebra $\widehat{\mathfrak{sl}}_n$. This group contains the finite torus $\exp(\mathfrak{h})$ and the two-dimensional torus $(\mathbb{C}^*)^2 = \exp(\mathbb{C}K \oplus \mathbb{C}d)$. We denote by $P_i \subset \widehat{SL_n}$, $i = 0, \dots, n-1$ the maximal parabolic subgroups. Then the affine Grassmannians are defined as the quotients $\widehat{SL_n}/P_i$.

Let $\mathfrak{B} \subset \widehat{SL_n}$ be the Iwahori subgroup. More precisely, $\mathfrak{B}$ consists of matrices $A(t) \in SL_n(\mathbb{C}[t]) \subset \widehat{SL_n}$ such that $A(0)$ is upper triangular. In particular, the Lie algebra of $\mathfrak{B}$ is $\mathfrak{b} \otimes 1 \oplus \mathfrak{sl}_n \otimes t\mathbb{C}[t]$

Let $\mathcal{AF}_n \simeq \widehat{SL_n}/\mathfrak{B}$ be the affine flag variety for the group $\widehat{SL_n}$. One has the natural embedding of the affine flag variety into the product of affine Grassmannians $\mathcal{AF}_n \subset \prod_{i=0}^{n-1} \widehat{SL_n}/P_i$. We note that the fixed points of $\mathcal{AF}_n$ with respect to the torus of $\widehat{SL_n}$ are labeled by the elements of the group W_n. For $w \in W_n$ let $p_w \in \mathcal{AF}_n$ be the corresponding torus fixed point.

The affine flag variety is an ind-variety, i.e. the inductive limit of finite-dimensional projective algebraic varieties. Namely, for an element $w \in W_n$ we denote the corresponding (finite-dimensional) affine Schubert variety $\overline{\mathfrak{B}.p_w}$ by X_w. Then $\mathcal{AF}_n = \bigcup_{w \in W_n} X_w$.

5.2 Sato Grassmannians

The affine Grassmannians enjoy explicit embeddings to the Sato Grassmannians. Hence the affine flag variety can be realized inside the product of Sato Grassmannians. We provide some details below.

The Sato Grassmannian $\mathrm{SGr}^{(i)}$, $i \in \mathbb{Z}$ consists of subspaces $V \subset \mathbb{C}[t, t^{-1}]$ such that

- $t^N\mathbb{C}[t^{-1}] \supset V \supset t^{-N}\mathbb{C}[t^{-1}]$ for some $N \in \mathbb{Z}_{>0}$,
- $\dim V/t^{-N}\mathbb{C}[t^{-1}] = i + N$.

For example, the subspace $\mathring{V}^{(i)} = \mathrm{span}\{t^j : j \le i\}$ belongs to $\mathrm{SGr}^{(i)}$.

Remark 5.1 We opt to use $\mathbb{C}[t^{-1}]$ instead of $\mathbb{C}[t]$ to make notation compatible with our notation for the affine groups and Lie algebras.

Sato Grassmannians are ind varieties (in particular, they can be realized as inductive limits of finite-dimensional Grassmann varieties). They enjoy a Plücker embedding into the projective space $F = \Lambda^{\infty/2}(\mathbb{C}[t, t^{-1}])$ of semi-infinite forms. The space F is spanned by infinite wedge products

$$t^L = t^{l_1} \wedge t^{l_2} \wedge \dots, \qquad L = (l_1, l_2, \dots),$$

where $l_1 > l_2 > \dots$ and $l_{s+1} = l_s - 1$ for s large enough. One has the charge decomposition $F = \bigoplus_{i \in \mathbb{Z}} F^{(i)}$, where $F^{(i)}$ is spanned by wedges t^L such that $l_s = i - s + 1$ for s large enough. Then one has the Plücker embedding $\mathrm{SGr}^{(i)} \hookrightarrow \mathbb{P}(F^{(i)})$.

Remark 5.2 Let $|i\rangle \in F^{(i)}$ be the charge i vacuum vector, explicitly given by $|i\rangle = t^i \wedge t^{i-1} \wedge \dots$. Then the image of the space $\mathring{V}^{(i)} \in \mathrm{SGr}^{(i)}$ inside $\mathbb{P}(F^{(i)})$ coincides with the line containing $|i\rangle$. We also note that each space $F^{(i)}$ is endowed with the action of the infinite-dimensional Heisenberg algebra. As a module over the Heisenberg algebra $F^{(i)}$ is isomorphic to a Fock module.

