Asymptotic completeness in a class of massless relativistic quantum field theories

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Abstract

This paper presents the first examples of massless relativistic quantum field theories which are interacting and asymptotically complete. These two-dimensional models are obtained by an application of a deformation procedure, introduced recently by Grosse and Lechner, to chiral conformal quantum field theories. The resulting models may not be strictly local, but they contain observables localized in spacelike wedges.

1 Introduction

Interpretation of quantum field theories in terms of particles is a long-standing fundamental problem. The last two decades witnessed significant progress on this issue, both on the side of structural analysis [Bu90, Po04.1, Po04.2, Dy05, Dy09] and in the study of concrete models [Sp97, DG99, FGS04, Le08]. In particular, the first examples of local, relativistic quantum field theories, which are interacting and asymptotically complete, have been constructed in [Le08]. As this class contains only massive models, the question of asymptotic completeness in the presence of massless particles is open to date in the local, relativistic framework. This can be partly attributed to the infamous *infrared problem*, which hinders rigorous construction and analysis of interacting massless theories by traditional methods (see however [CRW85, BFM04]). It is therefore remarkable that more recent constructive

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tools, developed in [BLS10], give rise to massless models which are asymptotically complete and interacting. We exhibit such theories in the present work.

We recall that a new class of relativistic quantum field theories, including both massive and massless models, has been obtained recently by a certain deformation procedure akin to the Rieffel deformation [GL08, BS08, BLS10, DLM10]. These theories are wedge-local i.e. observables can be localized in (unbounded) wedge-shaped regions extending in spacelike directions. In the massive case this remnant of locality suffices for a canonical construction of the two-body scattering matrix, as shown in [BBS01]. Exploiting this fact, it was demonstrated in [GL08, BS08] that the deformed theory is interacting even if the original theory is not. As in general only two-body scattering states are available, it may seem that the problem of asymptotic completeness cannot be addressed in the framework of wedge-local theories. However, in the case of two-dimensional massless theories such a conclusion would be pre-mature, as we demonstrate in this paper.

Our first task is to provide a scattering theory for such models. We recall that for *local* two-dimensional theories of massless excitations a scattering theory was developed in [Bu75]. The basic building blocks of this construction are the subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ in the physical Hilbert space $\mathcal{H}$, corresponding to the right and left branch of the lightcone in momentum space. These subspaces carry representations of the Poincaré group which are in general highly reducible. Thus vectors $\Psi_\pm \in \mathcal{H}_\pm$ do not describe particles in the Wigner sense, but rather composite objects, called in [Bu75] 'waves'. In view of their dispersionless motion, a composition of several waves travelling in the same direction (say elements of $\mathcal{H}_+$), gives rise to another wave from $\mathcal{H}_+$. Thus it suffices to consider scattering states $\Psi_+ \overset{\mathrm{out}}{\times} \Psi_-$ (resp. $\Psi_+ \overset{\mathrm{in}}{\times} \Psi_-$) which describe two waves travelling in the opposite directions in the remote future (resp. past). They span the subspaces $\mathcal{H}^{\mathrm{out}}$ (resp. $\mathcal{H}^{\mathrm{in}}$) of the outgoing (resp. incoming) states. The scattering matrix $S : \mathcal{H}^{\mathrm{out}} \to \mathcal{H}^{\mathrm{in}}$ can be defined as an isometry mapping $\Psi_+ \overset{\mathrm{out}}{\times} \Psi_-$ into $\Psi_+ \overset{\mathrm{in}}{\times} \Psi_-$. If $\mathcal{H}^{\mathrm{out}} = \mathcal{H}^{\mathrm{in}} = \mathcal{H}$, then we say that the theory is asymptotically complete. As we will show, there exists a large class of non-interacting massless theories in two-dimensional spacetime which have this property: it includes all chiral conformal quantum field theories.

In the light of the above discussion, it is not surprising that the scattering theory from [Bu75] can be generalized to the wedge-local context. Indeed, observables localized in two opposite spacelike wedges suffice to separate two waves travelling in opposite directions. We demonstrate this fact in Section 2 after some introductory remarks on wedge-local quantum field theories. In Section 3 we express the scattering matrix S_κ of the deformed theory (with a deformation parameter κ), by the scattering matrix S of the original one. We obtain

$$S_\kappa = e^{i\kappa M^2} S, \tag{1}$$

where M is the mass operator. Hence, similarly as in the massive case, the deformed theory is interacting, even if the original theory is not. Moreover, the property of asymptotic completeness is preserved by the deformation procedure. Thus deformations of chiral conformal field theories give rise to wedge-local theories which are interacting and asymptotically complete, as we show in Section 4. We summarize our results in Section 5, where also some open questions are discussed.

2 Scattering theory

A convenient framework for a study of wedge-local theories is provided by the concept of a Borchers triple [Bo92]. We recall that a Borchers triple $(\mathcal{R}, U, \Omega)$, (relative to the wedge $\mathcal{W} = \{ x = (x^0, x^1) \in \mathbb{R}^2 \,|\, x^1 \geq |x^0| \}$), consists of:

(a) a von Neumann algebra $\mathcal{R} \subset B(\mathcal{H})$,

(b) a strongly continuous unitary representation U of $\mathbb{R}^2$ on $\mathcal{H}$, whose spectrum $\mathrm{sp}\, U$ is contained in the closed forward lightcone $V_+ = \{ p = (p^0, p^1) \in \mathbb{R}^2 \,|\, p^0 \geq |p^1| \}$ and which satisfies $\alpha_x(\mathcal{R}) \subset \mathcal{R}$, for $x \in \mathcal{W}$, where $\alpha_x(\,\cdot\,) = U(x) \,\cdot\, U(x)^{-1}$,

(c) a unit vector $\Omega \in \mathcal{H}$ which is invariant under the action of U and is cyclic and separating for $\mathcal{R}$. It will be called the vacuum vector.

