Commun. Math. Phys. 386, 305–358 (2021)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-021-04046-6

Communications in **Mathematical Physics**

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Covariant Homogeneous Nets of Standard Subspaces

Vincenzo Morinelli[1], **Karl-Hermann Neeb**[2]

[1] Dipartimento di Matematica, Universita di Roma "Tor Vergata", via della Ricerca Scientica 1, 00133 Roma, Italy.
E-mail: morinell@mat.uniroma2.it

[2] Department Mathematik, FAU Erlangen-Nürnberg, Erlangen, Germany.
E-mail: neeb@math.fau.de

Received: 23 October 2020 / Accepted: 17 February 2021
Published online: 30 March 2021 –

Abstract: Rindler wedges are fundamental localization regions in AQFT. They are determined by the one-parameter group of boost symmetries fixing the wedge. The algebraic canonical construction of the free field provided by Brunetti–Guido–Longo (BGL) arises from the wedge-boost identification, the BW property and the PCT Theorem. In this paper we generalize this picture in the following way. Firstly, given a $\mathbb{Z}_2$-graded Lie group we define a (twisted-)local poset of abstract wedge regions. We classify (semisimple) Lie algebras supporting abstract wedges and study special wedge configurations. This allows us to exhibit an analog of the Haag–Kastler one-particle net axioms for such general Lie groups without referring to any specific spacetime. This set of axioms supports a first quantization net obtained by generalizing the BGL construction. The construction is possible for a large family of Lie groups and provides several new models. We further comment on orthogonal wedges and extension of symmetries.

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1. Introduction

Quantum Field Theory (QFT) lives in a tension between the locality principle and the underlying group of symmetries characterizing the theory. On one hand, it is a physical principle that every interesting quantity of a theory should be deducible by local measurements, namely—in the language of Algebraic Quantum Field Theory (AQFT)—by the structure of the local algebras (see e.g. [Ha96]). On the other hand, the symmetries of a theory provide a feature to describe physical objects, a "key to nature's secrets," as it happens in the standard model [We05,We11].

In AQFT, models are specified by a net of von Neumann algebras associated to causally complete spacetime regions satisfying fundamental quantum and relativistic principles, such as isotony, locality, covariance, positivity of the energy, and existence of a vacuum state. An important bridge between the geometry and the algebraic structure is the Bisognano–Wichmann (BW) property of (A)QFT claiming that the modular group of the algebra associated to any Rindler wedge W inside Minkowski spacetime with respect to the vacuum state implements unitarily the covariant one-parameter group of boosts fixing the wedge W. As a consequence, the algebraic structure of the model, through the Tomita–Takesaki theory, contains the information about the symmetry group acting on the model. Starting with the BW property, one can enlarge the symmetry group of a QFT [GLW98,MT18], find new relations among field theories [GLW98,LMPR19,MR20], establish proper relations among spin and statistics [GL95], and compute entropy in QFT [LX18,Wi18]. For recent results on this property we refer to [Gu19,DM20].

Particles are field-derived concepts that can be described as unitary positive energy representations of the symmetry group. They are building blocks to construct Quantum Field Theories. The operator-valued distribution Φ_U defining the free field associated to any particle U is not provided by a canonical construction, see e.g. [BGL02,LMR16]. On the other hand, the von Neumann algebra net generated by Φ_U satisfies the Bisognano–Wichmann property and the PCT Theorem.[1] These properties provide the tools for a canonical construction of the free algebra net [BGL02]: Segal's second quantization gives the vacuum representation of the Weyl algebra on the Fock space associated with the one-particle Hilbert space. The Araki lattice of von Neumann algebras is uniquely determined by the local one-particle structure encoded in the lattice of closed real subspaces, the first quantization [Ar63]. As a result of the Tomita–Takesaki modular theory

[1] The spacetime reflection $j_1(t, x_1, x_2, \ldots, x_n) = (-t, -x_1, x_2, \ldots, x_n)$ is implemented by the modular conjugation corresponding to the standard right wedge W_1.

for real subspaces, the set of real states for a particle U localized in a wedge region is uniquely determined by the couple $(e^{-2\pi K_W}, U(j_W))$ where $U(j_W)$ is the antiunitary implementation of the wedge reflection and K_W is the generator of the one-parameter group of boosts associated to the wedge W. They satisfy the Tomita relation $U(j_W)e^{2\pi K_W}U(j_W) = e^{-2\pi K_W}$. The one-particle states and the local algebra associated to bounded causally complete regions are obtained by wedge state spaces and algebra intersection, respectively.

Conversely, every pair (x, σ), consisting of an element x of the Poincaré–Lie algebra and an involution σ satisfying $\mathrm{Ad}(\sigma)x = x$ specifies for every (anti-)unitary representation $(U, \mathcal{H})$ of the Poincaré group a pair $(\Delta, J) = (e^{2\pi i \partial U(x)}, U(\sigma))$ that in turn defines a standard subspace $\mathtt{V} \subseteq \mathcal{H}$. This construction, called the *BGL construction*, was introduced in [BGL02] and allows us to observe: *The algebraic construction of the free fields is uniquely determined by its symmetries and the correspondence between spacetime regions and their relative position with symmetries*. In this sense, due to the one-to-one correspondence between boosts and the corresponding wedges, one should be able to specify the underlying symmetry structure of a quantum field theory without any reference to the spacetime. Then one can reconstruct the spacetime features, such as locality and region inclusions from the symmetry group.

With this claim in mind, we generalize the above picture as follows. Given a suitable Lie group G, we first define an abstract wedge space. We then endow the wedge space with a G-action, a notion of causal complement and an order structure. Eventually, starting from an (anti-)unitary representation of a graded Lie group G, we construct the analogue of the BGL one-particle net by the abstract setting.

We now collect the motivation and additional explanations of the fundamental structure we will use. In order to obtain a one-particle net by the Tomita–Takesaki theory we need to start with a graded Lie group $G = G^{\uparrow} \rtimes \mathbb{Z}_2$, such as the improper Möbius group $\mathrm{PGL}_2(\mathbb{R})$ or the proper Poincaré group $\mathcal{P}_+$. For the moment, we assume that $Z(G^{\uparrow}) = \{e\}$ and that $G^{\uparrow}$ is connected.

The key features of our approach are the following:

• *Abstract boost generator.* The abstract one-parameter group of boosts are generated by elements x in the Lie algebra $\mathfrak{g}$ of G defining a three grading $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ in the adjoint representation by $\mathfrak{g}_j = \ker(\mathrm{ad}\, x - j\, \mathrm{id}_{\mathfrak{g}})$. To see how this complies with the well known models, see Examples 2.10. We call such elements $x \in \mathfrak{g}$ Euler elements because they corresponds to the linear Euler vector field on the open embedding $\mathfrak{g}_1 \hookrightarrow G/\mathbf{P}$, $s \mapsto \exp(s)\mathbf{P}$, where $\mathbf{P} \subseteq G$ is the connected subgroup corresponding to the Lie algebra $\mathfrak{g}_0 + \mathfrak{g}_{-1}$. For more on the underlying geometry of theses spaces, we refer to [BN04].

• *The wedge reflection* is obtained by analytic continuation of the one-parameter group of boosts associated to the wedge at $i\pi$. For instance, on Minkowski space, the wedge reflection $j_1 = \Lambda_1(i\pi)$ is obtained by analytic extension of the one-parameter group of boosts in the first direction $\Lambda_1(t) = \exp(t\sigma_1)$ where $(\sigma_i)_{i=1,2,3}$ are the Pauli matrices. In our general setting, the reflection σ, called Euler involution, associated to an Euler element x is determined by the analytic continuation of the one-parameter group in the adjoint representation of the Lie algebra via $\mathrm{Ad}(\sigma) = e^{\pi i\, \mathrm{ad}\, x}$ (see (2.10)).

• *Euler wedge.* An Euler wedge is defined as a couple $W = (x_W, \sigma_W)$ of an Euler element and the related Euler involution. The need to use the couple is to implement the G-action on the wedge space (see (2.6)) and to establish the relation with the standard subspaces $\mathtt{V}$ and the corresponding modular objects $(\Delta_{\mathtt{V}}, J_{\mathtt{V}})$. We further remark that, in principle, it is not necessary to assume that the involution σ_W satisfies

$$\mathrm{Ad}(\sigma_W) = e^{\pi i \,\mathrm{ad}\, x_W}$$

in the adjoint representation, but only that it satisfies the proper commutation relation $\mathrm{Ad}(\sigma_W)x_W = x_W$, cf. Proposition 2.1.

• $G^{\uparrow}$*-covariance*. There is an action of the group G on the wedge space given by an adjoint action on both components that takes care of the grading (see (2.8)). In this way the language of Euler wedges is consistent with the one of the standard subspaces, cf. Sect. 2.

• *Locality*. Complementary wedges correspond to inverted one-parameter groups of boosts. For instance dilations associated to causally complementary intervals in chiral theory or boosts associated to complementary wedges are inverse to each other. On the abstract wedge space this is captured by defining the complementary wedge of $W = (x, \sigma)$ by $W' = (-x, \sigma)$.

• *Isotony*. By the existence of a (positive) invariant cone C in the Lie algebra $\mathfrak{g}$, it is possible to define a wedge endomorphism semigroup defining the wedge inclusion relation. Given an Euler wedge $W = (x, \sigma)$, the generators in the positive cone lying in the subspaces $\mathfrak{g}_{\pm 1}$ define proper wedge inclusions as each of them generates with x a translation-dilation group (isomorphic to the affine group of the real line); see [Bo92, Wi92,Wi93] and in particular [Bo00]. This is the case of wedge endomorphisms in Minkowski spacetime given by lightlike shifting or Möbius transformations mapping an interval into itself as the translations do for the half-lines. *These properties define a local partially ordered set of wedges that can support key features of an AQFT structure.*

It is important to note that the wedge space **only depends on the Lie group and its Lie algebra**, and the order structure given by the invariant cone $C \subseteq \mathfrak{g}$. The relations among the wedges specify the abstract spacetime structure to a large extent. For example, $\mathrm{PSL}_2(\mathbb{R})$ is the symmetry group for the 2-dimensional de Sitter spacetime and for the chiral circle. If one considers $\mathrm{PSL}_2(\mathbb{R})$ with the trivial cone in $\mathfrak{sl}_2(\mathbb{R})$—no proper inclusions of wedges—then it describes a QFT on de Sitter spacetime; if one considers $C \subset \mathfrak{sl}_2(\mathbb{R})$ as in (2.17), inclusion relations among wedges arise, and we obtain the wedge space on $\mathbb{S}^1$.[2] This correspondence between isotony and and positivity of the energy was also studied in [GL03]; see also [Bo92,Bo00,Wi92,Wi93] and [NÓ17,Ne19,Ne19b]. For recent classification results for the triples $(\mathfrak{g}, x, C)$, we refer to [Oeh20,Oeh21].

There is more interesting structure on the abstract wedge space:

• *Orthogonal wedges*: We call two abstract wedges $W_1 = (x_1, \sigma_1)$ and $W_2 = (x_2, \sigma_2)$ orthogonal if $\sigma_1(x_2) = -x_2$, i.e., W_2 is reflected into its complement W_2'. Examples of orthogonal wedges are coordinate wedges on Minkowski spacetime,[3] or the upper and the right half-circle in chiral theories on $\mathbb{S}^1$. This notion, which immediately generalizes to the abstract setting, plays a central role in spin-statistics relations [GL95] and the nuclearity property in conformal field theory [BDL07].

• *Symmetric wedges*. A wedge W is called symmetric if there exists $g \in G^{\uparrow}$, such that $g.W = W'$. For instance, any couple of wedge regions, in $1+s$-dimensional Minkowski spacetime with $s \geq 2$, are transformed one into the other by the action of the Poincaré group $G^{\uparrow} = \mathcal{P}_+^{\uparrow}$. On the other hand, in $1+1$-dimensional Minkowski space, the right and the left wedges are not symmetric. Indeed

$$W_R = \{(t, x) \in \mathbb{R}^{1+1} : |t| < x\} \quad \text{and} \quad W_L = \{(t, x) \in \mathbb{R}^{1+1} : |t| < -x\}$$

[2] In [GL03] it is used that the 2-dimensional de Sitter space $\mathrm{dS}^2 \cong \mathrm{SO}_{1,2}(\mathbb{R})^{\uparrow}/\mathrm{SO}_{1,1}(\mathbb{R})^{\uparrow}$ has the same abstract wedge space as the circle $\mathrm{SO}_{1,2}(\mathbb{R})/\mathbf{P}$ to set up a dS/CFT correspondence.

[3] For instance W_i and W_j for $i \neq j$, where $W_i = \{(t, x) \in \mathbb{R}^{1+s} : |t| < x_i\}$.

belong to disjoint transitive families with respect to the $\mathcal{P}_+^\uparrow$-action. Further examples of symmetric wedges are intervals in conformal theories on the circle. Half-lines in the real line are not symmetric wedges with respect to the translation-dilation group. A transitive family of wedges has the feature that algebras associated to complementary wedges are - by covariance - unitary equivalent. On the other side, there is no contradiction in having a $G^\uparrow$-covariant net of von Neumann algebras on a transitive family of non-symmetric wedges with trivial algebras associated to the family of complements.

In the first part of the paper we define and investigate the abstract structure we have described. When the center $Z(G^\uparrow)$ is non trivial, for instance when covering groups are considered, a generalized notion of complementary wedges has to be introduced. Indeed, while Euler elements are uniquely determined as generators of one-parameter groups in $G^\uparrow$, several involutions σ satisfying $\mathrm{Ad}(\sigma) = e^{\pi i \,\mathrm{ad}\, x}$ can be associated to the same Euler element x. In an analogous way, different wedge complements can be labeled by central elements. We classify wedge orbits and define a notion of a central wedge complement. Furthermore, if W' does not belong to the $G^\uparrow$-orbit of W, a new action of G on the wedge space is defined. This happens for instance in fermionic nets.

Having specified the abstract structures, we are prepared to answer the following question:
"*Which Lie algebras/groups support such a structure?*" To this end, we first classify Euler elements in real simple Lie algebras in Theorem 3.10. The key point of this classification is that Euler elements are conjugate under inner automorphisms to elements in any given Cartan subspace of hyperbolic elements. Here the restriction to simple Lie algebras is not restrictive because any symmetric Euler element is contained in a semi-simple Lie subalgebra. Furthermore, an Euler element is symmetric if and only if it is contained in an $\mathfrak{sl}_2(\mathbb{R})$-subalgebra (see Theorem 3.13 for these results). As a consequence, there is a large family of real Lie algebras supporting such wedge structures which properly contains the well known models. Note that, for a Lie algebra $\mathfrak{g}$ containing an Euler element $x \in \mathfrak{g}$, there always exists a graded Lie group G with Lie algebra $\mathfrak{g}$ and a corresponding Euler wedge (x, σ).

The second part of the paper is devoted to nets of standard subspaces.
Is it possible to construct one-particle models supporting this abstract setting? Starting with a $G^\uparrow$-orbit $\mathcal{W}_+$ in the wedge space, we describe a set of axioms which, for the well known models, reflect fundamental quantum and relativistic principles corresponding to the one-particle Haag–Kastler axioms. This set of axioms is fulfilled by extending the BGL construction to every graded Lie group G, supporting a suitable wedge space. A twisted locality relation among complementary wedges is introduced in order to relate central complementary wedges.

Do we get any new models out of this general construction? The answer is affirmative. All the simple Lie algebras whose restricted root system appears in Theorem 3.10 correspond to a graded Lie group with a non-trivial wedge space. There are for instance Lie algebras of type E_7 that do not correspond to any known models. In this context the Jordan spacetimes of Günaydin [Gu93,Gu00,Gu01] and the simple spacetime manifolds in the sense of Mack–de Riese [MdR07] are homogeneous spaces of simple hermitian Lie groups whose Lie algebras contain Euler elements, and the corresponding abstract wedges correspond to domains in these causal manifolds. These Lie groups have many (anti-)unitary representations, some of them with positive energy with respect to a non-trivial invariant cone C in the Lie algebra. As a consequence, they support many one-particle nets [NÓ20] and second quantization models of von Neumann algebras whose physical meaning has to be investigated.

The structure of this paper is as follows: In Sect. 2 the wedge space is defined and its properties are studied. A number of examples are discussed in detail to show how the abstract setting applies to the known models and realizes the well known structure. In Sect. 3 we study the Euler elements in Lie algebras. We relate orthogonal and symmetric wedges and provide a classification of Lie algebras supporting (symmetric) Euler elements. In Sect. 4 we apply this structure to define and construct one-particle nets associated to graded Lie groups supporting a wedge structure. We further stress new models, orthogonal wedges and extension of symmetries. An outlook on the construction is contained in Sect. 5. We hope this paper is approachable for the Lie Theory community as well as the Algebraic Quantum Field Theory community.

2. The Abstract Setting

In this section we develop an abstract perspective on wedge domains in spacetimes, phrased completely in group theoretic terms. As wedge domains are supposed to correspond to standard subspaces in Hilbert spaces, we orient our approach on how standard subspaces are parametrized.

Let $\mathrm{Stand}(\mathcal{H})$ denote the set of standard subspaces of the complex Hilbert space $\mathcal{H}$. In Sect. 4 we shall see that every standard subspace $\mathtt{V}$ determines a pair $(\Delta_{\mathtt{V}}, J_{\mathtt{V}})$ of modular objects and that $\mathtt{V}$ can be recovered from this pair by $\mathtt{V} = \mathrm{Fix}(J_{\mathtt{V}}\Delta_{\mathtt{V}}^{1/2})$. This observation can be used to obtain a representation theoretic parametrization of $\mathrm{Stand}(\mathcal{H})$: each standard subspace $\mathtt{V}$ specifies a continuous homomorphism

$$U^{\mathtt{V}} \colon \mathbb{R}^\times \to \mathrm{AU}(\mathcal{H}) \quad \text{by} \quad U^{\mathtt{V}}(e^t) := \Delta_{\mathtt{V}}^{-it/2\pi}, \quad U^{\mathtt{V}}(-1) := J_{\mathtt{V}}. \tag{2.1}$$

We thus obtain a bijection between $\mathrm{Stand}(\mathcal{H})$ and the set $\mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, \mathrm{AU}(\mathcal{H}))$ of continuous morphisms of graded topological groups.

The space $\mathrm{Stand}(\mathcal{H})$ carries three important features:

- an order structure, defined by set inclusion
- a duality operation $\mathtt{V} \mapsto \mathtt{V}' = \{\xi \in \mathcal{H} \colon (\forall v \in \mathtt{V})\ \mathrm{Im}\langle \xi, v\rangle = 0\}$
- the action of $\mathrm{AU}(\mathcal{H})$ as a symmetry group.

The order structure is hard to express in terms of the modular groups (see [Ne19b] for some first steps in this direction), but the duality operation corresponds to inversion

$$U^{\mathtt{V}'}(r) = U^{\mathtt{V}}(r^{-1}) \quad \text{for} \quad r \in \mathbb{R}^\times, \tag{2.2}$$

and the action of $\mathrm{AU}(\mathcal{H})$ translates into

$$U^{g\mathtt{V}}(r) = gU^{\mathtt{V}}(r^{\varepsilon(g)})g^{-1} \quad \text{for} \quad g \in \mathrm{AU}(\mathcal{H}), r \in \mathbb{R}^\times, \tag{2.3}$$

where $\varepsilon(g) = 1$ if g is unitary and $\varepsilon(g) = -1$ otherwise. So unitary operators $g \in \mathrm{U}(\mathcal{H})$ simply act by conjugation, but antiunitary operators also involve inversion. In particular, $J_{\mathtt{V}}\mathtt{V} = \mathtt{V}'$ corresponds to

$$U^{\mathtt{V}'}(r) = J_{\mathtt{V}}U^{\mathtt{V}}(r^{-1})J_{\mathtt{V}} = U^{\mathtt{V}}(r^{-1}) \quad \text{for} \quad r \in \mathbb{R}^\times.$$

We now develop the corresponding structures by replacing $\mathrm{AU}(\mathcal{H})$ by a finite dimensional graded Lie group.

2.1. Group theoretical setting. The basic ingredient of our approach is a finite dimensional *graded Lie group* (G, ε_G), i.e., G is a Lie group and $\varepsilon_G : G \to \{\pm 1\}$ a continuous homomorphism. We write

$$G^\uparrow = \varepsilon_G^{-1}(1) \quad \text{and} \quad G^\downarrow = \varepsilon_G^{-1}(-1),$$

so that $G^\uparrow \trianglelefteq G$ is a normal subgroup of index 2 and $G^\downarrow = G \setminus G^\uparrow$. We also fix a pointed closed convex cone $C \subseteq \mathfrak{g}$ satisfying

$$\mathrm{Ad}(g)C = \varepsilon_G(g)C \quad \text{for} \quad g \in G. \tag{2.4}$$

As we shall see in the following, for graded Lie groups, it is more natural to work with the *twisted adjoint action*

$$\mathrm{Ad}^\varepsilon : G \to \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{Ad}^\varepsilon(g) := \varepsilon_G(g)\,\mathrm{Ad}(g), \tag{2.5}$$

so that (2.4) actually means that C is invariant under the twisted adjoint action. The cone C will play a role in specifying an order structure. It is related to positive spectrum conditions on the level of unitary representations. We also allow $C = \{0\}$. For instance, the Lie algebra $\mathfrak{g} = \mathfrak{so}_{1,d}(\mathbb{R})$ of the Lorentz group $G = \mathrm{O}_{1,d}(\mathbb{R})$, the isometry group of de Sitter space time dS^d, contains no non-trivial invariant cone.

2.2. The space $\mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, G)$ *and abstract wedges.* In this section we define the fundamental objects we will need in the forthcoming discussion. We write $\mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, G)$ for the space of continuous morphisms of graded Lie groups $\mathbb{R}^\times \to G$, where $\mathbb{R}^\times$ is endowed with its canonical grading by $\varepsilon(r) := \mathrm{sgn}(r)$. On this space G acts by

$$(g.\gamma)(r) := g\gamma(r^{\varepsilon_G(g)})g^{-1}, \tag{2.6}$$

where the twist is motivated by formula (2.2). Elements of $G^\uparrow$ simply act by conjugation.

Since we are dealing with Lie groups, we also have the following simpler description of the space $\mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, G)$ by the set

$$\mathcal{G} := \{(x, \sigma) \in \mathfrak{g} \times G^\downarrow : \sigma^2 = e, \mathrm{Ad}(\sigma)x = x\}.$$

Proposition 2.1. *The map*

$$\Psi : \ \mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, G) \to \mathcal{G}, \quad \gamma \mapsto (\gamma'(1), \gamma(-1)) \tag{2.7}$$

is a bijection. It is equivariant with respect to the action of G on $\mathcal{G}$ by

$$g.(x, \sigma) := (\mathrm{Ad}^\varepsilon(g)x, g\sigma g^{-1}). \tag{2.8}$$

Note that center $Z(G^\uparrow)$ of $G^\uparrow$ acts trivially on the Lie algebra but it may act non-trivially on involutions in $G^\downarrow$.

Remark 2.2. For every involution $\sigma \in G^\downarrow$, the involutive automorphism $\sigma_G(g) := \sigma g \sigma$ defines the structure of a symmetric Lie group $(G^\uparrow, \sigma_G)$, and $G \cong G^\uparrow \rtimes \{\mathrm{id}, \sigma\}$, so that we can translate between G as a graded Lie group and the pair $(G^\uparrow, \sigma_G)$, without loosing information.

To indicate the analogy of elements of $\mathcal{G}$ with the wedge domains in QFT, we shall often denote the elements of $\mathcal{G}$ by $W = (x, \sigma)$.

Definition 2.3. (a) We assign to $W = (x, \sigma) \in \mathcal{G}$ the one-parameter group

$$\lambda_W : \mathbb{R} \to G^{\uparrow} \quad \text{by} \quad \lambda_W(t) := \exp(tx) \tag{2.9}$$

Then we have the graded homomorphism

$$\gamma_W : \mathbb{R}^{\times} \to G, \quad \gamma_W(e^t) := \lambda_W(t), \qquad \gamma_W(-1) := \sigma.$$

Note that $\Psi(\gamma_W) = W$ in terms of (2.7).

Definition 2.4. (a) We call an element x of the finite dimensional real Lie algebra $\mathfrak{g}$ an *Euler element* if $\mathrm{ad}\, x$ is diagonalizable with $\mathrm{Spec}(\mathrm{ad}\, x) \subseteq \{-1, 0, 1\}$, so that the eigenspace decomposition with respect to $\mathrm{ad}\, x$ defines a 3-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_1(x) \oplus \mathfrak{g}_0(x) \oplus \mathfrak{g}_{-1}(x), \quad \text{where} \quad \mathfrak{g}_\nu(x) = \ker(\mathrm{ad}\, x - \nu\, \mathrm{id}_{\mathfrak{g}})$$

(see [BN04] for more details on Euler elements in more general Lie algebras). Then $\sigma_x(y_j) = (-1)^j y_j$ for $y_j \in \mathfrak{g}_j(x)$ defines an involutive automorphism of $\mathfrak{g}$.

For an Euler element we write $\mathcal{O}_x = \mathrm{Inn}(\mathfrak{g})x \subseteq \mathfrak{g}$ for the orbit of x under the group $\mathrm{Inn}(\mathfrak{g}) = \langle e^{\mathrm{ad}\,\mathfrak{g}} \rangle$ of inner automorphisms.[4] We say that x is *symmetric* if $-x \in \mathcal{O}_x$.

We write $\mathcal{E}(\mathfrak{g})$ for the set of non-zero Euler elements in $\mathfrak{g}$ and $\mathcal{E}_{\mathrm{sym}}(\mathfrak{g}) \subseteq \mathcal{E}(\mathfrak{g})$ for the subset of symmetric Euler elements.

(b) An element $(x, \sigma) \in \mathcal{G}$ is called an *Euler couple* or *Euler wedge* if

$$\mathrm{Ad}(\sigma) = e^{\pi i\, \mathrm{ad}\, x}. \tag{2.10}$$

Then σ is called an *Euler involution* and $\sigma = \sigma_x$ as introduced before. We write $\mathcal{G}_E \subseteq \mathcal{G}$ for the subset of Euler couples and note that the relation $e^{\pi i\, \mathrm{ad}\, x} = e^{-\pi i\, \mathrm{ad}\, x}$ implies that the subset $\mathcal{G}_E$ is invariant under the G-action.

For an Euler element $x \in \mathcal{E}(\mathfrak{g})$, the relation (2.10) only determines σ up to an element $z \in G^{\uparrow} \cap \ker(\mathrm{Ad})$ for which $(\sigma z)^2 = e$, i.e., $\sigma z \sigma = z^{-1}$. Note that, if $G^{\uparrow}$ is connected, then $G^{\uparrow} \cap \ker(\mathrm{Ad}) = Z(G^{\uparrow})$ is the center of $G^{\uparrow}$. The couples (x, σ) that we have seen in the physics literature are all Euler couples (cf. [NÓ17, Ex. 5.15]). This ensures many properties, such as the proper relation between spin and statistics, see for instance [GL95].

Definition 2.5. (a) (*Duality operation*) For $W = (x, \sigma) \in \mathcal{G}$, we define $W' := (-x, \sigma)$. Under Ψ, this operation corresponds to inverting the homomorphism $\mathbb{R}^{\times} \to G$ pointwise. Note that $(W')' = W$ and $(gW)' = gW'$ for $g \in G$ by (2.8).

