The Bałaban variational problem in the non-linear sigma model

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Dedicated to the memory of Huzihiro Araki.

Abstract

The minimization of the action of a QFT with a constraint dictated by the block averaging procedure is an important part of Bałaban's approach to renormalization. It is particularly interesting for QFTs with non-trivial target spaces, such as gauge theories or non-linear sigma models on a lattice. We analyze this step for the $O(4)$ non-linear sigma model in two dimensions and demonstrate, in this case, how various ingredients of Bałaban's approach play together. First, using variational calculus on Lie groups, the equation for the critical point is derived. Then, this non-linear equation is solved by the Banach contraction mapping theorem. This step requires detailed control of lattice Green functions and their integral kernels via random walk expansions.

Contents

1 Introduction

The traditional strategy towards the construction of a non-trivial QFT in four dimensions had been to start from the $P(\phi)_2$ models, then tackle the ϕ^4-interaction in three dimensions, and finally attack the ϕ^4_4 model as the seemingly simplest interacting theory in spacetime of physical dimension. However, since the early 80s evidence started accumulating that ϕ^4_4 is actually trivial [Ai82, AD21, Fr82]. The attention shifted towards asymptotically free theories, such as the Gross-Neveu model in two dimensions (GN_2), the non-linear sigma model in two dimensions (NLSM_2), and the Yang-Mills theory in four dimensions (YM_4). While the GN_2 is by now mathematically well understood [DR00, FMRS86, GK85, DY23, Du24], only partial results on the latter two models have been obtained [GK86, PR91, BJ86, MRS93]. Interestingly, the NLSM_2 bears some similarity both with the good old $P(\phi)_2$ and with the elusive YM_4. To indicate the similarity to the $P(\phi)_2$, recall that the action of the $O(N)$ non-linear sigma model is given by

$$\mathcal{A}[\phi] = \frac{1}{2}\int d^2x\, \partial_\mu\phi(x)\cdot\partial^\mu\phi(x), \quad |\phi|^2 = 1, \tag{1.1}$$

i.e., it differs from the free field theory only by the constraint restricting the field $\phi : \mathbb{R}^2 \to \mathbb{R}^N$ to the sphere S^{N-1}. The Dirac delta implements this constraint in the functional measure. If we approximate this delta by Gaussian functions, we obtain a family of two-dimensional models with polynomial interactions, see, e.g., [BZ72, MR89]. As for the similarity of the NLSM_2 to the YM_4, it is particularly striking for $N = 4$. In this case, for any $\phi \in S^3$ we have $U := \phi_0 1 + \mathrm{i}\vec{\phi}\cdot\vec{\sigma} \in SU(2)$, where $\vec{\sigma}$ are the Pauli matrices. In these variables, we obtain from (1.1) a principal chiral model, cf. [Po87, p.19], [Mo98, p. 129]. Its formal discretization gives the following action

$$\mathcal{A}(U) = \sum_{b\in\Omega'} \mathrm{Tr}(1 - \partial U(b)), \tag{1.2}$$

where $\Omega \subset \mathbb{Z}^2$ is a finite unit lattice, the sum is over bonds $b = (b_-, b_+)$ in Ω and $\partial U(b) := U(b_-)U(b_+)^*$, which is analogous to the 'holonomy' along b. The theory has a global $SU(2)\times SU(2)$ symmetry given by $\mathcal{A}(u_1 U u_2) = \mathcal{A}(U)$ for any x-independent $u_1, u_2 \in SU(2)$. The expression (1.2) resembles the Wilson action of the YM_4, where the sum is over plaquettes in $\mathbb{Z}^4$, ∂U are the corresponding plaquette variables and local gauge symmetry holds. The two theories also share some important qualitative properties, such as perturbative asymptotic freedom [Po87, MR89] and, more speculatively, mass transmutation [Po87, Fa02, Ku80, Ko99]. Thus, a convincing strategy toward a construction of a non-trivial QFT in four dimensions has the form

$$P(\phi)_2 \to \mathrm{NLSM}_2 \to \mathrm{YM}_4. \tag{1.3}$$

This motivates our paper, which prepares the ground for non-perturbative renormalization of the non-linear sigma model.

The utility of the NLSM_2 as a toy model for the YM_4 was pointed out, in particular, by Bałaban in [Ba87]. Apparently, Bałaban worked out his proof of the UV stability of the YM_4 first in the case of the NLSM_2, but these considerations remained unpublished[1]. As Bałaban's papers on the YM_4 are not easily accessible (cf. [MRS93, p.326]), we find it worthwhile to work out in the NLSM_2 one aspect of Bałaban's method, which is the variational problem. To put the variational problem into perspective, we recall that the context of the entire construction is the Wilson-Kadanoff renormalization as sketched in [BJ86]. That is, the unit lattice Ω is divided into boxes $B_1(y)$, which determine the coarse lattice with a spacing $L > 1$. At each point of the coarse lattice, there lives a field $\mathcal{C}(U)(y)$, which is a suitable

[1] Actually, Bałaban published a series of papers about sigma models in the 90's, starting with [Ba95], but they concerned the IR problem.

average of the fields U inside the box $B_1(y)$. Given this data, one computes the effective action $\mathcal{A}_1$ after one step of the renormalization group:

$$\mathrm{e}^{-\frac{1}{g_1^2}\mathcal{A}_1(V)} := \int dU\, \chi_\varepsilon\, \delta(\mathcal{C}(U)V^{-1})\, \mathrm{e}^{-\frac{1}{g^2}\mathcal{A}(U)}, \tag{1.4}$$

where dU is the product of the Haar measures on $SU(2)$ over all lattice sites, the Dirac delta restricts the integration region to configurations U block-averaging to V [Ya01], and we ignored additive counterterms for simplicity. The characteristic function χ_ε imposes the small field condition, which requires that the differences between fields at neighboring points are bounded by some $\varepsilon > 0$. We note that the effective action also enjoys the global $SU(2)\times SU(2)$ symmetry since our averaging satisfies $\mathcal{C}(u_1 U u_2) = u_1 \mathcal{C}(U) u_2$ for $u_1, u_2 \in SU(2)$, cf. (1.21) below. In the regime of small g, the expression (1.4) can be studied by the method of steepest descent: the leading contribution to the integral is due to the critical points of $U \mapsto \mathcal{A}(U)$ subject to the constraint $\mathcal{C}(U) = V$ and the small field condition. This is the Bałaban variational problem. It prepares the ground for rewriting equation (1.4) as a perturbation of a Gaussian measure, which is tractable by the usual methods of constructive QFT. Our solution to the variational problem is quite different from Bałaban's discussion of the corresponding problem for the YM_4 in [Ba85b]. We put additional emphasis on the clarity of the presentation, in particular on the separation of the geometric and analytic considerations. We intend to demonstrate that the Bałaban variational problem is an elegant topic in the variational calculus on Lie groups, cf. Subsection 3.1. We hope that our paper will be a useful addition to a growing library of accessible literature on Bałaban's method, see, e.g., [BJ86, Ya01, Di13, DY23, Di18, DST24].

To support the above comments, let us outline our solution of the Bałaban variational problem. We first change the variables from $U(x)$ to $U'(x) := U(x)V^{-1}(y_x)$, $x \in B_1(y_x)$, which describe fluctuations around the value dictated by the constraint. Then the problem is to find the critical points of

$$\mathcal{A}'(U') := \sum_{b\in\Omega'} \mathrm{Tr}(1 - \underbrace{U'(b_-)\partial V(y_b)U'(b_+)^*}_{=:W(b)}) \quad \text{with the constraint} \quad \mathcal{C}(U') = 1, \tag{1.5}$$

where $\partial V(y_b) := V(y_{b_-})V(y_{b_+})^*$. Both U' and W are elements of $SU(2)$, thus can be parametrized by vectors $\vec{A}$ and $\vec{W}$ multiplying the Pauli matrices, as indicated above (1.2). Now, we consider the system of equations

$$\mathcal{L}_X \mathcal{C}(U') = 0, \quad \mathcal{L}_X \mathcal{A}'(U') = 0, \tag{1.6}$$

where $\mathcal{L}_X$ is the Lie derivative in the direction of the tangent vector field X. From the first equation in (1.6) we determine the tangent space of the constraint manifold, from the second one we obtain the critical points of the action on the constraint manifold. To describe the solution, we introduce a transformation $R(x)\vec{v} = A_0(x)\vec{v} + \vec{A}(x) \times \vec{v}$ which is a sum of a rotation in the plane orthogonal to $\vec{A}$ and a rescaling determined by the length of $\vec{A}$. We also introduce a derivation ∂ which maps functions on lattice sites into functions on bonds according to $(\partial f)(b) = f(b_-) - f(b_+)$. Then $-\partial^*\partial$ coincides with the lattice Laplacian Δ_Ω with Neumann boundary conditions. As a direct consequence of (1.6), the family of vectors

$$\vec{C}(x) := R^{-1}(x)^* \partial^* \vec{W}(x) \tag{1.7}$$

is constant on each block $B_1(y)$ at the critical point. We proceed from this relation to an equation for the critical point via the following steps: First, we note that in the variables $\vec{A}$ the constraint has the simple form $Q(\vec{A}) = 0$, where Q is the arithmetic mean over boxes. Second, we decompose the variable $\vec{W}$, appearing in (1.5), as follows

$$\vec{W} = \partial\vec{A} + \vec{r}, \tag{1.8}$$

where $\partial\vec{A}$ is the leading term and $\vec{r}_{\vec{A}}$ the remainder with respect to the parameter ε appearing in the small field condition. Then, using (1.7), (1.8), we obtain in Theorem 3.9 the following equation for the critical point:

$$\vec{A} = G(\Omega)R^*Q^*[QG(\Omega)R^*Q^*]^{-1}QG(\Omega)\partial^*\vec{r} - G(\Omega)\partial^*\vec{r}, \tag{1.9}$$

where $G(\Omega) := (-\Delta_\Omega + Q^*Q)^{-1}$ is a lattice Green function. Recalling that both R and $\vec{r}$ depend on $\vec{A}$, this is a highly non-linear equation.

The problem of existence and uniqueness of solutions of equation (1.9) constitutes the analytic part of our considerations. As the equation (1.9) has the schematic form $\vec{A} = \mathrm{T}(\vec{A})$, we apply the Banach contraction mapping theorem. The respective metric space is given by

$$\mathrm{X}_\varepsilon := \{\, \vec{A} \;|\; Q(\vec{A}) = 0,\ \sup_{b\in\Omega'} \|(\partial U)(b) - 1\| \le \varepsilon\}, \tag{1.10}$$

which is dictated by the constraint and the small field condition (stated explicitly here). We choose the $\mathcal{L}^\infty$-metric on this space as suggested by the supremum over b defining the small field condition. The choice of any other $\mathcal{L}^p$-metric would lead to a mismatch with the small field condition and, thus, to estimates depending on the number of lattice points n^2. We stress that we want to prove the existence and uniqueness of solutions of equation (1.9) for $\varepsilon > 0$, which is small depending on L but not on n as the latter should ultimately tend to infinity in the continuum limit.

Thus, to obtain that T is a contraction, we need $\mathcal{L}^\infty$-bounds on the relevant operators appearing in (1.9). In particular, we have to show

$$\|G(\Omega)f\|_\infty \le c\|f\|_\infty, \quad \|(QG(\Omega)Q^*)^{-1}f\|_\infty \le c\|f\|_\infty. \tag{1.11}$$

It is a simple and general fact that such bounds hold for operators whose integral kernels have an exponential decay. It is less well known, but also true for strictly positive operators on $\mathcal{L}^2(\mathbb{Z}^d)$, that exponential decay of the integral kernel implies the exponential decay of the integral kernel of *the inverse operator*. Bałaban and Jaffe showed the latter fact in [BJ86] using the method of random walk expansions. We reproduce their argument in Appendix A and use it in combination with the method of images to prove (1.11).

Another important step of the proof that T is a contraction is to show that $\sup_{x\in\Omega}|\vec{A}(x)| \le c\varepsilon$ for $\vec{A} \in \mathrm{X}_\varepsilon$. We stress that this bound cannot follow from the small field condition alone, as the latter only controls differences of fields at neighboring points. By exploiting in addition the constraint $Q(\vec{A}) = 0$, we obtain the required bound in Theorem 2.5. Interestingly, in the case of the YM_4, the corresponding bound would follow quite easily using a gauge fixing condition which switches off the fields on many bonds in each box, cf. [Ba85a, Lemma 1]. This demonstrates that the similarity of the two models has its limitations, and a rigorous analysis of the NLSM_2 must not rely on papers on the YM_4 for technical material.

Regarding future directions, we plan to expand the expression on the r.h.s. of (1.4) around the obtained critical point. After changing variables to the Lie algebra elements $\vec{A}$, it should be possible to rewrite the measure after one step as a perturbation of a Gaussian measure. It is an interesting question if the quadratic form defining this measure is strictly positive. As a matter of fact, the $P(\phi)_2$ models approximating the NLSM_2, which we mentioned above, have massless Goldstone bosons in their actions. We expect, however, that in (1.4), the resulting infrared problems will be eliminated by the Dirac delta imposing the constraint. Then, using the cluster expansion, we should be able to determine the behavior of the coupling constant $g \to g_1$ after the first step of the renormalization group. On the other hand, if the infrared problems persist, we may have to introduce a gauge fixing in the functional measure, similar to the one considered in [Da80].

In the present paper, we study only one step of the renormalization group, from a theory on a unit lattice to a theory on an L-lattice. Ultimately, we would like to understand k steps of the renormalization group, starting from a theory on an L^{-k}-lattice and going up to a theory on a unit lattice. Using the semigroup property of the renormalization group transformations (1.4) it is easy to guess that the Bałaban variational problem consists in minimizing $U \mapsto \mathcal{A}(U)$ with the constraint $\mathcal{C}^k(U) = V$ in this case. This is actually how the variational problem was originally formulated in [Ba85b], disregarding additional complications related to the large field problem. We hope to come back to this problem in the case of the NLSM$_2$ in future work. One complication is that the linearization of the constraint to $Q(\vec{A}) = 0$ is no longer automatic but requires an additional application of the Banach contraction mapping theorem. Furthermore, one needs to establish the exponential decay of integral kernels for more complicated operators than those appearing in (1.11). It is likely, however, that the Bałaban-Jaffe lemma mentioned above will solve a substantial part of this problem. It should also be added that principal chiral models (1.2) can be defined for more general Lie groups than $SU(2)$. They may not be related to sigma models in the sense of (1.1), but are interesting in their own right [Po87]. We believe that our analysis could be extended to such models at a cost of some technical complications. Specifically, the formula $U := \phi_0 1 + \mathrm{i}\vec{\phi} \cdot \vec{\sigma}$, which appeared above (1.2) and is used to multiply group elements in the following, is specific for $SU(2)$. For general Lie groups more cumbersome Baker-Campbell-Hausdorff type expansions would have to be employed.

Our paper is organized as follows: In Subsection 1.1, we introduce our setting and state the main result. In Section 2, we simplify the constraints by a change of variables and study the relations between spaces of configurations. In Section 3, we characterize the constraint manifold and write the equation for the critical point. In Section 4, we show the existence and uniqueness of the solution of the critical point equation using the Banach contraction mapping theorem. We supplement the paper with results about the exponential decay of integral kernels of certain operators in Appendix A and some more technical results in Appendix B.

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Notation

1. We introduce an odd positive integer $L > 1$ and set $I = [0, 1, \dots, n-1]$, $n - 1 = L^m$, so that the parameter m controls the size of the interval.

2. We denote by $\Omega \subset \mathbb{Z}^2$ the finite lattice $\Omega := I^{\times 2} = [0, 1, \dots, L^m]^{\times 2}$.

3. We denote by $\Omega_1 \subset L\mathbb{Z}^d$ the coarse lattices of the form $\Omega_1 = L[0, 1, \dots, L^{m-1}]^{\times 2}$.

4. We denote by Ω' the set of oriented bonds on Ω.

5. We denote by $|\cdot|$ the length of a vector in $\mathbb{R}^\ell$ and by $\|\cdot\|$ the operator norm on $\ell \times \ell$ matrices. (We will only need cases $\ell = 1, 3$).

6. The elements of the Hilbert spaces $\mathcal{L}^2(\Omega; \mathbb{R}^\ell), \mathcal{L}^2(\Omega_1; \mathbb{R}^\ell), \mathcal{L}^2(\Omega'; \mathbb{R}^\ell)$ are complex-valued functions

on the respective sets, denoted f, f', g, g'. The scalar products have the form

$$\langle f, g\rangle_\Omega = \sum_{x\in\Omega} f(x)\cdot g(x), \quad \langle f, g\rangle_{\Omega_1} = L^2 \sum_{x\in\Omega_1} f(x)\cdot g(x), \quad \langle f, g\rangle_{\Omega'} = \sum_{b\in\Omega'} f(b)\cdot g(b). \tag{1.12}$$

We set $\|f\|^2_{2,\Omega} = \langle f, f\rangle_\Omega$ and similarly in other cases. The dot above is the canonical scalar product in $\mathbb{R}^\ell$.

7. We will write $\|f\|_{\infty;\Omega} := \sup_{x\in\Omega}|f(x)|$. If there is no risk of confusion, we will drop Ω. We denote by $\mathcal{L}^\infty(\Omega)$ the Banach space of functions on Ω equipped with the norm $\|\cdot\|_{\infty;\Omega}$. Analogous definitions will be used for Ω_1.

8. The operator norm of a map $M : \mathcal{L}^p(\Omega;\mathbb{R}^\ell) \to \mathcal{L}^q(\Omega;\mathbb{R}^\ell)$ will be denoted $\|M\|_{p,q;\Omega}$. Explicitly,

$$\|M\|_{p,q;\Omega} = \sup_{\|f\|_{p;\Omega}\leq 1} \|Mf\|_{q;\Omega}. \tag{1.13}$$

An analogous definition will be used for Ω_1. (We will only need cases $p, q \in \{2, \infty\}$).

9. We define the boxes in the lattice Ω for $y \in \Omega_1$

$$B_1(y) := \{\, x \in \mathbb{Z}^2 \,|\, y_\mu \leq x_\mu < y_\mu + L,\ \mu = 0, 1\} \tag{1.14}$$

and for any x denote by y_x the label y of the box s.t. $x \in B_1(y)$.

10. We denote by Δ_Ω the Laplacian on Ω with Neumann boundary conditions.

11. $|x - x'| = \left(\sum_{\mu=0}^1 (x_\mu - x'_\mu)^2\right)^{1/2}$, $|x - x'|_\infty := \sup_{\mu=0,1} |x_\mu - x'_\mu|$.

12. $\mathbb{1}_O$ denotes the characteristic function of a set O.

13. By $c, c', c_1, c_2 \ldots$ we denote numerical constants, independent of any parameters. Unless stated otherwise, by C, C', C_1, C_2 we denote constants which may depend on L but independent of any other parameters (in particular independent of n). All these constants may change from line to line.

14. We denote a scalar multiple of the identity operator on various vector spaces by the scalar.

15. We assume summation over repeated indices e.g. $X_j\sigma_j := \sum_{j=1}^3 X_j\sigma_j$.

1.1 The setting and results

We set $I := [0, 1, \ldots, n-1]$ and denote by $\Omega = I^{\times 2} \subset \mathbb{Z}^2$ a finite lattice on which the model will be defined. Let Ω' be the set of oriented bonds on Ω denoted by $b = (b_-, b_+)$. Let $L > 1$ be an odd integer and $\Omega_1 := (L\Omega) \cap \Omega$ be the coarse lattice. For every $y \in \Omega_1$ we define a box in the original lattice

$$B_1(y) := \{\, x \in \Omega \,|\, y_\mu \leq x_\mu < y_\mu + L, \quad \mu = 0, 1\,\}, \tag{1.15}$$

whose label y is the left bottom corner. For $x \in \Omega$ we denote by y_x the label of the box containing x. We denote by Ω'_1 the set of oriented bonds on Ω_1.

Let a Lie group G_0 be a subgroup of the unitary group $U(2)$. We introduce the set of all configurations

$$\mathrm{Conf}(\Omega) := \mathrm{G}_0^{\times n^2} \tag{1.16}$$

whose elements have the form $U := \{U(x)\}_{x\in\Omega}$, $U(x) \in \mathrm{G}_0$. For future reference, we note that $\mathrm{Conf}(\Omega)$ corresponds to G from the general discussion in Subsection 3.1. Now we define the action of the model as a function on $\mathrm{Conf}(\Omega)$:

$$\mathcal{A}(U) = \sum_{b\in\Omega'} \mathrm{ReTr}(1 - \partial U(b)), \quad \partial U(b) := U(b_-)U(b_+)^*. \tag{1.17}$$

It has an important symmetry property: For any x-independent unitaries u, v we have

$$\mathcal{A}(uUv) = \mathcal{A}(U). \tag{1.18}$$

We are going to find critical points of this action with a constraint dictated by the block-averaging procedure. We follow the averaging method from [Iw85]. As a first step we define a function

$$\mathcal{C}_0(U)(y) := \frac{1}{L^2} \sum_{x\in B_1(y)} U(x) \tag{1.19}$$

which maps every configuration U into a family of matrices $\{M(y)\}_{y\in\Omega_1}$ on the coarse lattice. As these matrices need not be unitary, we take a polar decomposition at each $y \in \Omega_1$

$$\mathcal{C}_0(U)(y) = \mathcal{C}(U)(y)|\mathcal{C}_0(U)(y)| \tag{1.20}$$

and let the partial isometry $\mathcal{C}$ be our averaging operation. We set $\mathcal{C}(U)(y) = 1$ whenever $\mathcal{C}_0(U)(y) = 0$. As shown in Lemma B.1, such averaging operation is well defined for $\mathrm{G}_0 = SU(2)$ and we have $\mathcal{C}(U)(y) \in \mathrm{G}_0$.

We note that the averaging operation is consistent with the symmetry property (1.18) of the action. In fact, for any block-constant families of unitaries u, v

$$\mathcal{C}(uUv) = u\mathcal{C}(U)v. \tag{1.21}$$

This guarantees a consistent transformation of a constraint of the form

$$\mathcal{C}(U) = V, \tag{1.22}$$

where $V \in \mathrm{Conf}(\Omega_1)$ is a given configuration on the coarse lattice.

