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Outlier-free spline spaces for isogeometric discretizations of biharmonic and polyharmonic eigenvalue problems

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Dedicated to Thomas J. R. Hughes on the occasion of his 80th birthday.

Abstract

We present outlier-free isogeometric Galerkin discretizations of eigenvalue problems related to the biharmonic and the polyharmonic operator in the univariate setting. These are Galerkin discretizations in certain spline subspaces that provide accurate approximations of all eigenfrequencies and eigenmodes, without the occurrence of spurious outliers. The theoretical cornerstone is the use of spline subspaces that are optimal in the sense of L^2 Kolmogorov n-widths for special function classes related to the eigenvalue problems under investigation. This work is a continuation and extension of a similar study, recently presented for the Laplace operator, towards higher-order problems. As for the Laplace operator, the considered optimal spline spaces are identified by additional homogeneous boundary conditions and by specific sequences of break points. For fourth- and higher-order problems, however, optimal spline spaces are not known for all degrees and the optimal break points are not explicitly given. A careful analysis of the properties of optimal spaces allows us to determine spline subspaces of practical interest that are outlier-free. In particular, in the biharmonic case, we are able to construct such outlier-free spline spaces, for any odd and even degree, by considering break points that are uniform, or just a minor modification of the uniform ones. This improves upon the (numerical) results available in the literature. The theoretical findings are validated by a selection of numerical tests.
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Keywords: Biharmonic and polyharmonic problems; Eigenvalue problems; Isogeometric analysis; Outlier-free discretizations; Optimal spline subspaces

A tribute to Tom Hughes

We met Tom for the first time at the Curves and Surfaces conference in Tønsberg, in June 2008, where he gave a very inspiring invited talk on isogeometric analysis. Few years later, in June 2012, during the CIME summer school in Cetraro, we had the opportunity to extensively discuss with him and we became closer.

From 2013, we visited Tom in Austin several times over the years. Each visit was very pleasant from the personal side, because he is a perfect host and thanks to the nice atmosphere in his group. Unforgettable

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are the dinners at the restaurant Asti, always enriched with good wine, often a personal bottle from Tom's exceptional wine collection. With a tremendous supply of anecdotes, he is able to sketch vivid pictures of many of the most prominent researchers in the area of computational engineering, finite elements, and related fields.

Our visits were also very stimulating from the professional side, as Tom always had great advices for us, not only related to the research topics we were working on together. We are extremely grateful to him for his mathematical and personal support. He made us self-confident when coming to the new field of isogeometric analysis and gave us the opportunity of truly remarkable collaborations, thanks to his passion for bringing together different research fields. He fundamentally shaped our research in the last ten years.

In particular, Tom initiated the investigation of outliers in isogeometric discretizations and made substantial contributions to this topic. It is our pleasure to dedicate this paper to him.

Carla and Hendrik

1. Introduction

Isogeometric analysis was introduced by Hughes et al. in the seminal paper [1] with the aim of providing a true design-through-analysis methodology, by unifying computer aided design and finite element analysis. Its core idea is to use spline representation tools not only for design but also for analysis, in an isoparametric environment [1,2]. The inherent high smoothness of spline spaces leads to a high accuracy per degree of freedom in isogeometric analysis [3,4] and is beneficial in several contexts. In particular, it is the root of a key advantage of isogeometric discretizations: their accurate and well-behaved eigenfrequencies and eigenmodes.

Isogeometric discretizations of eigenvalue problems related to the Laplace operator have been widely investigated; see, e.g., [5–8]. It has been observed that the isogeometric elements improve the overall accuracy of the spectral approximation significantly compared to C^0 finite element analysis. Maximally smooth spline discretizations of degree p on uniform grids do not exhibit so-called spurious optical branches in the discrete spectrum and hence are an excellent choice for addressing eigenvalue problems. Yet, they still present a flaw: a very small portion of the frequencies is poorly approximated. For these frequencies, the errors are significantly larger than for the others and the corresponding computed values are much larger than the exact ones. These spurious values are usually referred to as *outliers*.

The number of outliers increases with the spline degree p. For fixed p, however, it is independent of the degrees of freedom for univariate problems, while a "thin layer" of outliers is observed for multivariate problems. Outliers depend on the treatment of boundary conditions [6] and are not confined to polynomial splines, e.g., they also appear in isogeometric discretizations based on Tchebycheffian splines [9,10]. In the context of multi-patch discretizations, optical branches of spurious outlier frequencies and modes may appear due to boundaries or reduced continuity at patch interfaces [11].

The presence of (few) largely overestimated outliers is an issue that weakens the power of high-order isogeometric methods in explicit dynamics [12–15].

Outlier-free discretizations for the spectrum of the Laplace operator are completely understood: they are constructed, and theoretically explained, by considering spline subspaces that are optimal in the sense of the L^2 Kolmogorov n-widths for special function classes related to the Laplace eigenvalue problem [16–19]. These optimal spline spaces are identified by additional homogeneous boundary conditions and by suitable sequences of break points. Spline subspaces based on uniform break points that satisfy the additional boundary conditions turn out to be close (and often equal) to the optimal ones and they also provide discretizations where the spurious outlier frequencies and modes are removed in a fully satisfactory way [15]. Alternatively, weak imposition of these additional boundary conditions can be used for numerical outlier removal and does not change the test and solution spaces from the standard spline space; see [20] and references therein. These methods are referred to as *outlier-free IGA* and have been applied, e.g., to reduce the condition number of the linear system resulting from the discretization [21]. A special class of Toeplitz-plus-Hankel matrices that lead to outlier-free solutions has been studied in [22].

For higher-order differential operators, the analysis of the outliers' behavior and of outlier-free discretizations is much less investigated in the literature. Interesting results have been presented in [15] for fourth-order problems. Similar to the Laplace operator, standard isogeometric discretizations for the biharmonic operator lead to an

overestimation of a small portion of the highest frequencies in the univariate setting. Again, the number of outliers increases with the spline degree p and for fixed p it is independent of the degrees of freedom. In the same work, it has also been numerically shown that isogeometric discretizations in reduced spline spaces with suitable additional homogeneous boundary conditions completely remove (for odd degrees) or substantially curtail (for even degrees) the spurious outliers.

In this paper, we address the eigenvalue problem for the biharmonic and the polyharmonic operator in the univariate setting. We continue the theoretical study in [19] and the more applied perspective discussed in [18], and we exploit the theory of optimal spline spaces to construct outlier-free discretizations. We prove that the use of optimal spline spaces leads to accurate outlier-free isogeometric Galerkin discretizations for the spectrum of the biharmonic operator, not only for the simpler cases of periodic and Laplacian-type boundary conditions, but also for the perhaps more interesting Dirichlet/Neumann/mixed boundary conditions. Moreover, we outline the extension of the above results to general polyharmonic operators. The considered optimal spline spaces are characterized by additional homogeneous boundary conditions and by suitable sequences of break points. The starting point of our investigation is the family of optimal spline spaces presented in [16] for the polyharmonic operator with Neumann boundary conditions.

To the best of our knowledge, this is the first time that outlier-free discretizations are addressed for the general polyharmonic problem with relevant boundary conditions. Confining our discussion to the biharmonic operator subject to Dirichlet/Neumann/mixed boundary conditions, we point out that its treatment presents some substantial differences with respect to the case of the Laplace operator:

- only odd-degree optimal spline spaces are available, and
- the break points for these optimal spline spaces are not explicitly known.

Nevertheless, we show that a simple approximation of the optimal break points, consisting of (a minor modification of) equally spaced points, still leads to spline subspaces where the spurious outlier frequencies and modes are completely removed. Furthermore, by exploiting conditions that are necessary for a space being optimal, we also deduce the structure of spline subspaces of even degree that provide outlier-free discretizations in practice. It turns out that the spaces we are using for our discretizations are similar to, and in some cases exactly agree with, those presented in [15]. However, the solid mathematical investigation allows us to explain the different behavior between odd and even degrees and to identify also even-degree outlier-free spline spaces.

1.1. Notation

Throughout the paper, for real-valued functions f and g we denote the norm and inner product on $L^2 := L^2(a,b)$ by

$$\|f\|^2 := (f,f), \quad (f,g) := \int_a^b f(x)g(x)\,\mathrm{d}x,$$

and we consider the Sobolev spaces

$$H^r := H^r(a,b) = \{u \in L^2 : u^{(\alpha)} \in L^2(a,b),\ \alpha = 1,\dots,r\}.$$

For the sake of simplicity, we set $(a,b) = (0,1)$.

Suppose $\boldsymbol{\tau} := (\tau_0,\dots,\tau_{n_{\mathrm{el}}})$ is a sequence of (break) points that partition the interval $[0,1]$ in n_{el} elements, i.e.,

$$0 =: \tau_0 < \tau_1 < \cdots < \tau_{n_{\mathrm{el}}-1} < \tau_{n_{\mathrm{el}}} := 1,$$

and let $I_j := [\tau_{j-1},\tau_j)$, $j = 1,\dots,n_{\mathrm{el}}-1$, and $I_{n_{\mathrm{el}}} := [\tau_{n_{\mathrm{el}}-1},\tau_{n_{\mathrm{el}}}]$. Let $\mathbb{P}_p$ be the space of polynomials of degree at most p. For $0 \le k \le p-1$, we define the space $\mathbb{S}^k_{p,\boldsymbol{\tau}}$ of splines of degree p and smoothness k by

$$\mathbb{S}^k_{p,\boldsymbol{\tau}} := \{s \in C^k[0,1] : s|_{I_j} \in \mathbb{P}_p,\ j = 1,\dots,n_{\mathrm{el}}\},$$

and we set

$$\mathbb{S}_{p,\boldsymbol{\tau}} := \mathbb{S}^{p-1}_{p,\boldsymbol{\tau}}. \tag{1.1}$$

The grid size is the value $h := \max_{j=1,\dots,n_{\mathrm{el}}} \tau_j - \tau_{j-1}$.

1.2. Overview

The remainder of the paper is divided in six sections and is organized as follows. In Section 2, we state the fourth-order eigenvalue problems we are interested in, we summarize some properties of the analytical solutions, and we briefly describe their Galerkin discretizations. Section 3 is the theoretical core of the paper. After reviewing the known results for periodic and Laplacian-type boundary conditions, we focus on other (more difficult) boundary conditions that are also relevant from the application point of view. For odd degrees, we present suitable spline subspaces that are optimal spaces in the sense of the L^2 Kolmogorov n-widths for special function classes related to the eigenvalue problems under investigation, and we prove that they ensure outlier-free discretizations while enjoying full accuracy. For even degrees, we exhibit spline subspaces that are good candidates to be outlier-free. In Section 4, we extend the results of the previous section based on optimal spaces to the general polyharmonic problem. Numerical tests validating the theoretical findings are gathered in Section 5, while a multivariate tensor-product setting is considered in Section 6. We conclude in Section 7 with some final remarks. Appendix collects the technical details and the proofs of the error estimates used in Section 3.

2. Fourth-order eigenvalue problems: Univariate case

In this section, we describe the eigenvalue problem related to the biharmonic operator in the univariate case subject to various boundary conditions. We also summarize some properties of the analytical solutions and we briefly outline the construction of the corresponding Galerkin discretizations.

2.1. Problem setting

We consider the fourth-order equation

$$u^{(4)} = \vartheta^4 u, \quad \text{in } (0,1), \tag{2.1}$$

and distinguish between three types of boundary conditions: periodic boundary conditions, those coming from the standard Laplacian eigenvalue problem, and those that do not.

The periodic boundary conditions are

$$u(0) = u(1), \quad u^{(1)}(0) = u^{(1)}(1), \quad u^{(2)}(0) = u^{(2)}(1), \quad u^{(3)}(0) = u^{(3)}(1). \tag{2.2}$$

Up to a proper normalization, the non-trivial solutions of (2.1) subject to (2.2) form the trigonometric system

$$1, \cos(2\pi x), \sin(2\pi x), \ldots, \cos(2l\pi x), \sin(2l\pi x), \ldots \tag{2.3}$$

and the eigenvalues are the fourth power of even multiples of π.

The Laplacian boundary conditions are

$$u(0) = u^{(2)}(0) = u(1) = u^{(2)}(1) = 0; \tag{2.4}$$

$$u^{(1)}(0) = u^{(3)}(0) = u^{(1)}(1) = u^{(3)}(1) = 0; \tag{2.5}$$

$$u(0) = u^{(2)}(0) = u^{(1)}(1) = u^{(3)}(1) = 0. \tag{2.6}$$

Up to a proper normalization, the non-trivial exact solutions of (2.1) subject to one of the above Laplacian boundary conditions (2.4)–(2.6) form a numerable set of trigonometric functions, respectively,

$$u_{l,\mathrm{Lap}_0}(x) := \sin(\vartheta_{l,\mathrm{Lap}_0} x), \quad \vartheta_{l,\mathrm{Lap}_0} := l\pi, \qquad l = 1, 2, \ldots \tag{2.7}$$

$$u_{l,\mathrm{Lap}_1}(x) := \cos(\vartheta_{l,\mathrm{Lap}_1} x), \quad \vartheta_{l,\mathrm{Lap}_1} := l\pi, \qquad l = 0, 1, 2, \ldots \tag{2.8}$$

$$u_{l,\mathrm{Lap}_2}(x) := \sin(\vartheta_{l,\mathrm{Lap}_2} x), \quad \vartheta_{l,\mathrm{Lap}_2} := \frac{2l-1}{2}\pi, \quad l = 1, 2, \ldots \tag{2.9}$$

These are exactly the same solutions as the ones of the standard Laplace eigenvalue problem with proper boundary conditions; see, e.g., [18, Section 2.1].

There are also three kinds of other relevant boundary conditions, and we collectively refer to them as non-Laplacian boundary conditions. We have full Dirichlet boundary conditions (also referred to as fixed or clamped boundary conditions),

$$u(0) = u^{(1)}(0) = u(1) = u^{(1)}(1) = 0, \tag{2.10}$$

full Neumann boundary conditions (also referred to as free or natural boundary conditions),

$$u^{(2)}(0) = u^{(3)}(0) = u^{(2)}(1) = u^{(3)}(1) = 0, \tag{2.11}$$

and a combination of the previous ones (also referred to as mixed boundary conditions),

$$u(0) = u^{(1)}(0) = u^{(2)}(1) = u^{(3)}(1) = 0. \tag{2.12}$$

The non-trivial exact solutions of (2.1) subject to one of the boundary conditions (2.10)–(2.12) form a numerable set of combinations of (polynomial) trigonometric and hyperbolic functions. More precisely, a direct computation gives the following results; see also [23, Section 5.5] and [15, Table 2]. In case of full Dirichlet boundary conditions the eigenfunctions are (non-trivial multiples of)

$$u_{l,\mathrm{Dir}}(x) := \sinh(\vartheta_{l,\mathrm{Dir}}x) - \sin(\vartheta_{l,\mathrm{Dir}}x) + a_{l,\mathrm{Dir}}(\cosh(\vartheta_{l,\mathrm{Dir}}x) - \cos(\vartheta_{l,\mathrm{Dir}}x)), \quad l = 1, 2, \ldots \tag{2.13}$$

where

$$a_{l,\mathrm{Dir}} := \frac{\sinh(\vartheta_{l,\mathrm{Dir}}) - \sin(\vartheta_{l,\mathrm{Dir}})}{\cos(\vartheta_{l,\mathrm{Dir}}) - \cosh(\vartheta_{l,\mathrm{Dir}})},$$

and $\vartheta_{l,\mathrm{Dir}}$ are the positive solutions (ordered increasingly) of the equation

$$\cos(\vartheta)\cosh(\vartheta) = 1. \tag{2.14}$$

In case of full Neumann boundary conditions, we have that any non-trivial element of $\mathbb{P}_1$ is an eigenfunction associated with the eigenvalue 0. The remaining eigenfunctions are (non-trivial multiples of)

$$u_{l,\mathrm{Neu}}(x) := \sinh(\vartheta_{l,\mathrm{Neu}}x) + \sin(\vartheta_{l,\mathrm{Neu}}x) + a_{l,\mathrm{Neu}}(\cosh(\vartheta_{l,\mathrm{Neu}}x) + \cos(\vartheta_{l,\mathrm{Neu}}x)), \quad l = 1, 2, \ldots \tag{2.15}$$

where

$$a_{l,\mathrm{Neu}} := a_{l,\mathrm{Dir}}, \quad \vartheta_{l,\mathrm{Neu}} := \vartheta_{l,\mathrm{Dir}}.$$

Finally, in case of mixed boundary conditions, the eigenfunctions are (non-trivial multiples of)

$$u_{l,\mathrm{mix}}(x) := \sinh(\vartheta_{l,\mathrm{mix}}x) - \sin(\vartheta_{l,\mathrm{mix}}x) + a_{l,\mathrm{mix}}(\cosh(\vartheta_{l,\mathrm{mix}}x) - \cos(\vartheta_{l,\mathrm{mix}}x)), \quad l = 1, 2, \ldots \tag{2.16}$$

where

$$a_{l,\mathrm{mix}} := -\frac{\sinh(\vartheta_{l,\mathrm{mix}}) + \sin(\vartheta_{l,\mathrm{mix}})}{\cosh(\vartheta_{l,\mathrm{mix}}) + \cos(\vartheta_{l,\mathrm{mix}})},$$

and $\vartheta_{l,\mathrm{mix}}$ are the positive solutions (ordered increasingly) of the equation

$$\cos(\vartheta)\cosh(\vartheta) = -1.$$

Remark 2.1. The exact value of $\vartheta_{l,\mathrm{Dir}}$ in (2.13) is not explicitly known. Let

$$\tilde{\vartheta}_{l,\mathrm{Dir}} := \frac{2l+1}{2}\pi, \quad l = 1, 2, \ldots$$

From (2.14) it is easy to see that $\tilde{\vartheta}_{l,\mathrm{Dir}}$ is an upper bound of $\vartheta_{l,\mathrm{Dir}}$ for l even and a lower bound otherwise. Moreover, it turns out that $\tilde{\vartheta}_{l,\mathrm{Dir}}$ is a very accurate approximation to $\vartheta_{l,\mathrm{Dir}}$ (and hence also to $\vartheta_{l,\mathrm{Neu}}$). Indeed, by applying one iteration of the Newton method starting from $\tilde{\vartheta}_{l,\mathrm{Dir}}$, we have

$$\frac{|\tilde{\vartheta}_{l,\mathrm{Dir}} - \vartheta_{l,\mathrm{Dir}}|}{\vartheta_{l,\mathrm{Dir}}} \leq \frac{1}{\vartheta_{l,\mathrm{Dir}}\cosh(\tilde{\vartheta}_{l,\mathrm{Dir}})} < 10^{-15}, \quad l \geq 10.$$

Similarly, for large l, a good approximation of the eigenvalues in case of mixed boundary conditions is given by

$$\tilde{\vartheta}_{l,\mathrm{mix}} := \frac{2l-1}{2}\pi, \quad l = 1, 2, \ldots.$$

2.2. Galerkin approximation

For the sake of brevity and simplicity of presentation, we only discuss the Galerkin approximation in the case of full Dirichlet boundary conditions; the treatment of the other boundary conditions is completely analogous. Therefore, we focus on the problem

$$\begin{cases} u^{(4)} = \vartheta^4 u, \quad \text{in } (0,1), \\ u(0) = u(1) = u^{(1)}(0) = u^{(1)}(1) = 0. \end{cases} \tag{2.17}$$

Let $H_0^2 := \{u \in H^2 : u(0) = u(1) = u^{(1)}(0) = u^{(1)}(1) = 0\}$. The weak form of the problem (2.17) reads as follows: find non-trivial $u \in H_0^2$ and $\vartheta^4 \in \mathbb{R}$ such that

$$(u^{(2)}, v^{(2)}) = \vartheta^4 (u, v), \quad \forall v \in H_0^2.$$

According to the Galerkin approach, we choose a finite-dimensional subspace $\mathbb{V}_h$ of H_0^2 spanned by the basis $\{\varphi_1, \dots, \varphi_{n_h}\}$ and we find approximate values ϑ_h to ϑ by solving

$$S_h \mathbf{u}_h = \vartheta_h^4 M_h \mathbf{u}_h,$$

where the matrices S_h and M_h consist of the elements

$$S_{h,i,j} := \int_0^1 \varphi_j^{(2)}(t) \varphi_i^{(2)}(t)\, \mathrm{d}t, \quad M_{h,i,j} := \int_0^1 \varphi_j(t) \varphi_i(t)\, \mathrm{d}t, \quad i, j = 1, \dots, n_h.$$

This means that each ϑ_h^4 is an eigenvalue of the matrix $M_h^{-1} S_h$. Then, for $l = 1, \dots, n_h$, an approximation of the frequency $\omega_{l,\mathrm{Dir}} := \vartheta_{l,\mathrm{Dir}}^2$ is obtained by considering the square root of the l-th eigenvalue of $M_h^{-1} S_h$, denoted by $\omega_{h,l,\mathrm{Dir}} := \vartheta_{h,l,\mathrm{Dir}}^2$. Here we assume that those eigenvalues are given in increasing order. Similarly, for $l = 1, \dots, n_h$, an approximation of the eigenfunction $u_{l,\mathrm{Dir}}$ is obtained by considering

$$u_{h,l,\mathrm{Dir}}(x) := \sum_{i=1}^{n_h} u_{h,l,\mathrm{Dir},i} \varphi_i(x), \tag{2.18}$$

where $\mathbf{u}_{h,l,\mathrm{Dir}} := (u_{h,l,\mathrm{Dir},1}, \dots, u_{h,l,\mathrm{Dir},n_h})$ is the l-th eigenvector of $M_h^{-1} S_h$. Of course, a proper normalization is needed. We are interested in selecting discretization spaces of dimension n_h such that the first n_h eigenfunctions and eigenvalues are all approximated well.