One interprets $\mathfrak{A}(\mathcal{W}) := \mathcal{R}$ as the algebra of all observables localized in the wedge $\mathcal{W}$. In view of (c), one can apply to $(\mathcal{R}, \Omega)$ the Tomita-Takesaki theory and we denote by (Δ, J) the modular operator and the conjugation. As shown in [Bo92], with the help of the modular objects one can construct an (anti-)unitary representation $\lambda \to \tilde{U}(\lambda)$ of the proper Poincaré group $\mathcal{P}_+$ which extends the original representation of translations. In particular, J implements the spacetime reflection, i.e.

$$JU(x)J = U(-x), \quad x \in \mathbb{R}^2. \tag{2}$$

Thus with any wedge $\lambda\mathcal{W}$ one can associate the algebra of observables $\mathfrak{A}(\lambda\mathcal{W}) = \tilde{U}(\lambda)\mathcal{R}\tilde{U}(\lambda)^{-1}$. Since, by the Tomita-Takesaki theory, $J\mathcal{R}J = \mathcal{R}'$, the resulting net is wedge-local i.e $\mathfrak{A}((\lambda\mathcal{W})') = \mathfrak{A}(\lambda\mathcal{W})'$, where $(\lambda\mathcal{W})'$ is the causal complement of $\lambda\mathcal{W}$ and a prime over an algebra denotes the commutant. Hence this net gives rise to a (two-dimensional) wedge-local, relativistic quantum field theory. See [Bo92, Fl98] for proofs of the above statements and [BLS10] for a more detailed overview.

Let (H, P) be the generators of U i.e. $U(x^0, x^1) = e^{iHx^0 - iPx^1}$. We set $\mathcal{H}_\pm = \ker(H \mp P)$ and denote by $P_\pm$ the corresponding projections. We assume that $\mathcal{H}_+ \cap \mathcal{H}_- = [c\Omega]$ i.e. Ω is the unique (up to a phase) vector which is invariant under translations. This implies that $\mathcal{H}_+ \cap [c\Omega]^\perp$ is orthogonal to $\mathcal{H}_- \cap [c\Omega]^\perp$. We assume that the latter two subspaces are non-trivial, to ensure that the theory contains massless excitations. Let us now describe briefly their collision theory. The construction follows closely [Bu75].

For any $F \in B(\mathcal{H})$ and $x \in \mathbb{R}^2$ we denote $F(x) := \alpha_x(F)$ and define the sequences of operators

$$F_\pm(h_T) = \int dt\, h_T(t) F(t_\pm) \text{ with } t_\pm = (t, \pm t), \tag{3}$$

where $h_T(t) = |T|^{-\varepsilon} h(|T|^{-\varepsilon}(t-T))$, $0 < \varepsilon < 1$ and $h \in C_0^\infty(\mathbb{R})$ is a non-negative, symmetric function s.t. $\int dt\, h(t) = 1$. With the help of these approximating sequences we construct the asymptotic fields corresponding to the wedge $\mathcal{W}$.

Lemma 2.1 *Let $F \in \mathcal{R}$. Then the limits*

$$\Phi_+^{\mathrm{out}}(F) := \underset{T\to\infty}{\text{s-}\lim}\ F_+(h_T), \quad \Phi_-^{\mathrm{in}}(F) := \underset{T\to-\infty}{\text{s-}\lim}\ F_-(h_T), \tag{4}$$

exist and are elements of $\mathcal{R}$. They depend only on the respective vectors $\Phi_+^{\mathrm{out}}(F)\Omega = P_+F\Omega$, $\Phi_-^{\mathrm{in}}(F)\Omega = P_-F\Omega$ and satisfy

(a) $\Phi_+^{\mathrm{out}}(F)\mathcal{H}_+ \subset \mathcal{H}_+$, $\quad \Phi_-^{\mathrm{in}}(F)\mathcal{H}_- \subset \mathcal{H}_-$,

(b) $\alpha_x(\Phi_+^{\mathrm{out}}(F)) = \Phi_+^{\mathrm{out}}(\alpha_x(F))$, $\quad \alpha_x(\Phi_-^{\mathrm{in}}(F)) = \Phi_-^{\mathrm{in}}(\alpha_x(F))$ *for $x \in \mathcal{W}$.*

Proof. Let us consider the first limit in (4). Since there holds the estimate $\|F_+(h_T)\| \le \|F\| \int dt\, |h(t)|$, it suffices to show the convergence on the dense set of vectors $R'\Omega$. First, one checks using the mean ergodic theorem

$$\underset{T\to\infty}{\text{s-}\lim}\ F_+(h_T)\Omega = P_+F\Omega. \tag{5}$$

In view of part (b) of the definition of the Borchers triple and the fact that $t_+ \in \mathcal{W}$ there holds $F_+(h_T) \in \mathcal{R}$ for T sufficiently large. Hence, for any $F' \in R'$,

$$\underset{T\to\infty}{\text{s-}\lim}\ F_+(h_T)F'\Omega = F'P_+F\Omega, \tag{6}$$

which proves convergence. Since $\mathcal{R}$ is a von Neumann algebra, the limit $\Phi_+^{\mathrm{out}}(F)$ is an element of $\mathcal{R}$. Since Ω is separating for $\mathcal{R}$, this operator depends only on $\Phi_+^{\mathrm{out}}(F)\Omega = P_+F\Omega$.

The second part of (4) is proven analogously. Property (a) follows by application of the mean ergodic theorem, similarly as in (5). Property (b) is obvious from the definitions of Φ_+^{out}, Φ_-^{in}. $\square$

Let us now define the asymptotic fields corresponding to the wedge $\mathcal{W}'$. Keeping in mind that $J\mathcal{R}'J = \mathcal{R}$, we set for any $F' \in \mathcal{R}'$

$$\Phi_+^{\mathrm{in}}(F') := J\Phi_+^{\mathrm{out}}(JF'J)J, \quad \Phi_-^{\mathrm{out}}(F') := J\Phi_-^{\mathrm{in}}(JF'J)J. \tag{7}$$

Making use of formula (2), we easily obtain the following counterpart of Lemma 2.1.