(b) (Order structure on $\mathcal{G}$) We now define an order structure on $\mathcal{G}$ that depends on the invariant cone C from (2.4). We associate to $W = (x, \sigma) \in \mathcal{G}$

- the Lie wedge

$$L_W := L(x, \sigma) := C_+(W) \oplus \underbrace{(\mathfrak{g}^{\sigma} \cap \ker(\mathrm{ad}\, x))}_{\mathfrak{g}_W :=} \oplus C_-(W),$$

  where

$$C_{\pm}(W) = \pm C \cap \mathfrak{g}^{-\sigma} \cap \ker(\mathrm{ad}\, x \mp \mathbf{1}) \quad \text{and} \quad \mathfrak{g}^{\pm\sigma} := \{y \in \mathfrak{g} \colon \mathrm{Ad}(\sigma)(y) = \pm y\}.$$

- $\mathfrak{g}(W) := L_W - L_W$, the Lie algebra generated by L_W.

[4] For a Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$, we write $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s}) = \langle e^{\mathrm{ad}\,\mathfrak{s}} \rangle \subseteq \mathrm{Aut}(\mathfrak{g})$ for the subgroup generated by $e^{\mathrm{ad}\,\mathfrak{s}}$.

- the semigroup associated to the triple (C, x, σ):

$$\mathcal{S}_W := \exp(C_+(W))G^{\uparrow}_W \exp(C_-(W)) = G^{\uparrow}_W \exp\big(C_+(W) + C_-(W)\big),$$

where

$$G^{\uparrow}_W = \{g \in G^{\uparrow} : g.W = W\} = \{g \in G^{\uparrow} : \sigma_G(g) = g, \mathrm{Ad}(g)x = x\}$$

is the stabilizer of $W = (x, \sigma)$ in $G^{\uparrow}$ (cf. [Ne19b, Thm. 3.4]).[5]
- the subgroups $G^{\uparrow}(W) := \langle \exp \mathfrak{g}(W)\rangle G^{\uparrow}_W$ and $G(W) := G^{\uparrow}(W)\{e, \sigma\}$ with Lie algebra $\mathfrak{g}(W)$.

As the unit group of $\mathcal{S}_W$ is given by $\mathcal{S}_W \cap \mathcal{S}_W^{-1} = G^{\uparrow}_W$ ([Ne19b, Thm. III.4]), the semigroup $\mathcal{S}_W$ defines a $G^{\uparrow}$-invariant partial order on the orbit $G^{\uparrow}.W \subseteq \mathcal{G}$ by

$$g_1.W \leq g_2.W \quad :\Longleftrightarrow \quad g_2^{-1}g_1 \in \mathcal{S}_W. \tag{2.11}$$

In particular, $g.W \leq W$ is equivalent to $g \in \mathcal{S}_W$.

We have the following relations among these objects:

Lemma 2.6. *For every $W = (x_W, \sigma_W) \in \mathcal{G}$, $g \in G$, and $t \in \mathbb{R}$, the following assertions hold:*

(i) $\lambda_W(t)W = W$, $\lambda_W(t)W' = W'$ *and* $\sigma_W.W = W'$.
(ii) $\sigma_{W'} = \sigma_W$ *and* $\lambda_{W'}(t) = \lambda_W(-t)$.
(iii) σ_W *commutes with* $\lambda_W(\mathbb{R})$.
(iv) $L_{W'} = -L_W$ *and* $\mathcal{S}_{W'} = \mathcal{S}_W^{-1}$.
(v) $C_\pm(g.W) = \mathrm{Ad}(g)C_{\pm\varepsilon_G(g)}(W)$, $L_{g.W} = \mathrm{Ad}(g)L_W$, *and* $\mathcal{S}_{g.W} = g\mathcal{S}_W g^{-1}$.
(vi) *For* $W_1, W_2 \in \mathcal{G}$, *the relation* $W_1 \leq W_2$ *in* $\mathcal{G}$ *implies* $g.W_1 \leq g.W_2$.

Proof. (i) For $W = (x, \sigma) \in \mathcal{G}$, the first two relations follow from the fact that $\exp(\mathbb{R}x)$ commutes with x and σ. The second follows from $\sigma_W.W = \sigma.(x, \sigma) = (-\mathrm{Ad}(\sigma)x, \sigma) = (-x, \sigma) = W'$.
(ii) is clear from the definition of W'.
(iii) follows from (i).
(iv) follows from $C_\pm(W') = -C_\mp(W)$.
(v) The assertion is clear for $g \in G^{\uparrow}$. For $g \in G^{\downarrow}$, we have $g\sigma \in G^{\uparrow}$, so that

$$\begin{aligned} C_\pm(g.W) &= C_\pm(g\sigma.W') = \mathrm{Ad}(g\sigma)C_\pm(W') = -\mathrm{Ad}(g\sigma)C_\mp(W) = \mathrm{Ad}(g)C_\mp(W) \\ &= \mathrm{Ad}(g)C_{\pm\varepsilon_G(g)}(W). \end{aligned}$$

This implies in particular that $L_{g.W} = \mathrm{Ad}(g)L_W$. From $G^{\uparrow}_{g.W} = gG^{\uparrow}_W g^{-1}$, we thus obtain $\mathcal{S}_{g.W} = g\mathcal{S}_W g^{-1}$.
(vi) If $W_1 \leq W_2$, then $W_1 = s.W_2$ for $s \in \mathcal{S}_{W_2}$. Then $g.W_1 = gs.W_2 = gsg^{-1}.(g.W_2)$ with $gsg^{-1} \in g\mathcal{S}_{W_2}g^{-1} = \mathcal{S}_{g.W_2}$ implies $g.W_1 \leq g.W_2$. □

[5] In [Ne19b] it is shown that the different descriptions as a product of two sets (polar decomposition) and a product of two abelian subsemigroups and a group yield the same set $\mathcal{S}_W$ which actually is a subsemigroup.

In this discussion we started with a Lie group. We remark that one can also start with a Lie algebra as follows: Consider a quadruple $(\mathfrak{g}, \sigma_{\mathfrak{g}}, h, C)$ of a Lie algebra $\mathfrak{g}$, an involutive automorphism $\sigma_{\mathfrak{g}}$ of $\mathfrak{g}$, fixing the Euler element h and a pointed closed convex invariant cone $C \subseteq \mathfrak{g}$ with $\sigma_{\mathfrak{g}}(C) = -C$. Then $\sigma_{\mathfrak{g}}$ integrates to an automorphism σ_G of the 1-connected Lie group $G^{\uparrow}$ with Lie algebra $\mathfrak{g}$, so that we obtain all the data required above with $G := G^{\uparrow} \rtimes \{\mathrm{id}_G, \sigma_G\}$.

For two such quadruples $(\mathfrak{g}_j, \tau_{\mathfrak{g},j}, h_j, C_j)_{j=1,2}$, a homomorphism $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ of Lie algebras is compatible with this structure if

$$\varphi \circ \tau_{\mathfrak{g},1} = \tau_{\mathfrak{g},2} \circ \varphi, \qquad \varphi(h_1) = h_2 \quad \text{and} \quad \varphi(C_1) \subseteq C_2.$$

We thus obtain a category whose objects are the quadruples $(\mathfrak{g}, \tau_{\mathfrak{g}}, h, C)$ and its morphisms are the compatible homomorphisms.

A similar category can be defined on the group level, but there are some subtle ambiguities concerning the possible extensions of the group structure from $G^{\uparrow}$ to G.

Remark 2.7. (Twisted extensions of $G^{\uparrow}$ to G) We start with a graded group G for which $G^{\downarrow}$ contains an involution σ, so that $G \cong G^{\uparrow} \rtimes \{e, \sigma\}$, where σ acts on $G^{\uparrow}$ by the automorphism $\sigma_G(g) := \sigma g \sigma$. This defines a split group extension

$$G^{\uparrow} \to G \to \mathbb{Z}_2$$

and we are now asking for other group extensions

$$G^{\uparrow} \to \widehat{G} \to \mathbb{Z}_2$$

for which the elements in $\widehat{G}^{\downarrow}$ define the same element in the group $\mathrm{Out}(G^{\uparrow}) = \mathrm{Aut}(G^{\uparrow})/\mathrm{Inn}(G^{\uparrow})$ of outer automorphisms of $G^{\uparrow}$. These extensions are parametrized by the group

$$Z(G^{\uparrow})^+ := \{z \in Z(G^{\uparrow})\colon \sigma_G(z) = z\},$$

by assigning to $z \in Z(G^{\uparrow})^+$ the group structure on $G^{\uparrow} \times \{1, -1\}$ given by

$$(g, 1)(g', \varepsilon') = (gg', \varepsilon'), \quad (e, -1)(g', 1) = (\sigma_G(g'), -1) \quad \text{and} \quad (e, -1)^2 = (z, 1). \tag{2.12}$$

We write $\widehat{G}_z$ for the corresponding Lie group. Basically, this means that the element $\widehat{\sigma} := (e, -1)$ has the same commutation relations with $G^{\uparrow}$ but its square is z instead of e:

$$\widehat{\sigma} g \widehat{\sigma}^{-1} = \sigma_G(g) \quad \text{for} \quad g \in G \quad \text{and} \quad \widehat{\sigma}^2 = z. \tag{2.13}$$

For two elements $z, z' \in Z(G^{\uparrow})^+$, the corresponding extensions are equivalent if and only if

$$z^{-1}z' \in B := \{w\sigma_G(w)\colon w \in Z(G^{\uparrow})\}. \tag{2.14}$$

This follows from [HN12, Thm 18.1.13], combined with [HN12, Ex. 18.3.5(b)].
(a) For $G = \mathrm{O}_n(\mathbb{R})$, $n > 3$, and $G^{\uparrow} = \mathrm{SO}_n(\mathbb{R})$, the situation depends on the parity of n. If n is odd, then $Z(G^{\uparrow}) = \{e\}$ and no twists exist. If n is even, then $Z(G^{\uparrow}) = \{\pm\mathbf{1}\} = Z(G)$. Therefore $Z(G^{\uparrow})^+ = \{\pm\mathbf{1}\} \neq B = \{e\}$. We therefore have one twisted

group $\widehat{G} = \mathrm{SO}_n(\mathbb{R})\{e, \widehat{\sigma}\}$, where $\sigma \in \mathrm{O}_n(\mathbb{R})$ corresponds to a hyperplane reflection, and $\widehat{\sigma}^2 = -\mathbf{1}$ in $\widehat{G}$.
(b) The same phenomenon occurs for Spin groups. Let $G := \mathrm{Pin}_n(\mathbb{R}) \cong \mathrm{Spin}_n(\mathbb{R}) \rtimes \{e, \sigma\}$, where σ corresponds to a hyperplane reflection. If n is odd, then $Z(\mathrm{Spin}_n(\mathbb{R})) = \{e, z\}$ contains two elements, and we have a twisted group

$$\widehat{G} = \mathrm{Spin}_n(\mathbb{R})\{e, \widehat{\sigma}\} \quad \text{with} \quad \widehat{\sigma}^2 = z$$

(cf. [HN12, Rem. B.3.25]). If n is even, then the situation is more complicated because the center of $\mathrm{Spin}_n(\mathbb{R})$ has order 4.
(c) For $G = \widetilde{\mathsf{M\ddot{o}b}} \rtimes \{e, \sigma\}$, where σ corresponds to a reflection $\sigma(x) = -x$ on $\mathbb{R}^\infty \cong \mathbb{S}^1$, we have $Z(G^\uparrow) \cong \mathbb{Z}$ and $\sigma_G(z) = z^{-1}$ for $z \in Z(G^\uparrow)$. Hence $Z(G^\uparrow)^+ = \{e\}$, so that there are no twists.
(d) If $G = \mathsf{M\ddot{o}b}^{(2n)} \rtimes \{e, \sigma\}$, where $\mathsf{M\ddot{o}b}^{(2n)}$ is the covering of $\mathsf{M\ddot{o}b}$ of even order, then $Z(G^\uparrow) \cong \mathbb{Z}_{2n}$ and $\sigma_G(z) = z^{-1}$ for $z \in Z(G^\uparrow)$. Therefore $Z(G^\uparrow)^+ = \{e, \gamma\}$, where γ is the unique non-trivial involution in $Z(G^\uparrow)$ and $B = \{e\}$. Hence there exists a non-trivial twist $\widehat{G} = G^\uparrow\{e, \widehat{\sigma}\}$ with $\widehat{\sigma}^2 = \gamma$.
(e) As we shall see in Example 2.13 below, it may happen that, for the twisted groups $\widehat{G}_z$, the coset $\widehat{G}_z^\downarrow$ contains no involutions. In this example $G^\uparrow = \mathrm{SL}_2(\mathbb{R})$ and $G = G^\uparrow\{e, \gamma\}$ with $\gamma^2 = -\mathbf{1}$.

In general, elements in $\widehat{G}_z^\downarrow$ are of the form $g\widehat{\sigma}$ with $g \in G^\uparrow$, and then

$$(g\widehat{\sigma})^2 = g\widehat{\sigma}g\widehat{\sigma} = g\sigma_G(g)\widehat{\sigma}^2 = g\sigma_G(g)z. \tag{2.15}$$

Hence $\widehat{G}_z^\downarrow$ contains an involution if and only if

$$z \in \{\sigma_G(g)^{-1}g^{-1} : g \in G^\uparrow\} = \{g\sigma_G(g) : g \in G^\uparrow\}.$$

If $z = g\sigma_G(g)$ for some $g \in G^\uparrow$, then conjugating with g implies that g and $\sigma_G(g)$ commute.

The discussion in Example 2.13 shows that (2.15) is not satisfied for $z = -\mathbf{1}$ and the Euler involution of $G^\uparrow = \mathrm{SL}_2(\mathbb{R})$. For any odd degree covering $\mathrm{SL}_2(\mathbb{R})^{(2k+1)} \to \mathrm{SL}_2(\mathbb{R})$, the central involution is mapped onto $-\mathbf{1}$, so that this observation carries over to odd coverings of $\mathrm{SL}_2(\mathbb{R})$.

The situation changes if we consider $G^\uparrow = \mathrm{SL}_2(\mathbb{C})$ instead. Then $g := i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ satisfies $g^2 = -\mathbf{1}$, so that the group $\widehat{G} = G^\uparrow\{\mathbf{1}, \widehat{\sigma}\}$ with $\widehat{\sigma}^2 = -\mathbf{1}$ contains the non-trivial involution $g\widehat{\sigma} \in \widehat{G}^\downarrow$. As this involution is central, $\widehat{G} \cong \widehat{G}^\uparrow \times \mathbb{Z}_2$ is a direct product.

2.3. The abstract wedge space, some fundamental examples.

Definition 2.8 *(The abstract wedge space).* From here on, we always assume that $\mathcal{G} \neq \emptyset$, i.e., that $G^\downarrow$ contains an involution σ. Then

$$G \cong G^\uparrow \rtimes \{\mathrm{id}, \sigma\}$$

(cf. Remark 2.2). For a fixed couple $W_0 = (h, \sigma) \in \mathcal{G}$, the orbits

$$\mathcal{W}_+(W_0) := G^\uparrow.W_0 \subseteq \mathcal{G} \quad \text{and} \quad \mathcal{W}(W_0) := G.W_0 \subseteq \mathcal{G}$$

are called the *positive* and the *full wedge space containing* W_0.

Remark 2.9. (a) As $\sigma.W_0 = (-h,\sigma) = W_0'$, we have $\mathcal{W}(W_0) = \mathcal{W}_+(W_0) \cup \mathcal{W}_+(W_0')$, and $\mathcal{W}(W_0)$ coincides with $\mathcal{W}_+(W_0)$ if and only if $W_0' = (-h,\sigma) \in \mathcal{W}_+(W_0)$. This is equivalent to the existence of an element $g \in G^\uparrow$ with $g.W_0 = W_0'$, i.e., $g \in (G^\uparrow)^\sigma$ with $\mathrm{Ad}(g)h = -h$.
(b) If W_0 is an Euler couple, then $\mathcal{W}(W_0)$ is a family of Euler couples, and we shall see below that in this case we have $\mathcal{W}(W_0) = \mathcal{W}_+(W_0)$ in many important cases.

We collect some fundamental examples, starting from the low dimensional cases, that we shall refer to throughout the paper.

Examples 2.10. (a) The smallest example is the abelian group $G = \mathbb{R} \times \{\pm 1\}$, where $G^\uparrow = \mathbb{R}$, $C = \{0\}$ and $L = \mathfrak{g}$. For $W_0 = (h,\sigma)$ with $h = 1$ and $\sigma = (0,1)$, we then have the one-point set $\mathcal{W}_+ = \{(h,\sigma)\}$, and $\mathcal{W} = \{(h,\sigma),(-h,\sigma)\}$.
(b) **The affine group** $G := \mathrm{Aff}(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{R}^\times$ of the real line is two-dimensional. Its elements are denoted (b,a), and they act by $(b,a)x = ax + b$ on the real line. The identity component $G^\uparrow = \mathbb{R} \rtimes \mathbb{R}^\times_+$ acts by orientation preserving maps, and $G^\downarrow$ consists of reflections $r_p(x) = 2p - x$, $p \in \mathbb{R}$.
Let $\zeta(t) = (t,1)$ and $\delta(t) = (0,e^t)$ be the translation and dilation one-parameter groups, respectively. We write $\lambda = (0,1) \in \mathfrak{g} = \mathbb{R} \rtimes \mathbb{R}$ for the infinitesimal generator of δ, which is an Euler element. Therefore $W := (\lambda, r_0)$ is an Euler couple.

The cone $C = \mathbb{R}_+ \times \{0\} \subseteq \mathfrak{g}$ satisfies the invariance condition (2.4) and the corresponding semigroup $\mathcal{S}_W$ is

$$\mathcal{S}_W = [0,\infty) \rtimes \mathbb{R}^\times_+ = \{g = (b,a) \colon g.0 = b \geq 0, a > 0\} = \{g \in G^\uparrow \colon g\mathbb{R}_+ \subseteq \mathbb{R}_+\}.$$

Therefore the map

$$\mathcal{W}_+(W) \ni g.(\lambda, r_0) \mapsto g(0,+\infty)$$

defines an order preserving bijection between the abstract wedge space $\mathcal{W}_+(W) \subseteq \mathcal{G}$ and the set $\mathcal{I}_+(\mathbb{R}) = \{(t,\infty) \colon t \in \mathbb{R}\}$ of lower bounded open intervals in $\mathbb{R}$. Accordingly, we may write $W_{(t,\infty)} = (\Lambda_{(t,\infty)}, r_t) := \zeta(t)W = (\mathrm{Ad}(\zeta(t))\lambda, r_t)$ for $t \in \mathbb{R}$. Acting with reflections, we also obtain the couples

$$W_{(-\infty,t)} := (\Lambda_{(-\infty,t)}, r_t) = r_t.W_{(t,\infty)} = (-\mathrm{Ad}(\zeta(t))\lambda, r_t)$$

corresponding to past pointing half-lines $(-\infty,t) \subset \mathbb{R}$. We thus obtain a bijection between the full wedge space $\mathcal{W}(W)$ and the set $\mathcal{I}(\mathbb{R})$ of open semibounded intervals in $\mathbb{R}$. We shall denote with δ_I the one-parameter group of dilations with generator λ_I corresponding to the half line I.

The set $\mathcal{E}(\mathfrak{g}) = \mathrm{Ad}(G^\uparrow)\{\pm\lambda\}$ of non-zero Euler elements in $\mathfrak{g}$ consists of two $G^\uparrow$-orbits and, for each non-zero Euler element $\pm\mathrm{Ad}(\zeta(t))\lambda \in \mathcal{E}(\mathfrak{g})$, the reflection r_t is the unique partner for which $(\pm\mathrm{Ad}(\zeta(t))\lambda, r_t) \in \mathcal{G}$. Accordingly, Euler couples in $\mathcal{G}$ are in one-to-one correspondence with semi-infinite open intervals in $\mathbb{R}$.
(c) **The Möbius group** $G := \mathsf{M\ddot{o}b}_2 := \mathrm{PGL}_2(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^\times$ acts on the compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of the real line by

$$g.x := \frac{ax+b}{cx+d} \quad \text{on} \quad \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}, \qquad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}).$$

We write $G^{\uparrow} = \mathsf{M\ddot{o}b} \cong \mathrm{PSL}_2(\mathbb{R})$ for the subgroup of orientation preserving maps. The Cayley transform

$$C : \overline{\mathbb{R}} \to \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}, \quad C(x) := \frac{i - x}{i + x}, \qquad C(\infty) := -1,$$

is a homeomorphism, identifying $\overline{\mathbb{R}}$ with the circle. Its inverse is the stereographic map

$$C^{-1} : \mathbb{S}^1 \to \overline{\mathbb{R}}, \quad C^{-1}(z) = i\frac{1 - z}{1 + z}.$$

It maps the upper semicircle $\{z \in \mathbb{S}^1 : \ \mathrm{Im}\, z > 0\}$ to the positive half line $(0, +\infty)$. The Cayley transform intertwines the action of $\mathsf{M\ddot{o}b}$ on $\overline{\mathbb{R}}$ with the action of $\mathrm{PSU}_{1,1}(\mathbb{C}) = \mathrm{SU}_{1,1}(\mathbb{C})/\{\pm\mathbf{1}\}$, given by

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}.z := \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \quad \text{for} \quad z \in \mathbb{S}^1, \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}_{1,1}(\mathbb{C}).$$

The three-dimensional Lie group $\mathsf{M\ddot{o}b}$ is generated by the following one-parameter subgroups:

- Rotations: $\rho(\theta)(x) = \frac{\cos(\theta/2)x + \sin(\theta/2)}{-\sin(\theta/2)x + \cos(\theta/2)}$ for $\theta \in \mathbb{R}$; note that $C(\rho(\theta)x) = e^{i\theta} C(x)$.
- Dilations: $\delta(t)(x) = e^t x$ for $t \in \mathbb{R}$.
- Translation: $\zeta(t)x = x + t$ for $t \in \mathbb{R}$.

In the circle picture δ and ζ will be denoted by $\delta_{\cap}$ and $\zeta_{\cap}$, referring to the upper semicircle with endpoints $\{-1, 1\} = C(\{0, \infty\})$. Note that -1 is the unique fixed point of $\zeta_{\cap}$ and one of the two fixed points $\{\pm 1\}$ of $\delta_{\cap}$. On the circle, $\rho(\pi)$ maps 1 to -1 and exchanges the upper and the lower semicircle. Accordingly, $\zeta_{\cup} = \rho(\pi)\zeta\rho(\pi)$ is the subgroup of conjugated translations fixing the point $1 \in \mathbb{S}^1$.

We write $\mathbf{K} = \rho(\mathbb{R})$, $\mathbf{A} = \delta(\mathbb{R})$, $\mathbf{N}^+ = \zeta(\mathbb{R})$ and $\mathbf{N}^- = \zeta_{\cup}(\mathbb{R})$ for the corresponding one-dimensional subgroups of $\mathsf{M\ddot{o}b}$, and $\mathbf{P}^+ = \mathbf{A}\mathbf{N}^+ = \mathsf{M\ddot{o}b}_{\infty}$, $\mathbf{P}^- := \mathbf{A}\mathbf{N}^- = \mathsf{M\ddot{o}b}_0$ for the stabilizer groups of ∞ and 0 in $\mathsf{M\ddot{o}b}$. We observe that $\overline{\mathbb{R}} \cong \mathsf{M\ddot{o}b}/\mathbf{P}^-$ and that the circle group $K = \mathrm{PSO}_2(\mathbb{R})$ acts simply transitively on $\overline{\mathbb{R}}$.

On the compactified line, the point reflection $\tau(x) = -x$ in 0 acts on the Lie algebra by

$$\mathrm{Ad}(\tau)\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix}. \tag{2.16}$$

Note that $\tau \in G^{\downarrow}$.

The infinitesimal generator $h := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ of δ is an Euler element and $W := (h, \tau)$ is an Euler couple. Since $\mathsf{M\ddot{o}b}_2 \cong \mathrm{PGL}_2(\mathbb{R}) \cong \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{R}))$, for any Euler couple (x, τ), the involution τ is determined by the requirement that it acts on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ by $e^{\pi i\, \mathrm{ad}\, x}$. We conclude that the action of $G^{\uparrow} = \mathsf{M\ddot{o}b}$ on the set of Euler couples is transitive, i.e., $\mathcal{G}_E = G^{\uparrow}.(h, \tau)$.

To see the geometric side of Euler couples, let us call a non-dense, non-empty open connected subset $I \subseteq \mathbb{S}^1$ an *interval* and write $\mathcal{I}(\mathbb{S}^1)$ for the set of intervals in $\mathbb{S}^1$. It is easy to see that $\mathsf{M\ddot{o}b}$ acts transitively on $\mathcal{I}(\mathbb{S}^1)$. To determine the stabilizer of an interval, we consider the upper half circle, which corresponds to the half line $(0, \infty) \subseteq \overline{\mathbb{R}}$. Each element $g \in \mathsf{M\ddot{o}b}$ mapping $(0, \infty)$ onto itself fixes 0 and ∞. Since it is completely determined by the image of a third point, it is of the form $\delta(t)$ if $g.1 = e^t$. Therefore the stabilizer of $(0, \infty)$ in $\mathsf{M\ddot{o}b}$ is the subgroup $\delta(\mathbb{R})$, which coincides with the stabilizer of h under the adjoint action. This already shows that $\mathcal{W}_+(W)$ and $\mathcal{I}(\mathbb{S}^1)$ are isomorphic homogeneous spaces of $\mathsf{M\ddot{o}b}$. In particular, we can associate to an interval $I = g(0, \infty)$ the reflection $\tau_I = g\tau g^{-1}$ and the one-parameter group $\delta_I := g\delta g^{-1}$. Note that τ_I is an orientation reversing involution mapping I to the complementary open interval I'. We write $x_I := \mathrm{Ad}(g)h$ for the infinitesimal generator of δ_I, so that the assignment $I \mapsto x_I$ defines an equivariant bijection $\mathcal{I}(\mathbb{S}^1) \to \mathcal{E}(\mathfrak{g})$. The anticlockwise orientation of $\mathbb{S}^1$, which can also be considered as a causal structure, is used here to pick the sign of x_I in such a way that the flow δ_I is counter clockwise (future pointing) on I. Accordingly, $x_{I'} = -x_I$ corresponds to the complementary interval I'.

To identify the natural order on the abstract wedge space $\mathcal{G}_E = \mathcal{W}_+(W)$, we consider for $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ the corresponding fundamental vector field

$$V_X(x) = \frac{d}{dt}\Big|_{t=0} \exp(tX).x = (a-d)x + b - cx^2 = b + 2ax - cx^2.$$

This shows that

$$C := \{X \in \mathfrak{g} \colon V_X \geq 0\} = \left\{X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : b \geq 0, c \leq 0, a^2 \leq -bc\right\} \tag{2.17}$$

is a pointed generating invariant cone in $\mathfrak{g}$. The Lie wedge specified by the triple (h, τ, C) is

$$L_W = L(h, \tau, C) = \underbrace{\mathbb{R}_+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{C_+} \oplus \mathbb{R}h \oplus \underbrace{\mathbb{R}_+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{C_-} = \left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a \in \mathbb{R}, b \geq 0, c \geq 0\right\}.$$

We further have $G(W) = G^{\uparrow}$, and the associated semigroup is

$$\mathcal{S}_W = \exp(C_+)\exp(\mathbb{R}h)\exp(C_-) = \{g \in G^{\uparrow} \colon g(0, \infty) \subseteq (0, \infty)\}.$$

Therefore the map

$$\mathcal{G}_E = \mathcal{W}_+(W) = \mathcal{W}(W) \to \mathcal{I}(\mathbb{S}^1), \quad g.W \mapsto g(0, \infty) \tag{2.18}$$

defines an order preserving bijection between the abstract wedge space $\mathcal{W}(W)$ and the ordered set $\mathcal{I}(\mathbb{S}^1)$.