A natural discretization space for (2.17) is the subspace of H_0^2 given by maximally smooth splines with vanishing function and first derivative values at the two ends of the unit interval, i.e.,

$$\mathbb{S}_{p,\boldsymbol{\tau},\mathrm{Dir}}^{\mathrm{bih}} := \{s \in \mathbb{S}_{p,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = 0\}. \tag{2.19}$$

The dimension of such space equals

$$n_h = n_{\mathrm{el}} + p - 4.$$

Galerkin approximations in the space (2.19) with uniformly spaced break points are an excellent choice for addressing the eigenvalue problem (2.17). However, a small portion of the frequencies, and corresponding eigenfunctions, are poorly approximated; see Example 5.1 (and also Figs. 1 and 2). These are called *outliers* and their number is independent of n_{el} but increases with p. We refer to [15, Table 4] for the counting of the observed outliers.

Outlier-free discretizations can be achieved by identifying spline subspaces of $\mathbb{S}_{p,\boldsymbol{\tau}}$ of dimension n_h that ensure convergence in p for the approximations of the first n_h eigenfunctions of the problem we are dealing with. In Section 3 we propose a solution to this problem based on the properties of optimal spline spaces.

3. Outlier-free spline spaces

In this section, we extend the results of [18, Section 4] (see also [19, Section 7]) to prove that there are no outliers in the Galerkin eigenvalue approximation for the biharmonic operator subject to various boundary conditions when using spline subspaces that are optimal with respect to the Kolmogorov n-width in L^2-norm.

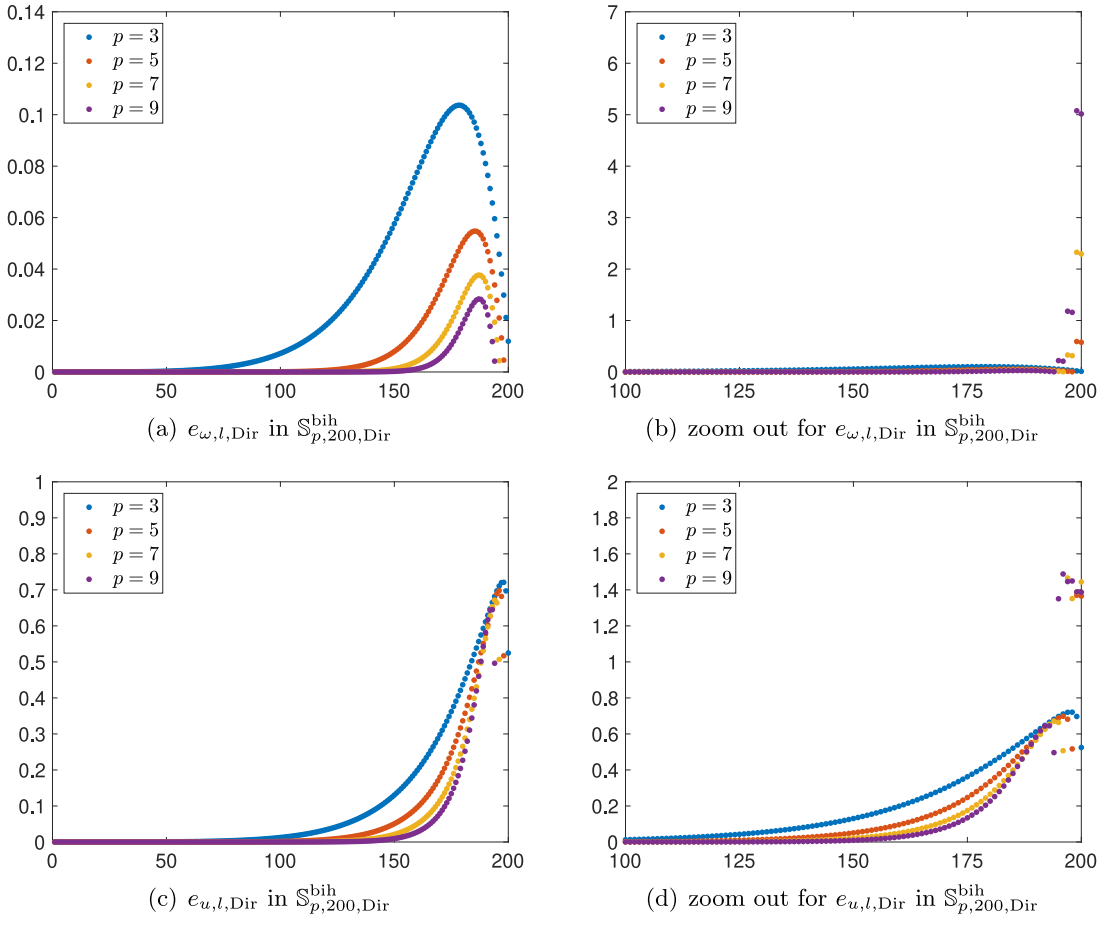


Fig. 1. Example 5.1: Relative frequency errors $e_{\omega,l,\mathrm{Dir}}$ in (5.1) and L^2 relative eigenfunction errors $e_{u,l,\mathrm{Dir}}$ in (5.2) corresponding to different spline spaces $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$ in (5.3) for odd degrees p and $n = 200$. The errors are ordered according to increasing exact frequencies. Few outliers are observed and their number depends on p. They are outside the visible range in (a) and (c), but a zoom out is given in (b) and (d), respectively.

We denote by X the L^2-projector onto a finite-dimensional subspace $\mathbb{X}$ of L^2. For a subset A of L^2, the Kolmogorov L^2 n-width of A is defined by

$$d_n(A) := \inf_{\substack{\mathbb{X}\subset L^2 \\ \dim \mathbb{X}=n}} E(A,\mathbb{X}),$$

where $E(A,\mathbb{X}) := \sup_{u\in A} \|u - Xu\|$ is the distance to A from $\mathbb{X}$ relative to the L^2-norm. We call $\mathbb{X}$ an *optimal* subspace for $d_n(A)$ if it has dimension at most n and satisfies

$$d_n(A) = E(A,\mathbb{X}).$$

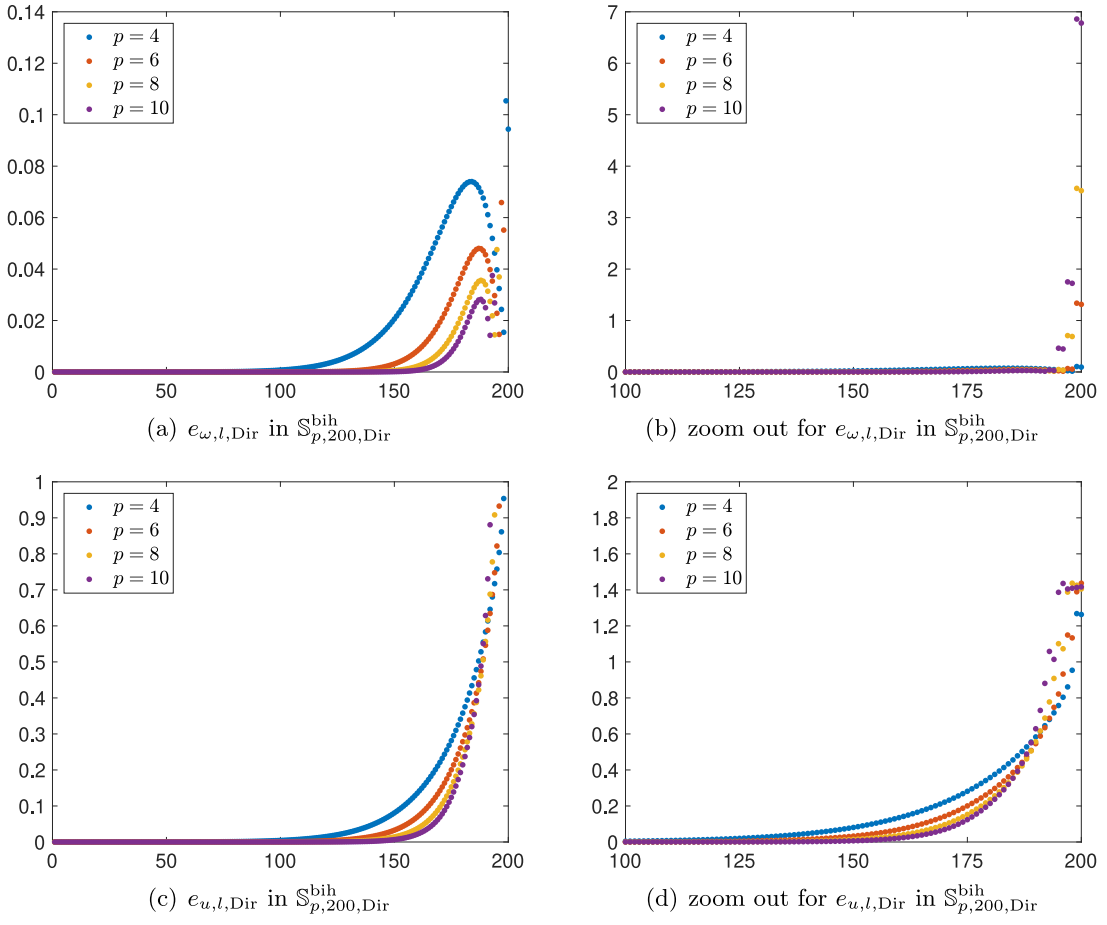
Example 3.1. Let $A = \{u \in H^r : \|u^{(r)}\| \le 1\}$. Then, by looking at $u/\|u^{(r)}\|$, the quantity $E(A,\mathbb{X})$ gives rise to a sharp estimate for functions $u \in H^r$,

$$\|u - Xu\| \le E(A,\mathbb{X})\|u^{(r)}\|.$$

Moreover, if $\mathbb{X}$ is optimal for the n-width of A, then

$$\|u - Xu\| \le d_n(A)\|u^{(r)}\|, \tag{3.1}$$

and $d_n(A)$ is the smallest possible constant over all n-dimensional subspaces $\mathbb{X}$.



(a) $e_{\omega,l,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{bih}}_{p,200,\mathrm{Dir}}$ (b) zoom out for $e_{\omega,l,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{bih}}_{p,200,\mathrm{Dir}}$

(c) $e_{u,l,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{bih}}_{p,200,\mathrm{Dir}}$ (d) zoom out for $e_{u,l,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{bih}}_{p,200,\mathrm{Dir}}$

Fig. 2. Example 5.1: Relative frequency errors $e_{\omega,l,\mathrm{Dir}}$ in (5.1) and L^2 relative eigenfunction errors $e_{u,l,\mathrm{Dir}}$ in (5.2) corresponding to different spline spaces $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$ in (5.3) for even degrees p and $n = 200$. The errors are ordered according to increasing exact frequencies. Few outliers are observed and their number depends on p. Most of them are outside the visible range in (a) and (c), but a zoom out is given in (b) and (d), respectively.

3.1. Periodic boundary conditions

For $r \geq 1$ let us consider the Sobolev space of periodic functions,

$$H^r_{\mathrm{per}} := \{u \in H^{2r} : u^{(\alpha)}(0) = u^{(\alpha)}(1);\ \alpha = 0, 1, \ldots, 2r-1\},$$

and the function class

$$A^r_{\mathrm{per}} := \{u \in H^r_{\mathrm{per}} : \|u^{(2r)}\| \leq 1\}.$$

We define the periodic spline space $\mathbb{S}_{p,\boldsymbol{\tau},\mathrm{per}}$ by

$$\mathbb{S}_{p,\boldsymbol{\tau},\mathrm{per}} := \{s \in \mathbb{S}_{p,\boldsymbol{\tau}} : s^{(\alpha)}(0) = s^{(\alpha)}(1);\ \alpha = 0, 1, \ldots, p-1\}. \tag{3.2}$$

Taking the uniform sequence of break points

$$\boldsymbol{\tau}_{n,\mathrm{per}} := \left(0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right),$$

the space in (3.2) has dimension n. From [19,24] we know that such spaces are optimal (for n even) or close to optimal (for n odd) for A^r_{per}, for $p = 2\ell - 1, 2\ell$ and $\ell \geq r$. Moreover, in all these cases, for $p \geq 3$ they

provide outlier-free discretizations to the eigenvalue problem (2.1) subject to the boundary conditions (2.2); see [19, Section 4].

3.2. Laplacian boundary conditions

For the Laplacian boundary conditions (2.4)–(2.6), the exact solutions (2.7)–(2.9) coincide with those of the standard Laplace eigenvalue problem with proper boundary conditions. The full analysis of the corresponding outlier-free discretizations has been presented in [18]. For the sake of completeness, here we shortly rephrase those results in the context of fourth-order problems. For $r \geq 1$ let us consider the function spaces

$$
\begin{aligned}
H^r_{\mathrm{Lap}_0} &:= \{u \in H^{2r} : u^{(\alpha)}(0) = u^{(\alpha)}(1) = 0;\ 0 \leq \alpha < 2r;\ \alpha \text{ even}\},\\
H^r_{\mathrm{Lap}_1} &:= \{u \in H^{2r} : u^{(\alpha)}(0) = u^{(\alpha)}(1) = 0;\ 0 \leq \alpha < 2r;\ \alpha \text{ odd}\},\\
H^r_{\mathrm{Lap}_2} &:= \{u \in H^{2r} : u^{(\alpha_0)}(0) = u^{(\alpha_1)}(1) = 0;\ 0 \leq \alpha_0, \alpha_1 < 2r;\ \alpha_0 \text{ even},\ \alpha_1 \text{ odd}\},
\end{aligned}
$$

and the function classes

$$
\begin{aligned}
A^r_{\mathrm{Lap}_0} &:= \{u \in H^r_{\mathrm{Lap}_0} : \|u^{(2r)}\| \leq 1\},\\
A^r_{\mathrm{Lap}_1} &:= \{u \in H^r_{\mathrm{Lap}_1} : \|u^{(2r)}\| \leq 1\},\\
A^r_{\mathrm{Lap}_2} &:= \{u \in H^r_{\mathrm{Lap}_2} : \|u^{(2r)}\| \leq 1\}.
\end{aligned}
$$

From [17] and [18, Section 4] (see also Example A.2 in Appendix) we know that the spline subspaces

$$
\begin{aligned}
\mathbb{S}_{p,\boldsymbol{\tau},\mathrm{Lap}_0} &:= \{s \in \mathbb{S}_{p,\boldsymbol{\tau}} : s^{(\alpha)}(0) = s^{(\alpha)}(1) = 0;\ 0 \leq \alpha \leq p;\ \alpha \text{ even}\},\\
\mathbb{S}_{p,\boldsymbol{\tau},\mathrm{Lap}_1} &:= \{s \in \mathbb{S}_{p,\boldsymbol{\tau}} : s^{(\alpha)}(0) = s^{(\alpha)}(1) = 0;\ 0 \leq \alpha \leq p;\ \alpha \text{ odd}\},\\
\mathbb{S}_{p,\boldsymbol{\tau},\mathrm{Lap}_2} &:= \{s \in \mathbb{S}_{p,\boldsymbol{\tau}} : s^{(\alpha_0)}(0) = s^{(\alpha_1)}(1) = 0;\ 0 \leq \alpha_0, \alpha_1 \leq p;\ \alpha_0 \text{ even},\ \alpha_1 \text{ odd}\},
\end{aligned}
$$

with the break points $\boldsymbol{\tau}^{\mathrm{opt}}_{p,n,i}$ defined as in [18, Section 4], are optimal for the function classes $A^r_{\mathrm{Lap}_i}$, $i = 0, 1, 2$, for $p = 2\ell - 1, 2\ell$ and $\ell \geq r$. For $p \geq 3$ they provide accurate outlier-free discretizations to the eigenvalue problem (2.1) subject to the boundary conditions (2.4)–(2.6); see [18, Section 4].

3.3. Non-Laplacian boundary conditions

Contrary to the types of boundary conditions discussed in Sections 3.1 and 3.2, removal of spurious outliers for the eigenvalue problem (2.1) with the boundary conditions (2.10)–(2.12) has not yet been investigated from the theoretical point of view in the literature. In this section, we identify spline subspaces that ensure accurate outlier-free discretizations for these relevant cases. This will again be achieved by using suitable optimal spaces and their properties.

For $r \geq 1$ let us consider the function spaces

$$
\begin{aligned}
H^r_{\mathrm{Dir}} &:= \{u \in H^{2r} : u^{(\beta)}(0) = u^{(\beta)}(1) = 0;\ \beta = 4\alpha, 4\alpha + 1;\ \alpha = 0, 1, \ldots;\ \beta < 2r\},\\
H^r_{\mathrm{Neu}} &:= \{u \in H^{2r} : u^{(\beta)}(0) = u^{(\beta)}(1) = 0;\ \beta = 4\alpha + 2, 4\alpha + 3;\ \alpha = 0, 1, \ldots;\ \beta < 2r\},\\
H^r_{\mathrm{mix}} &:= \{u \in H^{2r} : u^{(\beta_0)}(0) = u^{(\beta_1)}(1) = 0;\ \beta_0 = 4\alpha_0, 4\alpha_0 + 1;\ \alpha_0 = 0, 1, \ldots;\ \beta_0 < 2r;\\
&\qquad\qquad \beta_1 = 4\alpha_1 + 2, 4\alpha_1 + 3;\ \alpha_1 = 0, 1, \ldots;\ \beta_1 < 2r\},
\end{aligned}
$$

and the function classes

$$
\begin{aligned}
A^r_{\mathrm{Dir}} &:= \{u \in H^r_{\mathrm{Dir}} : \|u^{(2r)}\| \leq 1\},\\
A^r_{\mathrm{Neu}} &:= \{u \in H^r_{\mathrm{Neu}} : \|u^{(2r)}\| \leq 1\},\\
A^r_{\mathrm{mix}} &:= \{u \in H^r_{\mathrm{mix}} : \|u^{(2r)}\| \leq 1\}.
\end{aligned}
\tag{3.3}
$$

By using the representation of these function classes in terms of repeated applications of suitable integral operators, it can be shown [25] that the n-dimensional spaces consisting of the first n eigenfunctions of (2.1) satisfying the boundary conditions (2.10)–(2.12) are optimal for A^r_{Dir}, A^r_{Neu}, A^r_{mix}, respectively; see also Example A.3 in Appendix.

Moreover, the corresponding n-widths are given in terms of the $(n+1)$-st related eigenvalues. More precisely, for $n \geq 2$ we have

$$d_n(A^r_{\mathrm{Dir}}) = \left(\frac{1}{\vartheta^2_{n+1,\mathrm{Dir}}}\right)^r, \quad d_n(A^r_{\mathrm{Neu}}) = \left(\frac{1}{\vartheta^2_{n-1,\mathrm{Neu}}}\right)^r, \quad d_n(A^r_{\mathrm{mix}}) = \left(\frac{1}{\vartheta^2_{n+1,\mathrm{mix}}}\right)^r; \tag{3.4}$$

see [16,17,25,26]. Note that $\vartheta^4_{n-1,\mathrm{Neu}}$ is the $(n+1)$-st eigenvalue of (2.1) with the boundary conditions (2.11) when counting the eigenvalue 0 with its double multiplicity.

In Section 3.3.1, we describe certain spline spaces of odd degree, which are optimal for the n-width of the classes in (3.3), and we show that they are outlier-free. The starting point is the family of optimal spline spaces presented in [16] associated with the Neumann boundary conditions. The other boundary conditions can be treated in a similar way following [17]. Afterwards, in Section 3.3.2, we describe candidates for outlier-free spline spaces of even degree.

3.3.1. Odd degrees: Optimal and outlier-free spline spaces

For $\ell \geq 1$ let us consider the following subspaces of $\mathbb{S}_{2\ell-1,\boldsymbol{\tau}}$ identified by certain derivatives vanishing at the boundary:

$$\begin{aligned}
\mathbb{S}_{2\ell-1,\boldsymbol{\tau},\mathrm{Dir}} &:= \{s \in \mathbb{S}_{2\ell-1,\boldsymbol{\tau}} : s^{(\beta)}(0) = s^{(\beta)}(1) = 0;\ \beta = 4\alpha, 4\alpha+1;\ \alpha = 0,1,\ldots;\ \beta \leq 2\ell-1\},\\
\mathbb{S}_{2\ell-1,\boldsymbol{\tau},\mathrm{Neu}} &:= \{s \in \mathbb{S}_{2\ell-1,\boldsymbol{\tau}} : s^{(\beta)}(0) = s^{(\beta)}(1) = 0;\ \beta = 4\alpha+2, 4\alpha+3;\ \alpha = 0,1,\ldots;\ \beta \leq 2\ell-1\},\\
\mathbb{S}_{2\ell-1,\boldsymbol{\tau},\mathrm{mix}} &:= \{s \in \mathbb{S}_{2\ell-1,\boldsymbol{\tau}} : s^{(\beta_0)}(0) = s^{(\beta_1)}(1) = 0;\ \beta_0 = 4\alpha_0, 4\alpha_0+1;\ \alpha_0 = 0,1,\ldots;\ \beta_0 \leq 2\ell-1;\\
&\qquad\qquad \beta_1 = 4\alpha_1+2, 4\alpha_1+3;\ \alpha_1 = 0,1,\ldots;\ \beta_1 \leq 2\ell-1\}.
\end{aligned} \tag{3.5}$$

Note that the lowest degree is 1, but only degrees 3 and up can be used for the discretization of the biharmonic eigenvalue problem.

