informs
http://pubsonline.informs.org/journal/mnsc

MANAGEMENT SCIENCE
Vol. 68, No. 8, August 2022, pp. 6219–6234
ISSN 0025-1909 (print), ISSN 1526-5501 (online)

The Distortion in Prices Due to Passive Investing

Shmuel Baruch,[a] Xiaodi Zhang[b]

[a] Department of Economics and Finance, University of Rome Tor Vergata, Rome 00133, Italy; [b] School of Finance, Shanghai University of Finance and Economics, Shanghai 200433, China

Contact: shmuel.baruch@uniroma2.it, https://orcid.org/0000-0002-6129-7181 (SB); zhang.xiaodi@sufe.edu.cn, https://orcid.org/0000-0002-8759-1477 (XZ)

Received: May 28, 2019
Revised: October 28, 2020; April 8, 2021
Accepted: April 15, 2021
Published Online in Articles in Advance: November 24, 2021

https://doi.org/10.1287/mnsc.2021.4114

Abstract. In the capital asset pricing model (CAPM), it is ex post optimal to index. To examine the implications of market indexing, we develop a conditional CAPM with costless private information in which some investors are, for exogenous reasons, ex ante indexers. We show that, as more nonindexers become indexers, the price efficiency of stocks diminishes, asset prices comove, and the statistical fit (measured by R^2) of the CAPM regression decreases. We also report asset prices at the limit, when 100% of the investors are market indexers.

History: Accepted by Tyler Shumway, finance.

Keywords: partially revealing rational expectation equilibrium • conditional CAPM • market indexing • comovement • R^2 • 100% indexing

> What each prudent investor must do is to decide what fraction of savings he can afford, in this age of inflation, to keep in equities and in other things. An unmanaged, low-turnover, low-fee index fund is merely an efficient way of holding that part deemed appropriate for equities.
>
> (Samuelson 1976)

1. Introduction

The separation theorem is the theoretical underpinning of passive investment, and the capital asset pricing model (CAPM) risk-return relation is the most important pricing implication of the separation theorem. The CAPM, however, is silent about the impact of market indexing on asset prices. In this paper, we extend mean-variance equilibrium analysis to examine the implications of market indexing when (i) the separation result holds and (ii) a conditional CAPM holds. Thus, we frame the results in the standard CAPM terminology.

Our starting point is a multiasset version of Grossman's rational expectation model (Grossman 1976, 1978). Uncertainty is modeled using the multivariate normal distribution, and investors have negative exponential utilities. Investors costlessly observe private signals about asset payoffs and combine their private information with information contained in equilibrium prices. In that setting, Grossman (1978) shows that the equilibrium prices aggregate all payoff-relevant private information about asset payoffs (the equilibrium is fully revealing). Moreover, a conditional CAPM holds, whereby everyone selects an investment portfolio on the capital market line (combinations of the risk-free asset and market portfolio).

We modify the classical Grossman framework by supposing that some investors are market indexers—those who restrict their investment portfolios to the capital market line. Even though, in Grossman's model, this restriction is ex post nonbinding, having it imposed ex ante means that, whereas everyone participates in the price discovery process of the market portfolio and the risk-free asset, only nonindexers participate in the price discovery of individual stocks. Consequently, the equilibrium is not fully revealing, and it is meaningful to study the implications of indexing on asset prices.

After removing market indexers' demand, the group of nonindexers is still left to hold, in aggregate, the remaining fraction of the market portfolio. As in the classic CAPM, the equilibrium outcome is that each nonindexer's investment portfolio is also located on the capital market line. Thus, all investment portfolios—those of indexers and those of nonindexers—are located on the capital market line. The resulting equilibrium result is that allocations, the price of the market portfolio, and the risk-free interest rate are all unaffected by the level of market indexing.

We note that equilibrium prices depend on the realizations of signals and, furthermore, that the prices of individual stocks also depend on the specific partition of investors into indexers and nonindexers. Hence, prices are themselves random variables. Our comparative statics analysis is based on statistical properties of these prices. Those statistics do not depend on the specific data sample (i.e., the realization of the signals); they depend only on the number of indexers in the economy.

As we already stated, market indexing does not impact the price of the market portfolio, and, hence, indexing does not affect the price efficiency of the market portfolio. However, we show that indexing reduces the price efficiency of individual stocks. This result naturally follows from the assumption that indexers do not participate in the price discovery of individual assets. Another way to express this result is in terms of the systematic and idiosyncratic payoff risk of individual assets. The systematic risk is unaffected, whereas the idiosyncratic risk increases with the level of indexing.

The next result is that, although the CAPM risk-return relation holds in equilibrium, the statistical strength of the relation, measured by R^2, is reduced. This reduction in the explanatory power of the market portfolio is solely driven by the increase in idiosyncratic payoff risk.

The last result we report is that indexing causes comovement in stock prices. We start by noting that the price of the market portfolio is a public source of information that relays some of the indexers' information about individual assets. This information is noisy. To see why, suppose that an indexer sees a high signal for some asset. Trade by this indexer drives up the price of the market portfolio (assuming it is positively correlated with the portfolio of other assets), but this trade is not associated with a particular asset; it raises the prices of all assets. Thus, the price of the market portfolio is a public, but noisy, signal. As we have already pointed out, the information content of the market portfolio is unaffected by the level of indexing. However, as the number of indexers increases, the novelty of the information contained in the market portfolio increases. At one extreme, when everyone is a nonindexer, the equilibrium is fully revealing, and, accordingly, the information in the market portfolio's price is redundant. At the other extreme, when everyone is an indexer, the market portfolio is the only source of public information. Thus, the greater the level of indexing is, the more heavily nonindexers rely on the market portfolio when they price individual assets. This reliance translates to comovement in prices.

In equilibrium, the exogenous restriction of market indexers to the capital market line is ex post binding. Because equilibrium prices do not fully reflect indexers' information, any indexer can do better by trading individual assets at the equilibrium prices. Our modeling choice to impose the restriction is in the spirit of Levy (1978) and Merton (1987), who study equilibrium outcomes that result from an exogenous restriction on portfolio holdings. In these studies, the restricted investors forgo the full benefits of diversification. In our model, indexers forgo the full benefits of their costless private information. Thus, index traders in our model play the role of noise traders, except that, as in Levy (1978) and Merton (1987), we give them the discretion to optimize within the exogenous constraint. Such exogenous investment constraints exist in some employer-sponsored retirement accounts and state-sponsored college savings accounts.

Our paper contributes to the study of index investment in several aspects. Bond and Garcia (2021) and Liu and Wang (2018) also model mean-variance competitive economies in which a group of investors is confined to an index. These models focus on the impact of index investment on information production and welfare, whereas our model focuses on asset pricing. In addition, in the Bond and Garcia (2021) model, investors also face a participation cost: it is costly to participate in markets but cheaper to index, allowing their model to capture the notion that indexing democratizes the investment world. For Liu and Wang (2018), the cost of acquiring information about assets is an increasing, convex, multidimensional function of signal precisions. This cost structure allows their model to demonstrate how a rise in index investment can have different impacts on information production in the index asset and the nonindex asset.

Interestingly, Campbell et al. (2001) find that from 1962 to 1997, market variance was stable while firms' variances more than doubled, implying an increase in the idiosyncratic variance. These authors provide several possible explanations for this finding, such as the breakup of conglomerates. Also, Irvine and Pontiff (2009) report that idiosyncratic return volatility increases over time due to economy-wide competition, whereas Campbell (2017) and Rawley et al. (2018) find that the volatility of a highly diversified portfolio stays constant. The present paper provides a new explanation to these findings, namely that the rise of passive investment can explain the higher idiosyncratic variance without the increase in market volatility.

Our paper also relates to the study of the price efficiency of individual stocks. Both Bennett et al. (2020) and Billett et al. (2020) find that the informativeness of individual stock prices declines after joining the S&P 500 index. Our theory supports their finding: as index trading increases, stock prices impound less idiosyncratic information, thus becoming less informative.

Our model also adds to the literature that examines price comovements in different settings. According to Barberis and Shleifer (2003), rational traders take advantage of the extrapolative expectations of switchers who reallocate holdings from one set of assets to another. Barberis et al. (2005) review additional theories of comovement that stem from market frictions or noise traders' sentiment. Comovement also shows up in market structure–type models. According to Bhattacharya and O'Hara (2018), comovement stems from the inability to precisely tease out of the exchange-traded funds information relevant to individual assets. Cong and Xu (2019) present a model in which a nonredundant composite security is designed to cater to the needs of factor investors. They show that the introduction of the composite security increases comovement in prices. On the empirical side, Glosten et al. (2021) find that exchange traded funds activity increases return comovement due to the timely incorporation of systematic earnings information.

In the literature on active asset management, the term *indexing* is sometimes used as shorthand for benchmark indexing. Whenever investors are exposed to a benchmark index that is not the market portfolio, benchmarking distorts assets pricing. This point was demonstrated in a static, mean-variance setting with proportional fees by Brennan (1993) and in a dynamic setting with symmetric fees by Cuoco and Kaniel (2011). In our model, the index is the market portfolio, and we nevertheless find distortions due to indexing.[1]

Our model, finally, contributes to the literature on partially revealing equilibria. To avoid the fully revealing outcome, this literature relies on noise trading (Kyle 1985), supply uncertainty (Hellwig 1980, Admati 1985), extrinsic noise (DeMarzo and Skiadas 1998), or on preference uncertainty (see Ausubel 1990 and the dynamic model of Detemple 2002). Our model relies on none of these. Instead, the equilibrium we compute is partially revealing because index investors only participate in the price discovery process for the market portfolio.

The remainder of this paper is organized as follows. Section 2 describes the model. In Section 3, we define statistical sufficiency. In Section 4, we compute a partially revealing equilibrium with index investors. Section 5 presents, as a limiting case of our model, an equilibrium in which all investors are indexers. In Section 6, we study the asset pricing implications of the model. Section 7 concludes.

2. The Model

We study a two-period, single-good exchange economy with a risk-free financial asset (a bond in zero net supply), n risky real assets (stocks), and m investors. Risky assets are labeled $i=1,\dots,n$, and investors are labeled $k=1,\dots,m$. Investors have exponential utility, and we model uncertainty using the normal distribution. Thus, wealth plays no role in investment decisions, the economy is a mean-variance economy, and conditional variance-covariance matrices are deterministic.

For each risky asset, we normalize the number of outstanding shares to one. A *stock portfolio* is a vector $\mathbf{x}=[x_1 \;\; \dots \;\; x_n]'$ in R^n with the interpretation that x_i is the number of shares of the ith risky asset. Let $\mathbf{1}\in R^n$ denote the vector of all ones, so $\mathbf{1}$ is the *market portfolio*. Whenever an investor holds a stock portfolio that is a strictly positive scalar multiplication of $\mathbf{1}$, we can say, interchangeably, that the investor holds the market or that the *investment portfolio* (a portfolio of bonds and stocks) is on the capital market line.

The prices of the assets are denominated in units of the time-zero consumption good, and the assets' payoffs are denominated in units of the time-one consumption good. The consumption good is perishable, so the only way to transfer consumption between periods is through the capital market.

The payoff of the bond is one. The price of the bond is denoted by p_f. The vector of random payoffs, per share, of the risky assets is denoted by $\mathbf{v}=[v_1 \;\; \dots \;\; v_n]'$, where $\mathbf{v}\sim N(\mu_{\mathbf{v}},\Sigma_{\mathbf{vv}})$.[2] Thus, the random payoff of a stock portfolio $\mathbf{x}$ is $\mathbf{x}'\mathbf{v}$. The vector of share prices is denoted by $\mathbf{p}=[p_1 \;\; \dots \;\; p_n]'$, so the cost of $\mathbf{x}$ is $\mathbf{x}'\mathbf{p}$. When we need to work with scalars, we use the dot product. Thus, the payoff and the cost of the stock portfolio are sometimes expressed as $\mathbf{x}\cdot\mathbf{v}$ and $\mathbf{x}\cdot\mathbf{p}$, respectively. In particular, we let $v_{\text{mkt}}\equiv\mathbf{1}\cdot\mathbf{v}$ and $p_{\text{mkt}}\equiv\mathbf{1}\cdot\mathbf{p}$ denote the payoff and the price of the market portfolio.

Investors costlessly observe the realizations of private signals centered around $\mathbf{v}$. The signals are

$$\forall k=1\dots m,\quad \mathbf{s}_k=\mathbf{v}+\epsilon_k,$$
$$\epsilon_k\sim N(\mathbf{0},\Sigma_{\epsilon\epsilon}).$$

The random vectors $\{\mathbf{v},\epsilon_1,\dots,\epsilon_m\}$ are nondegenerate, jointly normally distributed, and mutually independent. The nondegeneracy assumption implies that the two matrices of variance-covariance, $\Sigma_{\mathbf{vv}}$ and $\Sigma_{\epsilon\epsilon}$, are positive definite.

The exponential utility assumption implies that investors' initial endowments are not relevant for their investment decisions. For simplicity of exposition, we turn off heterogeneity in endowments. Specifically, we assume that each investor is endowed with $\bar{c}$ units of the time-zero consumption good, zero bonds, and the stock portfolio $(1/m)\mathbf{1}$.

Subject to (s.t.) the budget constraint, the kth investor chooses the number of time-zero consumption units, c; the number of bonds, b; and a stock portfolio, $\mathbf{x}$, to maximize the expected value of the utility function

$$U_k(c,b,\mathbf{x}\cdot\mathbf{v})=-e^{-\rho_k c}-e^{-\rho_k(b+\mathbf{x}\cdot\mathbf{v})}.$$

Let

$$\bar{\rho}\equiv\left(\frac{1}{m}\sum_{k=1}^{m}\rho_k^{-1}\right)^{-1}$$

denote the harmonic mean of the coefficients of risk aversion. Investors are either market indexers or nonindexers. Market indexers are exogenously confined to holding the market. Nonindexers solve a complete portfolio selection problem. Later, we will show that this exogenous partition entails no welfare implications, in the sense that the equilibrium expected utility and allocations are unaffected by whether an investor is an indexer or not.

We denote the set of indices of market indexers by $\mathcal{I}$ and the set of indices of the nonindexers by $\mathcal{NI}$. We have $|\mathcal{I}|+|\mathcal{NI}|=m$. For now, we assume that both types of investors are present. In Section 5, we relax this assumption.