The affine flag variety $\mathcal{AF}_n$ is realized inside the product $\prod_{i\in\mathbb{Z}} \mathrm{SGr}^{(i)}$ as the set of collections $(V_i)_{i\in\mathbb{Z}}$ such that

- $V_i \subset V_{i+1}$,
- $V_{i+n} = t^n V_i$.

In particular, the collection $(\mathring{V}^{(i)})_i$ corresponds to the coset of the identity in $\mathcal{AF}_n \simeq \widehat{SL_n}/\mathfrak{B}$.

Remark 5.3 Given an element $w \in W_n$ the corresponding torus fixed point $p_w \in \mathcal{AF}_n$ is given by

$$\prod_{i\in\mathbb{Z}} \mathrm{span}(t^{w(j)}, j \le i) \in \prod_{i\in\mathbb{Z}} \mathrm{SGr}^{(i)}.$$

6 Quiver Grassmannians inside affine flags

The goal of this section is to construct explicitly the embedding of the quiver Grassmannians $X(k,n,\omega)$ into the affine flag varieties. The image is described as a union of certain Schubert varieties. We note that due to the identification of our quiver Grassmannians with the local models of Shimura varieties (c.f. Remark 2.3) such an embedding can be found e.g. in [16, 28]. Our goal in this section is to construct the embedding explicitly in the language of quiver Grassmannians and Sato Grassmannians and to label the Schubert varieties showing up in the image via generalized juggling patterns and bounded affine permutations.

6.1 The construction

By definition, the quiver Grassmannian $X(k,n,\omega)$ sits inside the product $\prod_{i=0}^{n-1} \mathrm{Gr}_{k\omega}(M^{(i)})$, where $M^{(i)}$ are $n\omega$-dimensional vector spaces with bases $v_j^{(i)}$, $j \in [n\omega]$ (here we identify $\mathbb{Z}/n\mathbb{Z}$ with the set $0,\dots,n-1$). Using the Plücker embeddings for $\mathrm{Gr}_{k\omega}(M^{(i)})$ we obtain the embedding

$$X(k,n,\omega) \subset \prod_{i=0}^{n-1} \mathbb{P}(\Lambda^{k\omega}(M^{(i)})) \tag{6.1}$$

Now let us construct the embeddings $\Psi^{(i)} : \mathrm{Gr}_{k\omega}(M^{(i)}) \hookrightarrow \mathrm{SGr}^{(i)}$. We define the map $\psi^{(i)} : M^{(i)} \to \mathbb{C}[t,t^{-1}]$ by the formula

$$\psi^{(i)} v_{n\omega+1-j}^{(i)} = t^{i-k\omega+j}, \; j \in [n\omega]. \tag{6.2}$$

In particular, the image of $\psi^{(i)}$ is spanned by $t^{i-k\omega+1},\dots,t^{i-k\omega+n\omega}$.

Remark 6.1 Let us briefly explain the shifts of indices showing up in (6.2) (the details are given in the proofs below). Recall the map τ_1 defined by $v_j^{(i)} \mapsto v_{j+1}^{(i+1)}$. The subscript $n\omega+1-j$ in the left hand side and the power $i-k\omega+j$ are chosen in such a way that the map τ_1 induces the identity map when translated to $\mathbb{C}[t,t^{-1}]$. We also note that the term $-k\omega$ in the power of t in the right hand side guaranties that the induced map $\Lambda^{k\omega}(M^{(i)}) \to F$, defined by

$$\mathrm{span}\{w_1,\dots,w_{k\omega}\} \mapsto w_1 \wedge \dots \wedge w_{k\omega} \wedge |i-k\omega\rangle,$$

lands in the subspace $F^{(i)}$ of the charge i forms.

Now the maps $\Psi^{(i)}$ are defined as follows:

$$\Psi^{(i)}(U^{(i)}) = \psi^{(i)}U^{(i)} \oplus \mathrm{span}\{t^j \ : \ j \le i - k\omega\}. \tag{6.3}$$

The following lemma is obvious.