(d) We now consider **the universal covering of the Möbius group** $\mathsf{M\ddot{o}b}$. Concretely, we put $G := \widetilde{\mathsf{M\ddot{o}b}} \rtimes \{\mathbf{1}, \widetilde{\tau}\}$, where $\widetilde{\tau}$ acts on $\widetilde{\mathsf{M\ddot{o}b}}$ by integrating $\mathrm{Ad}(\tau)$ from (2.16) to an automorphism of $\widetilde{\mathsf{M\ddot{o}b}}$. The group G is a graded Lie group and $G^{\uparrow} := \widetilde{\mathsf{M\ddot{o}b}}$ is its identity component. We have a covering homomorphism $q_G : G \to \mathsf{M\ddot{o}b}_2$ whose kernel $Z(\widetilde{\mathsf{M\ddot{o}b}}) \cong \mathbb{Z}$ is discrete cyclic. We write $\widetilde{\rho}, \widetilde{\delta}, \widetilde{\zeta}$ and $\widetilde{\zeta}_\cup$ for the canonical lifts of the one-parameter groups $\rho, \delta, \zeta, \zeta_\cup$ of $\mathsf{M\ddot{o}b}$, $\widetilde{\mathbf{P}}^+ := \widetilde{\delta}(\mathbb{R})\widetilde{\zeta}(\mathbb{R})$, and $\widetilde{\mathbf{P}}^- := \widetilde{\delta}(\mathbb{R})\widetilde{\zeta}_\cup(\mathbb{R})$.

The action of $\mathsf{M\ddot{o}b}$ on $\mathbb{S}^1$ lifts canonically to an action of the connected group $G^{\uparrow} = \widetilde{\mathsf{M\ddot{o}b}}$ on the universal covering $\widetilde{\mathbb{S}^1} \cong \mathbb{R}$, where we fix the covering map $q_{\mathbb{S}^1} : \mathbb{R} \to \widetilde{\widetilde{\mathbb{R}}}$, defined by $q_{\mathbb{S}^1}(\theta) = \widetilde{\rho}(\theta).0$, which corresponds to the map $\theta \mapsto e^{i\theta} = C(\widetilde{\rho}(\theta).0)$ in the circle picture. We may thus identify $\widetilde{\mathbb{S}^1}$ with the homogeneous space $\widetilde{\mathsf{M\ddot{o}b}}/\widetilde{\mathbf{P}}^- \cong \mathbb{R}$. As conjugation with $\widetilde{\tau}$ on $\widetilde{\mathsf{M\ddot{o}b}}$ preserves the subgroup $\widetilde{\mathbf{P}}^-$, it also acts on $\widetilde{\mathbb{S}^1}$. From (2.16) it follows that it simply acts by the point reflection $\widetilde{\tau}.x = -x$ in the base point 0. We also note that $Z := \ker(q_G) = \widetilde{\rho}(2\pi\mathbb{Z})$ is the group of deck transformations of the covering $q_{\mathbb{S}^1}$, which acts by

$$\widetilde{\rho}(2\pi n).x = x + 2\pi n \quad \text{for} \quad n \in \mathbb{Z}. \tag{2.19}$$

We call a non-empty interval $I \subseteq \mathbb{R}$ *admissible* if its length is strictly smaller than 2π and write $\mathcal{I}(\mathbb{R})$ for the set of admissible intervals. An interval $I \subseteq \mathbb{R}$ is admissible if and only if there exists an interval $\underline{I} \in \mathcal{I}(\mathbb{S}^1)$ such that I is a connected component of $q_{\mathbb{S}^1}^{-1}(\underline{I})$. The group Z acts transitively on the set of these connected components. As $\mathsf{M\ddot{o}b}$ acts transitively on $\mathcal{I}(\mathbb{S}^1)$, it follows that the group $\widetilde{\mathsf{M\ddot{o}b}}$ acts transitively on the set $\mathcal{I}(\mathbb{R})$, and that composition with $q_{\mathbb{S}^1}$ yields an equivariant covering map

$$\mathcal{I}(\mathbb{R}) \cong \widetilde{\mathsf{M\ddot{o}b}}/\widetilde{\delta}(\mathbb{R}) \to \mathcal{I}(\mathbb{S}^1) \cong \mathsf{M\ddot{o}b}/\delta(\mathbb{R}), \quad I \mapsto q_{\mathbb{S}^1}(I). \tag{2.20}$$

We further have:

- The group $\widetilde{\mathbf{P}}^+ = \widetilde{\delta}(\mathbb{R})\widetilde{\zeta}(\mathbb{R})$ fixes the points $\{(2k+1)\pi : k \in \mathbb{Z}\}$.
- For $I \in \mathcal{I}(\mathbb{S}^1)$, let $\widetilde{\delta}_I$ be the lift of the one-parameter group δ_I. Then $\widetilde{\delta}_I$ preserves every interval in the preimage $q_{\mathbb{S}^1}^{-1}(I)$.
- The inverse images of $\tau \in \mathsf{M\ddot{o}b}_2$ in $\widetilde{\mathsf{M\ddot{o}b}_2}$ are the elements $\widetilde{\tau}_n := \widetilde{\rho}(2\pi n)\widetilde{\tau}$, $n \in \mathbb{Z}$. These are involutions, acting by

$$\widetilde{\tau}_n(x) = 2\pi n - x \quad \text{for} \quad x \in \mathbb{R} \tag{2.21}$$

which is a point reflection in the point πn. All pairs $(h, \widetilde{\tau}_n)$ are Euler couples in $\mathcal{G}(\widetilde{\mathsf{M\ddot{o}b}_2})$, and from the discussion of the set of Euler couples $\mathcal{G}_E(\mathsf{M\ddot{o}b}_2)$ under (c), we know that the involutions $\widetilde{\tau}_n$ exhaust all possibilities for supplementing h to an Euler couple.

There is an interesting difference to the situation for $\mathsf{M\ddot{o}b}_2$, where $\mathsf{M\ddot{o}b}$ acts transitively on the set $\mathcal{G}_E(\mathsf{M\ddot{o}b}_2)$ of Euler couples. To see what happens for $\widetilde{\mathsf{M\ddot{o}b}_2}$, recall that the stabilizer of the element $(h, \tau) \in \mathcal{G}_E(\mathsf{M\ddot{o}b}_2)$ in $\mathsf{M\ddot{o}b}$ is the subgroup $\delta(\mathbb{R})$. Its inverse image is the group

$$\widetilde{\delta}(\mathbb{R})\widetilde{\rho}(2\pi\mathbb{Z}) \cong \mathbb{R} \times \mathbb{Z}.$$

An element $g \in \widetilde{\mathsf{M\ddot{o}b}}$ fixes $(h, \widetilde{\tau}_n)$ if and only if $\mathrm{Ad}(g)h = h$ and $g\widetilde{\tau}_n g^{-1} = \widetilde{\tau}_n$. The first condition is equivalent to g being of the form

$$g = \widetilde{\delta}(t)\widetilde{\rho}(2\pi k) \quad \text{for some} \quad t \in \mathbb{R}, k \in \mathbb{Z}.$$

The second condition is equivalent to $\widetilde{\tau} g \widetilde{\tau} = \widetilde{\tau}_n g \widetilde{\tau}_n = g$, which takes the form

$$\widetilde{\delta}(t)\widetilde{\rho}(-2\pi k) = \widetilde{\delta}(t)\widetilde{\rho}(2\pi k),$$

and this is equivalent to $k = 0$. We conclude that the stabilizer of $(h, \widetilde{\tau}_n)$ is

$$\widetilde{\mathsf{M\ddot{o}b}}_{(h,\widetilde{\tau}_n)} = \widetilde{\delta}(\mathbb{R}). \tag{2.22}$$

We also note that

$$\widetilde{\rho}(\pi k).(h,\widetilde{\tau}_n) = ((-1)^k h, \widetilde{\rho}(\pi k)\widetilde{\tau}_n\widetilde{\rho}(-\pi k)) = ((-1)^k h, \widetilde{\rho}(2\pi k)\widetilde{\tau}_n) = ((-1)^k h, \widetilde{\tau}_{n+k}).$$

We conclude that the group $\widetilde{\mathsf{M\ddot{o}b}}$ does not act transitively on the set $\mathcal{G}_E$ of Euler couples. It has two orbits:

$$\mathcal{G}_E(\widetilde{\mathsf{M\ddot{o}b}}_2) = G^{\uparrow}.W_0\dot{\cup}G^{\uparrow}.W_1 = \mathcal{W}_+(W_0)\dot{\cup}\mathcal{W}_+(W_1) \quad \text{for} \quad W_0 := (h,\widetilde{\tau}_0), W_1 := (h,\widetilde{\tau}_1). \tag{2.23}$$

We also refer to Example 2.14 for a discussion of this issue from a different perspective.

- The subgroup $\widetilde{\delta}(\mathbb{R})$ preserves every interval which is a non-trivial orbit of $\widetilde{\delta}(\mathbb{R})$, acting on $\mathbb{R}$. If, conversely, $g \in \widetilde{\mathsf{M\ddot{o}b}}$ preserves such an interval, then its image in $\mathsf{M\ddot{o}b}$ is contained in $\delta(\mathbb{R})$, so that

$$g = \widetilde{\delta}(t)\widetilde{\rho}(2\pi k) \quad \text{for some} \quad t \in \mathbb{R}, k \in \mathbb{Z}.$$

As every open orbit of $\widetilde{\delta}(\mathbb{R})$ is an interval of length π, the element g can only preserve such an orbit if $k = 0$. This shows that $\widetilde{\mathsf{M\ddot{o}b}}_{(h,\widetilde{\tau}_n)}$ also is the stabilizer group of any open $\widetilde{\delta}(\mathbb{R})$-orbit in $\mathbb{R}$. We conclude that, for the Euler couple $W_0 = (h,\widetilde{\tau}_0)$, the map

$$\Phi\colon \mathcal{W}_+(W_0) \to \mathcal{I}(\mathbb{R}), \quad g.(h,\widetilde{\tau}_0) \mapsto g(0,\pi) \tag{2.24}$$

defines a $G^{\uparrow}$-equivariant bijection between the abstract wedge space $\mathcal{W}_+(W_0) \subseteq \mathcal{G}$ and the set $\mathcal{I}(\mathbb{R})$ of admissible intervals in $\mathbb{R}$. Since the full group G acts on the space $\mathcal{I}(\mathbb{R})$ of intervals, Φ can be used to transport this action to a G-action on the space $\mathcal{W}_+(W_0)$, extending the action of the subgroup $G^{\uparrow}$. Since $\tau_0(0,\pi) = (-\pi, 0) = \rho(-\pi)(0,\pi)$, we have

$$\Phi^{-1}(\tau_0(0,\pi)) = \Phi^{-1}(\rho(-\pi)(0,\pi)) = \rho(-\pi).\Phi((0,\pi))^{-1} = (-h, \rho(-2\pi)\tau_0),$$

so that $\tau_0.W_0 := (-h, \rho(-2\pi)\tau_0)$. By $G^{\uparrow}$-equivariance of the map Φ, we conclude that the action of $G^{\downarrow}$ on $\mathcal{W}_+(W_0)$ is given by

$$g *_{\rho(-2\pi)} (x,\sigma) := (\mathrm{Ad}^{\varepsilon}(g), \rho(-2\pi)g\sigma g^{-1}) \quad \text{for every} \quad g \in G^{\downarrow}. \tag{2.25}$$

Here we use that $\widetilde{\rho}(-2\pi) \in Z(G^{\uparrow})$. Note that we have chosen $(0,\pi)$ to be the image of W_0 through Φ. Further possible actions come from the identifications

$$\Phi_n\colon \mathcal{W}_+(W_n) \to \mathcal{I}(\mathbb{R}), \quad g.(h,\widetilde{\tau}_n) \mapsto g(0,\pi) \quad \text{with} \quad W_n = (h,\tau_n), \tag{2.26}$$

and one can likewise see that

$$g *_{\alpha_n} (x,\sigma) := (\mathrm{Ad}^{\varepsilon}(g), \alpha_n g\sigma g^{-1}) \quad \text{for} \quad g \in G^{\downarrow} \quad \text{and} \quad \alpha_n = \widetilde{\rho}((2n-1)2\pi) \in Z(G^{\uparrow}),$$

extends the action of $G^{\uparrow}$ on $\mathcal{W}_+(W_n)$ to G and $\Phi = \Phi_0$ for $n = 0$ (see also (2.37) and Section 2.4.2 for this kind of action).

(e) Let $q\colon \mathsf{M\ddot{o}b}^{(n)} \to \mathsf{M\ddot{o}b}$ be **the n-fold covering group** of $\mathsf{M\ddot{o}b}$ and $\rho^{(n)}, \delta^{(n)}, \zeta^{(n)}$ and $\zeta_{\cup}^{(n)}$ be the lifts of the corresponding one-parameter groups of $\mathsf{M\ddot{o}b}$. We further put $\mathbf{P}^{-,(n)} := \delta^{(n)}(\mathbb{R})\zeta_{\cup}^{(n)}(\mathbb{R})$, so that we obtain an n-fold covering

$$q_n\colon \mathbb{S}^1_n := \mathsf{M\ddot{o}b}^{(n)}/\mathbf{P}^{-,(n)} \to \mathbb{S}^1 = \mathsf{M\ddot{o}b}/\mathbf{P}^-, \quad g\mathbf{P}^{-,(n)} \mapsto q(g)\mathbf{P}^-$$

of the circle, and the action of the one-parameter group $\rho^{(n)}$ induces a diffeomorphism

$$\mathbb{R}/2\pi n\mathbb{Z} \to \mathbb{S}^1_n, \quad [t] \mapsto \rho^{(n)}(t).0$$

The set of wedges can be described analogously to the case (d), but there is a difference depending on the parity of n. If n is even, the group $G^{\uparrow}$ has two orbits in the set $\mathcal{G}_E$ of Euler couples, but if n is odd, there is only one. Indeed, for $n = 2k$, the element $\rho^{(n)}(2\pi k)$ acts as an involution on $\mathbb{S}^1_n$. So it fixes all Euler couples $(h, \widetilde{\tau}_n)$, even if it does NOT fix any proper interval in $\mathbb{S}^1_n$ (see also Example 2.14).
(f) The example arising most prominently in physics is the proper **Poincaré group**

$$G := \mathcal{P}_+ := \mathbb{R}^{1,d} \rtimes \mathrm{SO}_{1,d}(\mathbb{R}), \qquad G^{\uparrow} := \mathcal{P}_+^{\uparrow} := \mathbb{R}^{1,d} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})^{\uparrow}.$$

It acts on $1+d$-dimensional Minkowski space $\mathbb{R}^{1,d}$ as an isometry group of the Lorentzian metric given by $(x, y) = x_0y_0 - \mathbf{xy}$ for $x = (x_0, \mathbf{x}) \in \mathbb{R}^{1,d}$. Writing

$$V_+ := \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d} \colon x_0 > 0, x_0^2 > \mathbf{x}^2\}$$

for the open future light cone, the grading on G is specified by time reversal, i.e., $gV_+ = \varepsilon(x, g)V_+$. In particular $C := \overline{V_+}$ is a pointed closed convex cone satisfying (2.4). For $d > 1$, this is, up to sign, the only non-zero pointed invariant cone in the Lie algebra $\mathfrak{g}$.

The generator $k_1 \in \mathfrak{so}_{1,d}(\mathbb{R})$ of the Lorentz boost on the (x_0, x_1)-plane

$$k_1(x_0, x_1, x_2, \ldots, x_d) = (x_1, x_0, x_2, \ldots, x_d)$$

is an Euler element. It combines with the spacetime reflection $j_1(x) = (-x_0, -x_1, x_2, \ldots, x_d)$ to the Euler couple (k_1, j_1). We associate to (k_1, j_1) the spacetime region

$$W_1 = \{x \in \mathbb{R}^{1+d} : |x_0| < x_1\},$$

the *standard right wedge*, and note that W_1 is invariant under $\exp(\mathbb{R}k_1)$. It turns out that the semigroup $\mathcal{S}_{(k_1,j_1)}$ associated to the couple (k_1, j_1) in Definition 2.5 satisfies

$$\mathcal{S}_{(k_1,j_1)} = \{g \in G\colon gW_1 \subseteq W_1\} =: \mathcal{S}_{W_1} \tag{2.27}$$

(see [NÓ17, Lemma 4.12]). From (2.27) it follows that the map

$$\mathcal{W}_+ = \mathcal{W} = G^{\uparrow}.(k_1, j_1) \ni g.(k_1, j_1) \mapsto gW_1 \tag{2.28}$$

defines an order preserving bijection between the abstract wedge space $\mathcal{W} \subseteq \mathcal{G}$ and the set of wedge domains in Minkowski space $\mathbb{R}^{1+d}$. For an abstract wedge $W = (k_W, j_W) \in \mathcal{W}$, the Euler element k_W is the corresponding boost generator. For an axial wedge $W_i := \{x \in \mathbb{R}^{1+d} : |x_0| < x_i\}$, $i = 1, \ldots, n$, the corresponding Euler couple will be denoted (k_i, j_i).

2.4. Nets of wedges, isotony, central locality and covering groups. In the following sections we will focus on the description of relative positions of wedges, in particular wedge inclusions and the locality principle.

2.4.1. Wedge inclusion Firstly consider this wedge inclusion configuration called *half-sided modular inclusion*:

Definition 2.11. Let $W_0 = (x, \sigma) \in \mathcal{G}$ and $y \in \pm C$ with $[x, y] = \pm y$. Then $\exp(y) \in \mathcal{S}_{W_0}$ (Definition 2.5(b)), so that

$$W_1 := \exp(y).W_0 \leq W_0.$$

We then call $W_1 \leq W_0$ a $\pm$*half-sided modular inclusion*.

The next lemma shows that any wedge inclusion can be described in terms of positive and negative half-sided modular inclusions.

Lemma 2.12. *If $W_1 \leq W_3$ in $\mathcal{G}$, then there exists an element $W_2 \in \mathcal{G}$ with $W_1 \leq W_2 \leq W_3$ for which the inclusion $W_1 \leq W_2$ is +half-sided modular and the inclusion $W_2 \leq W_3$ is −half-sided modular.*

Proof. That $W_1 \leq W_3$ means that $W_1 = sW_3$ for some

$$s \in \mathcal{S}_{W_3} = \exp(C_-(W_3)) \exp(C_+(W_3)) G^{\uparrow}_{W_3}.$$

Accordingly, we write $s = g_- g_+ g_0$ and observe that $W_1 = g_- g_+ W_3$ because $g_0 W_3 = W_3$. Put $W_2 := g_- W_3$. Then $W_2 \leq W_3$ and $g_+ W_3 \leq W_3$ implies $W_1 = g_- g_+ W_3 \leq g_- W_3 = W_2$.

Further, the inclusion $W_2 \leq W_3$ is −half-sided modular because $g_- \in \exp(C_-(W_3))$. Likewise the inclusion $g_+ W_3 \leq W_3$ is +half-sided modular, and therefore $W_1 \leq W_2$ is also +half-sided modular. □

2.4.2. Central locality For a wedge $W = (x, \sigma)$, the dual wedge $W' = (-x, \sigma)$ need not be contained in the orbit $\mathcal{W}_+ = G^{\uparrow}.W$. If, however, $G^{\uparrow}$ has a non-trivial central subgroup Z such that, modulo Z, the complement W' is contained in $\mathcal{W}_+$, then we use central elements $\alpha \in Z$ to define "twisted complements" W'^{α} which are contained in $\mathcal{W}_+$, and this in turn leads to a twisted action of the full group G on $\mathcal{W}_+$. We also obtain on $\mathcal{W}_+$ a complementation map $W \mapsto W'^{\alpha}$.

Let $Z \subseteq Z(G^{\uparrow})$ be a closed normal subgroup of G, and $q \colon G \to \underline{G} := G/Z$ be the corresponding surjective morphism of graded Lie groups with kernel Z. If Z is discrete, then q is a covering map. The morphism of graded Lie groups q induces a natural map

$$q_{\mathcal{G}} \colon \mathcal{G}(G) \to \underline{\mathcal{G}} := \{(x, \underline{\sigma}) \in \mathfrak{g} \times \underline{G}^{\downarrow} \colon \underline{\sigma}^2 = e, \mathrm{Ad}_{\mathfrak{g}}(\underline{\sigma})x = x\}, \quad (x, \sigma) \mapsto (x, q(\sigma)), \tag{2.29}$$

where $\mathrm{Ad}_{\mathfrak{g}} \colon \underline{G} \to \mathrm{Aut}(\mathfrak{g})$ denotes the factorized adjoint action which exists because $Z = \ker(q)$ acts trivially on $\mathfrak{g}$. It restricts to a map

$$\mathcal{G}_E(G) \to \underline{\mathcal{G}}_E := \{(x, \underline{\sigma}) \in \mathcal{E}(\mathfrak{g}) \times \underline{G}^{\downarrow} \colon \underline{\sigma}^2 = e, \mathrm{Ad}_{\mathfrak{g}}(\underline{\sigma}) = e^{\pi i \,\mathrm{ad}\, x}\}. \tag{2.30}$$

As the following example shows, neither of these maps is always surjective. The main obstruction is that, although the differential $\mathbf{L}(q) \colon \mathbf{L}(G) \to \mathbf{L}(\underline{G})$ is surjective, there may be involutions $\tau \in \underline{G}^{\downarrow}$ for which no involution $\sigma \in G^{\downarrow}$ with $q(\sigma) = \tau$ exists. This phenomenon is tightly related to the twisted groups $\widehat{G}_z$ discussed in Remark 2.7 because these twists disappear for $z \in Z$ in $\widehat{G}/Z \cong G/Z$.

Example 2.13. We consider the graded Lie group

$$G := \mathrm{SL}_2(\mathbb{R})\{\mathbf{1}, \gamma\} \subseteq \mathrm{SL}_2(\mathbb{C}), \quad \text{where} \quad \gamma := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{satisfies} \quad \gamma^2 = -\mathbf{1}.$$

It has two connected component and $G^{\uparrow} = \mathrm{SL}_2(\mathbb{R})$.[6] The subgroup $Z := \{\pm\mathbf{1}\}$ is central and the quotient map $q\colon G \to \underline{G} := G/Z$ is a 2-fold covering. The Euler element $x := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ combines with the involution $q(\gamma) \in \underline{G}^{\downarrow}$ to the Euler couple $(x, q(\gamma)) \in \underline{\mathcal{G}}$. However, the set $\mathcal{G}(G)$ is empty because $G^{\downarrow}$ contains no involution. In fact, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, the condition that $g\gamma$ is an involution is equivalent to

$$\begin{pmatrix} -a & b \\ c & -d \end{pmatrix} = \gamma g \gamma = g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This is equivalent to $a = -d$ and $b = c = 0$, contradicting that $1 = \det(g) = -a^2$. We conclude in particular that the maps $\mathcal{G}(G) \to \underline{\mathcal{G}}$ and $\mathcal{G}_E(G) \to \mathcal{G}_E(\underline{G})$ are not surjective.

We now discuss $G^{\uparrow}$-orbits in $\mathcal{G}(G)$. In the examples we have in mind, the central subgroup Z is discrete.

Involution lifts and central wedge orbit. Each element $\sigma \in G^{\downarrow}$ acts in the same way on the abelian normal subgroup Z by the involution

$$\sigma_Z\colon Z \to Z, \quad \gamma \mapsto \gamma^{\sigma} := \sigma\gamma\sigma$$

which restricts to an involution $\sigma_Z \in \mathrm{Aut}(Z)$ because Z is central in $G^{\uparrow}$ and a normal subgroup of G. In the following we shall need the subgroups

$$Z^- := \{\gamma \in Z\colon \gamma^{\sigma} = \gamma^{-1}\} \supseteq Z_1 := \{\gamma^{\sigma}\gamma^{-1}\colon \gamma \in Z\}. \tag{2.31}$$

For $\gamma \in Z^-$, the element $\gamma^2 = (\gamma^{\sigma}\gamma^{-1})^{-1}$ is contained in Z_1, so that the quotient group Z^-/Z_1 is an elementary abelian 2-group, i.e., isomorphic to $\mathbb{Z}_2^{(B)}$ for some index set B.

For an involution $\sigma \in G^{\downarrow}$ and $\beta \in Z(G^{\uparrow})$, the element $\beta\sigma \in G^{\downarrow}$ is an involution if and only if $\beta \in Z^-$. Therefore

$$\alpha * (x, \sigma) := (x, \alpha\sigma) \tag{2.32}$$

defines an action of Z^- on $\mathcal{G}(G)$, commuting with the conjugation action of $G^{\uparrow}$ and satisfying

$$g.(\alpha * (x, \sigma)) = \alpha^{-1} * (g.(x, \sigma)) \quad \text{for} \quad g \in G^{\downarrow}, \alpha \in Z^-. \tag{2.33}$$

For $W = (x, \sigma) \in \mathcal{G}(G)$, the fiber over $\underline{W} := (x, q(\sigma))$ is thus given by

$$Z^- * W := \{(x, \alpha\sigma)\colon \alpha \in Z^-\}. \tag{2.34}$$

The subgroup $Z \subseteq G^{\uparrow}$ acts by conjugation on the fiber $Z^- * W$:

$$\gamma.(x, \sigma) = (x, \gamma\sigma\gamma^{-1}) = (x, \gamma(\gamma^{\sigma})^{-1}\sigma),$$

---

[6] This is the twisted version of the 2-fold cover of the extended Möbius groups; see Remark 2.7(d).

so that the quotient group Z^-/Z_1 parametrizes the Z-conjugation orbits in the fiber $Z^- * W$.[7] Here is an example.

Example 2.14. (a) If $Z \cong \mathbb{Z}$ and $n^\sigma = -n$, then $Z^- = \mathbb{Z}$ and $Z_1 = 2\mathbb{Z}$, so that $Z^-/Z_1 \cong \mathbb{Z}/2\mathbb{Z}$.
(b) If $Z = \mathbb{Z}_n$ and $\overline{n}^\sigma = -\overline{n}$, then $Z^- = \mathbb{Z}_n$ and $Z_1 = 2\mathbb{Z}_n$, so that

$$Z^-/Z_1 \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is even} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

Wedge $G^\uparrow$-orbits. Let $W = (x, \sigma) \in \mathcal{G}_E(G)$ and $\underline{W} = (x, q(\sigma)) \in \underline{\mathcal{G}}$. In general the group $G^\uparrow$ does not act transitively on the inverse image of the orbit $\underline{\mathcal{W}_+} := \underline{G}^\uparrow.\underline{W} \subseteq \underline{\mathcal{G}}$ under $q_{\mathcal{G}}$. We now describe how this set decomposes into orbits. By the transitivity of the $\underline{G}^\uparrow$-action on $\underline{\mathcal{W}_+}$, it suffices to consider the orbits of the stabilizer

$$G^\uparrow_{\underline{W}} = \{g \in G^\uparrow : q(g).\underline{W} = \underline{W}\}$$

on the fiber $Z^- * W$. That $g \in G^\uparrow$ fixes $\underline{W}$ implies in particular that $g\sigma g^{-1}\sigma = g(g^\sigma)^{-1} \in Z$. This leads to a homomorphism

$$\partial \colon G^\uparrow_{\underline{W}} \to Z^-, \quad g \mapsto g(g^\sigma)^{-1} \quad \text{with} \quad g.(x, \sigma) = (\mathrm{Ad}(g)x, g\sigma g^{-1}) = (x, \partial(g)\sigma). \tag{2.35}$$

As $Z \subseteq G^\uparrow_{\underline{W}}$, the image $Z_2 := \partial(G^\uparrow_{\underline{W}})$ is a subgroup containing Z_1.