We can now write the investors' problems as follows:

$$\begin{aligned} &\forall k\in\mathcal{NI}, \ \max_{c,b,\mathbf{x}} \ E[U_k(c,b,\mathbf{x}\cdot\mathbf{v})|\mathbf{s}_k,\mathbf{p},p_f] \\ &\text{s.t.} \qquad \bar{c}-c+(0-b)p_f+((1/m)\mathbf{1}-\mathbf{x})\cdot\mathbf{p}=0, \end{aligned}$$

$$\begin{aligned} &\forall k\in\mathcal{I}, \ \max_{c,b,q} \ E[U_k(c,b,qv_{\text{mkt}})|\mathbf{s}_k,p_{\text{mkt}},p_f] \\ &\text{s.t.} \qquad \bar{c}-c+(0-b)p_f+(1/m-q)p_{\text{mkt}}=0. \end{aligned}$$

Let c_k, b_k, and $\mathbf{x}_k$ denote the optimal solutions of the previous maximization problems for k in either $\mathcal{NI}$ or $\mathcal{I}$, and $\mathbf{x}_k\equiv q_k\mathbf{1}$, whenever $k\in\mathcal{I}$. The solutions of the maximization problems that investors face depend on the realization of the prices and signals, but for brevity our notation does not show this dependency.

A *rational expectation equilibrium* (REE) is a random pair $(\mathbf{p},p_f)$ such that for each joint realization of prices and signals, the market for the consumption good, the market for debt, and the market for risky assets clear:

$$\sum_{k=1}^{m} c_k=m\bar{c}, \qquad \sum_{k=1}^{m} b_k=0, \qquad \sum_{k=1}^{m}\mathbf{x}_k=\sum_{k\in\mathcal{NI}}\mathbf{x}_k+\sum_{k\in\mathcal{I}}q_k\mathbf{1}=\mathbf{1}.$$

The definition of a rational expectation equilibrium is too permissive. For example, there is a rational expectation equilibrium in which $p_i=v_i/(1+r_f)$. In this unreasonable equilibrium, prices reveal the future. Furthermore, even in a noisy environment that includes random numbers of shares or noise trading, these prices form a rational expectation equilibrium. To rule out this future-telling equilibrium, we can follow Grossman (1976, 1978) and require that equilibrium prices be expressed as functions of the signals. However, in our model, only the nonindexers participate in the price discovery of individual assets. Therefore, we are interested in an equilibrium in which prices of individual assets depend only on information available to nonindexers.

The information available to nonindexers consists of their private signals and two additional public signals: the bond price and the price of the market portfolio. These prices are informative because indexers base their trades on them. Given that these prices are endogenous, we define the equilibrium implicitly.

We denote by $\mathbf{s}$ and $\mathbf{s}_{NI}$ the concatenations of all investors' signals and all nonindexers' signals, respectively. We say that an REE is an *REE in the presence of indexers* if $(\mathbf{p},p_f)$ is a (measurable) function of $\mathbf{s}$, and, in addition, $\mathbf{p}$ can be expressed as a (measurable) function of p_f, p_{mkt}, and $\mathbf{s}_{NI}$.

Note that knowing $\mathbf{s}$ alone is sufficient to compute equilibrium prices. Thus, we rule out future-telling equilibria or the equilibria with extrinsic noise studied by DeMarzo and Skiadas (1998). Furthermore, the definition requires that knowing $\mathbf{s}_{NI}$, p_f, and p_{mkt} be sufficient to compute the prices of individual assets. Thus, we also rule out Grossman's fully revealing equilibrium.

3. Sufficiency

Grossman (1976, 1978) uses the notion of statistical sufficiency to define informational efficiency in rational expectation equilibria. This statistical metaphor requires that we think of the realization of signals as a data sample that the market uses to estimate the unknown future payoff of assets. Under this interpretation, the market prices, themselves functions of the signals, are treated as a sample statistic.

Let $\mathbf{u}$ (representing the unknown) and $\mathbf{w}$ (representing the data sample) be two random vectors with a given joint distribution. Let $t(\mathbf{w})$ be a statistic. Following Grossman (1978), we say that $t(\mathbf{w})$ is sufficient for $\mathbf{u}$ if, for any measurable function F, we have[3]

$$E[F(\mathbf{u})|\mathbf{w}]=E[F(\mathbf{u})|t(\mathbf{w})].$$

For example, if we let $t(\cdot)$ be the identity, we see that $\mathbf{w}$ itself is a sufficient statistic. In general, however, the role of the statistic is to reduce the dimensionality of the data. Thus, $t(\cdot)$ is typically a many-to-one function. The statistic is sufficient if it perfectly aggregates information in $\mathbf{w}$ about the unknown $\mathbf{u}$. In particular, if the statistic $t(\mathbf{w})$ is a sufficient for $\mathbf{u}$, then

$$E[\mathbf{u}|\mathbf{w}]=E[\mathbf{u}|t(\mathbf{w})], \tag{1}$$
$$\text{var}(\mathbf{u}|\mathbf{w})=\text{var}(\mathbf{u}|t(\mathbf{w})). \tag{2}$$

To see that (1) holds, we take $F(\cdot)$ (in the definition of sufficiency) to be the identity. To see that (2) holds, we take $F(\mathbf{u})=\mathbf{u}\mathbf{u}'$ and, in addition, use (1). That is,

$$\begin{aligned} \text{var}(\mathbf{u}|\mathbf{w}) &= E[\mathbf{u}\mathbf{u}'|\mathbf{w}]-E[\mathbf{u}|\mathbf{w}]E[\mathbf{u}|\mathbf{w}]' \\ &= E[\mathbf{u}\mathbf{u}'|t(\mathbf{w})]-E[\mathbf{u}|t(\mathbf{w})]E[\mathbf{u}|t(\mathbf{w})]' \\ &= \text{var}(\mathbf{u}|t(\mathbf{w})). \end{aligned}$$

We need the following lemma.

Lemma 1. *Assume* $\mathbf{u}$ *and* $\mathbf{w}$ *are jointly normal. If a statistic* $t(\mathbf{w})$ *conveys the conditional expectation* $E[\mathbf{u}|\mathbf{w}]$ *in the sense that there is a measurable function* $\Psi(\cdot)$ *such that*

$$E[\mathbf{u}|\mathbf{w}]=\Psi(t(\mathbf{w})),$$

then $t(\mathbf{w})$ *is sufficient for* $\mathbf{u}$.

The proof of Lemma 1 is in the Appendix.

Note that by taking $\Psi(\cdot)$ to be the identity function, we see that the conditional expectation itself is a sufficient statistic. The lemma is useful because, once we show that a statistic conveys the conditional expectation, we can use the two identities, (1) and (2), saving ourselves lengthy computations.

4. Equilibrium in the Presence of Indexers

In a rational expectation equilibrium, prices play two roles: they relay information and determine budgets. Grossman's (1978) approach to finding the equilibrium prices is to disentangle the two roles. Following this approach, we first guess the information that equilibrium prices relay. To compute those prices, we then consider a symmetric information economy, called the artificial economy, that is otherwise identical to the actual economy. In the artificial economy, prices only determine budgets, and, therefore, equilibrium prices are relatively easy to compute. The public information in the artificial economy can be any statistic that conveys the information we have guessed the prices in the REE relay. If our guess was right, it is straightforward to verify that the prices in the artificial economy form an REE in the actual economy.

For example, when all investors are nonindexers (as in Grossman's models), we guess that the equilibrium is fully revealing, and to find the equilibrium prices, we use an artificial economy in which the public information is a statistic sufficient for $\mathbf{v}$. Grossman (1976, 1978) uses $\mathbf{s}$, which is the most trivial statistic sufficient for $\mathbf{v}$. But we can also use other sufficient statistics, such as $E[\mathbf{v}|\mathbf{s}]$. Grossman's approach is a heuristic, and as such, we do not have a proof for it. This is why our last step, when following this approach, is a verification step.

In a model with indexers, our educated guess is that prices relay all of the information that is in $\mathbf{s}_{NI}$ about $\mathbf{v}$. Therefore, in the artificial economy, everyone observes a statistic, $t(\mathbf{s}_{NI})$, sufficient for $\mathbf{v}$. Our educated guess has a second component: because all investors, indexers and nonindexers, participate in the price discovery of the market portfolio, prices relay all the information there is in $\mathbf{s}$ about v_{mkt}. Therefore, in the artificial economy, everyone also observes another statistic, $t(\mathbf{s})$, sufficient for v_{mkt}.

Formally, let

$$\mathbf{g} := (m\Sigma_{\mathbf{vv}} + \Sigma_{\epsilon\epsilon})^{-1}\Sigma_{\mathbf{vv}}\mathbf{1}, \tag{3}$$

and define

$$\mathbf{y} := \begin{bmatrix} \dfrac{1}{|\mathcal{NI}|}\displaystyle\sum_{k\in\mathcal{NI}} \mathbf{s}_k \\ \mathbf{g}'\displaystyle\sum_{k=1}^{m}\mathbf{s}_k \end{bmatrix}_{(n+1)\times 1}. \tag{4}$$

The realization of $\mathbf{y}$ is going to be the public information in the artificial economy that we are about to define and study. Each of the first n coordinates of $\mathbf{y}$ is the sample average of the nonindexers' signals. With normality, it is well known that the sample average is sufficient for $\mathbf{v}$. The next theorem shows that y_{n+1}, a statistic computed from $\mathbf{s}$, is sufficient for v_{mkt}.

Proposition 1. *Treating* $\mathbf{y}$ *and* y_{n+1} *as statistics computed from* $\mathbf{s}$, *each of them is sufficient for* v_{mkt}. *In particular,*

$$\begin{aligned} E[v_{\text{mkt}}|\mathbf{s}] &= E[v_{\text{mkt}}|\mathbf{y}] = E[v_{\text{mkt}}|y_{n+1}] \\ &= (\mathbf{1} - m\mathbf{g})'\mu_{\mathbf{v}} + y_{n+1}, \end{aligned} \tag{5}$$

$$\begin{aligned} \text{var}(v_{\text{mkt}}|\mathbf{s}) &= \text{var}(v_{\text{mkt}}|\mathbf{y}) = \text{var}(v_{\text{mkt}}|y_{n+1}) \\ &= (\mathbf{1} - m\mathbf{g})'\Sigma_{\mathbf{vv}}\mathbf{1}. \end{aligned} \tag{6}$$

The proof of Proposition 1 is in the Appendix.

For future reference, we state the identities

$$\text{cov}(\mathbf{v},\mathbf{y}) \equiv \Sigma_{\mathbf{vy}} = [\,\Sigma_{\mathbf{vv}} \quad m\Sigma_{\mathbf{vv}}\mathbf{g}\,]_{n\times(n+1)}, \tag{7}$$

$$\text{var}(\mathbf{y}) \equiv \Sigma_{\mathbf{yy}} = \begin{bmatrix} \Sigma_{\mathbf{vv}} + \dfrac{1}{|\mathcal{NI}|}\Sigma_{\epsilon\epsilon} & \Sigma_{\mathbf{vv}}\mathbf{1} \\ \mathbf{1}'\Sigma_{\mathbf{vv}} & m\mathbf{1}'\Sigma_{\mathbf{vv}}\mathbf{g} \end{bmatrix}_{(n+1)\times(n+1)}, \tag{8}$$

where the derivations of (7) and (8) are in the Appendix.

We note that $\text{cov}(\mathbf{v},\mathbf{y})$ is independent of the partition of investors into indexers and nonindexers, whereas the dependency of $\text{var}(\mathbf{y})$ on the partition is straightforward and involves only the number of nonindexers, $|\mathcal{NI}|$.

We define the artificial economy to be a symmetric information economy in which everything is as in the actual economy, except that everyone observes the realization of $\mathbf{y}$. An equilibrium in the artificial economy is a pair $(\mathbf{p}, p_f)$ such that for each realization of $\mathbf{y}$, investors solve

$$\begin{aligned} &\forall k \in \mathcal{NI}, \ \max_{c,b,\mathbf{x}} \ E[U_k(c,b,\mathbf{x}\cdot\mathbf{v})|\mathbf{y}] \\ &\text{s.t.} \qquad \bar{c} - c + (0-b)p_f + \Big((1/m)\mathbf{1} - \mathbf{x}\Big)\cdot\mathbf{p} = 0, \end{aligned}$$

$$\begin{aligned} &\forall k \in \mathcal{I}, \ \max_{c,b,q} \ E[U_k(c,b,q\mathbf{1}\cdot\mathbf{v})|\mathbf{y}] \\ &\text{s.t.} \qquad \bar{c} - c + (0-b)p_f + (1/m - q)\mathbf{1}\cdot\mathbf{p} = 0, \end{aligned}$$

and their optimal solutions clear the markets:

$$\sum_{k=1}^{m} c_k = m\bar{c}, \qquad \sum_{k=1}^{m} b_k = 0, \qquad \sum_{k\in\mathcal{NI}} \mathbf{x}_k + \sum_{k\in\mathcal{I}} q_k\mathbf{1} = \mathbf{1}.$$

The next theorem presents the equilibrium prices andallocations. Later, we verify that these prices form an REE in the presence of indexers in the actual economy.

Theorem 1. *The artificial economy has an equilibrium. The equilibrium prices are given by*

$$p_f = \exp\left(-\frac{\bar\rho}{m}\left(E[v_{\text{mkt}}|y_{n+1}] - \frac{\bar\rho}{2m}\underbrace{\text{var}(v_{\text{mkt}}|y_{n+1})}_{\text{deterministic}}\right) + \bar\rho\bar c\right), \tag{9}$$

$$\mathbf{p} = p_f\left(E[\mathbf{v}|\mathbf{y}] - \frac{\bar\rho}{m}\,\text{cov}(\mathbf{v}, \mathbf{1}'\mathbf{v}|\mathbf{y})\right) \Longleftrightarrow E[\mathbf{v}|\mathbf{y}] = \frac{1}{p_f}\mathbf{p} + \frac{\bar\rho}{m}\underbrace{\text{cov}(\mathbf{v}, \mathbf{1}'\mathbf{v}|\mathbf{y})}_{\text{deterministic}}, \tag{10}$$

and the price of the market portfolio is

$$p_{\text{mkt}} = p_f\left(E[v_{\text{mkt}}|y_{n+1}] - \frac{\bar\rho}{m}\underbrace{\text{var}(v_{\text{mkt}}|y_{n+1})}_{\text{deterministic}}\right). \tag{11}$$

In equilibrium, the optimal stock portfolio, bond holding, and consumption of the kth investor are

$$\mathbf{x}_k = \frac{1}{m}\frac{\bar\rho}{\rho_k}\mathbf{1}, \tag{12}$$

$$b_k = \left(1 - \frac{\bar\rho}{\rho_k}\right)\frac{\bar c + p_{\text{mkt}}/m}{1 + p_f}, \tag{13}$$

$$c_k = b_k + \frac{\bar\rho}{\rho_k}\bar c. \tag{14}$$

The proof of Theorem 1 is in the Appendix.