Lemma 6.2 *The image $\Psi^{(i)}Gr_{k\omega}(M^{(i)})$ belongs to $SGr^{(i)}$.*

Using the maps $\Psi^{(i)}$ one obtains the embedding

$$\Psi : X(k, n, \omega) \to \prod_{i\in\mathbb{Z}} \mathrm{SGr}^{(i)}. \tag{6.4}$$

To be precise, $\Psi(X(k, n, \omega))$ sits inside the product $\prod_{i=0}^{n-1} \mathrm{SGr}^{(i)}$. We consider a larger product $\prod_{i\in\mathbb{Z}} \mathrm{SGr}^{(i)}$ for all integers i via the obvious extension rule $M^{(i+n)} = M^{(i)}$, $\mathrm{Gr}_{k\omega}(M^{(i)}) = \mathrm{Gr}_{k\omega}(M^{(i+n)})$. This gives the desired realization (6.4).

Remark 6.3 Let us consider the point $\mathring{U} = (\mathring{U}^{(i)})_{i\in\mathbb{Z}}$ defined by $\mathring{U}^{(i)} = \mathrm{span}\{v_j^{(i)},\ (n - k)\omega + 1 \le j \le n\omega\}$. Then clearly $\mathring{U} \in X(k, n, \omega)$ and $\Psi\mathring{U}$ is the line passing through the product of highest weight vectors $|i\rangle$.

Recall the embeddings $\mathcal{AF}_n \subset \prod_{i\in\mathbb{Z}} \mathrm{SGr}^{(i)} \subset \prod_{i\in\mathbb{Z}} \mathbb{P}(F^{(i)})$ from Sect. 5. In what follows we identify the affine flag variety with its image inside the product of Sato Grassmannians.

Lemma 6.4 *The image of Ψ belongs to $\mathcal{AF}_n$.*

Proof Let $U = (U^{(i)})_{i\in\mathbb{Z}}$ be a point in $X(k, n, \omega)$ (recall that we use the extension $U^{(i)} = U^{(i+n)}$ inside $M^{(i)} = M^{(i+n)}$). We have to show that

- $\Psi^{(i)}U^{(i)} \subset \Psi^{(i+1)}U^{(i+1)}$,
- $\Psi^{(i+n)}U^{(i+n)} = t^n\Psi^{(i)}U^{(i)}$.

By definition,

$$\psi^{(i+n)} = t^n\psi^{(i)} \text{ and } \mathrm{span}\{t^j \ : \ j \le i + n - k\omega\} = t^n\mathrm{span}\{t^j \ : \ j \le i - k\omega\}.$$

Hence the second property $\Psi^{(i+n)}U^{(i+n)} = t^n\Psi^{(i)}U^{(i)}$ holds.

To prove the first property we need to show that

$$\psi^{(i)}U^{(i)} \oplus \mathrm{span}\{t^j \ : \ j \le i - k\omega\} \subset \psi^{(i+1)}U^{(i+1)} \oplus \mathrm{span}\{t^j \ : \ j \le i + 1 - k\omega\}.$$

Clearly it suffices to show that

$$\psi^{(i)}U^{(i)} \subset \psi^{(i+1)}U^{(i+1)} \oplus \mathrm{span}\{t^j \ : \ j \le i + 1 - k\omega\}. \tag{6.5}$$

Recall that $\tau_1U^{(i)} \subset U^{(i+1)}$, where $\tau_1v_j^{(i)} = v_{j+1}^{(i+1)}$ unless $j = n\omega$ and $\tau_1v_{n\omega}^{(i)} = 0$. For a vector $u = \sum_{j=1}^{n\omega} r_jv_{n\omega+1-j}^{(i)} \in U^{(i)}$ one has:

$$\psi^{(i)}u = \sum_{j=1}^{n\omega} r_jt^{i-k\omega+j}, \quad \psi^{(i+1)}(\tau_1u) = \sum_{j=2}^{n\omega} r_jt^{i-k\omega+j}.$$