Example 2.15 (An example where $Z_1 \neq Z_2$.) We consider the group $G = \widetilde{\mathsf{M\ddot{o}b}} \rtimes \{\mathbf{1}, \widetilde{\tau}\}$ from Example 2.10(d) and the canonical homomorphism

$$q \colon G \to \underline{G} := \mathrm{SL}_2(\mathbb{R}) \rtimes \{\mathbf{1}, \sigma\}, \quad \sigma := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

whose kernel is the central subgroup $Z := 2Z(G^\uparrow) \subseteq Z(G^\uparrow) \cong \mathbb{Z}$ of index two. Now $W = (h, \widetilde{\tau}) \in \mathcal{G}(G)$ is an Euler couple mapped to $\underline{W} = (h, \sigma) \in \underline{\mathcal{G}}$. As $z^{\widetilde{\tau}} = z^{-1}$ for every $z \in Z$, we have $Z = Z^-$ and $Z_1 = 2Z$ is a subgroup of index 2. To calculate Z_2, we observe that

$$\underline{G}^\uparrow_{\underline{W}} = \underline{G}_h = \exp(\mathbb{R}h)\{\pm\mathbf{1}\} \quad \text{and} \quad G^\uparrow_{\underline{W}} = \exp(\mathbb{R}h)Z(G^\uparrow).$$

We conclude that

$$Z_2 = \partial\big(G^\uparrow_{\underline{W}}\big) = \partial(Z(G^\uparrow)) = 2Z(G^\uparrow) = Z^- \neq Z_1.$$

The situation changes if we consider $Z = Z(G^\uparrow)$ and the center-free group $\underline{G} = \mathsf{M\ddot{o}b} \rtimes \{\mathbf{1}, \tau\}$ instead. Then $Z = Z^- = Z(G^\uparrow)$ and $Z_1 = Z_2 = 2Z$.

As the $G^\uparrow$ orbits in $q_{\mathcal{G}}^{-1}(\underline{G}^\uparrow.\underline{W}) = q_{\mathcal{G}}^{-1}(\underline{\mathcal{W}_+})$ correspond to the $G^\uparrow_{\underline{W}}$-orbits in the fiber $q_{\mathcal{G}}^{-1}(\underline{W}) = Z^- * W$, we obtain the following lemma.

[7] Considering Z as a module $\mathbb{Z}_2$-module via the involution σ_Z, we have $Z^1(\mathbb{Z}_2, Z) \cong Z^-$ and $B^1(\mathbb{Z}_2, Z) \cong Z_1$, so that the cohomology group is $H^1(\mathbb{Z}_2, Z) := Z^1(\mathbb{Z}_2, Z)/B^1(\mathbb{Z}_2, Z) \cong Z^-/Z_1$. We refer to [HN12, Ex. 18.3.15] or [ML63, Thm. IV.7.1] for more on group cohomology.

Lemma 2.16. *The quotient group Z^-/Z_2 parametrizes the set of $G^\uparrow$-orbits in $q_{\mathcal{G}}^{-1}(\underline{\mathcal{W}}_+)$.*

α-twisted complement. The following definition generalizes the notion of complementary wedge given in Definition 2.5 (a).

Definition 2.17. For $\alpha \in Z^-$, we define the *α-twisted complement* of $W = (x, \sigma) \in \mathcal{G}(G)$ by

$$(x, \sigma)^{'\alpha} := (-x, \alpha\sigma).$$

We will refer to couples of the form $W^{'\alpha}$ as complementary wedges. We consider $W^{'\alpha}$ as a "complement" of W because $q_{\mathcal{G}}$ maps $W^{'\alpha}$ to W' (see item (a) below).

Lemma 2.18. *For each $\alpha \in Z^-$, the α-twisted complementation $W \mapsto W^{'\alpha}$ satisfies:*

(a) *For $\alpha \in Z^-$, $W^{'\alpha}$ is mapped by $q_{\mathcal{G}}$ onto the complement $W' = (-x, q(\sigma))$ of $\underline{W} = (x, q(\sigma))$.*
(b) *The α-twisted complementation is not involutive if $\alpha^2 \neq e$.*
(c) *The map $'^\alpha : \mathcal{G}(G) \to \mathcal{G}(G)$, $(x, \sigma) \mapsto (-x, \alpha\sigma)$ is $G^\uparrow$-equivariant.*
(d) *In terms of the action (2.32) of Z^- on $\mathcal{G}(G)$, we have*

$$W^{'\alpha} = \alpha * W' \quad \text{for} \quad W \in \mathcal{G}(G), \alpha \in Z^-. \tag{2.36}$$

(e) *The prescription*

$$g *_\alpha (x, \sigma) := \begin{cases} g.(x, \sigma) & \text{for } g \in G^\uparrow \\ g.(\alpha^{-1} * (x, \sigma)) = \alpha * (g.(x, \sigma)) & \text{for } g \in G^\downarrow. \end{cases} \tag{2.37}$$

defines an action of G on $\mathcal{G}(G)$. This action satisfies

$$W^{'\alpha} = \sigma *_\alpha W \quad \text{for} \quad W = (x, \sigma) \in \mathcal{G}(G), \alpha \in Z^-. \tag{2.38}$$

If $W^{'\alpha} \in G^\uparrow.W$, then $\mathcal{W}_+ = G^\uparrow.W$ is invariant under the full group G with respect to the α-twisted action.
(f) *There exists an $\alpha \in Z^-$ with $W^{'\alpha} \in \mathcal{W}_+$ if and only if $\underline{W}' := (-x, q(\sigma)) \in G^\uparrow.\underline{W}$. If this is the case, then $W^{'\beta} \in \mathcal{W}_+$ for $\beta \in Z^-$ if and only if $\beta^{-1}\alpha \in Z_2$. In this case, the twisted actions of $g \in G^\downarrow$ are related by $g*_\beta = (\beta\alpha^{-1}) * g*_\alpha$.*

Proof. (a) and (b) are easy to see.
(c) follows from $\alpha \in Z(G^\uparrow)$ and the $G^\uparrow$-equivariance of the complementation map.
(d) is immediate from the definition of $\alpha * W$.
(e) That the prescription defines an action follows easily from the fact that $g_1 *_\alpha (g_2 *_\alpha W) = (g_1 g_2).W$ for $g_1, g_2 \in G^\downarrow$ (cf. (2.33))). The relation (2.38) follows from $\sigma.W = \sigma.(x, \sigma) = (-x, \sigma)$. For the last statement, we note that by (2.38), the relation $W^{'\alpha} \in \mathcal{W}_+$ implies

$$G *_\alpha \mathcal{W}_+ = \mathcal{W}_+ \cup \sigma *_\alpha \mathcal{W}_+ = \mathcal{W}_+ \cup G^\uparrow.W^{'\alpha} = \mathcal{W}_+ \cup G^\uparrow \mathcal{W}_+ = \mathcal{W}_+.$$

(f) As $q_{\mathcal{G}}(\mathcal{W}_+) = \underline{\mathcal{W}}_+ = G^\uparrow.\underline{W}$ and $q_{\mathcal{G}}(W^{'\alpha}) = \underline{W}'$, the inclusion $W^{'\alpha} \in \mathcal{W}_+$ implies that $\underline{W}' \in \underline{\mathcal{W}}_+$. If, conversely, $\underline{W}' \in \underline{\mathcal{W}}_+$, then there exists a $g \in G^\uparrow$ with

$$(-x, q(\sigma)) = g.(x, q(\sigma)) = (\mathrm{Ad}(g)x, q(g\sigma g^{-1})),$$

so that $\alpha := g\sigma g^{-1}\sigma \in \ker(q) = Z$ satisfies

$$\mathcal{W}_+ \ni g.W = g.(x, \sigma) = (-x, g\sigma g^{-1}) = (-x, \alpha\sigma) = \alpha * W' = W'^{\alpha}.$$

Now suppose that $W'^{\alpha} = \alpha * W' \in \mathcal{W}_+$. Then $W'^{\beta} = \beta * W' \in \mathcal{W}_+$ is equivalent to $\beta\alpha^{-1} * W'^{\alpha} = W'^{\beta} \in \mathcal{W}_+$, and this is equivalent to $\beta^{-1}\alpha * \mathcal{W}_+ = \mathcal{W}_+$. Next we observe that the relation $\beta\alpha^{-1} * W \in \mathcal{W}_+$ is equivalent to the existence of some $g \in G^{\uparrow}_{\underline{W}}$ with $g.W = (x, \beta^{-1}\alpha\sigma)$, which means that $\beta\alpha^{-1} \in Z_2 = \partial(G^{\uparrow}_{\underline{W}})$. □

Example 2.19. We show that for $G = \widetilde{\mathsf{M\ddot{o}b}} \rtimes \{\mathbf{1}, \widetilde{\tau}\}$ as in Example 2.10(d), we have to use twisted complements to obtain a $G^{\uparrow}$-orbit in $\mathcal{G}_E(G)$ invariant under complementation. We have already seen that $\mathcal{G}_E(\widetilde{\mathsf{M\ddot{o}b}})$ contains two $G^{\uparrow}$-orbits, represented by the couples $W_0 = (h, \widetilde{\tau})$ and $W_1 = (h, \widetilde{\tau}_1)$. The complement $W_0' = (-h, \widetilde{\tau})$ satisfies

$$\widetilde{\rho}(\pi)W_0' = (h, \widetilde{\rho}(\pi)\widetilde{\tau}\widetilde{\rho}(-\pi)) = (h, \widetilde{\rho}(2\pi)\widetilde{\tau}) = (h, \widetilde{\tau}_1) = W_1,$$

so that complementation exchanges the two $G^{\uparrow}$-orbits in $\mathcal{G}_E(\widetilde{\mathsf{M\ddot{o}b}})$. On the other hand, for the action $*_\alpha$ defined in (2.37), the full group G preserves both $G^{\uparrow}$-orbits.

Since $\mathrm{Ad}(\rho(-\pi))h = -h$, the element $g := \widetilde{\rho}(-\pi)$ can be used to define a suitable α-twisted conjugation as follows. We note that

$$\alpha := g(g^{\widetilde{\tau}})^{-1} = \widetilde{\rho}(-\pi)\widetilde{\rho}(-\pi) = \widetilde{\rho}(-2\pi)$$

is a generator of $Z := Z(\widetilde{\mathsf{M\ddot{o}b}}) = Z^-$. We now have

$$W_0'^{\alpha} = (-h, \alpha\widetilde{\tau}) = \widetilde{\rho}(-\pi).(h, \widetilde{\tau}) = \widetilde{\rho}(-\pi).W_0 \in G^{\uparrow}.W_0.$$

Thus $\mathcal{G}_E(\widetilde{\mathsf{M\ddot{o}b}}_2)$ consists of two $G^{\uparrow}$-orbits, none of which is invariant under complementation, but both are invariant under α-complementation. An analogous computation leads to the same picture for even coverings of $\mathsf{M\ddot{o}b}$, in particular for the fermionic case.

3. Euler Elements and 3-graded Lie Algebras

In this section we exhibit a general relation between two notions that are a priori unrelated: complementary and orthogonal wedges. For the sake of simplicity we consider in this introductory part the case of the Poincaré group $G = \mathcal{P}_+$ on $\mathbb{R}^{1+2}$ (cf. Example 2.10). We have seen that if $W = (k_W, j_W)$ is a wedge of the group G, then $W' = (-k_W, j_W)$ is the opposite wedge. The π-spatial rotation $\rho(\pi)$ takes W onto W' and vice versa. Thus there exists a group element $g \in G^{\uparrow} = \mathcal{P}_+^{\uparrow}$ such that $\mathrm{Ad}(g)k_W = -k_W$, and in this sense k_W is *symmetric*. This ensures a symmetry between a wedge and its opposite wedge, which corresponds to its causal complement in Minkowski spacetime.

Typical pairs of orthogonal wedges are the coordinate wedges

$$W_i = \{(t, x) \in \mathbb{R}^{1+2} : |t| < x_i\} \equiv (k_i, j_i) \in \mathcal{G}_E(G) \quad \text{for} \quad i = 1, 2. \tag{3.1}$$

The importance of this couple of wedges comes by the clear geometric relation: the wedge reflection of W_1 acts on the orthogonal wedge as

$$j_1.W_2 = W_2 \quad \text{resp.} \quad \mathrm{Ad}(j_1)(k_2) = -k_2.$$

In [GL95] the authors study the orthogonality relation in order to extend the unitary covariance representation of the Poincaré group $\mathcal{P}_+^\uparrow$ to an (anti-)unitary representation of the graded group $\mathcal{P}_+$ and establish the Spin–Statistics Theorem. In this extension process, orthogonal Euler wedges play a crucial role. This point will be discussed from our abstract perspective in Section 4.4 below.

In this section we will see how, in our setting, the existence of a symmetric Euler element in the Lie algebra ensures the existence of an orthogonal pair. For symmetric Euler elements, the orthogonality relation for Euler elements is symmetric, and orthogonal pairs of Euler elements generate a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ in $\mathfrak{g}$.

3.1. Preliminaries on Lie algebras and algebraic groups. In this subsection we collect some basic facts on finite dimensional real Lie algebras and on real algebraic groups (see [HN12] for Lie algebras and [Ho81] for algebraic groups).

A Lie algebra $\mathfrak{g}$ is called *simple* if $\mathfrak{g}$ and $\{0\}$ are the only ideals of $\mathfrak{g}$. It is called *semisimple* if it is a direct sum of simple ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$. On the other side of the spectrum, we have *solvable* Lie algebras. These are the ones for which the derived series defined by $D^0(\mathfrak{g}) := \mathfrak{g}$ and $D^{n+1}(\mathfrak{g}) := [D^n(\mathfrak{g}), D^n(\mathfrak{g})]$ satisfies $D^N(\mathfrak{g}) = \{0\}$ for some $N \in \mathbb{N}$. Here

$$[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{[x, y] \colon x, y \in \mathfrak{g}\}$$

is the *commutator algebra* of $\mathfrak{g}$.

The fundamental theorem on the Levi decomposition asserts that, if $\mathfrak{r}$ is the maximal solvable ideal of $\mathfrak{g}$, then there exists a semisimple subalgebra $\mathfrak{s}$ (a Levi complement), such that

$$\mathfrak{g} \cong \mathfrak{r} \rtimes \mathfrak{s}$$

is a semidirect sum, i.e., a vector space direct sum of the ideal $\mathfrak{r}$ and the subalgebra $\mathfrak{s}$.

A key feature in the structure theory of semisimple real Lie algebras is the concept of a compactly embedded subalgebra. A subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ is said to be *compactly embedded* if the subgroup $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k}) = \langle e^{\mathrm{ad}\,\mathfrak{k}} \rangle \subseteq \mathrm{Aut}(\mathfrak{g})$ has compact closure. We write $\mathrm{Inn}(\mathfrak{g}) := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{g})$ for the subgroup of *inner automorphisms* of $\mathfrak{g}$.

An element $x \in \mathfrak{g}$ is called

- *elliptic*, if $\mathrm{ad}\, x$ is semisimple with purely imaginary eigenvalues, which is equivalent to the one-dimensional Lie subalgebra $\mathbb{R}x$ being compactly embedded.
- *hyperbolic*, if $\mathrm{ad}\, x$ is diagonalizable.
- *nilpotent*, if $\mathrm{ad}\, x$ is nilpotent, i.e., $(\mathrm{ad}\, x)^n = 0$ for some $n \in \mathbb{N}$.

The Cartan–Killing form

$$\kappa \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad \kappa(x, y) := \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$$

is a symmetric bilinear form on $\mathfrak{g}$ invariant under the automorphism group $\mathrm{Aut}(\mathfrak{g})$. Recall that a finite dimensional real Lie algebra is semisimple if and only if κ is non-degenerate (Cartan's criterion). Note that $\kappa(x, x) = \mathrm{tr}((\mathrm{ad}\, x)^2) \geq 0$ if x is hyperbolic and $\kappa(x, x) \leq 0$ if x is elliptic.

In the proof of Proposition 3.2 below we shall use some results from the theory of linear algebraic groups. We now recall the basic concepts. If V is a finite dimensional real vector space, then $\mathrm{GL}(V)$ denotes the group of linear automorphisms of V. Any polynomial function on the linear space $\mathrm{End}(V)$ defines a function on the group $\mathrm{GL}(V)$ and we call a subgroup $G \subseteq \mathrm{GL}(V)$ *algebraic* if it is the zero set of a family of polynomial functions $p_j \colon \mathrm{End}(V) \to \mathbb{R}$. An algebraic group G is said to be

- *reductive*, if each G-invariant subspace $V_1 \subseteq V$ has a G-invariant linear complement V_2.
- *unipotent*, if there exists a flag of linear subspaces

$$F_0 = \{0\} \subseteq F_1 \subseteq \cdots \subseteq F_n = V$$

such that $(g - \mathbf{1})F_j \subseteq F_{j-1}$ for $j = 1, \ldots, n$ and $g \in G$.

In this context one has a decomposition theorem (the Levi decomposition), asserting that every algebraic subgroup $G \subseteq \mathrm{GL}(V)$ is a semidirect product $G \cong U \rtimes L$, where U is unipotent and L is reductive. Moreover, for every reductive subgroup $L_1 \subseteq G$ there exists an element $g \in G$ with $gL_1g^{-1} \subseteq L$ ([Ho81, Thm. VIII.4.3]).

3.2. Symmetric and orthogonal Euler elements.

Definition 3.1. A pair (h, x) of Euler elements is called *orthogonal* if $\sigma_h(x) = -x$ (cf. Definition 2.4).

Proposition 3.2. *The following assertions hold:*

(i) *An Euler element $h \in \mathfrak{g}$ is symmetric, i.e., $-h \in \mathcal{O}_h$, if and only if h is contained in a Levi complement $\mathfrak{s}$ and h is a symmetric Euler element in $\mathfrak{s}$.*

(ii) *Let $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ be a Levi decomposition.*

  (a) *If $h \in \mathfrak{g}$ is a symmetric Euler element, then $\mathcal{O}_h = \mathrm{Inn}(\mathfrak{g})(\mathcal{O}_h \cap \mathfrak{s}) = \mathcal{O}_{q(h)}$, where $q \colon \mathfrak{g} \to \mathfrak{s}$ is the projection map.*

  (b) *Two symmetric Euler elements are conjugate under* $\mathrm{Inn}(\mathfrak{g})$ *if and only if their images in $\mathfrak{s}$ are conjugate under* $\mathrm{Inn}(\mathfrak{s})$.

Proof. (i) As $\mathcal{O}_h \subseteq h + [\mathfrak{g}, \mathfrak{g}]$ follows from the invariance of the affine subspace $h + [\mathfrak{g}, \mathfrak{g}]$ under $\mathrm{Inn}(\mathfrak{g})$, the relation $-h \in \mathcal{O}_h$ implies $h \in [\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ be a Levi decomposition of $\mathfrak{g}$. As $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$, the commutator algebra is adapted to this decomposition:

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{r} + \mathfrak{s}, \mathfrak{r} + \mathfrak{s}] = [\mathfrak{g}, \mathfrak{r}] + \mathfrak{s} \cong [\mathfrak{g}, \mathfrak{r}] \rtimes \mathfrak{s}.$$

Now h is an Euler element in the ideal $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{r}] \rtimes \mathfrak{s}$. This is the Lie algebra of an algebraic group for which $[\mathfrak{g}, \mathfrak{r}]$ is the Lie algebra of the unipotent radical and $\mathfrak{s}$ the Lie algebra of a reductive complement ([Ho81, Thm. VIII.3.3]). As the algebraic group generated by $\exp(\mathbb{R} \,\mathrm{ad}\, h)$ is reductive, the conjugacy of Levi decompositions ([Ho81, Thm. VIII.4.3]) implies that $\mathrm{ad}\, h$ is contained in some Levi complement $\mathrm{ad}\, \mathfrak{s}$ of $\mathrm{ad}([\mathfrak{g}, \mathfrak{g}]) = [\mathrm{ad}\, \mathfrak{g}, \mathrm{ad}\, \mathfrak{g}]$. Replacing h by another element in $\mathcal{O}_h$, we may thus assume that $h \in \mathfrak{z}(\mathfrak{g}) + \mathfrak{s}$ for some Levi complement $\mathfrak{s}$ of $\mathfrak{g}$. Then $\mathfrak{r}$ and $\mathfrak{s}$ are $\mathrm{ad}\, h$-invariant, so that the $\mathrm{ad}\, h$-eigenspaces of the restrictions satisfy

$$\mathfrak{r} = \mathfrak{r}_1(h) + \mathfrak{r}_0(h) + \mathfrak{r}_{-1}(h) \quad \text{and} \quad \mathfrak{s} = \mathfrak{s}_1(h) + \mathfrak{s}_0(h) + \mathfrak{s}_{-1}(h),$$

and define 3-gradings of $\mathfrak{r}$ and $\mathfrak{s}$. Further $\mathfrak{g}_{\pm 1}(h) \subseteq [h, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ imply that $\mathfrak{g} = \mathfrak{r}_0(h) + [\mathfrak{g}, \mathfrak{g}]$. As $[\mathfrak{g}, \mathfrak{g}]$ is an ideal and $\mathfrak{r}_0(h)$ a subalgebra of $\mathfrak{g}$, the subgroup $\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{g}, \mathfrak{g}])$ of $\mathrm{Inn}(\mathfrak{g})$ is normal, and $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Inn}_{\mathfrak{g}}([\mathfrak{g}, \mathfrak{g}])\,\mathrm{Inn}(\mathfrak{r}_0(h))$. As $\mathrm{Inn}(\mathfrak{r}_0(h))$

fixes h, this in turn shows that $\mathcal{O}_h = \mathrm{Inn}_{\mathfrak{g}}([\mathfrak{g},\mathfrak{g}])h = \mathrm{Inn}_{\mathfrak{g}}([\mathfrak{g},\mathfrak{r}])\,\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s})h$. Writing $h = h_z + h_s$ with $h_z \in \mathfrak{z}(\mathfrak{g})$ and $h_s \in \mathcal{E}(\mathfrak{s})$, we thus find $x \in [\mathfrak{g},\mathfrak{r}]$ and $s \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s})$ such that[8]

$$-h_z - h_s = -h = e^{\mathrm{ad}\,x}s.h = h_z + e^{\mathrm{ad}\,x}s.h_s.$$

Applying the Lie algebra homomorphism q to both sides, we derive from $q(h_z) = 0$ and $q \circ e^{\mathrm{ad}\,x} = q$ that $-h_s = s.h_s$, and therefore

$$e^{\mathrm{ad}\,x}h_s = h_s + 2h_z.$$

We conclude that the unipotent linear map $e^{\mathrm{ad}\,x}$ preserves the linear subspace $\mathbb{R}h_s + \mathbb{R}h_z$, and this implies that $\mathrm{ad}\,x = \log(e^{\mathrm{ad}\,x})$ also has this property. We thus arrive at

$$[h, x] = [h_s, x] \subseteq \mathbb{R}h_s + \mathbb{R}h_z \subseteq \mathfrak{g}_0(h),$$

so that we must have $x \in \mathfrak{g}_0(h) = \mathfrak{g}_0(h_s)$, which in turn leads to $0 = e^{\mathrm{ad}\,x}h_s - h_s = 2h_z$, i.e., $h = h_s \in \mathfrak{s}$.

To prove the second assertion of (i), we observe that the homomorphism $q \colon \mathfrak{g} \to \mathfrak{s} \cong \mathfrak{g}/\mathfrak{r}$ satisfies

$$q(\mathcal{O}_x) = \mathcal{O}^{\mathfrak{s}}_{q(x)} \quad \text{for} \quad x \in \mathfrak{g}. \tag{3.2}$$

Hence $q(\mathcal{E}_{\mathrm{sym}}(\mathfrak{g})) \subseteq \mathcal{E}_{\mathrm{sym}}(\mathfrak{s})$. If, conversely, $h \in \mathcal{E}_{\mathrm{sym}}(\mathfrak{s})$, then we clearly have $-h \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s})h \subseteq \mathrm{Inn}(\mathfrak{g})h$, so that $h \in \mathcal{E}_{\mathrm{sym}}(\mathfrak{g})$.
(ii)(a) As $\mathcal{O}_h$ intersects $\mathfrak{s}$ by (i), $q(\mathcal{O}_h) \cap \mathcal{O}_h \neq \emptyset$, and since $\mathrm{Inn}(\mathfrak{s})$ acts transitively on $q(\mathcal{O}_h)$ by (3.2), we obtain $q(\mathcal{O}_h) \subseteq \mathcal{O}_h$ and thus $q(\mathcal{O}_h) = \mathcal{O}_h \cap \mathfrak{s}$. This further leads to

$$\mathcal{O}_h = \mathrm{Inn}(\mathfrak{g})(\mathcal{O}_h \cap \mathfrak{s}) = \mathrm{Inn}(\mathfrak{g})q(\mathcal{O}_h) = \mathrm{Inn}(\mathfrak{g})\mathcal{O}^{\mathfrak{s}}_{q(h)} = \mathcal{O}_{q(h)}.$$

(ii)(b) follows immediately from (a). □

Proposition 3.2 reduces the description of symmetric Euler elements up to conjugation by inner automorphisms to the case of simple Lie algebras.

Remark 3.3. Suppose that $\mathfrak{g}$ is a finite dimensional Lie algebra containing a pointed generating invariant cone C. If $\mathfrak{g}$ is not reductive, then $C \cap \mathfrak{z}(\mathfrak{g}) \neq \{0\}$ ([Ne99, Thm. VII.3.10]). If $\tau = \sigma_h$ is an involution defined by a symmetric Euler element h, then τ fixes every central element, so that we cannot have $\tau(C) = -C$ if $\mathfrak{g}$ is not reductive.

Examples 3.4. (a) If $\mathfrak{s}$ is a semisimple Lie algebra and $h \in \mathfrak{s}$ an Euler element, then it also is an Euler element in the semidirect sum $T\mathfrak{s} := |\mathfrak{s}| \rtimes \mathfrak{s}$, where $|\mathfrak{s}|$ is the linear subspace underlying $\mathfrak{s}$, endowed with the $\mathfrak{s}$-module structure defined by the adjoint representation.
(b) In the simple Lie algebra $\mathfrak{g} := \mathfrak{sl}_n(\mathbb{R})$, we write $n \times n$-matrices as block 2×2-matrices according to the partition $n = k + (n-k)$. Then

$$h_k := \frac{1}{n}\begin{pmatrix} (n-k)\mathbf{1}_k & 0 \\ 0 & -k\mathbf{1}_{n-k} \end{pmatrix}$$

8 Here we use that the Lie algebra $[\mathfrak{g},\mathfrak{r}]$ is nilpotent, so that the exponential function of the corresponding group $\mathrm{Inn}_{\mathfrak{g}}([\mathfrak{g},\mathfrak{r}])$ is surjective, see [HN12].

is diagonalizable with the two eigenvalue $\frac{n-k}{n} = 1 - \frac{k}{n}$ and $-\frac{k}{n}$. Therefore h_k is an Euler element whose 3-grading is given by

$$\mathfrak{g}_0(h) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in \mathfrak{gl}_k(\mathbb{R}), d \in \mathfrak{gl}_{n-k}(\mathbb{R}), \operatorname{tr}(a) + \operatorname{tr}(d) = 0 \right\},$$
$$\mathfrak{g}_1(h) = \begin{pmatrix} 0 & M_{k,n-k}(\mathbb{R}) \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_{-1}(h) \cong \begin{pmatrix} 0 & 0 \\ M_{n-k,k}(\mathbb{R}) & 0 \end{pmatrix}.$$

Example 3.5. For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, the Euler element

$$h := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{satisfies} \quad \sigma_h \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Any element in $\operatorname{Fix}(-\sigma_h)$ is of the form $x = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$, and it is an Euler element if and only if $bc = -\det(x) = \frac{1}{4}$. If $g \in \mathrm{SL}_2(\mathbb{R})$ commutes with h, then it is diagonal, i.e., $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, and thus

$$\operatorname{Ad}(g)\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^2 b \\ a^{-2} c & 0 \end{pmatrix}.$$

We thus obtain two representatives

$$x_\pm = \pm \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of conjugacy classes of orthogonal pairs (h, x) of Euler elements for $\mathfrak{sl}_2(\mathbb{R})$. The involution corresponding to $x_\pm$ is given by

$$\sigma_{x_\pm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{\pi i x_\pm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{-\pi i x_\pm} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix},$$

which shows in particular that

$$\sigma_{x_\pm}(h) = -h. \tag{3.3}$$

As a consequence of the preceding discussion, we see that the orthogonality relation on $\mathcal{E}(\mathfrak{sl}_2(\mathbb{R}))$ is symmetric:

Lemma 3.6. *If (x, y) is an orthogonal pair of Euler elements in $\mathfrak{sl}_2(\mathbb{R})$, then $\sigma_y(x) = -x$, so that (y, x) is also orthogonal.*

Example 3.7. For $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$, the Euler element

$$h := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{satisfies} \quad \sigma_h \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

and we see, as for $\mathfrak{sl}_2(\mathbb{R})$, that the orthogonal Euler elements are given by

$$x_\pm = \pm \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{with} \quad \sigma_{x_\pm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

This shows that

$$\sigma_{x_\pm}(h) \neq -h. \tag{3.4}$$

Therefore $\mathfrak{gl}_2(\mathbb{R})$ contains a pair (h, x) of orthogonal Euler elements for which $\sigma_x(h) \neq -h$. From $h \notin [\mathfrak{g}, \mathfrak{g}]$ it immediately follows that h is not symmetric. We shall see in Theorem 3.13 below that this pathology of the orthogonality relation on the set of Euler elements does not occur for symmetric Euler elements.