Lemma 2.2 *Let $F' \in \mathcal{R}'$. Then there holds*

$$\Phi_+^{\mathrm{in}}(F') = \underset{T\to-\infty}{\text{s-}\lim}\ F'_+(h_T), \quad \Phi_-^{\mathrm{out}}(F') = \underset{T\to\infty}{\text{s-}\lim}\ F'_-(h_T). \tag{8}$$

These operators depend only on the respective vectors $\Phi_+^{\mathrm{in}}(F')\Omega = P_+F'\Omega$, $\Phi_-^{\mathrm{out}}(F')\Omega = P_-F'\Omega$ and satisfy

(a) $\Phi_+^{\mathrm{in}}(F')\mathcal{H}_+ \subset \mathcal{H}_+$, $\quad \Phi_-^{\mathrm{out}}(F')\mathcal{H}_- \subset \mathcal{H}_-$,

(b) $\alpha_x(\Phi_+^{\mathrm{in}}(F'))=\Phi_+^{\mathrm{in}}(\alpha_x(F'))$, $\quad \alpha_x(\Phi_-^{\mathrm{out}}(F'))=\Phi_-^{\mathrm{out}}(\alpha_x(F'))$ *for $x \in \mathcal{W}'$.*

Let us now proceed to the construction of scattering states. Clustering properties of the asymptotic fields are of importance here. Proceeding as in [Bu75], we note that for any $F, G \in \mathcal{R}$, $F', G' \in \mathcal{R}'$ there holds

$$
\begin{aligned}
&(\Phi_+^{\mathrm{out}}(F)\Phi_-^{\mathrm{out}}(F')\Omega|\Phi_+^{\mathrm{out}}(G)\Phi_-^{\mathrm{out}}(G')\Omega) \\
&\qquad = (\Phi_+^{\mathrm{out}}(G)^*\Phi_+^{\mathrm{out}}(F)\Omega|\Phi_-^{\mathrm{out}}(F')^*\Phi_-^{\mathrm{out}}(G')\Omega) \\
&\qquad = (\Phi_+^{\mathrm{out}}(F)\Omega|\Phi_+^{\mathrm{out}}(G)\Omega)(\Phi_-^{\mathrm{out}}(F')\Omega|\Phi_-^{\mathrm{out}}(G')\Omega),
\end{aligned} \tag{9}
$$

where in the last step we made use of Lemma 2.1 (a), Lemma 2.2 (a) and of the fact that $\mathcal{H}_+ \cap [c\Omega]^\perp$ is orthogonal to $\mathcal{H}_- \cap [c\Omega]^\perp$. Now for any $\Psi_+ \in \mathcal{H}_+$, (resp. $\Psi_- \in \mathcal{H}_-$) we choose, with the help of property (c) of the Borchers triple, a sequence $\{F_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{R}$ (resp. a sequence $\{F'_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{R}'$) s.t. $\operatorname*{s-lim}_{n\to\infty} P_+F_n\Omega = \Psi_+$ (resp. $\operatorname*{s-lim}_{n\to\infty} P_+F'_n\Omega = \Psi_-$). By relation (9), the limit

$$
\Psi_+ \overset{\mathrm{out}}{\times} \Psi_- := \operatorname*{s-lim}_{n\to\infty} \Phi_+^{\mathrm{out}}(F_n)\Phi_-^{\mathrm{out}}(F'_n)\Omega \tag{10}
$$

exists and does not depend on the choice of the sequences within the above restrictions. We will call it the outgoing scattering state. Next, we define the incoming scattering states as follows

$$
\Psi_+ \overset{\mathrm{in}}{\times} \Psi_- := J((J\Psi_+) \overset{\mathrm{out}}{\times} (J\Psi_-)). \tag{11}
$$

This definition is meaningful, since relation (2) gives $J\mathcal{H}_+ \subset \mathcal{H}_+$ and $J\mathcal{H}_- \subset \mathcal{H}_-$. It is easily seen, that for suitable sequences $\{G_n\}_{n\in\mathbb{N}}$ (resp. $\{G'_n\}_{n\in\mathbb{N}}$) of elements of $\mathcal{R}$ (resp. of $\mathcal{R}'$), there holds

$$
\Psi_+ \overset{\mathrm{in}}{\times} \Psi_- = \operatorname*{s-lim}_{n\to\infty} \Phi_+^{\mathrm{in}}(G'_n)\Phi_-^{\mathrm{in}}(G_n)\Omega, \tag{12}
$$

similarly as in (10). The states constructed above have the following basic properties which justify their interpretation as scattering states:

Lemma 2.3 *For any $\Psi_\pm, \Psi'_\pm \in \mathcal{H}_\pm$ there holds:*

(a) $(\Psi_+ \overset{\mathrm{out}}{\times} \Psi_-, \Psi'_+ \overset{\mathrm{out}}{\times} \Psi'_-) = (\Psi_+, \Psi'_+)(\Psi_-, \Psi'_-)$,

(b) $U(x)(\Psi_+ \overset{\mathrm{out}}{\times} \Psi_-) = (U(x)\Psi_+) \overset{\mathrm{out}}{\times} (U(x)\Psi_-)$, *for* $x \in \mathbb{R}^2$.

Analogous relations hold for the incoming scattering states.