The fact that equilibrium prices, $(\mathbf{p}, p_f)$, are functions of $\mathbf{y}$ means that we can think of $\mathbf{y}$ as a data sample and the prices as a statistic computed from $\mathbf{y}$. We can see that these prices convey $E[\mathbf{v}|\mathbf{y}]$ (to see this, take $\Psi(\mathbf{p})$ in Lemma 1 to be the right-hand side of (10)). Therefore, the pair $(\mathbf{p}, p_f)$ is a sufficient statistic for $\mathbf{v}$. That is, the prices in the artificial economy contain all the information about $\mathbf{v}$ contained in $\mathbf{y}$.[4] In particular,

$$E[\mathbf{v}|\mathbf{y}] = E[\mathbf{v}|\mathbf{p}, p_f], \tag{15}$$

$$\text{var}(\mathbf{v}|\mathbf{y}) = \text{var}(\mathbf{v}|\mathbf{p}, p_f). \tag{16}$$

The next theorem shows that these prices are informationally superior to investors' own private signals.

Theorem 2. *When presented with the artificial economy's equilibrium prices, individuals view their own signals as redundant in the sense that*

$$\forall k \in \mathcal{NI},\ \forall c, b \in R,\ \forall \mathbf{x} \in R^n,$$

$$E[U_k(c, b, \mathbf{x}\cdot\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f] = E[U_k(c, b, \mathbf{x}\cdot\mathbf{v})|\mathbf{p}, p_f] = E[U_k(c, b, \mathbf{x}\cdot\mathbf{v})|\mathbf{y}], \tag{17}$$

and $\forall k \in \mathcal{I}, \forall c, b, q \in R,$

$$E[U_k(c, b, q\mathbf{1}\cdot\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f] = E[U_k(c, b, q\mathbf{1}\cdot\mathbf{v})|\mathbf{p}, p_f] = E[U_k(c, b, q\mathbf{1}\cdot\mathbf{v})|\mathbf{y}]. \tag{18}$$

The proof of Theorem 2 is in the Appendix.

We are now ready to show that the prices in Theorem 1 form an REE in the actual economy. First, the left-hand sides of (17) and (18) are the investors' objectives in the actual economy, and the right-hand sides of (17) and (18) are the investors' objectives in the artificial economy. Thus, the objectives are equal. Second, the prices are the same, so the investors' budget constraints are the same in both economies. Hence, the optimal allocations are also identical in both economies. Finally, since these allocations clear the artificial economy market, they also clear the market in the actual economy. We conclude that the artificial economy's equilibrium prices form a rational expectation equilibrium in the actual economy. Furthermore, the equilibrium allocations are also identical.

To see that the REE we reported in Theorem 1 satisfies the definition of an REE in the presence of indexers, we need to verify two additional conditions. The first is that the prices are functions of the signals. Because prices are functions of $\mathbf{y}$, and $\mathbf{y}$ is a function of the signals, the first condition is indeed satisfied. The second condition that we need to verify is that $\mathbf{p}$ can be expressed as a function of $\mathbf{s}_{NI}$, p_f, and p_{mkt}. We already know that $\mathbf{p}$ is a function of $\mathbf{y}$. Each of the first n coordinates are a function of $\mathbf{s}_{NI}$. Equations (5)–(6) and (11) imply that $y_{n+1} = \frac{p_{\text{mkt}}}{p_f} +$ a constant. Therefore, the prices in Theorem 1 form an REE in the presence of indexers.

We can find other rational expectations equilibria using Grossman's apparatus of artificial economies. We could start with an artificial economy in which everyone observes $\mathbf{v}$ and $\mathbf{s}$, or even just replace the first n coordinates of $\mathbf{y}$ with a sample mean of any group of investors, provided that this group contains all the nonindexers. The prices in each of those artificial economies would form a rational expectation equilibrium. Nevertheless, the prices in Theorem 1 are the only ones we know of that satisfy our definition of REE in the presence of indexers.

Returning our attention to Theorem 1, we see that price of the bond, (9), and the price of the market portfolio, (11), do not depend on the specific partition of investors into indexers and nonindexers. This is because the conditional expected payoff and conditional variance of the market portfolio do not depend on the partition (see (5) and (6)). Therefore, the equilibrium allocations, (12)–(14), also do not depend on the specific partition. In particular, all investment portfolios are located on the capital market line, and the location on the line is independent of the specific partition. We conclude that, in terms of welfare implications, indexing in this economy is benign.

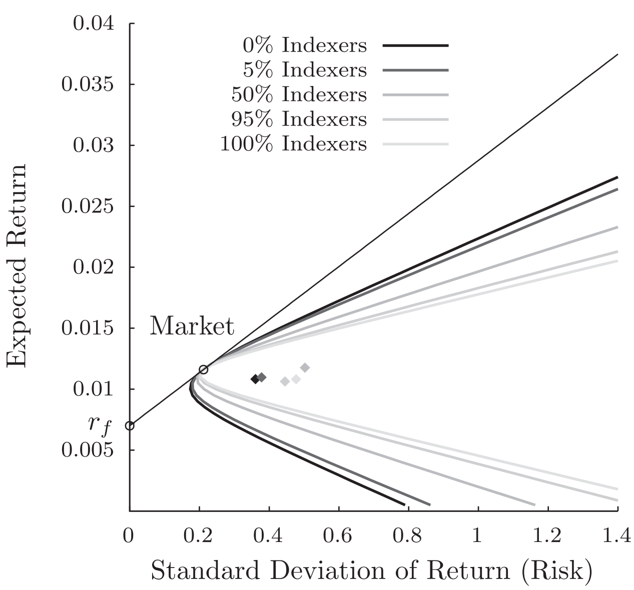
5. Asset Pricing When All Investors Are Indexers

In deriving the rational expectation equilibrium in the presence of indexers, we have explicitly assumed that some investors are indexers and some are nonindexers. When all investors are indexers, it is unclear how individual assets are priced. It is conceivable that there are infinitely many equilibria, all of which agree on the price of the market portfolio but disagree on the prices of individual assets. Our model can be used to pick one of these equilibria, the equilibrium that corresponds to the limiting case of our model in which all investors are indexers. This limiting case is the one in which we set $\mathbf{y}=y_{n+1}$, and we consider any statement about nonindexers to be trivially satisfied because the set of nonindexers is empty. The analysis in the previous section is then valid, providing us with asset prices in a model in which all investors are indexers.

For completeness of discussion, we also note that, when all investors are nonindexers, the last coordinate of $\mathbf{y}$ is redundant. We can remove this last coordinate from $\mathbf{y}$ and carry the analysis exactly as in the previous section, considering any statement about indexers to be trivially satisfied because the set of indexers is empty.

To visualize the two corner cases, we consider the mean-variance frontier for different levels of index investment. That is, we define the risk-free return as $r_f=(1-p_f)/p_f$ and the return on asset i as $r_i=(v_i-p_i)/p_i$. The equilibrium expected return and variance of return are $E[r_i|\mathbf{p},p_f]$ and $\mathrm{var}(r_i|\mathbf{p},p_f)$.

Figure 1. Frontiers and Capital Market Line for Different Levels of Market Indexing



Notes. These are the realized mean-variance frontiers in an example with six risky assets and 10,000 investors. The capital market line and the position of the market portfolio (the tangency point) do not depend on the fraction of indexers. In contrast, the five points in the center of the frontiers stand for the same individual risky asset, illustrating that the indexing affects the expected return and risk of the individual assets.

Figure 1 shows an example in which the realizations of the signals are fixed, and the only thing that we change is the partition of investors into indexers and nonindexers. The figure depicts, in the volatility-return plane, the capital market line and the mean-variance frontiers for different partitions of the set of investors. The capital market line is the same, regardless of the partition.

The figure shows the most common situation we found in the many simulations we tried: the mean-variance frontier that corresponds to a partition of the investors with a large set of indexers is nested in the mean-variance frontier that corresponds to a partition with a smaller set of indexers. To further investigate the hierarchy of the frontiers, we study the following example.

Example. Let $\mathbf{P}=\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}$ be a correlation matrix, and set $n=2$, $\Sigma_{\mathbf{vv}}=\sigma_v^2\mathbf{P}$ and $\Sigma_{\mathbf{vv}}=\sigma_\varepsilon^2\mathbf{P}$. Note that in this example, the assets are not necessarily identically distributed because we did not make any assumption of $\mu_{\mathbf{v}}$. The assets are also not necessarily independently distributed because ρ may be different from zero.

We say that the frontiers are nested when the mean-variance frontier that corresponds to the equilibrium when all investors are indexers is nested in the mean-variance frontier that corresponds to a partition with all investors nonindexers. We state, without showing our work, our findings: if the realized signals equal their expected values, then the frontiers are nested. Moreover, we have

$$\text{prob.(Frontiers are nested)}=$$

$$\Phi\Bigg(\underbrace{\frac{1}{\sigma_v^2\sqrt{2m(1-\rho)}}\left(-|E(v_1-v_2)|\sigma_\varepsilon+E(v_1-v_2)\sqrt{\sigma_\varepsilon^2+m\sigma_v^2}\right)}_{r_1}\Bigg)$$

$$+\Phi\Bigg(\underbrace{\frac{1}{\sigma_v^2\sqrt{2m(1-\rho)}}\left(-|E(v_1-v_2)|\sigma_\varepsilon+E(v_2-v_1)\sqrt{\sigma_\varepsilon^2+m\sigma_v^2}\right)}_{r_2}\Bigg),$$

where Φ is the standard normal distribution. Because $\max\{r_1,r_2\}>0$, the probability that the frontiers are nested is strictly greater than half. Because $r_1+r_2<0$, this probability is also strictly smaller than one.

As we let Ev_1 get closer to Ev_2, r_1 and r_2 get closer to zero, so both $\Phi(r_1)$ and $\Phi(r_2)$ approach $1/2$, and consequently their sum, the probability that the frontiers are nested, approaches one. To see why, we note that when the mean returns do not vary much from one another, the mean-variance frontier is thin, and consequently more likely to be nested. Now, in this example, when $Ev_1 = Ev_2$ and all investors are indexers, assets' returns are ex post returns are identically distributed, and in particular, they have the same mean.

Finally, when the correlation coefficient approaches one, $\min\{r_1, r_2\}$ goes to $-\infty$ while their max goes to ∞, and consequently the probability that the frontiers are nested approaches one.

6. Implications for Asset Pricing

In this section, we present some implications of our model for asset pricing. First, we show that indexing reduces the efficiency of prices. Second, we show that indexing causes comovements in asset prices. Lastly, we show that the CAPM holds even in the presence of indexing. However, the statistical strength of the CAPM diminishes as we increase the level of index investment.

The implications we report are no more than corollaries to the fact that though indexers do not participate in the price discovery of individual assets, they do participate in the price discovery of the market portfolio. To algebraically express this dichotomy between individual assets and the market portfolio, we rely on two technical lemmas.

Lemma 2. *Let $\mathbf{x} \in R^n$ be an arbitrary stock portfolio. By means of matching terms, define $\theta \in R^n$ and the scalar q to be*

$$[\,\theta' \quad q\,]_{1\times(n+1)} := \mathbf{x}'\Sigma_{\mathbf{vy}}\Sigma_{\mathbf{yy}}^{-1}. \tag{19}$$

The following three statements are equivalent:

a. *The stock portfolio $\mathbf{x}$ is a scalar multiplication of the market portfolio.*

b. $\mathbf{x} = q\mathbf{1}$.

c. $\theta = \mathbf{0}$.

That (b) implies (a) is a tautology. The complete proof of Lemma 2 is in the Appendix. Building on this result, we now show that for every risky asset i, $\text{cov}(v_i, v_{\text{mkt}}|\mathbf{p}, p_f)$ is independent of the partition of investors into indexers and nonindexers. We state this result in a matrix form.

Lemma 3. *The vector $\text{cov}(\mathbf{v}, \mathbf{1}'\mathbf{v}|\mathbf{p}, p_f) = \text{cov}(\mathbf{v}, \mathbf{1}'\mathbf{v}|\mathbf{y})$ is independent of the level of index investment.*

The proof of Lemma 3 is in the Appendix.

6.1. Price Efficiency

A natural way to measure the efficiency of prices is to consider the variance of the payoff, conditional on prices. The more information prices convey, the lower is the conditional variance. Due to the joint normality of $\mathbf{v}$ and $\mathbf{y}$, we have

$$\text{var}(\mathbf{v}|\mathbf{p}, p_f) \underset{(16)}{=} \text{var}(\mathbf{v}|\mathbf{y}) = \Sigma_{\mathbf{vv}} - \Sigma_{\mathbf{vy}}\Sigma_{\mathbf{yy}}^{-1}\Sigma_{\mathbf{yv}}. \tag{20}$$

The next theorem tells us that market indexing does not hamper the price efficiency of the market portfolio. However, indexing reduces the efficiency of prices of individual assets. The intuition behind these results is as follows. Prices aggregate payoff-relevant information. Since everyone invests in the market portfolio and the bond, a partial equilibrium result is that the information content of both p_m and p_f is insensitive to the level of indexing. On the other hand, only nonindexers invest in individual assets, giving us a second partial equilibrium result: prices aggregate information in $\mathbf{s}_{NI}$ about the payoff of individual assets. Since our definition of equilibrium precludes prices from revealing information beyond that in $\mathbf{s}_{NI}$, p_m, and p_f, we conclude that an increase in the level of indexing reduces the price efficiency of individual assets. In fact, the theorem shows even more. Not only does the distortion to price efficiency not diminish, but also, as the level of indexing increases, the distortion increases. Although at first glance this result may seem surprising, it is natural if we think in terms of increasing the number of nonindexers in the economy. As we increase the number of nonindexers in the economy, the additional information revealed through the price discovery process is diminished.[5] But this is the same as saying that prices lose their information content at a greater rate as we increase the number of indexers. We present the theorem in a matrix form.