Since $\tau_1u \in U^{(i+1)}$ we obtain (6.5). □

6.2 Description of the image

Now our goal is to identify the image $\Psi X(k,n,\omega)$ inside the affine flag variety. Recall (see Theorems 2.11 and 2.13) that $X(k,n,\omega)$ has $\binom{n}{k}$ irreducible components $X_I(k,n,\omega)$ labeled by the cardinality k subsets $I \subset [n]$. Each irreducible component is a closure of a cell C_I containing a unique torus fixed point p_I. The cell C_I is equal to the $\mathrm{Aut}_{\Delta_n}(U_{n\omega})$ orbit passing through p_I. Our goal here is two-fold. First, we show that after the embedding of $X(k,n,\omega)$ into the affine flag variety the action of the automorphism group translates into the action of the Iwahori subgroup $\mathfrak{B}$. Second, we compute the images Ψp_I of the torus fixed points, i.e. we find the affine Weyl group elements $w(I)$ such that $\Psi p_I = p_{w(I)}$. As a corollary we conclude that the embedding Ψ realizes the quiver Grassmannian $X(k,n,\omega)$ as the union of the Schubert varieties $X_{w(I)}$ inside the affine flag variety $\mathcal{AF}_n$.

Lemma 6.5 *The point p_I is determined by the condition $v_1^{(i)} \in p_I^{(i)}$ for all $i \in I$.*

Proof We note that, since $\tau_1 U^{(i)} \subset U^{(i+1)}$ for any point $U \in X(k,n,\omega)$ with $U = (U^{(i)})_{i\in\mathbb{Z}}$, the condition $v_1^{(i)} \in p_I^{(i)}$ for $i \in I$ says that

$$p_I^{(i_0)} \ni v^{(i_0)}_{1+(i_0-i)+nr} \text{ for all } i \in I, i \le i_0 \text{ and } r = 0,\dots,\omega-1,$$
$$p_I^{(i_0)} \ni v^{(i_0)}_{1+(i-i_0)+n(r+1)} \text{ for all } i \in I, i > i_0 \text{ and } r = 0,\dots,\omega-1.$$

Hence for each pair $i \in I$, $r = 0,\dots,\omega-1$ we obtain a basis vector in $U^{(i_0)}$. Since $|I| = k$ and $\dim U^{(i_0)} = k\omega$ we obtain the desired claim. □

Remark 6.6 One easily sees that a point p_I as above does not belong to the closure of the $\mathrm{Aut}_{\Delta_n}(U_{n\omega})$ orbit of any other torus fixed point. In fact, the explicit form of the automorphism group elements from Proposition 2.10 shows that if $p_I \in \mathrm{Aut}_{\Delta_n}(U_{n\omega}).p$ for a torus fixed point p, then $p^{(i)} \ni v_1^{(i)}$, which implies $p = p_I$. We conclude that the closure of the the orbit $\mathrm{Aut}_{\Delta_n}(U_{n\omega})p_I$ is an irreducible component for any I.

Remark 6.7 The (ω-generalized) juggling pattern $(J_1,\dots,J_n)$ corresponding to the point p_I is given by

$$\begin{aligned} J_{i_0} =& \{1+(i_0-i)+n(r-1),\ i \in I, i \le i_0, r \in [\omega]\} \\ &\cup \{1+(i-i_0)+nr,\ i \in I, i > i_0, r \in [\omega]\} \end{aligned}$$

for all $i_0 \in [n]$.

For $I \in \binom{[n]}{k}$ we denote by $w(I) \in W_n$ the Weyl group element such that $\Psi p_I = p_{w(I)}$.

Corollary 6.8 *The element $w(I) \in W_n$ is defined by*

$$w(I) : i \mapsto \begin{cases} i - k\omega, & i \notin I, \\ i - k\omega + n\omega, & i \in I \end{cases}$$

for all $i = 1,\dots,n$.