Proof. Part (a) follows immediately from relation (9). As for part (b), for any $x \in \mathbb{R}^2$ we can choose such $y \in \mathcal{W}$ and $y' \in \mathcal{W}'$ that $x + y \in \mathcal{W}$ and $x + y' \in \mathcal{W}'$. We choose a sequence $\{F_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{R}$ and $\{F'_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{R}'$ s.t. $\operatorname*{s-lim}_{n\to\infty} P_+F_n(y)\Omega = \Psi_+$ and $\operatorname*{s-lim}_{n\to\infty} P_-F'_n(y')\Omega = \Psi_-$. Then

$$
\begin{aligned}
U(x)(\Psi_+ \overset{\mathrm{out}}{\times} \Psi_-) &= \operatorname*{s-lim}_{n\to\infty} \alpha_x(\Phi_+^{\mathrm{out}}(F_n(y)))\alpha_x(\Phi_-^{\mathrm{out}}(F'_n(y')))\Omega \\
&= \operatorname*{s-lim}_{n\to\infty} \alpha_{x+y}(\Phi_+^{\mathrm{out}}(F_n))\alpha_{x+y'}(\Phi_-^{\mathrm{out}}(F'_n))\Omega \\
&= \operatorname*{s-lim}_{n\to\infty} \Phi_+^{\mathrm{out}}(F_n(x+y))\Phi_-^{\mathrm{out}}(F'_n(x+y'))\Omega,
\end{aligned} \tag{13}
$$

where we applied Lemma 2.1 (b) and Lemma 2.2 (b) in the second and third step. We note that the last state on the r.h.s. above is just $(U(x)\Psi_+) \overset{\text{out}}{\times} (U(x)\Psi_-)$, completing the proof of (b). The statement concerning the incoming states follows immediately from the properties of the outgoing states and from definition (11). □

After this preparation, we introduce the scattering subspaces

$$\mathcal{H}^{\text{in}} = \mathcal{H}_+ \overset{\text{in}}{\times} \mathcal{H}_- \text{ and } \mathcal{H}^{\text{out}} = \mathcal{H}_+ \overset{\text{out}}{\times} \mathcal{H}_-, \tag{14}$$

which are spanned by the respective scattering states. In view of Lemma 2.3, they are canonically isomorphic to the tensor product $\mathcal{H}_+ \otimes \mathcal{H}_-$. Similarly as in [Bu75], we define the scattering operator $S : \mathcal{H}^{\text{out}} \to \mathcal{H}^{\text{in}}$, extending by linearity the following relation:

$$S(\Psi_+ \overset{\text{out}}{\times} \Psi_-) = \Psi_+ \overset{\text{in}}{\times} \Psi_-. \tag{15}$$

In view of Lemma 2.3 this map is an isometry and it is invariant under translations. If S differs from (a constant multiple of) the identity transformation on $\mathcal{H}^{\text{in}}$, then we say that the theory is interacting. If $\mathcal{H}^{\text{in}} = \mathcal{H}^{\text{out}} = \mathcal{H}$, then we say that the theory is asymptotically complete. In the next two sections we will exhibit a class of theories which have both properties.

To conclude this section, we point out that the asymptotic fields form new Borchers triples, which are non-interacting and asymptotically complete. In view of Lemma 2.1 (a) and Lemma 2.2 (a), we can define the following von Neumann algebras acting on $\mathcal{H}^{\text{as}} := \mathcal{H}_+ \otimes \mathcal{H}_-$:

$$\mathcal{R}^{\text{as}} := \{\Phi_+^{\text{out}}(F)|_{\mathcal{H}_+} \otimes \Phi_-^{\text{in}}(G)|_{\mathcal{H}_-} \mid F, G \in \mathcal{R}\}'', \tag{16}$$

$$(\mathcal{R}')^{\text{as}} := \{\Phi_+^{\text{in}}(F')|_{\mathcal{H}_+} \otimes \Phi_-^{\text{out}}(G')|_{\mathcal{H}_-} \mid F', G' \in \mathcal{R}'\}''. \tag{17}$$

Moreover, we set $U^{\text{as}}(x) = U(x)|_{\mathcal{H}_+} \otimes U(x)|_{\mathcal{H}_-}$ and $\Omega^{\text{as}} = \Omega \otimes \Omega$. Clearly, there holds

$$\operatorname{sp} U^{\text{as}} = \operatorname{sp} U|_{\mathcal{H}_+} + \operatorname{sp} U|_{\mathcal{H}_-} \subset V_+. \tag{18}$$

and Ω^{as} is the unique (up to a phase) vector which is invariant under the action of U^{as}. Since Ω^{as} is cyclic for $\mathcal{R}^{\text{as}}$ and $(\mathcal{R}')^{\text{as}}$, and $(\mathcal{R}')^{\text{as}} \subset (\mathcal{R}^{\text{as}})'$, we obtain that $(\mathcal{R}^{\text{as}}, U^{\text{as}}, \Omega^{\text{as}})$ is a Borchers triple w.r.t. $\mathcal{W}$. We call it the asymptotic Borchers triple of (R, U, Ω). It has the following properties:

Proposition 2.4 *The Borchers triple* $(\mathcal{R}^{\text{as}}, U^{\text{as}}, \Omega^{\text{as}})$ *defined above gives rise to an asymptotically complete and non-interacting wedge-local quantum field theory. Moreover,* $\operatorname{sp} U^{\text{as}} = V_+$.

Proof. Making use of the fact that $(\mathcal{R}')^{\text{as}} \subset (\mathcal{R}^{\text{as}})'$, we obtain the equalities

$$\Phi_+^{\text{out}}(\Phi_+^{\text{out}}(F)|_{\mathcal{H}_+} \otimes I)\Omega^{\text{as}} = P_+ F\Omega \otimes \Omega, \tag{19}$$

$$\Phi_-^{\text{out}}(I \otimes \Phi_-^{\text{out}}(F')|_{\mathcal{H}_-})\Omega^{\text{as}} = \Omega \otimes P_- F'\Omega, \tag{20}$$

valid for any $F \in \mathcal{R}$, $F' \in \mathcal{R}'$. Thus we conclude that $\mathcal{H}_+^{\text{as}} \supset \mathcal{H}_+ \otimes [c\Omega]$ and $\mathcal{H}_-^{\text{as}} \supset [c\Omega] \otimes \mathcal{H}_-$. Let $\Psi_\pm \in \mathcal{H}_\pm$ and let $\{F_n\}_{n\in\mathbb{N}}$ (resp. $\{F_n'\}_{n\in\mathbb{N}}$)