Example 3.2. For $\ell = 1,2,3,4,5$ the first type of spaces in (3.5) reads as

$$\begin{aligned}
\mathbb{S}_{1,\boldsymbol{\tau},\mathrm{Dir}} &= \{s \in \mathbb{S}_{1,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = 0\},\\
\mathbb{S}_{3,\boldsymbol{\tau},\mathrm{Dir}} &= \{s \in \mathbb{S}_{3,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = 0\},\\
\mathbb{S}_{5,\boldsymbol{\tau},\mathrm{Dir}} &= \{s \in \mathbb{S}_{5,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = s^{(4)}(0) = s^{(4)}(1) = s^{(5)}(0) = s^{(5)}(1) = 0\},\\
\mathbb{S}_{7,\boldsymbol{\tau},\mathrm{Dir}} &= \{s \in \mathbb{S}_{7,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = s^{(4)}(0) = s^{(4)}(1) = s^{(5)}(0) = s^{(5)}(1) = 0\},\\
\mathbb{S}_{9,\boldsymbol{\tau},\mathrm{Dir}} &= \{s \in \mathbb{S}_{9,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = s^{(4)}(0) = s^{(4)}(1) = s^{(5)}(0) = s^{(5)}(1) = 0,\\
&\qquad\qquad s^{(8)}(0) = s^{(8)}(1) = s^{(9)}(0) = s^{(9)}(1) = 0\}.
\end{aligned}$$

Example 3.3. For $\ell = 1,2,3,4,5$ the second type of spaces in (3.5) reads as

$$\begin{aligned}
\mathbb{S}_{1,\boldsymbol{\tau},\mathrm{Neu}} &= \mathbb{S}_{1,\boldsymbol{\tau}},\\
\mathbb{S}_{3,\boldsymbol{\tau},\mathrm{Neu}} &= \{s \in \mathbb{S}_{3,\boldsymbol{\tau}} : s^{(2)}(0) = s^{(2)}(1) = s^{(3)}(0) = s^{(3)}(1) = 0\},\\
\mathbb{S}_{5,\boldsymbol{\tau},\mathrm{Neu}} &= \{s \in \mathbb{S}_{5,\boldsymbol{\tau}} : s^{(2)}(0) = s^{(2)}(1) = s^{(3)}(0) = s^{(3)}(1) = 0\},\\
\mathbb{S}_{7,\boldsymbol{\tau},\mathrm{Neu}} &= \{s \in \mathbb{S}_{7,\boldsymbol{\tau}} : s^{(2)}(0) = s^{(2)}(1) = s^{(3)}(0) = s^{(3)}(1) = s^{(6)}(0) = s^{(6)}(1) = s^{(7)}(0) = s^{(7)}(1) = 0\},\\
\mathbb{S}_{9,\boldsymbol{\tau},\mathrm{Neu}} &= \{s \in \mathbb{S}_{9,\boldsymbol{\tau}} : s^{(2)}(0) = s^{(2)}(1) = s^{(3)}(0) = s^{(3)}(1) = s^{(6)}(0) = s^{(6)}(1) = s^{(7)}(0) = s^{(7)}(1) = 0\}.
\end{aligned}$$

Example 3.4. For $\ell = 1,2,3,4,5$ the third type of spaces in (3.5) reads as

$$\begin{aligned}
\mathbb{S}_{1,\boldsymbol{\tau},\mathrm{mix}} &= \{s \in \mathbb{S}_{1,\boldsymbol{\tau}} : s(0) = s^{(1)}(0) = 0\},\\
\mathbb{S}_{3,\boldsymbol{\tau},\mathrm{mix}} &= \{s \in \mathbb{S}_{3,\boldsymbol{\tau}} : s(0) = s^{(1)}(0) = s^{(2)}(1) = s^{(3)}(1) = 0\},\\
\mathbb{S}_{5,\boldsymbol{\tau},\mathrm{mix}} &= \{s \in \mathbb{S}_{5,\boldsymbol{\tau}} : s(0) = s^{(1)}(0) = s^{(2)}(1) = s^{(3)}(1) = s^{(4)}(0) = s^{(5)}(0) = 0\},\\
\mathbb{S}_{7,\boldsymbol{\tau},\mathrm{mix}} &= \{s \in \mathbb{S}_{7,\boldsymbol{\tau}} : s(0) = s^{(1)}(0) = s^{(2)}(1) = s^{(3)}(1) = s^{(4)}(0) = s^{(5)}(0) = s^{(6)}(1) = s^{(7)}(1) = 0\},\\
\mathbb{S}_{9,\boldsymbol{\tau},\mathrm{mix}} &= \{s \in \mathbb{S}_{9,\boldsymbol{\tau}} : s(0) = s^{(1)}(0) = s^{(2)}(1) = s^{(3)}(1) = s^{(4)}(0) = s^{(5)}(0) = s^{(6)}(1) = s^{(7)}(1) = 0,\\
&\qquad\qquad s^{(8)}(0) = s^{(9)}(0) = 0\}.
\end{aligned}$$

The optimal spaces in the family (3.5) are obtained by selecting certain (degree-dependent) sequences of break points; see [16,17,26]. They are in fact given by the zeros of the $(n+1)$-st eigenfunction of the problem (2.1) satisfying the boundary conditions (2.10)–(2.12). It was shown in [27] that any n-dimensional optimal space for A^r_{BC}, $\mathrm{BC} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{mix}\}$ has to be orthogonal to the $(n+1)$-st eigenfunction of (2.1) with the corresponding boundary conditions. One can check that this property uniquely determines the break points of the optimal spline spaces for each degree.

First, consider the eigenfunctions $u_{l,\mathrm{Dir}}$, $l = 1, 2, \ldots$ defined in (2.13), i.e., the non-trivial solutions of (2.17). Note that $u^{(2)}_{l,\mathrm{Dir}}$, $l = 1, 2, \ldots$ are solutions of the eigenvalue problem (2.1) with the Neumann boundary conditions (2.11) and that $u^{(4)}_{l,\mathrm{Dir}}$, $l = 1, 2, \ldots$ are again solutions of (2.17). Let us now look at the zeros of the $(n+1)$-st eigenfunction. From [26] we deduce that $u_{n+1,\mathrm{Dir}}$ has n simple zeros in $(0, 1)$,

$$u_{n+1,\mathrm{Dir}}(\tau^0_{i,n,\mathrm{Dir}}) = 0, \quad 0 < \tau^0_{1,n,\mathrm{Dir}} < \cdots < \tau^0_{n,n,\mathrm{Dir}} < 1,$$

and that $u^{(2)}_{n+1,\mathrm{Dir}}$ has $n+2$ simple zeros in $(0, 1)$,

$$u^{(2)}_{n+1,\mathrm{Dir}}(\tau^2_{i,n,\mathrm{Dir}}) = 0, \quad 0 < \tau^2_{1,n,\mathrm{Dir}} < \cdots < \tau^2_{n+2,n,\mathrm{Dir}} < 1.$$

Second, consider the eigenfunctions for mixed boundary conditions, i.e., $u_{l,\mathrm{mix}}$, $l = 1, 2, \ldots$ defined in (2.16). Note that $u^{(2)}_{l,\mathrm{mix}}$, $l = 1, 2, \ldots$ are solutions of the eigenvalue problem (2.1) with mirrored mixed boundary conditions. Moreover, from Example A.3 and from a theorem of Kellog [25, page 109], see also [17, Section 5], it follows that $u_{n+1,\mathrm{mix}}$ and $u^{(2)}_{n+1,\mathrm{mix}}$ have n simple zeros in $(0, 1)$,

$$\begin{aligned} &u_{n+1,\mathrm{mix}}(\tau^0_{i,n,\mathrm{mix}}) = u^{(2)}_{n+1,\mathrm{mix}}(\tau^2_{i,n,\mathrm{mix}}) = 0, \\ &0 < \tau^0_{1,n,\mathrm{mix}} < \cdots < \tau^0_{n,n,\mathrm{mix}} < 1, \quad 0 < \tau^2_{1,n,\mathrm{mix}} < \cdots < \tau^2_{n,n,\mathrm{mix}} < 1. \end{aligned}$$

Now, setting

$$\boldsymbol{\tau}^0_{n,\mathrm{Dir}} := (0, \tau^0_{1,n,\mathrm{Dir}}, \ldots, \tau^0_{n,n,\mathrm{Dir}}, 1), \quad \boldsymbol{\tau}^2_{n,\mathrm{Dir}} := (0, \tau^2_{1,n,\mathrm{Dir}}, \ldots, \tau^2_{n+2,n,\mathrm{Dir}}, 1) \tag{3.6}$$

and

$$\boldsymbol{\tau}^0_{n,\mathrm{mix}} := (0, \tau^0_{1,n,\mathrm{mix}}, \ldots, \tau^0_{n,n,\mathrm{mix}}, 1), \quad \boldsymbol{\tau}^2_{n,\mathrm{mix}} := (0, \tau^2_{1,n,\mathrm{mix}}, \ldots, \tau^2_{n,n,\mathrm{mix}}, 1), \tag{3.7}$$

we consider the following sequences of break points:

$$\boldsymbol{\tau}^{\mathrm{opt}}_{\ell,n,\mathrm{Dir}} := \begin{cases} \boldsymbol{\tau}^0_{n,\mathrm{Dir}}, & \ell \text{ even}, \\ \boldsymbol{\tau}^2_{n,\mathrm{Dir}}, & \ell \text{ odd}, \end{cases}$$

$$\boldsymbol{\tau}^{\mathrm{opt}}_{\ell,n,\mathrm{Neu}} := \begin{cases} \boldsymbol{\tau}^0_{n-2,\mathrm{Dir}}, & \ell \text{ odd}, \\ \boldsymbol{\tau}^2_{n-2,\mathrm{Dir}}, & \ell \text{ even}, \end{cases}$$

$$\boldsymbol{\tau}^{\mathrm{opt}}_{\ell,n,\mathrm{mix}} := \begin{cases} \boldsymbol{\tau}^0_{n,\mathrm{mix}}, & \ell \text{ even}, \\ \boldsymbol{\tau}^2_{n,\mathrm{mix}}, & \ell \text{ odd}. \end{cases}$$

Then, we define the spaces

$$\mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}} := \mathbb{S}_{2\ell-1,\boldsymbol{\tau}^{\mathrm{opt}}_{\ell,n,\mathrm{BC}},\mathrm{BC}}, \quad \mathrm{BC} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{mix}\}. \tag{3.8}$$

An elementary counting shows that all the above spaces have dimension n. From [16, Theorem 2] and [17] it follows that for $\ell \geq r$ and $r \geq 1$ the spaces $\mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Dir}}$, $\mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Neu}}$, and $\mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{mix}}$ are optimal for the function classes A^r_{Dir}, A^r_{Neu}, and A^r_{mix}, respectively.

Let $S^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}}$ be the L^2-projector onto $\mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}}$, with $\mathrm{BC} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{mix}\}$. Then, as a consequence of (3.1) and (3.4), we have for $\ell \geq r$, $r \geq 1$, and $n \geq 2$,

$$\begin{aligned} \|u - S^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}} u\| &\leq \left(\frac{1}{\vartheta^2_{n+1,\mathrm{BC}}}\right)^r \|u^{(2r)}\|, \quad u \in H^r_{\mathrm{BC}}, \quad \mathrm{BC} \in \{\mathrm{Dir}, \mathrm{mix}\}, \\ \|u - S^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Neu}} u\| &\leq \left(\frac{1}{\vartheta^2_{n-1,\mathrm{Neu}}}\right)^r \|u^{(2r)}\|, \quad u \in H^r_{\mathrm{Neu}}. \end{aligned} \tag{3.9}$$

To establish convergence of the standard Galerkin approximation we need error estimates for derivatives, analogous to (3.9). The next theorem is an extension to the biharmonic operator of the results proved in [18, Theorem 4.2] for second-order eigenvalue problems.

Theorem 3.1. *Let $u \in H^r_{\mathrm{BC}}$, $\mathrm{BC} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{mix}\}$ for $r \geq 1$ be given. Then, for all $\ell \geq \max\{2, r\}$ there exists*

$$R^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}} u \in \mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}},$$

such that for $\mathrm{BC} \in \{\mathrm{Dir}, \mathrm{mix}\}$,

$$\|u - R^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}} u\| \leq \left(\frac{1}{\vartheta^2_{n+1,\mathrm{BC}}}\right)^r \|u^{(2r)}\|, \quad \|(u - R^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}} u)^{(2)}\| \leq \left(\frac{1}{\vartheta^2_{n+1,\mathrm{BC}}}\right)^{r-1} \|u^{(2r)}\|,$$

and

$$\|u - R^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Neu}} u\| \leq \left(\frac{1}{\vartheta^2_{n-1,\mathrm{Neu}}}\right)^r \|u^{(2r)}\|, \quad \|(u - R^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Neu}} u)^{(2)}\| \leq \left(\frac{1}{\vartheta^2_{n-1,\mathrm{Neu}}}\right)^{r-1} \|u^{(2r)}\|.$$

Proof. The result follows from Example A.3 and Proposition A.1 (see Appendix) with $q = 2$ and with

$$\begin{aligned}
R^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Dir}} &\coloneqq R_{\mathbb{Y}_{\ell-1}}, & \mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Dir}} &= \mathbb{Y}_{\ell-1},\\
R^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Neu}} &\coloneqq P_1 \oplus R_{\mathbb{X}_{\ell-1}}, & \mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Neu}} &= \mathbb{P}_1 \oplus \mathbb{X}_{\ell-1},\\
R^{\mathrm{opt}}_{2\ell-1,n,\mathrm{mix}} &\coloneqq R_{\mathbb{X}_{\ell-1}}, & \mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{mix}} &= \mathbb{X}_{\ell-1},
\end{aligned}$$

for the integral operators specified in Example A.3 and taking into account (3.4) and (A.5). Here P_1 stands for the L^2-projector onto $\mathbb{P}_1$. □

Then, noting that $\mathbb{P}_1 \subset \mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Neu}}$, Corollaries A.2 and A.3 (in Appendix) imply the following result.

Proposition 3.1. *For any $l = 1, \ldots, n$, let u_l, ϑ_l be an exact solution of the problem (2.1) with the boundary conditions (2.10), (2.11), or (2.12), where the 0 value with its double multiplicity is counted in the case (2.11). Let $u_{h,l}, \vartheta_{h,l}$ be their approximation obtained by finding $u \in \mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}}$ and $\vartheta^4 \in \mathbb{R}$ such that*

$$(u^{(2)}, v^{(2)}) = \vartheta^4 (u, v), \quad \forall v \in \mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}}, \quad \ell \geq 2,$$

for $\mathrm{BC} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{mix}\}$, *respectively. Then,*

$$\vartheta_l \leq \vartheta_{h,l} \leq \frac{\vartheta_l}{\left(1 - \left(\frac{\vartheta_l}{\vartheta_{n+1}}\right)^{2\ell}\right)^{1/2}}. \tag{3.10}$$

Moreover,

$$\frac{\|u_l - u_{h,l}\|}{\|u_l\|} \leq 2(1 + \rho_l) \left(\frac{\vartheta_l}{\vartheta_{n+1}}\right)^{2\ell},$$

where $\|u_l\| = \|u_{h,l}\|$, $(u_l, u_{h,l}) > 0$, and

$$\rho_l \coloneqq \max_{\substack{i=1,\ldots,n\\ i\neq l}} \frac{\vartheta_l^4}{|\vartheta_l^4 - \vartheta_{h,i}^4|}$$

is the l-th separation constant.

Note that the error estimate for the approximation of the exact values ϑ_l by the $\vartheta_{h,l}$ in (3.10) only depends on the exact values ϑ_l and ϑ_{n+1}. Since $\vartheta_n < \vartheta_{n+1}$ (see Remark 2.1), we have that $\vartheta_{h,l} \to \vartheta_l$ for all $l = 1, \ldots, n$ as the degree $p = 2\ell - 1$ of the optimal spline spaces increases. Moreover, $\vartheta_{n+1} < \vartheta_{n+2} < \ldots$, so for a fixed index l we have that $\vartheta_{h,l} \to \vartheta_l$ as the grid size h decreases, since $h \to 0$ is equivalent to $n \to \infty$. There is thus convergence both in p and h.

We can summarize the above results as follows. Consider the problem (2.1) with the boundary conditions (2.10), (2.11), or (2.12). The approximations of the eigenvalues obtained by finding $u \in \mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}}$ and $\vartheta^4 \in \mathbb{R}$ such that

$$(u^{(2)}, v^{(2)}) = \vartheta^4 (u, v), \quad \forall v \in \mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}}, \quad \ell \geq 2,$$

for BC $\in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{mix}\}$, respectively, have no outliers.

Remark 3.1. From (2.13) it turns out that the sequence of break points $\boldsymbol{\tau}^0_{n,\mathrm{Dir}}$ in (3.6) is very close to being uniform. The sequence $\boldsymbol{\tau}^2_{n,\mathrm{Dir}}$ in (3.6), on the other hand, is close to $\boldsymbol{\tau}^0_{n,\mathrm{Dir}}$ but with two additional break points near the boundary; see (2.15). Actually, from (2.13) and taking into account Remark 2.1, a direct computation shows that the two sets of zeros enjoy the following simple approximations:

$$\tau^0_{i,n,\mathrm{Dir}} \simeq \frac{4i+1}{4n+6} \simeq \frac{i}{n+1}, \quad i = 1, \ldots, n,$$
$$\tau^2_{i+1,n,\mathrm{Dir}} \simeq \frac{4i+1}{4n+6} \simeq \frac{i}{n+1}, \quad i = 0, \ldots, n+1.$$

The above approximations are very accurate for sufficiently large n ($n \geq 10$) and $i = 1, \ldots, n$. For $i = 0, n+1$ they can be improved by considering

$$\tau^2_{1,n,\mathrm{Dir}} = 1 - \tau^2_{n+2,n,\mathrm{Dir}} \simeq \frac{2}{3(2n+3)} \simeq \frac{1}{3(n+1)}.$$

Similarly, the sequences $\boldsymbol{\tau}^0_{n,\mathrm{mix}}$ and $\boldsymbol{\tau}^2_{n,\mathrm{mix}}$ in (3.7) are very close to being uniform with a shift towards the first or the second extreme of the interval; see (2.16).

Remark 3.2. The spline subspaces $\mathbb{S}_{2\ell-1,\boldsymbol{\tau},\mathrm{Dir}}$, with ℓ even and uniform break points $\boldsymbol{\tau}$, were considered in [15] for outlier removal in spline discretizations of the biharmonic eigenvalue problem (2.1) with Dirichlet boundary conditions. On the other hand, for ℓ odd and uniform break points $\boldsymbol{\tau}$, the spaces $\mathbb{S}_{2\ell-1,\boldsymbol{\tau},\mathrm{Dir}}$ are proper subspaces of the spline spaces of the same degree considered in [15, Section 5.2]. Similarly, for Neumann boundary conditions, the spline subspaces $\mathbb{S}_{2\ell-1,\boldsymbol{\tau},\mathrm{Neu}}$, with ℓ odd and uniform break points $\boldsymbol{\tau}$, agree with the spline subspaces in [15, Section 5.2] while, for ℓ even, they are proper subspaces of the corresponding ones in [15, Section 5.2].

3.3.2. Even degrees: Outlier-free spline spaces

The results in Section 3.3.1 concern odd-degree splines. For even-degree splines, a complete theory of optimal spaces is missing so that we cannot state the same results as for the odd-degree case. However, we can argue about the structure of outlier-free spline spaces of even degree by considering a fundamental property of optimal spaces. As already pointed out in Section 3.3.1, an n-dimensional optimal space for A^r_{BC}, BC $\in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{mix}\}$ has to be orthogonal to the $(n+1)$-st eigenfunction of the problem (2.1) satisfying the boundary conditions (2.10)–(2.12), respectively. Therefore, we intend to exploit this property to identify even-degree spline subspaces that are appropriate for removal of spurious outliers.

Let us first consider Neumann boundary conditions. For $n \geq 2$ let $u_{n-1,\mathrm{Neu}}$ be as in (2.15). Note that this is the $(n+1)$-st eigenfunction of the problem (2.1) with the boundary conditions (2.11) when also counting the eigenfunctions associated with the zero eigenvalue. From (2.15) we can see that for $n \geq 2$, $u^{(1)}_{n-1,\mathrm{Neu}}$ and $u^{(3)}_{n-1,\mathrm{Neu}}$ have $n-1$ simple zeros in $(0, 1)$,

$$u^{(1)}_{n-1,\mathrm{Neu}}(\tau^1_{i,n,\mathrm{Neu}}) = 0, \quad 0 < \tau^1_{1,n,\mathrm{Neu}} < \cdots < \tau^1_{n-1,n,\mathrm{Neu}} < 1,$$
$$u^{(3)}_{n-1,\mathrm{Neu}}(\tau^3_{i,n,\mathrm{Neu}}) = 0, \quad 0 < \tau^3_{1,n,\mathrm{Neu}} < \cdots < \tau^3_{n-1,n,\mathrm{Neu}} < 1.$$

We set

$$\boldsymbol{\tau}^1_{n,\mathrm{Neu}} := (0, \tau^1_{1,n,\mathrm{Neu}}, \ldots, \tau^1_{n-1,n,\mathrm{Neu}}, 1), \quad \boldsymbol{\tau}^3_{n,\mathrm{Neu}} := (0, \tau^3_{1,n,\mathrm{Neu}}, \ldots, \tau^3_{n-1,n,\mathrm{Neu}}, 1). \tag{3.11}$$

Using these sequences of break points, in the next proposition we identify two spline spaces that are orthogonal to the $(n+1)$-st eigenfunction of the problem (2.1) with Neumann boundary conditions.