Proof Let $p_I = (U^{(i)})_{i\in[n]}$. Then one has $U^{(i)} = \mathrm{span}\{v_j^{(i)} : j \in R_i\}$ for certain subsets $R_i \subset [\omega n]$. We note that the following holds true:

$$R_i = \begin{cases} \{a+1,\ a \in R_{i-1}\}, & i \notin I, \\ \{a+1,\ a \in R_{i-1}, a \ne \omega n\} \cup \{1\}, & i \in I. \end{cases} \tag{6.6}$$

Now let $\Psi p_I = (\Psi p_I)_i$ $(i \in \mathbb{Z})$. Then each space $(\Psi p_I)_i$ is a vector space spanned by the elements t^a, $a \in S_i$ for a subset $S_i \subset \mathbb{Z}$. By definition $w(I)(i) = S_i \backslash S_{i-1}$ (see Remarks 5.3). Formulas (6.2) and (6.3) say that

$$S_i = \{i - k\omega + n\omega + 1 - r, r \in R_i\} \cup \mathbb{Z}_{\leq i-k\omega}.$$

Taking into account equality (6.6) we obtain

$$S_i \backslash S_{i-1} = \begin{cases} i - k\omega, & i \notin I, \\ i - k\omega + n\omega, & i \in I \end{cases}.$$

□

Remark 6.9 Since $|I| = k$, one has $\sum_{i \in [n]}(w(I)(i) - i) = 0$.

Let $\widetilde{\mathfrak{B}} \subset GL_n(\mathbb{C}[z])$ be the Iwahori subgroup consisting of such matrices $\tilde{g}(z)$ that $\tilde{g}(0)$ is lower triangular.

Proposition 6.10 *The map* Ψ *translates the action of* $Aut_{\Delta_n}(U_{n\omega})$ *on* $X(k, n, \omega)$ *to the* $\widetilde{\mathfrak{B}}$ *action on its image.*

Proof Let us recall the explicit action of the Iwahori group on the affine flag variety sitting inside the product of Sato Grassmannians. Let us fix an n-dimensional space W with a basis $w_1, \ldots, w_n$. We identify the space $\mathbb{C}[t, t^{-1}]$ with $W \otimes \mathbb{C}[z, z^{-1}]$ via the map

$$\varphi : t^{(n-k)\omega - rn - s + 1} \mapsto w_s \otimes z^r, \ r \in \mathbb{Z}, s = 0, \ldots, n-1.$$

Let us comment on the choice of the map φ. Recall (see (6.2)) the mappings $\psi^{(i)} : M^{(i)} \to \mathbb{C}[t, t^{-1}]$, sending $v^{(i)}_{n\omega+1-j}$ to $t^{i-k\omega+j}$. The map φ is chosen in such a way that the composition map $\varphi\psi^{(0)} : M^{(0)} \to W \otimes \mathbb{C}[z, z^{-1}]$ (see (6.2)) is given by

$$\varphi\psi^{(0)} : v^{(0)}_{rn+s} \mapsto w_s \otimes z^r.$$

The general composition map $\varphi\psi^{(i)} : M^{(i)} \to W \otimes \mathbb{C}[z, z^{-1}]$ reads as

$$\varphi\psi^{(i)} : v^{(i)}_{rn+s+i} \mapsto w_s \otimes z^r.$$

The natural action of $GL_n(\mathbb{C}[z])$ (and thus of the Iwahori subgroup $\widetilde{\mathfrak{B}}$) on the space $W \otimes \mathbb{C}[z, z^{-1}]$ induces the action on $\mathbb{C}[t, t^{-1}]$ via the identification above. Each Iwahori group element $\tilde{g}(z)$ induces the linear endomorphism of $F^{(i)}$ and a map on $\mathcal{AF}_n$ embedded into the product of the projective spaces $\mathbb{P}(F^{(i)})$.

Recall (see Proposition 2.10) that the elements of the automorphism group $\mathrm{Aut}_{\Delta_n}(U_{n\omega})$ are the collections of maps $A_i \in \mathrm{End}(M^{(i)})$ with

$$A_i = \begin{pmatrix} a^{(i)}_{1,1} & & & & \\ a^{(i)}_{2,1} & a^{(i-1)}_{1,1} & & & \\ \vdots & \vdots & \ddots & & \\ a^{(i)}_{n\omega-1,1} & a^{(i-1)}_{n\omega-2,1} & \cdots & a^{(i-n\omega+2)}_{1,1} & \\ a^{(i)}_{n\omega,1} & a^{(i-1)}_{n\omega-1,1} & \cdots & a^{(i-n\omega-2)}_{2,1} & a^{(i-n\omega+1)}_{1,1} \end{pmatrix}$$

written in the basis $v^{(i)}_j$.