be a sequence of elements of $\mathcal{R}$ (resp. $\mathcal{R}'$) s.t. $\operatorname{s-lim}_{n\to\infty} P_+F_n\Omega = \Psi_+$ (resp. $\operatorname{s-lim}_{n\to\infty} P_-F'_n\Omega = \Psi_-$). Then we get

$$\begin{aligned}&(\Psi_+\otimes\Omega) \overset{\text{out}}{\times} (\Omega\otimes\Psi_-)\\ &= \operatorname{s-lim}_n \Phi_+^{\text{out}}(\Phi_+^{\text{out}}(F_n)|_{\mathcal{H}_+}\otimes I)\Phi_-^{\text{out}}(I\otimes\Phi_-^{\text{out}}(F'_n)|_{\mathcal{H}_-})\Omega^{\text{as}}\\ &= \Psi_+\otimes\Psi_-. \end{aligned} \tag{21}$$

By an analogous argument, we verify that

$$(\Psi_+\otimes\Omega) \overset{\text{in}}{\times} (\Omega\otimes\Psi_-) = \Psi_+\otimes\Psi_-. \tag{22}$$

We infer from equalities (21) and (22) that $(\mathcal{H}^{\text{as}})^{\text{out}} = (\mathcal{H}^{\text{as}})^{\text{in}} = \mathcal{H}^{\text{as}}$ (i.e. asymptotic completeness holds) and $S = I$ (i.e. the theory is non-interacting).

To justify the statement concerning the spectrum of U^{as}, we recall that $\mathcal{H}_+\cap[c\Omega]^\perp$ and $\mathcal{H}_-\cap[c\Omega]^\perp$ are assumed to be non-trivial. Consequently, $\operatorname{sp} U|_{\mathcal{H}_+}$ and $\operatorname{sp} U|_{\mathcal{H}_-}$ have some non-zero elements. From the existence of the unitary representation of the Poincaré group $\tilde{U}$, associated with the triple $(\mathcal{R}, U, \Omega)$, we conclude that these two spectra coincide with the right and left branch of the lightcone, respectively. Since $\operatorname{sp}(U|_{\mathcal{H}_+}\otimes U|_{\mathcal{H}_-}) = \operatorname{sp} U|_{\mathcal{H}_+} + \operatorname{sp} U|_{\mathcal{H}_-}$, the statement follows. □

3 Deformations and interaction

In the previous section we showed that for any Borchers triple in two-dimensional spacetime (with a unique vacuum state) we can canonically construct the scattering matrix S which describes collisions of massless particles. In this section we consider a class of deformations of Borchers triples, introduced in [BLS10] and study their effect on the scattering matrix. Similarly as in the massive case [GL08, BS08], the deformed theory turns out to be interacting, even if the original one is not. Moreover, we show that the property of asymptotic completeness is preserved under these deformations.

Let us recall briefly the deformation procedure of [BLS10]. Let $(\mathcal{R}, U, \Omega)$ be a Borchers triple w.r.t. the wedge $\mathcal{W}$. We denote by $\mathcal{R}^\infty$ the subset of elements of $\mathcal{R}$ which are smooth under the action of α in the norm topology. (It is easy to see that $\mathcal{R}^\infty$ is a dense subalgebra of $\mathcal{R}$ in the strong operator topology). Let $\mathcal{D}$ be the dense domain of vectors which are smooth w.r.t. to the action of U. Then, as shown in [BLS10], one can define for any $F\in\mathcal{R}^\infty$, and a matrix Q, antisymmetric w.r.t. the Minkowski scalar product $(x, y)\to xy$, the warped convolution

$$F_Q = \int dE(x)\,\alpha_{Qx}(F) := \lim_{\varepsilon\searrow 0}(2\pi)^{-2}\int d^2x\,d^2y\, f(\varepsilon x,\varepsilon y)e^{-ixy}\alpha_{Qx}(F)U(y) \tag{23}$$

where dE is the spectral measure of U and $f\in S(\mathbb{R}^2\times\mathbb{R}^2)$ is s.t. $f(0,0) = 1$. The limit exists in the strong sense on vectors from $\mathcal{D}$ and is independent of the function f within the above restrictions. We set

$$\mathcal{R}_Q := \{ F_Q \,|\, F \in \mathcal{R}^\infty \}''. \tag{24}$$

Let us now restrict attention to the following family of antisymmetric matrices

$$Q_\kappa = \left(\begin{array}{cc} 0 & \kappa \\ \kappa & 0 \end{array} \right), \tag{25}$$

where $\kappa > 0$, and recall the following result from [BLS10]:

Theorem 3.1 *If $(\mathcal{R}, U, \Omega)$ is a Borchers triple w.r.t. $\mathcal{W}$, then $(\mathcal{R}_{Q_\kappa}, U, \Omega)$ is also a Borchers triple w.r.t. $\mathcal{W}$. Moreover, $(\mathcal{R}')_{-Q_\kappa} \subset (\mathcal{R}_{Q_\kappa})'$.*

Our goal is to express the scattering matrix S_κ of the deformed theory $(\mathcal{R}_{Q_\kappa}, U, \Omega)$ by the scattering matrix S of the original theory $(\mathcal{R}, U, \Omega)$. To this end, we prove the following fact.

Theorem 3.2 *For any $\Psi_\pm \in \mathcal{H}_\pm$ there hold the relations*

$$\Psi_+ \overset{\text{out}}{\times}_\kappa \Psi_- = e^{-i\frac{1}{2}\kappa(H^2-P^2)}(\Psi_+ \overset{\text{out}}{\times} \Psi_-), \tag{26}$$

$$\Psi_+ \overset{\text{in}}{\times}_\kappa \Psi_- = e^{i\frac{1}{2}\kappa(H^2-P^2)}(\Psi_+ \overset{\text{in}}{\times} \Psi_-), \tag{27}$$

where on the l.h.s. (resp. r.h.s.) there appear the scattering states of the deformed (resp. undeformed) theory.