Proposition 3.2. *The spaces*

$$\mathbb{S}_{2,\boldsymbol{\tau}^1_{n,\mathrm{Neu}},\mathrm{Neu}} := \{s \in \mathbb{S}_{2,\boldsymbol{\tau}^1_{n,\mathrm{Neu}}} : s^{(2)}(0) = s^{(2)}(1) = 0\},$$
$$\mathbb{S}_{4,\boldsymbol{\tau}^3_{n,\mathrm{Neu}},\mathrm{Neu}} := \{s \in \mathbb{S}_{4,\boldsymbol{\tau}^3_{n,\mathrm{Neu}}} : s^{(2)}(0) = s^{(2)}(1) = s^{(3)}(0) = s^{(3)}(1) = 0\}$$

are orthogonal to $u_{n-1,\mathrm{Neu}}$.

Proof. Let us set

$$\varphi_{n-1,1} := u^{(3)}_{n-1,\mathrm{Neu}}, \quad \varphi_{n-1,2} := u^{(2)}_{n-1,\mathrm{Neu}}, \quad \varphi_{n-1,3}(x) := u^{(1)}_{n-1,\mathrm{Neu}}.$$

It follows

$$\varphi^{(1)}_{n-1,1} = u^{(4)}_{n-1,\mathrm{Neu}} = \vartheta^4_{n-1,\mathrm{Neu}} u_{n-1,\mathrm{Neu}},$$

and $\varphi_{n-1,1}$ and $\varphi_{n-1,3}$ are solutions of the eigenvalue problem (2.1) with the boundary conditions

$$u(0) = u(1) = u^{(3)}(0) = u^{(3)}(1) = 0,$$
$$u^{(1)}(0) = u^{(1)}(1) = u^{(2)}(0) = u^{(2)}(1) = 0,$$

respectively. Similarly, $\varphi_{n-1,2}$ is a solution of the eigenvalue problem (2.1) with the Dirichlet boundary conditions (2.10).

Let $s \in \mathbb{S}_{2,\boldsymbol{\tau}^1_{n,\mathrm{Neu}},\mathrm{Neu}}$ be given. According to the notation in Section 1.1, we set $I_j := [\tau_{j-1}, \tau_j)$, where $\boldsymbol{\tau} := (\tau_0, \ldots, \tau_n) := \boldsymbol{\tau}^1_{n,\mathrm{Neu}}$. Taking into account that $s^{(2)}_{|I_j} = c_j$ for some $c_j \in \mathbb{R}$, we deduce

$$\begin{aligned}
\vartheta^4_{n-1,\mathrm{Neu}} \int_0^1 s(t) u_{n-1,\mathrm{Neu}}(t)\,\mathrm{d}t &= s(t)\varphi_{n-1,1}(t)\big|_0^1 - \int_0^1 s^{(1)}(t)\varphi_{n-1,1}(t)\,\mathrm{d}t \\
&= -s^{(1)}(t)\varphi_{n-1,2}(t)\big|_0^1 + \sum_{j=1}^n \int_{I_j} s^{(2)}(t)\varphi_{n-1,2}(t)\,\mathrm{d}t \\
&= \sum_{j=1}^n c_j \int_{I_j} \varphi_{n-1,2}(t)\,\mathrm{d}t = \sum_{j=1}^n c_j[\varphi_{n-1,3}(\tau_j) - \varphi_{n-1,3}(\tau_{j-1,})] = 0,
\end{aligned}$$

where the last equality follows from $s^{(2)}(0) = s^{(2)}(1) = 0$ and the fact that the interior break points are zeros of $\varphi_{n-1,3}$.

Now, let $s \in \mathbb{S}_{4,\boldsymbol{\tau}^3_{n,\mathrm{Neu}},\mathrm{Neu}}$ be given. With the same line of arguments, setting $\boldsymbol{\tau} := (\tau_0, \ldots, \tau_n) := \boldsymbol{\tau}^3_{n,\mathrm{Neu}}$ and taking into account that $s^{(4)}_{|I_j} = c_j$ for some $c_j \in \mathbb{R}$, we obtain

$$\begin{aligned}
\vartheta^8_{n-1,\mathrm{Neu}} \int_0^1 s(t) u_{n-1,\mathrm{Neu}}(t)\,\mathrm{d}t &= \vartheta^4_{n-1,\mathrm{Neu}} \int_0^1 s^{(2)}(t)\varphi_{n-1,2}(t)\,\mathrm{d}t \\
&= \vartheta^4_{n-1,\mathrm{Neu}} \left[s^{(2)}(t)\varphi_{n-1,3}(t)\big|_0^1 - s^{(3)}(t)u_{n-1,\mathrm{Neu}}(t)\big|_0^1 + \sum_{j=1}^n \int_{I_j} s^{(4)}(t)u_{n-1,\mathrm{Neu}}(t)\,\mathrm{d}t \right] \\
&= \sum_{j=1}^n c_j[\varphi_{n-1,1}(\tau_j) - \varphi_{n-1,1}(\tau_{j-1})] = 0.
\end{aligned}$$

The last two equalities follow from $s^{(2)}(0) = s^{(2)}(1) = s^{(3)}(0) = s^{(3)}(1) = 0$ and the fact that the break points are zeros of $\varphi_{n-1,1}$. □

By applying the construction in (A.6), it is then natural to consider for $\ell \geq 1$ the following subspaces of $\mathbb{S}_{2\ell,\boldsymbol{\tau}}$ identified by the same vanishing derivatives at the boundary as in (3.5):

$$\mathbb{S}_{2\ell,\boldsymbol{\tau},\mathrm{Neu}} := \{s \in \mathbb{S}_{2\ell,\boldsymbol{\tau}} : s^{(\beta)}(0) = s^{(\beta)}(1) = 0;\ \beta = 4\alpha + 2, 4\alpha + 3;\ \alpha = 0, 1, \ldots;\ \beta \leq 2\ell\}. \tag{3.12}$$

Setting

$$\boldsymbol{\tau}^{\mathrm{good}}_{\ell,n,\mathrm{Neu}} := \begin{cases} \boldsymbol{\tau}^1_{n,\mathrm{Neu}}, & \ell \text{ odd}, \\ \boldsymbol{\tau}^3_{n,\mathrm{Neu}}, & \ell \text{ even}, \end{cases}$$

it turns out that the spaces

$$\mathbb{S}^{\text{good}}_{2\ell,n,\text{Neu}} := \mathbb{S}_{2\ell,\boldsymbol{\tau}^{\text{good}}_{\ell,n,\text{Neu}},\text{Neu}} \tag{3.13}$$

have all dimension n and are good candidates to provide outlier-free discretizations for the problem (2.1) equipped with the boundary conditions (2.11).

Example 3.5. For $\ell = 1, 2, 3, 4, 5$ the spaces in (3.12) read as

$$\begin{aligned}
\mathbb{S}_{2,\boldsymbol{\tau},\text{Neu}} &= \{s \in \mathbb{S}_{2,\boldsymbol{\tau}} : s^{(2)}(0) = s^{(2)}(1) = 0\},\\
\mathbb{S}_{4,\boldsymbol{\tau},\text{Neu}} &= \{s \in \mathbb{S}_{4,\boldsymbol{\tau}} : s^{(2)}(0) = s^{(2)}(1) = s^{(3)}(0) = s^{(3)}(1) = 0\},\\
\mathbb{S}_{6,\boldsymbol{\tau},\text{Neu}} &= \{s \in \mathbb{S}_{6,\boldsymbol{\tau}} : s^{(2)}(0) = s^{(2)}(1) = s^{(3)}(0) = s^{(3)}(1) = s^{(6)}(0) = s^{(6)}(1) = 0\},\\
\mathbb{S}_{8,\boldsymbol{\tau},\text{Neu}} &= \{s \in \mathbb{S}_{8,\boldsymbol{\tau}} : s^{(2)}(0) = s^{(2)}(1) = s^{(3)}(0) = s^{(3)}(1) = s^{(6)}(0) = s^{(6)}(1) = s^{(7)}(0) = s^{(7)}(1) = 0\}.\\
\mathbb{S}_{10,\boldsymbol{\tau},\text{Neu}} &= \{s \in \mathbb{S}_{10,\boldsymbol{\tau}} : s^{(2)}(0) = s^{(2)}(1) = s^{(3)}(0) = s^{(3)}(1) = s^{(6)}(0) = s^{(6)}(1) = s^{(7)}(0) = s^{(7)}(1) = 0,\\
&\qquad\qquad s^{(10)}(0) = s^{(10)}(1) = 0\}.
\end{aligned}$$

Let us now focus on Dirichlet boundary conditions. In analogy with the odd-degree case, it is natural to consider the family of spaces

$$\mathbb{S}_{2\ell,\boldsymbol{\tau},\text{Dir}} := \{s \in \mathbb{S}_{2\ell,\boldsymbol{\tau}} : s^{(\beta)}(0) = s^{(\beta)}(1) = 0;\ \beta = 4\alpha, 4\alpha+1;\ \alpha = 0, 1, \ldots;\ \beta \le 2\ell\}. \tag{3.14}$$

Setting

$$\boldsymbol{\tau}^{\text{good}}_{\ell,n,\text{Dir}} := \begin{cases} \boldsymbol{\tau}^{1}_{n+2,\text{Neu}}, & \ell \text{ even},\\ \boldsymbol{\tau}^{3}_{n+2,\text{Neu}}, & \ell \text{ odd}, \end{cases}$$

we observe that the spaces

$$\mathbb{S}^{\text{good}}_{2\ell,n,\text{Dir}} := \mathbb{S}_{2\ell,\boldsymbol{\tau}^{\text{good}}_{\ell,n,\text{Dir}},\text{Dir}} \tag{3.15}$$

have all dimension n. From the above discussion we can also conclude that they are good candidates to provide outlier-free discretizations for the problem (2.1) equipped with the boundary conditions (2.10).

Example 3.6. For $\ell = 1, 2, 3, 4, 5$ the spaces in (3.14) read as

$$\begin{aligned}
\mathbb{S}_{2,\boldsymbol{\tau},\text{Dir}} &= \{s \in \mathbb{S}_{2,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = 0\},\\
\mathbb{S}_{4,\boldsymbol{\tau},\text{Dir}} &= \{s \in \mathbb{S}_{4,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = s^{(4)}(0) = s^{(4)}(1) = 0\},\\
\mathbb{S}_{6,\boldsymbol{\tau},\text{Dir}} &= \{s \in \mathbb{S}_{6,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = s^{(4)}(0) = s^{(4)}(1) = s^{(5)}(0) = s^{(5)}(1) = 0\},\\
\mathbb{S}_{8,\boldsymbol{\tau},\text{Dir}} &= \{s \in \mathbb{S}_{8,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = s^{(4)}(0) = s^{(4)}(1) = s^{(5)}(0) = s^{(5)}(1) = 0,\\
&\qquad\qquad s^{(8)}(0) = s^{(8)}(1) = 0\},\\
\mathbb{S}_{10,\boldsymbol{\tau},\text{Dir}} &= \{s \in \mathbb{S}_{10,\boldsymbol{\tau}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = s^{(4)}(0) = s^{(4)}(1) = s^{(5)}(0) = s^{(5)}(1) = 0,\\
&\qquad\qquad s^{(8)}(0) = s^{(8)}(1) = s^{(9)}(0) = s^{(9)}(1) = 0\}.
\end{aligned}$$

Suitable spline subspaces for mixed boundary conditions can be obtained in a similar way.

Remark 3.3. From (2.15) it turns out that the sequences of break points $\boldsymbol{\tau}^{1}_{n,\text{Neu}}$ and $\boldsymbol{\tau}^{3}_{n,\text{Neu}}$ in (3.11) are both very close to being uniform. Actually, taking into account Remark 2.1, a direct computation shows that the two sets of zeros enjoy the following simple approximations, which are very accurate for sufficiently large n:

$$\tau^{1}_{i+1,n,\text{Neu}} \simeq \tau^{3}_{i+1,n,\text{Neu}} \simeq \frac{4i+3}{4n-2} \simeq \frac{i+1/2}{n-1}, \quad i = 0, \ldots, n-2. \tag{3.16}$$

Similar simple approximations can be found in case of mixed boundary conditions.

Remark 3.4. The spline subspaces $\mathbb{S}_{2\ell,\boldsymbol{\tau},\text{Dir}}$, with ℓ odd and uniform break points $\boldsymbol{\tau}$, were considered in [15] for outlier removal in spline discretizations of the biharmonic eigenvalue problem (2.1) with Dirichlet boundary

conditions. On the other hand, for ℓ even and uniform break points $\boldsymbol{\tau}$, the spaces $\mathbb{S}_{2\ell,\boldsymbol{\tau},\mathrm{Dir}}$ are proper subspaces of the spline subspaces of the same degree considered in [15, Section 5.2]. Similarly, for Neumann boundary conditions, the spline subspaces $\mathbb{S}_{2\ell,\boldsymbol{\tau},\mathrm{Neu}}$, with ℓ even and uniform break points $\boldsymbol{\tau}$, agree with the spline subspaces in [15, Section 5.2] while, for ℓ odd, they are proper subspaces of the corresponding ones in [15, Section 5.2].

4. High-order eigenvalue problems: Univariate case

In this section, we briefly outline the construction of outlier-free spline spaces for general polyharmonic problems. The results extend those presented in detail in Section 3 for the biharmonic operator. Dealing with general q, it is necessary to consider a more heavy notation than in Section 3. In particular, a subscript q is added to the notation of all the objects. Of course, for $q = 2$ we obtain the corresponding classes/spaces considered in Section 3. For $q = 1$ we reproduce the results from [18, Section 4].

For $q \geq 1$ we consider the $(2q)$th-order equation

$$(-1)^q u^{(2q)} = \vartheta^{2q} u, \quad \text{in } (0,1). \tag{4.1}$$

Several types of boundary conditions can be associated with it. For the sake of brevity, we focus on the following interesting boundary conditions:

- full Dirichlet boundary conditions,

$$u^{(\beta)}(0) = u^{(\beta)}(1) = 0, \quad \beta = 0, 1, \ldots, q-1; \tag{4.2}$$

- full Neumann boundary conditions,

$$u^{(\beta)}(0) = u^{(\beta)}(1) = 0, \quad \beta = q, q+1, \ldots, 2q-1; \tag{4.3}$$

- mixed boundary conditions,

$$u^{(\beta_0)}(0) = u^{(\beta_1)}(1) = 0, \quad \beta_0 = 0, 1, \ldots, q-1, \quad \beta_1 = q, q+1, \ldots, 2q-1. \tag{4.4}$$

For $r \geq 1$ let us consider the function spaces

$$\begin{aligned}
H^r_{q,\mathrm{Dir}} &:= \{u \in H^{qr} : u^{(\beta)}(0) = u^{(\beta)}(1) = 0; \\
&\qquad \beta = 2\alpha q, \ldots, (2\alpha+1)q - 1;\ \alpha = 0, 1, \ldots;\ (2\alpha+1)q - 1 < qr\}, \\
H^r_{q,\mathrm{Neu}} &:= \{u \in H^{qr} : u^{(\beta)}(0) = u^{(\beta)}(1) = 0; \\
&\qquad \beta = (2\alpha+1)q, \ldots, (2\alpha+2)q - 1;\ \alpha = 0, 1, \ldots;\ (2\alpha+2)q - 1 < qr\}, \\
H^r_{q,\mathrm{mix}} &:= \{u \in H^{qr} : u^{(\beta_0)}(0) = u^{(\beta_1)}(1) = 0; \\
&\qquad \beta_0 = 2\alpha_0 q, \ldots, (2\alpha_0+1)q - 1;\ \alpha_0 = 0, 1, \ldots;\ (2\alpha_0+1)q - 1 < qr; \\
&\qquad \beta_1 = (2\alpha_1+1)q, \ldots, (2\alpha_1+2)q - 1;\ \alpha_1 = 0, 1, \ldots;\ (2\alpha_1+2)q - 1 < qr\},
\end{aligned}$$

and the function classes

$$\begin{aligned}
A^r_{q,\mathrm{Dir}} &:= \{u \in H^r_{q,\mathrm{Dir}} : \|u^{(qr)}\| \leq 1\}, \\
A^r_{q,\mathrm{Neu}} &:= \{u \in H^r_{q,\mathrm{Neu}} : \|u^{(qr)}\| \leq 1\}, \\
A^r_{q,\mathrm{mix}} &:= \{u \in H^r_{q,\mathrm{mix}} : \|u^{(qr)}\| \leq 1\}.
\end{aligned} \tag{4.5}$$

By using the representation of these function classes in terms of repeated applications of suitable integral operators, it can be shown [25] that the n-dimensional spaces consisting of the first n eigenfunctions of (4.1) satisfying the boundary conditions (4.2)–(4.4) are optimal for $A^r_{q,\mathrm{Dir}}$, $A^r_{q,\mathrm{Neu}}$, $A^r_{q,\mathrm{mix}}$, respectively; see also Example A.4 in Appendix. Moreover, denoting the corresponding positive eigenvalues (ordered increasingly) by $\vartheta^{2q}_{q,l,\mathrm{Dir}}$, $\vartheta^{2q}_{q,l,\mathrm{Neu}}$, $\vartheta^{2q}_{q,l,\mathrm{mix}}$, $l = 1, 2, \ldots$, for $n \geq q$ we have

$$d_n(A^r_{q,\mathrm{Dir}}) = \left(\frac{1}{\vartheta^q_{q,n+1,\mathrm{Dir}}}\right)^r, \quad d_n(A^r_{q,\mathrm{Neu}}) = \left(\frac{1}{\vartheta^q_{q,n+1-q,\mathrm{Neu}}}\right)^r, \quad d_n(A^r_{q,\mathrm{mix}}) = \left(\frac{1}{\vartheta^q_{q,n+1,\mathrm{mix}}}\right)^r;$$

see [16,17,26].

From [16, Theorem 2] and Example A.4, with a similar line of arguments as in [17], we can prove that the following subspaces of $\mathbb{S}_{p,\boldsymbol{\tau}}$,

$$\begin{aligned}
\mathbb{S}_{q,q\ell-1,\boldsymbol{\tau},\mathrm{Dir}} &:= \{s \in \mathbb{S}_{q\ell-1,\boldsymbol{\tau}} : s^{(\beta)}(0) = s^{(\beta)}(1) = 0; \\
&\qquad \beta = 2\alpha q, \dots, (2\alpha+1)q - 1;\ \alpha = 0, \dots, \lceil \ell/2 \rceil - 1\}, \\
\mathbb{S}_{q,q\ell-1,\boldsymbol{\tau},\mathrm{Neu}} &:= \{s \in \mathbb{S}_{q\ell-1,\boldsymbol{\tau}} : s^{(\beta)}(0) = s^{(\beta)}(1) = 0; \\
&\qquad \beta = (2\alpha+1)q, \dots, (2\alpha+2)q - 1;\ \alpha = 0, \dots, \lfloor \ell/2 \rfloor - 1\}, \\
\mathbb{S}_{q,q\ell-1,\boldsymbol{\tau},\mathrm{mix}} &:= \{s \in \mathbb{S}_{q\ell-1,\boldsymbol{\tau}} : s^{(\beta_0)}(0) = s^{(\beta_1)}(1) = 0; \\
&\qquad \beta_0 = 2\alpha_0 q, \dots, (2\alpha_0+1)q - 1;\ \alpha_0 = 0, 1, \dots, \lceil \ell/2 \rceil - 1; \\
&\qquad \beta_1 = (2\alpha_1+1)q, \dots, (2\alpha_1+2)q - 1;\ \alpha_1 = 0, 1, \dots, \lfloor \ell/2 \rfloor - 1\},
\end{aligned} \tag{4.6}$$

are optimal spaces for $A^r_{q,\mathrm{Dir}}$, $A^r_{q,\mathrm{Neu}}$, and $A^r_{q,\mathrm{mix}}$, respectively, once a proper sequence of break points $\boldsymbol{\tau}$ is considered. Here $\lceil \cdot \rceil$ is the ceiling function and $\lfloor \cdot \rfloor$ is the floor function.