Example 3.8. For $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$, the Euler element

$$h_1 := \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{satisfies} \quad \sigma_{h_1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

where we write matrices as 2×2-block matrices according to the partition $3 = 1 + 2$. Up to conjugacy under the centralizer of h_1, the symmetric matrices in $\mathrm{Fix}(-\sigma_{h_1})$ are represented by

$$x = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}.$$

These matrices have three different eigenvalues, so that $\operatorname{ad} x$ has five eigenvalues, and thus x cannot be an Euler elements of $\mathfrak{sl}_3(\mathbb{R})$. We conclude that there exists no Euler element $x \in \mathcal{E}(\mathfrak{sl}_3(\mathbb{R}))$ for which (h_1, x) is orthogonal.

We shall see in Theorem 3.13(b) below that this never happens for symmetric Euler elements, but h_1 is not symmetric. It corresponds to h_1 for the root system A_2 in the notation of Section 3.3.

Example 3.9. For $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{R})$, the Euler element

$$h_1 := \frac{1}{4}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{satisfies} \quad \sigma_{h_1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

where we write matrices as 2×2-block matrices according to the partition $4 = 1 + 3$. Up to conjugacy under the centralizer of h_1, the symmetric matrices in $\mathrm{Fix}(-\sigma_{h_1})$ are represented by

$$x = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}.$$

They all have three different eigenvalues and $\operatorname{ad} x$ has five eigenvalues, so that they are not Euler elements. We conclude that there exists no Euler element $x \in \mathcal{E}(\mathfrak{sl}_4(\mathbb{R}))$ for which (h_1, x) is orthogonal.

This is different for the symmetric Euler element

$$h_2 := \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{with} \quad \sigma_{h_2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

where we write matrices as 2×2-block matrices according to the partition $4 = 2 + 2$. Up to conjugacy under the centralizer of h_2, the symmetric matrices in $\mathrm{Fix}(-\sigma_{h_2})$ are represented by

$$x = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix},$$

and, for $a = b = \frac{1}{2}$, these are Euler elements orthogonal to h_2.

3.3. Euler elements in simple real Lie algebras. In this section we take a systematic look at Euler elements in simple real Lie algebras. In particular we determine which of them are symmetric and show that pairs of orthogonal ones generate $\mathfrak{sl}_2$-subalgebras (Theorem 3.13). For the classification of 3-gradings of simple Lie algebras, we refer to [KA88], the concrete list of the 18 types in [Kan98, p. 600] which is also listed below, and Kaneyuki's lecture notes [Kan00].

Let $\mathfrak{g}$ is a real semisimple Lie algebra. An involutive automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$ is called a *Cartan involution* if its eigenspaces

$$\mathfrak{k} := \mathfrak{g}^\theta = \{x \in \mathfrak{g} \colon \theta(x) = x\} \quad \text{and} \quad \mathfrak{p} := \mathfrak{g}^{-\theta} = \{x \in \mathfrak{g} \colon \theta(x) = -x\}$$

have the property that they are orthogonal with respect to κ, which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{3.5}$$

is called a *Cartan decomposition*. Cartan involutions always exist and two such involutions are conjugate under the group $\mathrm{Inn}(\mathfrak{g})$ of inner automorphism, so they produce isomorphic decompositions ([HN12, Thm. 13.2.11]).

If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, then $\mathfrak{k}$ is a maximal compactly embedded subalgebra of $\mathfrak{g}$, $x \in \mathfrak{g}$ is elliptic if and only if its adjoint orbit $\mathcal{O}_x = \mathrm{Inn}(\mathfrak{g})x$ intersects $\mathfrak{k}$, and $x \in \mathfrak{g}$ is hyperbolic if and only if $\mathcal{O}_x \cap \mathfrak{p} \neq \emptyset$.

For the finer structure theory, and also for classification purposes, one starts with a Cartan involution θ and fixes a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. As $\mathfrak{a}$ is abelian, $\mathrm{ad}\,\mathfrak{a}$ is a commuting set of diagonalizable operators, hence simultaneously diagonalizable. For a linear functional $0 \neq \alpha \in \mathfrak{a}^*$, the simultaneous eigenspaces

$$\mathfrak{g}_\alpha := \{y \in \mathfrak{g} \colon (\forall x \in \mathfrak{a})\ [x, y] = \alpha(x)y\}$$

are called *root spaces* and

$$\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}) := \{\alpha \in \mathfrak{a}^* \setminus \{0\} \colon \mathfrak{g}_\alpha \neq 0\}$$

is called the set of *restricted roots*. We pick a set

$$\Pi := \{\alpha_1, \ldots, \alpha_n\} \subseteq \Sigma$$

of *simple roots*. This is a subset with the property that every root $\alpha \in \Sigma$ is a linear combination $\alpha = \sum_{j=1}^n n_j \alpha_j$, where the coefficients are either all in $\mathbb{Z}_{\geq 0}$ or in $\mathbb{Z}_{\leq 0}$. The convex cone

$$\Pi^\star := \{x \in \mathfrak{a} \colon (\forall \alpha \in \Pi)\ \alpha(x) \geq 0\}$$

is called the *positive (Weyl) chamber corresponding to* Π.

We have the *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}, \quad \text{where} \quad \mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}.$$

Now $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, and for a non-zero element $x_\alpha \in \mathfrak{g}_\alpha$, the 3-dimensional subspace spanned by $x_\alpha, \theta(x_\alpha)$ and $[x_\alpha, \theta(x_\alpha)] \in \mathfrak{a}$ is a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. In particular, it contains a unique element $\alpha^\vee \in \mathfrak{a}$ with $\alpha(\alpha^\vee) = 2$. Then

$$r_\alpha \colon \mathfrak{a} \to \mathfrak{a}, \quad r_\alpha(x) := x - \alpha(x)\alpha^\vee$$

is a reflection, and the subgroup

$$\mathcal{W} := \langle r_\alpha \colon \alpha \in \Sigma \rangle \subseteq \mathrm{GL}(\mathfrak{a})$$

is called the *Weyl group*. Its action on $\mathfrak{a}$ provides a good description of the adjoint orbits of hyperbolic elements: Every hyperbolic element in $\mathfrak{g}$ is conjugate to a unique element in $\Pi^\star$ and, for $x \in \mathfrak{a}$, the intersection $\mathcal{O}_x \cap \mathfrak{a} = \mathcal{W}x$ is the Weyl group orbit ([KN96, Thm. III.10]).

From now on we assume that $\mathfrak{g}$ is simple. Then Σ is an irreducible root system, hence of one of the following types:

$$A_n, \qquad B_n, \qquad C_n, \qquad D_n, \qquad E_6, E_7, E_8, \quad F_4, \quad G_2 \quad \text{or} \quad BC_n, n \geq 1$$

(cf. [Bo90a]). If $\mathfrak{g}$ is a complex simple Lie algebra, then it is also simple as a real Lie algebra, and a Cartan decomposition takes the form

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k},$$

where $\mathfrak{k} \subseteq \mathfrak{g}$ is a compact real form. Then $\mathfrak{a} = i\mathfrak{t}$, where $\mathfrak{t} \subseteq \mathfrak{k}$ is maximal abelian. In particular, the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ coincides with the root system of the complex Lie algebra $\mathfrak{g}$. This leads to a one-to-one correspondence between isomorphy classes of simple complex Lie algebras and the irreducible reduced root systems. If $\mathfrak{g}$ is not complex, then neither the isomorphy class of $\mathfrak{g}$ nor of $\mathfrak{g}_\mathbb{C}$ is determined by the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. For instance all Lie algebras $\mathfrak{so}_{1,n}(\mathbb{R})$ have the restricted root system A_1 with $\dim \mathfrak{a} = 1$, but their complexifications $\mathfrak{so}_{n+1}(\mathbb{C})$ have the root systems B_k for $n = 2k$ and D_k for $n = 2k - 1$.

The adjoint orbit of an Euler element in $\mathfrak{g}$ contains a unique $h \in \Pi^\star$. For any Euler element $h \in \Pi^\star$, we have $\alpha(h) \in \{0, 1\}$ for $\alpha \in \Pi$ because the values of the roots on h are the eigenvalues of $\operatorname{ad} h$. If such an element exists, then the irreducible root system Σ must be reduced. Otherwise, for any root α with $2\alpha \in \Sigma$, we must have $\alpha(h) = 0$ because $\operatorname{ad} x$ has only three eigenvalues. As the set of such roots generates the same linear space as Σ, this leads to the contradiction $h = 0$. This excludes the non-reduced simple root systems of type BC_n.

To see how many possibilities we have for Euler elements in $\mathfrak{a}$, we recall that Π is a linear basis of $\mathfrak{a}$, so that, for each $j \in \{1, \ldots, n\}$, there exists a uniquely determined element

$$h_j \in \mathfrak{a}, \quad \text{satisfying} \quad \alpha_k(h_j) = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

A simple Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called *hermitian* if the center

$$\mathfrak{z}(\mathfrak{k}) = \{x \in \mathfrak{k} \colon [x, \mathfrak{k}] = \{0\}\}$$

of a maximal compactly embedded subalgebra $\mathfrak{k}$ is non-zero. For hermitian Lie algebras, the restricted root system Σ is either of type C_r or BC_r (cf. Harish Chandra's Theorem [Ne99, Thm. XII.1.14]), and we say that $\mathfrak{g}$ is *of tube type* if the restricted root system is of type C_r.

The following theorem lists for each irreducible root system Σ the possible Euler elements in the positive chamber $\Pi^\star$. Since every adjoint orbit in $\mathcal{E}(\mathfrak{g})$ has a unique representative in $\Pi^\star$, this classifies the $\mathrm{Inn}(\mathfrak{g})$-orbits in $\mathcal{E}(\mathfrak{g})$ for any non-compact simple real Lie algebra. For semisimple algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, an element $x = (x_1, \ldots, x_k)$ is an Euler element if and only if its components $x_j \in \mathfrak{g}_j$ are Euler elements, and its orbit is

$$\mathcal{O}_x = \mathcal{O}_{x_1} \times \cdots \times \mathcal{O}_{x_k}.$$

Therefore it suffices to consider simple Lie algebras, and for these the root system Σ is irreducible. As every complex simple Lie algebra $\mathfrak{g}$ is also a real simple Lie algebra, our discussion also covers complex Lie algebras.

Theorem 3.10. *Suppose that $\mathfrak{g}$ is a non-compact simple real Lie algebra, with restricted root system $\Sigma \subseteq \mathfrak{a}^*$ of type X_n. We follow the conventions of the tables in* [Bo90a] *for the classification of irreducible root systems and the enumeration of the simple roots $\alpha_1, \ldots, \alpha_n$. Then every Euler element $h \in \mathfrak{a}$ on which Π is non-negative is one of $h_1, \ldots, h_n$, and for every irreducible root system, the Euler elements among the h_j are the following:*

$$A_n : h_1, \ldots, h_n, \quad B_n : h_1, \quad C_n : h_n, \quad D_n : h_1, h_{n-1}, h_n, \quad E_6 : h_1, h_6, \quad E_7 : h_7. \tag{3.7}$$

For the root systems BC_n, E_8, F_4 and G_2 no Euler element exists (they have no 3-grading). The symmetric Euler elements are

$$A_{2n-1} : h_n, \quad B_n : h_1, \quad C_n : h_n, \quad D_n : h_1, \quad D_{2n} : h_{2n-1}, h_{2n}, \quad E_7 : h_7. \tag{3.8}$$

Proof. Writing the highest root in Σ with respect to the simple system Π as $\alpha_{\max} = \sum_{j=1}^n c_j \alpha_j$, we have $c_j \in \mathbb{Z}_{>0}$ for each j. If $h \in \Pi^\star$ is an Euler element, then $\Pi(h) \subseteq \{0, 1\}$, and $1 = \alpha_{\max}(h) = \sum_{j=1}^n c_j \alpha_j(h)$ implies that at most one value $\alpha_j(h)$ can be 1, and then the others are 0, i.e., $h = h_j$ for some $j \in \{1, \ldots, n\}$. Moreover, h_j is an Euler element if and only if $c_j = 1$. Consulting the tables on the irreducible root systems in [Bo90a], we obtain the Euler elements listed in (3.7).

To determine the symmetric ones, let $w_0 \in \mathcal{W}$ be the longest element of the Weyl group, which is uniquely determined by $w_0^* \Pi = -\Pi$ for the dual action of $\mathcal{W}$ on $\mathfrak{a}^*$. Then $h_j' := w_0(-h_j)$ is the Euler element in the positive chamber representing the orbit $\mathcal{O}_{-h_j}$. Therefore h_j is symmetric if and only if $-h_j \in \mathcal{W} h_j$, which is equivalent to $h_j' = h_j$. Using the description of w_0 and the root systems in [Bo90a], now leads to

$$A_{n-1} : h_j' = h_{n-j}, \quad B_n : h_1' = h_1, \quad C_n : h_n' = h_n, \tag{3.9}$$

$$D_n : h_1' = h_1, h_n' = \begin{cases} h_{n-1} & \text{for } n \text{ odd}, \\ h_n & \text{for } n \text{ even}, \end{cases} \tag{3.10}$$

$$E_6 : h_1' = h_6, \quad E_7 : h_7' = h_7. \tag{3.11}$$

Hence the symmetric Euler elements are given by the list (3.8). □

This theorem requires some interpretation. So let us first see what it says about complex simple Lie algebras $\mathfrak{g}$. In (3.7) we see that only if $\mathfrak{g}$ is not of type E_8, F_4 or G_2, the Lie algebra $\mathfrak{g}$ contains an Euler element. Euler elements correspond to 3-gradings of the root system and these in turn to hermitian real forms $\mathfrak{g}^\circ$, where $ih_j \in \mathfrak{z}(\mathfrak{k}^\circ)$ generates the center of a maximal compactly embedded subalgebra $\mathfrak{k}^\circ$ ([Ne99, Thm. A.V.1]). We thus obtain the following possibilities. In Table 1, we write $\mathfrak{g}^\circ$ for the hermitian real form, $\mathfrak{g}$ for the complex Lie algebra, Σ for its root system, and h_j for the corresponding Euler element:

Table 1. Simple hermitian Lie algebras $\mathfrak{g}^\circ$

$\mathfrak{g}^\circ$ (hermitian)	$\Sigma(\mathfrak{g}^\circ, \mathfrak{a}^\circ)$	$\mathfrak{g} = (\mathfrak{g}^\circ)_\mathbb{C}$	$\Sigma(\mathfrak{g}, \mathfrak{a})$	Euler element
$\mathfrak{su}_{p,q}(\mathbb{C}), 1 \leq p \leq q$	$BC_p(p < q), C_p(p = q)$	$\mathfrak{sl}_{p+q}(\mathbb{C})$	A_{p+q-1}	h_p
$\mathfrak{so}_{2,2n-1}(\mathbb{R}), n > 1$	C_2	$\mathfrak{so}_{2n+1}(\mathbb{C})$	B_n	h_1
$\mathfrak{sp}_{2n}(\mathbb{R})$	C_n	$\mathfrak{sp}_{2n}(\mathbb{C})$	C_n	h_n
$\mathfrak{so}_{2,2n-2}(\mathbb{R}), n > 2$	C_2	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_1
$\mathfrak{so}^*(2n)$	$BC_m(n = 2m+1), C_m(n = 2m)$	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_{n-1}, h_n
$\mathfrak{e}_{6(-14)}$	BC_2	$\mathfrak{e}_6$	E_6	$h_1 = h_6'$
$\mathfrak{e}_{7(-25)}$	C_3	$\mathfrak{e}_7$	E_7	h_7

Note that $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}_{2,1}(\mathbb{R}) \cong \mathfrak{su}_{1,1}(\mathbb{C})$. More expectional isomorphisms are discussed in some detail in [HN12, Sect. 17].

In this correspondence, those hermitian simple Lie algebras corresponding to symmetric Euler elements are of particular interest. Comparing with the list of hermitian simple Lie algebras of tube type (cf. [FK94, p. 213]), we see that they correspond precisely to the 3-gradings specified by symmetric Euler elements, as listed in (3.8). Since the Euler elements h_{n-1} and h_n for the root system of type D_n are conjugate under a diagram automorphism, they correspond to isomorphic hermitian real forms.

Table 2. Simple hermitian Lie algebras $\mathfrak{g}^\circ$ of tube type

$\mathfrak{g}^\circ$ (hermitian)	$\Sigma(\mathfrak{g}^\circ, \mathfrak{a}^\circ)$	$\mathfrak{g} = (\mathfrak{g}^\circ)_\mathbb{C}$	$\Sigma(\mathfrak{g}, \mathfrak{a})$	symm.Eulerelement h
$\mathfrak{su}_{n,n}(\mathbb{C})$	C_n	$\mathfrak{sl}_{2n}(\mathbb{C})$	A_{2n-1}	h_n
$\mathfrak{so}_{2,2n-1}(\mathbb{R}), n > 1$	C_2	$\mathfrak{so}_{2n+1}(\mathbb{C})$	B_n	h_1
$\mathfrak{sp}_{2n}(\mathbb{R})$	C_n	$\mathfrak{sp}_{2n}(\mathbb{C})$	C_n	h_n
$\mathfrak{so}_{2,2n-2}(\mathbb{R}), n > 2$	C_2	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_1
$\mathfrak{so}^*(4n)$	C_n	$\mathfrak{so}_{4n}(\mathbb{C})$	D_{2n}	h_{2n-1}, h_{2n}
$\mathfrak{e}_{7(-25)}$	C_3	$\mathfrak{e}_7$	E_7	h_7

In our context hermitian simple Lie algebras are of particular interest. We therefore collect some of their main properties in the following proposition.

Proposition 3.11. *For a simple real Lie algebra, the following assertions hold:*

(a) $\mathfrak{g}$ *is hermitian if and only if there exists a closed convex* $\mathrm{Inn}(\mathfrak{g})$*-invariant cone* $C \neq \{0\}, \mathfrak{g}$.
(b) *A simple hermitian Lie algebra contains an Euler element if and only if it is of tube type, and in this case* $\mathrm{Inn}(\mathfrak{g})$ *acts transitively on* $\mathcal{E}(\mathfrak{g})$.

Proof. (a) is a consequence of the Kostant–Vinberg Theorem (cf. [HÓ96, Lemma 2.5.1]). (b) Since the restricted root system of a hermitian simple Lie algebra is of type C_r or BC_r, and the first case characterizes the algebras of tube type, the assertion follows from Theorem 3.10 because C_r only permits one class of Euler elements. □

There are many types of simple 3-graded Lie algebras that are neither complex nor hermitian of tube type; for instance the Lorentzian algebras $\mathfrak{so}_{1,n}(\mathbb{R})$. We refer to [Kan98, p. 600] or [Kan00]. for the list of all 18 types which is reproduced below.

Table 3. Simple 3-graded Lie algebras

	$\mathfrak{g}$	$\Sigma(\mathfrak{g},\mathfrak{a})$	h	$\mathfrak{g}_1(h)$
1	$\mathfrak{sl}_n(\mathbb{R})$	A_{n-1}	$h_j, 1 \leq j \leq n-1$	$M_{j,n-j}(\mathbb{R})$
2	$\mathfrak{sl}_n(\mathbb{H})$	A_{n-1}	$h_j, 1 \leq j \leq n-1$	$M_{j,n-j}(\mathbb{H})$
3	$\mathfrak{su}_{n,n}(\mathbb{C})$	C_n	h_n	$\mathrm{Herm}_n(\mathbb{C})$
4	$\mathfrak{sp}_{2n}(\mathbb{R})$	C_n	h_n	$\mathrm{Sym}_n(\mathbb{R})$
5	$\mathfrak{u}_{n,n}(\mathbb{H})$	C_n	h_n	$\mathrm{Aherm}_n(\mathbb{H})$
6	$\mathfrak{so}_{p,q}(\mathbb{R})$	$B_p\ (p<q),\ D_p\ (p=q)$	h_1	$\mathbb{R}^{p+q-2}$
7	$\mathfrak{so}^*(4n)$	C_n	h_n	$\mathrm{Herm}_n(\mathbb{H})$
8	$\mathfrak{so}_{n,n}(\mathbb{R})$	C_n	h_n	$\mathrm{Alt}_n(\mathbb{R})$
9	$\mathfrak{e}_6(\mathbb{R})$	E_6	$h_1 = h_6'$	$M_{1,2}(\mathbb{O}_{\mathrm{split}})$
10	$\mathfrak{e}_{6(-26)}$	A_2	h_1	$M_{1,2}(\mathbb{O})$
11	$\mathfrak{e}_7(\mathbb{R})$	E_7	h_7	$\mathrm{Herm}_3(\mathbb{O}_{\mathrm{split}})$
12	$\mathfrak{e}_{7(-25)}$	C_3	h_3	$\mathrm{Herm}_3(\mathbb{O})$
13	$\mathfrak{sl}_n(\mathbb{C})$	A_{n-1}	$h_j, 1 \leq j \leq n-1$	$M_{j,n-j}(\mathbb{C})$
14	$\mathfrak{sp}_{2n}(\mathbb{C})$	C_n	h_n	$\mathrm{Sym}_n(\mathbb{C})$
15a	$\mathfrak{so}_{2n+1}(\mathbb{C})$	B_n	h_1	$\mathbb{C}^n$
15b	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_1	$\mathbb{C}^n$
16	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_{n-1}, h_n	$\mathrm{Alt}_n(\mathbb{C})$
17	$\mathfrak{e}_6(\mathbb{C})$	E_6	$h_1 = h_6'$	$M_{1,2}(\mathbb{O})_{\mathbb{C}}$
18	$\mathfrak{e}_7(\mathbb{C})$	E_7	h_7	$\mathrm{Herm}_3(\mathbb{O})_{\mathbb{C}}$

Remark 3.12. As $h \in \mathfrak{a}$ implies $\theta(h) = -h$, the Cartan involution θ always maps h into $-h$, but this only implies that h is symmetric if $\theta \in \mathrm{Inn}(\mathfrak{g})$. This is the case if $\mathfrak{g}$ is hermitian, so that in these Lie algebras all Euler elements are symmetric.

We conclude this section with some finer results concerning orthogonality and symmetry of Euler elements.

Theorem 3.13. *If $\mathfrak{g}$ is simple and $h \in \mathcal{E}(\mathfrak{g})$, then the following assertions hold:*

(a) *If $x \in \mathcal{E}(\mathfrak{g})$ is such that (h, x) is orthogonal, then*
 (i) *h and x are symmetric,*
 (ii) *the Lie algebra generated by h and x is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, and*
 (iii) *$\sigma_x(h) = -h$, so that (x, h) is also orthogonal.*

(b) *There exists an Euler element x such that (h, x) is orthogonal if and only if h is symmetric.*

Proof. (a) We split the proof into the two cases, according to whether $\mathfrak{g}$ is a complex Lie algebra or not. We then reduce the second case to the first one.

Case 1: $\mathfrak{g}$ is complex: A simple complex Lie algebra $\mathfrak{g}$ contains an Euler element, i.e., it possesses a 3-graded root system, if and only if it has a real form $\mathfrak{g}^\circ$ which is hermitian, i.e., $\mathfrak{g} = (\mathfrak{g}^\circ)_{\mathbb{C}} = \mathfrak{g}^\circ \oplus i\mathfrak{g}^\circ$. This follows for example by comparing the list of irreducible root systems for which Euler elements exist (see (3.7)) with the classification of hermitian simple Lie algebras $\mathfrak{g}^\circ$ (see [Ne99, Thm. A.V.1] and Table 1). In this case the real Lie algebra $\mathfrak{g}^\circ$ has a Cartan decomposition $\mathfrak{g}^\circ = \mathfrak{k}^\circ \oplus \mathfrak{p}^\circ$ and the center $\mathfrak{z}(\mathfrak{k}^\circ)$ is one-dimensional and generated by an element z with $\mathrm{Spec}(\mathrm{ad}\, z) = \{0, \pm i\}$ ([Ne99, Thm. A.V.1]). Then $h = iz$ is an Euler element in the complexification $\mathfrak{g}$ for which $\mathfrak{k}^\circ = \ker(\mathrm{ad}\, z) \cap \mathfrak{g}^\circ$ and $[z, \mathfrak{g}^\circ] = \mathfrak{p}^\circ$, where $\mathrm{ad}\, z|_{\mathfrak{p}^\circ}$ is a complex structure on the real vector space $\mathfrak{p}^\circ$. The corresponding Euler involution $\sigma_h = e^{\pi i\, \mathrm{ad}\, h} = e^{\pi\, \mathrm{ad}\, z} \in \mathrm{Aut}_{\mathbb{C}}(\mathfrak{g})$ thus restricts to the Cartan involution on $\mathfrak{g}^\circ$, corresponding to the decomposition $\mathfrak{k}^\circ \oplus \mathfrak{p}^\circ$. Accordingly, we obtain

$$\mathfrak{h} := \mathrm{Fix}(\sigma_h) = (\mathfrak{k}^\circ)_{\mathbb{C}} \quad \text{and} \quad \mathfrak{q} := \mathrm{Fix}(-\sigma_h) = (\mathfrak{p}^\circ)_{\mathbb{C}}.$$

A Cartan decomposition of $\mathfrak{g}$ is obtained by $\mathfrak{k} = \mathfrak{k}^\circ + i\mathfrak{p}^\circ$ and $\mathfrak{p} = \mathfrak{p}^\circ + i\mathfrak{k}^\circ$. If $\mathfrak{t} \subseteq \mathfrak{k}^\circ$ is a maximal abelian Lie subalgebra, then $\mathfrak{a} := i\mathfrak{t} \subseteq \mathfrak{p}$ is a maximal abelian subspace which contains $h = iz \in i\mathfrak{z}(\mathfrak{k}^\circ) \subseteq i\mathfrak{t}$. The orthogonality of the pair (h, x) means that $x \in \mathfrak{q} = \mathrm{Fix}(-\sigma_h)$. By [KN96, Cor. III.9], $x \in \mathcal{E}(\mathfrak{g}) \cap \mathfrak{q}$ is conjugate under the centralizer of h to an element in $\mathfrak{q} \cap \mathfrak{p} = \mathfrak{p}^\circ$. Fixing a maximal abelian subspace $\mathfrak{a}^\circ \subseteq \mathfrak{p}^\circ$, we may therefore assume that x is an Euler element for the corresponding restricted root system $\Sigma^\circ := \Sigma(\mathfrak{g}^\circ, \mathfrak{a}^\circ) \subseteq (\mathfrak{a}^\circ)^*$, which is of type C_r or BC_r (cf. [Ne99, Thm. XII.1.14]). As we have already observed above, the existence of an Euler element $x \in \mathfrak{a}^\circ$ implies that the restricted root system Σ° is reduced, which excludes the case BC_r. Therefore $\mathfrak{g}^\circ$ is of tube type (cf. Proposition 3.11) and Table 2 thus implies that h is symmetric.