Theorem 3 (Efficiency of Prices). *Let $\mathbf{x} \in R^n$ be a stock portfolio, and consider changes as we increase $|\mathcal{I}|$. If $\mathbf{x}$ is a scalar multiplication of the market portfolio, then the conditional variance $\text{var}(\mathbf{x}'\mathbf{v}|p_f, \mathbf{p})$ does not change. For all other stock portfolios, $\text{var}(\mathbf{x}'\mathbf{v}|p_f, \mathbf{p})$ strictly increases at an increasing rate.*

Proof. Because increasing $|\mathcal{I}|$ is equivalent to decreasing $|\mathcal{NI}|$, we can examine the derivative of the conditional variance with respect to $|\mathcal{NI}|$. We fix a stock portfolio $\mathbf{x}$, and, by matching terms, define θ and q as in (19). Because $\mathbf{x}$, $\Sigma_{\mathbf{vv}}$, and $\Sigma_{\mathbf{vy}}$ are independent of the number $|\mathcal{NI}|$, we have[6]

$$\begin{aligned} \frac{\partial \text{var}(\mathbf{x}'\mathbf{v}|\mathbf{p}, p_f)}{\partial|\mathcal{NI}|} &= \mathbf{x}'\Sigma_{\mathbf{vy}}\Sigma_{\mathbf{yy}}^{-1}\frac{\partial \Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|}\Sigma_{\mathbf{yy}}^{-1}\Sigma_{\mathbf{yv}}\mathbf{x} \\ &= [\,\theta' \quad q\,]\underbrace{\begin{bmatrix} -\frac{1}{|\mathcal{NI}|^2}\Sigma_{\epsilon\epsilon} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}}_{\frac{\partial \Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|}}\begin{bmatrix} \theta \\ q \end{bmatrix} \\ &= -\frac{1}{|\mathcal{NI}|^2}\theta'\Sigma_{\epsilon\epsilon}\theta \le 0. \end{aligned} \tag{21}$$

The inequality arises because $\Sigma_{\epsilon\epsilon}$ is positive definite. The inequality is strict if and only if $\theta \neq \mathbf{0}$. According to Lemma 2, $\theta = \mathbf{0}$ if and only if the stock portfolio $\mathbf{x}$ is a scalar multiplication of $\mathbf{1}$.

Since $|\mathcal{I}| = m - |\mathcal{NI}|$, we also have $\frac{\partial^2 \text{var}(\mathbf{x}'\mathbf{v}|\mathbf{p}, p_f)}{\partial|\mathcal{I}|^2} = \frac{\partial^2 \text{var}(\mathbf{x}'\mathbf{v}|\mathbf{p}, p_f)}{\partial|\mathcal{NI}|^2}$. Thus, to prove that the rate of change is increasing as we increase $|\mathcal{I}|$, it is enough to show that the second derivative with respect to $|\mathcal{NI}|$ is increasing.

We differentiate both sides of the first equality in (21), relying on the fact that $\mathbf{x}$ and $\Sigma_{\mathbf{vy}}$ are independent of the number $|\mathcal{NI}|$:

$$\frac{\partial}{\partial|\mathcal{NI}|}\frac{\partial \mathrm{var}(\mathbf{x}'\mathbf{v}|\mathbf{p},p_f)}{\partial|\mathcal{NI}|}=\mathbf{x}'\Sigma_{\mathbf{vy}}\frac{\partial\left(\Sigma_{\mathbf{yy}}^{-1}\frac{\partial\Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|}\Sigma_{\mathbf{yy}}^{-1}\right)}{\partial|\mathcal{NI}|}\Sigma_{\mathbf{yv}}\mathbf{x}$$
$$=\mathbf{x}'\Sigma_{\mathbf{vy}}\left(\frac{\partial\Sigma_{\mathbf{yy}}^{-1}}{\partial|\mathcal{NI}|}\frac{\partial\Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|}\Sigma_{\mathbf{yy}}^{-1}+\Sigma_{\mathbf{yy}}^{-1}\frac{\partial^2\Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|^2}\Sigma_{\mathbf{yy}}^{-1}+\Sigma_{\mathbf{yy}}^{-1}\frac{\partial\Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|}\frac{\partial\Sigma_{\mathbf{yy}}^{-1}}{\partial|\mathcal{NI}|}\right)\Sigma_{\mathbf{yv}}\mathbf{x}$$
$$=\mathbf{x}'\Sigma_{\mathbf{vy}}\Sigma_{\mathbf{yy}}^{-1}\frac{\partial\Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|}\underbrace{\Sigma_{\mathbf{yy}}^{-1}}_{\text{PD}}\frac{\partial\Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|}\Sigma_{\mathbf{yy}}^{-1}\Sigma_{\mathbf{yv}}\mathbf{x}+\mathbf{x}'\Sigma_{\mathbf{vy}}\Sigma_{\mathbf{yy}}^{-1}\underbrace{\frac{\partial^2\Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|^2}}_{\text{PSD}}\Sigma_{\mathbf{yy}}^{-1}\Sigma_{\mathbf{yv}}\mathbf{x}$$
$$+\mathbf{x}'\Sigma_{\mathbf{vy}}\Sigma_{\mathbf{yy}}^{-1}\frac{\partial\Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|}\underbrace{\Sigma_{\mathbf{yy}}^{-1}}_{\text{PD}}\frac{\partial\Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|}\Sigma_{\mathbf{yy}}^{-1}\Sigma_{\mathbf{yv}}\mathbf{x}\geq 0.$$

To see that $\Sigma_{\mathbf{yy}}^{-1}$ is positive definite (PD), we note that $\mathbf{y}$ is nondegenerate, and hence $\Sigma_{\mathbf{yy}}$ is positive definite. The inverse of a positive definite is positive definite. Finally,

$$\frac{\partial^2\Sigma_{\mathbf{yy}}}{\partial|\mathcal{NI}|^2}=\frac{2}{|\mathcal{NI}|^3}\begin{bmatrix}\Sigma_{\epsilon\epsilon} & \mathbf{0}_{n\times 1}\\ \mathbf{0}_{1\times n} & 0\end{bmatrix}$$

is positive semidefinite (PSD) because $\Sigma_{\epsilon\epsilon}$ is positive definite. Q.E.D.

Example. The result in Theorem 3 is qualitative. To get a quantitative result, we need to impose stronger distributional assumptions. We let $\Sigma_{\mathbf{vv}}=\sigma_v^2\mathbf{P}$ and $\Sigma_{\mathbf{vv}}=\sigma_\varepsilon^2\mathbf{P}$, where $\mathbf{P}$ is a correlation matrix with all off-diagonal elements equal to ρ. Thus, ρ measures the correlation in the economy, σ_v^2 measures the ex ante payoff uncertainty, and σ_ε^2 measures the noise in the signals. For the covariance matrices to be positive definite, we must also assume $\rho>-1/(n-1)$. The distributional assumptions are as in the example we presented in Section 5, except that now n is arbitrary. Under these simplifying assumptions, we can analytically compute equilibrium outcomes. In particular, the conditional variance of v_{mkt} is

$$\mathrm{var}(v_{\mathrm{mkt}}|\mathbf{p},p_f)=\frac{n((n-1)\rho+1)\sigma_v^2\sigma_\varepsilon^2}{m\sigma_v^2+\sigma_\varepsilon^2},$$

and the conditional variance of individual payoffs is identical for all assets and equal to

$$\mathrm{var}(v_i|\mathbf{p},p_f)=\sigma_v^2-\frac{\sigma_v^4\left(\left((1-\rho)|\mathcal{NI}|+\left(\frac{1}{(n-1)}+\rho\right)m\right)\sigma_\varepsilon^2(n-1)+\sigma_v^2|\mathcal{NI}|mn\right)}{n(m\sigma_v^2+\sigma_\varepsilon^2)(\sigma_v^2|\mathcal{NI}|+\sigma_\varepsilon^2)}.$$

Using the identity $|\mathcal{NI}|=m-|\mathcal{I}|$ and the chain rule, a simple differentiation with respect to $|\mathcal{I}|$ shows that

$$\frac{\partial\mathrm{var}(v_i|\mathbf{p},p_f)}{\partial|\mathcal{I}|}=\frac{(n-1)}{n}\frac{(1-\rho)\sigma_v^4\sigma_\varepsilon^2}{(|\mathcal{NI}|\sigma_v^2+\sigma_\varepsilon^2)^2}>0,$$

and

$$\frac{\partial^2\mathrm{var}(v_i|\mathbf{p},p_f)}{\partial|\mathcal{I}|^2}=\frac{2(n-1)}{n}\frac{(1-\rho)\sigma_v^6\sigma_\varepsilon^2}{(|\mathcal{NI}|\sigma_v^2+\sigma_\varepsilon^2)^3}>0,$$

consistent with Theorem 3.

We can use this example to find how the interaction between indexing and ex ante correlation affects price efficiency. We expect that the greater the correlation among assets is, the more informative the market portfolio will be about individual assets and, consequently, the smaller the impact of indexing on asset prices will be. To see whether our model provides support for this intuition, we examine the cross derivative. Indeed, we find that

$$\frac{\partial^2\mathrm{var}(v_i|\mathbf{p},p_f)}{\partial\rho\partial|\mathcal{I}|}=-\frac{(n-1)}{n}\frac{\sigma_v^4\sigma_\varepsilon^2}{(|\mathcal{NI}|\sigma_v^2+\sigma_\varepsilon^2)^2}<0.$$

Thus, while increasing the level of indexing increases the conditional variance (and thereby reduces price efficiency), this increase in the variance is smaller when the correlation coefficient increases.

6.2. Comovement

Index trading can cause comovement of asset prices for a variety of reasons. The most obvious is that commonality in demand translates to commonality in prices. In our model, indexing does not impact equilibrium allocations, and therefore there is no commonality in demand. Nevertheless, indexing in our model causes asset prices to comove.

As we have already shown, market indexing does not influence the price of the market portfolio or its information content. However, the more indexers are present, the more heavily nonindexers rely on the price of the market portfolio as a source of information. At one extreme, when all investors are nonindexers, the information in the price of the market portfolio is redundant. At the other extreme, as we approach 100% indexers, the price of the market portfolio becomes the sole source of information for pricing all assets.

We use the notion of correlation to measure comovement. Pearson's correlation is designed to measure linear dependency, and nominal prices in our model are related to each other in a nonlinear manner. On the other hand, the discounted prices, $p_f^{-1}\mathbf{p}$, depend linearly on $\mathbf{y}$, and hence depend linearly on each other. We therefore use the discounted prices to demonstrate the comovement.

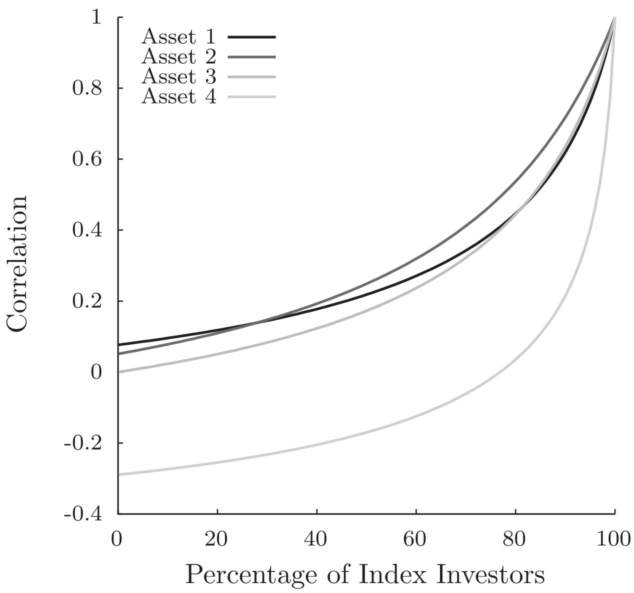
For any asset i, let $\mathbf{e}_i$ and $\mathbf{1}-\mathbf{e}_i$ be the portfolios that include only asset i and all assets except for asset i, respectively. This yields the following.

Theorem 4 (Comovement in Asset Prices). *If* $\mathrm{corr}(\mathbf{e}_i{}'p_f^{-1}\mathbf{p}, (\mathbf{1}-\mathbf{e}_i)'p_f^{-1}\mathbf{p})>0$, *then increasing the level of indexing strictly increases the correlation.*

For the proof of Theorem 4, see the Appendix.

As the percentage of indexers approaches 100%, the information content of $\mathbf{y}$ approaches the information content of y_{n+1} (see the discussion in Section 5). At the

Figure 2. Correlation in Discounted Prices Becomes Perfect When the Percentage of Index Investors is 100%



Notes. In this example, there are four risky assets, and for each we compute the correlation between it and the portfolio of remaining assets. The payoff of one of the assets is negatively correlated with the payoff of the remaining assets. The parameters in this example are

$$n=4,\quad m=10{,}000,\quad \Sigma_{\mathbf{vv}}=\begin{pmatrix} 5 & 1 & 1 & -1 \\ 1 & 20 & 2 & -2 \\ 1 & 2 & 10 & -3 \\ -1 & -2 & -3 & 10 \end{pmatrix},\ \text{and}\ \Sigma_{\epsilon\epsilon}=10{,}000\times\Sigma_{\mathbf{vv}}.$$

The values of $\bar{p}$, $\mu_{\mathbf{v}}$, and $\bar{c}$ are irrelevant for the purpose of computing these correlations.

limit, discounted prices are perfectly correlated because discounted prices of individual assets are expressed as a linear function of y_{n+1}. We note that Cong and Xu (2019) and Glosten et al. (2021) also find that the presence of investors that trade baskets of securities leads to price comovement.

In Figure 2, we provide an example of how indexing can distort asset prices. In this example, the payoff of one asset is negatively correlated with the remaining assets' payoff. One thinks of an asset as a natural hedge against market movements. Indeed, at a low level of indexing, this asset's price typically moves in the opposite direction of the market. However, as we can see from the figure, even though the conditions in Theorem 4 do not hold, this asset's price comoves with the market at a high level of indexing.

6.3. The CAPM

In general, to show that the CAPM risk-return relationship holds in an economy, it is sufficient that one nonindexer, with mean-variance risk preferences, holds the market portfolio. And we have shown that all nonindexers hold the market portfolio. Therefore, we can easily prove the following.

Theorem 5 (Conditional CAPM). *In the economy with index investors, the CAPM holds. That is, let*

$$\beta_i \equiv \frac{\text{cov}(r_{\text{mkt}}, r_i|p_f, \mathbf{p})}{\text{var}(r_{\text{mkt}}|p_f, \mathbf{p})}.$$

Then the following risk-return relation holds:

$$E[r_i|p_f, \mathbf{p}]=r_f+\beta_i(E[r_{\text{mkt}}|p_f, \mathbf{p}]-r_f). \tag{22}$$

The proof of Theorem 5 is in the Appendix.

Because the conditional CAPM holds in our model, the theory of linear regression is applicable, and it is meaningful to use R^2 as a measure of the strength of the CAPM regression of r_i-r_f on $r_{\text{mkt}}-r_f$. In practice, one computes the sample R^2. Here, we compute the population R^2. We have the following.