More precisely, in a complete similarity with the case $q = 2$ discussed in Section 3.3.1, let $u_{q,l,\mathrm{Dir}}$, $l = 1, 2, \dots$ be the non-trivial solutions of (4.1) with the Dirichlet boundary conditions (4.2). Then, $u^{(q)}_{q,l,\mathrm{Dir}}$, $l = 1, 2, \dots$ are solutions of the eigenvalue problem (4.1) with the Neumann boundary conditions (4.3). Furthermore, it can be verified that $u_{q,n+1,\mathrm{Dir}}$ has n simple zeros in $(0, 1)$, denoted by $\tau^0_{q,i,n,\mathrm{Dir}}$, $i = 1, 2, \dots, n$, and that $u^{(q)}_{q,n+1,\mathrm{Dir}}$ has $n+q$ simple zeros in $(0, 1)$, denoted by $\tau^q_{q,i,n,\mathrm{Dir}}$, $i = 1, 2, \dots, n+q$; see [16,26,28]. Similarly, $u_{q,n+1,\mathrm{mix}}$ and $u^{(q)}_{q,n+1,\mathrm{mix}}$ have n simple zeros in $(0, 1)$, denoted by $\tau^0_{q,i,n,\mathrm{mix}}$ and $\tau^q_{q,i,n,\mathrm{mix}}$, $i = 1, 2, \dots, n$, respectively; see [17,25]. Let us set

$$\boldsymbol{\tau}^0_{q,n,\mathrm{Dir}} := (0, \tau^0_{q,1,n,\mathrm{Dir}}, \dots, \tau^0_{q,n,n,\mathrm{Dir}}, 1), \quad \boldsymbol{\tau}^q_{q,n,\mathrm{Dir}} := (0, \tau^q_{q,1,n,\mathrm{Dir}}, \dots, \tau^q_{q,n+q,n,\mathrm{Dir}}, 1)$$

and

$$\boldsymbol{\tau}^0_{q,n,\mathrm{mix}} := (0, \tau^0_{q,1,n,\mathrm{mix}}, \dots, \tau^0_{q,n,n,\mathrm{mix}}, 1), \quad \boldsymbol{\tau}^q_{q,n,\mathrm{mix}} := (0, \tau^q_{q,1,n,\mathrm{mix}}, \dots, \tau^q_{q,n,n,\mathrm{mix}}, 1).$$

Then, the optimal break points are given by

$$\begin{aligned}
\boldsymbol{\tau}^{\mathrm{opt}}_{q,\ell,n,\mathrm{Dir}} &:= \begin{cases} \boldsymbol{\tau}^0_{q,n,\mathrm{Dir}}, & \ell \text{ even}, \\ \boldsymbol{\tau}^q_{q,n,\mathrm{Dir}}, & \ell \text{ odd}, \end{cases} \\
\boldsymbol{\tau}^{\mathrm{opt}}_{q,\ell,n,\mathrm{Neu}} &:= \begin{cases} \boldsymbol{\tau}^0_{q,n-q,\mathrm{Dir}}, & \ell \text{ odd}, \\ \boldsymbol{\tau}^q_{q,n-q,\mathrm{Dir}}, & \ell \text{ even}, \end{cases} \\
\boldsymbol{\tau}^{\mathrm{opt}}_{q,\ell,n,\mathrm{mix}} &:= \begin{cases} \boldsymbol{\tau}^0_{q,n,\mathrm{mix}}, & \ell \text{ even}, \\ \boldsymbol{\tau}^q_{q,n,\mathrm{mix}}, & \ell \text{ odd}, \end{cases}
\end{aligned} \tag{4.7}$$

and the corresponding spaces by

$$\mathbb{S}^{\mathrm{opt}}_{q,q\ell-1,n,\mathrm{BC}} := \mathbb{S}_{q,q\ell-1,\boldsymbol{\tau}^{\mathrm{opt}}_{q,\ell,n,\mathrm{BC}},\mathrm{BC}}, \quad \mathrm{BC} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{mix}\}. \tag{4.8}$$

All the spline spaces in (4.8) have dimension n and they are optimal spaces for the function classes (4.5). From Proposition A.1, Corollaries A.1 and A.2, and Example A.4 in Appendix, we obtain that the spline spaces in (4.8), with $\ell \geq 2$, provide outlier-free Galerkin discretizations for the polyharmonic problem (4.1) equipped with the boundary conditions (4.2), (4.3), or (4.4).

Remark 4.1. The break points in (4.7) were numerically studied in [29]. It turns out that their qualitative behavior is very similar to the case $q = 2$ discussed in Remark 3.1.

Remark 4.2. For $q = 1$ we reproduce the analysis made in [18] for optimal spline spaces leading to outlier-free discretizations for the eigenvalue problem related to the Laplace operator. In this case, the spaces in (4.8) are those in [18, Section 4]. For a proper comparison with [18], note that for $q = 1$ the degree of the spline space is $p = \ell - 1$.

5. Numerical examples

In this section, we consider some numerical tests to show the potential of outlier-free spline spaces. For the sake of brevity, we mainly focus on fourth-order problems with Dirichlet boundary conditions. We will solve the problem (2.17) by means of Galerkin discretizations in different spaces with the same dimension n, so as to make a fair comparison. Let $\omega_{h,l,\mathrm{Dir}}$ be the approximate value of the frequency $\omega_{l,\mathrm{Dir}} := \vartheta^2_{l,\mathrm{Dir}}$ for $l = 1, \ldots, n$; see (2.13). Here we assume that the exact frequencies and the approximate ones are sorted in increasing order to retrieve their matching. For each space, we will compute the relative frequency errors

$$e_{\omega,l,\mathrm{Dir}} := \frac{\omega_{h,l,\mathrm{Dir}} - \omega_{l,\mathrm{Dir}}}{\omega_{l,\mathrm{Dir}}}, \quad l = 1, \ldots, n, \tag{5.1}$$

and the corresponding L^2 relative errors for the eigenfunction approximations

$$e_{u,l,\mathrm{Dir}} := \frac{\|u_{l,\mathrm{Dir}} - u_{h,l,\mathrm{Dir}}\|}{\|u_{l,\mathrm{Dir}}\|}, \quad l = 1, \ldots, n. \tag{5.2}$$

Let us start by considering the standard spline space $\mathbb{S}^{\mathrm{bih}}_{p,\boldsymbol{\tau},\mathrm{Dir}}$ defined in (2.19). Taking the uniform sequence of break points,

$$\boldsymbol{\tau}^{\mathrm{bih}}_{p,n,\mathrm{Dir}} := \left(0, \frac{1}{n_{\mathrm{el}}}, \frac{2}{n_{\mathrm{el}}}, \ldots, 1\right), \quad n_{\mathrm{el}} := n - p + 4,$$

the space

$$\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}} := \mathbb{S}^{\mathrm{bih}}_{p,\boldsymbol{\tau}^{\mathrm{bih}}_{p,n,\mathrm{Dir}},\mathrm{Dir}} \tag{5.3}$$

has dimension n.

Example 5.1. In Figs. 1(a,b) and 2(a,b), we depict the relative frequency errors (5.1) obtained by the Galerkin approximations of the problem (2.17) in the spline spaces $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$ with $n = 200$ and $p = 3, 4, \ldots, 10$. We clearly notice that there are few outliers and their number increases with the degree p. In Figs. 1(c,d) and 2(c,d), we report the corresponding L^2 relative errors for the eigenfunction approximations (5.2).

In Section 3.3, we have discussed several proposals for outlier-free spline spaces of dimension n, namely $\mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Dir}}$ in (3.8) for odd degrees $p = 2\ell - 1$ and $\mathbb{S}^{\mathrm{good}}_{2\ell,n,\mathrm{Dir}}$ in (3.15) for even degrees $p = 2\ell$. However, the break points needed for those spline spaces are not explicitly known. Therefore, for practical purposes, we suggest to use the following simple approximations of those points, inspired by the estimates in Remarks 3.1 and 3.3:

$$\boldsymbol{\tau}^{\mathrm{free}}_{p,n,\mathrm{Dir}} := \begin{cases} \left(0, \frac{1/2}{n+1}, \frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}, \frac{n+1/2}{n+1}, 1\right), & p = 4k-3, \\ \left(0, \frac{1/2}{n+1}, \frac{3/2}{n+1}, \ldots, \frac{n+1/2}{n+1}, 1\right), & p = 4k-2, \\ \left(0, \frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}, 1\right), & p = 4k-1, \\ \left(0, \frac{1/2}{n+1}, \frac{3/2}{n+1}, \ldots, \frac{n+1/2}{n+1}, 1\right), & p = 4k. \end{cases} \tag{5.4}$$

Note that for $p = 4k - 3$ we could have taken the break points $\frac{1/3}{n+1}$ and $\frac{n+2/3}{n+1}$ in correspondence with Remark 3.1. Yet, the choice of the break points $\frac{1/2}{n+1}$ and $\frac{n+1/2}{n+1}$ works equally well in practice and is more nicely distributed. Given the sequences of break points in (5.4), we define the spaces

$$\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}} := \mathbb{S}_{p,\boldsymbol{\tau}^{\mathrm{free}}_{p,n,\mathrm{Dir}},\mathrm{Dir}}. \tag{5.5}$$

Since $\mathbb{S}^{\mathrm{free}}_{2\ell-1,n,\mathrm{Dir}}$ is close to $\mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Dir}}$ and $\mathbb{S}^{\mathrm{free}}_{2\ell,n,\mathrm{Dir}}$ is close to $\mathbb{S}^{\mathrm{good}}_{2\ell,n,\mathrm{Dir}}$, they are good candidates to provide outlier-free discretizations and this is confirmed in the next example.

Example 5.2. In Figs. 3(a) and 4(a), we depict the relative frequency errors (5.1) obtained by the Galerkin approximations of the problem (2.17) in the spline spaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ with $n = 200$ and $p = 3, 4, \ldots, 10$. We clearly notice that these spaces capture all the frequencies without any outlier. In Figs. 3(c) and 4(c), we report the corresponding L^2 relative errors for the eigenfunction approximations (5.2). Finally, in Fig. 5, we show a comparison

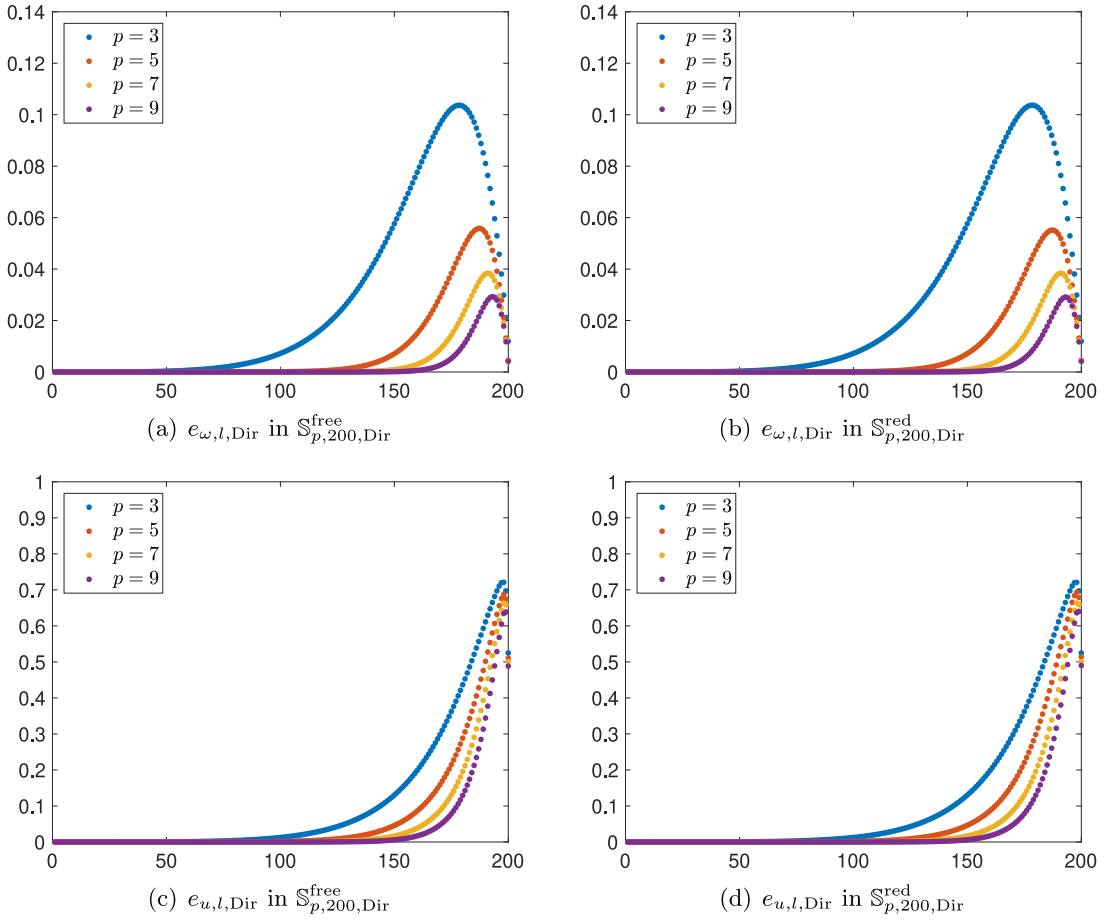


Fig. 3. Examples 5.2 and 5.3: Relative frequency errors $e_{\omega,l,\mathrm{Dir}}$ in (5.1) and L^2 relative eigenfunction errors $e_{u,l,\mathrm{Dir}}$ in (5.2) corresponding to different spline spaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ and $\mathbb{S}^{\mathrm{red}}_{p,n,\mathrm{Dir}}$ for odd degrees p and $n = 200$. The errors are ordered according to increasing exact frequencies. No outliers are observed for both $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ and $\mathbb{S}^{\mathrm{red}}_{p,n,\mathrm{Dir}}$.

of the relative frequency errors for the spaces $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$ and $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ in logarithmic scale, so as to visualize better the behavior in the first half of the spectrum for both types of spaces. Except for the outliers, we see a very good match for error values larger than about 10^{-10}, while the differences for smaller error values might be due to numerical imprecisions. The spaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ seem to suffer less from numerical imprecisions.

Now, we consider the reduced spline spaces used for outlier removal in [15]. Let $\overline{\mathbb{S}}_{p,\boldsymbol{\tau},\mathrm{Dir}}$ be the spline subspaces defined in a similar way as $\mathbb{S}_{p,\boldsymbol{\tau},\mathrm{Dir}}$ in (3.5) for $p = 2\ell - 1$ and in (3.14) for $p = 2\ell$ but where a strict inequality is used in the upper bound of β, i.e., $\beta < p$ instead of $\beta \leq p$. Then, the reduced spline spaces in [15] are defined as

$$\mathbb{S}^{\mathrm{red}}_{p,n,\mathrm{Dir}} := \overline{\mathbb{S}}_{p,\boldsymbol{\tau}^{\mathrm{red}}_{p,n,\mathrm{Dir}},\mathrm{Dir}}, \tag{5.6}$$

where

$$\boldsymbol{\tau}^{\mathrm{red}}_{p,n,\mathrm{Dir}} := \begin{cases} \left(0, \frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}, 1\right), & p = 4k-3, \\ \left(0, \frac{1}{n+2}, \frac{2}{n+2}, \ldots, \frac{n+1}{n+2}, 1\right), & p = 4k-2, \\ \left(0, \frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}, 1\right), & p = 4k-1, \\ \left(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right), & p = 4k. \end{cases}$$

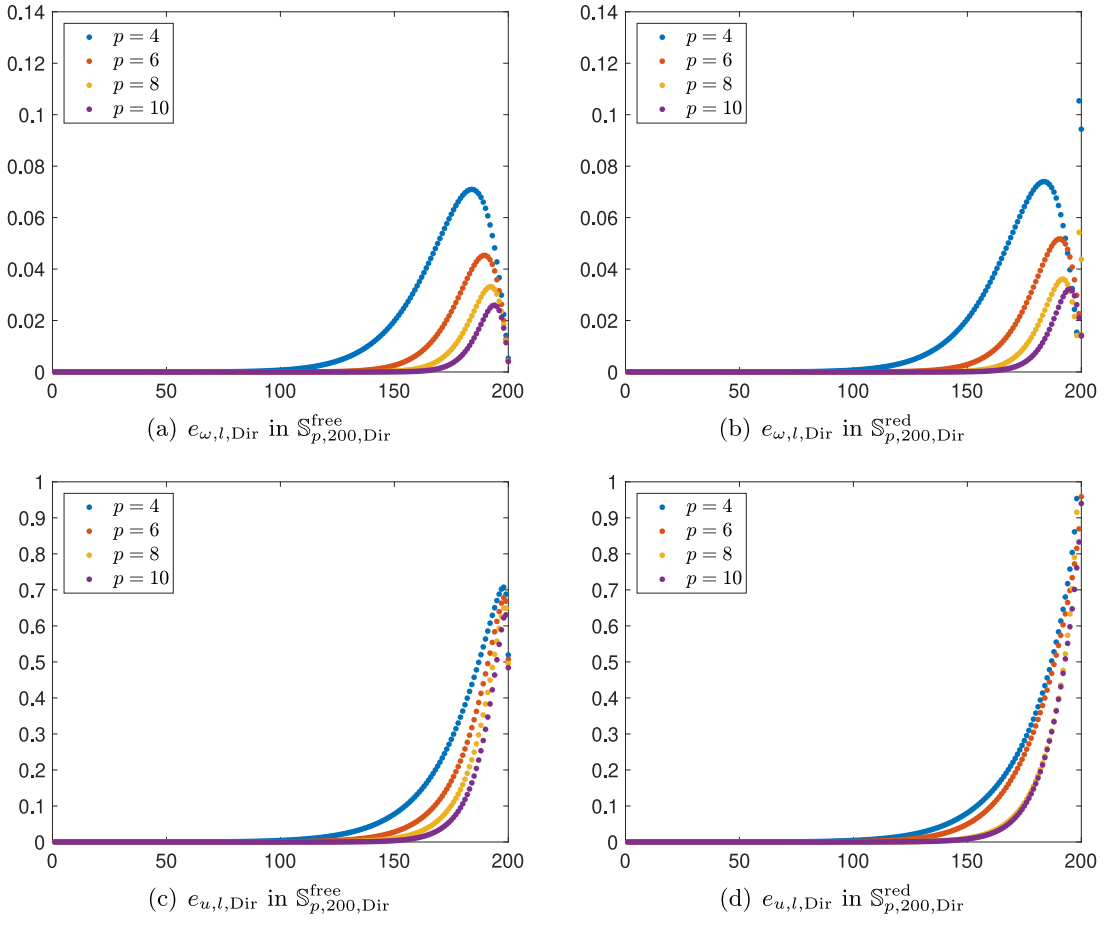


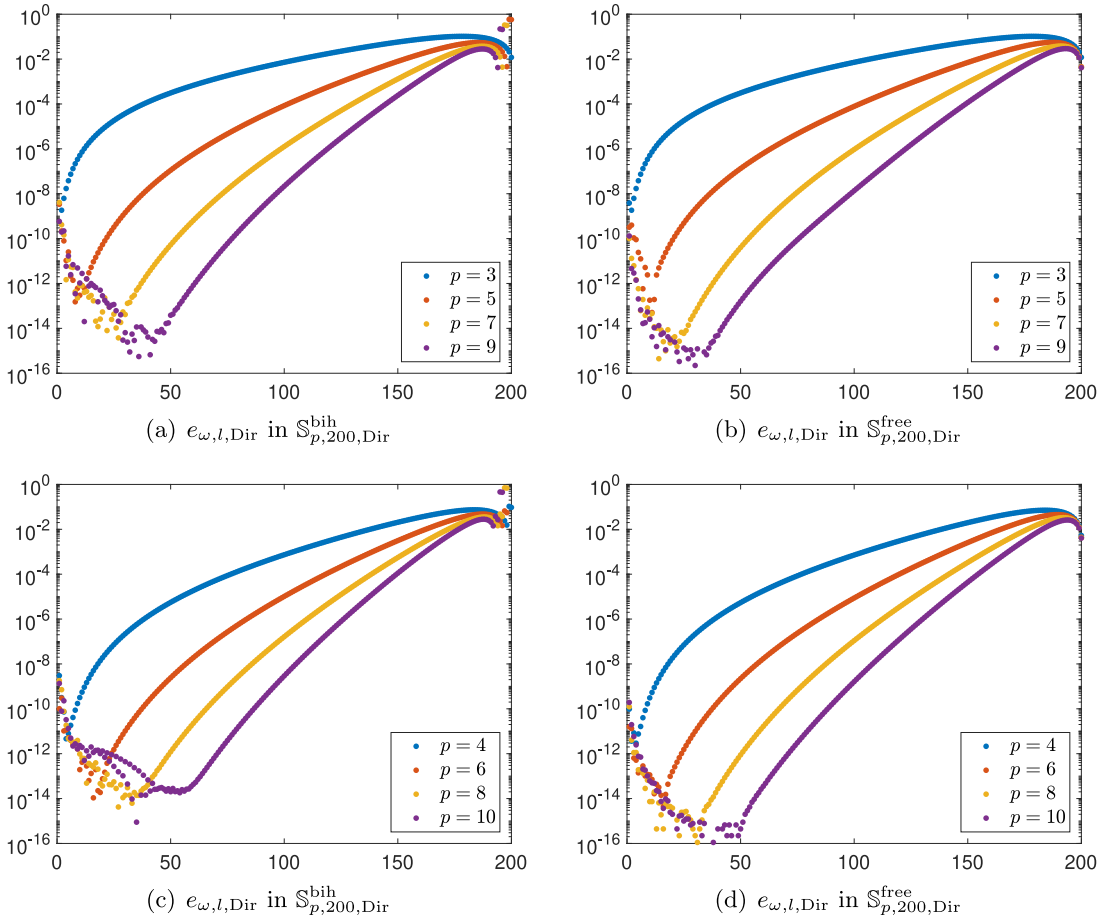
Fig. 4. Examples 5.2 and 5.3: Relative frequency errors $e_{\omega,l,\mathrm{Dir}}$ in (5.1) and L^2 relative eigenfunction errors $e_{u,l,\mathrm{Dir}}$ in (5.2) corresponding to different spline spaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ and $\mathbb{S}^{\mathrm{red}}_{p,n,\mathrm{Dir}}$ for even degrees p and $n = 200$. The errors are ordered according to increasing exact frequencies. No outliers are observed for $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$, while there are still very few outliers present for $\mathbb{S}^{\mathrm{red}}_{p,n,\mathrm{Dir}}$.