The fact that $\mathfrak{g}^\circ$ is of tube type implies that $x \in \mathfrak{a}^\circ$ corresponds to the unique Euler element h_r for the restricted root system Σ° of type C_r (see (3.7)). From (3.8) it now follows that x is symmetric (see also Proposition 3.11). This proves (i).

To verify (ii) and (iii), we observe that the root system C_r contains the maximal subset $\{2\varepsilon_1, \ldots, 2\varepsilon_r\}$ of strongly orthogonal roots, i.e., neither sums nor differences of these roots are roots. The multiplicities of these restricted roots are 1 ([Ne99, Thm. XII.1.14]), and

$$\mathfrak{s} := \bigoplus_{j=1}^{r} (\mathfrak{g}^\circ_{2\varepsilon_j} + \mathfrak{g}^\circ_{-2\varepsilon_j} + \mathbb{R}(2\varepsilon_j)^\vee) = \mathfrak{a}^\circ \oplus \bigoplus_{j=1}^{r} (\mathfrak{g}^\circ_{2\varepsilon_j} + \mathfrak{g}^\circ_{-2\varepsilon_j}) \cong \mathfrak{sl}_2(\mathbb{R})^r$$

(cf. [Ne99, Lemma XII.1.11], [Ta79, p. 12]). As the roots $2\varepsilon_j$ all take the value 1 on the Euler element $x \in \mathfrak{a}^\circ$, we have $x = \frac{1}{2}\sum_{j=1}^r (2\varepsilon_j)^\vee$, which is the diagonal element in $\mathfrak{sl}_2(\mathbb{R})^r$, corresponding to $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$. Likewise, ih is contained in $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})^r$ as the diagonal element corresponding to $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$. As the Lie subalgebra of $\mathfrak{gl}_2(\mathbb{C})$, generated by

$$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, the same holds for the real Lie subalgebra of $\mathfrak{g}$ generated by h and x. Now (ii) and (iii) follow from Lemma 3.6.

Case 2: $\mathfrak{g}$ is not complex: Then $\mathfrak{g}_{\mathbb{C}}$ is a simple complex Lie algebra to which all arguments in Case 1 apply. In particular, the real Lie subalgebra $\mathfrak{s}$ spanned by h, x and $[h, x]$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. This proves (ii) and (iii). As $\mathfrak{s} \subseteq \mathfrak{g}$ and all Euler elements in $\mathfrak{sl}_2(\mathbb{R})$ are symmetric, we also obtain (i).

(b) If there exists an Euler element x for which (h, x) is orthogonal, then (a)(i) implies that h is a symmetric Euler element. Suppose, conversely, that h is a symmetric Euler element. For a Cartan involution θ with $\theta(h) = -h$, we choose a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p} = \mathrm{Fix}(-\theta)$ containing h and choose in the subset

$$\Sigma_1 := \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})\colon \alpha(h) = 1\}$$

a maximal set $\{\gamma_1, \ldots, \gamma_r\}$ of strongly orthogonal roots (cf. [Ta79, p.13] or [Kan00, p. 134]). From these references we further infer the existence of elements $e_j \in \mathfrak{g}_{\gamma_j}$ such that, for each j, the subalgebra $\mathfrak{s}_j := \mathrm{span}_{\mathbb{R}}\{e_j, \theta(e_j), [e_j, \theta(e_j)]\}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. We normalize e_j in such a way that $x_j := e_j - \theta(e_j)$ is an Euler element of $\mathfrak{s}_j$. Then loc. cit. further implies that

$$\mathfrak{a}_{\mathfrak{q}} := \mathrm{span}\{x_j\colon j = 1, \ldots, r\}$$

is maximal abelian in $\mathfrak{q}_{\mathfrak{p}} = \mathfrak{q} \cap \mathfrak{p}$ for $\mathfrak{q} := \mathfrak{g}^{-\sigma_h}$. Since h is a symmetric Euler element and the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ is irreducible, h corresponds to some h_j in the list (3.8). The restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathfrak{q}})$ is always of type C_r. The explicit description of the restricted roots in [Kan98, p. 596] now implies that $x := \sum_{j=1}^r x_j \in \mathfrak{a}_{\mathfrak{q}}$ is an Euler element. By construction, it satisfies $\sigma_h(x) = -x$, so that (h, x) is orthogonal. This completes the proof. □

Corollary 3.14. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and (h, x) be orthogonal Euler elements such that h is also symmetric. Then the following assertions hold:*

(a) *There exists a Levi complement containing h and x.*
(b) *The Lie algebra generated by h and x is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.*
(c) *(x, h) is also orthogonal.*

Proof. By Proposition 3.2(i), there exists a Levi decompositions $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ with $h \in \mathfrak{s}$. We then have for $\mathfrak{q} := \mathrm{Fix}(-\sigma_h)$ the decompositions

$$\mathfrak{q} := \mathfrak{g}_1(h) \oplus \mathfrak{g}_{-1}(h) = \mathfrak{q}_{\mathfrak{r}} \oplus \mathfrak{q}_{\mathfrak{s}} \quad \text{with} \quad \mathfrak{q}_{\mathfrak{r}} = \mathfrak{q} \cap \mathfrak{r} \quad \text{and} \quad \mathfrak{q}_{\mathfrak{s}} = \mathfrak{q} \cap \mathfrak{s},$$

and $x \in \mathfrak{q}$ is an Euler element, hence in particular hyperbolic. Let $\mathfrak{a}_{\mathfrak{r}} \subseteq \mathfrak{q}_{\mathfrak{r}}$ be a maximal hyperbolic subspace, i.e., $\mathfrak{a}_{\mathfrak{r}}$ is abelian, consists of ad-diagonalizable elements and is maximal with respect to this property. Then $\mathfrak{a}_{\mathfrak{r}} \subseteq [h, \mathfrak{r}] \subseteq [\mathfrak{g}, \mathfrak{r}]$ consists also of ad-nilpotent elements, hence is central. As $\mathrm{ad}\, h|_{\mathfrak{q}}$ is injective, it follows that $\mathfrak{a}_{\mathfrak{r}} = \{0\}$. By [KN96, Prop. III.5], $\mathfrak{q}_{\mathfrak{s}}$ contains a maximal hyperbolic subspace $\mathfrak{a}$ of $\mathfrak{q}$ and x is conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ to an element of $\mathfrak{a} \subseteq \mathfrak{q}_{\mathfrak{s}}$. This proves(a).
(b) In view of (a), we may w.l.o.g. assume that $\mathfrak{g}$ is semisimple, and by Theorem 3.13, which applies to each simple ideal, even that $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{R})^r$ for some $r \in \mathbb{N}$. As $\mathrm{Aut}(\mathfrak{sl}_2(\mathbb{R})) \cong \mathrm{PGL}_2(\mathbb{R})$ acts transitively on the set of orthogonal pairs of Euler elements in $\mathfrak{sl}_2(\mathbb{R})$ (Example 3.5), we may further assume that

$$h = (h_0, \cdots, h_0) \quad \text{and} \quad x = (x_0, \cdots, x_0) \quad \text{for} \quad h_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad x_0 := \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$

so that the Lie subalgebra generated by x and h is the diagonal in $\mathfrak{sl}_2(\mathbb{R})^r$, hence isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.
(c) follows directly from (b) and Lemma 3.6. □

4. Covariant Nets of Real Subspaces

In this section we develop an axiomatic setting for covariant nets of standard subspaces parametrized by $G^{\uparrow}$-orbits in $\mathcal{G}_E(G)$.

4.1. Standard subspaces. Here we collect some fundamental notions concerning real subspaces of a complex Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$, linear in the second argument. We call a closed real subspace $\mathsf{H} \subseteq \mathcal{H}$ *cyclic* if $\mathsf{H}+i\mathsf{H}$ is dense in $\mathcal{H}$, *separating* if $\mathsf{H}\cap i\mathsf{H} = \{0\}$, and *standard* if it is cyclic and separating. The symplectic "complement" of a real subspace H is defined by the symplectic form $\mathrm{Im}\langle\cdot,\cdot\rangle$, namely

$$\mathsf{H}' = \{\xi \in \mathcal{H} : (\forall \eta \in \mathsf{H})\ \mathrm{Im}\langle \xi, \eta\rangle = 0\}.$$

Note that H is separating if and only if H' is cyclic, hence H is standard if and only if H' is standard. For a standard subspace H, we define the *Tomita operator* as the closed antilinear involution

$$S_{\mathsf{H}} : \mathsf{H} + i\mathsf{H} \to \mathsf{H} + i\mathsf{H}, \quad \xi + i\eta \mapsto \xi - i\eta.$$

The polar decomposition $S_{\mathsf{H}} = J_{\mathsf{H}}\Delta_{\mathsf{H}}^{\frac{1}{2}}$ defines an antiunitary involution J_{H} and the modular operator Δ_{H}. For the modular group $(\Delta_{\mathsf{H}}^{it})_{t\in\mathbb{R}}$, we then have

$$J_{\mathsf{H}}\mathsf{H} = \mathsf{H}' \quad \text{and} \quad \Delta_{\mathsf{H}}^{it}\mathsf{H} = \mathsf{H} \qquad \text{for every} \quad t \in \mathbb{R}$$

([Lo08, Thm. 3.4]). This construction leads to a one-to-one correspondence between Tomita operators and standard subspaces:

Proposition 4.1 ([Lo08, Prop. 3.2]). *The map $\mathsf{H} \mapsto S_{\mathsf{H}}$ is a bijection between the set of standard subspaces of $\mathcal{H}$ and the set of closed, densely defined, antilinear involutions on $\mathcal{H}$. Moreover, polar decomposition $S = J\Delta^{1/2}$ defines a one-to-one correspondence between such involutions and pairs (Δ, J), where J is a conjugation and $\Delta > 0$ selfadjoint with $J\Delta J = \Delta^{-1}$.*

The modular operators of symplectic complements satisfy the following relations

$$S_{\mathsf{H}'} = S_{\mathsf{H}}^*, \qquad \Delta_{\mathsf{H}'} = \Delta_{\mathsf{H}}^{-1}, \qquad J_{\mathsf{H}'} = J_{\mathsf{H}}.$$

From Proposition 4.1 we easily deduce:

Lemma 4.2 ([Mo18, Lemma 2.2]). *Let $\mathsf{H} \subset \mathcal{H}$ be a standard subspace and $U \in \mathrm{AU}(\mathcal{H})$ be a unitary or anti-unitary operator. Then $U\mathsf{H}$ is also standard and $U\Delta_{\mathsf{H}}U^* = \Delta_{U\mathsf{H}}^{\varepsilon(U)}$ and $UJ_{\mathsf{H}}U^* = J_{U\mathsf{H}}$.*

Lemma 4.3 ([Lo08, Cor. 2.1.8]). *Let $\mathsf{H} \subset \mathcal{H}$ be a standard subspace, and $\mathsf{K} \subset \mathsf{H}$ be a closed, real linear subspace of H. If $\Delta_{\mathsf{H}}^{it}\mathsf{K} = \mathsf{K}$ for all $t \in \mathbb{R}$, then K is a standard subspace of $\mathcal{K} := \overline{\mathsf{K} + i\mathsf{K}}$ and $\Delta_{\mathsf{H}}|_{\mathsf{K}}$ is the modular operator of K on $\mathcal{K}$. If, in addition, K is a cyclic subspace of $\mathcal{H}$, then $\mathsf{H} = \mathsf{K}$.*

The following theorem relates positive generators and inclusions of real subspaces.

Theorem 4.4 ([Lo08, Thms. 3.15, 3.17], [BGL02, Thm. 3.2]). *Let $\mathsf{H} \subset \mathcal{H}$ be a standard subspace and $U(t) = e^{itP}$ be a unitary one-parameter group on $\mathcal{H}$ with a generator P.*

(a) *If* $\pm P > 0$ *and* $U(t)\mathsf{H} \subset \mathsf{H}$ *for all* $t \geq 0$, *then*

$$\Delta_{\mathsf{H}}^{-is/2\pi} U(t) \Delta_{\mathsf{H}}^{is/2\pi} = U(e^{\pm s} t) \quad \text{and} \quad J_{\mathsf{H}} U(t) J_{\mathsf{H}} = U(-t) \quad \text{for all} \quad t, s \in \mathbb{R}. \tag{4.1}$$

(b) *If* $\Delta_{\mathsf{H}}^{-is/2\pi} U(t) \Delta_{\mathsf{H}}^{is/2\pi} = U(e^{\pm s} t)$ *for* $s, t \in \mathbb{R}$, *then the following are equivalent:*
 (1) $U(t)\mathsf{H} \subset \mathsf{H}$ *for* $t \geq 0$;
 (2) $\pm P$ *is positive.*

Part (a) is also called the One-particle Borchers Theorem. Borchers originally proved it for von Neumann algebras with a cyclic and separating vectors, see [Bo92]. Part (b) is in [BGL02].

With the notation introduced in Examples 2.10(b), we have seen that any couple (U, H) of a one-parameter group $(U_t)_{t\in\mathbb{R}}$ with positive (resp. negative) generator and a standard subspace H satisfying the assumptions of Theorem 4.4(a) defines a unitary, positive energy representation of the affine group $\mathrm{Aff}(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{R}^\times$ implemented by

$$U(\zeta(t)) = U(t), \quad U(\delta(t)) = \Delta_{\mathsf{H}}^{-\frac{it}{2\pi}}, \quad U(r_0) = J_{\mathsf{H}} \quad \text{for} \quad t \in \mathbb{R}.$$

A representation of $\mathrm{Aff}(\mathbb{R})$ can also be obtained by looking at some peculiar relative positions of standard subspaces: The half-sided modular inclusions.

Definition 4.5. An inclusion $\mathsf{K} \subseteq \mathsf{H}$ of standard subspaces of $\mathcal{H}$ is called a $\pm$*half-sided modular inclusion* ($\pm$HSMI) if

$$\Delta_{\mathsf{H}}^{-it}\mathsf{K} \subseteq \mathsf{K} \quad \text{for} \quad \pm t \geq 0.$$

Theorem 4.6 ([Lo08, Cor. 3.6.6.], [NÓ17, Thm. 3.15]). $\mathsf{K} \subseteq \mathsf{H}$ *is a positive half-sided modular inclusion if and only if there exists an (anti-)unitary positive energy representation* $(U, \mathcal{H})$ *of* $\mathrm{Aff}(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{R}^\times$ *with* $U(\delta(t)) = \Delta_{\mathsf{H}}^{-\frac{it}{2\pi}}$, $U(r_0) = J_{\mathsf{H}}$, $U(\delta_{(1,\infty)}(t)) = \Delta_{\mathsf{K}}^{-\frac{it}{2\pi}}$, $U(r_1) = J_{\mathsf{K}}$. *Suppose that* $W_0 = (\lambda, r_0)$ *corresponds to the half-line* $(0, \infty)$. *For* $g \in \mathrm{Aff}(\mathbb{R})$, *let* $\mathsf{N}(g.W_0)$ *be the standard subspace associated to* $g.W_0 = (x_{g.W_0}, \sigma_{g.W_0})$ *through the (anti-)unitary representation* U. *In this picture,*

$$\mathsf{K} = \mathsf{N}((1, 1).W_0) \quad \text{and} \quad \mathsf{H} = \mathsf{N}(W_0),$$

and the translations satisfy

$$\mathsf{K} = U(1)\mathsf{H} \quad \text{and} \quad U(1 - e^t) = \Delta_{\mathsf{K}}^{-it/2\pi} \Delta_{\mathsf{H}}^{it/2\pi} \quad \text{for} \quad t \in \mathbb{R}.$$

As a consequence, negative half-sided modular inclusions $\mathsf{K} \subseteq \mathsf{H}$ are in in 1-1 correspondence with (anti-)unitary negative energy representation $(U, \mathcal{H})$ of $\mathrm{Aff}(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{R}^\times$ with

$$\mathsf{K} = \mathsf{N}((-1, 1).W_0') = U(-1)\mathsf{N}(W_0)' \quad \text{and} \quad \mathsf{H} = \mathsf{N}(W_0') = \mathsf{N}(W_0)'$$

and with $U(\delta(-t)) = \Delta_{\mathsf{H}}^{-\frac{it}{2\pi}}$, $U(r_0) = J_{\mathsf{H}}$, $U(\delta_{(-\infty,-1)}(t)) = \Delta_{\mathsf{K}}^{-\frac{it}{2\pi}}$, $U(r_{-1}) = J_{\mathsf{K}}$.

Corollary 4.7 ([Lo08, Corollary 2.4.3.]). *If* $\mathsf{K} \subset \mathsf{H}$ *is +HSMI, then* $\mathsf{H}' \subset \mathsf{K}'$ *is -HSMI*

4.2. The axiomatics of abstract covariant nets. Hereafter we will make the following assumption on the group G.

Assumption 1. We assume that $\mathcal{G}_E(G) \neq \emptyset$ and write $G = G^{\uparrow} \rtimes \{e, \sigma\}$ for some Euler involution σ.

Example 4.8. Note that $G^{\downarrow}$ may contain involutions which are not Euler.

We consider the graded Lie group $G := \mathrm{SO}_{1,n}(\mathbb{R})$ with the identity component $G^{\uparrow} = \mathrm{SO}_{1,n}(\mathbb{R})^{\uparrow}$. For $n \geq 2$, the Lie algebra $\mathfrak{g} = \mathfrak{so}_{1,n}(\mathbb{R})$ is simple, $\theta(x) = -x^{\top}$ is a Cartan involution, and $\mathfrak{a} := \mathfrak{so}_{1,1}(\mathbb{R}) \subseteq \mathfrak{p}$ (acting on the first two components) is a maximal abelian subspace. As the corresponding restricted root system is of type A_1, our classification scheme (see (3.8) in Theorem 3.10) implies that all Euler elements in $\mathfrak{g}$ are conjugate to the one corresponding to the boost generator

$$h(x_0, \ldots, x_n) = (x_1, x_0, 0, \ldots, 0).$$

Accordingly, an involution $\sigma \in G$ is Euler if and only if σ or $-\sigma$ is the orthogonal reflection in a 2-dimensional Lorentzian plane.

However, $G^{\downarrow}$ contains all reflections of the type

$$\tau(x) = (\varepsilon_0 x_0, \ldots, \varepsilon_n x_n) \quad \text{with} \quad \varepsilon_j \in \{\pm 1\} \quad \text{satisfying} \quad \prod_{j=0}^{n} \varepsilon_j = 1.$$

In particular neither $\mathrm{Fix}(\tau)$ nor $\mathrm{Fix}(-\tau)$ must have dimension 2.

We now present the analogs of the one-particle Haag–Kastler axioms and further fundamental properties in our general setting.

Definition 4.9. Let $G = G^{\uparrow} \rtimes \{e, \sigma\}$ be as above, $C \subseteq \mathfrak{g}$ be a closed convex $\mathrm{Ad}^{\varepsilon}(G)$-invariant cone in $\mathfrak{g}$, and fix a $G^{\uparrow}$-orbit $\mathcal{W}_+ = G^{\uparrow}.W \subseteq \mathcal{G}_E(G)$. Let $(U, \mathcal{H})$ be a unitary representation of $G^{\uparrow}$ and

$$\mathsf{N} \colon \mathcal{W}_+ \to \mathrm{Stand}(\mathcal{H}) \tag{4.2}$$

be a map, also called a *net of standard subspaces*. In the following we denote this data as $(\mathcal{W}_+, U, \mathsf{N})$. We consider the following properties:

(HK1) **Isotony:** $\mathsf{N}(W_1) \subseteq \mathsf{N}(W_2)$ for $W_1 \leq W_2$.
(HK2) **Covariance:** $\mathsf{N}(gW) = U(g)\mathsf{N}(W)$ for $g \in G^{\uparrow}$, $W \in \mathcal{W}_+$.
(HK3) **Spectral condition:** $C \subseteq C_U := \{x \in \mathfrak{g} \colon -i\partial U(x) \geq 0\}$. We then say that U is *C-positive*.
(HK4) **Central twisted locality:** For $\alpha \in Z(G^{\uparrow})^{-}$ and $W \in \mathcal{W}_+$ with $W'^{\alpha} \in \mathcal{W}_+$, there exists a unitary $Z_\alpha \in U(G^{\uparrow})'$ satisfying

$$Z_\alpha^2 = U(\alpha) \quad \text{and} \quad J_{\mathsf{N}(W)} Z_\alpha J_{\mathsf{N}(W)} = Z_\alpha^{-1}, \tag{4.3}$$

such that

$$\mathsf{N}(W'^{\alpha}) \subset Z_\alpha \mathsf{N}(W)'. \tag{4.4}$$

Moreoever, such an α exists.

When Z_α is trivial, for instance when $\partial(G^{\uparrow}_{\underline{W}}) = \{e\}$, then the central twisted locality reduces to the more familiar locality relation.

(HK4$_e$) **Locality:** If $W \in \mathcal{W}_+$ is such that $W' \in \mathcal{W}_+$, then $\mathsf{N}(W') \subset \mathsf{N}(W)'$.

Concerning (HK3), note that C_U is pointed if and only if $\ker(U)$ is discrete. Therefore the assumption that C is pointed is compatible with the possible existence of representations with discrete kernel satisfying (HK3). Furthermore, if $C = \{0\}$, then (HK3) trivially holds.

The following property will be central in our discussion because it connects the modular groups of standard subspaces to the unitary representation U of $G^\uparrow$.

(HK5) **Bisognano–Wichmann (BW) property:** $U(\lambda_W(t)) = \Delta_{\mathsf{N}(W)}^{-it/2\pi}$ for $W \in \mathcal{W}_+$, $t \in \mathbb{R}$.

We will see in Proposition 4.19 that a consequence of (HK1-5) is the following stronger form of (HK4):

(HK6) **Central Haag Duality**: $\mathsf{N}(W'^\alpha) = Z_\alpha \mathsf{N}(W)'$ for $\alpha \in Z(G^\uparrow)^-$, $W \in \mathcal{W}_+$ with $W'^\alpha \in \mathcal{W}_+$ and Z_α as in (4.3).

If the representation U extends antiunitarily to G we can further require:

(HK7) **G-covariance:** For any $\alpha \in Z(G^\uparrow)^-$ such that $W'^\alpha \in \mathcal{W}_+$, there exists an (anti-)unitary extension U^α of U from $G^\uparrow$ to G such that the following condition is satisfied:

$$\mathsf{N}(g *_\alpha W) = U^\alpha(g)\mathsf{N}(W) \quad \text{for} \quad g \in G, \tag{4.5}$$

where $*_\alpha$ is the α-twisted action (2.37) of G on $\mathcal{W}_+$ defined in Lemma 2.18(e).

It is enough to provide an extension U^α w.r.t. one $\alpha \in Z(G^\uparrow)^-$ such that $W'^\alpha \in \mathcal{W}_+$. All the other extensions come as described in Lemma 2.18(f). The modular conjugation of standard subspaces can have a geometric meaning when the extension U^α from (HK7) has the following specific form:

(HK8) **Modular reflection:** $U^\alpha(\sigma_W) = Z_\alpha J_{\mathsf{N}(W)}$ for $\alpha \in Z(G^\uparrow)^-$, $W \in \mathcal{W}_+$ with $W'^\alpha \in \mathcal{W}_+$ and Z_α as in (4.3).

In the next sections we will show that there exist nets of standard subspaces satisfying all the above assumptions. It is the analog of the BGL construction in this general setting.

4.2.1. Wedge isotony and half-sided modular inclusions Taking the wedge modular inclusion defined in Section 2.4.1 into account, we now prove that isotony can be deduced from covariance, the Bisognano–Wichmann property and the C-spectral condition. On specific models this has been checked in [BGL02,Lo08].

Proposition 4.10. *Let* $(\mathcal{W}_+, \mathsf{N}, U)$ *be a net of standard subspaces. Then the spectral condition* (HK3)*, the BW property* (HK5) *and* $G^\uparrow$*-covariance* (HK2) *imply isotony* (HK1).

Proof. Let $W_0 = (h, \tau) \in \mathcal{G}_E$ and $\mathsf{H}_0 = \mathsf{N}(W_0)$. By covariance, the net N is isotone if and only if

$$\mathcal{S}_{W_0} = G^\uparrow_{W_0} \exp(C_+) \exp(C_-) \subseteq \mathcal{S}_{\mathsf{H}_0} := \{g \in G^\uparrow : U(g)\mathsf{H}_0 \subseteq \mathsf{H}_0\}.$$

As the stabilizer $G^\uparrow_{W_0}$ stabilizes H_0 by covariance, isotony is equivalent to $\exp(x) \in \mathcal{S}_{\mathsf{H}_0}$ for every $x \in C_+ \cup C_-$.

By the spectral condition (HK3), we have $\mp i\partial U(x) \geq 0$. Therefore Theorem 4.4 shows that isotony is equivalent to

$$U^{\mathsf{H}_0}(e^s)U(\exp tx)U^{\mathsf{H}_0}(e^{-s}) = U(\exp e^{\pm s}tx) \quad \text{for} \quad s, t \in \mathbb{R}, x \in C_\pm. \tag{4.6}$$

By the BW property (HK5), $U^{\mathsf{H}_0}(e^s) = \Delta_{\mathsf{H}_0}^{-is/2\pi} = U(\exp sh)$, so that $[h, x] = \pm x$ for $x \in C_\pm$ implies (4.6). □

4.2.2. The Brunetti–Guido–Longo (BGL) construction We have seen in the introduction to Section 2 that each standard subspace H specifies a homomorphism

$$U^{\mathsf{H}} \colon \mathbb{R}^\times \to \mathrm{AU}(\mathcal{H}) \quad \text{by} \quad U^{\mathsf{H}}(e^t) := \Delta_{\mathsf{H}}^{-it/2\pi}, \quad U^{\mathsf{H}}(-1) := J_{\mathsf{H}}, \tag{4.7}$$

and that this leads to a bijection

$$\Phi\colon \mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, \mathrm{AU}(\mathcal{H})) \to \mathrm{Stand}(\mathcal{H}), \quad U^{\mathsf{H}} \mapsto \mathsf{H}$$

between continuous (anti-)unitary representations of the graded Lie group $\mathbb{R}^\times$ and standard subspaces ([NÓ17, Prop. 3.2]). By Lemma 4.2, Φ is equivariant with respect to the natural action of $\mathrm{AU}(\mathcal{H})$ on $\mathrm{Stand}(\mathcal{H})$ and the action (2.3) on $\mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, \mathrm{AU}(\mathcal{H}))$.

Now every (anti-)unitary representation $U\colon G \to \mathrm{AU}(\mathcal{H})$ defines by composition a natural G-equivariant map

$$\mathcal{G} \xrightarrow{\Psi^{-1}} \mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, G) \xrightarrow{U\circ} \mathrm{Hom}_{\mathrm{gr}}(\mathbb{R}^\times, \mathrm{AU}(\mathcal{H})), \qquad W \mapsto U \circ \gamma_W.$$

Combining this with Φ leads to the so-called Brunetti–Guido–Longo (BGL) construction:

Definition 4.11 (Brunetti–Guido–Longo (BGL) net.) If (U, G) is an (anti-)unitary representation, then we obtain a G-equivariant map $\mathsf{N}_U\colon \mathcal{G} \to \mathrm{Stand}(\mathcal{H})$ determined for $W = (k_W, \sigma_W)$ by

$$J_{\mathsf{N}_U(W)} = U(\sigma_W) \quad \text{and} \quad \Delta_{\mathsf{N}_U(W)}^{-it/2\pi} = U(\exp tk_W) \quad \text{for} \quad t \in \mathbb{R}. \tag{4.8}$$

This means that, with respect to Definition 2.3, $U^{\mathsf{N}_U(W)} = U \circ \gamma_W$ for $W \in \mathcal{G}$ (see [BGL02], [NÓ17, Prop. 5.6]).