Theorem 6 (Conditional R^2). *For every asset i, R^2 of the CAPM risk-return relation decreases with $|\mathcal{I}|$.*

Proof. We have

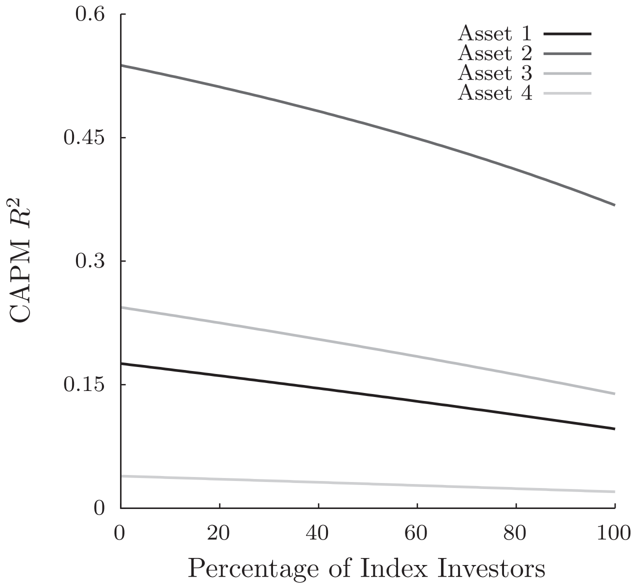
$$\begin{aligned} R^2 &= \text{corr}^2(r_i, r_f+\beta_i(r_{\text{mkt}}-r_f)|p_f, \mathbf{p}) \\ &= \frac{\text{cov}^2(r_i, \beta_i r_{\text{mkt}}|p_f, \mathbf{p})}{\text{var}(r_i|p_f, \mathbf{p})\times\text{var}(\beta_i r_{\text{mkt}}|p_f, \mathbf{p})} \\ &= \frac{\text{cov}^2(v_i, v_{\text{mkt}}|p_f, \mathbf{p})}{\text{var}(v_i|p_f, \mathbf{p})\times\text{var}(v_{\text{mkt}}|p_f, \mathbf{p})} \\ &\underset{(16)}{=} \frac{\text{cov}^2(v_i, v_{\text{mkt}}|p_f, \mathbf{p})}{\text{var}(v_i|p_f, \mathbf{p})\times\text{var}(v_{\text{mkt}}|\mathbf{y})}. \end{aligned}$$

From Lemma 3, we know the numerator is independent of the partition. From (6), we know $\text{var}(v_{\text{mkt}}|\mathbf{y})$ is independent of the partition. Thus, whether R^2 decreases as the level of indexing increases depends on whether $\text{var}(v_i|p_f, \mathbf{p})$ increases with the level of indexing. The latter is true because we can apply Theorem 3 with $\mathbf{x}=\mathbf{e}_i$. Q.E.D.

One can read the result to mean that, as index trading increases, the market portfolio's explanatory power decreases. However, we already know that prices perfectly aggregate market-level information, regardless of indexing. Thus, the result we reported in Theorem 6 is driven solely by the fact that idiosyncratic risk increases with indexing. Figure 3 depicts an example.

In the international finance literature, researchers document that R^2s are lower in developed countries than in developing countries. Morck et al. (2000) attribute their empirical findings to better protection of property rights in developed markets. Jin and Myers (2006) attribute their empirical findings to better corporate transparency in developed markets. To the extent that indexing is more common in developed countries, our model provides a new explanation for the lower R^2s in developed countries.

Figure 3. R^2 of the CAPM Risk-Return Relation Decreases When the Percentage of Index Investors Increases



Notes. In this example there are four assets, and for each one we compute its R^2. The parameters are the same as those used in Figure 2.

As in other conditional CAPM models, betas are random variables. In fact, a bit of algebra shows that

$$\beta_i = \frac{p_{\text{mkt}}}{p_i} \times \frac{\text{cov}(v_i, v_{\text{mkt}} | p_f, \mathbf{p})}{\text{var}(v_{\text{mkt}} | \mathbf{y})}.$$

The only term that depends on the level of indexing is p_i, the price of the asset. Multiplying and dividing β_i by p_f, and using (10), we see that β_i is a ratio of two Gaussian random variables. Therefore, moments of betas do not exist in our model. Consequently, it is meaningless in our model to talk about how the mean of beta (or any other moment) changes with the level of indexing. This result is an artifact of our technical assumption that payoffs are normally distributed.

7. Concluding Remarks

Markowitz (1952) studies mean-variance portfolio selection and discovers the efficient frontier. Tobin (1958) adds the risk-free asset, discovers the separation theorem, and explains that "Markowitz's main interest is prescription of rules of rational behaviour for investors; [whereas] the main concern of [my] paper is the implications for economic theory, mainly comparative statics, that can be derived from assuming that investors do in fact follow [Markowitz's] rules" (p. 85). We add costless private signals, and the main concern of our paper is the implications—mainly comparative statics—derived from assuming that some investors follow Tobin's rule.

The essence of the model is that index investors do not participate in the price discovery process of individual assets but do participate in the price discovery process of the market portfolio. Therefore, there is a dichotomy between the impact of indexing on the price of the market portfolio and the prices of individual assets. We have expressed this dichotomy in terms of price efficiency, price comovement, and the statistical strength of the CAPM.

Acknowledgments

The authors thank Kerry Back, Dan Bernhardt, Joel Hasbrouck, Yud Izhakian, Davar Khoshnevisan, Matt Ringgenberg, Yajun Wang, two anonymous referees, and seminar participants at Baruch College, Brigham Young University, National Bureau of Economic Research, the U.S. Securities and Exchange Commission, Roma Tor Vergata, Sienna, the University of Utah, and the University of Zurich for their helpful comments. This is a revised version of an earlier paper circulated under the title "Is Index Trading Benign?"

Appendix. Equilibrium: Proofs

Proof of Lemma 1. Lemma 2 of Grossman (1978) shows that the conditional expectation is a sufficient statistic. Then, in the appendix, Grossman (1978) shows that the definition of sufficiency via conditional expectations is equivalent to the classic factorization criterion. Using the factorization criterion, it is trivial to complete the proof. To keep this paper self-contained, we provide an alternative proof.

We start by proving that if the conditions in the lemma hold, then (1) and (2) hold.

To prove (1), we do not need to use the assumption of joint normality. We have

$$E[\mathbf{u}|\mathbf{w}] = \Psi(t(\mathbf{w})) = E[\Psi(t(\mathbf{w}))|t(\mathbf{w})] = E[E[\mathbf{u}|\mathbf{w}]|t(\mathbf{w})] = E[\mathbf{u}|t(\mathbf{w})].$$

To prove (2), we use the assumption that $\mathbf{u}$ and $\mathbf{w}$ are jointly normal. That is, the conditional variance is deterministic, and hence

$$\begin{aligned}
\text{var}(\mathbf{u}|\mathbf{w}) &= E[\text{var}(\mathbf{u}|\mathbf{w})|t(\mathbf{w})] \\
&\quad \text{(since var}(\mathbf{u}|\mathbf{w}) \text{ is deterministic)} \\
&= E[E[(\mathbf{u} - E[\mathbf{u}|\mathbf{w}])(\mathbf{u} - E[\mathbf{u}|\mathbf{w}])'|\mathbf{w}]|t(\mathbf{w})] \\
&\quad \text{(definition of conditional variance)} \\
&= E[(\mathbf{u} - E[\mathbf{u}|\mathbf{w}])(\mathbf{u} - E[\mathbf{u}|\mathbf{w}])'|t(\mathbf{w})] \\
&\quad \text{(the law of iterated conditional expectations)} \\
&= E[(\mathbf{u} - E[\mathbf{u}|t(\mathbf{w})])(\mathbf{u} - E[\mathbf{u}|t(\mathbf{w})])'|t(\mathbf{w})] \\
&\quad \text{(we already proved that } E[\mathbf{u}|\mathbf{w}] = E[\mathbf{u}|t(\mathbf{w})]) \\
&= \text{var}(\mathbf{u}|t(\mathbf{w})) \\
&\quad \text{(definition of conditional variance).}
\end{aligned}$$

Having proved (1) and (2), we need to show that $t(\mathbf{w})$ is sufficient. We first show that this is the case when $\Psi(\cdot)$ is the identity function. In other words, we want to show that statistic $E[\mathbf{u}|\mathbf{w}]$ is sufficient for $\mathbf{u}$. Because $\mathbf{u}$ and $\mathbf{w}$ are jointly normal, $\mathbf{u}$ and $E[\mathbf{u}|\mathbf{w}]$ are jointly normal as well. Thus, the mean and variance fully characterize both the distributions of $\mathbf{u}$, conditional on $\mathbf{w}$, and the distribution of $\mathbf{u}$, conditional on $E[\mathbf{u}|\mathbf{w}]$. We have shown that the mean and variance are identical, and therefore we conclude that these two conditional distributions are identical. Hence, by the law of the unconscious statistician, the conditional expected value of an arbitrary $F(\mathbf{u})$ is also identical. We have proved that $E[\mathbf{u}|\mathbf{w}]$ is sufficient for $\mathbf{u}$.

We now consider an arbitrary measurable function $\Psi(\cdot)$. The assumption of the lemma is that $E[\mathbf{u}|\mathbf{w}]=\Psi(t(\mathbf{w}))$. Thus, having already proved that the conditional expectation is sufficient, we know $\Psi(t(\mathbf{w}))$ is sufficient. Take an arbitrary $F(\mathbf{u})$. We have

$$\begin{aligned}
E[F(\mathbf{u})|\mathbf{w}] &= E[F(\mathbf{u})|\Psi(t(\mathbf{w}))] \\
&\quad \big(\Psi(t(\mathbf{w})) \text{ is sufficient}\big) \\
&= E[E[F(\mathbf{u})|\Psi(t(\mathbf{w}))]|t(\mathbf{w})] \\
&\quad \big(\text{since } E[F(\mathbf{u})|\Psi(t(\mathbf{w}))] \text{ is a mesurable function of } t(\mathbf{w})\big) \\
&= E[E[F(\mathbf{u})|\mathbf{w})]|t(\mathbf{w})] \\
&\quad \big(\text{again, we already know } \Psi(t(\mathbf{w})) \text{ is sufficient}\big) \\
&= E[F(\mathbf{u})|t(\mathbf{w})] \\
&\quad \text{(the law of iterated conditional expectations).} \quad \text{Q.E.D}
\end{aligned}$$

Proof of Proposition 1. From Lemma 1, to show that $\mathbf{y}$ and y_{n+1} are sufficient statistics, it is enough to show that y_{n+1} conveys $E[\mathbf{1}'\mathbf{v}|\mathbf{s}]$. This is our first goal. Because $\mathbf{v}$ and $\mathbf{s}$ are jointly normally distributed, we have

$$\begin{aligned}
&E[\mathbf{v}|\mathbf{s}]=\mu_{\mathbf{v}}+\text{cov}(\mathbf{v},\mathbf{s})\text{var}(\mathbf{s})^{-1}(\mathbf{s}-E[\mathbf{s}]) \quad \Rightarrow \\
&E[\mathbf{1}'\mathbf{v}|\mathbf{s}]=\mathbf{1}'\mu_{\mathbf{v}}+\mathbf{1}'\text{cov}(\mathbf{v},\mathbf{s})\text{var}(\mathbf{s})^{-1}(\mathbf{s}-E[\mathbf{s}]).
\end{aligned}$$

To express $E[\mathbf{1}'\mathbf{v}|\mathbf{s}]$ in terms of the primitives of the model, we use the definition of the signals to write

$$\text{cov}(\mathbf{v},\mathbf{s})=[\,\Sigma_{\mathbf{vv}} \ \ \cdots \ \ \Sigma_{\mathbf{vv}}\,],$$

and

$$\text{var}(\mathbf{s})=\begin{bmatrix} \Sigma_{\mathbf{vv}}+\Sigma_{\epsilon\epsilon} & \Sigma_{\mathbf{vv}} & \cdots & \Sigma_{\mathbf{vv}} \\ \Sigma_{\mathbf{vv}} & \Sigma_{\mathbf{vv}}+\Sigma_{\epsilon\epsilon} & \cdots & \Sigma_{\mathbf{vv}} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{\mathbf{vv}} & \Sigma_{\mathbf{vv}} & \cdots & \Sigma_{\mathbf{vv}}+\Sigma_{\epsilon\epsilon} \end{bmatrix}_{nm\times nm}.$$

We then note that we can write the identity

$$\begin{aligned}
&[\,\mathbf{I}_{n\times n} \ \ \cdots \ \ \mathbf{I}_{n\times n}\,]_{n\times nm} \\
&\quad =[\,(m\Sigma_{\mathbf{vv}}+\Sigma_{\epsilon\epsilon})^{-1} \ \ \cdots \ \ (m\Sigma_{\mathbf{vv}}+\Sigma_{\epsilon\epsilon})^{-1}\,]_{n\times nm}\text{var}(\mathbf{s}).
\end{aligned} \tag{A.1}$$

Therefore, by multiplying each side of (A.1) on the right by $\text{var}(\mathbf{s})^{-1}$ and on the left by $\Sigma_{\mathbf{vv}}$, we obtain

$$\text{cov}(\mathbf{v},\mathbf{s})\text{var}(\mathbf{s})^{-1}=[\,\Sigma_{\mathbf{vv}}(m\Sigma_{\mathbf{vv}}+\Sigma_{\epsilon\epsilon})^{-1} \ \ \cdots \ \ \Sigma_{\mathbf{vv}}(m\Sigma_{\mathbf{vv}}+\Sigma_{\epsilon\epsilon})^{-1}\,]_{n\times nm}.$$

Multiplying both sides of the previous equation on the left by $\mathbf{1}'$, and recalling the definition of $\mathbf{g}$ (see (3)), we conclude that

$$\mathbf{1}'\text{cov}(\mathbf{v},\mathbf{s})\text{var}(\mathbf{s})^{-1}=[\,\mathbf{g}' \ \ \cdots \ \ \mathbf{g}'\,]_{1\times nm}.$$

Thus,

$$\begin{aligned}
E[\mathbf{1}'\mathbf{v}|\mathbf{s}] &= \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}}+\mathbf{1}'\text{cov}(\mathbf{v},\mathbf{s})\text{var}(\mathbf{s})^{-1}(\mathbf{s}-E[\mathbf{s}]) \\
&= \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}}+\mathbf{g}'\sum_{k=1}^{m}(\mathbf{s}_k-\boldsymbol{\mu}_{\mathbf{v}}) \\
&= (\mathbf{1}-m\mathbf{g})'\boldsymbol{\mu}_{\mathbf{v}}+\underbrace{\mathbf{g}'\sum_{k=1}^{m}\mathbf{s}_k}_{y_{n+1}}
\end{aligned} \tag{A.2}$$

depends on $\mathbf{s}$ only through y_{n+1}. That is, in Lemma 1 we take $\Psi(y_{n+1})$ to be the right-hand side of (A.2) to conclude y_{n+1} is sufficient for $\mathbf{1}'\mathbf{v}$. Also, in Lemma 1 we take $\Psi(\mathbf{y})$ to be the right-hand side of (A.2) to conclude that $\mathbf{y}$ is also sufficient for $\mathbf{1}'\mathbf{v}$.