Given a collection of maps $A = (A_i) \in \mathrm{Aut}_{\Delta_n}(U_{n\omega})$ we construct the Iwahori group element $\tilde{g}(z) \in GL_n(\mathbb{C}[z])$ such that $\Psi A = \tilde{g}(z)$. Recall that the map $\Psi : X(k,n,\omega) \to \mathcal{AF}_n$ is induced by the maps (6.2). Then one shows that the following $\tilde{g}(z)$ does the job:

$$\begin{pmatrix} \sum_{l=0}^{\omega-1} z^l a_{1+nl,1}^{(0)} & z\sum_{l=0}^{\omega-1} z^l a_{n+nl,1}^{(-1)} & \dots & z\sum_{l=0}^{\omega-1} z^l a_{2+nl,1}^{(-n+1)} \\ \sum_{l=0}^{\omega-1} z^l a_{2+nl,1}^{(0)} & \sum_{l=0}^{\omega-1} z^l a_{1+nl,1}^{(-1)} & \dots & z\sum_{l=0}^{\omega-1} z^l a_{3+nl,1}^{(-n+1)} \\ \vdots & \vdots & \ddots & \\ \sum_{l=0}^{\omega-1} z^l a_{n-1+nl,1}^{(0)} & \sum_{l=0}^{\omega-1} z^l a_{n-2+nl,1}^{(-1)} & \dots & z\sum_{l=0}^{\omega-1} z^l a_{n+nl,1}^{(-n+1)} \\ \sum_{l=0}^{\omega-1} z^l a_{n+nl,1}^{(0)} & \sum_{l=0}^{\omega-1} z^l a_{n-1+nl,1}^{(-1)} & \dots & \sum_{l=0}^{\omega-1} z^l a_{1+nl,1}^{(-n+1)} \end{pmatrix}$$

In fact, let us examine the image $\tilde{g}(z)w_1$. By definition, one gets

$$\tilde{g}(z)w_1 = a_{1,1}^{(0)}w_1 + a_{2,1}^{(0)}w_2 + \dots + a_{n,1}^{(0)}w_n + a_{n+1,1}^{(0)}w_1 \otimes z + \dots + a_{2n,1}^{(0)}w_n \otimes z \\ + \dots + a_{n(\omega-1)+1,1}^{(0)}w_n \otimes z^{\omega-1} + \dots + a_{n\omega,1}^{(0)}w_n \otimes z^{\omega-1},$$

which agrees with the image $\Psi A w_1$. The same argument works for $w_2, \dots, w_n$. □

Example 6.11 Let $n = \omega = 2$. Then

$$A_0 = \begin{pmatrix} a_{1,1}^{(0)} & 0 & 0 & 0 \\ a_{2,1}^{(0)} & a_{1,1}^{(1)} & 0 & 0 \\ a_{3,1}^{(0)} & a_{2,1}^{(1)} & a_{1,1}^{(0)} & 0 \\ a_{4,1}^{(0)} & a_{3,1}^{(1)} & a_{2,1}^{(0)} & a_{1,1}^{(1)} \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_{1,1}^{(1)} & 0 & 0 & 0 \\ a_{2,1}^{(1)} & a_{1,1}^{(0)} & 0 & 0 \\ a_{3,1}^{(1)} & a_{2,1}^{(0)} & a_{1,1}^{(1)} & 0 \\ a_{4,1}^{(1)} & a_{3,1}^{(0)} & a_{2,1}^{(1)} & a_{1,1}^{(0)} \end{pmatrix}$$

and

$$\tilde{g}(z) = \begin{pmatrix} a_{1,1}^{(0)} + za_{3,1}^{(0)} & za_{2,1}^{(1)} + z^2 a_{4,1}^{(1)} \\ a_{2,1}^{(0)} + za_{4,1}^{(0)} & a_{1,1}^{(1)} + za_{3,1}^{(1)} \end{pmatrix}.$$

Corollary 6.12 *For every $I \in \binom{[n]}{k}$ one has an equality*

$$\Psi Aut_{\Delta_n}(U_{n\omega})p_I = \mathfrak{B}p_{w(I)}.$$

Proof Thanks to Proposition 6.10 the desired equality holds if we use the GL_n version of the Iwahori group instead of $\mathfrak{B}$. However, $\mathfrak{B}p_{w(I)}$ coincides with the GL_n Iwahori group orbit of the point $p_{w(I)}$. □

Remark 6.13 Corollary above claims that for any $A \in \mathrm{Aut}_{\Delta_n}(U_{n\omega})$ and any $I \in \binom{[n]}{k}$ there exists an element $g(z) \in \mathfrak{B}$ such that $Ap_I = g(z)p_{w(I)}$. We note that $g(z)$ does depend on I, but if we extend $\mathfrak{B}$ to the Iwahori subgroup for GL_n (as we do in Proposition 6.10), then $g(z)$ depends only on A.