Finally we define the following subset of the set of configurations (1.16)

$$\mathrm{Conf}_\varepsilon(\Omega) = \{ U \in \mathrm{Conf}(\Omega) \,|\, \|\partial U(b) - 1\| \leq \varepsilon \text{ for all } b \in \Omega' \}, \tag{1.23}$$

for $0 < \varepsilon \leq 1$, which encodes the small field condition. Now we state our main theorem:

Theorem A. *Let* $\mathrm{G}_0 = SU(2)$. *Then there exist* $0 < \varepsilon, \varepsilon_1 \leq 1$ *s.t. for* $V \in \mathrm{Conf}_{\varepsilon_1}(\Omega_1)$ *the action* $\mathcal{A}$ *has a unique critical point over* $\mathrm{Conf}_\varepsilon(\Omega)$ *with the constraint* $\mathcal{C}(U) = V$. *The parameters* $\varepsilon, \varepsilon_1$ *are independent of* n *but may depend on* L.

We will prove this as Theorem 4.26 below.

Remark 1.1. *We provide several comments on Theorem A and the method of proof:*

1. *It is clear from Definitions (1.15), (1.17), (1.19) that the variational problem formulated above is in fact independent of the lattice spacing. For this reason we formulated the problem on a unit lattice from the beginning. This simplification is due to the fact that we look only at one step of the renormalization group. In the full variational problem, mentioned in the Introduction, there are* k *steps of the renormalization group, from the* L^{-k}*-lattice to the unit lattice. Then the dependence on the lattice spacing is encoded in the parameter* k *and it persists after rescaling.*

2. *To prove Theorem A we will follow the strategy from Subsection 3.1. As the open set* G_ε*, on which the Lie derivatives will be computed, we take the interior of* $\mathrm{Conf}_\varepsilon(\Omega)$*.*

3. *By setting* $\varepsilon_1 < \varepsilon^2$ *one can ensure that the critical point is not at the boundary of* $\mathrm{Conf}_\varepsilon(\Omega)$*. Indeed, a critical point in* $\mathrm{Conf}_\varepsilon(\Omega)$ *is also a critical point for* $\mathrm{Conf}_{\varepsilon'}(\Omega)$ *s.t.* $\varepsilon_1 < (\varepsilon')^2 < \varepsilon^2$*. Thus the unique critical point must belong to the interior of* $\mathrm{Conf}_\varepsilon(\Omega)$*.*

4. *An essential step of the proof of Theorem A consists in showing that configurations from* $\mathrm{Conf}_\varepsilon(\Omega)$*, satisfying the constraint, can be written as* $U(x) = U'(x)V(y_x)$*, where* $U'(x)$ *belongs to*

$$\mathrm{Conf}^{\varepsilon'}(\Omega) {:=} \{\, U \in \mathrm{Conf}(\Omega) \,|\, \|U(x) - 1\| \leq \varepsilon' \ \textit{for all } x \in \Omega \,\} \tag{1.24}$$

for certain ε*,* ε'*. We note the crucial role of the constraint in this step, which is performed in Subsection 2.2: the small field condition (1.23) by itself does not imply the condition from (1.24), as it concerns only differences of fields at neighbouring points.*

In the course of our analysis we will often use a parametrization of elements of $SU(2)$ by an axis $\hat{n}$ and an angle a so that

$$\mathrm{e}^{\mathrm{i}a(\hat{n}\cdot\vec{\sigma})} = I\cos(a) + \mathrm{i}(\hat{n}\cdot\vec{\sigma})\sin(a), \tag{1.25}$$

where $\vec{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices (see e.g. [CDD82, Chapter III, Problem 5]). Consequently, we can write

$$U'(x) = sA_0(x)I + \mathrm{i}\vec{A}(x)\cdot\vec{\sigma}, \tag{1.26}$$

where $s := \mathrm{sgn}(\cos(a))$, $|\vec{A}(x)| \leq 1$, $A_0(x) := \sqrt{1 - |\vec{A}(x)|^2}$ and U' was introduced above (1.24). It is therefore convenient to define for $0 < \varepsilon \leq 1$ the corresponding family of configurations:

$$\overrightarrow{\mathrm{Conf}}^{\varepsilon}(\Omega) := \{\{\vec{A}(x)\}_{x\in\Omega} \,|\, \vec{A}(x) \in \mathbb{R}^3, \ \sup_{x\in\Omega} |\vec{A}(x)| \leq \varepsilon\,\}. \tag{1.27}$$

For any $\vec{A} \in \overrightarrow{\mathrm{Conf}}^{\varepsilon}(\Omega)$ we have two distinct configurations U', corresponding to the two signs $s = \{\pm 1\}$. However, as we will see in Theorem 2.5, only $s = 1$ plays a role in our discussion as we work close to the unity in G_0.

2 Preparations

2.1 Linearization of the constraint

In this subsection we simplify the constraint. As we explained in Section 1, we make a change of variables $U(x) = U'(x)V(y_x)$, where y_x is the label of the box to which x belongs, i.e., $x \in B_1(y_x)$. The action in the new variables has the form

$$\mathcal{A}'(U') = \sum_{b\in\Omega} \mathrm{ReTr}(1 - U'(b_-)\partial V(y_b)U'(b_+)^*), \quad \partial V(y_b) := V(y_{b_-})V(y_{b_+})^*, \tag{2.1}$$

with the constraint

$$\mathcal{C}(U'V) = V \quad \Leftrightarrow \quad \mathcal{C}_0(U')(y) = (\mathcal{C}_0(U')(y)^*\mathcal{C}_0(U')(y))^{\frac{1}{2}}, \tag{2.2}$$

where we made use of (1.21). By Lemma B.1 and the uniqueness of the polar decomposition the second relation in (2.2) is equivalent to $\mathrm{Re}\,\mathcal{C}_0(U') \geq 0$ and $\mathrm{Im}\,\mathcal{C}_0(U') = 0$. The condition $\mathrm{Re}\,\mathcal{C}_0(U') \geq 0$ is satisfied automatically for U' close enough to the identity (cf. Lemma 2.3). Thus there remains the constraint

$$\mathcal{C}'(U')(y) := 2\mathrm{i}\,\mathrm{Im}\,\mathcal{C}_0(U')(y) = 0, \quad y \in \Omega_1, \tag{2.3}$$

which reads as follows

$$\mathcal{C}'(U')(y) = \sum_{x\in B_1(y)} (U'(x) - U'(x)^*) = 0, \quad y \in \Omega_1. \tag{2.4}$$

Using this, decomposition (1.26) and Theorem 2.5 we obtain

$$\sum_{x\in B_1(y)} \vec{A}(x) = 0, \quad s = 1, \tag{2.5}$$

thus we linearized the constraint. It is therefore convenient to define a linear averaging map $Q : \mathcal{L}^2(\Omega;\mathbb{R}^\ell) \to \mathcal{L}^2(\Omega_1;\mathbb{R}^\ell)$ by

$$(Qf)(y) := \frac{1}{L^2} \sum_{x\in B_1(y)} f(x) \tag{2.6}$$

to state the constraint (2.5) as $Q(\vec{A}) = 0$. We note for future reference that

$$(Q^*f)(x) \;=\; f(y_x), \quad (Q^*Qf)(x) = \frac{1}{L^2} \sum_{x'\in B_1(y_x)} f(x'), \tag{2.7}$$

i.e., Q^*Q is the projection on block-constant functions.

2.2 Sets of configurations

Recall the sets of configurations defined in (1.23), (1.24), (1.27).

Lemma 2.1. *For $U \in \mathrm{Conf}_\varepsilon(\Omega)$ and $x, x' \in B_1(y)$ we have*

$$\|U(x)U(x')^* - 1\| \le (2L)\varepsilon. \tag{2.8}$$

Proof. Let $b_1 \circ \cdots \circ b_\ell$ be the shortest oriented path of bonds s.t. $b_{1,-} = x$ and $b_{\ell,+} = x'$. (We can always find such a path, possibly with x, x' exchanged). Thus we can write

$$\begin{aligned} U(x)U(x')^* - 1 \;&=\; \partial U(b_1)\ldots\partial U(b_\ell) - 1 \\ &=\; \partial U(b_1)\ldots\partial U(b_{\ell-1})(\partial U(b_\ell) - 1) + \partial U(b_1)\ldots\partial U(b_{\ell-1}) - 1 \end{aligned} \tag{2.9}$$

where $\|\partial U(b_\ell) - 1\| \le \varepsilon$. After ℓ steps we estimate the norms and get

$$\|U(x)U(x')^* - 1\| \le \ell\varepsilon \le (2L)\varepsilon. \tag{2.10}$$

This completes the proof. □

Lemma 2.2. *Let $U \in \mathrm{Conf}_\varepsilon(\Omega)$, $0 < \varepsilon \le 1/(4L)$. Then $|\mathcal{C}_0(U)(y)|$, $y \in \Omega_1$, are strictly positive.*

Proof. We come back to definitions (1.19), (1.20) and compute

$$\begin{aligned}
\mathcal{C}_0(U)(y)^*\,\mathcal{C}_0(U)(y) &= \frac{1}{L^4}\sum_{x,x'\in B_1(y)} U(x)^*U(x') \\
&= \frac{1}{L^2}\Big(1+\frac{1}{L^2}\sum_{\substack{x,x'\in B_1(y)\\ x\neq x'}} U(x)^*U(x')\Big) \qquad (2.11)\\
&= \Big(\frac{1}{L^2}+\frac{1}{L^4}\sum_{\substack{x,x'\in B_1(y)\\ x\neq x'}} (U(x)^*U(x')-1)+\frac{1}{L^4}(L^4-L^2)\Big) \qquad (2.12)\\
&= \Big(1+\frac{1}{L^4}\sum_{\substack{x,x'\in B_1(y)\\ x\neq x'}} (U(x)^*U(x')-1)\Big). \qquad (2.13)
\end{aligned}$$

In view of Lemma 2.1, for $(2L)\varepsilon \le 1/2$ the claim follows. □

We recall from (2.2) that

$$\mathcal{C}(U'V)=V \quad \Leftrightarrow \quad \mathcal{C}_0(U')(y)=(\mathcal{C}_0(U')(y)^*\mathcal{C}_0(U')(y))^{\frac{1}{2}}. \qquad (2.14)$$

This will be used in the following lemma.

Lemma 2.3. *Let $0<\varepsilon\le 1/(4L)$ and suppose that $U\in \mathrm{Conf}_\varepsilon(\Omega)$ satisfies the constraint (1.22). Then $\|U'(x)-1\|\le 4c_{\frac{1}{2}}L\varepsilon$.*

Proof. Using (2.13), Lemma 2.1 and Lemma B.2 we have

$$\|(\mathcal{C}_0(U')(y)^*\mathcal{C}_0(U')(y))^{\frac{1}{2}}-1\|\le c_{\frac{1}{2}}(2L)\varepsilon. \qquad (2.15)$$

Only in this lemma we denote by $O(\epsilon)$ any operator satisfying $\|O(\epsilon)\|\le\epsilon$. Thus starting from (2.14),

$$\mathcal{C}_0(U')(y)=1+O(c_{\frac{1}{2}}2L\varepsilon),\ \text{ i.e., } \frac{1}{L^2}\sum_{x\in B_1(y)}U'(x)=1+O(c_{\frac{1}{2}}2L\varepsilon). \qquad (2.16)$$

Fix some $x,x'\in B_1(y)$ and let $b_1\circ b_2\circ\ldots\circ b_\ell$ be a chain of bonds linking x' to x, i.e. $b_{1,-}=x'$ and $b_{\ell,+}=x$. We have, by Lemma 2.1,

$$U'(x)+U'(x')=(1+U'(x')U'(x)^*)U'(x)=(2+O(2L\varepsilon))U'(x). \qquad (2.17)$$

Proceeding analogously,

$$U'(x)+U'(x')+U'(x'')=(2+O(2L\varepsilon)+U'(x'')U'(x)^*)U'(x)=(3+2O(2L\varepsilon))U'(x). \qquad (2.18)$$

Repeating the procedure L^2 times to account for all the sites of $B_1(y)$, we have

$$\frac{1}{L^2}\sum_{x\in B(y)}U'(x)=\frac{1}{L^2}(L^2+(L^2-1)O(2L\varepsilon))U'(x). \qquad (2.19)$$

Coming back to (2.16), we have the following equality

$$(1+O(2L\varepsilon))U'(x)=1+O(c_{\frac{1}{2}}2L\varepsilon) \qquad (2.20)$$

Hence $U'(x)-1=O(4c_{\frac{1}{2}}L\varepsilon)$, which we wanted to prove. □

Lemma 2.4. *Let $0 < \varepsilon \leq 1$. Then the following implication holds true:*

$$U' \in \mathrm{Conf}^{\varepsilon}(\Omega) \quad \Rightarrow \quad (\vec{A} \in \overrightarrow{\mathrm{Conf}}^{\varepsilon}(\Omega) \text{ and } s = 1). \tag{2.21}$$

Proof. We have

$$\begin{aligned} \|\mathrm{e}^{\mathrm{i}a(\hat{n}\cdot\sigma)} - 1\|^2 &= \|(\cos(a) - 1) + \mathrm{i}(\hat{n}\cdot\vec{\sigma})\sin(a)\|^2 \\ &= (\cos(a)-1)^2 + \sin^2(a) \\ &= 2 - 2\cos(a) = 2(1-\cos(a)) = 2\frac{\sin^2(a)}{1 + s\sqrt{1-\sin^2(a)}}. \end{aligned} \tag{2.22}$$

Considering that $\|\mathrm{e}^{\mathrm{i}a(\hat{n}\cdot\sigma)} - 1\| \leq 1$, we have $s = 1$ and

$$\|U' - 1\|^2 = 2\frac{|\vec{A}|^2}{1 + \sqrt{1 - |\vec{A}|^2}}. \tag{2.23}$$

Thus $\|U' - 1\|^2 \leq \varepsilon^2$ implies $|\vec{A}|^2 \leq \varepsilon^2$. □

Theorem 2.5. *Let $0 < \varepsilon, \varepsilon_1 \leq 1$, $U(x) = U'(x)V(y_x)$, $V \in \mathrm{Conf}_{\varepsilon_1}(\Omega_1)$. Then*

$$U \in \mathrm{Conf}_{\varepsilon}(\Omega) \text{ and } \mathcal{C}(U) = V \quad \Rightarrow \quad U' \in \mathrm{Conf}^{(4c_{\frac{1}{2}}L\varepsilon)}(\Omega), \tag{2.24}$$

$$U' \in \mathrm{Conf}^{(4c_{\frac{1}{2}}L\varepsilon)}(\Omega) \quad \Rightarrow \quad (\vec{A} \in \overrightarrow{\mathrm{Conf}}^{(4c_{\frac{1}{2}}L\varepsilon)}(\Omega) \text{ and } s = 1). \tag{2.25}$$

The second implication requires $4c_{\frac{1}{2}}L\varepsilon \leq 1$. (The numerical constant $c_{\frac{1}{2}}$ appears in Lemma B.2).

Proof. The first implication follows from Lemma 2.3. The second from Lemma 2.4. □

3 Geometric considerations

3.1 Variational calculus on Lie groups

Consider a smooth manifold M of dimension $\tilde{n}$ with a vector field X which is a linear map on $C^\infty(M)$. Its flow is $\mathbb{R} \times M \ni (t, x) \mapsto \gamma_t(x) \in M$ s.t. $\gamma_0 = \mathrm{id}_M$ and $\gamma_s \circ \gamma_t = \gamma_{s+t}$ for $s, t \in \mathbb{R}$. The vector field is characterized by

$$X(f) = \frac{d}{dt} f \circ \gamma_t|_{t=0}. \tag{3.1}$$

Given a vector field X, the flow satisfies

$$\frac{d}{dt}\gamma_t(x) = X(\gamma_t(x)). \tag{3.2}$$

Now the Lie derivative of a function $f : M \to \mathbb{R}$ w.r.t. X is

$$(\mathcal{L}_X f)(x) = \lim_{t\to 0} \frac{f(\gamma_t(x)) - f(x)}{t} : \quad M \to \mathbb{R}. \tag{3.3}$$

Clearly if a function f has a minimum at x_0 then $(\mathcal{L}_X f)(x_0) = X(f)(x_0) = 0$. Let us now recall the standard setting for minimisation with constraints:

Definition 3.1. *Let $0 < k < \tilde{n}$. A subset $M_C \subset M$ is a k-dimensional equation-defined C^1-manifold if there is an open set $O \subset M$, functions $C_j \in C^1(O;\mathbb{R})$, $j = 1, \dots \tilde{n} - k$, s.t. $M_C = \{ x \in O \,|\, C_1(x) = \cdots = C_{\tilde{n}-k}(x) = 0\}$ and the differential forms on M*

$$dC_1(x), \dots dC_{\tilde{n}-k}(x), \quad x \in M_C, \tag{3.4}$$

are linearly independent. Apart from this, we define the space of normal forms

$$N_x(M) := \mathrm{Span}\{ dC_1(x), \dots dC_{\tilde{n}-k}(x) \}. \tag{3.5}$$

Theorem 3.2. *Let M_C as above and $F \in C^1(O;\mathbb{R})$ so that $F|_{M_C}$ has a local minimum in $x_0 \in M_C$, i.e., there is a neighbourhood $U \subset O$ of x_0 s.t.*

$$F(x) \geq F(x_0) \text{ for all } x \in U \cap M. \tag{3.6}$$

Then $dF(x_0) \in N_{x_0}(M_C)$ or, equivalently, $X(F)(x_0) = 0$ for any $X \in TM_C$.

Proof. Let $\gamma_t^{M_C}$ be the flow of a tangent vector X understood as an element of $T_{x_0}M_C$ and γ_t be the flow of X understood as an element of $T_{x_0}M$. Then

$$0 = \frac{d}{dt}F(\gamma_t^{M_C}(x_0))|_{t=0} = \frac{d}{dt}F(\gamma_t(x_0))|_{t=0} = X(F)(x_0) = \langle dF(x_0), X_{x_0}\rangle, \tag{3.7}$$

where the first equality follows from the fact that $t \mapsto \gamma_t^{M_C}(x_0) \in M_C$ near $t = 0$ and $F|_{M_C}$ has a minimum there. The second equality follows from the definition of a tangent vector via an equivalence class of curves. □

Now consider the special case of a Lie group G and a Lie algebra $\mathfrak{g}$. For an element $\mathrm{i}X \in \mathfrak{g}$ we define a flow on G by

$$\gamma_t(U) = \mathrm{e}^{\mathrm{i}tX}U, \quad U \in \mathrm{G}. \tag{3.8}$$

Considering (3.7), we conclude that a necessary condition for U_0 to be a minimum is

$$(\mathcal{L}_X F)(U_0) = \frac{d}{dt}F(\mathrm{e}^{\mathrm{i}tX}U_0)|_{t=0} = 0, \tag{3.9}$$

for all $X \in T_{U_0}M_C$. Such X can be characterized by the conditions

$$(\mathcal{L}_X C_j)(U_0) = \frac{d}{dt}C_j(\mathrm{e}^{\mathrm{i}tX}U_0)|_{t=0} = 0. \tag{3.10}$$

Clearly, by the same method one can look for minima in any open subset $\mathrm{G}_\varepsilon \subset \mathrm{G}$ cf. parts 2., 3. of Remark 1.1. The linear independence condition from Definition 3.1 is easily checked in our case using the linearized formulation of the constraint stated below (2.6). For more on variational calculus on groups see [Ch12].

The plan of the remaining part of the paper is to first determine the tangent vectors $X \in M_C$ from (3.10) in Subsection 3.2 and then derive equation (3.9) in Subsection 3.3. This equation will be solved using the Banach contraction mapping theorem in Section 4.

3.2 Tangent space of the constraint manifold

To describe the tangent space of the constraint manifold, we need some preparations. Given any $U \in \mathrm{Conf}_\varepsilon(\Omega)$, we define a family of linear transformations on $\mathbb{R}^3$ by

$$R_{\vec{A}}(x)\vec{v} = sA_0(x)\vec{v} + \vec{A}(x) \times \vec{v} \tag{3.11}$$

and denote their inverses, which exist for $A_0 \neq 0$, by $R^{-1}_{\vec{A}}(x)$. Using $\vec{w} \cdot (\vec{A}(x) \times \vec{v}) = -(\vec{A}(x) \times \vec{w}) \cdot \vec{v}$, we easily obtain that

$$R^*_{\vec{A}}(x)\vec{v} = sA_0(x)\vec{v} - \vec{A}(x) \times \vec{v}. \tag{3.12}$$

Using (1.25) to identify $sA_0 = \cos(a)$ and $\vec{A} = \sin(a)\hat{n}$, we obtain

$$R_{\vec{A}}\vec{v} \;=\; \cos(a)\vec{v} + \sin(a)(\hat{n} \times \vec{v}) = \cos(a)P_{\hat{n}}\vec{v} + \cos(a)P^{\perp}_{\hat{n}}\vec{v} + \sin(a)(\hat{n} \times P^{\perp}_{\hat{n}}\vec{v}), \tag{3.13}$$

$$R^{-1}_{\vec{A}}\vec{v} \;=\; \frac{1}{\cos(a)}P_{\hat{n}}\vec{v} + \cos(a)P^{\perp}_{\hat{n}}\vec{v} - \sin(a)(\hat{n} \times P^{\perp}_{\hat{n}}\vec{v}), \tag{3.14}$$

where $P_{\hat{n}}$ is the orthogonal projection on $\hat{n}$. Thus $R_{\vec{A}}(x)$ is a sum of a rotation in the plane orthogonal to $\hat{n}$ and a scaling transformation in the direction of $\hat{n}$. We will often write $R := R_{\vec{A}}$ for brevity. Now we follow the procedure from Subsection 3.1:

Proposition 3.3. *Suppose that $U \in \mathrm{Conf}_\varepsilon(\Omega)$ satisfies the constraint (1.22). Then for any vector $\mathrm{i}X \in \mathfrak{g}_0^{\oplus n^2}$ the property $\mathcal{L}_X\mathcal{C}'(U')(y) = 0$ is equivalent to*

$$\sum_{x \in B_1(y)} R(x)\vec{X}(x) = 0, \quad y \in \Omega_1. \tag{3.15}$$

Proof. Using (3.10), (2.4), we compute

$$\mathcal{L}_X\mathcal{C}'(U')(y) = \mathrm{i}\sum_{x \in B_1(y)} (X(x)U'(x) + U'(x)^*X(x)) = 0. \tag{3.16}$$

Let us recall that for $SU(2)$ we have $X(x) = X_j(x)\sigma_j \in \mathfrak{su}(2)$, $U'(x) = A_0(x) + \mathrm{i}A_j\sigma_j \in SU(2)$, where the coefficients X_j, A_0, A_j are real. By Theorem 2.5, we could set $s = 1$ in (1.26), because we are in $\mathrm{Conf}_\varepsilon(\Omega)$ and we are differentiating at a point U' of the constraint manifold. Thus equation (3.16) gives, omitting the dependence on x,

$$\begin{aligned} 0 \;&=\; \sum_{x \in B_1(y)} X_j(\sigma_j(A_0 + \mathrm{i}A_k\sigma_k) + \text{h.c.}) \\ &=\; \sum_{x \in B_1(y)} X_j(A_0\sigma_j + \mathrm{i}A_k(\delta_{j,k} + i\varepsilon_{j,k,\ell}\sigma_\ell) + \text{h.c.}) \\ &=\; \sum_{x \in B_1(y)} X_j(A_0\sigma_j + \mathrm{i}A_j - A_k\varepsilon_{j,k,\ell}\sigma_\ell) + \text{h.c.}) \\ &=\; \sum_{x \in B_1(y)} 2(A_0X_j - A_kX_{j'}\varepsilon_{j',k,j})\sigma_j. \end{aligned} \tag{3.17}$$

Thus we get that for any y

$$\sum_{x \in B_1(y)} (A_0(x)\vec{X}(x) + \vec{A}(x) \times \vec{X}(x)) = 0 \tag{3.18}$$

which concludes the proof. □

Let us now describe more explicitly families of vectors $\{\vec{X}(x)\}_{x\in\Omega}$ satisfying (3.15). We note that for each y we obtain an independent condition which depends only on $\vec{X}(x), \vec{A}(x)$ for $x \in B_1(y)$. Thus it suffices to solve (3.15) in one block: We relabel the points in this block as $x_1, \dots, x_{L^2}$ and ask for the kernel of the matrix:

$$[\mathcal{C}] = \begin{bmatrix} R(x_1) & R(x_2) & \dots & R(x_{L^2}) \end{bmatrix}, \tag{3.19}$$

where each $R(x_j)$ symbolizes a 3×3 matrix, see (3.11). This kernel coincides with the range of the following matrix, as we show in Lemma 3.5 below:

$$[\mathcal{D}] = \begin{bmatrix} R^{-1}(x_1) & 0 & 0 & 0 & 0 \\ -R^{-1}(x_2) & R^{-1}(x_2) & 0 & 0 & 0 \\ 0 & -R^{-1}(x_3) & \ddots & 0 & 0 \\ 0 & 0 & 0 & R^{-1}(x_{L^2-2}) & 0 \\ 0 & 0 & 0 & -R^{-1}(x_{L^2-1}) & R^{-1}(x_{L^2-1}) \\ 0 & 0 & 0 & 0 & -R^{-1}(x_{L^2}) \end{bmatrix}. \tag{3.20}$$

Recall that for $U \in \mathrm{Conf}_\varepsilon(\Omega)$, $\mathcal{C}(U) = V$, we have $|\vec{A}(x)|^2 \leq 4c_{\frac{1}{2}} L\varepsilon$ by Theorem 2.5. Thus $A_0(x) \neq 0$ and $R^{-1}(x)$ above exist. Now the tangent space of the constraint manifold can be described as follows:

Theorem 3.4. *Fix $y \in \Omega_1$ and consider a spanning tree $T(y)$ of the box $B_1(y)$ as in Figure 1. Then for every bond $c \in T(y)$, whose orientation equals the tree's orientation, there is a three-dimensional space of vectors tangent to the constraint manifold*

$$\vec{X}_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ R^{-1}(c_-)\vec{v}_c \\ -R^{-1}(c_+)\vec{v}_c \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{v}_c \in \mathbb{R}^3. \tag{3.21}$$

If the two orientations are opposite, then c_-, c_+ should be exchanged in (3.21).