The BGL net associates to every wedge $W \in \mathcal{G}$ a standard subspace $\mathsf{N}_U(W)$. We shall denote with $(\mathcal{W}_+, \mathsf{N}_U, U)$ the restriction of the BGL net to the $G^\uparrow$-orbit $\mathcal{W}_+ \subseteq \mathcal{G}_E(G)$.

Theorem 4.12. *The restriction of the BGL net N_U associated to an (anti-)unitary C-positive representation U of $G = G^\uparrow \rtimes \{e, \sigma\}$ to a $G^\uparrow$-orbit $\mathcal{W}_+ \subseteq \mathcal{G}_E$ satisfies all the axioms* (HK1)-(HK3) *and* (HK5).

We shall see in Proposition 4.16 that the twisted locality (HK4), Central Haag Duality (HK6) and (HK7-8) are also satisfied.

Proof. Let $\mathcal{W}_+ \subseteq \mathcal{G}_E(G)$ be a $G^\uparrow$-orbit. By construction, the restriction of the BGL net N_U to $\mathcal{W}_+$ satisfies (HK2) and by construction it satisfies (HK5). By Proposition 4.10, isotony (HK1) follows from the Spectral Condition (HK3), which is the C-positivity of U. □

As a last remark in this section we stress that two (anti-)unitary extensions of a unitary representation $(U, \mathcal{H})$ of $G^\uparrow$ are unitarily equivalent, but the corresponding BGL nets depend on the choice of the (anti-)unitary extension. The following proposition makes this dependence explicit and provides a natural parameter space.

Proposition 4.13 (The space of (anti-)unitary extensions). *Fix $(h, \tau) \in \mathcal{G}$, let $U : G \to \mathrm{AU}(\mathcal{H})$ be an (anti-)unitary representation and let $\mathcal{M} := U(G^\uparrow)'$. Then the following assertions hold:*

(i) *All (anti-)unitary representations $(\widetilde{U}, \mathcal{H})$ extending $U|_{G^\uparrow}$ are of the form $\widetilde{U} = TUT^*$ for some $T \in \mathrm{U}(\mathcal{M})$. The corresponding BGL nets are related by*

$$\mathsf{N}_{\widetilde{U}}(W) = T\mathsf{N}_U(W) \quad \textit{for} \quad W \in \mathcal{G}. \tag{4.9}$$

(ii) (Parametrization of (anti-)unitary extensions) *Let $J := U(\tau)$, $\tau \in G^\downarrow$. For every*

$$N \in \mathrm{U}(\mathcal{M})^- := \{M \in \mathrm{U}(\mathcal{M}) : JMJ = M^{-1}\},$$

there exists a unique (anti-)unitary extension $\widetilde{U}$ of $U|_{G^\uparrow}$ with $\widetilde{U}(\tau) = NJ$, and we thus obtain a bijection between the set $\mathrm{U}(\mathcal{M})^-$ and the set of (anti-)unitary extensions of $U|_{G^\uparrow}$ to G.

Proof. (i) follows from Proposition A.1 and the assertion on the BGL nets is an immediate consequence of the definitions.
(ii) Let $T \in \mathrm{U}(\mathcal{M})$, so that $\widetilde{U} = TUT^{-1} : G \to \mathrm{AU}(\mathcal{H})$ is an (anti-)unitary extension of $U|_{G^\uparrow}$ with $\widetilde{J} := \widetilde{U}(\tau) = TJT^{-1}$. Since $\widetilde{U}$ and U extend the same representation of $G^\uparrow$,

$$N := \widetilde{J}J = \widetilde{J}J^{-1} \in \mathrm{U}(\mathcal{M}).$$

This element satisfies $JNJ = J\widetilde{J} = N^{-1}$, so that $N \in \mathrm{U}(\mathcal{M})^-$ and $\widetilde{J} = NJ$.

If, conversely, $N \in \mathrm{U}(\mathcal{M})^-$, then Lemma A.2 implies the existence of an $X = -X^* \in \mathcal{M}$ with $N = e^{2X}$ and $JXJ = -X$. For $T := e^X \in \mathrm{U}(\mathcal{M})$ and $\widetilde{J} := TJT^{-1}$, we then have

$$\widetilde{J}J = TJT^{-1}J = T^2 = e^{2X} = N.$$

Therefore the manifold $\mathrm{U}(\mathcal{M})^-$ parametrizes the (anti-)unitary extensions of $U|_{G^\uparrow}$. □

4.2.3. Twisted locality We have seen in Section 2.4.2 that it can happen that $W' \notin \mathcal{W}_+ = G^\uparrow.W$. One can anyway attach to W' a real subspace by the BGL net and by construction obtain the relation $\mathsf{H}(W') = \mathsf{H}(W)'$. On the other hand one can define natural complementary wedges $W^{'\alpha}$ indexed by central elements α. In this section we will see that in the BGL construction, the complementary wedge subspaces satisfy the central Haag duality condition (HK6), hence the twisted locality relation (HK4). We start with a lemma on standard subspaces.

Lemma 4.14. *Let $\mathsf{H} \subset \mathcal{H}$ be a standard subspace, and $U \in \mathrm{U}(\mathcal{H})$ be a unitary operator commuting with Δ_H and satisfying $J_\mathsf{H}UJ_\mathsf{H} = U^{-1}$. Let H_1 be the standard subspace defined by $(\Delta_\mathsf{H}, UJ_\mathsf{H})$. There exists a unitary square root Z of U commuting with Δ_H such that $J_\mathsf{H}ZJ_\mathsf{H} = Z^{-1}$ and $Z\mathsf{H} = \mathsf{H}_1$. The standard subspace H_1 does not depend on this choice of Z.*

Proof. The existence of the square root and the commutation relation with the modular conjugation and the modular operator follows by Lemma A.3. Then

$$Z(J_{\mathsf{H}}\Delta_{\mathsf{H}}^{1/2})Z^{-1} = ZJ_{\mathsf{H}}Z^{-1}\Delta_{\mathsf{H}}^{1/2} = Z^2 J_{\mathsf{H}}\Delta_{\mathsf{H}}^{1/2} = UJ_{\mathsf{H}}\Delta_{\mathsf{H}}^{1/2}$$

implies that $\mathsf{H}_1 = Z\mathsf{H}$. It is clear that H_1 does not depend on the choice of Z. □

In order to conclude (HK6), hence the central locality condition on a BGL net N_U, we will need an analogous statement relating complementary wedge subspaces.

Proposition 4.15. *Let $(U, \mathcal{H})$ be an (anti-)unitary representation of the graded group $G = G^{\uparrow} \rtimes \{e, \sigma\}$ and $\alpha \in Z(G^{\uparrow})^{-}$. Then the commutant $U(G^{\uparrow})'$ contains a unitary square root Z_α of $U(\alpha)$ satisfying*

$$U(g)Z_\alpha U(g)^{-1} = Z_\alpha^{-1} \quad \textit{for every} \quad g \in G^{\downarrow}. \tag{4.10}$$

Proof. First we note that $U(\alpha) \in \mathcal{M} := U(G^{\uparrow})'$. We fix $\sigma_0 \in \mathrm{Inv}(G^{\downarrow})$ and observe that conjugation with $U(\sigma_0)$ defines an antilinear isomorphism β of $\mathcal{M}$. As $\beta(U(\alpha)) = U(\alpha)^{-1}$ follows from $\alpha \in Z(G^{\uparrow})^{-}$, we find with Lemma A.3(c) in the appendix, a unitary square root Z_α of $U(\alpha)$ satisfying

$$U(\sigma_0)Z_\alpha U(\sigma_0) = \beta(Z_\alpha) = Z_\alpha^{-1}. \tag{4.11}$$

For any other $\sigma \in G^{\downarrow}$ we have $\sigma = \sigma_0 g$ with $g \in G^{\uparrow}$, so that

$$U(\sigma)Z_\alpha U(\sigma) = U(\sigma)Z_\alpha U(\sigma)^{-1} = U(\sigma_0)U(g)Z_\alpha U(g)^{-1}U(\sigma_0) = U(\sigma_0)Z_\alpha U(\sigma_0) = Z_\alpha^{-1}.$$

□

We are now ready to verify that the BGL net is compatible with the twistings appearing in (HK4), (HK6) and (HK7).

Proposition 4.16. *For every (anti-)unitary representation $(U, \mathcal{H})$ of G, the BGL net N_U satisfies* (HK4) *and* (HK6)*. Moreover, for $\alpha \in Z(G^{\uparrow})^{-}$, $W \in \mathcal{W}_+$ with $W'^{\alpha} \in \mathcal{W}_+$ and $Z_\alpha \in U(G^{\uparrow})'$ satisfying (4.3), the (anti-)unitary extension $(U^\alpha, \mathcal{H})$ of $U|_{G^{\uparrow}}$ to G, determined by $U^\alpha(\sigma_W) := Z_\alpha U(\sigma_W)$, satisfies* (HK7) *and* (HK8).

Proof. Let $\alpha \in Z(G^{\uparrow})^{-}$ and $W = (x, \sigma) \in \mathcal{W}_+$ be such that $W'^{\alpha} = (-x, \alpha\sigma) \in \mathcal{W}_+$. Proposition 4.15 implies the existence of $Z_\alpha \in U(G^{\uparrow})'$ satisfying (4.3). Then

$$\Delta_{\mathsf{N}_U(W'^{\alpha})}^{-it/2\pi} = U(\exp(-tx)) \quad \text{and} \quad J_{\mathsf{N}_U(W'^{\alpha})} = U(\alpha\sigma) = Z_\alpha^2 J_{\mathsf{N}_U(W)} = Z_\alpha J_{\mathsf{N}_U(W)} Z_\alpha^{-1}$$

imply that $\mathsf{N}_U(W'^{\alpha}) = Z_\alpha \mathsf{N}_U(W)'$. This shows that (HK6), hence also (HK4) are satisfied. We also have

$$\mathsf{N}_U(\sigma *_\alpha W) = \mathsf{N}_U(W'^{\alpha}) = Z_\alpha \mathsf{N}_U(W)' = Z_\alpha U(\sigma)\mathsf{N}_U(W).$$

Since N_U is G-equivariant on $\mathcal{G}$, this leads for $g \in G^{\uparrow}$ to

$$\begin{aligned}\mathsf{N}_U(g\sigma *_\alpha W) &= \mathsf{N}_U(g.(\sigma *_\alpha W)) = U(g)\mathsf{N}_U(\sigma *_\alpha W) = U(g)Z_\alpha \mathsf{N}_U(W)' \\ &= U(g)Z_\alpha U(\sigma)\mathsf{N}_U(W) = U(g)U^\alpha(\sigma)\mathsf{N}_U(W) = U^\alpha(g\sigma)\mathsf{N}_U(W).\end{aligned}$$

This proves (HK7). As $J_{\mathsf{N}_U(W)} = U(\sigma_W)$ by definition, we also have

$$U^\alpha(\sigma_W) = Z_\alpha U(\sigma_W) = Z_\alpha J_{\mathsf{N}_U(W)},$$

so that U^α also satisfies (HK8). □

Remark 4.17. (a) If $U|_{G^\uparrow}$ is irreducible, then $U(Z(G^\uparrow)) \subseteq \mathbb{T}\mathbf{1}$, so that, we find for any $\alpha \in Z(G^\uparrow)$ that $U(\alpha) = \zeta\mathbf{1}$ with $|\zeta| = 1$. We may thus put $Z_\alpha := z\mathbf{1}$ for any complex number z with $z^2 = \zeta$. In this case $JZ_\alpha J = Z_\alpha^*$ holds for any antiunitary operator J.
(b) Let $(U, \mathcal{H})$ be an (anti-)unitary representation of G. For any other square root Z of $U(\alpha)$ satisfying the same requirements as Z_α, the unitary operator $Z^{-1}Z_\alpha$ is an involution commuting with $U(G)$, so that it leaves all standard subspaces $\mathsf{N}(W)$ of the BGL net invariant.
(c) If $\alpha \in Z(G^\uparrow)$ satisfies $\alpha^\sigma = \alpha$ for $\sigma \in G^\downarrow$, then α acts trivially on $\mathcal{G}(G)$ and, by covariance of N, leaves all standard subspaces $\mathsf{N}(W)$ invariant. This happens in particular if $\alpha^2 = e$. Then also $\alpha \in Z(G^\uparrow)^-$, so that α-twisted complements are useful in the context of fermionic theories. Here $U(\alpha)$ is an involution and one choice of a square root of $U(\alpha)$ is given by

$$Z_\alpha := \frac{\mathbf{1} + iU(\alpha)}{1+i}. \tag{4.12}$$

Given a net satisfying (HK1)-(HK5), the commutation relation among twist operators and the wedge modular operators immediately hold.

Proposition 4.18. *Let $(\mathcal{W}_+, U, \mathsf{N})$ be a G-covariant net satisfying* (HK1)-(HK5)*, suppose that U extends to an (anti-)unitary representation of G, and let $Z_\alpha \in U(G^\uparrow)'$ as in (4.3). Then, for every $W \in \mathcal{W}_+$, we have*

$$Z_\alpha \Delta_{\mathsf{N}(W)} Z_\alpha^{-1} = \Delta_{\mathsf{N}(W)}.$$

The latter proposition allows to conclude that (HK6) is a consequence of (HK1)-(HK5).

Proposition 4.19. *Let $(\mathcal{W}_+, \mathsf{N}, U)$ be a net of standard subspace satisfying* (HK1)-(HK5)*. Then it also satisfies central Haag duality* (HK6)*:*

$$\mathsf{N}(W'^{\alpha}) = Z_\alpha \mathsf{N}(W)' \quad \textit{for} \quad \alpha \in Z(G^\uparrow)^-, W \in \mathcal{W}_+, W'^{\alpha} \in \mathcal{W}_+.$$

In particular, the right hand side does not depend on the choice of Z_α.

Proof. By (HK5), the unitary operator $Z_\alpha \in U(G^\uparrow)'$ commutes with the modular operator of $\mathsf{N}(W)$, by Proposition 4.18. Therefore the two standard subspaces $\mathsf{N}(W'^{\alpha})$ and $Z_\alpha\mathsf{N}(W)'$ have the same modular operator. By twisted locality $\mathsf{N}(W'^{\alpha}) \subseteq Z_\alpha\mathsf{N}(W)'$, so that Lemma 4.3 implies that they coincide. □

Remark 4.20. Let $(\mathcal{W}_+, \mathsf{N}, U)$ be a net of standard subspaces with a unitary C-positive representation $(U, \mathcal{H})$ of $G^\uparrow$. Let $W_0 = (x, \sigma) \in \mathcal{W}_+ \subset \mathcal{G}_E$ and $\mathsf{H}_0 := \mathsf{N}(W_0)$. We claim that (HK1-3) imply that

$$\widetilde{U}(\sigma) := J_{\mathsf{H}_0}$$

defines an (anti-)unitary extension of $U|_{G^\uparrow(W_0)}$ to the graded subgroup $G(W_0) = G^\uparrow(W_0) \rtimes \{e, \sigma\}$ of G. In fact, J_{H_0} commutes with $G^\uparrow_{W_0}$ by Lemma 4.2. Further, the C-positivity and Theorem 4.4(b) imply that it also has the correct commutation relation with $\exp(C_\pm)$, hence also with $G^\uparrow(W_0)$. We shall see in Section 4.4, when we actually obtain an extension to the full group G.

Example 4.21 (The Poincaré case). Let $\underline{G} := \mathcal{P}_+ := \mathbb{R}^{1,3} \rtimes \mathrm{SO}_{1,3}(\mathbb{R})_0^\uparrow$ be the proper Poincaré group and

$$G = \widetilde{\mathcal{P}}_+ = \mathbb{R}^{1,d-1} \rtimes \mathrm{Spin}_{1,3}(\mathbb{R})_0$$

be its simply connected covering. We write $\lambda_{\underline{W}}$ for the one-parameter group lifting the boost group $\Lambda_{\underline{W}}$ associated to a wedge $\underline{W} \in \mathcal{W} = \underline{G}.W_1$ (see e.g. [Mo18]). For $\underline{G}^\uparrow$, a wedge is defined by a pair $\underline{W} = (x, r_x)$, where x generates $\Lambda_{\underline{W}}$ and $r_x = e^{\pi i x}$ is the spacetime reflection in the direction of the wedge. Since $Z = Z(G) = \{\pm\mathbf{1}\}$ is a 2-element group, a wedge $W \in \underline{\mathcal{G}}$ has two lifts which belong to two different $G^\uparrow$-orbits in $\mathcal{G}(G)$. To see this, we note that $Z = Z^-$ and $Z_2 = \{e\}$. For the second equality we use the isomorphism $\mathrm{Spin}_{1,3}(\mathbb{R})$ with $\mathrm{SL}_2(\mathbb{C})$ and note that the centralizer of any Euler element x, which may be assumed to be $x = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, is connected and isomorphic to the multiplicative group $\mathbb{C}^\times$, on which the involution σ_x acts trivially. Therefore the central elements $\partial(g) = g^\sigma g^{-1}$, $g \in G^\uparrow_{(x,\sigma_x)}$, are all trivial, which leads to $Z_2 = \{e\}$.

For $\alpha := -\mathbf{1}$, the twisted complement of $W = (k_W, \sigma_W)$ is $W^{'-1} = (-k_W, -\sigma_W)$. Any lift $\widetilde{r} \colon \mathbb{R} \to G^\uparrow$ of a rotation one-parameter group $\rho \colon \mathbb{R} \to \mathrm{SO}_2(\mathbb{R}) \hookrightarrow \mathrm{SO}_{1,3}(\mathbb{R})$ in $\underline{G}^\uparrow$ satisfying $\mathrm{Ad}(\rho(\pi))k_W = -k_W$ now satisfies $\widetilde{\rho}(2\pi) = -\mathbf{1}$. This shows that, $W' = (-k_W, \sigma_W) \notin G^\uparrow.W$, but that $W^{'-1} = (-k_W, -\sigma_W) \notin G^\uparrow.W$.

Let $(U, \mathcal{H})$ be an irreducible unitary positive energy representation of $G^\uparrow$ for which $U(-1) \neq \mathbf{1}$, then $U(-1) = -\mathbf{1}$ by Schur's Lemma. For the BGL net $\mathsf{N} \colon \mathcal{G}(G) \to \mathrm{Stand}(\mathcal{H})$ we therefore have $\mathsf{N}(W^{'-1}) = i\mathsf{N}(W)'$ and $Z_\alpha = i\mathbf{1}$ is a suitable twist operator (cf. [Mo18, Thm. 2.8]).

Example 4.22 (Finite coverings of the Möbius group). Consider the n-fold covering of the Möbius group $G^\uparrow := \mathsf{M\ddot{o}b}^{(n)} \subseteq G := \mathsf{M\ddot{o}b}_2^{(n)}$, where $\underline{G} = \mathsf{M\ddot{o}b}_2$ (cf. Example 2.10(e)). This group is obtained from $\widetilde{\mathsf{M\ddot{o}b}}_2$ by factorization of the subgroup $nZ(\widetilde{\mathsf{M\ddot{o}b}})$. Then $Z := Z(G^\uparrow) \cong \mathbb{Z}_n$ is a cyclic group of order n. Let $\alpha := \widetilde{\rho}(2\pi) \in Z$ be a generator, where $\widetilde{\rho} \colon \mathbb{R} \to G^\uparrow$ is the lift of the rotation group.

Let $(U, \mathcal{H})$ be an (anti-)unitary representation of G whose restriction to $G^\uparrow$ is irreducible. Then, by Schur's Lemma, $U(\alpha^n) = U(\widetilde{\rho}(\pi n))$ is an involution in $\mathbb{T}\mathbf{1}$, hence $\pm\mathbf{1}$. We now define *n-twisted local nets of real subspaces* as follows:

- n **is even.** As $\beta^\tau = \beta^{-1}$ for $\beta \in Z$, we have $Z^- = Z$ and $Z_1 \cong \mathbb{Z}_{n/2}$ is a subgroup of index 2. As for $\widetilde{\mathsf{M\ddot{o}b}}_2$, we have $Z_2 = Z_1$. We therefore obtain for every Euler couple $W = (x, \sigma) \in \mathcal{G}_E(G)$ two $G^\uparrow$-orbits $G^\uparrow.(\pm x, \sigma)$ covering $\underline{G}^\uparrow.\underline{W} \subseteq \mathcal{G}_E(\underline{G})$. Choosing $G^\uparrow.(x, \sigma)$, one obtains with the BGL construction a net of real subspaces $I \mapsto \mathsf{N}(I)$, where I denotes an interval of length smaller than 2π in the $\frac{n}{2}$-covering $\mathbb{S}^1_{(n/2)} \simeq \mathbb{R}/\pi n\mathbb{Z}$ of $\mathbb{S}^1$. We can realize the net on intervals in $\mathbb{S}^1_{(n/2)}$ because $U(\widetilde{\rho}(n\pi))\mathsf{N}(I) = \pm\mathsf{N}(I) = \mathsf{N}(I)$. For the central element $\alpha = \widetilde{\rho}(-2\pi) \in Z$, twisted complements look as follows. For $I = (a, b) \subset \mathbb{R}/\pi n\mathbb{Z}$ with $b - a < 2\pi$, we have $I^{'\alpha} = I^c$, where $I^c = (b - 2\pi, a)$ is the "complement" obtained by conformal reflection on the left endpoint, cf. (2.25). All the other twisted complement, belonging to the same orbit, are obtained by covariance.

The locality relation then is given by

$$\mathsf{N}(I^{'\alpha}) = \omega^k \mathsf{N}(I)', \qquad k \in \mathbb{Z},$$

where $\alpha = \widetilde{\rho}(2\pi k)$ and $\omega \in \mathbb{T}$ satisfies $\omega^2 \mathbf{1} = U(\widetilde{\rho}(2\pi))$. Since U is irreducible and Z is a cyclic group of order n, $U(\widetilde{\rho}(2\pi))$ is an n-th root of the unity, hence $\omega^{2n} = 1$ and $Z_\alpha = \omega^k \mathbf{1}$.

- n **is odd.** Then $Z^- = Z_1$ implies that $G^\uparrow$ acts transitively on the inverse images of $\underline{G}^\uparrow$-orbits in $\underline{\mathcal{G}}$. Fixing the orbit $G^\uparrow.(x, \sigma)$, we have by the BGL construction a net of real subspaces $I \mapsto \mathsf{N}(I)$, where, again, I is an interval of length smaller than 2π in the n-fold covering of $\mathbb{S}^1$. Here the locality relation is

$$\mathsf{N}(I'^{\alpha}) = \omega^k \mathsf{N}(I)', \qquad k \in \mathbb{Z},$$

where $\alpha = \widetilde{\rho}(2\pi k)$ and $\omega^{2n} = 1$, I'^{α} and Z_α are as above.

4.3. New models. Theorem 3.10 provides the list of restricted root systems for real simple Lie algebras containing (symmetric) Euler elements, hence supporting (symmetric) Euler wedges. Any such Lie algebra $\mathfrak{g}$ is the Lie algebra of a simply connected Lie group $G^\uparrow$. Then (2.10) defines an Euler involution on the group $G^\uparrow$, so that we obtain the extension to $G = G^\uparrow \rtimes \{\mathrm{id}, \sigma\}$.

Such a Lie group $G^\uparrow$ has many unitary representations, possibly with positive energy if the Lie algebra $\mathfrak{g}$ is hermitian. By unitary induction, one can construct a unitary representation of $G^\uparrow$ from a unitary representation of a subgroup, for instance from a covering of $\mathrm{PSL}_2(\mathbb{R}) \subset G^\uparrow$ [Ma52]. It is always possible, to extend a unitary representation $(U, \mathcal{H}_U)$ of $G^\uparrow$ to an (anti-)unitary representation of G by doubling the Hilbert space, if the representation does not extend on $\mathcal{H}_U$ itself. Indeed, we can choose any conjugation C on $\mathcal{H}_U$ and observe that the representation defined by $\widetilde{U}(g) = U(g) \oplus CU(\sigma g \sigma)C$ on $\mathcal{H}_U \oplus \mathcal{H}_U$ extends to G by $U(\sigma) = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$. By the BGL construction there exists a (twisted-)local one-particle net satisfying (HK1-8).

As a consequence we have the theorem:

Theorem 4.23. *Let $\mathfrak{g}$ be a simple real Lie algebra containing an Euler element, i.e., whose restricted root system occurs in* Theorem 3.10*. Then there exists a graded Lie group $G = G^\uparrow \cup G^\downarrow$ as in* Section 2.1 *with an (anti-)unitary representations U, and these in turn define twisted G-covariant BGL nets $(\mathcal{W}_+, U, \mathsf{N})$.*

This theorem shows, for instance, that it is possible to associate a covariant homogeneous net of standard subspace to a Lie algebra $\mathfrak{g}$ with restricted root system E_7. The subgroups $G_{\pm 1} = \exp(\mathfrak{g}_{\pm 1}(x)) \subseteq G^\uparrow$ are closed, and if $G_0 := \{g \in G^\uparrow : \mathrm{Ad}(g)x = x\}$, then so is $\mathbf{P} := G_0 G_{-1}$. Then $M := G^\uparrow/\mathbf{P}$ is a homogeneous space whose tangent space in the base point can be identified with the eigenspace $\mathfrak{g}_1(x) = \ker(\mathrm{ad}\, x - \mathbf{1})$. If $\mathfrak{g}$ is simple hermitian of tube type and $C \subseteq \mathfrak{g}$ is a pointed generating invariant cone, then $C_+ := C \cap \mathfrak{g}_1(x)$ defines a $G^\uparrow$-invariant causal structure on M. The so-obtained manifolds include the Jordan space-times of Günaydin [Gu93,Gu00,Gu01] and the simple spacetime manifolds in the sense of Mack–de Riese [MdR07]. If the rank of the restricted root system Σ of $(\mathfrak{g}, \mathfrak{a})$ is 2, then M is a Lorentzian manifold, but in general it is not. As a consequence of Proposition 3.11 and Table 2, there exists a real form with a non-trivial positive cone i.e., $\mathfrak{g}$ is hermitian of tube type, for every root system appearing in Theorem 3.10. Thus models with a proper notion of positive energy can be associated to every root system supporting symmetric Euler elements.

Recently, in [NÓ20] it has been shown that irreducible (anti-)unitary representations $(U, \mathcal{H})$ of G which are of positive energy in the sense that $-i\partial U(y) \geq 0$ for $y \in C$, lead to G-covariant nets $(\mathtt{V}_{\mathcal{O}})$ of real subspace of $\mathcal{H}$, indexed by open subsets $\mathcal{O} \subseteq M$. If $\mathcal{O} \neq \emptyset$, then $\mathtt{V}_{\mathcal{O}}$ is generating, and it is standard if $\mathcal{O}$ is not "too big". In particular, the open subset $\mathcal{O} = \exp(C_+^0)\mathbf{P} \subseteq M$ corresponds to a standard subspace with the Bisognano–Wichmann property for which the modular group is represented by the one-parameter group $(\exp tx)_{t\in\mathbb{R}}$ of G (see [NÓ20, §5.2]).

4.4. The SL_2*-problem, symmetry extension.* In Section 3.2 we have seen that the existence of orthogonal Euler wedges corresponds to the existence of an $\mathfrak{sl}_2$-subalgebra containing both Euler elements. In this section we will discuss when we can extend a covariant net of standard subspaces $(\mathcal{W}_+, \mathsf{N}, U)$ of Euler wedges satisfying (HK1)-(HK5) to a G-covariant net.