We summarize the discussion above as follows.

Theorem 6.14 *The quiver Grassmannian $X(k,n,\omega)$ is isomorphic to the union of Schubert varieties $X_{w(I)} \subset \mathcal{AF}_n$ for all $I \in \binom{[n]}{k}$.*

The embedded varieties $X(k,n,\omega)$ exhaust the affine flags. More precisely, the following proposition holds.

Proposition 6.15 *For all k and n one has $\Psi(X(k,n,\omega)) \subset \Psi(X(k,n,\omega+1))$. The union $\bigcup_{\omega\geq 1}\Psi(X(k,n,\omega))$ is equal to $\mathcal{AF}_n$.*

Proof By definition, $X(k,n,\omega) = \mathrm{Gr}_{(k\omega,\ldots,k\omega)}(U_{\omega n})$. The embedding of Δ_n-modules $U_{\omega n} \subset U_{(\omega+1)n}$, defined by sending $v_j^{(i)}$ to $v_{j+n}^{(i)}$, induces the embedding of quiver Grassmannians $X(k,n,\omega) \subset X(k,n,\omega+1)$. Formula (6.2) shows that this embedding is compatible with the embeddings into the affine flag variety. Hence $\Psi(X(k,n,\omega)) \subset \Psi(X(k,n,\omega+1))$.

To prove the second claim it suffices to show that for any $f \in W_n$ the Schubert variety X_f sits inside $X_{w(I)}$ for some $I \in \binom{[n]}{k}$ for large enough ω. We rephrase the condition in the following way: let $p_f = (p_{f,i})_{i=1}^n$, $p_{w(I)} = (p_{w(I),i})_{i=1}^n$ and let

$$p_{f,i} = \mathbb{C}.t^{a_1} \wedge t^{a_2} \wedge \ldots,\ a_1 > a_2 > \ldots,$$
$$p_{w(I),i} = \mathbb{C}.t^{b_1} \wedge t^{b_2} \wedge \ldots,\ b_1 > b_2 > \ldots.$$

Then

$$X_f \subset X_{w(I)} \text{ if and only if } b_1 \geq a_1, b_2 \geq a_2, \ldots. \tag{6.7}$$

First, let $k = 1$. Let M_1, M_2 be positive integers such that

- $a_s \leq i + M_2 n$ for all s,
- for any $j \leq i - M_1$ there exists s such that $a_s = j$.

In particular, since $p_{f,i} = \mathbb{C}.t^{a_1} \wedge t^{a_2} \wedge \cdots \in \mathrm{SGr}^{(i)}$ the number of $a_\bullet$ in between $i - M_1 + 1$ and $i + M_2 n$ is equal to M_1. Now assume that ω satisfies $(n-1)\omega > (M_1 + M_2)n$. Then formula (6.2) guaranties that for any i one has $\psi(v_1^{(i)}) = t^L$ with $L > i + M_2 n + M_1 n$. Therefore the number of elements $b_\bullet$ which are larger than $i + M_2 n$ is at least M_1. Hence the condition in (6.7) holds true. Now let us consider the case of arbitrary k. Then for any ω and any cardinality one subset $I \subset [n]$ one finds an ω' and $J \subset \binom{[n]}{k}$ such that $X_{w(I)} \subset X_{w(J)}$. Hence for any k the union of the Ψ images of $X(k,n,\omega)$ is equal to $\mathcal{AF}_n$. □

Remark 6.16 We note that in general $\Psi(X(k,n,\omega))$ is not contained in $\Psi(X(k+1,n,\omega))$. The simplest example is $n = 3$, $k = 1$, $\omega = 1$. Then $X(1,3,1)$ and $X(2,3,1)$ are unions of three two-dimensional Schubert varieties inside $\mathcal{AF}_3$ and all these six Schubert varieties are different.