Lemma 3.5. *Suppose R_i, $i = 1, \dots, \tilde{m}$ are invertible transformations on $\mathbb{R}^3$. Let $R := (R_1, \dots, R_{\tilde{m}})$ be a transformation from $\mathbb{R}^{3\tilde{m}}$ to $\mathbb{R}^3$. Define mappings*

$$D_i = (0, \dots, R_i^{-1}, -R_{i+1}^{-1}, 0, \dots, 0)^T, \quad i = 1, \dots, \tilde{m}-1, \tag{3.22}$$

from $\mathbb{R}^3$ to $\mathbb{R}^{3\tilde{m}}$, where R_i^{-1} denotes the inverse of R_i. Then the kernel of R equals the range of $D := (D_1, \dots, D_{\tilde{m}-1})$.

Proof. We note that if $(\vec{v}_1, \dots, \vec{v}_{\tilde{m}}) \in \mathbb{R}^{3\tilde{m}}$ is in the kernel of R then

$$-\vec{v}_1 = R_1^{-1}R_2\vec{v}_2 + \dots + R_1^{-1}R_{\tilde{m}}\vec{v}_{\tilde{m}}. \tag{3.23}$$

Thus we can parametrize this kernel with $3(\tilde{m}-1)$ parameters. Hence, the dimension of the kernel is maximally $3(\tilde{m}-1)$. To show that it is exactly $3(\tilde{m}-1)$, one notes that $RD = 0$, $\mathrm{Ran}D_i \cap \mathrm{Ran}D_j = \{0\}$ for $i \neq j$ and the dimension of each $\mathrm{Ran}D_i$ is 3. The last claim follows from the linear independence of $D_i(R_i e_j)$, where e_j are unit vectors in $\mathbb{R}^3$. □

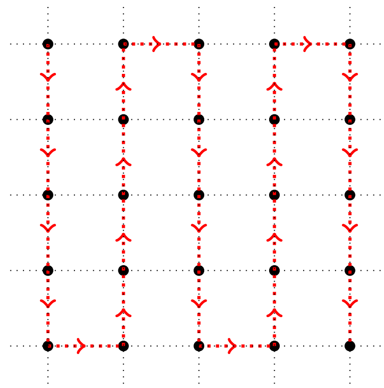

Figure 1: The spanning tree $T(y)$ of the box $B_1(y)$ is indicated in red together with its orientation. The orientation of the bonds of the lattice is fixed by the axes of the coordinate frame.

3.3 Derivation of the critical point equation

Before we derive the critical point equation, we need some definitions.

Definition 3.6. *We define a map $\partial : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega')$ by*

$$(\partial f)(b) := f(b_-) - f(b_+). \tag{3.24}$$

The adjoint map $\partial^* : \mathcal{L}^2(\Omega') \to \mathcal{L}^2(\Omega)$ is defined by the relation

$$\langle \partial f, g \rangle_{\Omega'} = \langle f, \partial^* g \rangle_{\Omega} \tag{3.25}$$

for $f \in \mathcal{L}^2(\Omega)$, $g \in \mathcal{L}^2(\Omega')$. It is given explicitly by

$$(\partial^* f)(x) = \sum_{b \in \Omega', b \ni x} \sigma_b(x) f(b), \tag{3.26}$$

where $\sigma_b(x) = +1/-1$ for bonds incoming/outgoing from x, according to the orientation of Ω. Finally, $\Delta_\Omega := -\partial^* \partial$ coincides with the lattice Laplacian on Ω with Neumann boundary conditions and thus satisfies

$$\langle f, (-\Delta_\Omega) f \rangle = \sum_{b \in \Omega'} \langle \partial f(b), \partial f(b) \rangle, \tag{3.27}$$

cf. e.g. [DST24, Lemma 2.5]. The above definitions extend naturally to vector-valued functions.

Now we fix a configuration $U \in \mathrm{Conf}_\varepsilon(\Omega)$ satisfying the constraint (1.22) with some $V \in \mathrm{Conf}_{\varepsilon_1}(\Omega_1)$. We recall that $\partial U(b) = U'(b_-) \partial V(y_b) U'(b_+)^*$, $\partial V(y_b) := V(y_{b_-}) V(y_{b_+})^*$ and $y_{b_\pm}$ was defined below (1.15). In the following definition we set $W(b) := \partial U(b)$ for brevity, decompose it into the Pauli matrices and extract the leading part of the resulting vector $\vec{W}(b)$.

Definition 3.7. *For any $b \in \Omega'$ we define $W(b) := U'(b_-) \partial V(y_b) U'(b_+)^* = s_W W_0(b) + \mathrm{i} \vec{W}(b) \cdot \vec{\sigma}$ as in formula (1.26). We also define a remainder $\vec{r}_{\vec{A}}(b)$ by the decomposition*

$$\vec{W}(b) =: \partial \vec{A}(b) + \vec{r}_{\vec{A}}(b), \tag{3.28}$$

We will often write $\vec{r} := \vec{r}_{\vec{A}}$ for brevity. For an explicit formula for $\vec{r}_{\vec{A}}(b)$ see (3.46) below.

Let us now move towards the derivation of the critical point equation. The key step is the following proposition. It identifies quantities which are constant along the spanning trees of the respective boxes, see Figure 1.

Proposition 3.8. *Fix $y \in \Omega_1$ and consider a spanning tree $T(y)$ of the box $B(y)$ as in Figure 1. Then, $\vec{A}$ is a critical point of the action (1.17) in $\mathrm{Conf}_\varepsilon(\Omega)$, with the constraint (1.22), if and only if we have*

$$R^{-1}(c_-)^*\partial^*\vec{W}(c_-) = R^{-1}(c_+)^*\partial^*\vec{W}(c_+) \tag{3.29}$$

for any bond $c \in T(y)$.

Proof. We proceed as explained in Subsection 3.1. We consider the functional:

$$\mathcal{A}'(U') = \sum_{b\in\Omega'} \mathrm{ReTr}(1 - U'(b_-)\partial V(y_b)U'(b_+)^*), \quad \partial V(y_b) := V(y_{b_-})V(y_{b_+})^*, \tag{3.30}$$

and compute the Lie derivative

$$\begin{aligned}
\mathcal{L}_X\mathcal{A}'(U') &= -\frac{d}{dt}\sum_{b\in\Omega'} \mathrm{ReTr}(\mathrm{e}^{\mathrm{i}tX(b_-)}U'(b_-)\partial V(y_b)U'(b_+)^*\mathrm{e}^{-\mathrm{i}tX(b_+)})|_{t=0} \\
&= \sum_{b\in\Omega'} \mathrm{ImTr}(X(b_-)U'(b_-)\partial V(y_b)U'(b_+)^* - U'(b_-)\partial V(y_b)U'(b_+)^*X(b_+)) \\
&= \sum_{b\in\Omega'} \mathrm{ImTr}(\partial X(b)U'(b_-)\partial V(y_b)U'(b_+)^*).
\end{aligned} \tag{3.31}$$

Thus, referring to Definition 3.7, we have

$$\mathcal{L}_X\mathcal{A}'(U') = 2\sum_{b\in\Omega'}(\partial X_k(b))W_k(b) = 2\sum_{x\in\Omega} X_k(x)\partial^*W_k(b), \tag{3.32}$$

where the first equality follows from

$$(\vec{X}\cdot\vec{\sigma})(\vec{Y}\cdot\vec{\sigma}) = (\vec{X}\cdot\vec{Y}) + \mathrm{i}(\vec{X}\times\vec{Y})\cdot\vec{\sigma} \tag{3.33}$$

and the fact that the Pauli matrices are traceless. The second equality follows from (3.25).

Now we come back to the tangent vectors $\vec{X}_c$ of (3.21) and consider a bond c on a spanning tree $T(y)$. Formula (3.32) gives

$$\mathcal{L}_{X_c}\mathcal{A}'(U') = -2\vec{v}_c\cdot(R^{-1}(c_-)^*\partial^*\vec{W}(c_-) - R^{-1}(c_+)^*\partial^*\vec{W}(c_+)), \tag{3.34}$$

possibly with c_-, c_+ interchanged depending on the orientation, cf. Theorem 3.4. As the Lie derivatives vanish at a critical point and $\vec{v}_c \in \mathbb{R}^3$ are arbitrary, this concludes the proof. □

Next, we will use the conservation property from Proposition 3.8 to derive the critical point equation. To this end, we recall the linear averaging operator Q stated in (2.6) and define the lattice Green function

$$G(\Omega) := (-\Delta_\Omega + Q^*Q)^{-1}. \tag{3.35}$$

The existence of the inverse defining $G(\Omega)$ is a standard fact, see Lemma 4.4 below. In the statement of the following theorem we also use that $QG(\Omega)R^*Q^*$ is an invertible map on $\mathcal{L}^2(\Omega_1;\mathbb{R}^3)$, which is shown in Lemma 4.20.

Theorem 3.9. *At a critical point of the action (1.17) in $\mathrm{Conf}_\varepsilon(\Omega)$, with the constraint (1.22), the following equation holds*

$$\vec{A} = G(\Omega)R^*_{\vec{A}}Q^*[QG(\Omega)R^*_{\vec{A}}Q^*]^{-1}QG(\Omega)\partial^*\vec{r}_{\vec{A}} - G(\Omega)\partial^*\vec{r}_{\vec{A}}. \tag{3.36}$$

Conversely, any solution of this equation is a critical point.

Proof. By Proposition 3.8, we can write for some block-constant family of vectors $\vec{C}(y_x)$

$$\begin{aligned} R^{-1}(x)^*(-\partial^*\vec{W})(x) &= \vec{C}(y_x), & (3.37)\\ (\Delta_\Omega \vec{A})(x) - (\partial^*\vec{r})(x) &= R(x)^*\vec{C}(y_x), & (3.38)\end{aligned}$$

where we used (3.28). As we checked in Section 2.1, the constraint has the form $Q(\vec{A}) = 0$. Thus we can replace Δ_Ω with $\Delta_\Omega - Q^*Q$ in (3.38), which gives, in terms of the Green function (3.35),

$$\vec{A} = -G(\Omega)R^*\vec{C} - G(\Omega)\partial^*\vec{r}. \tag{3.39}$$

Now by applying Q to both sides, we obtain

$$QG(\Omega)R^*\vec{C} + QG(\Omega)\partial^*\vec{r} = 0. \tag{3.40}$$

Since Q^*Q is the projection on block-constant vectors, cf. (2.7), we have $\vec{C} = Q^*Q\vec{C}$. Since the map $QG(\Omega)R^*Q^*$ is invertible, cf. Lemma 4.20, we have

$$Q\vec{C} = -[QG(\Omega)R^*Q^*]^{-1}QG(\Omega)\partial^*\vec{r}. \tag{3.41}$$

Substituting this to (3.39), and using $C = Q^*QC$, we have

$$\vec{A} = G(\Omega)R^*Q^*[QG(\Omega)R^*Q^*]^{-1}QG(\Omega)\partial^*\vec{r} - G(\Omega)\partial^*\vec{r}. \tag{3.42}$$

This concludes the proof of the first part of the theorem. The last statement is shown by reversing the steps and recalling that Proposition 3.8 is an if and only if statement. □

3.4 Structure of the expressions $\overrightarrow{W}$

In this subsection we will derive a formula of the form:

$$\vec{W}(b) = \vec{A}(b_-) - \vec{A}(b_+) + \vec{r}(b) \tag{3.43}$$

and state a formula for the remainder $\vec{r} = \vec{r}(b) = \vec{r}(b_-, b_+)$. Differently than in the main part of the paper, we will denote a vector in $\mathbb{R}^3$ corresponding to $U, V, Z \in SU(2)$ by $\vec{U}$, $\vec{V}$, $\vec{Z}$, respectively. In the representation (1.26) we have $U = U^0 + \mathrm{i}\vec{U}\cdot\vec{\sigma}$, where $U^0 := s_U U_0$. The multiplication table for a product UV is:

$$\begin{aligned} (UV)^0 &= U^0V^0 - \vec{U}\cdot\vec{V}, & (3.44)\\ (\vec{UV}) &= U^0\vec{V} + V^0\vec{U} - (\vec{U}\times\vec{V}), & (3.45)\end{aligned}$$

where we used (3.33). Let us write for brevity $A_\pm := A(b_\pm)$, $B := B(b) := \partial V(y_b)$ and $\delta(M) := 1 - M$ for any $M \in \mathbb{R}$. Then, by a straightforward application of (3.44), (3.45), postponed to Lemma B.4, we obtain

$$\begin{aligned} \vec{r}(b) = {} & -\delta(A_-^0B^0)\vec{A}_+ + \delta(A_+^0B^0)\vec{A}_- + A_+^0A_-^0\vec{B}\\ & -A_+^0(\vec{A}_-\times\vec{B}) + A_-^0(\vec{B}\times\vec{A}_+) + B^0(\vec{A}_-\times\vec{A}_+)\\ & +\vec{A}_+(\vec{A}_-\cdot\vec{B}) - \vec{A}_-(\vec{B}\cdot\vec{A}_+) + \vec{B}(\vec{A}_-\cdot\vec{A}_+). \qquad (3.46)\end{aligned}$$

From this representation and Theorem 2.5 it is clear that for $U \in \mathrm{Conf}_\varepsilon(\Omega)$ satisfying the constraint (1.22) and $V \in \mathrm{Conf}_{\varepsilon_1}(\Omega_1)$, for $\varepsilon, \varepsilon_1$ sufficiently small (uniformly in n) we can lower all the 0-superscripts, since the corresponding signs s equal 1. Therefore, we have

$$\delta(A_{0,\pm}B_0) = 1 - \sqrt{1 - (\vec{A}_{0,\pm})^2}\sqrt{1 - \vec{B}^2} = \frac{\vec{B}^2 + (\vec{A}_{0,\pm})^2 - \vec{B}^2(\vec{A}_{0,\pm})^2}{1 + A_{0,\pm}B_0}. \tag{3.47}$$

4 Analytic considerations

In this section we prove the existence and uniqueness of solutions of the critical point equation (3.36). In Subsection 4.1 we provide the main line of the argument based on the Banach contraction mapping theorem. One important ingredient here are $\mathcal{L}^\infty$-bounds on the inverse of $QG(\Omega)R^*Q^*$, which are established in Subsections 4.2 - 4.5 and in Appendix A. Another ingredient are $\mathcal{L}^\infty$-bounds on the remainder $\vec{r}$, which are derived in Subsection 4.6.

4.1 Existence of solutions of the critical point equation

Theorem 4.1. ***(Banach contraction mapping theorem)*** *Let* X *be a non-empty complete space with metric* d*. Suppose that* $\mathrm{T}:\mathrm{X}\to\mathrm{X}$ *is a contraction, i.e.,*

$$\mathrm{d}(\mathrm{Tx},\mathrm{Tx}')\leq q\mathrm{d}(\mathrm{x},\mathrm{x}') \tag{4.1}$$

for some $q\in[0,1)$*. Then there is a unique* $\mathrm{x}^*\in\mathrm{X}$ *s.t.* $\mathrm{Tx}^*=\mathrm{x}^*$*. Given the sequence* $\mathrm{x}_n=\mathrm{Tx}_{n-1}$ *we have* $\mathrm{x}^*=\lim_{n\to\infty}\mathrm{Tx}_n$*.*

The space X_ε**.** Suppose $\varepsilon,\varepsilon_1$ are sufficiently small, uniformly in n. Then, for any $U\in\mathrm{Conf}_\varepsilon(\Omega)$, satisfying the constraint (1.22) with $V\in\mathrm{Conf}_{\varepsilon_1}(\Omega_1)$, we have $\vec{A}\in\overrightarrow{\mathrm{Conf}}^{4c_{\frac{1}{2}}L\varepsilon}(\Omega)$ by (2.5) . Furthermore, the relation (1.26) is invertible in this case, if we specify to $s=1$, cf. Subsection 2.1. Thus we can define the map, denoted by the square bracket,

$$[\vec{A}(x)]=U'(x). \tag{4.2}$$

We note that the inverse is simple: $\frac{1}{2}\mathrm{Tr}(U'(x)\cdot\vec{\sigma})=\vec{A}(x)$. Now we define the following notion of distance from zero

$$\mathrm{d}_0(\vec{A},0)=\sup_{b\in\Omega}\|[\vec{A}](b_-)\partial V(y_b)[\vec{A}](b_+)^*-1\|=\sup_{b\in\Omega}\|(\partial U)(b)-1\|, \tag{4.3}$$

which is dictated by the small field condition (1.23). (Clearly, it may vanish for $\vec{A}\neq 0$ as it probes only differences of fields at neighbouring points). Now we define the space

$$\mathrm{X}_\varepsilon:=\left\{\vec{A}\in\overrightarrow{\mathrm{Conf}}^{4c_{1/2}L^2\varepsilon}(\Omega)\,|\,Q(\vec{A})=0,\ \mathrm{d}_0(\vec{A},0)\leq\varepsilon\right\} \tag{4.4}$$

which, considering (2.24), (2.25), consists precisely of configurations satisfying the small field condition and the constraint. While it is clear that any configuration from Theorem A is contained in X_ε, let us comment on the opposite inclusion: Any configuration $\vec{A}\in\overrightarrow{\mathrm{Conf}}^{4c_{1/2}L^2\varepsilon}(\Omega)$ gives one element $U'=[\vec{A}]$, since we specified $s=1$ in (4.2). This provides us with $U=U'V\in\mathrm{Conf}(\Omega)$. Then the condition $\mathrm{d}_0(\vec{A},0)\leq\varepsilon$ is only satisfied if $U\in\mathrm{Conf}_\varepsilon(\Omega)$. Now by reversing the steps in Subsection 2.1 we check that this configuration satisfies the constraint $\mathcal{C}(U)=V$. This equality of configuration spaces is important, for the following reason: If we made the set of configurations smaller than in Theorem A we could not conclude the uniqueness of solutions in the original set. If we made it larger we could not conclude the existence of solutions in the original set.

The metric d**.** We equip X_ε with the following metric:

$$\mathrm{d}(\vec{A}_1,\vec{A}_2):=\|\vec{A}_1-\vec{A}_2\|_{\infty;\Omega}=\sup_{x\in\Omega}|\vec{A}_1(x)-\vec{A}_2(x)|. \tag{4.5}$$

Since X_ε is a closed subset of $\mathbb{R}^{3n^2}$, the completeness of X_ε in the metric d is clear.

The map T. Now the map T is dictated by equation (3.36)

$$\mathrm{T}(\vec{A}) := (G(\Omega)R^*_{\vec{A}}Q^*[QG(\Omega)R^*_{\vec{A}}Q^*]^{-1}Q - 1)G(\Omega)\partial^*\vec{r}_{\vec{A}}. \tag{4.6}$$

We have to check that T maps X_ε into itself and satisfies

$$\|\mathrm{T}(\vec{A}_1) - \mathrm{T}(\vec{A}_2)\|_{\infty;\Omega} \leq q\|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega} \tag{4.7}$$

for some $0 \leq q < 1$. We start by checking that T maps X_ε into itself for sufficiently small ε, ε_1.