We already computed the conditional expectation. To conclude the proof, we now compute the conditional variance. Because $\mathbf{v}$ and $\mathbf{s}$ are jointly normally distributed, we have

$$\begin{aligned}
&\text{var}(\mathbf{v}|\mathbf{s})=\Sigma_{\mathbf{vv}}-\text{cov}(\mathbf{v},\mathbf{s})\text{var}(\mathbf{s})^{-1}\text{cov}(\mathbf{s},\mathbf{v}) \quad \Rightarrow \\
&\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s})=\mathbf{1}'\Sigma_{\mathbf{vv}}\mathbf{1}-\mathbf{1}'\text{cov}(\mathbf{v},\mathbf{s})\text{var}(\mathbf{s})^{-1}\text{cov}(\mathbf{s},\mathbf{v})\mathbf{1} \\
&\qquad\qquad =\mathbf{1}'\Sigma_{\mathbf{vv}}\mathbf{1}-m\mathbf{g}'\Sigma_{\mathbf{vv}}\mathbf{1}. \quad \text{Q.E.D}
\end{aligned}$$

Derivation of Equation (7).

$$\begin{aligned}
\text{cov}(\mathbf{v},\mathbf{y}) \equiv \Sigma_{\mathbf{vy}} &= \text{cov}\left(\mathbf{v},\begin{bmatrix}\frac{1}{|\mathcal{NI}|}\sum_{k\in\mathcal{NI}}\mathbf{s}_k \\ \mathbf{g}'\sum_{k=1}^{m}\mathbf{s}_k\end{bmatrix}\right) \\
&= \left[\text{cov}\left(\mathbf{v},\mathbf{v}+\frac{1}{|\mathcal{NI}|}\sum_{k\in\mathcal{NI}}\boldsymbol{\epsilon}_k\right) \ \ \text{cov}\left(\mathbf{v},m\mathbf{g}'\mathbf{v}+\mathbf{g}'\sum_{k=1}^{m}\boldsymbol{\epsilon}_k\right)\right] \\
&= [\,\Sigma_{\mathbf{vv}} \quad m\Sigma_{\mathbf{vv}}\mathbf{g}\,]. \quad \square
\end{aligned}$$

Derivation of Equation (8).

$$\begin{aligned}
\text{var}(\mathbf{y}) \equiv \Sigma_{\mathbf{yy}} &= \begin{bmatrix} \text{var}\left(\frac{1}{|\mathcal{NI}|}\sum_{k\in\mathcal{NI}}\mathbf{s}_k\right) & \text{cov}\left(\frac{1}{|\mathcal{NI}|}\sum_{k\in\mathcal{NI}}\mathbf{s}_k,\mathbf{g}'\sum_{k=1}^{m}\mathbf{s}_k\right) \\ \text{cov}\left(\mathbf{g}'\sum_{k=1}^{m}\mathbf{s}_k,\frac{1}{|\mathcal{NI}|}\sum_{k\in\mathcal{NI}}\mathbf{s}_k\right) & \text{var}\left(\mathbf{g}'\sum_{k=1}^{m}\mathbf{s}_k\right) \end{bmatrix} \\
&= \begin{bmatrix} \Sigma_{\mathbf{vv}}+\frac{1}{|\mathcal{NI}|}\Sigma_{\epsilon\epsilon} & \Sigma_{\mathbf{vv}}\mathbf{1} \\ \mathbf{1}'\Sigma_{\mathbf{vv}} & m\mathbf{1}'\Sigma_{\mathbf{vv}}\mathbf{g} \end{bmatrix}. \quad \square
\end{aligned}$$

Proof of Theorem 1. The equilibrium allocations (12)–(14) clear the markets. We obtain the price of the market portfolio, (11), by multiplying (10) from the left by $\mathbf{1}'$, using Proposition 1 to replace conditioning on $\mathbf{y}$ with conditioning on y_{n+1}.

Thus, all we need to show is that given the bond price, (9), and the price of risky assets, (10), the allocations are optimal.

Whether the investor is an indexer, the stock allocation, (12), is a scalar multiplication of $\mathbf{1}$. Thus, to prove the optimality of (12), we can ignore the constraint of indexers to hold the market portfolio. The problem of the kth investor is

$$\begin{aligned}
J_k &\equiv \max_{c,b,\mathbf{x}} E[U_k(c,b,\mathbf{x}\cdot\mathbf{v})|\mathbf{y}] \\
&\text{subject to } \bar{c}-c+(0-b)p_f+((1/m)\mathbf{1}-\mathbf{x})\cdot\mathbf{p}=0 \\
&= \max_{c\in R,\mathbf{x}\in R^n} E\left[U_k\left(c,(\bar{c}-c)\frac{1}{p_f}+((1/m)\mathbf{1}-\mathbf{x})\cdot\frac{1}{p_f}\mathbf{p},\mathbf{x}\cdot\mathbf{v}\right)\middle|\mathbf{y}\right] \\
&= \max_{c\in R,\mathbf{x}\in R^n} -e^{-\rho_k c}-e^{-\rho_k\left((\bar{c}-c)\frac{1}{p_f}+((1/m)1-\mathbf{x})\cdot\frac{1}{p_f}\mathbf{p}+\mathbf{x}'E[\mathbf{v}|\mathbf{y}]-\frac{\rho_k}{2}\mathbf{x}'\mathrm{var}(\mathbf{v}|\mathbf{y})\mathbf{x}\right)} \\
&= \max_{c\in R} -e^{-\rho_k c}+e^{-\rho_k\left((\bar{c}-c)\frac{1}{p_f}+\frac{1}{m}\frac{1}{p_f}p_{\mathrm{mkt}}\right)}\times\max_{\mathbf{x}\in R^n} \\
&\qquad -e^{-\rho_k\left(\mathbf{x}'\left(E[\mathbf{v}|\mathbf{y}]-\frac{1}{p_f}\mathbf{p}\right)-\frac{\rho_k}{2}\mathbf{x}'\mathrm{var}(\mathbf{v}|\mathbf{y})\mathbf{x}\right)}.
\end{aligned} \tag{A.3}$$

We solve this problem recursively. The first-order condition (f.o.c.) with respect to $\mathbf{x}$ is

$$-\frac{1}{p_f}\mathbf{p}+E[\mathbf{v}|\mathbf{y}]-\frac{\rho_k}{2}\left(\mathrm{var}(\mathbf{v}|\mathbf{y})+\mathrm{var}(\mathbf{v}|\mathbf{y})'\right)\mathbf{x}=0.$$

We replace $\mathbf{x}$ with $\mathbf{x}_k$ to emphasize that this is the optimal stock portfolio of the kth investor. Using the symmetry of the variance matrix, we rearrange and obtain

$$\mathbf{x}_k=\frac{1}{\rho_k}\mathrm{var}(\mathbf{v}|\mathbf{y})^{-1}\left(E[\mathbf{v}|\mathbf{y}]-\frac{1}{p_f}\mathbf{p}\right). \tag{A.4}$$

Plugging the prices stated in the theorem, (10), back into the f.o.c., (A.4), we see that the optimal stock portfolio is indeed (12).

Next, inserting (10) and (12) back into (A.3), we get

$$\begin{aligned}
J_k &= \max_{c\in R} -e^{-\rho_k c}-e^{-\rho_k\left((\bar{c}-c)\frac{1}{p_f}+\frac{1}{m}\frac{1}{p_f}p_{\mathrm{mkt}}+\frac{\bar{\rho}}{2m}\frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathrm{var}(v_{\mathrm{mkt}}|\mathbf{y})\right)} \\
&\underset{\text{Proposition 1}}{=} \max_{c\in R} -e^{-\rho_k c}-e^{-\rho_k\left((\bar{c}-c)\frac{1}{p_f}+\frac{1}{m}\frac{1}{p_f}p_{\mathrm{mkt}}+\frac{\bar{\rho}}{2m}\frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathrm{var}(v_{\mathrm{mkt}}|y_{n+1})\right)}.
\end{aligned}$$

We take the f.o.c. with respect to c, and replace c with c_k:

$$\rho_k e^{-\rho_k c_k}=\frac{\rho_k}{p_f}e^{-\rho_k\left((\bar{c}-c_k)\frac{1}{p_f}+\frac{1}{m}\frac{1}{p_f}p_{\mathrm{mkt}}+\frac{\bar{\rho}}{2m}\frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathrm{var}(v_{\mathrm{mkt}}|y_{n+1})\right)}. \tag{A.5}$$

To find the optimal bond holding, we use the budget constraint:

$$b_k p_f=\bar{c}-c_k+\left(\frac{1}{m}\mathbf{1}-\mathbf{x}_k\right)\cdot\mathbf{p}\underset{(12)}{=}\bar{c}-c_k+\frac{1}{m}\left(1-\frac{\bar{\rho}}{\rho_k}\right)p_{\mathrm{mkt}}. \tag{A.6}$$

Now, solving (A.5) and (A.6) for b_k and c_k gives us the equilibrium bond holding, (13), and consumption, (14). Q.E.D.

Proof of Theorem 2. We first consider (17). Fix $k\in\mathcal{NI}$. We are going to prove a stronger result. Let $F(\cdot)$ be an arbitrary measurable function. We want to prove

$$E[F(\mathbf{v})|\mathbf{s}_k,\mathbf{p},p_f]=E[F(\mathbf{v})|\mathbf{p},p_f]=E[F(\mathbf{v})|\mathbf{y}]. \tag{A.7}$$

Indeed, since c, b, and $\mathbf{x}$ are constants, (17) is a special case of (A.7) with $F(\mathbf{v})=U_k(c,b,\mathbf{x}\cdot\mathbf{v})$.

Equation (10) shows that the pair $(\mathbf{p},p_f)$ conveys the conditional expectation $E[\mathbf{v}|\mathbf{y}]$, and hence from Lemma 1, the pair $(\mathbf{p},p_f)$ is sufficient for $\mathbf{v}$. This proves that the second equality of (A.7) holds.

To prove the first equality, we start by showing that $\mathbf{y}$ conveys the conditional expectation $E[\mathbf{v}|\mathbf{y},\mathbf{s}_k]$.

From the definitions of $\mathbf{y}$ and $\mathbf{s}_k$, we see that

$$\mathrm{cov}(\mathbf{y},\mathbf{s}_k)=\mathrm{cov}(\mathbf{y},\mathbf{y})\begin{bmatrix}\mathbf{I}_{n\times n}\\ \mathbf{0}_{1\times n}\end{bmatrix}_{(n+1)\times n}.$$

Multiplying both sides of this identity from the left by $\mathrm{cov}(\mathbf{y},\mathbf{y})^{-1}$, we get

$$\mathrm{cov}(\mathbf{y},\mathbf{y})^{-1}\mathrm{cov}(\mathbf{y},\mathbf{s}_k)=\begin{bmatrix}\mathbf{I}_{n\times n}\\ \mathbf{0}_{1\times n}\end{bmatrix}_{(n+1)\times n}. \tag{A.8}$$

Because all random variables are jointly normal, we have

$$\begin{aligned}
E[\mathbf{v}|\mathbf{s}_k,\mathbf{y}] &= \mu_{\mathbf{v}}+[\,\mathrm{cov}(\mathbf{v},\mathbf{s}_k)\quad \mathrm{cov}(\mathbf{v},\mathbf{y})\,] \\
&\quad \begin{bmatrix}\mathrm{cov}(\mathbf{s}_k,\mathbf{s}_k) & \mathrm{cov}(\mathbf{s}_k,\mathbf{y})\\ \mathrm{cov}(\mathbf{y},\mathbf{s}_k) & \mathrm{cov}(\mathbf{y},\mathbf{y})\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{s}_k-E\mathbf{s}_k\\ \mathbf{y}-E\mathbf{y}\end{bmatrix} \\
&= \mu_{\mathbf{v}}+[\,\mathrm{cov}(\mathbf{v},\mathbf{v})\quad \mathrm{cov}(\mathbf{v},\mathbf{v})\quad m\,\mathrm{cov}(\mathbf{v},\mathbf{v})\mathbf{g}\,] \\
&\quad \begin{bmatrix}\mathrm{cov}(\mathbf{s}_k,\mathbf{s}_k) & \mathrm{cov}(\mathbf{s}_k,\mathbf{y})\\ \mathrm{cov}(\mathbf{y},\mathbf{s}_k) & \mathrm{cov}(\mathbf{y},\mathbf{y})\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{s}_k-E\mathbf{s}_k\\ \mathbf{y}-E\mathbf{y}\end{bmatrix},
\end{aligned} \tag{A.9}$$

where for the second equality we have used the definitions of $\mathbf{s}_k$ and $\mathbf{y}$ to compute $\mathrm{cov}(\mathbf{v},\mathbf{s}_k)$ and $\mathrm{cov}(\mathbf{v},\mathbf{y})$.