We first look at the (anti-)unitary extensions of unitary representations of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. In $\mathfrak{sl}_2(\mathbb{R})$, we consider the two Euler elements

$$h := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad k := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.13}$$

Let $(U, \mathcal{H})$ be a unitary representation of the group $G := \widetilde{\mathrm{SL}}_2(\mathbb{R})$ and consider the two selfadjoint operators

$$H := -2\pi i \partial U(h) \quad \text{and} \quad K := -2\pi i \partial U(k).$$

Theorem 4.24. *Every continuous unitary representation of* $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ *extends to an (anti-)unitary representation of the group*

$$\widetilde{\mathrm{GL}}_2(\mathbb{R}) := \widetilde{\mathrm{SL}}_2(\mathbb{R}) \rtimes \{\mathbf{1}, \tau_G\},$$

where τ_G *is the involutive automorphism of* $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ *induced by the Lie algebra automorphism*

$$\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

corresponding to the Euler element h*.*

In [GL95, Lo08] this theorem was proved for $\mathrm{SL}_2(\mathbb{R})$-representations of the principal and discrete series. Here the argument does not depend on the family of the representation.

Proof. Since $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is a type I group, every unitary representation has a unique direct integral decomposition into irreducible unitary representations. This reduces the problem to the irreducible case. We have to show that $U \circ \tau_G \cong U^*$ (the dual representation). Let

$$u := [h, k] = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then h, k, u is a basis of $\mathfrak{sl}_2(\mathbb{R})$ and

$$\omega := h^2 + k^2 - u^2 \in \mathcal{U}(\mathfrak{sl}_2(\mathbb{R}))$$

is a Casimir element, so that

$$\partial U(\omega) = c\mathbf{1} \quad \text{for some} \quad c \in \mathbb{R}.$$

The antilinear extension $\overline{\tau}$ of τ to $\mathfrak{sl}_2(\mathbb{C})$ satisfies $\overline{\tau}(iu) = iu$ and the operator $i\partial U(u)$ is selfadjoint and diagonalizable. We have

$$\partial U^*(u) = -\partial U(u) = \partial U(\tau(u)),$$

so that $U^* \circ \tau_G$ is an irreducible with the same u-weights and the same Casimir eigenvalue c. Below we argue that U is uniquely determined by any pair (μ, c), where μ is an eigenvalue of $i\partial U(u)$ occurring in the representation ([Sa67], [Lo08]), and this implies that $U \circ \tau_G \cong U^*$.

To see that U is determined by the pair (μ, c), we first recall that $\mathcal{H}$ decomposes into one-dimensional eigenspaces of $i\partial U(u)$ and, by irreducibility, it is generated by any eigenvector ξ_μ of eigenvalue μ. Let $\mathcal{U}(\mathfrak{g})$ denote the complex enveloping algebra of $\mathfrak{g}$. Then $V_\mu := \mathcal{U}(\mathfrak{g})\xi_\mu$ is a dense subspace consisting of analytic vectors, so that the representation U is determined by the $\mathfrak{g}$-representation on this space. In $\mathcal{U}(\mathfrak{g})$ the centralizer $\mathcal{C}_u$ of u is generated by u and the Casimir element. Therefore ξ_μ is a $\mathcal{C}_u$-eigenvector and the corresponding homomorphism $\chi : \mathcal{C}_u \to \mathbb{C}$ is determined by $\chi(u) = \mu$ and $\chi(\omega) = c$. It is now easy to verify that these two values determine the $\mathcal{U}(\mathfrak{g})$-module structure on V_μ, hence the unitary representation U. □

Remark 4.25. Here the determination of the representation is obtained by considering in the enveloping algebra $\mathcal{U}(\mathfrak{sl}_2(\mathbb{R}))$, the centralizer subalgebra $\mathbb{C}[\omega, u]$ of u. Any cyclic weight vector $\xi_{\mu,c}$ defines a character χ of this subalgebra by $\chi(iu) = \mu$ and $\chi(\omega) = c$, and $\mathcal{U}(\mathfrak{g})\xi_{\mu,c}$ is isomorphic to the quotient of $\mathcal{U}(\mathfrak{g})$ by the left ideal generated by $\mu\mathbf{1} - iu$ and $\omega - c\mathbf{1}$.

Now, we consider the positive selfadjoint operator

$$\Delta_h := e^{-H} = e^{2\pi i \partial U(h)}.$$

By Theorem 4.24, U extends to an (anti-)unitary representation of $\widetilde{\mathrm{GL}}_2(\mathbb{R})$, and we put

$$J := U(\tau_G), \quad S := J\Delta_h^{-1/2} = Je^{-\pi i \partial U(h)} \quad \text{and} \quad \mathtt{V} := \mathrm{Fix}(S).$$

Lemma 4.26. *For a unitary operator $T \in \mathrm{U}(\mathcal{H})$, the following assertions are equivalent:*

(a) *$STS \subseteq T$ holds on a dense subspace.*
(b) *$T^{-1}\mathtt{V} \cap \mathtt{V}$ is standard.*

If these conditions are satisfied, then (a) *holds on $T^{-1}\mathtt{V} \cap \mathtt{V}$.*

Proof. If (b) holds, then any $\xi \in T^{-1}\mathtt{V} \cap \mathtt{V}$ satisfies $STS\xi = ST\xi = T\xi$, so that (a) holds.

Conversely, assume that

$$\mathcal{D} := \{\xi \in \mathcal{D}(S) : STS\xi = T\xi\}$$

is dense in $\mathcal{H}$. For any $\xi \in \mathcal{D}$ we then have $T\xi \in \mathcal{R}(S) = \mathcal{D}(S)$ and

$$STS(S\xi) = ST\xi = S(STS\xi) = T(S\xi),$$

so that $\mathcal{D}$ is S-invariant. This implies that $\mathcal{D} = (\mathcal{D} \cap \mathtt{V}) + i(\mathcal{D} \cap \mathtt{V})$, so that $\mathcal{D} \cap \mathtt{V}$ is standard. For $\xi \in \mathtt{V}$, we have $T\xi \in \mathtt{V}$ if and only if $\xi \in \mathcal{D}$, so that $\mathcal{D} \cap \mathtt{V} = T^{-1}\mathtt{V} = \mathtt{V}$. This proves the lemma. □

Proposition 4.27. *The following assertions are equivalent:*

(a) $\Delta_h^{-1/2} e^{itK} \Delta_h^{1/2} \subseteq e^{-itK}$ *holds for every* $t \in \mathbb{R}$ *on a dense subspace of* $\mathcal{H}$.
(b) $Se^{itK} S \subseteq e^{itK}$ *holds for every* $t \in \mathbb{R}$ *on a dense subspace of* $\mathcal{H}$.
(c) $e^{-itK}\mathbb{V} \cap \mathbb{V}$ *is standard for every* $t \in \mathbb{R}$.

If these conditions are satisfied, then (a) *holds on* $J(e^{-itK}\mathbb{V} \cap \mathbb{V})$ *and* (b) *on* $e^{-itK}\mathbb{V} \cap \mathbb{V}$.

Proof. (a) $\Leftrightarrow$ (b): From $\tau(k) = -k$ it follows that

$$JU(\exp tk)J = U(\tau_G(\exp tk)) = U(\exp(-tk)),$$

so that conjugating with J translates (a) into (b).
(b) $\Leftrightarrow$ (c) follows from Lemma 4.26. □

From [GL95, Thm. 1.1, Cor. 1.3(c)] one can deduce that the equivalent conditions in Proposition 4.27 are satisfied for principal series representations and lowest and highest weight representations, but it is not known for complementary series representations.

The following theorem shows that an isotone, central twisted local $G^{\uparrow}$-covariant net of standard subspaces satisfying the BW property is actually G-covariant. The argument needs the density property described in Proposition 4.27 for $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. The extension is done by (HK8).The proof generalizes the argument in [GL95].

Theorem 4.28 (Extension Theorem). *Let* $G = G^{\uparrow} \rtimes \{\mathrm{id}, \sigma\}$ *be a graded Lie group, where* σ *is an Euler involution. Let* $(U, \mathcal{H})$ *be a unitary* C*-positive representation of* $G^{\uparrow}$, $\mathcal{W}_+ \subseteq \mathcal{G}_E(G)$ *be a* $G^{\uparrow}$*-orbit, and* $(\mathcal{W}_+, \mathsf{N}, U)$ *be a net of standard subspaces satisfying* (HK1-4) *and the BW property* (HK5). *If* $h_1, \ldots, h_n$, $n \geq 2$, *is a pairwise orthogonal family of Euler elements generating the Lie algebra* $\mathfrak{g}$, *and the conditions in* Proposition 4.27 *hold for the representations of the connected subgroups corresponding to the* $\mathfrak{sl}_2$*-subalgebras generated by* h_1 *and* h_j *for* $j = 2, \ldots, n$, *then* U *extends to an (anti-)unitary representation of* G *such that* G*-covariance* (HK7) *and modular reflection* (HK8) *hold.*

Proof. Let $(\mathcal{W}_+, \mathsf{N}, U)$ be a net of standard subspaces satisfying (HK1-5). The Bisognano–Wichmann property (HK5) implies Central Haag Duality (HK6) by Proposition 4.19. Let $H_j := -i\partial U(h_j)$ be the selfadjoint generators of the unitary one-parameter group corresponding to h_j. By Corollary 3.14, every pair (h_1, h_j) generates a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ and the generators H_1 and H_j integrate to a representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$. Consider the Euler wedges $W_1, W_j \in \mathcal{W}_+$ associated to h_1 and h_j, respectively.

We claim that Proposition 4.27 implies that $U(\sigma_{h_1}) := J_{\mathsf{N}(W_1)}$ associated to the standard subspace $\mathsf{N}(W_1)$ extends the $\widetilde{\mathrm{SL}}_2(\mathbb{R})$-representation to an (anti-) unitary representation of $\widetilde{\mathrm{PGL}}_2(\mathbb{R})$. Indeed, by Proposition 4.27(b) we have that

$$\Delta_{h_1}^{-1/2} e^{itH_j} \Delta_{h_1}^{1/2} \subset J_{\mathsf{N}(W_1)} e^{itH_j} J_{\mathsf{N}(W_1)} \tag{4.14}$$

on the dense domain $J_{\mathsf{N}(W_1)}(e^{-itH_j}\mathbb{V} \cap \mathbb{V})$ with $\mathbb{V} = \mathrm{Fix}(J_{\mathsf{N}(W_1)}\Delta_{\mathsf{N}(W_1)}^{1/2}) = \mathsf{N}(W_1)$, cf. condition (c) in Proposition 4.27. On the previous domain we then have

$$\Delta_{h_1}^{-1/2} e^{itH_j} \Delta_{h_1}^{1/2} \subset U(\sigma_{h_1}) e^{itH_j} U(\sigma_{h_1}).$$

With Proposition 4.27(a) we can now conclude that

$$U(\sigma_{h_1}) e^{itH_j} U(\sigma_{h_1}) = e^{-itH_j} \quad \text{for} \quad t \in \mathbb{R} \tag{4.15}$$

because both sides are bounded operators which coincide on a dense subspace. Since the Lie algebra $\mathfrak{g}$ is generated by $h_1, \ldots, h_n$, we obtain

$$U(\sigma_{h_1})U(g)U(\sigma_{h_1}) = U(\sigma_{h_1} g \sigma_{h_1}) \quad \text{for all} \quad g \in G^\uparrow. \tag{4.16}$$

In particular, U defines an (anti-)unitary representation of G. Pick $\alpha \in Z(G^\uparrow)^-$ such that (HK4) is satisfied and consider the twisted representation of G defined by $U^\alpha(\sigma_{h_1}) := Z_\alpha J_{\mathsf{N}(W_1)} = Z_\alpha U(\sigma_{h_1})$. Since N coincides with the restriction to $\mathcal{W}_+$ of the BGL net of the (anti-)unitary representation U of G, the representation U^α satisfies (HK7) and (HK8) by Proposition 4.16. □

Note that the density property as well as the existence of orthogonal wedges are sufficient but not necessary to have a G-covariant action: Consider the BGL net associated to the unique irreducible positive energy representation U of the $G = \mathrm{Aff}(\mathbb{R})$ on the real line. Then the standard subspaces $\mathsf{N}_U(a, \infty)$ and $\mathsf{N}_U(-\infty, b)$ are associated to positive and negative half-lines and satisfy (HK1)-(HK5). There are no-orthogonal wedges in this case but the extension to an (anti-)unitary representation of G is given by

$$U(\sigma_W) = J_{\mathsf{N}_U(W)}.$$

We further remarks that in this case σ_W does not preserve the wedge family $\mathcal{W}_+$.

For the Poincaré group, with the identification of wedge regions and Euler elements (see (2.28)), the axial wedges

$$W_j = \{(t, x) \in \mathbb{R}^{1+d} : |t| < x_j\}, \quad j = 1, \ldots, d,$$

define a family of orthogonal wedge regions, namely wedge regions associated to orthogonal Euler elements. Considering wedges as subsets of Minkowski spaces one can define further regions by wedge intersection. Spacelike cones are particularly important: they are defined, up to translations by finite intersection of wedges obtained by Lorentz transforms of W_1. Analogously one can define, by intersecting wedge subspaces, subspaces associated to any spacelike cone. In principle this can also be trivial, but if they are standard, the cyclicity assumption of 4.27(c) is ensured, cf. [GL95].

Consider $G = \widetilde{\mathsf{M\ddot{o}b}} \rtimes \{\mathrm{id}, \widetilde{\tau}\}$. Let $(\mathcal{W}_+, U, \mathsf{N})$ be a net of standard subspaces satisfying (HK1)-(HK5). Let $\widetilde{I}_\supset \subseteq \mathbb{R}$ be an interval with $q(\widetilde{I}_\supset) = I_\supset$ where the latter is the right semicircle with endpoints $(-i, i) \subset \mathbb{S}^1$. Then the dilation generators $\widetilde{\delta}_\cap$ and $\widetilde{\delta}_\supset$ define orthogonal Euler elements generating $\widetilde{\mathsf{M\ddot{o}b}}$. Considering the wedges $W_\cap = (x_\cap, \sigma_\cap)$ and $W_\supset = (x_\supset, \sigma_\supset)$ with $W_\cap = \widetilde{\rho}(\pi/2)W_\supset$, the intersection is again a wedge interval $\widetilde{I} = \widetilde{I}_\cap \cap \widetilde{I}_\supset$. In particular, by isotony, $\mathsf{N}(\widetilde{I}_\cap) \cap \mathsf{N}(\widetilde{I}_\supset) \supset \mathsf{N}(\widetilde{I})$ is standard and condition (c) in Proposition 4.27 holds.

5. Outlook

5.1. Nets of von Neumann algebras and standard subspaces. The fundamental objects in algebraic quantum field theory are the local algebras of observables depicted by von Neumann algebras. In this setting standard subspaces arise naturally in the modular theory of von Neumann algebras when the vacuum state is specified. Indeed, if $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra and $\Omega \in \mathcal{H}$ is a cyclic and separating vector, then the modular conjugation $J_{\mathcal{M},\Omega}$ and the modular operator $\Delta_{\mathcal{M},\Omega}$ specified by the Tomita–Takesaki Theorem coincide respectively with the modular conjugation $J_{\mathtt{V}}$ and the modular operator $\Delta_{\mathtt{V}}$ determined by the standard subspace $\mathtt{V} := \overline{\{M\Omega \colon M = M^* \in \mathcal{M}\}}$. However, our discussion aims at another realization of von Neumann algebra nets. One can use the Second Quantization procedure to associate to each standard subspace $\mathtt{V} \subseteq \mathcal{H}$ a pair $(\mathcal{R}_+(\mathtt{V}), \Omega)$, where $\mathcal{R}_+(\mathtt{V})$ is a von Neumann algebra on the bosonic Fock space $\mathcal{F}_+(\mathcal{H})$, see e.g. [LRT78], [NÓ17, §6]. For other statistics, such as fermions and anyons, we refer to [EO73,BJL02,Sc97]. This correspondence permits to translate between results on configurations of standard subspaces and configurations of von Neumann algebras and to construct a net of von Neumann algebras out of a one particle net of standard subspaces (see for instance [LRT78]).

This procedure can be applied in our case. For instance it is straightforward to construct the bosonic second quantization of the BGL net associated to an (anti-)unitary positive energy representation of a $\mathbb{Z}_2$-graded Lie group G supporting Euler wedges, provided $Z(G) = e$. In the general case an appropriate symmetrization of the Fock space has to be carried out, which requires a finer analysis of the twisted locality in the second quantization picture. This first example suggests that a generalized axiomatic framework for Haag–Kastler nets of von Neumann algebras on the abstract set of Euler wedges is possible and deserves to be explored.

Once wedge localization of one particle states or von Neumann algebra has been properly defined, the next step is to look for finer localization properties in our abstract setting. In general, due to the a priori lack of a reference spacetime for the orbits of Euler wedges, a possible definition of wedge intersection is an intriguing step to face. Indeed, very different configurations are visible in the literature for the well known models. For example fundamental localization regions on the Minkowski spacetime are obtained by intersecting wedges. They can be bounded, for instance doublecones,[9] or unbounded. The latter are called spacelike cones when—up to a Poincaré transformation—we consider an intersection of a family of wedges through the origin. In this case the spacelike cones are characterized by a semigroup of translations sending the cone into itself. In a chiral theory, wedges are intervals on the circle and connected regions obtained by wedge intersections are intervals again. In this case, intersections can be described again in terms of Euler wedges. On the de Sitter spacetime, wedge intersections are still possible. On the other hand it is not clear how to define them at the group level: for instance there is no Lorentz transformation fixing or sending intersections into themselves. Once a proper notion of spacelike cone is determined in our abstract setting, it will be possible to explore fundamental properties and notions in AQFT such as spin-statistics relations or nuclearity properties [GL95,DF84,BDL07].

5.2. 3-graded Lie algebras. For an orbit $\mathcal{W}_+ := G^\uparrow.(x, \sigma)$ in the abstract wedge space of G, any (anti-)unitary representation leads by the BGL construction to an equivariant map $\mathsf{N} \colon \mathcal{W}_+ \to \mathrm{Stand}(\mathcal{H})$, which can be considered as a homogeneous net of standard subspaces. To see which of these nets are most interesting, one may ask which of the orbits $U(G^\uparrow)\mathtt{V} \subseteq \mathrm{Stand}(\mathcal{H})$ carry a non-degenerate order structure in the sense that the semigroup

$$S_{\mathtt{V}} := \{g \in G^\uparrow \colon U(g)\mathtt{V} \subseteq \mathtt{V}\}$$

is "large", meaning that its interior clusters in e. From [Ne19,Ne19b] we know that this happens if and only if $(\operatorname{ad} x)^3 = \operatorname{ad} x$, i.e., if $\operatorname{ad} x$ defines a 3-grading on the Lie algebra $\mathfrak{g}$ (cf. Definition 2.4) and the positive cone $C_U := \{x \in \mathfrak{g} \colon -i\partial U(x) \geq 0\}$ is such that

[9] Up to Poincaré transformations, these are the causal closures of a ball in the time-zero surface.

the intersections $C_U \cap \mathfrak{g}_{\pm 1}(x)$ generate $\mathfrak{g}_{\pm 1}(x)$. This motivates our focus on pairs (x, σ), where x is an Euler element and $\mathrm{Ad}(\sigma)$ the corresponding Lie algebra involution σ_x.

On the abstract level, this leads to configurations represented by quadruples $(\mathfrak{g}, \sigma, C, x)$, where $\mathfrak{g}$ is a Lie algebra, $\sigma \in \mathrm{Aut}(\mathfrak{g})$ an involution, $x \in \mathcal{E}(\mathfrak{g})$ an Euler element with $\sigma(x) = x$, and $C \subseteq \mathfrak{g}$ an invariant convex cone with $\sigma(C) = -C$. For classification results concerning such configurations, we refer to the recent work of D. Oeh ([Oeh20,Oeh21]).

Now consider a general net of standard subspaces N undergoing an action of the group G satisfying the assumptions (HK1-4). It is a consequence of Lemma 4.2 and Theorem 4.4 that

$$Z_{\mathsf{N}(W)}(t) = \Delta_{\mathsf{N}(W)}^{it} U(\lambda_W(2\pi t))$$

is a one-parameter group in $U(S_{\mathsf{N}(W)})'$. So the Bisognano-Wichmann property holds if $Z_{\mathsf{N}(W)}(t) \equiv 1$. In particular an analysis of the gauge space $U(S_{\mathsf{N}(W)})'$ will allow us to understand possible counterexamples to the BW property, possibly generalizing the ones contained in [Mo18,LMR16], to the case where the modular theory of the net does not implement any geometric global action.

5.3. Geometric realization of wedge spaces. In the present paper, we studied the abstract wedge spaces $\mathcal{W}_+ = G^\uparrow.(x, \sigma)$ on an algebraic level, having in mind that there are many interesting concrete situations, where the elements of this space correspond to "wedge domains" in various kinds of space-time manifolds.

A systematic descriptions of "wedge domains" in causal symmetric spaces has to be undertaken. A causal symmetric space is a homogeneous space $M = G/H$, where G is a connected Lie group,[10] and there exists an involution τ on G for which $H \subseteq G^\tau$ is an open subgroup (i.e., a union of connected components) such that M carries a G-invariant field of pointed open cones $V_+(m) \subseteq T_m(M)$ (the causal structure). The non-flat causal symmetric spaces come in two flavors, the *compactly causal* ones with closed causal geodesics, and the *non-compactly causal* ones for which M carries a global order structure defined by the causal curves (see [HÓ96] for details on these concepts). Important examples are de Sitter space $\mathrm{dS}^d \cong \mathrm{SO}_{1,d}^\uparrow(\mathbb{R})/\mathrm{SO}_{1,d-1}(\mathbb{R})^\uparrow$ (non-compactly causal) and anti de Sitter space $\mathrm{AdS}^d \cong \mathrm{SO}_{2,d-1}(\mathbb{R})_e/\mathrm{SO}_{1,d-1}(\mathbb{R})^\uparrow$ (compactly causal).

In particular, any Lie group G is a symmetric space with respect to the transitive action of $G \times G$ by $(g_1, g_2).g = g_1 g g_2^{-1}$, and any pointed generating invariant cone $C \subseteq \mathfrak{g} \cong T_e(G)$ defines a $(G \times G)$-invariant causal structure. In this case any Euler element h for which the cones $C_\pm := \pm C \cap \mathfrak{g}_{\pm 1}(h)$ are generating, leads to the semigroup

$$S := \exp(C_+^0) G_e^h \exp(C_-^0),$$

which plays in the symmetric space G the role of a wedge domain. If $G = (\mathbb{R}^{1,d-1}, +)$ is d-dimensional Minkowski space, $C = V_+$ is the open light cone and h the generator of a Lorentz boost, then S coincides with the corresponding Rindler wedge.

It is also possible to describe for any reductive causal symmetric space $M = G/H$ and Euler elements $h \in \mathfrak{g}$ generating a flow on M fixing the base point, a wedge domain $W_M(h)$. Then $\mathcal{W}_M(h) := \{g W_M(h) \colon g \in G\}$ is a *geometric wedge* space. Up to coverings, this construction leads in particular to a geometric realisation of all abstract

[10] In this subsection we write G instead of $G^\uparrow$ to simplify notation.

wedge spaces corresponding to Euler couples of semisimple Lie algebras. The geometric wedge spaces have a natural order if M is compactly causal (such as Minkowski space or anti-de Sitter space), but if M is non-compactly causal, there is no natural order structure (as for de Sitter space). This picture can be used to obtain, for compactly causal spaces, from certain unitary representations of G satisfying a spectral condition covariant nets of real subspace $\mathtt{V}(\mathcal{O}) \subseteq \mathcal{H}$, $\mathcal{O} \subseteq M$ open, that assign to the wedge domain $W_M(h)$ the standard subspace $\mathtt{V}$ associated to the pair (h, τ). In this sense, they have the Bisognano–Wichmann property ([NÓ20]). Several aspects of the underlying geometry will be discussed in the forthcoming papers [NÓ21a,NÓ21b].

These constructions are only first steps in a theory that needs to be developed. Important open questions concern characterizations of (irreducible) representations for which such geometric nets exist. In particular, the case of non-compactly causal spaces, which corresponds to non-ordered wedge spaces, is largely open, but there are well known nets on de Sitter space (such as the ones obtained from covariant quantum fields) that fit into this picture.

Acknowledgments VM was supported by the European Research Council Advanced Grant 669240 QUEST and by Indam from March 2019 to February 2020. VM acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome “Tor Vergata”, CUP E83C18000100006 and the University of Rome “Tor Vergata” funding scheme “Beyond Borders”, CUP E84I19002200005.
KHN acknowledges support by DFG-grant NE 413/10-1.

Funding Open access funding provided by Università degli Studi di Roma Tor Vergata within the CRUI-CARE Agreement.

A. Toolbox

Proposition A.1 ([NÓ17, Thm. 2.11(a)]). *If $(U, \mathcal{H})$ is a unitary representation of $G^\uparrow$, then any two (anti-)unitary extensions $(\widetilde{U}_j, \mathcal{H})$, $j = 1, 2$, of U to G are unitarily equivalent, i.e., there exists $\Gamma \in U(G^\uparrow)'$ with*

$$\Gamma \circ \widetilde{U}_1(g) = \widetilde{U}_2(g) \circ \Gamma \quad \textit{for} \quad g \in G.$$

Lemma A.2. *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $J \in \mathrm{Conj}(\mathcal{H})$ such that $J\mathcal{M}J = \mathcal{M}$. Then the exponential function of the Banach symmetric space*

$$\mathrm{U}(\mathcal{M})^{J,-} := \{U \in \mathrm{U}(\mathcal{M}) \colon JUJ = U^{-1}\}$$

is surjective, i.e., for every $U \in \mathrm{U}(\mathcal{M})^{J,-}$ there exists an element $X = -X^* \in \mathcal{M}$ with $JXJ = -X$ such that $U = e^X$.

Proof. We consider the antilinear automorphism

$$\alpha\colon \mathcal{M} \to \mathcal{M}, \quad \alpha(M) := JMJ$$

of the von Neumann algebra $\mathcal{M}$. Let $\mathcal{N} \subseteq \mathcal{M}$ be the abelian von Neumann algebra generated by a fixed element $U \in \mathrm{U}(\mathcal{M})^{J,-}$. Then $\alpha(U) = U^{-1} = U^*$ implies that $\alpha(\mathcal{N}) = \mathcal{N}$ with $\alpha(A) = A^*$ for every $A \in \mathcal{N}$. Any spectral resolution of U in $\mathcal{N}$ and any bounded measurable function $f\colon \mathbb{T} \to i\mathbb{R}$ with $e^f = \mathrm{id}_{\mathbb{T}}$ yields an element $X := f(U) \in \mathcal{N}$ with $X^* = -X$ and $e^X = U$. Then $JXJ = \alpha(X) = X^* = -X$. $\square$

The following lemma is [NÓ17, Lemma A.1]:

Lemma A.3. *Let $\mathcal{M} \subseteq \mathcal{H}$ be a von Neumann algebra, $\alpha\colon \mathcal{M} \to \mathcal{M}$ a real-linear weakly continuous automorphism and $U \in \mathrm{U}(\mathcal{M})$ be a unitary element. Then the following assertions hold:*

(a) *If α is complex linear and $\alpha(U) = U$, then there exists a $V \in \mathrm{U}(\mathcal{M})$ with $\alpha(V) = V$ and $V^2 = U$.*
(b) *If α is complex linear and $\alpha(U) = U^{-1}$ with $\ker(U + \mathbf{1}) = \{0\}$, then there exists a $V \in \mathrm{U}(\mathcal{M})$ with $\alpha(V) = V^{-1}$ and $V^2 = U$.*
(c) *If α is antilinear and $\alpha(U) = U^{-1}$, then there exists a $V \in \mathrm{U}(\mathcal{M})$ with $\alpha(V) = V^{-1}$ and $V^2 = U$.*
(d) *If α is antilinear and $\alpha(U) = U$ with $\ker(U + \mathbf{1}) = \{0\}$, then there exists a $V \in \mathrm{U}(\mathcal{M})$ with $\alpha(V) = V$ and $V^2 = U$.*

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Communicated by Y. Kawahigashi