Note that all the spaces in (5.6) have dimension n. It can be verified that $\mathbb{S}^{\mathrm{red}}_{p,n,\mathrm{Dir}} = \mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ only for $p = 4k - 1$; the other spaces are different from our outlier-free spline spaces (see also Remarks 3.2 and 3.4). Finally, note that $\mathbb{S}^{\mathrm{red}}_{4,n,\mathrm{Dir}} = \mathbb{S}^{\mathrm{bih}}_{4,n,\mathrm{Dir}}$.

Example 5.3. In Figs. 3(b) and 4(b), we depict the relative frequency errors (5.1) obtained by the Galerkin approximations of the problem (2.17) in the spline spaces $\mathbb{S}^{\mathrm{red}}_{p,n,\mathrm{Dir}}$ with $n = 200$ and $p = 3, 4, \ldots, 10$. We see that the odd-degree spaces do not exhibit outliers, very similar to the spaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ of the same degree. However, the even-degree spaces are not completely outlier-free (there are still two outliers for $p = 4$ and $p = 8$). Moreover, the latter spaces present slightly higher errors for the high frequencies than the spaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ of the same degree (especially for $p = 6$ and $p = 10$). In Figs. 3(d) and 4(d), we report the corresponding L^2 relative errors for the eigenfunction approximations (5.2).

Lastly, we consider a triharmonic problem, namely the problem (4.1) with $q = 3$, subject to the Dirichlet boundary conditions (4.2). As before, we will compare the performance of standard spline spaces against outlier-free spline spaces of the same degree p and the same dimension n. Taking the uniform sequence of break points,

$$\boldsymbol{\tau}^{\mathrm{trih}}_{p,n,\mathrm{Dir}} := \left(0, \frac{1}{n_{\mathrm{el}}}, \frac{2}{n_{\mathrm{el}}}, \ldots, 1\right), \quad n_{\mathrm{el}} := n - p + 6,$$



(a) $e_{\omega,l,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{bih}}_{p,200,\mathrm{Dir}}$ (b) $e_{\omega,l,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{p,200,\mathrm{Dir}}$

(c) $e_{\omega,l,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{bih}}_{p,200,\mathrm{Dir}}$ (d) $e_{\omega,l,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{p,200,\mathrm{Dir}}$

Fig. 5. Example 5.2: Relative frequency errors $e_{\omega,l,\mathrm{Dir}}$ in (5.1) corresponding to different spline spaces $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$ and $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ for odd and even degrees p and $n = 200$ in logarithmic scale. The errors are ordered according to increasing exact frequencies. The spaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ seem to suffer less from numerical imprecisions.

we define $\mathbb{S}^{\mathrm{trih}}_{p,n,\mathrm{Dir}}$ as

$$\mathbb{S}^{\mathrm{trih}}_{p,n,\mathrm{Dir}} := \{s \in \mathbb{S}_{p,\boldsymbol{\tau}^{\mathrm{trih}}_{p,n,\mathrm{Dir}}} : s(0) = s(1) = s^{(1)}(0) = s^{(1)}(1) = s^{(2)}(0) = s^{(2)}(1) = 0\}. \tag{5.7}$$

We focus only on the degrees $p = 3\ell - 1$ with $\ell \geq 2$ as prescribed by the optimal spaces $\mathbb{S}^{\mathrm{opt}}_{3,p,n,\mathrm{Dir}}$ in (4.8). Moreover, since the break points needed for those spline spaces are not explicitly known, we suggest to use the following simple choice,

$$\boldsymbol{\tau}^{\mathrm{free}}_{3,p,n,\mathrm{Dir}} := \begin{cases} \left(0, \frac{1/2}{n+2}, \frac{1}{n+2}, \frac{2}{n+2}, \ldots, \frac{n+1}{n+2}, \frac{n+3/2}{n+2}, 1\right), & p = 6k-4, \\ \left(0, \frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}, 1\right), & p = 6k-1, \end{cases} \tag{5.8}$$

in the same spirit as the odd-degree cases in the biharmonic scenario described in (5.4). Given the sequences of break points in (5.8), we define the spaces

$$\mathbb{S}^{\mathrm{free}}_{3,p,n,\mathrm{Dir}} := \mathbb{S}_{3,p,\boldsymbol{\tau}^{\mathrm{free}}_{3,p,n,\mathrm{Dir}},\mathrm{Dir}} \tag{5.9}$$

as our outlier-free candidates. Note that all the spaces in (5.7) and (5.9) have dimension n. We will solve the triharmonic problem under consideration by means of Galerkin discretizations in all those spaces. Let $\omega_{l,\mathrm{Dir}}$ be the approximate value of $\omega_{l,\mathrm{Dir}} := \vartheta^{q}_{l,\mathrm{Dir}}$ in (4.1) with $q = 3$. For each space, we will compute the relative frequency errors (5.1) and the corresponding L^2 relative errors for the eigenfunction approximations (5.2).

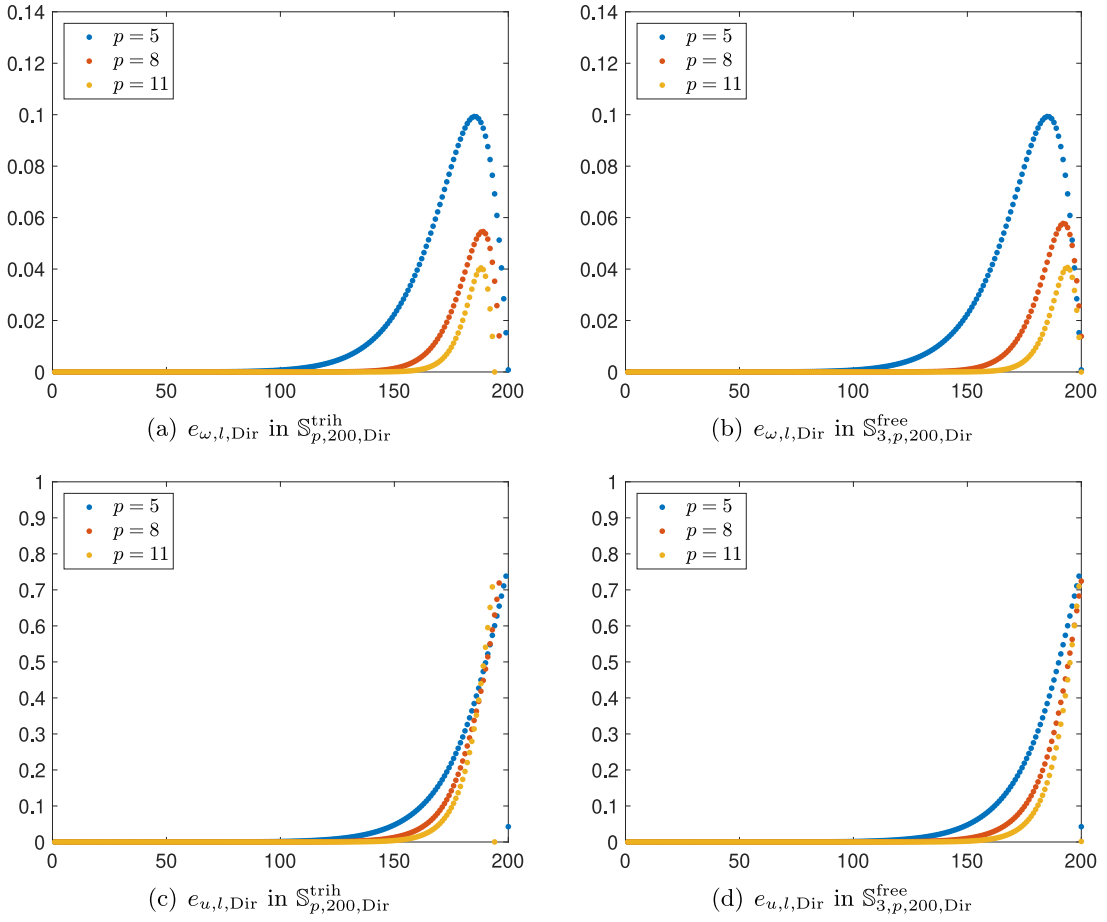


Fig. 6. Example 5.4 (triharmonic problem): Relative frequency errors $e_{\omega,l,\mathrm{Dir}}$ in (5.1) corresponding to different spline spaces $\mathbb{S}^{\mathrm{trih}}_{p,n,\mathrm{Dir}}$ and $\mathbb{S}^{\mathrm{free}}_{3,p,n,\mathrm{Dir}}$ for degrees $p = 3\ell - 1$ and $n = 200$. The errors are ordered according to increasing exact frequencies. No outliers are observed for $\mathbb{S}^{\mathrm{free}}_{3,p,n,\mathrm{Dir}}$, while there are few outliers present for $\mathbb{S}^{\mathrm{trih}}_{p,n,\mathrm{Dir}}$. The outliers are outside the visible range in (a) and (c); they are not shown for clarity of the figure and comparison.

Example 5.4. In Fig. 6(a), we depict the relative frequency errors (5.1) obtained by the Galerkin approximations of the triharmonic problem under consideration in the spline spaces $\mathbb{S}^{\mathrm{trih}}_{p,n,\mathrm{Dir}}$ with $n = 200$ and $p = 5, 8, 11$. Few outliers are present in the cases $p = 8, 11$, but they are outside the visible range of the figure (similar to Example 5.1). Note that their number increases with the degree p, namely there are 4 outliers for $p = 8$ and 6 outliers for $p = 11$. In Fig. 6(b), we depict the relative frequency errors for the spline spaces $\mathbb{S}^{\mathrm{free}}_{3,p,n,\mathrm{Dir}}$ for the same values of n and p. We clearly notice that these spaces capture all the frequencies without any outlier. In Figs. 6(c,d), we report the corresponding L^2 relative errors for the eigenfunction approximations (5.2).

6. Fourth-order eigenvalue problems: Multivariate tensor-product case

The eigenvalue problem related to the biharmonic operator in the unit cube of $\mathbb{R}^d$, $d \geq 2$, namely

$$\Delta^2 u = \vartheta^4 u, \quad \text{in } (0,1)^d,$$

has explicit analytical solutions only for special boundary conditions, e.g., for the so-called simply supported boundary conditions corresponding to the first set of Laplacian boundary conditions described in (2.4).

To illustrate the behavior of the outlier-free spaces in higher dimensions, following [15, Section 3.4], we consider a simplified fourth-order eigenvalue problem where the biharmonic operator is replaced by

$$\sum_{k=1}^{d} \frac{\partial^4}{\partial x_k^4}, \tag{6.1}$$

for which we can easily derive analytical solutions. For the sake of brevity and simplicity of presentation, we only focus on Dirichlet boundary conditions and $d = 2$,

$$\begin{cases} \displaystyle\sum_{k=1}^{2} \frac{\partial^4}{\partial x_k^4} u = \vartheta^4 u, \quad \text{in } \Omega := (0,1)^2, \\ u_{|\partial\Omega} = \boldsymbol{\nu} \cdot \nabla u_{|\partial\Omega} = 0, \end{cases} \tag{6.2}$$

with $\boldsymbol{\nu}$ the unit outer normal to $\partial\Omega$. The non-trivial exact solutions are given by

$$u_{l_1,l_2,\mathrm{Dir}}(x_1,x_2) := u_{l_1,\mathrm{Dir}}(x_1) u_{l_2,\mathrm{Dir}}(x_2), \quad \vartheta^4_{l_1,l_2,\mathrm{Dir}} := \vartheta^4_{l_1,\mathrm{Dir}} + \vartheta^4_{l_2,\mathrm{Dir}}, \quad l_1, l_2 = 1, 2, \ldots \tag{6.3}$$

where $u_{l_1,\mathrm{Dir}}$, $u_{l_2,\mathrm{Dir}}$ and $\vartheta_{l_1,\mathrm{Dir}}$, $\vartheta_{l_2,\mathrm{Dir}}$ are the univariate counterparts in (2.13) and (2.14).

In order to discretize the problem (6.2) with the Galerkin method, it is natural to consider a finite-dimensional tensor-product discretization space $\mathbb{V}_{h_1,h_2} := \mathbb{V}_{h_1} \otimes \mathbb{V}_{h_2}$. Due to the special tensor-product structure of the discretization, very similar to (6.3), our approximations take the form

$$u_{h_1,h_2,l_1,l_2,\mathrm{Dir}}(x_1,x_2) = u_{h_1,l_1,\mathrm{Dir}}(x_1) u_{h_2,l_2,\mathrm{Dir}}(x_2), \quad \vartheta^4_{h_1,h_2,l_1,l_2,\mathrm{Dir}} = \vartheta^4_{h_1,l_1,\mathrm{Dir}} + \vartheta^4_{h_2,l_2,\mathrm{Dir}}, \tag{6.4}$$

where $u_{h_1,l_1,\mathrm{Dir}}$, $u_{h_2,l_2,\mathrm{Dir}}$ and $\vartheta_{h_1,l_1,\mathrm{Dir}}$, $\vartheta_{h_2,l_2,\mathrm{Dir}}$ are the univariate counterparts described in Section 2.2; see [18, Section 2.2] and [15, Appendix D]. The univariate results of Section 3 immediately extend towards higher dimensions for the operator in (6.1) in the unit cube.

Finally, we end with some numerical tests in the bivariate setting. We consider the eigenvalue problem (6.2) and approximately solve it by means of Galerkin discretizations in the tensor-product spline spaces $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$ and $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$, where the univariate spaces are defined in (5.3) and (5.5), respectively. All these spaces have the same dimension n^2. Let $\omega_{h_1,h_2,l_1,l_2,\mathrm{Dir}}$ be the approximate value of the frequency $\omega_{l_1,l_2,\mathrm{Dir}} := \vartheta^2_{l_1,l_2,\mathrm{Dir}}$ for $l_1, l_2 = 1, \ldots, n$; see (6.3). For each space, we will compute the relative frequency errors

$$e_{\omega,l_1,l_2,\mathrm{Dir}} := \frac{\omega_{h_1,h_2,l_1,l_2,\mathrm{Dir}} - \omega_{l_1,l_2,\mathrm{Dir}}}{\omega_{l_1,l_2,\mathrm{Dir}}}, \quad l_1, l_2 = 1, \ldots, n, \tag{6.5}$$

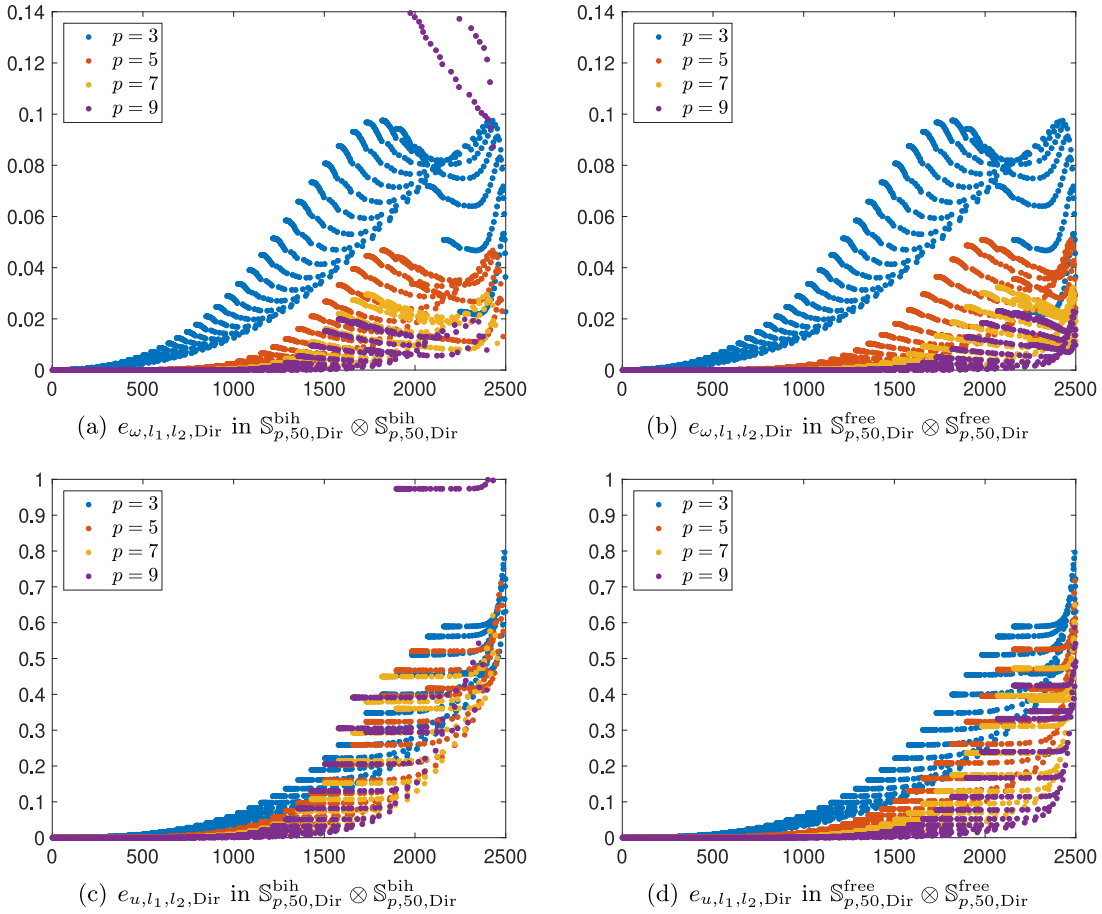
and the corresponding L^2 relative errors for the eigenfunction approximations

$$e_{u,l_1,l_2,\mathrm{Dir}} := \frac{\|u_{l_1,l_2,\mathrm{Dir}} - u_{h_1,h_2,l_1,l_2,\mathrm{Dir}}\|}{\|u_{l_1,l_2,\mathrm{Dir}}\|}, \quad l_1, l_2 = 1, \ldots, n. \tag{6.6}$$

We follow the matching strategy between the exact frequencies and the approximate ones explained in [18, Remark 7.1] (see also [15]).

Example 6.1. In Figs. 7(a) and 8(a), we depict the relative frequency errors (6.5) obtained by the Galerkin approximations of the problem (6.2) in the spline spaces $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$ with $n = 50$ and $p = 3, 4, \ldots, 10$. There is a "thin layer" of outliers and their number increases with the degree p. Note that most of the outliers are outside the visible range of the figures (see Example 5.1 for an illustration of the magnitude of the outliers in the univariate case). In Figs. 7(c) and 8(c), we report the corresponding L^2 relative errors for the eigenfunction approximations (6.6).

Example 6.2. In Figs. 7(b) and 8(b), we depict the relative frequency errors (6.5) obtained by the Galerkin approximations of the problem (6.2) in the spline spaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ with $n = 50$ and $p = 3, 4, \ldots, 10$. There are no outliers present. In Figs. 7(d) and 8(d), we report the corresponding L^2 relative errors for the eigenfunction approximations (6.6). Finally, we remark that the frequency errors and eigenfunction errors seem to approximately lie on several smooth subcurves instead of a single smooth curve as in the univariate case (see Example 5.2). This can be simply explained by the fact that they actually lie on a bivariate smooth surface due to the bivariate nature of the problem; see (6.3) and (6.4). This is illustrated in Figs. 9 and 10 for various degrees.



(a) $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{bih}}_{p,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{bih}}_{p,50,\mathrm{Dir}}$

(b) $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{p,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{p,50,\mathrm{Dir}}$

(c) $e_{u,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{bih}}_{p,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{bih}}_{p,50,\mathrm{Dir}}$

(d) $e_{u,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{p,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{p,50,\mathrm{Dir}}$

Fig. 7. Examples 6.1 and 6.2: Relative frequency errors $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in (6.5) and L^2 relative eigenfunction errors $e_{u,l_1,l_2,\mathrm{Dir}}$ in (6.6) corresponding to different spline spaces $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$ and $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ for odd degrees p and $n = 50$. The errors are ordered according to increasing exact frequencies. No outliers are observed for $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$, while there is a "thin layer" of outliers present for $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$. Most of the outliers are outside the visible range in (a) and (c); they are not shown for clarity of the figure and comparison.

7. Conclusion

In this paper, we have presented and analyzed outlier-free Galerkin discretizations of eigenvalue problems related to the biharmonic and the polyharmonic operator, subject to all standard boundary conditions, in the univariate setting. This is a continuation and extension of a similar study for the Laplace operator in [18]. The discretizations are based on suitable spline subspaces exploiting the property of optimality in the sense of the L^2 Kolmogorov n-widths for special function classes related to the eigenvalue problems under investigation. It turns out that, for a fixed number of degrees of freedom, there are no spurious values in the approximated spectrum and all the eigenfunctions and the corresponding eigenvalues are well approximated, without loss of accuracy in the whole spectrum when compared to the full spline space.