Proposition 4.2. *For $\varepsilon_1 \leq \varepsilon^2$, sufficiently small (uniformly in n),* T *maps* X_ε *into itself.*

Proof. It is manifest from definition (4.6) that T preserves the constraint, i.e., $Q(\mathrm{T}(\vec{A})) = 0$. Next, we observe that, by Lemmas 4.20, 4.23,

$$\|\mathrm{T}(\vec{A})\|_{\infty;\Omega} \leq C\|\partial^*\vec{r}\|_{\infty;\Omega} \leq 24C(\varepsilon^2 + \varepsilon_1). \tag{4.8}$$

Thus for ε sufficiently small relation (4.2) is well defined for $\mathrm{T}(\vec{A})$. Given this, we have to show that, if $\mathrm{d}_0(\vec{A}, 0) \leq \varepsilon$, then also

$$\mathrm{d}_0(\mathrm{T}(\vec{A}), 0) = \sup_{b\in\Omega'} \|[\mathrm{T}(\vec{A})](b_-)\partial V(y_b)[\mathrm{T}(\vec{A})](b_+)^* - 1\| \leq \varepsilon. \tag{4.9}$$

We have

$$\|[\mathrm{T}(\vec{A})](b_-)\partial V(y_b)[\mathrm{T}(\vec{A})](b_+)^* - 1\| \leq \varepsilon_1 + \|[\mathrm{T}(\vec{A})](b_-)[\mathrm{T}(\vec{A})](b_+)^* - 1\|. \tag{4.10}$$

Furthermore, by Lemma B.3,

$$\begin{aligned}\|[\mathrm{T}(\vec{A})](b_-)[\mathrm{T}(\vec{A})](b_+)^* - 1\| &= \|[\mathrm{T}(\vec{A})](b_-) - [\mathrm{T}(\vec{A})](b_+)\| \\ &\leq \sqrt{6}(|\mathrm{T}(\vec{A})(b_-)| + |\mathrm{T}(\vec{A})(b_+)|) \\ &\leq C'(\varepsilon^2 + \varepsilon_1),\end{aligned} \tag{4.11}$$

where in the last step we used (4.8). Now using that we have ε^2 (and not just ε) on the r.h.s. of (4.11), we conclude the proof. □

Proposition 4.3. *Under the assumptions of Proposition 4.2 and for $\varepsilon, \varepsilon_1$, sufficiently small (uniformly in n), the map* $\mathrm{T} : \mathrm{X}_\varepsilon \to \mathrm{X}_\varepsilon$ *is a contraction.*

Proof. We rewrite (4.6) as follows

$$\mathrm{T}(\vec{A}) = (M_{\vec{A}} - 1)G(\Omega)\partial^*\vec{r}_{\vec{A}}, \quad M_{\vec{A}} := G(\Omega)R^*_{\vec{A}}Q^*[QG(\Omega)R^*_{\vec{A}}Q^*]^{-1}Q. \tag{4.12}$$

We will divide the problem into two parts

$$\mathrm{T}(\vec{A}_1) - \mathrm{T}(\vec{A}_2) = (M_{\vec{A}_1} - 1)(G(\Omega)\partial^*\vec{r}_{\vec{A}_1} - G(\Omega)\partial^*\vec{r}_{\vec{A}_2}) \tag{4.13}$$

$$+(M_{\vec{A}_1} - M_{\vec{A}_2})G(\Omega)\partial^*\vec{r}_{\vec{A}_2}. \tag{4.14}$$

We consider first the shift (4.14). We have, by Lemma 4.23,

$$\|G(\Omega)\partial^*\vec{r}_{\vec{A}}\|_{\infty;\Omega} \leq \|G(\Omega)\|_{\infty,\infty;\Omega}24C(\varepsilon^2 + \varepsilon_1). \tag{4.15}$$

Next, we write

$$\begin{aligned}M_{\vec{A}_2} - M_{\vec{A}_1} &= G(\Omega)R^*_{\vec{A}_2}Q^*([QG(\Omega)R^*_{\vec{A}_2}Q^*]^{-1} - [QG(\Omega)R^*_{\vec{A}_1}Q^*]^{-1})Q \\ &\quad +G(\Omega)(R^*_{\vec{A}_2} - R^*_{\vec{A}_1})Q^*[QG(\Omega)R^*_{\vec{A}_1}Q^*]^{-1}Q.\end{aligned} \tag{4.16}$$

Thus, by Lemmas 4.22, 4.21, we have for $D_{\vec{A}} := QG(\Omega)R^*_{\vec{A}}Q^*$

$$\begin{aligned} \|M_{\vec{A}_2} - M_{\vec{A}_1}\|_{\infty,\infty;\Omega} \;\leq\; & 2\|G(\Omega)\|_{\infty,\infty;\Omega} 2\|D^{-1}_{\vec{A}_1}\|_{\infty,\infty;\Omega_1} \|D^{-1}_{\vec{A}_2}\|_{\infty,\infty;\Omega_1} \|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega_1} \\ & + \|G(\Omega)\|_{\infty,\infty;\Omega} \|D^{-1}_{\vec{A}_1}\|_{\infty,\infty;\Omega_1} \|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega}. \end{aligned} \tag{4.17}$$

Finally, making use of Lemmas 4.24, 4.21 we can estimate (4.13) as follows

$$\begin{aligned} \|(4.13)\|_{\infty;\Omega} \;&\leq\; (1 + \|M_{\vec{A}_1}\|_{\infty,\infty;\Omega}) \|G(\Omega)\|_{\infty,\infty;\Omega} \|\partial^* \vec{r}_{\vec{A}_1} - \partial^* \vec{r}_{\vec{A}_2}\|_{\infty;\Omega} & (4.18) \\ &\leq\; (1 + \|M_{\vec{A}_1}\|_{\infty,\infty;\Omega}) \|G(\Omega)\|_{\infty,\infty;\Omega} 96C(\varepsilon + \varepsilon_1) \|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega}. & (4.19) \end{aligned}$$

This concludes the proof, considering that all the $\| \dots \|_{\infty,\infty;\Omega}$-norms above are bounded uniformly in n by Lemmas 4.20, 4.19, 4.22. (We remark that Lemmas 4.24, 4.22, 4.21, which we used in the proof, require the constraint as they use implication (2.24)). □

4.2 Strict positivity of G^{-1}, G and QGQ^*

In this subsection we start working towards the $\mathcal{L}^\infty$-bounds on $QG(\Omega)Q^*$, $(QG(\Omega)Q^*)^{-1}$ which we used in the proofs of Propositions 4.2, 4.3. These bounds follow from the exponential decay of the integral kernels of these operators. This latter property is first shown for the counterparts of these operators on $\mathcal{L}^2(\mathbb{Z}^2)$ by the method of random walk expansions (see Appendix A, Proposition 4.5). In Subsection 4.4 we translate this property on the finite lattice using the method of images.

We will denote by $-\Delta$ the Laplacian with free boundary conditions on $\mathcal{L}^2(\mathbb{Z}^2)$, by Q the averaging operator (which is a natural extension of its finite lattice counterpart (2.6)) and write

$$G := (-\Delta + Q^*Q)^{-1} \tag{4.20}$$

for the lattice Green function on $\mathcal{L}^2(\mathbb{Z}^2)$ and note that QGQ^* is an operator on $\mathcal{L}^2(L\mathbb{Z}^2)$. To substantiate definition (4.20) and also to check one assumption of Proposition 4.5 below, we study the strict positivity of the relevant operators. The following lemma is standard, cf. [Di13, Lemma 29], [DST24, Lemma 2.10] for similar considerations.

Lemma 4.4. *The following points hold:*

1. *Let $y \in \Omega_1$ and denote by $\Delta_{B_1(y)}$ the Laplacian on $B_1(y)$ with Neumann boundary conditions, cf. [DST24, Section 2]. Then, as operators on $\mathcal{L}^2(B_1(y))$,*

$$-\Delta_{B_1(y)} + Q^*Q \geq C. \tag{4.21}$$

2. *The following inequalities hold as operators on $\mathcal{L}^2(\Omega)$*

$$\begin{aligned} -\Delta_\Omega + Q^*Q \;&\geq\; C, & (4.22) \\ (-\Delta_\Omega + Q^*Q)^{-1} \;&\geq\; c. & (4.23) \end{aligned}$$

3. *The following inequality holds as operators on $\mathcal{L}^2(\Omega_1)$*

$$Q(-\Delta_\Omega + Q^*Q)^{-1}Q^* \geq c. \tag{4.24}$$

4. *The bounds (4.22), (4.23), (4.24) also hold for the corresponding operators on $\mathcal{L}^2(\mathbb{Z}^2)$, resp. $\mathcal{L}^2(L\mathbb{Z}^2)$.*

Here $C, c > 0$ are independent of n, but C may depend on L.

Proof. As for 1., if $f \in \mathcal{L}^2(B_1(y))$ is constant then $-\Delta_{B_1(y)} f = 0$ and

$$\langle f, Q^*Qf \rangle = \|f\|^2_{2;B_1(y)}. \tag{4.25}$$

On the other hand, $-\Delta_{B_1(y)}$ is strictly positive on the orthogonal complement of the subspace of constant functions. Specifically, its lowest eigenvalue is given by (see e.g. [DST24, Lemma 2.3])

$$(-\lambda^{(1)}) = 4\sin^2\left(\frac{\pi}{2L}\right). \tag{4.26}$$

We have $(-\lambda^{(1)}) \geq C > 0$. Therefore,

$$\langle f, (-\Delta_{B_1(y)})f \rangle \geq C\|f\|^2_{2;B_1(y)} \tag{4.27}$$

and since Q^*Q is also positive this proves (4.21).

Regarding 2., let $f \in \mathcal{L}^2(\Omega)$ and set $f_{B_1(y)} := f|_{B_1(y)}$. We have

$$\begin{aligned} \langle f, (-\Delta_\Omega + Q^*Q)f \rangle \ &\geq \sum_{y\in\Omega_1} \langle f_{B_1(y)}, (-\Delta_{B_1(y)} + Q^*Q) f_{B_1(y)} \rangle \\ &\geq \ C \sum_{y\in\Omega_1} \|f_{B_1(y)}\|^2_{2;\Omega} = C\|f\|^2_{2;\Omega}. \end{aligned} \tag{4.28}$$

Here in the first inequality we used the Neumann boundary conditions and (3.27) to justify that we can drop the bonds linking different boxes $B_1(y)$. This gives (4.22). To justify (4.23), we use that on a unit lattice

$$\|\Delta_\Omega\|_{2,2;\Omega} \leq 4 \tag{4.29}$$

and Q^*Q has norm one as a projection.

As for 3., we note that, in general, if a Hermitian matrix M satisfies $M \geq c$ on $\mathcal{L}^2(\Omega)$, then $QMQ^* \geq c$ on $\mathcal{L}^2(\Omega_1)$. In fact, suppose $\langle f, QMQ^*f \rangle < c\|f\|^2_{2;\Omega_1}$ for some $f \in \mathcal{L}^2(\Omega_1)$. Then, since $QQ^* = 1$, we have $\|f\|^2_{2;\Omega_1} = \|Q^*f\|_{2;\Omega} = 0$. Setting $M{=}G(\Omega)$ and using item 2. we obtain the claim. Alternatively, this implication can be seen using that the map $N \mapsto QNQ^*$ on the set of Hermitian matrices is completely positive and unital. By setting $N = M - c$, the claim follows.

Regarding 4, we observe that the proofs of items 2. and 3. can be immediately adapted to infinite lattices. In particular, (4.28), (4.29) remain valid. □

4.3 Exponential decay of integral kernels of G, $(QGQ^*)^{-1}$

From Theorem A.6 and Lemma A.3 we obtain immediately the following fact:

Proposition 4.5. *Let M on $\mathcal{L}^2(\mathbb{Z}^2)$ be strictly positive, i.e., $M \geq \mathrm{m} > 0$, and let*

$$|M(x, x')| \leq C\,\mathrm{e}^{-C_1|x-x'|} \tag{4.30}$$

for some constants $C, C_1 > 0$. Then

$$|M^{-1}(x, x')| \leq C'\,\mathrm{e}^{-C_1'|x-x'|} \tag{4.31}$$

for some constants $C', C_1' > 0$. (These constants depend on C, C_1 and m).

Corollary 4.6. *Let N on $\mathcal{L}^2(L\mathbb{Z}^2)$ be strictly positive i.e., $M \geq \mathrm{m} > 0$, and let*

$$|N(y,y')| \leq C\,\mathrm{e}^{-C_1|y-y'|}, \tag{4.32}$$

for some constants $C, C_1 > 0$. Then

$$|N^{-1}(y,y')| \leq C'\,\mathrm{e}^{-C_1'|y-y'|} \tag{4.33}$$

for some constants $C', C_1' > 0$. (These constants depend on C, C_1, m and L).

Proof. Consider a unitary scaling transformation $S_L : \mathcal{L}^2(\mathbb{Z}^2) \to \mathcal{L}^2(L\mathbb{Z}^2)$ given by

$$S_L f := L^{-1} f_L, \quad f_L(x) := f(L^{-1}x). \tag{4.34}$$

We define

$$M := S_L^* N S_L, \tag{4.35}$$

which is strictly positive on $\mathcal{L}^2(\mathbb{Z}^2)$. We check the assumption (4.30) of Proposition 4.5:

$$|M(x,x')| = |\langle \delta_x, S_L^* N S_L \delta_{x'}\rangle| = |\langle \delta_{Lx}^L, N\delta_{Lx'}^L\rangle| = |N(Lx, Lx')| \leq c\,\mathrm{e}^{-C_1 L|x-x'|}, \tag{4.36}$$

where $\delta_y^L := \frac{1}{L^2}\delta_y$ is the delta function on the lattice $L\mathbb{Z}^2$. Thus Proposition 4.5 gives

$$|M^{-1}(x,x')| \leq C'\,\mathrm{e}^{-C_1'|x-x'|}. \tag{4.37}$$

Now analogous steps as in (4.36) give

$$|N^{-1}(Lx, Lx')| = |M^{-1}(x,x')| \leq C'\mathrm{e}^{-C_1'|x-x'|}, \tag{4.38}$$

which concludes the proof. □

Now we apply Corollary 4.6 to study the integral kernels of operators G and $(QGQ^*)^{-1}$.

Lemma 4.7. *The following properties hold true:*

$$|G(x,x')| \ \leq\ C\mathrm{e}^{-C_1|x-x'|}, \tag{4.39}$$
$$|(QGQ^*)(y,y')| \ \leq\ C\mathrm{e}^{-C_1|y-y'|}, \tag{4.40}$$
$$|(QGQ^*)^{-1}(y,y')| \ \leq\ C\mathrm{e}^{-C_1|y-y'|}, \tag{4.41}$$

for some constants $C, C_1 > 0$.

Proof. We consider the operator $M := -\Delta + Q^*Q$, which is strictly positive on $\mathcal{L}^2(\mathbb{Z}^2)$ by Lemma 4.4. and note that its integral kernel $M(x,x')$ vanishes unless $|x-x'| \leq 1$ or $x' \in B_1(y_x)$. Hence, since $L > 1$,

$$|M(x,x')| \leq c\chi(|x-x'|_\infty \leq L) \leq c'\mathrm{e}^{-L^{-1}|x-x'|_\infty} \leq c'\mathrm{e}^{-\sqrt{2}L^{-1}|x-x'|}, \tag{4.42}$$

where χ is the characteristic function. Thus Proposition 4.5 gives

$$|G(x,x')| \leq C\mathrm{e}^{-C_1|x-x'|}, \quad C_1 > 0, \tag{4.43}$$

which proves (4.39). Now we consider $N := QGQ^*$ which is strictly positive on $\mathcal{L}^2(L\mathbb{Z}^2)$ by Lemma 4.4. We check assumption (4.32) of Corollary 4.6:

$$\begin{aligned} |N(y,y')| \ &=\ |\langle Q^*\delta_y^L, GQ^*\delta_{y'}^L\rangle| \leq \sum_{x,x'} \mathbb{1}_{B_1(y)}(x)|G(x,x')|\mathbb{1}_{B_1(y')}(x') \\ &\leq\ C\sum_{x,x'} \mathbb{1}_{B_1(y)}(x)\mathrm{e}^{-C_1|x-x'|}\mathbb{1}_{B_1(y')}(x') \\ &\leq\ CL^4\mathrm{e}^{-C_1(|y-y'|-\sqrt{2}L)}, \end{aligned} \tag{4.44}$$

where we use that $Q^*\delta_y^L$ is the characteristic function of $B_1(y)$ and $|x-x'| \geq |y-y'| - \sqrt{2}L$. Now Corollary 4.6 gives (4.41). □

4.4 Method of images

Recall that $\Omega := I^{\times 2}$, where $I = [0, 1, 2, \dots, n-1]$. The boundary $\partial\Omega \subset \Omega$ consists of 4 faces

$$\partial\Omega = (\{0\} \times I) \cup (I \times \{0\}) \cup (\{n-1\} \times I) \cup (I \times \{n-1\}), \tag{4.45}$$

which we denote $\partial\Omega_0$, $\partial\Omega_1$, $\partial\Omega^0$, $\partial\Omega^1$, respectively. Next, we introduce the usual discrete partial derivatives on $\mathcal{L}^2(\mathbb{Z}^2)$ and their adjoints consistently with Definition 3.6

$$(\partial_\mu f)(x) := f(x + e_\mu) - f(x), \quad (\partial_\mu^* f)(x) := -(f(x) - f(x - e_\mu)), \tag{4.46}$$

where e_μ is the unit vector in the μ-th direction. With these definitions we can select the following subspace of functions in $\mathcal{L}^2(\mathbb{Z}^2)$:

Definition 4.8. *We say that a function $f \in \mathcal{L}^2(\mathbb{Z}^d)$ satisfies Neumann boundary conditions on Ω, if the following relations hold on the respective subsets of the boundary (4.45):*

$$(\partial_\mu^* f)(x) \ = \ 0 \quad \textit{for} \quad x \in \partial\Omega_\mu, \tag{4.47}$$

$$(\partial_\mu f)(x) \ = \ 0 \quad \textit{for} \quad x \in \partial\Omega^\mu, \tag{4.48}$$

$\mu = 0, 1$. *We denote the subspace of such functions D_Ω.*

Now we formulate an equivalent condition for $f \in \mathcal{L}^2(\mathbb{Z}^2)$ to be an element of D_Ω. We let P_μ (resp. $\overline{P}_\mu$) be reflections w.r.t. axes parallel to $\partial\Omega_\mu$ (resp. $\partial\Omega^\mu$) as indicated in Figure 2. (We refer to [DST24, Section 3] for formal definitions of these reflections). The following lemma is immediate:

Lemma 4.9. *Let $f \in \mathcal{L}^2(\mathbb{Z}^2)$, $\mu = 0, 1$. Then $f \in D_\Omega$ iff it satisfies*

$$(P_\mu f)(x) \ = \ f(x), \quad x \in \partial\Omega_\mu, \tag{4.49}$$

$$(\overline{P}_\mu f)(x) \ = \ f(x), \quad x \in \partial\Omega^\mu. \tag{4.50}$$

Proof. See [DST24, Lemma 3.10]. □

Lemma 4.10. *The following properties hold*

$$P_\mu Q^* Q P_\mu = Q^* Q, \quad \overline{P}_\mu Q^* Q \overline{P}_\mu = Q^* Q. \tag{4.51}$$

Consequently,

$$G(P_\mu x, P_\mu x') = G(\overline{P}_\mu x, \overline{P}_\mu x') = G(x, x'). \tag{4.52}$$

Proof. See [DST24, Lemma 4.1]. □

As a useful application of Lemmas 4.9, 4.10, we obtain the following:

Lemma 4.11. *Suppose that $f \in D_\Omega$. Then, for any $\ell \in \mathbb{N}$ we have*

$$((-\Delta_\Omega + Q^* Q)^\ell f_\Omega)(x) = ((-\Delta + Q^* Q)^\ell f)(x), \tag{4.53}$$

where $x \in \Omega$ and $f_\Omega := f|_\Omega$.

Proof. Property (4.53) clearly holds for $\ell = 1$ by definition of D_Ω. Using this, we write

$$\begin{aligned}\langle \delta_x, (-\Delta_\Omega + Q^*Q)^\ell f_\Omega\rangle &= \sum_{x'\in\Omega} \langle \delta_x, (-\Delta_\Omega + Q^*Q)^{\ell-1}\delta_{x'}\rangle \langle \delta_{x'}, (-\Delta + Q^*Q) f\rangle \\ &= \sum_{x'\in\Omega} \langle \delta_x, (-\Delta_\Omega + Q^*Q)^{\ell-1}\delta_{x'}\rangle \langle \delta_{x'}, ((-\Delta + Q^*Q) f)_\Omega\rangle \\ &= \langle \delta_x, (-\Delta_\Omega + Q^*Q)^{\ell-1}((-\Delta + Q^*Q) f)_\Omega\rangle. \end{aligned} \tag{4.54}$$

Now we obtain from Lemmas 4.9, 4.10 that if $f \in D_\Omega$ then also $(-\Delta + Q^*Q)f \in D_\Omega$. Thus we can iterate the argument (4.54) until we obtain (4.53). □

Using the Stone-Weierstrass theorem we can extend Lemma 4.11 from polynomials to continuous functions:

Lemma 4.12. *For any $f \in D_\Omega$ and $F \in C(\mathbb{R})$ we have*

$$(F(-\Delta_\Omega + Q^*Q) f_\Omega)(x) = (F(-\Delta + Q^*Q) f)(x), \quad x \in \Omega, \tag{4.55}$$

where $f_\Omega \in \mathcal{L}^2(\Omega)$ is the restriction of $f \in \mathcal{L}^2(\mathbb{Z}^2)$ to Ω.

Remark 4.13. *Referring to Lemma 4.4, we obtain from (4.55)*

$$(G(\Omega) f_\Omega)(x) = (Gf)(x), \quad f \in D_\Omega, \quad x \in \Omega. \tag{4.56}$$

Proof. Since $-\Delta_\Omega + Q^*Q$, $\Delta + Q^*Q$ are bounded operators, we can restrict F to a bounded interval $[a, b]$ containing their spectra. Then, for any ϵ we can find a polynomial F_ϵ s.t.

$$\sup_{\lambda\in[a,b]} |(F - F_\epsilon)(\lambda)| \leq \epsilon. \tag{4.57}$$

Using this, we estimate

$$\begin{aligned} &|(F(-\Delta_\Omega + Q^*Q) f_\Omega)(x) - (F(-\Delta + Q^*Q) f)(x)| \\ &\quad \leq |\langle \delta_x, (F - F_\epsilon)(-\Delta_\Omega + Q^*Q) f_\Omega\rangle| + |\langle \delta_x, (F - F_\epsilon)(-\Delta + Q^*Q) f\rangle| \\ &\qquad\qquad + |\langle \delta_x, F_\epsilon(-\Delta_\Omega + Q^*Q) f_\Omega\rangle - \langle \delta_x, F_\varepsilon(-\Delta + Q^*Q) f\rangle| \\ &\quad \leq \epsilon 2\|f\|_2 + |\langle \delta_x, F_\epsilon(-\Delta_\Omega + Q^*Q) f_\Omega\rangle - \langle \delta_x, F_\varepsilon(-\Delta + Q^*Q) f\rangle|. \end{aligned} \tag{4.58}$$

As the last term on the r.h.s. of (4.58) vanishes by Lemma 4.11, the proof is complete. □

After these preparations we can move on to the method of images. Define the set of image points $\mathrm{Img} := \{z_j\}_{j\in\mathbb{N}}$ on $\mathbb{Z}^2$ by the following two requirements ([GJ87][Section 7.4])

- $z \in \mathrm{Img}$,
- The set Img is invariant under the reflections P_μ, $\overline{P}_\mu$, $\mu = 0, 1$, defined above.