Proof. Let us first prove relation (26). To this end, we pick $F \in \mathcal{R}^\infty$, $F' \in (\mathcal{R}')^\infty$. We set $\Psi_+ = P_+ F\Omega = P_+ F_{Q_\kappa}\Omega$ and $\Psi_- = P_- F'\Omega = P_- F'_{-Q_\kappa}\Omega$, where we exploited the translational invariance of the state Ω. Since $F_{Q_\kappa} \in \mathcal{R}_{Q_\kappa}$ and, by Theorem 3.1, $F'_{-Q_\kappa} \in (\mathcal{R}_{Q_\kappa})'$, the outgoing state of the deformed theory is given by

$$\begin{aligned}\Psi_+ \overset{\text{out}}{\times}_\kappa \Psi_- &= \lim_{T\to\infty} F_{Q_\kappa,+}(h_T) F'_{-Q_\kappa,-}(h_T)\Omega = \lim_{T\to\infty} F_{Q_\kappa,+}(h_T) F'_-(h_T)\Omega \\ &= \lim_{T\to\infty} \lim_{\varepsilon\searrow 0} (2\pi)^{-2} \int d^2x\, d^2y\, f(\varepsilon x, \varepsilon y) e^{-ixy} \alpha_{Qx}(F_+(h_T)) F'_-(h_T)(y)\Omega,\end{aligned} \tag{28}$$

where in the last step we made use of the fact that $F'_-(h_T)\Omega \in \mathcal{D}$, and that Ω is invariant under translations. To exchange the order of the limits, we use methods from the proof of Lemma 2.1 of [BLS10]: We note that for each polynomial $(x,y) \to L(x,y)$, there exists a polynomial $(x,y) \to K(x,y)$ s.t

$$L(x,y)e^{-ixy} = K(-\partial_x, -\partial_y)e^{-ixy}. \tag{29}$$

We choose L so that L^{-1} and its derivatives are absolutely integrable. Denoting temporarily $\Psi_T(x,y) := \alpha_{Qx}(F_+(h_T))F'_-(h_T)(y)\Omega$, we obtain

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}(2\pi)^{-2} \int d^2x\, d^2y\, f(\varepsilon x, \varepsilon y) e^{-ixy} \Psi_T(x,y) \\ &= \lim_{\varepsilon\searrow 0}(2\pi)^{-2} \int d^2x\, d^2y\, e^{-ixy} K(\partial_x, \partial_y) f(\varepsilon x, \varepsilon y) L(x,y)^{-1} \Psi_T(x,y) \\ &\qquad = (2\pi)^{-2} \int d^2x\, d^2y\, e^{-ixy} K(\partial_x, \partial_y) L(x,y)^{-1} \Psi_T(x,y),\end{aligned} \tag{30}$$

where in the first step we integrated by parts and in the second step we applied the dominated convergence theorem. To obtain the last expression, we used the fact that derivatives of $(x, y) \to f(\varepsilon x, \varepsilon y)$ contain powers of ε and thus vanish in the limit. Substituting this expression to formula (28) and making use again of the dominated convergence theorem, we arrive at

$$\begin{aligned}&\Psi_+ \overset{\text{out}}{\times}_\kappa \Psi_- \\ &= (2\pi)^{-2} \int d^2x\, d^2y\, e^{-ixy} K(\partial_x, \partial_y) L(x,y)^{-1} (U(Qx)\Psi_+) \overset{\text{out}}{\times} (U(y)\Psi_-). \quad (31)\end{aligned}$$

To exchange the limit $T \to \infty$ with the action of the derivatives, we exploited the fact that for any $F_1 \in \mathcal{R}^\infty$, $\mu \in \{0, 1\}$, the derivative $\partial_{x^\mu} F_1 := (\partial_{x^\mu} F_1(x))|_{x=0}$ is an element of $\mathcal{R}^\infty$ and $\Phi_+^{\text{out}}(\partial_{x^\mu} F_1)(x) = \partial_{x^\mu} \Phi_+^{\text{out}}(F_1)(x)$. This equality (as well as its counterpart for Φ_-^{out}) follows immediately from the norm continuity of the respective map.

To analyze expression (31), we will exploit some special features of massless theories in two dimensions. First, we recall that $(H - P)\Psi_+ = 0$, and therefore

$$U(Q_\kappa x)\Psi_+ = e^{i\kappa(Hx^1 - Px^0)}\Psi_+ = e^{-i\frac{1}{2}\kappa(H+P)(x^0 - x^1)}\Psi_+. \quad (32)$$

Similarly, since $(H + P)\Psi_- = 0$, we obtain

$$U(y)\Psi_- = e^{i\frac{1}{2}(H-P)(y^0+y^1)}\Psi_-. \quad (33)$$

Hence, exploiting the equalities $(H \pm P)\Psi_\mp = 0$ and Lemma 2.3 (b), we get

$$\begin{aligned}(U(Qx)\Psi_+) \overset{\text{out}}{\times} (U(y)\Psi_-) &= e^{-\frac{i}{2}\kappa(H+P)(x^0-x^1)} e^{\frac{i}{2}(H-P)(y^0+y^1)} (\Psi_+ \overset{\text{out}}{\times} \Psi_-) \\ &= U(v(x,y))(\Psi_+ \overset{\text{out}}{\times} \Psi_-), \quad (34)\end{aligned}$$

where $v(x, y) = (\frac{1}{2}(y^0+y^1-\kappa x^0+\kappa x^1), \frac{1}{2}(y^0+y^1+\kappa x^0-\kappa x^1))$. We substitute this expression to formula (31), obtaining

$$\begin{aligned}&(\Psi_+ \overset{\text{out}}{\times}_\kappa \Psi_-) \\ &= (2\pi)^{-2} \int d^2x\, d^2y\, e^{-ixy} K(\partial_x, \partial_y) L(x,y)^{-1} U(v(x,y))(\Psi_+ \overset{\text{out}}{\times} \Psi_-) \\ &= \int \Big(\lim_{\varepsilon \searrow 0} (2\pi)^{-2} \int d^2x\, d^2y\, e^{-ixy} f(\varepsilon x, \varepsilon y) e^{ipv(x,y)} \Big) dE(p) (\Psi_+ \overset{\text{out}}{\times} \Psi_-). \quad (35)\end{aligned}$$