The starting point of our investigation was the family of optimal spline spaces presented in [16] for the polyharmonic operator with Neumann boundary conditions. Other kinds of boundary conditions can be treated in a similar way. The considered optimal spline spaces are identified by simple additional homogeneous boundary conditions and their break points are given in terms of the zeros of the eigenfunctions of the problems we are dealing with. Therefore, they are explicitly known only for second-order problems (unless special boundary conditions are considered). It turns out that uniform break points, or a slight variation of them, lead to spaces of practical interest that still offer outlier-free discretizations; see Section 5.

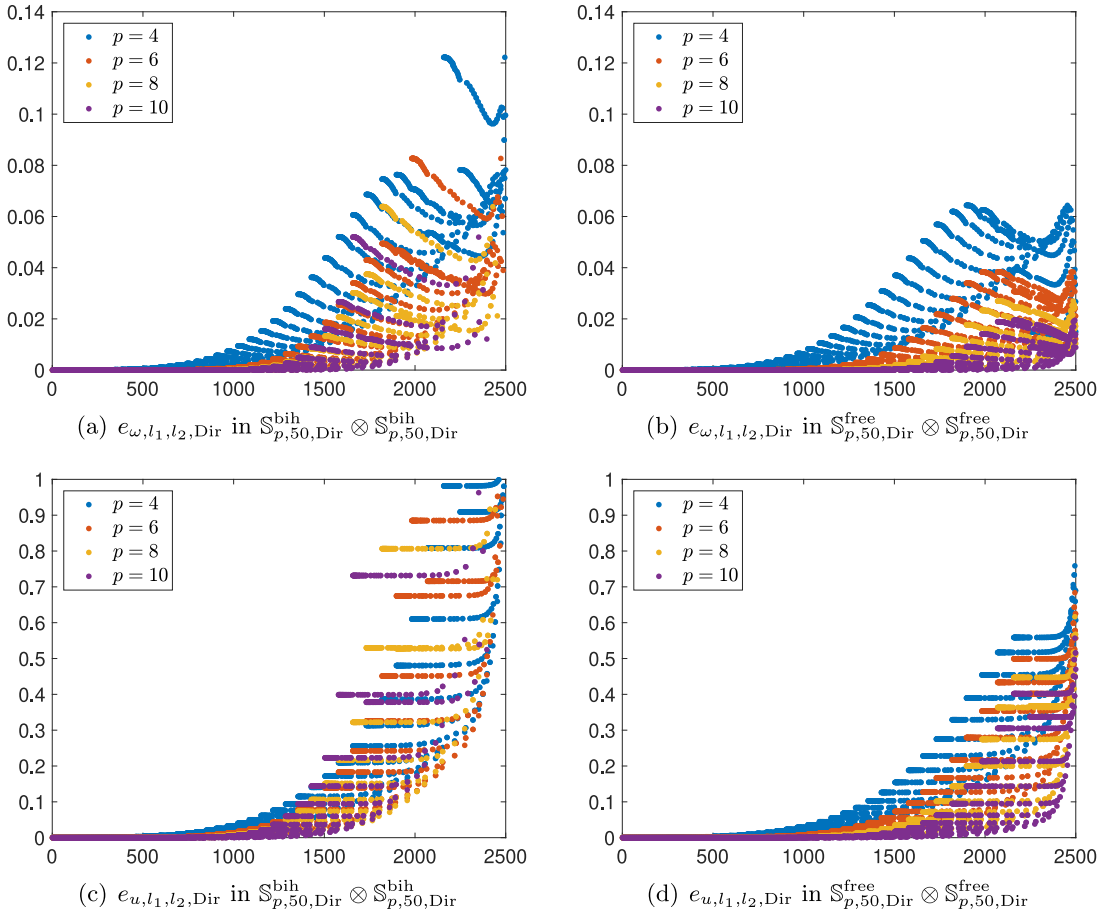
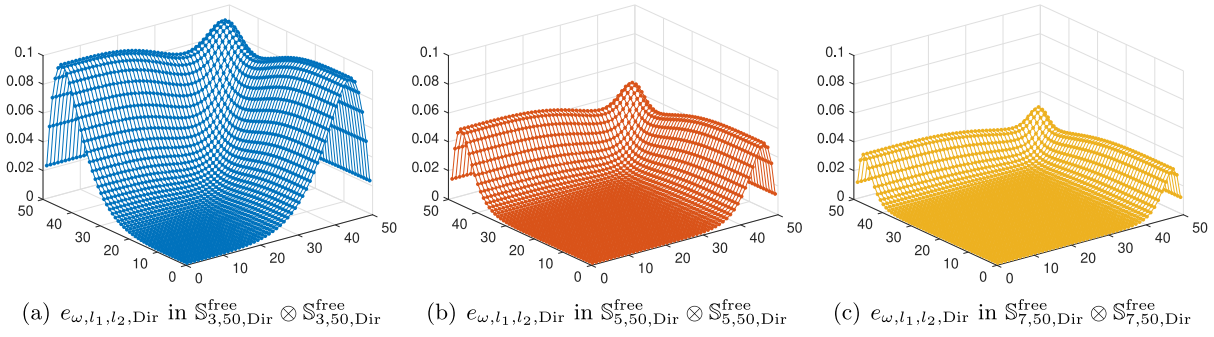


Fig. 8. Examples 6.1 and 6.2: Relative frequency errors $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in (6.5) and L^2 relative eigenfunction errors $e_{u,l_1,l_2,\mathrm{Dir}}$ in (6.6) corresponding to different spline spaces $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$ and $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ for even degrees p and $n = 50$. The errors are ordered according to increasing exact frequencies. No outliers are observed for $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$, while there is a "thin layer" of outliers present for $\mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{bih}}_{p,n,\mathrm{Dir}}$. Most of the outliers are outside the visible range in (a) and (c); they are not shown for clarity of the figure and comparison.

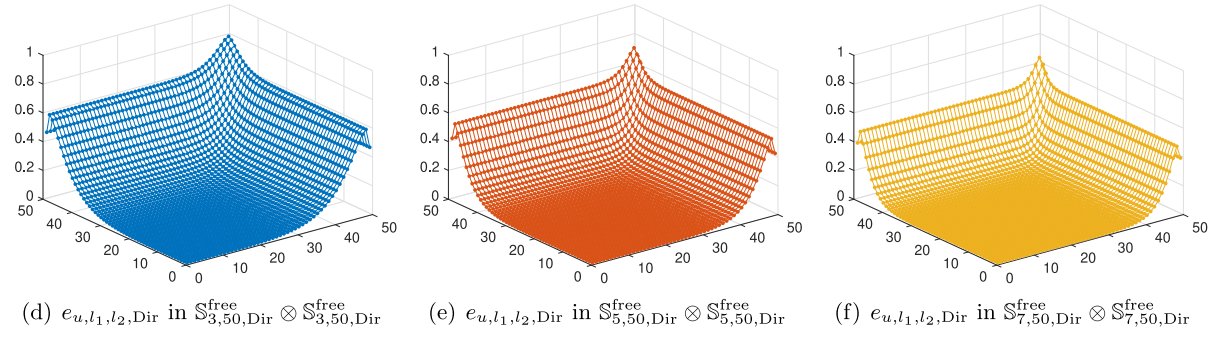
For the q-harmonic operator, subject to one of the boundary conditions (4.2)–(4.4), only optimal spline spaces of degree $q\ell - 1$, $\ell = 1, 2, \ldots$ are available. This implies that for the Laplace operator we have at our disposal optimal spline spaces of any degree, but this is not the case anymore (or at least it is not known) for higher-order problems. In particular, for the biharmonic operator, only optimal spline spaces of odd degree are available. Nevertheless, a careful analysis of necessary conditions for a space to be optimal has revealed also spline subspaces of even degree that provide outlier-free discretizations; see Sections 3 and 5. The latter spaces might be good candidates to be optimal or at least close to optimal.

The spline subspaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ in (5.5) – and which we propose for practical purposes – are very similar (and actually identical in some cases) to the reduced spline spaces used for outlier removal in [15]. For a given type of boundary conditions, however, the reduced spline spaces in [15] can differ from ours in the choice of break points and in the maximum order of vanishing derivatives at the boundary; see Sections 3 and 5. These differences motivate the improvement we achieve over some results in [15].

For practical implementations, a proper basis of the discretization spaces is of primal interest. A global basis for each optimal spline space $\mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}}$ in (3.8) can be obtained by using the Green's function of the related eigenvalue problem; see [16]. Local bases are usually preferred in practice. The full spline space $\mathbb{S}_{p,\boldsymbol{\tau}}$ in (1.1) is commonly represented in terms of the classical B-spline basis which is defined through a knot sequence and, when working with uniform grids, splines can be efficiently expressed in terms of cardinal B-splines. B-spline like

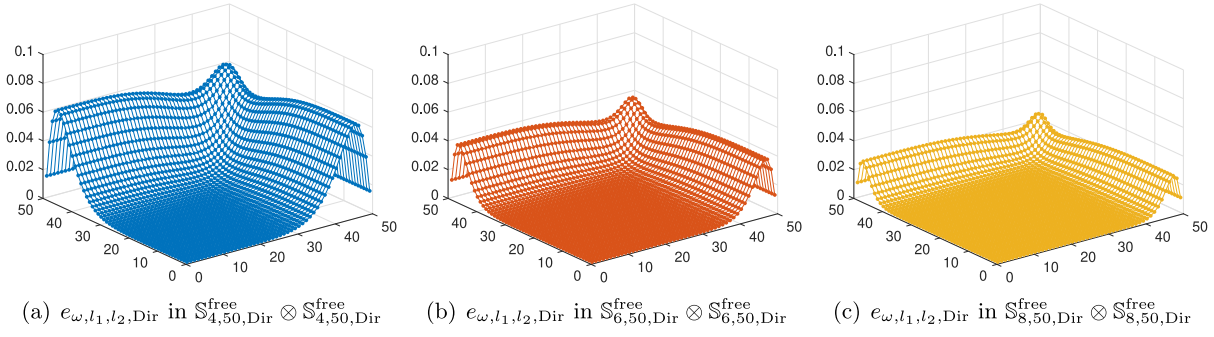


(a) $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{3,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{3,50,\mathrm{Dir}}$ (b) $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{5,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{5,50,\mathrm{Dir}}$ (c) $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{7,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{7,50,\mathrm{Dir}}$

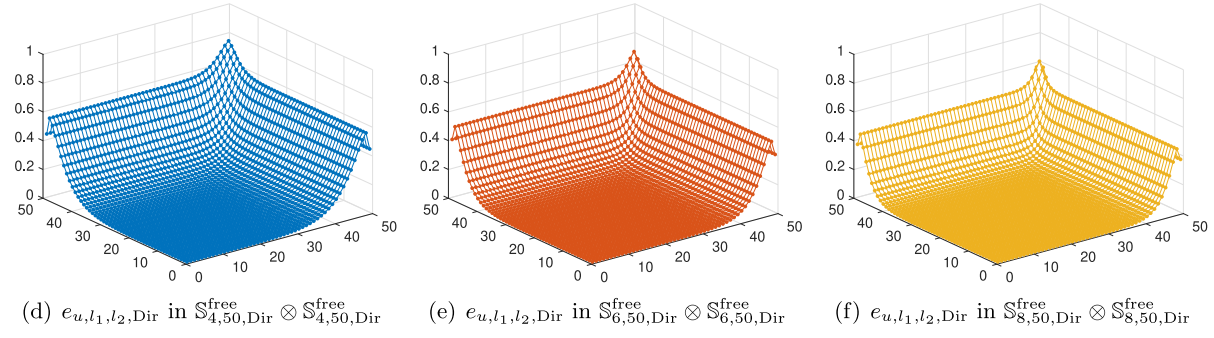


(d) $e_{u,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{3,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{3,50,\mathrm{Dir}}$ (e) $e_{u,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{5,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{5,50,\mathrm{Dir}}$ (f) $e_{u,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{7,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{7,50,\mathrm{Dir}}$

Fig. 9. Example 6.2: Relative frequency errors $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in (6.5) and L^2 relative eigenfunction errors $e_{u,l_1,l_2,\mathrm{Dir}}$ in (6.6) corresponding to different spline spaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ for odd degrees p and $n = 50$. The errors are ordered according to the decomposition of the exact frequencies in (6.3) in terms of increasing values of their univariate counterparts. No outliers are observed.



(a) $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{4,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{4,50,\mathrm{Dir}}$ (b) $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{6,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{6,50,\mathrm{Dir}}$ (c) $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{8,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{8,50,\mathrm{Dir}}$



(d) $e_{u,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{4,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{4,50,\mathrm{Dir}}$ (e) $e_{u,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{6,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{6,50,\mathrm{Dir}}$ (f) $e_{u,l_1,l_2,\mathrm{Dir}}$ in $\mathbb{S}^{\mathrm{free}}_{8,50,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{8,50,\mathrm{Dir}}$

Fig. 10. Example 6.2: Relative frequency errors $e_{\omega,l_1,l_2,\mathrm{Dir}}$ in (6.5) and L^2 relative eigenfunction errors $e_{u,l_1,l_2,\mathrm{Dir}}$ in (6.6) corresponding to different spline spaces $\mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}} \otimes \mathbb{S}^{\mathrm{free}}_{p,n,\mathrm{Dir}}$ for even degrees p and $n = 50$. The errors are ordered according to the decomposition of the exact frequencies in (6.3) in terms of increasing values of their univariate counterparts. No outliers are observed.

bases expressed in terms of cardinal B-splines were provided in [18, Section 6] in the case of the Laplace operator. Similar constructions can be provided for the spline spaces in (3.5) when using uniformly spaced break points. In the numerical tests we have used an algorithmic construction in the spirit of MDB-spline extraction as in [15]; see also [30,31].

The optimal spline spaces in (3.8) provide outlier-free approximations for the eigenfunctions of the corresponding eigenvalue problems still maintaining the accuracy of the full spline space. Indeed, the additional boundary conditions identifying such subspaces are satisfied by the exact eigenfunctions. However, it is clear that their direct use, or the use of the similar subspaces in Section 5, will result in a loss of accuracy for approximating solutions that do not satisfy such boundary conditions. Nevertheless, for general high-order problems, in the spirit of [18, Section 5] (see also [15, Appendix B] for the univariate case), suitable reduction strategies for boundary conditions can be applied so that the full approximation is achieved. Such strategies are a non-trivial generalization of the standard procedure to reduce non-homogeneous Dirichlet boundary conditions to homogeneous ones.

Contrary to the Laplace operator, the analysis of the multivariate extension of the biharmonic (or polyharmonic) operator is quite involved, even for simple domains, and analytical solutions are available only for very special boundary conditions. Therefore, while the tensor-product extensions of the spline subspaces we are proposing is straightforward, a comparison with the exact solutions is in general impossible. In this perspective, in [15] a simplified fourth-order operator as in (6.1) has been considered for the multivariate setting. For such a simplified operator, both the exact and approximated solutions enjoy a simple tensor-product structure, so the univariate results immediately extend to the multivariate case.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix. Convergence to eigenvalues and eigenfunctions

We extend the results of [18, Appendix A] (see also [19, Section 7]) in order to prove that there are no outliers in the Galerkin eigenvalue approximation for abstract polyharmonic eigenvalue problems when using optimal spline subspaces. We deliberately follow the presentation in [18, Appendix A] to stress the abstract similarities between the Laplacian and the general polyharmonic case. An important contribution of this appendix is a sequence of examples (see Examples A.1–A.4), where the specification of suitable integral operators allows us to frame instances of interest in the general abstract setting.

For $f \in L^2$, let K be the integral operator

$$Kf(x) := \int_a^b K(x, y) f(y)\, \mathrm{d}y.$$

As in [25], we use the notation $K(x, y)$ for the kernel of K. We will only consider kernels that are continuous or piecewise continuous. We denote by K^* the adjoint, or dual, of the operator K, defined by

$$(f, K^* g) = (Kf, g).$$

The operators K^*K and KK^* are self-adjoint, positive semi-definite, and have eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq \cdots \geq 0,$$

with corresponding orthonormal eigenfunctions

$$K^*K\phi_l = \lambda_l \phi_l, \quad l = 1, 2, \ldots, \tag{A.1}$$

$$KK^*\psi_l = \lambda_l \psi_l, \quad l = 1, 2, \ldots. \tag{A.2}$$

For $\lambda_l > 0$ we have

$$\psi_l = (\lambda_l)^{-1/2} K\phi_l, \quad \phi_l = (\lambda_l)^{-1/2} K^*\psi_l. \tag{A.3}$$

Following [17–19], for an arbitrary integral operator K, we consider the function classes

$$A_K^1 := A_K := K(B), \quad A_{K^*}^1 := A_{K^*} := K^*(B),$$

and, for $r \geq 2$,

$$A_K^r := K(A_{K^*}^{r-1}), \quad A_{K^*}^r := K^*(A_K^{r-1}), \tag{A.4}$$

where B denotes the unit ball in $L^2(a, b)$. As stated in [17, Theorem 3], we have that

$$d_n(A_{K^*}^r) = d_n(A_K^r) = d_n(A_K)^r = d_n(A_{K^*})^r = (\lambda_{n+1})^{r/2}, \tag{A.5}$$

and the space $[\psi_1, \ldots, \psi_n]$ is optimal for A_K^r, while the space $[\phi_1, \ldots, \phi_n]$ is optimal for $A_{K^*}^r$, for all $r \geq 1$. Moreover, let $\mathbb{X}_0$ and $\mathbb{Y}_0$ be any finite-dimensional subspaces of L^2 and define the subspaces $\mathbb{X}_{\bar{p}}$ and $\mathbb{Y}_{\bar{p}}$ in an analogous way to (A.4), by

$$\mathbb{X}_{\bar{p}} := K(\mathbb{Y}_{\bar{p}-1}), \quad \mathbb{Y}_{\bar{p}} := K^*(\mathbb{X}_{\bar{p}-1}), \tag{A.6}$$

for $\bar{p} \geq 1$. Finally, for any $\bar{p} \geq 0$, let $X_{\bar{p}}$ be the L^2-projector onto $\mathbb{X}_{\bar{p}}$ and let $Y_{\bar{p}}$ be the L^2-projector onto $\mathbb{Y}_{\bar{p}}$.

From [17, Theorem 4] we know that if $\mathbb{X}_0$ is optimal for the n-width of A_K and $\mathbb{Y}_0$ is optimal for the n-width of A_{K^*} then, for $r \geq 1$,

- the subspaces $\mathbb{X}_{\bar{p}}$ are optimal for the n-width of A_K^r, and
- the subspaces $\mathbb{Y}_{\bar{p}}$ are optimal for the n-width of $A_{K^*}^r$,

for all $\bar{p} \geq r - 1$. Therefore, if $\mathbb{X}_0$ is optimal for the n-width of A_K and $\mathbb{Y}_0$ is optimal for the n-width of A_{K^*} then from the definition of optimal space (see Section 3) and (A.5) we immediately get for all $\bar{p} \geq r - 1$,

$$\|(I - X_{\bar{p}})u\| \leq d_n(A_K)^r = (\lambda_{n+1})^{r/2}, \quad \forall u \in A_K^r, \tag{A.7}$$

$$\|(I - Y_{\bar{p}})u\| \leq d_n(A_{K^*})^r = (\lambda_{n+1})^{r/2}, \quad \forall u \in A_{K^*}^r. \tag{A.8}$$

In the following we consider the non-trivial case $\lambda_{n+1} > 0$. The relations (A.3) imply that

$$(\lambda_l)^{r/2} \psi_l \in A_K^r, \quad (\lambda_l)^{r/2} \phi_l \in A_{K^*}^r, \quad \forall r \geq 1. \tag{A.9}$$

Thus, when taking $r = \bar{p} + 1$, (A.7)–(A.8) give us the following result [19, Theorem 7.1].

Theorem A.1. *Suppose $\mathbb{X}_0$ is optimal for the n-width of A_K and $\mathbb{Y}_0$ is optimal for the n-width of A_{K^*}. Let $X_{\bar{p}}$ be the L^2-projector onto $\mathbb{X}_{\bar{p}}$ and $Y_{\bar{p}}$ be the L^2-projector onto $\mathbb{Y}_{\bar{p}}$. Then,*

$$\|(I - X_{\bar{p}})\psi_l\| \leq \left(\frac{\lambda_{n+1}}{\lambda_l}\right)^{(\bar{p}+1)/2}, \qquad \|(I - Y_{\bar{p}})\phi_l\| \leq \left(\frac{\lambda_{n+1}}{\lambda_l}\right)^{(\bar{p}+1)/2}.$$

Let ∂ denote the derivative operator. From now on, we consider integral operators K such that, for given $q \geq 1$,

$$\partial^q K\big|_{A_{K^*}^r} = I, \quad \partial^q K^*\big|_{A_K^r} = (-1)^q I, \quad r \geq 0, \tag{A.10}$$

where $A_K^0 := \{u \in B : u \perp \ker(K^*)\}$ and $A_{K^*}^0 := \{u \in B : u \perp \ker(K)\}$. We further assume

$$\mathbb{X}_0 \perp \ker(K^*), \quad \mathbb{Y}_0 \perp \ker(K). \tag{A.11}$$

This assumption is true in all non-trivial cases when dealing with optimal spaces.