This set is depicted in Figure 2. It is well known that the following relation between the Green functions with free and Neumann boundary conditions holds true:

Lemma 4.14. *For $x, z \in \Omega$ the following identity holds*

$$G(\Omega)(x, z) = \sum_{z_j\in\mathrm{Img}} G(x, z_j). \tag{4.59}$$

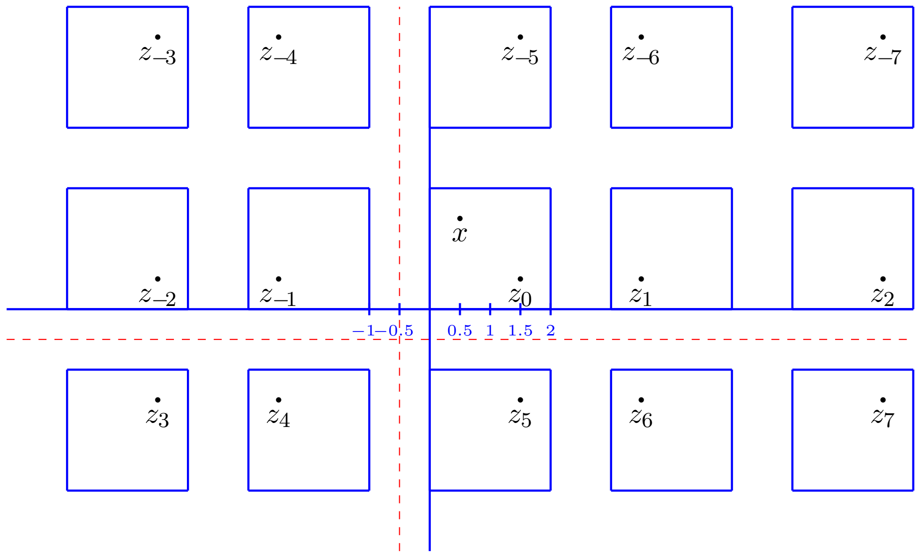

Figure 2: The square containing the origin is the set Ω. The reflections P_μ are defined w.r.t. the dashed lines from the figure. The points z_j of the argument z are as in formula (4.59).

Proof. See [DST24, Lemma 4.2]. □

Our goal in this subsection is to derive an analogous formula linking operators $(QG(\Omega)Q^*)^{-1}$ and $(QGQ^*)^{-1}$, acting on $\mathcal{L}^2(\Omega_1)$ and $\mathcal{L}^2(L\mathbb{Z}^2)$, respectively. For this purpose we define the reflections on the coarse lattice

$$P_{1,\mu} := QP_\mu Q^*, \quad \overline{P}_{1,\mu} := Q\overline{P}_\mu Q^*. \tag{4.60}$$

By acting on delta functions it is easy to see that the geometric action of these reflections is determined by

$$x \in B_1(P_{1,\mu}y) \quad \Leftrightarrow \quad P_\mu x \in B_1(y), \tag{4.61}$$

and analogously for $\overline{P}_\mu$, where we naturally extended the definition of $B_1(y)$ from Ω to $\mathbb{Z}^2$. Furthermore, we have by the first relation in (4.60) and $QQ^* = 1$

$$Q^*P_{1,\mu} = P_\mu Q^*. \tag{4.62}$$

This latter property combined with (4.52) immediately gives:

Lemma 4.15. *The following properties hold:*

$$(QGQ^*)(P_{1,\mu}y, P_{1,\mu}y') = (QGQ^*)(y, y'), \tag{4.63}$$
$$(QGQ^*)^{-1}(P_{1,\mu}y, P_{1,\mu}y') = (QGQ^*)^{-1}(y, y'), \tag{4.64}$$

and analogously for $\overline{P}_{1,\mu}$.

Now we define the set of image points $\mathrm{Img}_1 := \{z_j\}_{j\in\mathbb{N}}$ on the coarse lattice $L\mathbb{Z}^2$, by the following two requirements:

- $z \in \mathrm{Img}_1$,
- The set Img_1 is invariant under the reflections $P_{1,\mu}$, $\overline{P}_{1\mu}$, $\mu = 0, 1$, defined in (4.60).

The main result of this subsection is the following:

Lemma 4.16. *For $x, z \in \Omega$ the following identity holds*

$$(QG(\Omega)Q^*)^{-1}(y,z) = \sum_{z_j \in \mathrm{Img}_1} (QGQ^*)^{-1}(y,z_j). \tag{4.65}$$

Proof. Given the distribution of image points $z_j \in \mathrm{Img}_1$, and estimate (4.41) we obtain that the sum in (4.65) is convergent. It suffices to check that, in the sense of multiplication of operators on $\mathcal{L}^2(\Omega_1)$,

$$(QG(\Omega)Q^*)(\text{r.h.s. of (4.65)}) \; = 1. \tag{4.66}$$

For any $\ell \in \mathbb{N}$, let $I_1^\ell := L[-\ell, -\ell+1, \ldots 0, \ldots, n_1-1, \ldots, n_1+\ell-1]$, $n_1 - 1 := L^{m-1}$, and $\Omega_1^\ell := (I_1^\ell)^{\times 2}$ be a coarse finite lattice containing Ω_1. We will check that for each $z \in \Omega_1$ and $y \in L\mathbb{Z}^2$ the expression

$$F_z^\ell(y) := \chi_{\Omega_1^\ell}(y) \sum_{z_j \in \mathrm{Img}_1} (QGQ^*)^{-1}(y,z_j) \tag{4.67}$$

satisfies $Q^* F_z^\ell \in D_\Omega \subset \mathcal{L}^2(\mathbb{Z}^2)$ (cf. Definition 4.8). For this purpose we will use the criterion from Lemma 4.9: Let $\tilde{P}_\mu$ denote the reflections P_μ or $\overline{P}_\mu$ and similarly for $\tilde{P}_{1,\mu}$. We consider $x \in \partial\Omega$, so $\chi_{\Omega_1^\ell}(y_x) = \chi_{\Omega_1^\ell}(P_{1,\mu} y_x) = 1$. (We need $\chi_{\Omega_1^\ell}$ in (4.67) only to ensure that $x \mapsto F_z^\ell(x)$ is in $\mathcal{L}^2(L\mathbb{Z}^2)$). Then, since $Q\delta_x = \delta^L_{y_x}$,

$$\begin{aligned}(Q^*F_z^\ell)(\tilde{P}_\mu x) \;&=\; \langle Q\delta_{\tilde{P}_\mu x}, \sum_{z_j \in \mathrm{Img}_1} (QGQ^*)^{-1}\delta^L_{z_j}\rangle = \sum_{z_j \in \mathrm{Img}_1} (QGQ^*)^{-1}(y_{\tilde{P}_\mu x}, z_j) \\ &= \sum_{z_j \in \mathrm{Img}_1} (QGQ^*)^{-1}(\tilde{P}_{1,\mu} y_x, \tilde{P}_{1,\mu} z_j) = \sum_{z_j \in \mathrm{Img}_1} (QGQ^*)^{-1}(y_x, z_j) = (Q^*F_z^\ell)(x), \end{aligned} \tag{4.68}$$

where in the third step we used property (4.61) and the invariance of the set Img_1 under the reflections, and in the fourth step Lemma 4.15. Thus $Q^*F_z \in D_\Omega$ and we obtain from Lemma 4.12 that, for $x \in \Omega$, $F^\ell_{z,\Omega_1} := F_z^\ell|_{\Omega_1}$,

$$\begin{aligned}(QG(\Omega)Q^*F^\ell_{z,\Omega_1})(y) \;&=\; (QGQ^*F_z^\ell)(y) \\ &= \sum_{z_j \in \mathrm{Img}_1,\, y' \in L\mathbb{Z}^2} \langle \delta^L_y, QGQ^*\delta^L_{y'}\rangle \langle \delta^L_{y'}, \chi_{\Omega_1^\ell}(QGQ^*)^{-1}\delta^L_{z_j}\rangle \\ &= \delta^L_z(y) - \sum_{z_j \in \mathrm{Img}_1,\, y' \in L\mathbb{Z}^2} \langle \delta^L_y, QGQ^*\delta^L_{y'}\rangle \langle \delta^L_{y'}, (1-\chi_{\Omega_1^\ell})(QGQ^*)^{-1}\delta^L_{z_j}\rangle, \end{aligned} \tag{4.69}$$

where in the last step we made use of the fact that z is the only element of Img_1 inside Ω_1 to obtain $\delta^L_z(y)$. To conclude we note that the l.h.s. of (4.69) is independent of ℓ due to the restriction of F_z^ℓ to Ω_1 and that $\lim_{\ell\to\infty} \chi_{\Omega_1^\ell} = 1$ pointwise. To enter with the limit under the sums w.r.t. z_j, y' on the r.h.s. we use the exponential decay of the kernels of $QGQ^*, (QGQ^*)^{-1}$, shown in Lemma 4.7, and dominated convergence. □

We immediately obtain from Lemmas 4.14, 4.16, the exponential decay of kernels of the corresponding operators on a finite lattice. The following lemma is proven analogously to [DST24, Theorem 4.3].

Lemma 4.17. *The following properties hold true:*

$$|G(\Omega)(x,x')| \;\leq\; C\mathrm{e}^{-C_1|x-x'|}, \tag{4.70}$$

$$|(QG(\Omega)Q^*)^{-1}(y,y')| \;\leq\; C\mathrm{e}^{-C_1|y-y'|}, \tag{4.71}$$

for some constants $C, C_1 > 0$, independent of n.

Remark 4.18. *By a more careful analysis one can see that in (4.70) $C \sim L^4$ and C_1 is independent of L, cf. [DST24, Theorem A].*

4.5 $\mathcal{L}^\infty$-bounds on $G(\Omega)$, $(QG(\Omega)Q^*)^{-1}$, $(QG(\Omega)R^*Q^*)^{-1}$

In Subsections 4.2 – 4.4 we treated $G(\Omega)$, $(QG(\Omega)Q^*)^{-1}$, as operators acting on spaces $\mathcal{L}^2(\Omega), \mathcal{L}^2(\Omega_1)$ of scalar-valued functions. Let us now consider them as operators acting componentwise on functions with values in $\mathbb{R}^3$, as we did in Subsection 3.3. We recall that $\mathcal{L}^\infty(\Omega;\mathbb{R}^3)$ is equipped with the norm

$$\|\vec{v}\|_{\infty;\Omega} = \sup_{x\in\Omega} |\vec{v}(x)|, \tag{4.72}$$

where $|\cdot|$ is the Euclidean distance in $\mathbb{R}^3$. Now for $M : \mathcal{L}^\infty(\Omega;\mathbb{R}^3) \to \mathcal{L}^\infty(\Omega;\mathbb{R}^3)$, $N : \mathcal{L}^\infty(\Omega_1;\mathbb{R}^3) \to \mathcal{L}^\infty(\Omega_1;\mathbb{R}^3)$ the resulting operator norms satisfy

$$\|M\|_{\infty,\infty;\Omega} := \sup_{\|\vec{f}\|_{\infty;\Omega}\le 1} \|M\vec{f}\|_{\infty;\Omega} \le \sup_{x\in\Omega} \sum_{x'\in\Omega} \|M(x,x')\|, \tag{4.73}$$

$$\|N\|_{\infty,\infty;\Omega_1} := \sup_{\|\vec{f}\|_{\infty;\Omega_1}\le 1} \|N\vec{f}\|_{\infty;\Omega} \le \sup_{y\in\Omega_1} L^2 \sum_{y'\in\Omega_1} \|N(y,y')\|, \tag{4.74}$$

where $\|M(x,x')\|$ is the operator norm of the 3×3 matrix $M(x,x')$ w.r.t. $|\cdot|$. Using these relations we immediately obtain:

Lemma 4.19. *The following bounds hold true*

$$\|G(\Omega)\|_{\infty,\infty;\Omega} \le C, \tag{4.75}$$

$$\|(QG(\Omega)Q^*)^{-1}\|_{\infty,\infty;\Omega_1} \le C, \tag{4.76}$$

for C independent of n.

Proof. For $M = G(\Omega)$ we immediately obtain the bound referring to (4.73) and the decay of the kernel (4.70). For $N = (QG(\Omega)Q^*)^{-1}$ we use (4.74) and (4.71). □

In the next lemma we consider an operator on $\mathcal{L}^2(\Omega_1;\mathbb{R}^3)$ which has a non-trivial action on the target space $\mathbb{R}^3$.

Lemma 4.20. *The operator $D_{\vec{A}} := QG(\Omega)R^*_{\vec{A}}Q^*$ is invertible and the following estimate holds true for $\vec{A} \in \overrightarrow{\mathrm{Conf}}^\varepsilon(\Omega)$, $s = 1$ and some constant C independent of n and ε*

$$\|(QG(\Omega)R^*_{\vec{A}}Q^*)^{-1}\|_{\infty,\infty;\Omega_1} \le C, \tag{4.77}$$

provided that $0 < \varepsilon \le 1$ sufficiently small (uniformly in n).

Proof. We define $\delta R^*_{\vec{A}} := R^*_{\vec{A}} - 1$. It has the form

$$\delta R^*_{\vec{A}}(x)\vec{v}(x) = (A_0(x) - 1)\vec{v}(x) - \vec{A}(x) \times \vec{v}(x) \tag{4.78}$$

and satisfies, by assumption and the relation $A_0(x) = \sqrt{1 - |\vec{A}(x)|^2}$

$$\|\delta R_{\vec{A}}(x)\| \le |A_0(x) - 1| + |\vec{A}(x)| \le 2\varepsilon. \tag{4.79}$$

We have the following

$$QG(\Omega)R^*_{\vec{A}}Q^* = (QG(\Omega)Q^*)(1 + (QG(\Omega)Q^*)^{-1}QG(\Omega)\delta R^*_{\vec{A}}Q^*). \tag{4.80}$$

The first factor is invertible by part 3. of Lemma 4.4. The second factor is invertible provided that

$$\|(QG(\Omega)Q^*)^{-1}QG(\Omega)\delta R^*_{\vec{A}}Q^*\|_{\infty,\infty;\Omega_1} \le \|(QG(\Omega)Q^*)^{-1}\|_{\infty,\infty;\Omega_1}\|QG(\Omega)\delta R^*_{\vec{A}}Q^*\|_{\infty,\infty;\Omega_1} < 1. \tag{4.81}$$

Indeed, then the expansion of $(1+(QG(\Omega)Q^*)^{-1}QG(\Omega)\delta R^*_{\vec{A}}Q^*)^{-1}$ into the Neumann series is convergent. Given (4.76), it suffices to control $\|QG(\Omega)\delta R^*_{\vec{A}}Q^*\|_{\infty,\infty;\Omega_1}$ to show (4.81). In order to apply (4.74), we estimate

$$\begin{aligned}\|\langle\delta^L_y, QG(\Omega)\delta R^*_{\vec{A}}Q^*\delta^L_{y'}\rangle\| &\leq \sum_{x,x'}\mathbb{1}_{B_1(y)}(x)|G(\Omega)(x,x')|\,\|\delta R^*_{\vec{A}}(x')\|\mathbb{1}_{B_1(y')}(x') \\ &\leq \varepsilon\, C\sum_{x,x'}\mathbb{1}_{B_1(y)}(x)\mathrm{e}^{-C_1|x-x'|}\mathbb{1}_{B_1(y')}(x') \\ &\leq \varepsilon\, CL^4\mathrm{e}^{-C_1(|y-y'|-\sqrt{2}L)}, \end{aligned} \tag{4.82}$$

where we used in the first step that $Q^*\delta^L_{y'} = \mathbb{1}_{B_1(y')}$ and in the second step we applied (4.79). Now (4.74), (4.81) give

$$\|(QG(\Omega)Q^*)^{-1}QG(\Omega)\delta R^*_{\vec{A}}Q^*\|_{\infty,\infty;\Omega_1} \leq C'\varepsilon. \tag{4.83}$$

Consequently, for $C'\varepsilon \leq 1/2$ we can sum up and estimate the Neumann series as follows

$$\|(QG(\Omega)R^*_{\vec{A}}Q^*)^{-1}\|_{\infty,\infty;\Omega_1} \leq \frac{C''}{1-C'\varepsilon} \leq 2C''. \tag{4.84}$$

Thus we have chosen ε uniformly in n and established (4.77) with a constant independent of ε. □

Lemma 4.21. *The operator $D_{\vec{A}} := QG(\Omega)R^*_{\vec{A}}Q^*$ satisfies the following:*

$$\|(D_{\vec{A}_1})^{-1} - (D_{\vec{A}_2})^{-1}\|_{\infty,\infty;\Omega_1} \leq 2\|G(\Omega)\|_{\infty,\infty;\Omega}\|D^{-1}_{\vec{A}_1}\|_{\infty,\infty;\Omega_1}\,\|D^{-1}_{\vec{A}_2}\|_{\infty,\infty;\Omega_1}\|\vec{A}_1-\vec{A}_2\|_{\infty,\infty;\Omega}, \tag{4.85}$$

where $\vec{A}_1, \vec{A}_2 \in \overrightarrow{\mathrm{Conf}}^{\varepsilon}(\Omega)$ are two configurations (with $s_1 = s_2 = 1$) and $0 < \varepsilon \leq 1$ sufficiently small, uniformly in n.

Proof. We come back to (4.80), which reads in a short-hand notation

$$D^{-1}_{\vec{A}} = (1 + D_0^{-1}\delta D_{\vec{A}})^{-1}D_0^{-1}, \tag{4.86}$$

where $\delta D_{\vec{A}} := QG(\Omega)\delta R^*_{\vec{A}}Q^*$ and $\delta R^*_{\vec{A}} := R^*_{\vec{A}} - 1$. Next, by the resolvent identity,

$$\begin{aligned} D^{-1}_{\vec{A}_1} - D^{-1}_{\vec{A}_2} &= \Big\{(1 + D_0^{-1}\delta D_{\vec{A}_1})^{-1} - (1 + D_0^{-1}\delta D_{\vec{A}_2})^{-1}\Big\}D_0^{-1} \\ &= (1 + D_0^{-1}\delta D_{\vec{A}_1})^{-1}D_0^{-1}(\delta D_{\vec{A}_2} - \delta D_{\vec{A}_1})(1 + D_0^{-1}\delta D_{\vec{A}_2})^{-1}D_0^{-1}. \end{aligned} \tag{4.87}$$

Hence

$$\|D^{-1}_{\vec{A}_1} - D^{-1}_{\vec{A}_2}\|_{\infty,\infty;\Omega_1} \leq \|D^{-1}_{\vec{A}_1}\|_{\infty,\infty;\Omega_1}\,\|D^{-1}_{\vec{A}_2}\|_{\infty,\infty;\Omega_1}\,\|\delta D_{\vec{A}_2} - \delta D_{\vec{A}_1}\|_{\infty,\infty;\Omega_1}. \tag{4.88}$$

We can write

$$\begin{aligned} \|\delta D_{\vec{A}_2} - \delta D_{\vec{A}_1}\|_{\infty,\infty;\Omega_1} &= \|QG(\Omega)(\delta R^*_{\vec{A}_2} - \delta R^*_{\vec{A}_1})Q^*\|_{\infty,\infty;\Omega_1} \\ &\leq \|G(\Omega)\|_{\infty,\infty;\Omega}\|\delta R^*_{\vec{A}_2} - \delta R^*_{\vec{A}_1}\|_{\infty,\infty;\Omega} \\ &= \|G(\Omega)\|_{\infty,\infty;\Omega}\|R^*_{\vec{A}_2} - R^*_{\vec{A}_1}\|_{\infty,\infty;\Omega} \leq 2\|G(\Omega)\|_{\infty,\infty;\Omega}\|\vec{A}_1 - \vec{A}_2\|_{\infty,\infty;\Omega}, \end{aligned} \tag{4.89}$$

where in the last step we used (4.91) and the fact that $Q : \mathcal{L}^\infty(\Omega) \to \mathcal{L}^\infty(\Omega_1)$, $Q^* : \mathcal{L}^\infty(\Omega_1) \to \mathcal{L}^\infty(\Omega)$ have norms bounded by one. Now the statement follows from (4.88), (4.89) and Lemma 4.20. □

Lemma 4.22. *For $\vec{A}, \vec{A}_1, \vec{A}_2 \in \overrightarrow{\mathrm{Conf}}^{\varepsilon}(\Omega)$, $s = s_1 = s_2 = 1$, $0 < \varepsilon \le 1/2$, there hold the bounds*

$$\|R^*_{\vec{A}}\|_{\infty,\infty;\Omega} \;\le\; 2, \tag{4.90}$$

$$\|R^*_{\vec{A}_1} - R^*_{\vec{A}_2}\|_{\infty,\infty;\Omega} \;\le\; 2\|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega}. \tag{4.91}$$

Proof. Regarding (4.90), we recall (3.12) and write

$$\begin{aligned}\|R^*_{\vec{A}}\|_{\infty,\infty;\Omega} &= \sup_{\|\vec{v}\|_{\infty;\Omega}\le 1}\sup_{x\in\Omega}|R^*_{\vec{A}}(x)\vec{v}(x)| \\ &= \sup_{\|\vec{v}\|_{\infty;\Omega}\le 1}\sup_{x\in\Omega}|A_0(x)\vec{v}(x) - \vec{A}(x)\times\vec{v}(x)| \le 2,\end{aligned} \tag{4.92}$$

where we used the fact that $|\vec{A}(x)| \le 1$.