The block matrix inversion formula implies that we can write[7]

$$\begin{aligned}
\begin{bmatrix}\mathrm{cov}(\mathbf{s}_k,\mathbf{s}_k) & \mathrm{cov}(\mathbf{s}_k,\mathbf{y})\\ \mathrm{cov}(\mathbf{y},\mathbf{s}_k) & \mathrm{cov}(\mathbf{y},\mathbf{y})\end{bmatrix}^{-1} &= \begin{bmatrix}\mathbf{F} & \mathbf{G}\\ -\mathrm{cov}(\mathbf{y},\mathbf{y})^{-1}\mathrm{cov}(\mathbf{y},\mathbf{s}_k)\mathbf{F} & \mathbf{H}\end{bmatrix} \\
&\underset{(A.8)}{=} \begin{bmatrix}\mathbf{F} & \mathbf{G}\\ -\begin{bmatrix}\mathbf{I}_{n\times n}\\ \mathbf{0}_{1\times n}\end{bmatrix}\mathbf{F} & \mathbf{H}\end{bmatrix},
\end{aligned}$$

where the specifics of $\mathbf{F}_{n\times n},\mathbf{G}_{n\times(n+1)}$, and $\mathbf{H}_{(n+1)\times(n+1)}$ are not important. Inserting back into (A.9), we see that

$$E[\mathbf{v}|\mathbf{s}_k,\mathbf{y}]=\mu_{\mathbf{v}}+[\,\mathbf{0}_{n\times n}\quad \mathbf{J}_{n\times(n+1)}\,]\begin{bmatrix}\mathbf{s}_k-E\mathbf{s}_k\\ \mathbf{y}-E\mathbf{y}\end{bmatrix},$$

where the specifics of $\mathbf{J}_{n\times(n+1)}$ are not important. We conclude that $E[\mathbf{v}|\mathbf{s}_k,\mathbf{y}]$ depends only on $\mathbf{y}$ (and not on $\mathbf{s}_k$). That is, $\mathbf{y}$ conveys $E[\mathbf{v}|\mathbf{s}_k,\mathbf{y}]$. Applying Lemma 1, we get

$$E[F(\mathbf{v})|\mathbf{y},\mathbf{s}_k]=E[F(\mathbf{v})|\mathbf{y}]. \tag{A.10}$$

Thus,

$$\begin{aligned}
&E[F(\mathbf{v})|\mathbf{s}_k,\mathbf{p},p_f] \\
&=E[E[F(\mathbf{v})|\mathbf{s}_k,\mathbf{y}]|\mathbf{s}_k,\mathbf{p},p_f] \\
&\qquad \text{(by the law of iterated conditional expectations)} \\
&=E[E[F(\mathbf{v})|\mathbf{y}]|\mathbf{s}_k,\mathbf{p},p_f] \\
&\qquad \text{(A.10)} \\
&=E[E[F(\mathbf{v})|\mathbf{p},p_f]|\mathbf{s}_k,\mathbf{p},p_f] \\
&\qquad \left(\text{the second equality of (A.7), which we already proved}\right) \\
&=E[F(\mathbf{v})|\mathbf{p},p_f].
\end{aligned}$$

This shows that the first equality of (A.7) holds, and concludes the proof that (17) holds. We now turn our attention to (18). We fix $k\in\mathcal{I}$. Again, we prove a stronger result. Let

$F(\cdot)$ be an arbitrary (univariate) measurable function. We want to prove

$$E[F(\mathbf{1}\cdot\mathbf{v})|\mathbf{s}_k,\mathbf{p},p_f]=E[F(\mathbf{1}\cdot\mathbf{v})|\mathbf{p},p_f]=E[F(\mathbf{1}\cdot\mathbf{v})|\mathbf{y}]. \tag{A.11}$$

Indeed, since c, b, and q are constants, (18) is a special case of (A.11) with $F(\mathbf{1}\cdot\mathbf{v})=U_k(c,b,q\mathbf{1}\cdot\mathbf{v})$.

Repeating the same argument that we used earlier in the proof, (10) shows that the pair $(\mathbf{p},p_f)$ conveys the conditional expectation $E[\mathbf{v}|\mathbf{y}]$, and hence from Lemma 1, the pair $(\mathbf{p},p_f)$ is sufficient for $\mathbf{v}$. This proves that the second equality of (A.11) holds.

To prove the validity of the first equality in (A.11), we have

$$\begin{aligned}
&E[F(\mathbf{1}\cdot\mathbf{v})|\mathbf{s}_k,\mathbf{p},p_f]\\
&=E[E[F(\mathbf{1}\cdot\mathbf{v})|\mathbf{s}]|\mathbf{s}_k,\mathbf{p},p_f]\\
&\qquad\text{(by the law of iterated conditional expectations)}\\
&=E[E[F(\mathbf{1}\cdot\mathbf{v})|\mathbf{y}]|\mathbf{s}_k,\mathbf{p},p_f]\\
&\qquad\Big(\text{first equality in (5)}\Big)\\
&=E[E[F(\mathbf{1}\cdot\mathbf{v})|\mathbf{p},p_f]|\mathbf{s}_k,\mathbf{p},p_f]\\
&\qquad\Big(\text{the pair } (\mathbf{p}(\mathbf{y}),p_f(\mathbf{y})) \text{ is sufficient for } \mathbf{v}\Big)\\
&=E[F(\mathbf{1}\cdot\mathbf{v})|\mathbf{p},p_f].\quad \text{Q.E.D}
\end{aligned}$$

Proof of Lemma 2. Let $\mathbf{x}$ be an arbitrary stock portfolio, and multiply both sides of (19) by $\Sigma_{\mathbf{yy}}$ on the right to get the identity

$$[\,\boldsymbol{\theta}' \quad q\,]\Sigma_{\mathbf{yy}}=\mathbf{x}'\Sigma_{\mathbf{vy}}. \tag{A.12}$$

To prove that (a) implies both (b) and (c), we set $\mathbf{x}=\delta\mathbf{1}$ for some scalar δ. We then treat (A.12) as a matrix equation for the unknowns $[\,\boldsymbol{\theta}' \quad q\,]$.

Inserting the expressions for $\Sigma_{\mathbf{vy}}$ and $\Sigma_{\mathbf{yy}}$ (see (7) and (8)) into (A.12) yields

$$\begin{aligned}
&\left[\boldsymbol{\theta}'\left(\Sigma_{\mathbf{vv}}+\frac{1}{|\mathcal{NI}|}\Sigma_{\epsilon\epsilon}\right)+q\mathbf{1}'\Sigma_{\mathbf{vv}} \quad \boldsymbol{\theta}'\Sigma_{\mathbf{vv}}\mathbf{1}+mq\mathbf{1}'\Sigma_{\mathbf{vv}}\mathbf{g}\right]_{1\times(n+1)}\\
&\quad=[\,\delta\mathbf{1}'\Sigma_{\mathbf{vv}} \qquad m\delta\mathbf{1}'\Sigma_{\mathbf{vv}}\mathbf{g}\,]_{1\times(n+1)}.
\end{aligned} \tag{A.13}$$

Matching terms, it is clear that $[\,\mathbf{0}' \quad \delta\,]$ solves (A.12) (when $\mathbf{x}=\delta\mathbf{1}$). Because $\Sigma_{\mathbf{yy}}$ has full rank, we know that this is the only solution to (A.12), thus proving that if $\mathbf{x}$ is a scalar multiplication of the market portfolio, then the scalar must be q and that $\boldsymbol{\theta}=\mathbf{0}$.

That (b) implies (a) is a tautology. We are only left to show that (b) and (c) are equivalent.

To prove that (b) implies (c), we assume (b) holds. In other words, (A.12) is now

$$[\,\boldsymbol{\theta}' \quad q\,]\Sigma_{\mathbf{yy}}=q\mathbf{1}'\Sigma_{\mathbf{vy}}, \tag{A.14}$$

and our goal is to show $\boldsymbol{\theta}=\mathbf{0}$. Inserting the expressions for $\Sigma_{\mathbf{vy}}$ and $\Sigma_{\mathbf{yy}}$ (see (7) and (8)) into (A.14) yields

$$\begin{aligned}
&\left[\boldsymbol{\theta}'\left(\Sigma_{\mathbf{vv}}+\frac{1}{|\mathcal{NI}|}\Sigma_{\epsilon\epsilon}\right)+q\mathbf{1}'\Sigma_{\mathbf{vv}} \qquad \boldsymbol{\theta}'\Sigma_{\mathbf{vv}}\mathbf{1}+mq\mathbf{1}'\Sigma_{\mathbf{vv}}\mathbf{g}\right]_{1\times(n+1)}\\
&\quad=[\,q\mathbf{1}'\Sigma_{\mathbf{vv}} \qquad mq\mathbf{1}'\Sigma_{\mathbf{vv}}\mathbf{g}\,]_{1\times(n+1)}.
\end{aligned} \tag{A.15}$$

Matching terms, we see that

$$\boldsymbol{\theta}'\left(\Sigma_{\mathbf{vv}}+\frac{1}{|\mathcal{NI}|}\Sigma_{\epsilon\epsilon}\right)=\mathbf{0}_{1\times n}.$$

Multiplying on the right by θ, we get

$$\boldsymbol{\theta}'\left(\Sigma_{\mathbf{vv}}+\frac{1}{|\mathcal{NI}|}\Sigma_{\epsilon\epsilon}\right)\theta=0.$$

Noting that $\left(\Sigma_{\mathbf{vv}}+\frac{1}{|\mathcal{NI}|}\Sigma_{\epsilon\epsilon}\right)$ is a positive definite matrix, we conclude $\boldsymbol{\theta}=\mathbf{0}$.

To prove that (c) implies (b), we assume (c) holds. In other words, we assume $\boldsymbol{\theta}=\mathbf{0}$. This means that (A.12) is now

$$[\,\mathbf{0}' \quad q\,]\Sigma_{\mathbf{yy}}=\mathbf{x}'\Sigma_{\mathbf{vy}}, \tag{A.16}$$

and our goal is to show that $\mathbf{x}=q\mathbf{1}$. Inserting the expressions for $\Sigma_{\mathbf{vy}}$ and $\Sigma_{\mathbf{yy}}$ (see (7) and (8)) into (A.16) yields

$$\begin{aligned}
&\left[\mathbf{0}'\left(\Sigma_{\mathbf{vv}}+\frac{1}{|\mathcal{NI}|}\Sigma_{\epsilon\epsilon}\right)+q\mathbf{1}'\Sigma_{\mathbf{vv}} \qquad \mathbf{0}'\Sigma_{\mathbf{vv}}\mathbf{1}+mq\mathbf{1}'\Sigma_{\mathbf{vv}}\mathbf{g}\right]_{1\times(n+1)}\\
&\quad=[\,\mathbf{x}'\Sigma_{\mathbf{vv}} \qquad m\mathbf{x}'\Sigma_{\mathbf{vv}}\mathbf{g}\,]_{1\times(n+1)}.
\end{aligned}$$

After canceling the zero elements, we get the vector identity

$$[\,q\mathbf{1}'\Sigma_{\mathbf{vv}} \qquad mq\mathbf{1}'\Sigma_{\mathbf{vv}}\mathbf{g}\,]_{1\times(n+1)}=[\,\mathbf{x}'\Sigma_{\mathbf{vv}} \qquad m\mathbf{x}'\Sigma_{\mathbf{vv}}\mathbf{g}\,]_{1\times(n+1)}. \tag{A.17}$$

Obviously, $\mathbf{x}=q\mathbf{1}$ satisfies the vector identity. To see that this is the only $\mathbf{x}$ that satisfies the identity, we treat the first n elements of the vector identity in (A.17), $q\mathbf{1}'\Sigma_{\mathbf{vv}}=\mathbf{x}'\Sigma_{\mathbf{vv}}$, as a matrix equation in $\mathbf{x}$. Since $\Sigma_{\mathbf{vv}}$ is of full rank, we know the solution is unique. Thus, $\mathbf{x}=q\mathbf{1}$ is also the unique solution to the vector identity (A.17). Q.E.D.

Proof of Lemma 3. Because $\mathbf{v}$ and $\mathbf{y}$ are jointly normally distributed, we have

$$\text{var}(\mathbf{v}|\mathbf{y})=\text{var}(\mathbf{v})-\text{cov}(\mathbf{v},\mathbf{y})\text{var}(\mathbf{y},\mathbf{y})^{-1}\text{cov}(\mathbf{y},\mathbf{v}).$$

Now,

$$\begin{aligned}
\text{cov}(\mathbf{v},\mathbf{1}'\mathbf{v}|\mathbf{y})=\text{var}(\mathbf{v}|\mathbf{y})\mathbf{1}&=\Sigma_{\mathbf{vv}}\mathbf{1}-\text{cov}(\mathbf{v},\mathbf{y})\text{var}(\mathbf{y},\mathbf{y})^{-1}\text{cov}(\mathbf{y},\mathbf{v})\mathbf{1}\\
&=\Sigma_{\mathbf{vv}}\mathbf{1}-\text{cov}(\mathbf{v},\mathbf{y})\begin{bmatrix}\mathbf{0}_{1\times n}\\ 1\end{bmatrix}\\
&\underset{(7)}{=}\Sigma_{\mathbf{vv}}\mathbf{1}-[\Sigma_{\mathbf{vv}} \quad m\Sigma_{\mathbf{vv}}\mathbf{g}]\begin{bmatrix}\mathbf{0}_{1\times n}\\ 1\end{bmatrix}\\
&=\Sigma_{\mathbf{vv}}(\mathbf{1}-m\mathbf{g}),
\end{aligned}$$

where the second equality is a direct application of Lemma 2 to the portfolio $\mathbf{x}=\mathbf{1}$. This shows that $\text{var}(\mathbf{v}|\mathbf{y})\mathbf{1}$ is independent of the specific partition of investors into indexers and nonindexers. Q.E.D.

Proof of Theorem 4. Let $\mathbf{f}\equiv p_f^{-1}\mathbf{p}$ denote the discounted prices. The identity $\mathbf{1}'\mathbf{f}=\mathbf{e}_i{}'\mathbf{f}+(\mathbf{1}-\mathbf{e}_i)'\mathbf{f}$ implies

$$\text{var}(\mathbf{1}'\mathbf{f})=\text{var}(\mathbf{e}_i{}'\mathbf{f})+\text{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{f})+2\text{cov}(\mathbf{e}_i{}'\mathbf{f},(\mathbf{1}-\mathbf{e}_i)'\mathbf{f}). \tag{A.18}$$

Inserting (A.18) into the definition of correlation, we get

$$2\mathrm{corr}(\mathbf{e}_i{}'\mathbf{f},(\mathbf{1}-\mathbf{e}_i)'\mathbf{f}) \underset{\text{def.}}{=} \frac{2\mathrm{cov}(\mathbf{e}_i{}'\mathbf{f},(\mathbf{1}-\mathbf{e}_i)'\mathbf{f})}{\sqrt{\mathrm{var}(\mathbf{e}_i{}'\mathbf{f})}\sqrt{\mathrm{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{f})}} \underset{(A.18)}{=} \frac{\mathrm{var}(\mathbf{1}'\mathbf{f})-\mathrm{var}(\mathbf{e}_i{}'\mathbf{f})-\mathrm{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{f})}{\sqrt{\mathrm{var}(\mathbf{e}_i{}'\mathbf{f})}\sqrt{\mathrm{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{f})}}.$$

Because the denominator is strictly positive, the assumption that this correlation is strictly positive means that the numerator is also strictly positive. Thus, to show that the correlation increases with the level of indexing, it is sufficient to show that $\mathrm{var}(\mathbf{1}'\mathbf{f})$ does not change with indexing, whereas both $\mathrm{var}(\mathbf{e}_i{}'\mathbf{f})$ and $\mathrm{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{f})$ strictly decrease with the level of indexing. To demonstrate this, we now use the law of total variance:

$$\begin{aligned}\mathrm{var}(\mathbf{x}'\mathbf{v}) &= E\underbrace{\mathrm{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})}_{\text{deterministic}}+\mathrm{var}(E[\mathbf{x}'\mathbf{v}|\mathbf{y}])\\ &\underset{(10)}{=} \mathrm{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})+\mathrm{var}\left(\mathbf{x}'\left(\frac{1}{p_f}\mathbf{p}+\underbrace{\frac{\bar{\rho}}{m}\mathrm{cov}(\mathbf{v},\mathbf{1}'\mathbf{v}|\mathbf{y})}_{\text{deterministic}}\right)\right)\\ &= \mathrm{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})+\mathrm{var}(\mathbf{x}'p_f^{-1}\mathbf{p}).\end{aligned}$$

The left-hand side of the this equation is independent of the level of index investment. As we increase $|\mathcal{I}|$, according to Theorem 3, the first term on the right does not change when $\mathbf{x}=\mathbf{1}$, proving that $\mathrm{var}(\mathbf{1}'\mathbf{f})$ does not change with indexing. Repeating the same argument, once with $\mathbf{x}=\mathbf{e}_i$ and once with $\mathbf{x}=(\mathbf{1}-\mathbf{e}_i)$, we conclude that both $\mathrm{var}(\mathbf{e}_i{}'\mathbf{f})$ and $\mathrm{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{f})$ strictly decrease with the level of indexing. Q.E.D.