Here in the second step we expressed $U(v(\cdot, \cdot))$ as a spectral integral and used the Fubini theorem to exchange the order of integration. Then we reversed the steps which led to formula (30). Now we analyze the function in the bracket above. Setting $p^\pm = \frac{1}{2}(p^0 \pm p^1)$, we get

$$\begin{aligned}&(2\pi)^{-2} \int d^2x\, d^2y\, e^{-ixy} f(\varepsilon x, \varepsilon y) e^{ipv(x,y)} \\ &= (2\pi)^{-2} \int d^2x\, d^2y\, f(\varepsilon(x^0, x^1), \varepsilon(y^0, y^1)) e^{-i(\kappa p^+ + y^0)x^0} e^{i(\kappa p^+ + y^1)x^1} e^{ip^-(y^0+y^1)} \\ &= (2\pi)^{-1} \int d^2y\, \varepsilon^{-2} \hat{f}(-\varepsilon^{-1}(\kappa p^+ + y^0, \kappa p^+ + y^1), \varepsilon(y^0, y^1)) e^{ip^-(y^0+y^1)} \\ &= (2\pi)^{-1} \int d^2y\, \hat{f}(-(y^0, y^1), \varepsilon(\varepsilon y^0 - \kappa p^+, \varepsilon y^1 - \kappa p^+)) e^{ip^-((y^0+y^1)\varepsilon - 2\kappa p^+)}.\end{aligned} \quad (36)$$

Here $\hat{f}$ denotes the Fourier transform of f w.r.t. the x variable and in the last step we made use of the change of variables: $(y^0, y^1) \to \big(\varepsilon y^0 - \kappa p^+, \varepsilon y^1 - \kappa p^+)\big)$. Making use of the dominated convergence theorem, we can perform the limit $\varepsilon \searrow 0$, obtaining

$$\lim_{\varepsilon \searrow 0} (2\pi)^{-2} \int d^2x\, d^2y\, e^{-ixy} f(\varepsilon x, \varepsilon y) e^{ipv(x,y)} = e^{-i\frac{1}{2}\kappa((p^0)^2 - (p^1)^2)}. \tag{37}$$

In view of formula (35), this completes the proof of (26) for dense sets of vectors $\Psi_\pm \in \mathcal{H}_\pm$. For arbitrary $\Psi_\pm$ the statement follows by the limiting procedure (10).

The statement (27) concerning the incoming states can be shown using formula (12) and an obvious modification of the above argument. We obtain it however more directly, using formula (26) and definition (11):

$$\begin{aligned} \Psi_+ \overset{\text{in}}{\times}_\kappa \Psi_- = J((J\Psi_+) \overset{\text{out}}{\times}_\kappa (J\Psi_-)) &= J\big(e^{-\frac{i}{2}\kappa(H^2-P^2)}((J\Psi_+) \overset{\text{out}}{\times} (J\Psi_-))\big) \\ &= e^{\frac{i}{2}\kappa(H^2-P^2)}(\Psi_+ \overset{\text{in}}{\times} \Psi_-). \end{aligned} \tag{38}$$

Here in the last step we made use of the fact, shown in [BLS10], that the modular objects of the deformed and undeformed theory coincide. We also exploited the relation $Jg(H,P)J = g(-H,-P)$, valid for any bounded, measurable function g, which follows from formula (2). □

We immediately obtain the following corollary:

Corollary 3.3 *Let S be the scattering matrix of $(\mathcal{R}, U, \Omega)$ and let S_κ be the scattering matrix of $(\mathcal{R}_{Q_\kappa}, U, \Omega)$. Then*

$$S_\kappa = e^{i\kappa(H^2-P^2)} S. \tag{39}$$

In particular, if the original theory is asymptotically complete and non-interacting, and $\operatorname{sp} U = V_+$, *then the deformed theory is asymptotically complete and interacting.*

Proof. Making use of Theorem 3.2 and of the invariance of the scattering operator under translations, we obtain

$$\begin{aligned} S_\kappa(\Psi_+ \overset{\text{out}}{\times}_\kappa \Psi_-) &= \Psi_+ \overset{\text{in}}{\times}_\kappa \Psi_- \\ &= e^{i\frac{1}{2}\kappa(H^2-P^2)}(\Psi_+ \overset{\text{in}}{\times} \Psi_-) \\ &= e^{i\frac{1}{2}\kappa(H^2-P^2)} S(\Psi_+ \overset{\text{out}}{\times} \Psi_-) \\ &= e^{i\kappa(H^2-P^2)} S(\Psi_+ \overset{\text{out}}{\times}_\kappa \Psi_-). \end{aligned} \tag{40}$$

This proves formula (39). The property of asymptotic completeness is preserved under the deformation, since $e^{i\kappa(H^2-P^2)}$ is a unitary. If $\mathcal{H}^{\text{out}} = \mathcal{H}$, $S = I$ and $\operatorname{sp} U = V_+$ then $e^{i\kappa(H^2-P^2)}$ is not a constant multiple of identity on $\mathcal{H}^{\text{out}}$ i.e. the deformed theory is interacting. □

We have shown in Proposition 2.4 that any Borchers triple $(\mathcal{R}, U, \Omega)$ with a unique vacuum vector Ω and non-trivial single-particle subspaces $\mathcal{H}_+ \cap [c\Omega]^\perp$,

$\mathcal{H}_- \cap [c\Omega]^\perp$, gives rise to an asymptotic Borchers triple $(\mathcal{R}^{\mathrm{as}}, U^{\mathrm{as}}, \Omega^{\mathrm{as}})$ which is asymptotically complete, non-interacting and s.t. $\mathrm{sp}\, U^{\mathrm{as}} = V_+$. Hence, in view of the above corollary, the deformation of $(\mathcal{R}^{\mathrm{as}}, U^{\mathrm{as}}, \Omega^{\mathrm{as}})$ gives rise to an interacting, asymptotically complete theory.