Now we move on to (4.91). We have, by (3.12),

$$(R^*_{\vec{A}_1}(x) - R^*_{\vec{A}_2}(x))\vec{v}(x) = (A_{1,0}(x) - A_{2,0}(x))\vec{v}(x) - (\vec{A}_1(x) - \vec{A}_2(x))\times\vec{v}(x). \tag{4.93}$$

Making use of

$$|A_{1,0} - A_{2,0}| = \frac{|\,|\vec{A}_2|^2 - |\vec{A}_1|^2|}{\sqrt{1-|\vec{A}_1|^2} + \sqrt{1-|\vec{A}_2|^2}} \le 2\varepsilon|\vec{A}_1 - \vec{A}_2|, \tag{4.94}$$

we obtain the following estimate

$$|(R_{\vec{A}_1}(x) - R_{\vec{A}_2}(x))\vec{v}(x)| \le (2\varepsilon+1)|\vec{v}(x)||\vec{A}_1(x) - \vec{A}_2(x)|. \tag{4.95}$$

Hence,

$$\|(R_{\vec{A}_1} - R_{\vec{A}_2})\vec{v}\|_{\infty;\Omega} \le (2\varepsilon+1)\|\vec{v}\|_{\infty;\Omega}\|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega}, \tag{4.96}$$

which concludes the proof of (4.91). □

4.6 $\mathcal{L}^\infty$-bounds for the remainder $\vec{r}$

In this subsection we use the notation introduced above (3.46).

Lemma 4.23. *Suppose that $|\vec{B}(b)| \le \varepsilon_1$, $\vec{A} \in \overrightarrow{\mathrm{Conf}}^{\varepsilon}(\Omega)$, $s = 1$ and $0 \le \varepsilon, \varepsilon_1 \le 1/2$. Then*

$$\|\partial^*\vec{r}\|_{\infty;\Omega} \le 24(\varepsilon^2 + \varepsilon_1). \tag{4.97}$$

Proof. Given expressions (3.46), (3.47) we have, for $(x, x') := (b_-, b_+)$,

$$\begin{aligned}|\vec{r}(x,x')| \;\le\; & (|\vec{A}(x)|^2 + |\vec{B}(x,x')|^2)\,|\vec{A}(x')| \\ & + (|\vec{A}(x')|^2 + |\vec{B}(x,x')|^2)\,|\vec{A}(x)| \\ & + |\vec{B}(x,x')| \\ & + |\vec{B}(x,x')|\,(|\vec{A}(x)| + |\vec{A}(x')|) + |\vec{A}(x)|\,|\vec{A}(x')| \\ & + 3|\vec{A}(x')|\,|\vec{A}(x)|\,|\vec{B}(x,x')|.\end{aligned} \tag{4.98}$$

This gives, considering that $|\vec{A}(x)| \le \varepsilon \le 1$,

$$|\vec{r}(x,x')| \le 2(\varepsilon^2 + \varepsilon_1^2)\varepsilon + \varepsilon_1 + 2\varepsilon_1\varepsilon + \varepsilon^2 + 3\varepsilon^2\varepsilon_1 \le 6(\varepsilon^2 + \varepsilon_1). \tag{4.99}$$

Now we recall from Definition 3.6 that

$$(\partial^* \vec{r})(x) = -\vec{r}(x, x+e_0) - \vec{r}(x, x+e_1) + \vec{r}(x-e_0, x) + \vec{r}(x-e_1, x), \tag{4.100}$$

possibly with some terms omitted if x is close to the boundary of Ω. In any case,

$$|(\partial^* \vec{r})(x)| \le 24(\varepsilon^2 + \varepsilon_1), \tag{4.101}$$

which was to be shown. □

Lemma 4.24. *Suppose that* $|\vec{B}(b)| \le \varepsilon_1$, $\vec{A}_1, \vec{A}_2 \in \overrightarrow{\mathrm{Conf}}^\varepsilon(\Omega)$, $s_1 = s_2 = 1$, $0 < \varepsilon_1, \varepsilon \le 1/2$. *Then*

$$\|\partial^* \vec{r}_{\vec{A}_1} - \partial^* \vec{r}_{\vec{A}_2}\|_{\infty;\Omega} \le 96(\varepsilon + \varepsilon_1)\|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega}. \tag{4.102}$$

Proof. We recall formula (3.46). There we wrote for brevity $A_\pm = A(b_\pm)$ and $B = B(b) = \partial V(y_b)$. This gives, by Appendix 3.4,

$$\begin{aligned} \vec{r}(b) \ =\ & -\delta(A_{0,-}B_0)\vec{A}_+ + \delta(A_{0,+}B_0)\vec{A}_- + A_{0,+}A_{0,-}\vec{B} \\ & -A_{0,+}(\vec{A}_- \times \vec{B}) + A_{0,-}(\vec{B} \times \vec{A}_+) + B_0(\vec{A}_- \times \vec{A}_+) \\ & +\vec{A}_+(\vec{A}_- \cdot \vec{B}) - \vec{A}_-(\vec{B} \cdot \vec{A}_+) + \vec{B}(\vec{A}_- \cdot \vec{A}_+). \end{aligned} \tag{4.103}$$

We denote by $\vec{r}_1, \dots, \vec{r}_9$ the respective terms in (4.103). We note that $\vec{r}_1, \vec{r}_2$ have an analogous structure, thus it suffices to estimate one of them. Similarly, $\vec{r}_4, \vec{r}_5$ and $\vec{r}_7, \vec{r}_8, \vec{r}_9$ have analogous structures.

Now we recall from Definition 3.6 that

$$(\partial^* \vec{r})(x) = -\vec{r}(x, x+e_0) - \vec{r}(x, x+e_1) + \vec{r}(x-e_0, x) + \vec{r}(x-e_1, x), \tag{4.104}$$

possibly with some terms omitted if x is close to the boundary of Ω. We denote by $(\partial^* \vec{r}_j)_{j'}(x)$, $j' = 1, 2, 3, 4$, the four terms coming from (4.104) for any fixed j (or less if x is close to the boundary).

The items below give

$$\begin{aligned} \|\partial^* \vec{r}_{\vec{A}_1} - \partial^* \vec{r}_{\vec{A}_2}\|_{\infty;\Omega} \ \le\ & \big(24\varepsilon_1 + 8\varepsilon_1 + 16\varepsilon_1 + 16\varepsilon + 8(3\varepsilon^2 + \varepsilon_1^2)\big)\|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega} \\ \le\ & 24(\varepsilon_1 + \varepsilon_1 + \varepsilon_1 + \varepsilon + \varepsilon^2 + \varepsilon_1^2)\|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega} \\ \le\ & 24 \times 4(\varepsilon_1 + \varepsilon)\|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega} = 96(\varepsilon_1 + \varepsilon)\|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega}. \end{aligned} \tag{4.105}$$

- Contributions $\vec{r}_1, \vec{r}_2$. We have

$$(-\partial^* \vec{r}_{\vec{A}_1,1})_1(x) \ =\ -\delta\big(A_{1,0}(x)B_0(x, x+e_0)\big)\vec{A}_1(x+e_0), \tag{4.106}$$

$$(-\partial^* \vec{r}_{\vec{A}_2,1})_1(x) \ =\ -\delta\big(A_{2,0}(x)B_0(x, x+e_0)\big)\vec{A}_2(x+e_0). \tag{4.107}$$

We consider the difference

$$\begin{aligned} & (-\partial^* \vec{r}_{\vec{A}_1,1})_1(x) + (\partial^* \vec{r}_{\vec{A}_2,1})_1(x) \\ & = -\delta\big(A_{1,0}(x)B_0(x, x+e_0)\big)\vec{A}_1(x+e_0) - \delta\big(A_{2,0}(x)B_0(x, x+e_0)\big)\vec{A}_2(x+e_0) \\ & = -\delta\big(A_{1,0}(x)B_0(x, x+e_0)\big)(\vec{A}_1(x+e_0) - \vec{A}_2(x+e_0)) \qquad (4.108) \\ & \quad - \big(\delta\big(A_{1,0}(x)B_0(x, x+e_0)\big) - \delta\big(A_{2,0}(x)V_0(x, x+e_0)\big)\big)\vec{A}_2(x+e_0). \qquad (4.109) \end{aligned}$$

We have, by (4.133),

$$\delta(A_0 B_0) \le \varepsilon^2 + \varepsilon_1^2. \tag{4.110}$$

This gives

$$\begin{aligned}\sup_{x\in\Omega}|(4.108)(x)| &\le (\varepsilon^2+\varepsilon_1^2)\sup_{x\in\Omega}|\vec{A}_1(x+e_0)-\vec{A}_2(x+e_0)| \\ &\le (\varepsilon^2+\varepsilon_1^2)\|\vec{A}_1-\vec{A}_2\|_{\infty;\Omega}. \end{aligned} \tag{4.111}$$

Furthermore, by (4.134), we have

$$|\delta(A_{1,0}(x)B_0(x+e_0))-\delta(A_{2,0}(x)B_0(x+e_0))|\le 2\varepsilon|\vec{A}_1(x)-\vec{A}_2(x)|. \tag{4.112}$$

Therefore, taking into account the factor $\vec{A}_2(x+e_0)$ in (4.109)

$$\sup_{x\in\Omega}\|(4.109)(x)|\le 2\varepsilon^2\|\vec{A}_1-\vec{A}_2\|_{\infty;\Omega}. \tag{4.113}$$

Thus we have

$$\|(\partial^*\vec{r}_{\vec{A}_1,1})-(\partial^*\vec{r}_{\vec{A}_2,1})\|_{\infty;\Omega}\le 4(3\varepsilon^2+\varepsilon_1^2)\|\vec{A}_1-\vec{A}_2\|_{\infty;\Omega}, \tag{4.114}$$

where the factor 4 comes from (4.104).

- Contribution $\vec{r}_3$. We have

$$-(\partial^*\vec{r}_{\vec{A}_1,3})_1(x) = \delta(A_{1,0}(x+e_0)A_{1,0}(x))\vec{B}(x,x+e_0), \tag{4.115}$$

$$-(\partial^*\vec{r}_{\vec{A}_2,3})_1(x) = \delta(A_{2,0}(x+e_0)A_{2,0}(x))\vec{B}(x,x+e_0). \tag{4.116}$$

The difference gives, by (4.134),

$$\begin{aligned}&|(\partial^*\vec{r}_{\vec{A}_1,3})_1(x)-(\partial^*\vec{r}_{\vec{A}_2,3})_1(x)| \\ &= |\delta(A_{1,0}(x+e_0)A_{1,0}(x))-\delta(A_{2,0}(x+e_0)A_{2,0}(x))|\,|\vec{B}(x,x+e_0)| \\ &\le 2\varepsilon(|\vec{A}_2(x+e_0)-\vec{A}_1(x+e_0)|+|\vec{A}_2(x)-\vec{A}_1(x)|).\end{aligned} \tag{4.117}$$

This gives, taking into account the four terms from (4.104)

$$\|(\partial^*\vec{r}_{\vec{A}_1,3})-(\partial^*\vec{r}_{\vec{A}_2,3})\|_{\infty;\Omega}\le 16\varepsilon\|\vec{A}_2-\vec{A}_1\|_{\infty;\Omega}. \tag{4.118}$$

- Contributions $\vec{r}_4,\vec{r}_5$. We have

$$-(\partial^*\vec{r}_{\vec{A}_1,4})_1(x) = -A_{1,0}(x+e_0)(\vec{A}_1(x)\times\vec{B}(x,x+e_0)), \tag{4.119}$$

$$-(\partial^*\vec{r}_{\vec{A}_2,4})_1(x) = -A_{2,0}(x+e_0)(\vec{A}_2(x)\times\vec{B}(x,x+e_0)). \tag{4.120}$$

The difference has the form

$$|(\partial^*\vec{r}_{\vec{A}_1,4})_1(x)-(\partial^*\vec{r}_{\vec{A}_2,4})_1(x)| \le |A_{1,0}(x+e_0)-A_{2,0}(x+e_0)|\,|\vec{A}_1(x)\times\vec{B}(x,x+e_0)| \tag{4.121}$$

$$+\ |A_{2,0}(x+e_0)|\,|\vec{A}_1(x)-\vec{A}_2(x)|\,|\vec{B}(x,x+e_0)|. \tag{4.122}$$

Considering that $\varepsilon\le 1/2$,

$$\begin{aligned}|A_{1,0}-A_{2,0}| &\le |\sqrt{1-\vec{A}_1^2}-\sqrt{1-\vec{A}_2^2}| \\ &\le |\vec{A}_2-\vec{A}_1||\vec{A}_2+\vec{A}_1|\le 2\varepsilon|\vec{A}_2-\vec{A}_1|.\end{aligned} \tag{4.123}$$

Thus we can estimate

$$\begin{aligned}\|(\partial^*\vec{r}_{\vec{A}_1,4})-(\partial^*\vec{r}_{\vec{A}_2,4})\|_{\infty;\Omega} &\le 4(2\varepsilon^2\varepsilon_1+\varepsilon_1)\|\vec{A}_2-\vec{A}_1\|_{\infty;\Omega} \\ &\le 8\varepsilon_1\|\vec{A}_2-\vec{A}_1\|_{\infty;\Omega}.\end{aligned} \tag{4.124}$$

- Contribution $\vec{r}_6$. We have, for $B_0 := B_0(x, x+e_0)$,

$$-(\partial^* \vec{r}_{\vec{A}_1,6})_1(x) \; = \; B_0(\vec{A}_1(x) \times \vec{A}_1(x+e_0)), \tag{4.125}$$

$$-(\partial^* \vec{r}_{\vec{A}_2,6})_1(x) \; = \; B_0(\vec{A}_2(x) \times \vec{A}_2(x+e_0)). \tag{4.126}$$

  Hence,

$$|(\partial^* \vec{r}_{\vec{A}_1,6})_1(x) - \partial^*(\vec{r}_{\vec{A}_2,6})_1(x)| \leq \varepsilon_1(|\vec{A}_1(x) - \vec{A}_2(x)| + |\vec{A}_1(x+e_0) - \vec{A}_2(x+e_0)|). \tag{4.127}$$

  Consequently,

$$\|(\partial^* \vec{r}_{\vec{A}_1,6}) - (\partial^* \vec{r}_{\vec{A}_2,6})\|_{\infty;\Omega} \leq 8\varepsilon_1 \|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega}. \tag{4.128}$$

- Contributions $\vec{r}_7$, $\vec{r}_8$, $\vec{r}_9$. We have

$$-(\partial^* \vec{r}_{\vec{A}_1,7})_1(x) \; = \; \vec{A}_1(x+e_0)(\vec{A}_1(x) \cdot \vec{B}(x, x+e_0)), \tag{4.129}$$

$$-(\partial^* \vec{r}_{\vec{A}_2,7})_1(x) \; = \; \vec{A}_2(x+e_0)(\vec{A}_2(x) \cdot \vec{B}(x, x+e_0)). \tag{4.130}$$

  Hence,

$$|(\partial^* \vec{r}_{\vec{A}_1,7})_1(x) - (\partial^* \vec{r}_{\vec{A}_2,7})_1(x)| \leq \varepsilon_1(|\vec{A}_1(x+e_0) - \vec{A}_2(x+e_0)| + |\vec{A}_1(x) - \vec{A}_2(x)|). \tag{4.131}$$

  Consequently,

$$\|(\partial^* \vec{r}_{\vec{A}_1,7}) - (\partial^* \vec{r}_{\vec{A}_2,7})\|_{\infty;\Omega} \leq 8\varepsilon_1 \|\vec{A}_1 - \vec{A}_2\|_{\infty;\Omega}, \tag{4.132}$$

  where the factor 4 comes from (4.104).

This concludes the proof. □

Lemma 4.25. *Suppose $\vec{A}_1, \vec{A}_2, \vec{C}_1, \vec{C}_2 \in \mathbb{R}^3$ are such that $|\vec{A}_j| \leq \varepsilon \leq 1/2$ and $|\vec{C}_j| \leq \tilde{\varepsilon} \leq 1/2$. Then*

$$\delta(A_{j,0} C_{j',0}) \; \leq \; \varepsilon^2 + \tilde{\varepsilon}^2, \tag{4.133}$$

$$|\delta(A_{2,0} C_{2,0}) - \delta(A_{1,0} C_{1,0})| \; \leq \; 2\varepsilon|\vec{A}_2 - \vec{A}_1| + 2\tilde{\varepsilon}|\vec{C}_2 - \vec{C}_1|. \tag{4.134}$$

Proof. Concerning (4.133), we have by (3.47)

$$\delta(A_{j,0} C_{j',0}) = \frac{\vec{C}_{j'}^2 + \vec{A}_j^2 - \vec{C}_{j'}^2 \vec{A}_j^2}{1 + A_{j,0} C_{j',0}} \leq \varepsilon^2 + \tilde{\varepsilon}^2, \tag{4.135}$$

where we estimated $-\vec{C}_{j'}^2 \vec{A}_j^2 \leq 0$. Regarding (4.134) we have

$$\delta(A_{2,0} C_{2,0}) - \delta(A_{1,0} C_{1,0}) = \sqrt{1 - \vec{A}_1^2}\sqrt{1 - \vec{C}_1^2} - \sqrt{1 - \vec{A}_2^2}\sqrt{1 - \vec{C}_2^2}. \tag{4.136}$$

We write $\vec{A}_2 =: \vec{A}_1 + \delta\vec{A}$, $\vec{C}_2 =: \vec{C}_1 + \delta\vec{C}$ and obtain

$$\delta(A_{2,0} C_{2,0}) - \delta(A_{1,0} C_{1,0}) = \sqrt{1 - \vec{A}_1^2}\sqrt{1 - \vec{C}_1^2} - \sqrt{1 - (\vec{A}_1 + \delta\vec{A})^2}\sqrt{1 - (\vec{C}_1 + \delta\vec{C})^2}. \tag{4.137}$$

We define a function of $t \in [0, 1]$

$$F(t) := \sqrt{1 - \vec{A}_1^2}\sqrt{1 - \vec{C}_1^2} - \sqrt{1 - (\vec{A}_1 + t\delta\vec{A})^2}\sqrt{1 - (\vec{C}_1 + t\delta\vec{C})^2}, \tag{4.138}$$

which satisfies $F(0) = 0$. Thus we can write $F(1) = \int_0^1 dt \, \partial_t F(t)$ which gives

$$F(1) = \int_0^1 dt \Big(\frac{\delta\vec{A} \cdot (\vec{A}_1 + t\delta\vec{A})}{\sqrt{1 - (\vec{A}_1 + t\delta\vec{A})^2}} \sqrt{1 - (\vec{C}_1 + t\delta\vec{C})^2} + \{A \leftrightarrow C\} \Big). \tag{4.139}$$

Hence

$$|F(1)| \leq 2(|\delta\vec{A}|\varepsilon + |\delta\vec{C}|\tilde{\varepsilon}), \tag{4.140}$$

which concludes the proof. □

4.7 The main result

We summarize the considerations of this paper:

Theorem 4.26. *Let* $\mathrm{G}_0 = SU(2)$. *Then there exist* $0 < \varepsilon, \varepsilon_1 \leq 1$ *s.t. for* $V \in \mathrm{Conf}_{\varepsilon_1}(\Omega_1)$ *the action* $\mathcal{A}$ *has a unique critical point over* $\mathrm{Conf}_\varepsilon(\Omega)$ *with the constraint* $\mathcal{C}(U) = V$. *The parameters* $\varepsilon, \varepsilon_1$ *are independent of* n *but may depend on* L.

Proof. Follows from Theorem 3.9 and Propositions 4.2, 4.3. □

A Random walk expansions

In this appendix we reproduce Theorem 4.1 and its proof from Section IV of [BJ86] with some modifications. We work on an infinite lattice $\mathbb{Z}^d$ and keep the dimension d arbitrary. Also, in this appendix we simply write $\|\cdot\|_p$ for the norm in $\mathcal{L}^p(\mathbb{Z}^d)$, $1 \leq p \leq \infty$, and $\|\cdot\|_{\mathrm{op}} := \|\cdot\|_{2,2;\mathbb{Z}^d}$ for the C^*-norm on bounded operators on $\mathcal{L}^2(\mathbb{Z}^d)$.

A.1 Short range localizing functions

Definition A.1. *A function*[2] $a : \mathbb{R}^d \to \mathbb{R}$ *is short range localizing if*

(i) $0 < a(x)$.

(ii) $a(x) \leq c_\delta (1 + |x|)^{-2(d+\delta)}$ *for some* $\delta > 0, c_\delta \geq 0$.

(iii) For δ *as above, define*

$$b(x) := (1 + |x|)^{d+\delta} a(x). \tag{A.1}$$

We assume that for some constant $K < \infty$, *which may depend on* d *and* δ,

$$\frac{b(x+y)}{b(x)} \leq K \tag{A.2}$$

for all x *and for* $|y| \leq 2d^{1/2}$.

(iv) For δ *as above there are constants* c *and* $\epsilon > 0$ *s.t. for all* n

$$(\underbrace{b * b * \cdots * b}_{\mathrm{n}})(x) \leq c^{\mathrm{n}} b(\epsilon x), \tag{A.3}$$

where $(b * b)(x) := \sum_{y \in \mathbb{Z}^d} b(x-y) b(y)$, $x \in \mathbb{R}^d$.

(v) The function b *is eventually decreasing. That is, there exists* $M_0 \in \mathbb{N}$ *sufficiently large, possibly depending on* δ, *s.t.* $b(x') \leq b(x)$ *for all* $x, x' \in \mathbb{R}^d$ *satisfying* $M_0 \leq |x| \leq |x'|$.

In comparison with [BJ86] we removed a lower bound by K^{-1} in (A.2) and added item (v). The latter property has the following consequence which will be used in (A.29) below:

Lemma A.2. *Let* b *be as in Definition A.1 above. Then, for* $\tilde{M} \in \mathbb{N}$ *sufficiently large we have*

$$b(\tilde{M}x/3) \leq b(x), \quad x \in \mathbb{Z}^d. \tag{A.4}$$

[2] We choose $\mathbb{R}^d$ and not $\mathbb{Z}^d$ as a domain, because we want to consider $b(\epsilon x)$ in (A.3).