Proof of Theorem 5. Fix $k\in\mathcal{NI}$. Let c_k^*, b_k^*, and x_k^* be the investor's decisions in the equilibrium of the actual economy. From Theorems 1 and 2, we know that the investor holds the market. In other words, there is a scalar q_k^* such that $x_k^*=q_k^*\mathbf{1}$.

Define $w_k^*=b_k^*p_f+q_k^*p_{\mathrm{mkt}}$. We invoke a calculus of a variation-type argument. Instead of looking at the full problem, we restrict our attention to a subclass of feasible allocations that includes the optimal one. Specifically, let us say that the investor contemplates consuming the optimal c_k^* and investing a fraction φ of w_k^* in the market portfolio, a fraction κ of w_k^* in asset i, and the remaining $(1-\varphi-\kappa)w_k^*$ in bonds. In other words, the allocation the investor contemplates is to buy $w_k^*(1-\varphi-\kappa)/p_f$ bonds, a fraction $w_k^*\varphi/p_{\mathrm{mkt}}$ of the market portfolio, and an additional $w_k^*\kappa/p_i$ shares of asset i. The optimal fraction invested in asset i must satisfy $\kappa = 0$.

We can write the investor's problem as follows:

$$\begin{aligned}&\max_{c,b,\mathbf{x}} E[U_k(c,b,\mathbf{x}\cdot\mathbf{v})|p_f,\mathbf{p}],\text{ subject to }\bar{c}-c+(0-b)p_f+((1/m)\mathbf{1}-\mathbf{x})\cdot\mathbf{p}=0\\ =\ &\max_{b,\mathbf{x}\in R^n} E[U_k(c_k^*,b,\mathbf{x}\cdot\mathbf{v})|\mathbf{y}],\text{ subject to }bp_f+\mathbf{x}\cdot\mathbf{p}=w_k^*\\ =\ &-e^{-\rho_kc^*}+\max_{b,\mathbf{x}\in R^n}-E[e^{-\rho_k(b+\mathbf{x}\cdot\mathbf{v})}|\mathbf{y}],\text{ subject to }bp_f+\mathbf{x}\cdot\mathbf{p}=w_k^*\\ =\ &-e^{-\rho_kc^*}+\max_{\varphi,\kappa}-E\left[\exp\left(-\rho_kw_k^*\left(\frac{1-\varphi-\kappa}{p_f}+\frac{\varphi}{p_{\mathrm{mkt}}}v_{\mathrm{mkt}}+\frac{\kappa}{p_i}v_i\right)\right)\middle|\mathbf{y}\right].\end{aligned}$$

Thus, the maximization problem is equivalent to

$$\max_{\varphi,\kappa}(1-\varphi-\kappa)(1+r_f)+\varphi(1+E[r_{\mathrm{mkt}}|\mathbf{y}])+\kappa(1+E[r_i|\mathbf{y}]) -\frac{\rho_kw_k^*}{2}\left(\varphi^2\mathrm{var}(r_{\mathrm{mkt}}|\mathbf{y})+2\varphi\kappa\mathrm{cov}(r_{\mathrm{mkt}},r_i|\mathbf{y})+\kappa^2\mathrm{var}(r_i|\mathbf{y})\right).$$

Taking the first-order condition with respect to φ, and evaluating at $\kappa = 0$, yields

$$E[r_{\mathrm{mkt}}|\mathbf{y}]-r_f-\varphi\rho_kw_k^*\mathrm{var}(r_{\mathrm{mkt}}|\mathbf{y})=0 \;\Rightarrow\; \varphi\rho_kw_k^*=\frac{E[r_{\mathrm{mkt}}|\mathbf{y}]-r_f}{\mathrm{var}(r_{\mathrm{mkt}}|\mathbf{y})}.$$

Taking the first-order condition with respect to κ, and evaluating at $\kappa = 0$, yields

$$E[r_i|\mathbf{y}]-r_f-\varphi\rho_kw_k^*\mathrm{cov}(r_{\mathrm{mkt}},r_i|\mathbf{y})=0.$$

Combining both conditions, we obtain

$$E[r_i|\mathbf{y}]=r_f+\frac{\mathrm{cov}(r_{\mathrm{mkt}},r_i|\mathbf{y})}{\mathrm{var}(r_{\mathrm{mkt}}|\mathbf{y})}(E[r_{\mathrm{mkt}}|\mathbf{y}]-r_f). \tag{A.19}$$

To conclude, we have to show that instead of conditioning on $\mathbf{y}$, in (A.19), we can condition on prices.

From (15), we have $E[r_{\mathrm{mkt}}|\mathbf{y}]=p_{\mathrm{mkt}}^{-1}E[v_{\mathrm{mkt}}|\mathbf{y}]=p_{\mathrm{mkt}}^{-1}E[v_{\mathrm{mkt}}|p_f,\mathbf{p}]=E[r_{\mathrm{mkt}}|p_f,\mathbf{p}]$. We apply the same argument to the individual return and get $E[r_i|\mathbf{y}]=E[r_i|p_f,\mathbf{p}]$. Similarly, from (16), we obtain $\mathrm{cov}(r_{\mathrm{mkt}},r_i|\mathbf{y})=\mathrm{cov}(r_{\mathrm{mkt}},r_i|p_f,\mathbf{p})$ and $\mathrm{var}(r_{\mathrm{mkt}}|\mathbf{y})=\mathrm{var}(r_{\mathrm{mkt}}|p_f,\mathbf{p})$. Q.E.D.

Endnotes

[1] Basak and Pavlova (2013) offer an alternative view of what constitutes a distortion due to indexing. Index traders in their model (called institutional investors) are traders with a different utility function. However, the presence of institutional investors distorts prices for a reason more fundamental than benchmarking: preferences toward risk matter to asset pricing. Indeed, they report distortions in their model even when there is a single risky asset.

[2] Notation: Vectors are single-column matrices. The transpose operation is denoted by a prime symbol. Bold lowercase (Greek or upright Roman) letters are used for vectors. Bold uppercase (Greek or upright Roman) letters are used for matrices. We have no special notation to distinguish random variables from their realizations. The context should make our intention clear.

Given two random vectors, $\mathbf{z}=[z_1\ \ldots\ z_n]'$ and $\mathbf{y}=[y_1\ \ldots\ y_m]'$, we interchangeably use the notations $\mathrm{cov}(\mathbf{z},\mathbf{y})$ and $\Sigma_{\mathbf{zy}}$ to denote the $n\times m$ covariance matrix $[\mathrm{cov}(z_i,y_j)]_{n\times m}$. Consequently, using submatrix notation, we have

$$\mathrm{cov}\left(\mathbf{z},\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix}\right)=[\mathrm{cov}(\mathbf{z},\mathbf{y}_1)\quad \mathrm{cov}(\mathbf{z},\mathbf{y}_2)].$$

We routinely use the property that if $\mathbf{A}$ and $\mathbf{B}$ are deterministic, then $\mathrm{cov}(\mathbf{Ax},\mathbf{By})=\mathbf{A}\mathrm{cov}(\mathbf{x},\mathbf{y})\mathbf{B}'$.

[3] See Grossman (1978). In the classic definition of sufficiency, the unknown is treated as a parameter. In the Bayesian definition of sufficiency, the unknown is random, but the condition for sufficiency holds for any distribution. Here, the unknown is a random variable with a specific distribution.

[4] Prices are not informationally equivalent to $\mathbf{y}$, because knowing the prices one cannot infer $\mathbf{y}$. However, prices contain all the information about $\mathbf{v}$ contained in $\mathbf{y}$.

[5] For example, consider estimating an unobserved value, x, based on m noisy observations of x. If x and the errors in the observations are independently and normally distributed, then the conditional variance of x is $\sigma_x^2\sigma_e^2/(m\sigma_x^2+\sigma_e^2)$, which is a convex function in m that asymptotically approaches zero.

[6] We use the following standard notation. If $\mathbf{A}=[a_{ij}]$ is an arbitrary matrix with elements that depend on a parameter π, then $\frac{\partial\mathbf{A}}{\partial\pi}$ is the matrix $\left[\frac{\partial a_{ij}}{\partial\pi}\right]$. If $\mathbf{B}$ and $\mathbf{C}$ are arbitrary matrices with elements that do not depend on the parameter π, then

$$\frac{\partial \mathbf{BA}^{-1}\mathbf{C}}{\partial \pi} = -\mathbf{BA}^{-1}\frac{\partial \mathbf{A}}{\partial \pi}\mathbf{A}^{-1}\mathbf{C}.$$

[7] A note on the block inversion formula: the formula states that if one can partition a matrix into four blocks, $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, and both $\mathbf{D}$ and $\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C}$ are invertible, then the inverse is given by

$$\begin{bmatrix} (\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C})^{-1} & -(\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C})^{-1}\mathbf{BD}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1}+\mathbf{D}^{-1}\mathbf{C}(\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C})^{-1}\mathbf{BD}^{-1} \end{bmatrix}.$$

Here, the blocks are

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathrm{cov}(\mathbf{s}_k, \mathbf{s}_k) & \mathrm{cov}(\mathbf{s}_k, \mathbf{y}) \\ \mathrm{cov}(\mathbf{y}, \mathbf{s}_k) & \mathrm{cov}(\mathbf{y}, \mathbf{y}) \end{bmatrix}.$$

Clearly, $\mathbf{D} = \mathrm{cov}(\mathbf{y}, \mathbf{y})$ is invertible because it is the variance-covariance matrix of a nondegenerate multivariate random vector. From the definition of $\mathbf{s}_k$ and $\mathbf{y}$ and the help of (A.8), we can see that $\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C} = (1-1/|\mathcal{NI}|)\Sigma_{\epsilon\epsilon}$, which is invertible because we have assumed that ϵ_k is nondegenerate. Hence, we can use the block inversion formula.

References

Admati AR (1985) A noisy rational expectations equilibrium for multi-asset securities markets. *Econometrica* 53(3):629–657.

Ausubel LM (1990) Partially-revealing rational expectations equilibrium in a competitive economy. *J. Econom. Theory* 50(1):93–126.

Barberis N, Shleifer A (2003) Style investing. *J. Financial Econom.* 68(2):161–199.

Barberis N, Shleifer A, Wurgler J (2005) Comovement. *J. Financial Econom.* 75(2):283–317.

Basak S, Pavlova A (2013) Asset prices and institutional investors. *Amer. Econom. Rev.* 103(5):1728–1758.

Bennett B, Stulz R, Wang Z (2020) Does joining the S&P500 index hurt firms? Working paper, Tulane University, New Orleans, LA.

Bhattacharya A, O'Hara M (2018) Can ETFs increase market fragility? Effect of information linkages in ETF markets. Working paper, Baruch College, New York.

Billett M, Diep-Nguyen H, Garfinkel JA (2020) Index investing and corporate investment-price sensitivity. Working paper, Indiana University, Bloomington.

Bond P, Garcia D (2021) The equilibrium consequences of indexing. *Rev. Financial Stud*. Forthcoming.

Brennan MJ (1993) Agency and asset pricing. Working paper, University of California, Los Angeles.

Campbell JY (2017) Financial Decisions and Markets: A Course in Asset Pricing (Princeton University Press, Princeton, NJ).

Campbell JY, Lettau M, Malkiel BG, Xu Y (2001) Have individual stocks become more volatile? An empirical exploration of idiosyncratic risk. *J. Finance* 56(1):1–43.

Cong LW, Xu DX (2019) Rise of factor investing: asset prices, informational efficiency, and security design. Working paper, Cornell University, Ithaca, NY.

Cuoco D, Kaniel R (2011) Equilibrium prices in the presence of delegated portfolio management. *J. Financial Econom.* 101(2): 264–296.

DeMarzo P, Skiadas C (1998) Aggregation, determinacy, and informational efficiency for a class of economies with asymmetric information. *J. Econom. Theory* 80(1):123–152.

Detemple JB (2002) Asset pricing in an intertemporal partially-revealing rational expectations equilibrium. *J. Math. Econom.* 38(1–2):219–248.

Glosten L, Nallareddy S, Zou Y (2021) ETF activity and informational efficiency of underlying securities. *Management Sci.* 67(1): 22–47.

Grossman S (1976) On the efficiency of competitive stock markets where trades have diverse information. *J. Finance* 31(2):573–585.

Grossman S (1978) Further results on the informational efficiency of competitive stock markets. *J. Econom. Theory* 18(1):81–101.

Hellwig MF (1980) On the aggregation of information in competitive markets. *J. Econom. Theory* 22(3):477–498.

Irvine PJ, Pontiff J (2009) Idiosyncratic return volatility, cash flows, and product market competition. *Rev. Financial Stud.* 22(3): 1149–1177.

Jin L, Myers SC (2006) R^2 around the world: New theory and new tests. *J. Financial Econom.* 79(2):257–292.

Kyle AS (1985) Continuous auctions and insider trading. *Econometrica* 53(6):1315–1335.

Levy H (1978) Equilibrium in an imperfect market: A constraint on the number of securities in the portfolio. *Amer. Econom. Rev.* 68(4):643–658.

Liu H, Wang Y (2018) Index investing and price discovery. Working paper, Washington University in St. Louis. St. Louis, MO.

Markowitz H (1952) Portfolio selection. *J. Finance* 7(1):77–91.

Merton RC (1987) A simple model of capital market equilibrium with incomplete information. *J. Finance* 42(3):483–510.

Morck R, Yeung B, Yu W (2000) The information content of stock markets: why do emerging markets have synchronous stock price movements? *J. Financial Econom.* 58(1–2):215–260.

Rawley JJ, Hirt JM, Wang H (2018) Setting the Record Straight: Truths about Indexing (Vanguard Research, Valley Forge, PA).

Samuelson P (1976) Index-fund investing. *Newsweek*, August 16, http://johncbogle.com/wordpress/wp-content/uploads/2013/10/Samuelson-Newsweek-1976.pdf.

Tobin J (1958) Liquidity preference as behavior towards risk. *Rev. Econom. Stud.* 25(2):65–86.