Then, for $\bar{p} \geq 1$ we define the Ritz projectors

$$R_{\mathbb{X}_{\bar{p}}} : K(L^2(a,b)) \to \mathbb{X}_{\bar{p}}, \quad R_{\mathbb{X}_{\bar{p}}} := KY_{\bar{p}-1}\partial^q, \tag{A.12}$$

$$R_{\mathbb{Y}_{\bar{p}}} : K^*(L^2(a,b)) \to \mathbb{Y}_{\bar{p}}, \quad R_{\mathbb{Y}_{\bar{p}}} := (-1)^q K^* X_{\bar{p}-1}\partial^q. \tag{A.13}$$

Note that from (A.10), for any $u \in K(L^2(a,b))$, the projector $R_{\mathbb{X}_{\bar{p}}}$ satisfies

$$(\partial^q R_{\mathbb{X}_{\bar{p}}} u, \partial^q v) = (\partial^q u, \partial^q v), \quad \forall v \in \mathbb{X}_{\bar{p}},$$

and similarly, for any $u \in K^*(L^2(a,b))$,

$$(\partial^q R_{\mathbb{Y}_{\bar{p}}} u, \partial^q v) = (\partial^q u, \partial^q v), \quad \forall v \in \mathbb{Y}_{\bar{p}}.$$

The projections in (A.12) and (A.13) satisfy the stability estimates

$$\|\partial^q R_{\mathbb{X}_{\bar{p}}} u\| = \|Y_{\bar{p}-1}\partial^q u\| \leq \|\partial^q u\|,$$

$$\|\partial^q R_{\mathbb{Y}_{\bar{p}}} v\| = \|X_{\bar{p}-1}\partial^q v\| \leq \|\partial^q v\|,$$

for all $u \in K(L^2(a,b))$ and $v \in K^*(L^2(a,b))$.

Using Theorem A.1 we now arrive at the following proposition, whose proof is completely analogous to [18, Proposition A.1].

Proposition A.1. *Suppose $\mathbb{X}_0$ is optimal for the n-width of A_K and $\mathbb{Y}_0$ is optimal for the n-width of A_{K^*}. Let $R_{\mathbb{X}_{\bar{p}}}$ and $R_{\mathbb{Y}_{\bar{p}}}$ be the projectors in (A.12) and (A.13). Then, for $\bar{p} \geq \max\{1, r-1\}$,*

$$\|u - R_{\mathbb{X}_{\bar{p}}} u\| \leq (\lambda_{n+1})^{r/2}, \quad \|\partial^q u - \partial^q R_{\mathbb{X}_{\bar{p}}} u\| \leq (\lambda_{n+1})^{(r-1)/2}, \quad \forall u \in A_K^r,$$

$$\|u - R_{\mathbb{Y}_{\bar{p}}} u\| \leq (\lambda_{n+1})^{r/2}, \quad \|\partial^q u - \partial^q R_{\mathbb{Y}_{\bar{p}}} u\| \leq (\lambda_{n+1})^{(r-1)/2}, \quad \forall u \in A_{K^*}^r.$$

Observe that the case $r = 1$ of the above proposition implies the stability estimates

$$\|R_{\mathbb{X}_{\bar{p}}} u\| \leq \|u\| + \|R_{\mathbb{X}_{\bar{p}}} u - u\| \leq \|u\| + \lambda_{n+1}^{1/2} \|\partial^q u\|,$$

$$\|R_{\mathbb{Y}_{\bar{p}}} v\| \leq \|v\| + \|R_{\mathbb{Y}_{\bar{p}}} v - v\| \leq \|v\| + \lambda_{n+1}^{1/2} \|\partial^q v\|,$$

for all $u \in K(L^2(a,b))$ and $v \in K^*(L^2(a,b))$.

Using (A.3) we get

$$\partial^q \psi_l = \lambda_l^{-1/2} \phi_l, \quad \partial^q \phi_l = (-1)^q \lambda_l^{-1/2} \psi_l,$$

so that

$$\|\partial^q \psi_l\| = \|\partial^q \phi_l\| = \lambda_l^{-1/2}.$$

From (A.9) and Proposition A.1, when taking $r = \bar{p} + 1$, we obtain the following result.

Corollary A.1. *Suppose $\mathbb{X}_0$ is optimal for the n-width of A_K and $\mathbb{Y}_0$ is optimal for the n-width of A_{K^*}. Let $R_{\mathbb{X}_{\bar{p}}}$ and $R_{\mathbb{Y}_{\bar{p}}}$ be the projectors in (A.12) and (A.13). Then, for $\bar{p} \geq 1$,*

$$\frac{\|(I - R_{\mathbb{X}_{\bar{p}}})\psi_l\|}{\|\psi_l\|} \leq \left(\frac{\lambda_{n+1}}{\lambda_l}\right)^{(\bar{p}+1)/2}, \quad \frac{\|\partial^q (I - R_{\mathbb{X}_{\bar{p}}})\psi_l\|}{\|\partial^q \psi_l\|} \leq \left(\frac{\lambda_{n+1}}{\lambda_l}\right)^{\bar{p}/2},$$

$$\frac{\|(I - R_{\mathbb{Y}_{\bar{p}}})\phi_l\|}{\|\phi_l\|} \leq \left(\frac{\lambda_{n+1}}{\lambda_l}\right)^{(\bar{p}+1)/2}, \quad \frac{\|\partial^q (I - R_{\mathbb{Y}_{\bar{p}}})\phi_l\|}{\|\partial^q \phi_l\|} \leq \left(\frac{\lambda_{n+1}}{\lambda_l}\right)^{\bar{p}/2}.$$

Example A.1. Let $(a,b) = (0,1)$ and consider the function classes $A_{1,\mathrm{BC}}^r$, $\mathrm{BC} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{mix}\}$ defined in Section 4, with $q = 1$. This case was already discussed in [18, Example A.1]. Setting

$$K_1 f(x) := \int_0^x f(y)\,\mathrm{d}y, \tag{A.14}$$

we get

$$K_1^* f(x) = \int_x^1 f(y)\,\mathrm{d}y.$$

It was shown in [17] that the function classes $A^r_{1,\mathrm{mix}}$ in (4.5) are the function classes A^r_K in (A.4); see also [18, Section 4]. The remaining classes $A^r_{1,\mathrm{Dir}}$ and $A^r_{1,\mathrm{Neu}}$ associated with Dirichlet and Neumann boundary conditions, respectively, can be obtained by using the operator

$$Z_1 := (I - P_0)K_1, \tag{A.15}$$

where P_0 denotes the L^2-projector onto constant functions and K_1 is given in (A.14). Then, see [17, Section 8] and [18, Section 4],

$$A^r_{1,\mathrm{Neu}} = \mathbb{P}_0 \oplus A^r_{Z_1}, \quad A^r_{1,\mathrm{Dir}} = A^r_{Z_1^*}.$$

From [17, Theorem 2] we know that the spaces in (4.8), with $q = 1$, are optimal spaces of the form (A.6), with $\bar{p} = \ell - 1$, for the function classes $A^r_{1,\mathrm{Dir}}$ and $A^r_{1,\mathrm{mix}}$, for $r \geq 1$ and $\ell \geq r$; see also [18, Section 4]. On the other hand, the optimal spaces for Neumann boundary conditions are of the form $\mathbb{P}_0 \oplus \mathbb{X}_{\ell-1}$. Note that the integral operators in (A.14) and (A.15) satisfy the conditions in (A.10) and the spline spaces satisfy assumption (A.11). Therefore, Proposition A.1 holds for the function classes in Section 4 equipped with the optimal spaces (4.8) taking $q = 1$; see also [18, Section 4 and Example A.1].

Example A.2. Consider the function classes $A^r_{\mathrm{Lap}_i}$, $i = 1, 2, 3$ defined in Section 3.2. Here $q = 2$, however, the relevant integral operators are $K_1K_1^*$, $Z_1Z_1^*$, and $Z_1^*Z_1$ from the previous example. It thus follows from the definition in (A.4) that the function classes satisfy

$$A^r_{\mathrm{Lap}_0} = A^{2r}_{1,\mathrm{Dir}}, \quad A^r_{\mathrm{Lap}_1} = A^{2r}_{1,\mathrm{Neu}}, \quad A^r_{\mathrm{Lap}_2} = A^{2r}_{1,\mathrm{mix}},$$

and the optimal spaces are as described in Example A.1.

Example A.3. The function classes A^r_{Dir} and A^r_{mix} defined in Section 3.3 are examples of the function classes $A^r_{K^*}$ and A^r_K in (A.4), while A^r_{Neu} is of the form $\mathbb{P}_1 \oplus A^r_K$ for proper choices of the integral operator K. More precisely, considering $q = 2$, let K_2 be the integral operator

$$K_2 f(x) := \int_0^x \int_0^y f(z)\,\mathrm{d}z\mathrm{d}y = \int_0^x (x - z) f(z)\,\mathrm{d}z = \int_0^1 (x - z)_+ f(z)\,\mathrm{d}z, \tag{A.16}$$

where $w_+ := \max(w, 0)$. This implies

$$K_2^* f(x) = \int_x^1 (z - x) f(z)\,\mathrm{d}z = \int_0^1 (z - x)_+ f(z)\,\mathrm{d}z.$$

Then, the function class $A^r_{K_2}$ equals the class A^r_{mix} corresponding to the case of mixed boundary conditions described in Section 3. Now, let Z_2 be the operator defined by

$$Z_2 := (I - P_1)K_2, \tag{A.17}$$

where P_1 denotes the L^2-projector onto linear functions. Then, the function class $\mathbb{P}_1 \oplus A^r_{Z_2}$ equals the class A^r_{Neu} corresponding to Neumann boundary conditions. Finally, the class $A^r_{Z_2^*}$ equals the class A^r_{Dir} which corresponds to the case of Dirichlet boundary conditions described in Section 3; see also Section 4 taking $q = 2$. On this concern, it is important to note that

$$\begin{aligned} A^1_{\mathrm{Dir}} &= \{u \in H^2 : \|u^{(2)}\| \leq 1,\ u(0) = u(1) = u^{(1)}(0) = u^{(1)}(1) = 0\} \\ &= Z_2^*(\{f \in L^2 : \|f\| \leq 1,\ f \perp 1,\ f \perp x\}) = Z_2^*(B). \end{aligned}$$

From [16,17] we know that the spaces $\mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{BC}}$ in (3.8), with $\mathrm{BC} \in \{\mathrm{Dir}, \mathrm{mix}\}$, are optimal spaces of the form (A.6), with $\bar{p} = \ell - 1$, for $r \geq 1$ and $\ell \geq r$. On the other hand, the space $\mathbb{S}^{\mathrm{opt}}_{2\ell-1,n,\mathrm{Neu}}$ is of the form $\mathbb{P}_1 \oplus \mathbb{X}_{\ell-1}$; it is optimal for A^r_{Neu} for $\ell \geq r$. We observe that the integral operators in (A.16) and (A.17) satisfy the conditions in (A.10) with $q = 2$ and the spline spaces satisfy assumption (A.11). Therefore, Proposition A.1 holds for the function classes in Section 3 equipped with the optimal spaces (3.8); see also Section 4 taking $q = 2$.

Example A.4. Consider the function classes $A^r_{q,\mathrm{BC}}$, $\mathrm{BC} \in \{\mathrm{Dir}, \mathrm{Neu}, \mathrm{mix}\}$ defined in Section 4 for general $q \geq 1$. Let K_q be the integral operator

$$K_q f(x) := \frac{1}{(q-1)!} \int_0^x (x - z)^{q-1} f(z)\,\mathrm{d}z = \frac{1}{(q-1)!} \int_0^1 (x - z)_+^{q-1} f(z)\,\mathrm{d}z, \tag{A.18}$$

thus implying

$$K_q^* f(x) = \frac{1}{(q-1)!}\int_x^1 (z-x)^{q-1} f(z)\,\mathrm{d}z = \frac{1}{(q-1)!}\int_0^1 (z-x)_+^{q-1} f(z)\,\mathrm{d}z.$$

By a direct computation (see also [25]), we obtain for the function classes $A^r_{q,\mathrm{mix}}$ in (4.5) that

$$A^r_{q,\mathrm{mix}} = A^r_{K_q}.$$

Now, set

$$Z_q := (I - P_{q-1})K_q, \tag{A.19}$$

where P_q denotes the L^2-projector onto $\mathbb{P}_q$. From [16, Lemma 3] we deduce that $u := Z_q^* Z_q f$ is the unique solution to the boundary value problem

$$\begin{cases} (-1)^q u^{(2q)} = f, & \text{in } (0,1), \\ u^{(i)}(0) = u^{(i)}(1) = 0, & i = 0, 1, \dots, q-1. \end{cases}$$

Then, the function class $A^r_{Z_q^*}$ equals the class $A^r_{q,\mathrm{Dir}}$ corresponding to the case of Dirichlet boundary conditions described in Section 4. Similarly, from [16, Lemma 4] it follows that if f is orthogonal to $\mathbb{P}_{q-1}$ then $u := Z_q Z_q^* f$ is the unique solution to the boundary value problem

$$\begin{cases} (-1)^q u^{(2q)} = f, & \text{in } (0,1), \\ u^{(i)}(0) = u^{(i)}(1) = 0, & i = q, q+1, \dots, 2q-1. \end{cases}$$

Then, the function class $A^r_{q,\mathrm{Neu}}$ described in Section 4 is of the form $\mathbb{P}_{q-1} \oplus A^r_{Z_q}$. The spaces $\mathbb{S}^{\mathrm{opt}}_{q,q\ell-1,n,\mathrm{Neu}}$ in (4.6) are of the form $\mathbb{P}_{q-1} \oplus \mathbb{X}_{\ell-1}$. They are optimal for $A^r_{q,\mathrm{Neu}}$ for $\ell \geq r$; see [16, Theorem 2]. Similarly, the spaces $\mathbb{S}^{\mathrm{opt}}_{q,q\ell-1,n,\mathrm{Dir}}$ in (4.8) are optimal spaces of the form $\mathbb{Y}_{\bar{p}}$ in (A.6), with $\bar{p} = \ell - 1$, for $r \geq 1$ and $\ell \geq r$. We observe that the integral operators in (A.18) and (A.19) satisfy the conditions in (A.10) and the spline spaces satisfy assumption (A.11). Therefore, Proposition A.1 holds for the function classes in Section 4 equipped with the optimal spaces mentioned above.

Let us now focus on the following eigenvalue problem: find $\psi \in K(L^2(a,b))$ and $\mu \in \mathbb{R}$ such that

$$(\partial^q \psi, \partial^q v) = \mu(\psi, v), \quad \forall v \in K(L^2(a,b)). \tag{A.20}$$

Its discretization is given by: find $\psi_{\bar{p}} \in \mathbb{X}_{\bar{p}}$ and $\mu_{\bar{p}} \in \mathbb{R}$ such that

$$(\partial^q \psi_{\bar{p}}, \partial^q v) = \mu_{\bar{p}}(\psi_{\bar{p}}, v), \quad \forall v \in \mathbb{X}_{\bar{p}}. \tag{A.21}$$

Note that, if ϕ_l, λ_l are as in (A.1) then $K\phi_l$ and $\mu_l := \frac{1}{\lambda_l}$ solve the eigenvalue problem (A.20). We order the eigenvalues in (A.20) and (A.21) increasingly.

Combining Corollary A.1 with a classical argument from [32] we can prove an error estimate for the eigenvalues, but first we need the following lemma, whose proof is completely analogous to [18, Lemma A.1].

Lemma A.1. *Let $\mathbb{X}$ be any n-dimensional subspace such that the error estimate*

$$\|u - Pu\| \leq c\,\|\partial^q u\|, \quad \forall u \in [\psi_1, \dots, \psi_n],$$

holds for some projector P onto $\mathbb{X}$. Then, for any $l \leq n$ such that $\mu_l^{-1/2} > c$, the dimension of the range-space $P([\psi_1, \dots, \psi_l])$ is equal to l.

Corollary A.2. *For any $l = 1, \dots, n$, let μ_l be the l-th eigenvalue of (A.20) and let $\mu_{\bar{p},l}$ be the l-th eigenvalue of (A.21). Assume $\mu_1 \leq \cdots \leq \mu_n < \mu_{n+1}$. If $\mathbb{X}_0$ is optimal for the n-width of A_K and $\mathbb{Y}_0$ is optimal for the n-width of A_{K^*}, then*

$$\mu_l \leq \mu_{\bar{p},l} \leq \frac{\mu_l}{\left(1 - \left(\frac{\mu_l}{\mu_{n+1}}\right)^{(\bar{p}+1)/2}\right)^2}, \quad \bar{p} \geq 1. \tag{A.22}$$

Proof. As a consequence of the min–max theorem we have

$$\mu_l = \min_{\mathbb{E}\in\mathcal{V}_l} \max_{v\in\mathbb{E}} \frac{\|\partial^q v\|^2}{\|v\|^2}, \quad \mu_{\tilde{p},l} = \min_{\mathbb{E}\in\mathcal{V}_{\tilde{p},l}} \max_{v\in\mathbb{E}} \frac{\|\partial^q v\|^2}{\|v\|^2}, \tag{A.23}$$

where $\mathcal{V}_l$ is the set of all subspaces of $K(L^2(a,b))$ of dimension l, and $\mathcal{V}_{\tilde{p},l}$ is the set of all subspaces of $\mathbb{X}_{\tilde{p}}$ of dimension l. It immediately follows that $\mu_l \leq \mu_{\tilde{p},l}$. We note that for μ_l the minimum in (A.23) is achieved for $\mathbb{E} = [\psi_1, \ldots, \psi_l]$. Next we define the range-space $\mathbb{I} := R_{\mathbb{X}_{\tilde{p}}}([\psi_1, \ldots, \psi_l])$. Since $\mathbb{X}_{\tilde{p}}$ is an optimal subspace we deduce from Lemma A.1 that $\dim \mathbb{I} = l$. By choosing $\mathbb{I}$ in the formula for $\mu_{\tilde{p},l}$ in (A.23) we obtain

$$\begin{aligned}\mu_{\tilde{p},l} &\leq \max_{w\in\mathbb{I}} \frac{\|\partial^q w\|^2}{\|w\|^2} = \max_{v\in[\psi_1,\ldots,\psi_l]} \frac{\|\partial^q R_{\mathbb{X}_{\tilde{p}}} v\|^2}{\|R_{\mathbb{X}_{\tilde{p}}} v\|^2} \leq \max_{v\in[\psi_1,\ldots,\psi_l]} \frac{\|\partial^q v\|^2}{\|R_{\mathbb{X}_{\tilde{p}}} v\|^2} = \max_{v\in[\psi_1,\ldots,\psi_l]} \frac{\|\partial^q v\|^2}{\|v\|^2} \frac{\|v\|^2}{\|R_{\mathbb{X}_{\tilde{p}}} v\|^2} \\ &\leq \mu_l \max_{v\in[\psi_1,\ldots,\psi_l]} \frac{\|v\|^2}{\|R_{\mathbb{X}_{\tilde{p}}} v\|^2}.\end{aligned}$$

By Corollary A.1 we have that $\|v - R_{\mathbb{X}_{\tilde{p}}} v\| \leq (\mu_l/\mu_{n+1})^{(\tilde{p}+1)/2} \|v\|$ for all $v \in [\psi_1, \ldots, \psi_l]$, and so

$$\|R_{\mathbb{X}_{\tilde{p}}} v\| \geq \|v\| \left(1 - \left(\frac{\mu_l}{\mu_{n+1}}\right)^{(\tilde{p}+1)/2}\right),$$

which immediately gives the upper bound of $\mu_{\tilde{p},l}$ in (A.22). This completes the proof. □

For any $l = 1, \ldots, n$, we define the l-th separation constant by

$$\rho_l := \max_{\substack{i=1,\ldots,n \\ i\neq l}} \frac{\mu_l}{|\mu_l - \mu_{\tilde{p},i}|}.$$

From [32, Theorem 6.2] we obtain the following error estimate for the eigenfunctions.

Corollary A.3. *For any $l = 1, \ldots, n$, let ψ_l be the l-th eigenfunction of (A.20) and let $\psi_{\tilde{p},l}$ be the l-th eigenfunction of (A.21). Assume $\mu_1 < \cdots < \mu_n < \mu_{n+1}$. If $\mathbb{X}_0$ is optimal for the n-width of A_K and $\mathbb{Y}_0$ is optimal for the n-width of A_{K^*}, then*

$$\|\psi_l - \psi_{\tilde{p},l}\| \leq 2(1+\rho_l)\|\psi_l - R_{\mathbb{X}_{\tilde{p}}}\psi_l\| \leq 2(1+\rho_l)\left(\frac{\mu_l}{\mu_{n+1}}\right)^{(\tilde{p}+1)/2}.$$

Results analogous to Corollaries A.2 and A.3 hold for the following eigenvalue problem: find $\phi \in K^*(L^2(a,b))$ and $\mu \in \mathbb{R}$ such that

$$(\partial^q \phi, \partial^q v) = \mu(\phi, v), \quad \forall v \in K^*(L^2(a,b));$$

and its discretization given by: find $\phi_{\tilde{p}} \in \mathbb{Y}_{\tilde{p}}$ and $\mu_{\tilde{p}} \in \mathbb{R}$ such that

$$(\partial^q \phi_{\tilde{p}}, \partial^q v) = \mu_{\tilde{p}}(\phi_{\tilde{p}}, v), \quad \forall v \in \mathbb{Y}_{\tilde{p}}.$$

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