Proof. For $|x| \geq 3M_0$ from property (v) we clearly have that $b(\tilde{M}x/3) \leq b(x)$, for any $\tilde{M} \in \mathbb{N}$. Now the set $S := \{ x \in \mathbb{Z}^d \,|\, |x| \leq 3M_0 \}$ is finite and b tends to zero as $|x| \to \infty$ by property (ii). Thus we can find a sufficiently large $\tilde{M}$ s.t. $b(\tilde{M}x/3) \leq b(x)$ for all $x \in S$, $x \neq 0$. As for $x = 0$ the statement is trivial, this concludes the argument. □

Lemma A.3. *Fix $c_1 > 0$. The function $a(x) := \mathrm{e}^{-c_1|x|}$ is short range localizing in the sense of Definition A.1. A possible choice of parameters is $\delta = c_1$, $M_0 = 2/c_1$, $\epsilon = 1/4$.*

Proof. Property (i) is obvious. Regarding property (ii), we note that for any $\delta > 0$ there is such c_δ that

$$(1 + |x|)^{2(d+\delta)} \mathrm{e}^{-c_1|x|} \leq c_\delta. \tag{A.5}$$

To check (iii) we fix $\delta > 0$ and define $b(x) := (1 + |x|)^{d+\delta} a(x)$. We note that

$$\begin{aligned} \mathrm{e}^{-c_1|x+y|} &\leq \mathrm{e}^{-c_1|x|}\mathrm{e}^{c_1 2d^{1/2}}, & \text{(A.6)} \\ (1 + |x + y|)^{d+\delta} &\leq (1 + |x| + 2d)^{d+\delta}, & \text{(A.7)} \end{aligned}$$

which gives the bound in (A.2). Now we move on to property (iv). First, we note that

$$b(x) = (1 + |x|)^{d+\delta} a(x) = ((1 + |x|)^{d+\delta} \mathrm{e}^{-\frac{1}{2}c_1|x|})\mathrm{e}^{-\frac{1}{2}c_1|x|} \leq c_\delta^{1/2} a((1/2)x) =: c_\delta^{1/2} a_{1/2}(x). \tag{A.8}$$

Consequently, we have

$$\begin{aligned} (\underbrace{b * b * \cdots * b}_{\mathrm{n}})(x) &\leq c_\delta^{\mathrm{n}/2} (\underbrace{a_{1/2} * \ldots * a_{1/2}}_{\mathrm{n}})(x) \\ &= c_\delta^{\mathrm{n}/2} \sum_{x_1,\ldots,x_{\mathrm{n}-1}} \mathrm{e}^{-\frac{1}{2}c_1|x-x_1|}\mathrm{e}^{-\frac{1}{2}c_1|x_1-x_2|}\ldots\mathrm{e}^{-\frac{1}{2}c_1|x_{\mathrm{n}-2}-x_{\mathrm{n}-1}|}\mathrm{e}^{-\frac{1}{2}c_1|x_{\mathrm{n}-1}|}. \end{aligned} \tag{A.9}$$

We rewrite each exponential function above as $\mathrm{e}^{-\frac{1}{2}c_1|x_i - x_j|} = \mathrm{e}^{-\frac{1}{4}c_1|x_i-x_j|}\mathrm{e}^{-\frac{1}{4}c_1|x_i-x_j|}$ and treat the two factors separately. We observe that, by the inverse triangle inequality $|x - x_1| + |x_1 - x_2| \geq |x| - |x_1| + |x_1| - |x_2|$ and so on. Hence

$$\mathrm{e}^{-\frac{1}{4}c_1|x-x_1|}\mathrm{e}^{-\frac{1}{4}c_1|x_1-x_2|}\ldots\mathrm{e}^{-\frac{1}{4}c_1|x_{\mathrm{n}-2}-x_{\mathrm{n}-1}|}\mathrm{e}^{-\frac{1}{4}c_1|x_{\mathrm{n}-1}|} \leq \mathrm{e}^{-\frac{1}{4}c_1|x|}. \tag{A.10}$$

On the other hand $\|f_1 * f_2\|_\infty \leq \|f_1\|_\infty \|f_2\|_1$ and $\|g_1 * g_2\|_1 \leq \|g_1\|_1 \|g_2\|_1$. Hence

$$\| \underbrace{a_{1/4} * \ldots * a_{1/4}}_{\mathrm{n}} \|_\infty \leq \|a_{1/4}\|_\infty \|a_{1/4}\|_1^{\mathrm{n}-1}. \tag{A.11}$$

Since $\|a_{1/4}\|_\infty = 1$, this gives

$$(\underbrace{b * b * \cdots * b}_{\mathrm{n}})(x) \leq (c_\delta^{1/2}(1 + \|a_{1/4}\|_1))^{\mathrm{n}} \mathrm{e}^{-\frac{1}{4}c_1|x|} \leq (c'_\delta)^{\mathrm{n}} (1 + \frac{1}{4}|x|)^{d+\delta} \mathrm{e}^{-\frac{1}{4}c_1|x|} \tag{A.12}$$

and yields the proof of (iv) with $\varepsilon = 1/4$. Regarding (v), we consider the following function and its derivative

$$f(w) := (1 + w)^{d+\delta}\mathrm{e}^{-c_1 w}, \quad f'(w) = (d + \delta)(1 + w)^{d+\delta-1}\mathrm{e}^{-c_1 w} + (1 + w)^{d+\delta}(-c_1)\mathrm{e}^{-c_1 w}. \tag{A.13}$$

Thus f is decreasing, provided that

$$(d + \delta)(1 + w)^{-1} - c_1 \leq 0 \quad \Rightarrow \quad w \geq \frac{d + \delta}{c_1} - 1 =: M'_0. \tag{A.14}$$

We can choose as M_0 the smallest natural number larger than M'_0. This concludes the argument. □

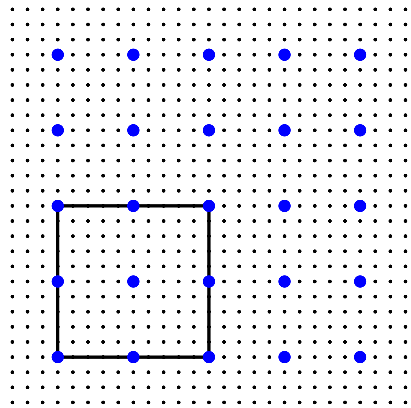

Figure 3: The centers of $2\tilde{M}$-boxes in $\mathbb{Z}^2$ as blue (thick) dots, with one of the $2\tilde{M}$-boxes indicated.

A.2 Exponential decay of integral kernels of operators and their inverses

We start with some definitions:

Definition A.4. *Let $\{\Box^j\}_{j\in\mathbb{Z}^d}$ be a cover of $\mathbb{R}^d$ by closed $2\tilde{M}$-cubes where $\Box^j$ has a center at $j\tilde{M}$, $j \in \mathbb{Z}^d$, and $\tilde{M} \in \mathbb{N}$. Thus cubes $\Box^i$, $\Box^j$ overlap if $|i-j| \leq 2d^{1/2}$. A given cube $\Box^i$ overlaps with 5^d neighbouring cubes, including itself. See Figure 3.*

We set $\Box_j := \mathbb{Z}^d \cap \Box^j$. We also let $\Box_j$ be the projection of $\mathcal{L}^2(\mathbb{Z}^d)$ onto $\mathcal{L}^2(\Box_j)$. The projections are orthogonal for $|i-j| > 2d^{1/2}$.

Definition A.5. *Let h_j^2, $j \in \mathbb{Z}^d$, be a smooth partition of unity on $\mathbb{R}^d$ constructed as follows: Let $h \in C_0^\infty(\mathbb{R})$ be positive and s.t.*

$$h(t) := \begin{cases} 1 & \text{for} \quad |t| \leq 1/3, \\ 0 & \text{for} \quad |t| \geq 2/3, \end{cases} \tag{A.15}$$

and

$$\sum_{j\in\mathbb{Z}} h^2(t-j) = 1, \quad \sup|h'(t)| \leq 10. \tag{A.16}$$

Now we define the functions on $\mathbb{Z}^d$

$$h_j(x) := \prod_{k=0}^{d-1} h\Big(\frac{x_k}{\tilde{M}} - j_k\Big) \tag{A.17}$$

which satisfy

$$h_j\Box_j = h_j, \quad \sum_{j\in\mathbb{Z}^d} h_j^2(x) = 1. \tag{A.18}$$

Theorem A.6. *Let A on $\mathcal{L}^2(\mathbb{Z}^d)$ be strictly positive and let*

$$|\mathrm{A}(x,x')| \leq a(x-x'), \tag{A.19}$$

where a is a short range localizing function (see Definition A.1). Now fix $\delta > 0$ s.t. $a(x) \leq c_\delta(1+|x|)^{-2(d+\delta)}$ for some finite c_δ and define $b(x) = (1+|x|)^{d+\delta}a(x)$. Then, for ϵ as in (A.3), and $\tilde{M}$ as in Definition A.4, sufficiently large, we have

$$|\mathrm{A}^{-1}(x,x')| \leq c\, b(\epsilon(x-x')/\tilde{M}). \tag{A.20}$$

Proof. By assumption, there exists $\mathrm{m}^2 > 0$ s.t. $\mathrm{m}^2 \le \mathrm{A}$. Consequently,

$$\mathrm{m}^2 \Box_j \le \Box_j \mathrm{A} \Box_j, \tag{A.21}$$

hence $\Box_j A \Box_j$ is invertible on $\mathcal{L}^2(\Box_j)$. We can define

$$\mathrm{C}_j := (\Box_j \mathrm{A} \Box_j)^{-1}|_{\mathcal{L}^2(\Box_j)}, \quad \mathrm{C} := \sum_j h_j \mathrm{C}_j h_j, \tag{A.22}$$

where h_j is a multiplication operator. We treat C as an approximate inverse of A and define the corresponding rest term R by the relation

$$\mathrm{AC} =: 1 - \mathrm{R}. \tag{A.23}$$

Now we study $\mathrm{A}^{-1}(x, x')$ using the series (A.35) from Proposition A.7. We can parametrize the terms in the expansion for $\mathrm{A}^{-1}(x, x')$ arising from (A.35) by paths $\omega = \{\omega_0, \omega_1, \ldots, \omega_{2\mathrm{n}}\}$, where $\omega_j \in \mathbb{Z}^d$ indexes a cube $\Box_{\omega_j}$ centered at $\tilde{M}\omega_j$. Here $x \in \Box_{\omega_0}$, $x' \in \Box_{\omega_{2\mathrm{n}}}$ and, using that $\mathrm{R} = \sum_{i,j} \mathrm{R}_{ij} \mathrm{C}_j h_j$, (see (A.38) below),

$$\begin{aligned} A^{-1} &= \sum_{\mathrm{n}=0}^{\infty} \mathrm{CR}^{\mathrm{n}} \\ &= \sum_{\mathrm{n}=0}^{\infty} \Big(\sum_{i_0} \sum_{i_1, j_1} \cdots \sum_{i_\mathrm{n}, j_\mathrm{n}}\Big)(h_{i_0} \mathrm{C}_{i_0} h_{i_0})(\mathrm{R}_{i_1 j_1} \mathrm{C}_{j_1} h_{j_1})(\mathrm{R}_{i_2 j_2} \mathrm{C}_{j_2} h_{j_2}) \ldots (\mathrm{R}_{i_\mathrm{n} j_\mathrm{n}} \mathrm{C}_{j_\mathrm{n}} h_{j_\mathrm{n}}) \\ &= \sum_{\mathrm{n}=0}^{\infty} \Big(\sum_{\omega_0, \omega_1, \ldots, \omega_{2\mathrm{n}-1}, \omega_{2\mathrm{n}}}\Big)(h_{\omega_0} \mathrm{C}_{\omega_0} h_{\omega_0})(\mathrm{R}_{\omega_1 \omega_2} \mathrm{C}_{\omega_2} h_{\omega_2})(\mathrm{R}_{\omega_3 \omega_4} \mathrm{C}_{\omega_4} h_{\omega_4}) \ldots (\mathrm{R}_{\omega_{2\mathrm{n}-1} \omega_{2\mathrm{n}}} \mathrm{C}_{\omega_{2\mathrm{n}}} h_{\omega_{2\mathrm{n}}}) \end{aligned} \tag{A.24}$$

The n-th summand vanishes, unless for all $i = 0, 1, \ldots, \mathrm{n} - 1$

$$|\omega_{2i} - \omega_{2i+1}| \le 2d^{1/2}. \tag{A.25}$$

Indeed, since $\mathrm{R}_{\omega_{2i+1}, \omega_{2i+2}} : \mathcal{L}^2(\Box_{2i+2}) \to \mathcal{L}^2(\Box_{2i+1})$, we have that $h_{\omega_{2i}} \Box_{\omega_{2i+1}} = 0$ if (A.25) fails, as then the cubes $\Box_{\omega_{2i}}$ and $\Box_{\omega_{2i+1}}$ are disjoint.

We have, using that $\|\mathrm{C}_j\|_{\mathrm{op}} \le \mathrm{m}^{-2}$ and $\|h_j\|_{\mathrm{op}} = 1$,

$$|\mathrm{A}^{-1}(x, x')| \le \sum_{\{\omega\}_{x,x'}} \mathrm{m}^{-(2\mathrm{n}+2)} \|\mathrm{R}_{\omega_1, \omega_2}\|_{\mathrm{op}} \|\mathrm{R}_{\omega_3, \omega_4}\|_{\mathrm{op}} \ldots \|\mathrm{R}_{\omega_{2\mathrm{n}-1}, \omega_{2\mathrm{n}}}\|_{\mathrm{op}} + |\mathrm{C}(x, x')|, \tag{A.26}$$

where the last term is the $\mathrm{n} = 0$ case, $\{\omega\}_{x,x'}$ are all paths satisfying the conditions (A.25) and

$$x \in \Box_{\omega_0}, \quad x' \in \Box_{\omega_{2\mathrm{n}}}. \tag{A.27}$$

We remark that for a fixed x there are less than 5^d boxes which may contain it. We also remark, that for $x \in \Box_{\omega_0}$ we have

$$\Big|\frac{x}{\tilde{M}} - \omega_0\Big|_\infty \le 1 \quad \Rightarrow \quad \Big|\frac{x}{\tilde{M}} - \omega_0\Big| \le \sqrt{d}. \tag{A.28}$$

Let us write explicitly the $\mathrm{n} = 4$ term of (A.26). Using that, by Lemma A.10 below, $\|\mathrm{R}_{i,j}\|_{\mathrm{op}} \le O(\tilde{M}^{-\delta}) b_{\tilde{M}/3}(i - j)$, we get

$$\begin{aligned} |\mathrm{A}^{-1}(x, x')^{(4)}| &\le \sum_{\{\omega_0, \ldots, \omega_8\}_{x,x'}} O(\tilde{M}^{-\mathrm{n}\delta/2}) \mathrm{m}^{-(2\mathrm{n}+2)} b_{\tilde{M}/3}(\omega_1 - \omega_2) b_{\tilde{M}/3}(\omega_3 - \omega_4) \times \\ &\qquad \times b_{\tilde{M}/3}(\omega_5 - \omega_6) b_{\tilde{M}/3}(\omega_7 - \omega_8) \\ &\le \sum_{\{\omega_0, \ldots, \omega_8\}_{x,x'}} O(\tilde{M}^{-\mathrm{n}\delta/2}) \mathrm{m}^{-(2\mathrm{n}+2)} b(\omega_1 - \omega_2) b(\omega_3 - \omega_4) b(\omega_5 - \omega_6) b(\omega_7 - \omega_8), \end{aligned} \tag{A.29}$$

where in the second step we exploited Definition A.1 (v) and Lemma A.2. In the next step we would like to express (A.29) as a convolution of functions b to be able to apply property (A.3):

$$
\begin{aligned}
(A.29) \;=\; & \sum_{\{\omega_0,\ldots,\omega_8\}_{x,x'}} O(\tilde{M}^{-\mathrm{n}\delta/2})\mathrm{m}^{-(2\mathrm{n}+2)} b(x/\tilde{M}-\omega_2)b(\omega_2-\omega_4)b(\omega_4-\omega_6)b(\omega_6-x'/\tilde{M}) \times \\
& \times \left\{\frac{b(\omega_0-\omega_2)}{b(x/\tilde{M}-\omega_2)}\right\}\left\{\frac{b(\omega_1-\omega_2)}{b(\omega_0-\omega_2)}\right\}\left\{\frac{b(\omega_3-\omega_4)}{b(\omega_2-\omega_4)}\right\} \times \\
& \times \left\{\frac{b(\omega_5-\omega_6)}{b(\omega_4-\omega_6)}\right\}\left\{\frac{b(\omega_7-\omega_8)}{b(\omega_6-\omega_8)}\right\}\left\{\frac{b(\omega_6-\omega_8)}{b(\omega_6-x'/\tilde{M})}\right\}. \qquad \text{(A.30)}
\end{aligned}
$$

Now we can use Definition A.1 (iii) to estimate the expressions in curly bracket. We have, for example,

$$
\left\{\frac{b(\omega_7-\omega_8)}{b(\omega_6-\omega_8)}\right\} = \left\{\frac{b((\omega_7-\omega_6)+\omega_6-\omega_8)}{b(\omega_6-\omega_8)}\right\} \le K, \qquad \text{(A.31)}
$$

since $|\omega_6-\omega_7| \le 2d^{1/2}$ by (A.25), and similarly for the remaining brackets. After estimating the curly brackets in (A.30), we can sum over ω_i, i-odd, and over the boundary terms ω_0, $\omega_{2\mathrm{n}}$. From (A.24) and the property that $\mathrm{R}_{i,j} : \mathcal{L}^2(\square_j) \to \mathcal{L}^2(\square_i)$ we see that, for example, the sum over ω_1 runs at most over 5^d boxes which overlap with ω_0. We can argue analogously for the boundary terms, using (A.27). Thus we have

$$
\begin{aligned}
|\mathrm{A}^{-1}(x,x')^{(4)}| \;\le\; & \sum_{\{\omega_2,\omega_4,\omega_6\}} O(\tilde{M}^{-\mathrm{n}\delta/2})\mathrm{m}^{-(2\mathrm{n}+2)}(K5^d)^{\mathrm{n}+2} b(x/\tilde{M}-\omega_2)b(\omega_2-\omega_4) \times \\
& \times b(\omega_4-\omega_6)b(\omega_6-x'/\tilde{M}). \qquad \text{(A.32)}
\end{aligned}
$$

Let us relax the conditions (A.25) in $\{\omega_2,\omega_4,\omega_6\}$. Now that ω_6 runs over $\mathbb{Z}^d$, we can change variables $\omega_6' = \omega_6 - x'/\tilde{M}$, which gives

$$
\begin{aligned}
|\mathrm{A}^{-1}(x,x')^{(4)}| \;\le\; & \sum_{\{\omega_2,\omega_4,\omega_6'\}} O(\tilde{M}^{-\mathrm{n}\delta/2})\mathrm{m}^{-(2\mathrm{n}+2)}(K5^d)^{\mathrm{n}+2} b(x/\tilde{M}-\omega_2)b(\omega_2-\omega_4) \times \\
& \times b(\omega_4-x'/\tilde{M}-\omega_6')b(\omega_6') \\
=\; & O(\tilde{M}^{-\mathrm{n}\delta/2})\mathrm{m}^{-(2\mathrm{n}+2)}(K5^d)^{\mathrm{n}+2}(b*b*b*b)((x-x')/\tilde{M}) \\
\le\; & O(\tilde{M}^{-\mathrm{n}\delta/2})\mathrm{m}^{-(2n+2)}(K5^d)^{\mathrm{n}+2}c^{\mathrm{n}} b(\epsilon(x-x')/\tilde{M}), \qquad \text{(A.33)}
\end{aligned}
$$

where we made analogous changes of variables in ω_4, ω_2 and in the last step we applied Definition A.1 (iv).

Guided by the above example, one immediately obtains in the general case

$$
|A^{-1}(x,x')| \le |\mathrm{C}(x,x')| + b(\epsilon(x-x')/\tilde{M}) \sum_{\mathrm{n}=1}^{\infty} O(\tilde{M}^{-\mathrm{n}\delta/2})\mathrm{m}^{-(2\mathrm{n}+2)}(K5^d)^{\mathrm{n}+2}c^{\mathrm{n}}. \qquad \text{(A.34)}
$$

Noting that the sum is finite for $\tilde{M}$ sufficiently large and that $\mathrm{C}(x,x') = 0$ unless x, x' belong to the same box, we conclude the proof. □

Proposition A.7. *For $\tilde{M}$ sufficiently large, we have $\|\mathrm{R}\|_{\mathrm{op}} \le O(\tilde{M}^{-\delta/2}) < 1$. Therefore,*

$$
\mathrm{A}^{-1} = \mathrm{C}(1-\mathrm{R})^{-1} = \sum_{\mathrm{n}=0}^{\infty} \mathrm{C}\mathrm{R}^{\mathrm{n}} \qquad \text{(A.35)}
$$

is a convergent series in the operator norm $\|\cdot\|_{\mathrm{op}}$.

Proof. Let us expand the product AC as

$$\begin{aligned}\mathrm{AC} &= \sum_j \mathrm{A}h_j\mathrm{C}_jh_j \\ &= \sum_j \Box_j\mathrm{A}h_j\mathrm{C}_jh_j + \sum_j(1-\Box_j)\mathrm{A}h_j\mathrm{C}_jh_j \\ &= \sum_j h_j\Box_j\mathrm{A}\Box_j\mathrm{C}_jh_j + \sum_j\Box_j[\mathrm{A},h_j]\Box_j\mathrm{C}_jh_j + \sum_j(1-\Box_j)\mathrm{A}h_j\mathrm{C}_jh_j, \end{aligned} \tag{A.36}$$

where in the last step we used $h_j\Box_j = \Box_jh_j = h_j$, cf. (A.18). By $\mathrm{C}_j := (\Box_j\mathrm{A}\Box_j)^{-1}|_{\mathcal{L}^2(\Box_j)}$ the first term in (A.36) sums up to one. Since $\mathrm{AC} = 1-\mathrm{R}$, the last two terms in (A.36) coincide with R. Thus we have

$$\mathrm{R} = \sum_j \Box_j[h_j,\mathrm{A}]\Box_j\mathrm{C}_jh_j + \sum_{j,i}(\Box_j-1)h_i^2\mathrm{A}h_j\mathrm{C}_jh_j. \tag{A.37}$$

We note that $(\Box_j-1)h_i$ vanishes for $i=j$. Thus we can write

$$\mathrm{R} = \sum_{i,j}\mathrm{R}_{i,j}\mathrm{C}_jh_j, \tag{A.38}$$

where

$$\mathrm{R}_{i,j} := \delta_{i,j}\Box_j[h_j,\mathrm{A}]\Box_j + (1-\delta_{i,j})(\Box_j-1)h_i^2\mathrm{A}h_j. \tag{A.39}$$

Now by Lemmas A.9, A.10 we obtain

$$\|\mathrm{R}\|_{\mathrm{op}} \le O(\tilde{M}^{-\delta/2})2^d\mathrm{m}^{-2}\sup_i\sum_j b(\tilde{M}(i-j)/3) \le O(\tilde{M}^{-\delta/2}), \tag{A.40}$$

where in the last step we used $b(x) := (1+|x|)^{d+\delta}a(x) \le (1+|x|)^{-(d+\delta)}$ to show that the sum is finite uniformly in $\tilde{M}$. □

Lemma A.8. *Let $T(x,x')$ be the kernel of T on $\mathcal{L}^2(\mathbb{Z}^d)$. Then*

$$\|T\|_{\mathrm{op}} \le \Big(\sup_x\sum_{x'}|T(x,x')|\Big)^{1/2}\Big(\sup_{x'}\sum_x|T(x,x')|\Big)^{1/2}. \tag{A.41}$$

Proof. The bound is standard and elementary. See e.g. [DG97, Lemma B.6.1]. □

Lemma A.9. *Let* R *be as above. Then*

$$\|\mathrm{R}\|_{\mathrm{op}} \le 2^d\mathrm{m}^{-2}\Big(\sup_i\sum_j\|\mathrm{R}_{i,j}\|_{\mathrm{op}}\Big)^{1/2}\Big(\sup_j\sum_i\|\mathrm{R}_{i,j}\|_{\mathrm{op}}\Big)^{1/2}. \tag{A.42}$$

Proof. First, we note that

$$\sum_i\langle f,\Box_if\rangle \le 3^d\|f\|_2^2. \tag{A.43}$$

In fact, as these are $2\tilde{M}$ boxes, in one dimension we could divide the sum into three families of shifted projections s.t. in each family the projections are mutually orthogonal. In arbitrary dimension we can do it in every direction. See Figure 4.