Example 6.17 We close this section with an example for $n = 2$. The quiver Grassmannians we are interested in are of the form $X(1,2,\omega)$. The Weyl group W_2 is generated by s_0 and s_1; the elements of W_2 are of the form $s_1 s_0 s_1 \ldots$ and $s_0 s_1 s_0 \ldots$. In particular, for each $\omega > 0$ there exist exactly two elements $\sigma_1(\omega), \sigma_2(\omega) \in W_2$ of length ω. The image of the quiver Grassmannian $X(1,2,\omega)$ inside the affine flag variety $\mathcal{AF}_2$ is equal to the union of the Schubert varieties $X_{\sigma_1(\omega)} \cup X_{\sigma_2(\omega)}$.

Appendix A Flatness: an example

Let us consider the degeneration of $\mathbb{P}^2 = \mathrm{Gr}(1,3)$ to the quiver Grassmannian $X(1,3) = X(1,3,1)$. It was claimed in [12] that this degeneration is not flat. More precisely, it was claimed in loc.cit. that the dimension of the degree $(1,1,1)$ component of the homogeneous coordinate ring of $X(1,3)$ is larger than 10 (which is the dimension of the degree three component of the homogeneous coordinate ring of $\mathrm{Gr}(1,3)$). However, the argument contains a mistake (we note that according to [14] the degeneration of any $\mathrm{Gr}(k,n)$ to $X(k,n)$ is flat). Below we correct this mistake.

The variety $X(1,3)$ sits inside $\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$. The corresponding homogeneous coordinate ring is triply-graded. We denote the homogeneous coordinates in the i-th $\mathbb{P}^2$ by x_i, y_i, z_i $(i=1,2,3)$. In particular, the degree $(1,1,1)$ homogeneous component of the coordinate ring sits inside the polynomial ring in variables $x_\bullet, y_\bullet, z_\bullet$; it consists of polynomials which are linear in each group of variables x_i, y_i, z_i $(i=1,2,3)$.

Recall that $X(1,3)$ has three irreducible components; the open cells in these irreducible components are of the form

$$\begin{pmatrix}1&0&0\\a_1&1&0\\b_1&a_1&1\end{pmatrix},\quad\begin{pmatrix}0&1&0\\0&a_2&1\\1&b_2&a_2\end{pmatrix},\quad\begin{pmatrix}0&0&1\\1&0&a_3\\a_3&1&b_3\end{pmatrix}.$$

Here the j-th matrix corresponds to the j-th irreducible component $(j=1,2,3)$; for $i=1,2,3$ the i-th column of the j-th matrix produces the homogeneous coordinates x_i, y_i, z_i of the points in the open cell of the j-th irreducible component (we note that $a_\bullet$ and $b_\bullet$ are free parameters).

The dimension of the degree three homogeneous component of the coordinate ring of the Grassmannian $\mathrm{Gr}(1,3)=\mathbb{P}^2$ is equal to ten. Hence we have to show that the dimension of the degree $(1,1,1)$ homogeneous component of the coordinate ring of $X(1,3)$ is also equal to 10. We claim that the following ten monomials produce a basis of the degree $(1,1,1)$ component:

$$\begin{aligned}y_1y_2z_3=(a_1,0,0),\quad y_1z_2y_3=(0,0,a_3),\quad y_1z_2z_3=(a_1^2,0,b_3),\\ z_1y_2y_3=(0,a_2,0),\quad z_1y_2z_3=(b_1,a_2a_3,0),\quad z_1z_2y_3=(0,b_2,a_1a_3),\\ z_1z_2z_3=(b_1a_2,b_2a_3,b_3a_1),\\ x_1y_2z_3=(1,0,0),\quad x_1y_3z_3=(0,1,0),\quad y_1z_2x_3=(0,0,1).\end{aligned}$$

Here the right hand side of each equality computes the value of the corresponding monomial on the three open cells. The monomials above are clearly linearly independent. The values of all other degree $(1,1,1)$ monomials are either zero or coincide with one of the above. For example, $x_1x_2z_3=0$ and $x_1z_2z_3=(a_1,0,0)=y_1y_2z_3$.

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