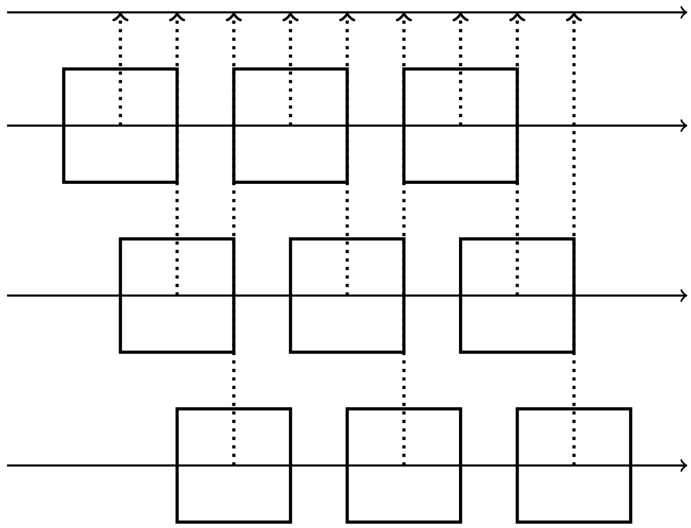

Figure 4: Three families of disjoint boxes in one dimension. Their union is the whole family of the $2\tilde{M}$-boxes centered at $\tilde{M}\mathbb{Z}$ described in Definition A.4.

Next, since $\mathrm{m}^2 \leq \mathrm{A}$, we have $\mathrm{m}^2\Box_j \leq \Box_j \mathrm{A} \Box_j$, hence $\mathrm{m}^2 \mathrm{C}_j^{1/2} \Box_j \mathrm{C}_j^{1/2} \leq \Box_j$. Thus $\|\mathrm{C}_j\|_{\mathrm{op}} \leq \mathrm{m}^{-2}$. We can write, using $\mathrm{R} = \sum_{i,j} \mathrm{R}_{ij} \mathrm{C}_j h_j$,

$$\begin{aligned} |\langle f, \mathrm{R} g\rangle| &\leq \sum_{i,j} \|\Box_i f\|_2 \, \|\mathrm{R}_{ij} \mathrm{C}_j h_j\|_{\mathrm{op}} \, \|\Box_j g\|_2 \\ &\leq \mathrm{m}^{-2} \sum_{i,j} (\|\Box_i f\|_2 \, \|\mathrm{R}_{ij}\|_{\mathrm{op}}^{1/2}) \, (\|\mathrm{R}_{ij}\|_{\mathrm{op}}^{1/2} \|\Box_j g\|_2) \\ &\leq 2^d \mathrm{m}^{-2} \Big(\sup_i \sum_j \|\mathrm{R}_{i,j}\|_{\mathrm{op}} \Big)^{1/2} \Big(\sup_j \sum_i \|\mathrm{R}_{i,j}\|_{\mathrm{op}} \Big)^{1/2} \|f\|_2 \|g\|_2, \end{aligned} \tag{A.44}$$

where we applied the Cauchy-Schwarz inequality in the last step. □

Lemma A.10. *Under the assumptions of Theorem A.6 and for $\tilde{M}$ sufficiently large*

$$\|\mathrm{R}_{i,j}\|_{\mathrm{op}} \leq O(\tilde{M}^{-\delta/2}) b(\tilde{M}(i-j)/3) \tag{A.45}$$

with δ, b of (A.1). (Here $O(\epsilon)$ denotes a number satisfying $|O(\epsilon)| \leq c\epsilon$).

Remark A.11. *This lemma only requires properties (i), (ii) and (v) from Definition A.1.*

Proof. We start with the $i = j$ case:

$$(\mathrm{R}_{i,i})(x, x') = (\Box_i [h_i, A] \Box_i)(x, x') = \mathrm{A}(x, x')(h_i(x) - h_i(x'))\Box_i(x)\Box_i(x'). \tag{A.46}$$

As h has a bounded derivative, we obtain

$$|h_i(x) - h_i(x')| \leq O(\tilde{M}^{-\delta/2}) |x - x'|^{\delta/2}. \tag{A.47}$$

Here we used the definition $h_j(x) := \prod_{k=0}^{d-1} h(\frac{x_k}{\tilde{M}} - j_k)$, the identity

$$\left| \frac{h_i(x) - h_i(x')}{|x - x'|^{\delta/2}} \right| = \left| \frac{h_i(x) - h_i(x')}{|x - x'|} \right|^{\delta/2} \left| h_i(x) - h_i(x') \right|^{1-\delta/2} \tag{A.48}$$

and the boundedness of h.

Now we recall that, by assumption, $|\mathrm{A}(x,x')| \le a(x-x')$ and $a(x-x') = (1+|x-x'|)^{-(d+\delta)} b(x-x')$. Hence, by (A.46), (A.47),

$$\begin{aligned} |\mathrm{R}_{i,i}(x,x')| &\le O(\tilde{M}^{-\delta})|x-x'|^{\delta/2}|\mathrm{A}(x,x')| \le O(\tilde{M}^{-\delta/2})|x-x'|^{\delta/2} a(x-x') \\ &\le O(\tilde{M}^{-\delta/2})(1+|x-x'|)^{-(d+\delta/2)} b(x-x'), \end{aligned} \tag{A.49}$$

where we expressed a by b in order to compensate the factor $|x-x'|^{\delta/2}$. Therefore, by Lemma A.8 and the fact that $b(x) \le c(1+|x|)^{-(d+\delta)}$

$$\|\mathrm{R}_{i,i}\|_{\mathrm{op}} \le O(\tilde{M}^{-\delta/2}) \sup_x \sum_{x'} b(x-x') = O(\tilde{M}^{-\delta/2}) \sum_x b(x) = O(\tilde{M}^{-\delta/2}). \tag{A.50}$$

This concludes the discussion of the $i=j$ case.

For $i \ne j$ we infer from (A.39) that the expression can only be different from zero if

$$x \in \operatorname{supp}(h_i) \cap (\mathbb{Z}^{\mathrm{d}} \backslash \square_j) \quad \textit{and} \quad x' \in \operatorname{supp} h_j. \tag{A.51}$$

With this restriction we have $|x-x'| > \frac{1}{3}\tilde{M}|i-j|$ by referring to (A.15). We infer from $\mathrm{R}_{i,j} := (\square_j - 1)h_i^2 A h_j$ that

$$|\mathrm{R}_{i,j}(x,x')| \le |\mathrm{A}(x,x')| \le a(x-x') = (1+|x-x'|)^{-(d+\delta)} b(x-x'). \tag{A.52}$$

In the next step we apply Lemma A.8

$$\begin{aligned} \|\mathrm{R}_{i,j}\|_{\mathrm{op}} &\le \Big(\sup_x \sum_{x'} |\mathrm{R}_{i,j}(x,x')|\Big)^{1/2} \Big(\sup_{x'} \sum_x |\mathrm{R}_{i,j}(x,x')|\Big)^{1/2} \\ &\le O(\tilde{M}^{-\delta/2}) b(\tilde{M}(i-j)/3), \end{aligned} \tag{A.53}$$

where the factors $(1+|x-x'|)^{-(d+\delta)}$ from (A.52) controlled the summation which is over $x \ne x'$ with the restriction $|x-x'| > \frac{1}{3}\tilde{M}|i-j|$. More precisely, we used that Definition A.1 (v) gives

$$\begin{aligned} \sum_{x'} |\mathrm{R}_{i,j}(x,x')| &\le \sum_{x'} (1+|x-x'|)^{-(d+\delta)} b(x-x') \\ &\le O(\tilde{M}^{-\delta/2}) b(\tilde{M}|i-j|/3) \sum_{x'} (1+|x'|)^{-(d+\delta/2)}. \end{aligned} \tag{A.54}$$

As the last sum is finite, this concludes the proof. □

B Some technical lemmas

Lemma B.1. *If* $U \in \mathrm{Conf}(\Omega)$ *then* $\mathcal{C}(U) \in \mathrm{Conf}(\Omega_1)$.

Proof. Let $c_1, c_2 > 0$, $U_1, U_2 \in SU(2)$. We show that there exists $c_3 \ge 0$ and $U_3 \in SU(2)$ s.t.

$$c_1 U_1 + c_2 U_2 = c_3 U_3. \tag{B.1}$$

Indeed, using representation (1.25), we write $U_j = s_j A_{j,0} 1 + \mathrm{i}\vec{A}_j \cdot \vec{\sigma}$, $A_{j,0} := \sqrt{1-|\vec{A}_j|^2}$, $s_j \in \{\pm 1\}$. Then

$$c_1 U_1 + c_2 U_2 = (c_1 s_1 A_{1,0} + c_2 s_2 A_{2,0}) 1 + \mathrm{i}(c_1 \vec{A}_1 + c_2 \vec{A}_2) \cdot \vec{\sigma}. \tag{B.2}$$

From this we read off conditions on c_3, U_3:

$$c_1 \vec{A}_1 + c_2 \vec{A}_2 = c_3 \vec{A}_3, \tag{B.3}$$

$$c_1 s_1 \sqrt{1 - |\vec{A}_1|^2} + c_2 s_2 \sqrt{1 - |\vec{A}_2|^2} = s_3 \sqrt{c_3^2 - (c_1 \vec{A}_1 + c_2 \vec{A}_2)^2}. \tag{B.4}$$

The latter equation gives

$$c_3^2 - (c_1 \vec{A}_1 + c_2 \vec{A}_2)^2 = c_1^2 (1 - |\vec{A}_1|^2) + c_2^2 (1 - |\vec{A}_2|^2) + 2 c_1 c_2 s_1 s_2 \sqrt{1 - |\vec{A}_1|^2} \sqrt{1 - |\vec{A}_2|^2}, \tag{B.5}$$

$$c_3^2 = c_1^2 + c_2^2 + 2 c_1 c_2 s_1 s_2 \sqrt{1 - |\vec{A}_1|^2} \sqrt{1 - |\vec{A}_2|^2} + 2 c_1 c_2 \vec{A}_1 \cdot \vec{A}_2. \tag{B.6}$$

Clearly the r.h.s. of the last equation is non-negative as the mixed terms sum up to a scalar product $A_1 \cdot A_2$ of two unit Euclidean four-vectors. Given c_3, we determine $\vec{A}_3$ via (B.3) and s_3 via (B.4). If $c_3 = 0$, by convention $\vec{A} = 0$ and $U_3 = 1$. This concludes the proof of (B.1). Now, by iterating this equality, we get

$$\mathcal{C}_0(U)(y) = \frac{1}{L^2} \sum_{x \in B_1(y)} U(x) = cU \tag{B.7}$$

for some $U \in SU(2)$ and $c \geq 0$. This completes the proof of the lemma. □

Lemma B.2. *Let M be a self-adjoint operator on a Hilbert space with $\|M\|_{\mathrm{op}} < 1/2$. Then*

$$\|(1+M)^{1/2} - 1\|_{\mathrm{op}} \leq c_{\frac{1}{2}} \|M\|_{\mathrm{op}}, \tag{B.8}$$

where $c_{\frac{1}{2}} \geq 1$ is a numerical constant and $\|\cdot\|_{\mathrm{op}}$ is the operator norm.

Proof. We recall the standard representation

$$(1+M)^{-1/2} = \frac{1}{\pi} \int_0^\infty dy \frac{1}{\sqrt{y}} \frac{1}{y+1+M}. \tag{B.9}$$

This gives

$$\begin{aligned} (1+M)^{1/2} - 1 &= \frac{1}{\pi} \int_0^\infty dy \frac{1}{\sqrt{y}} \left(\frac{1+M}{y+1+M} - \frac{1}{y+1} \right) \\ &= \frac{1}{\pi} \int_0^\infty dy \frac{1}{\sqrt{y}(y+1)} \left(\frac{(1+M)(y+1) - (y+1+M)}{y+1+M} \right) \\ &= \frac{1}{\pi} \int_0^\infty dy \frac{1}{\sqrt{y}(y+1)} \left(\frac{My}{y+1+M} \right). \end{aligned} \tag{B.10}$$

Hence

$$\begin{aligned} \|(1+M)^{1/2} - 1\|_{\mathrm{op}} &\leq \|M\|_{\mathrm{op}} \frac{1}{\pi} \int_0^\infty dy \, \frac{y}{\sqrt{y}(y+1)} \| \left(\frac{1}{y+1+M} \right) \|_{\mathrm{op}} \\ &\leq \|M\|_{\mathrm{op}} \int_0^\infty dy \, \frac{y}{\sqrt{y}(y+1)} \left(\frac{1}{y+1/2} \right) = c_{\frac{1}{2}} \|M\|_{\mathrm{op}}, \end{aligned} \tag{B.11}$$

which concludes the proof. □

Lemma B.3. *The following estimates hold true for* $|\vec{v}|, |\vec{w}_1|, |\vec{w}_2| \leq 1$

$$\|v_0 + \mathrm{i}\vec{v}\cdot\vec{\sigma}\|^2 \leq 2(v_0^2 + \vec{v}^2), \tag{B.12}$$

$$\|(\sqrt{1-\vec{w}_1^2} - \sqrt{1-\vec{w}_2^2}) + \mathrm{i}(\vec{w}_1 - \vec{w}_2)\cdot\vec{\sigma}\|^2 \leq 6(|\vec{w}_1|^2 + |\vec{w}_2|^2). \tag{B.13}$$

Proof. We note that, using $\|C\| \leq (\mathrm{Tr}(C^*C))^{1/2}$,

$$\begin{aligned} \|v_0 + \mathrm{i}\vec{v}\cdot\vec{\sigma}\|^2 &\leq \mathrm{Tr}((v_0 - \mathrm{i}\vec{v}\cdot\vec{\sigma})(v_0 + \mathrm{i}\vec{v}\cdot\vec{\sigma})) \\ &= \mathrm{Tr}(v_0^2 + (\vec{v}\cdot\vec{\sigma})^2) = 2(v_0^2 + \vec{v}^2). \end{aligned} \tag{B.14}$$

Now choosing $\vec{v} = \vec{w}_1 - \vec{w}_2$ and $v_0 = \sqrt{1-\vec{w}_1^2} - \sqrt{1-\vec{w}_2^2}$. Thus we have

$$\begin{aligned} \|(\sqrt{1-\vec{w}_1^2} - \sqrt{1-\vec{w}_2^2}) + \mathrm{i}(\vec{w}_1 - \vec{w}_2)\cdot\vec{\sigma}\|^2 &\leq 2[(\sqrt{1-\vec{w}_1^2} - \sqrt{1-\vec{w}_2^2})^2 + (\vec{w}_1 - \vec{w}_2)^2] \\ &= 2\frac{1-(1-\vec{w}_1^2)(1-\vec{w}_2^2)}{1+\sqrt{1-\vec{w}_1^2}\sqrt{1-\vec{w}_2^2}} - 2\vec{w}_1\cdot\vec{w}_2 \\ &\leq 2(\vec{w}_1^2 + \vec{w}_2^2 - \vec{w}_1^2\vec{w}_2^2) - 2\vec{w}_1\cdot\vec{w}_2 \leq 3(\vec{w}_1^2 + \vec{w}_2^2) \end{aligned} \tag{B.15}$$

which gives the required estimate. □

Finally, we provide computations leading to formula (3.46) for the remainder $\vec{r}$:

Lemma B.4. *Setting* $A_\pm = A(b_\pm)$, $B = \partial V(y_b)$ *and* $\delta(M) = 1 - M$ *for any* $M \in \mathbb{R}$, *we obtain*

$$\begin{aligned} \vec{r}(b) &= -\delta(A_-^0 B^0)\vec{A}_+ + \delta(A_+^0 B^0)\vec{A}_- + A_+^0 A_-^0 \vec{B} \\ &\quad -A_+^0(\vec{A}_- \times \vec{B}) + A_-^0(\vec{B}\times\vec{A}_+) + B^0(\vec{A}_- \times \vec{A}_+) \\ &\quad +\vec{A}_+(\vec{A}_-\cdot\vec{B}) - \vec{A}_-(\vec{B}\cdot\vec{A}_+) + \vec{B}(\vec{A}_-\cdot\vec{A}_+). \end{aligned} \tag{B.16}$$

Proof. In the notation introduced in Subsection 3.4 we recall the multiplication table for a product UV of two elements of $SU(2)$:

$$(UV)^0 = U^0V^0 - \vec{U}\cdot\vec{V}, \tag{B.17}$$

$$\overrightarrow{(UV)} = U^0\vec{V} + V^0\vec{U} - (\vec{U}\times\vec{V}), \tag{B.18}$$

where we used (3.33). Now we look at the product UVZ of three elements from $SU(2)$

$$\begin{aligned} (UVZ)^0 &= (UV)^0Z^0 - \overrightarrow{(UV)}\cdot\vec{Z}, \\ &= U^0V^0Z^0 - \vec{U}\cdot\vec{V}Z^0 - (U^0\vec{V} + V^0\vec{U} - (\vec{U}\times\vec{V}))\cdot\vec{Z} \\ &= U^0V^0Z^0 - Z^0(\vec{U}\cdot\vec{V}) - U^0(\vec{V}\cdot\vec{Z}) - V^0(\vec{U}\cdot\vec{Z}) + (\vec{U}\times\vec{V})\cdot\vec{Z} \end{aligned} \tag{B.19}$$

and similarly

$$\begin{aligned} \overrightarrow{(UVZ)} &= (UV)^0\vec{Z} + Z^0\overrightarrow{UV} - (\overrightarrow{UV}\times\vec{Z}) \\ &= (U^0V^0 - (\vec{U}\cdot\vec{V}))\vec{Z} + Z^0(U^0\vec{V} + V^0\vec{U} - (\vec{U}\times\vec{V})) \\ &\quad -(U^0\vec{V} + V^0\vec{U} - (\vec{U}\times\vec{V}))\times\vec{Z} \end{aligned} \tag{B.20}$$

$$\begin{aligned} &= U^0V^0\vec{Z} + Z^0U^0\vec{V} + Z^0V^0\vec{U} - Z^0(\vec{U}\times\vec{V}) - U^0(\vec{V}\times\vec{Z}) - V^0(\vec{U}\times\vec{Z}) \\ &\quad -(\vec{U}\cdot\vec{V})\vec{Z} - (\vec{U}\times\vec{V})\times\vec{Z} \end{aligned} \tag{B.21}$$

$$\begin{aligned} &= U^0V^0\vec{Z} + Z^0U^0\vec{V} + Z^0V^0\vec{U} - Z^0(\vec{U}\times\vec{V}) - U^0(\vec{V}\times\vec{Z}) - V^0(\vec{U}\times\vec{Z}) \\ &\quad -\vec{Z}(\vec{U}\cdot\vec{V}) + \vec{U}(\vec{Z}\cdot\vec{V}) - \vec{V}(\vec{Z}\cdot\vec{U}). \end{aligned} \tag{B.22}$$

Therefore, we have by substitution $\vec{Z} \to -\vec{Z}$.

$$\begin{aligned}\overrightarrow{(UVZ^*)} &= -U^0V^0\vec{Z} + Z^0U^0\vec{V} + Z^0V^0\vec{U} - Z^0(\vec{U}\times\vec{V}) + U^0(\vec{V}\times\vec{Z}) + V^0(\vec{U}\times\vec{Z}) \\ &\quad +\vec{Z}(\vec{U}\cdot\vec{V}) - \vec{U}(\vec{Z}\cdot\vec{V}) + \vec{V}(\vec{Z}\cdot\vec{U}). \end{aligned} \tag{B.23}$$

Setting $\overrightarrow{(UVZ^*)} =: \vec{U} - \vec{Z} + \vec{r}$, we obtain from (B.23)

$$\begin{aligned}\vec{r} &= \delta(U^0V^0)\vec{Z} - \delta(Z^0V^0)\vec{U} + Z^0U^0\vec{V} \\ &\quad -Z^0(\vec{U}\times\vec{V}) + U^0(\vec{V}\times\vec{Z}) + V^0(\vec{U}\times\vec{Z}) \\ &\quad +\vec{Z}(\vec{U}\cdot\vec{V}) - \vec{U}(\vec{Z}\cdot\vec{V}) + \vec{V}(\vec{Z}\cdot\vec{U}). \end{aligned} \tag{B.24}$$

We are interested in the case $U = U'(b_-)$, $V = \partial V(y_b)$, $Z = U'(b_+)$:

$$\begin{aligned}\vec{r}(b) &= -\delta(U'(b_-)^0\partial V(y_b)^0)\,\vec{U}'(b_+) + \delta(U'(b_+)^0\partial V(y_b)^0)\,\vec{U}'(b_-) + U'(b_+)^0U'(b_-)^0\partial\vec{V}(y_b) \\ &\quad -U'(b_+)^0(\vec{U}'(b_-)\times\partial\vec{V}(y_b)) + U'(b_-)^0(\partial\vec{V}(y_b)\times\vec{U}'(b_+)) + \partial V(y_b)^0(\vec{U}'(b_-)\times\vec{U}'(b_+)) \\ &\quad +\vec{U}'(b_+)(\vec{U}'(b_-)\cdot\partial\vec{V}(y_b)) - \vec{U}'(b_-)(\vec{U}'(b_+)\cdot\partial\vec{V}(y_b)) + \partial\vec{V}(y_b)(\vec{U}'(b_+)\cdot\vec{U}'(b_-)). \end{aligned} \tag{B.25}$$

This gives the formula from the statement of the lemma. ☐

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