Interestingly, there exists a large class of Borchers triples which are unitarily equivalent to their asymptotic Borchers triples (in the sense of [BLS10]). They give rise to interacting theories with a complete particle interpretation by a direct application of the deformation procedure. In the next section we show that the Borchers triples associated with chiral conformal field theories belong to this class.

4 Asymptotic completeness of chiral nets

In this section we consider a specific class of Borchers triples resulting from chiral nets. We will show that such triples are asymptotically complete, what is at first sight surprising in view of the rich family of superselection sectors in chiral conformal field theory [GF93]. We recall, however, that in the present case particles (or rather 'waves') are composite objects, so they may contain (pairs of) excitations from other sectors.

We start from the definition of a local net on $\mathbb{R}$, denoted by $(\hat{\mathcal{A}}, \hat{U}, \hat{\Omega})$. It consists of

(a) a map $\mathbb{R} \supset \mathcal{I} \to \hat{\mathcal{A}}(\mathcal{I}) \subset B(\hat{\mathcal{H}})$, from open, bounded intervals to von Neumann algebras on $\hat{\mathcal{H}}$ s.t.

$$\hat{\mathcal{A}}(\mathcal{I}) \subset \hat{\mathcal{A}}(\mathcal{J}) \text{ for } \mathcal{I} \subset \mathcal{J}, \tag{41}$$

$$[\hat{\mathcal{A}}(\mathcal{I}), \hat{\mathcal{A}}(\mathcal{J})] = 0 \text{ for } \mathcal{I} \cap \mathcal{J} = \phi; \tag{42}$$

(b) a unitary representation $\mathbb{R} \ni s \to \hat{U}(s)$ s.t.

$$\mathrm{sp}\, \hat{U} \subset \mathbb{R}_+, \tag{43}$$

$$\hat{U}(s)\hat{\mathcal{A}}(\mathcal{I})\hat{U}(s)^{-1} = \hat{\mathcal{A}}(\mathcal{I}+s) \ \text{ for } s \in \mathbb{R}; \tag{44}$$

(c) a unique (up to a phase) unit vector $\hat{\Omega}$, invariant under the action of $\hat{U}$, which is cyclic for any local algebra $\hat{\mathcal{A}}(\mathcal{I})$.

We remark, that there are many examples of local nets on $\mathbb{R}$. They arise, in particular, from conformal field theories on S^1 (see e.g. [BMT88, KL04] for concrete examples). Given a theory on S^1 one obtains a net on the compactified real line by means of the Cayley transform. Its restriction to the real line gives rise to a local net on $\mathbb{R}$ with properties specified above.

Let $(\hat{\mathcal{A}}_1, \hat{U}_1, \hat{\Omega}_1)$ and $(\hat{\mathcal{A}}_2, \hat{U}_2, \hat{\Omega}_2)$ be two local nets on $\mathbb{R}$, and let $\hat{\mathcal{H}}_1$, $\hat{\mathcal{H}}_2$ be the respective Hilbert spaces. We identify the two real lines with the lightlines $x + t = 0$ and $x - t = 0$ in $\mathbb{R}^2$. To construct a local net on $\mathbb{R}^2$, acting on the tensor product space $\mathcal{H} = \hat{\mathcal{H}}_1 \otimes \hat{\mathcal{H}}_2$, we first specify the unitary representation of translations:

$$U(x) := \hat{U}_1\left(\frac{1}{\sqrt{2}}(x^0 - x^1)\right) \otimes \hat{U}_2\left(\frac{1}{\sqrt{2}}(x^0 + x^1)\right). \tag{45}$$

Let $\alpha_x(\,\cdot\,) := U(x)\ \cdot\ U(x)^*$ be the corresponding group of translation automorphisms and let $\alpha_x^{(1/2)}(\,\cdot\,) := \hat{U}_{1/2}(x)\ \cdot\ \hat{U}_{1/2}(x)^*$. Then there holds

$$\alpha_x(A_1 \otimes A_2) = \alpha^{(1)}_{\frac{1}{\sqrt{2}}(x^0 - x^1)}(A_1) \otimes \alpha^{(2)}_{\frac{1}{\sqrt{2}}(x^0 + x^1)}(A_2), \quad A_1 \in \hat{\mathcal{A}}_1,\ A_2 \in \hat{\mathcal{A}}_2. \tag{46}$$

Any double cone in $\mathbb{R}^2$ can be written as a product of two intervals on lightlines $\mathcal{I}_1 \times \mathcal{I}_2$. We define the corresponding local algebra by $\mathfrak{A}(\mathcal{I}_1 \times \mathcal{I}_2) := \hat{\mathcal{A}}_1(\mathcal{I}_1) \otimes \hat{\mathcal{A}}_2(\mathcal{I}_2)$. Setting $\Omega = \hat{\Omega}_1 \otimes \hat{\Omega}_2$, we obtain a triple $(\mathfrak{A}, U, \Omega)$, which we call a chiral net on $\mathbb{R}^2$. Defining

$$\mathcal{R} := \bigvee_{\mathcal{I}_1 \times \mathcal{I}_2 \subset \mathcal{W}} \mathfrak{A}(\mathcal{I}_1 \times \mathcal{I}_2), \tag{47}$$

we arrive at a Borchers triple $(\mathcal{R}, U, \Omega)$ associated with $(\mathfrak{A}, U, \Omega)$.

We will show that this Borchers triple is unitarily equivalent to its asymptotic Borchers triple $(\mathcal{R}^{\mathrm{as}}, U^{\mathrm{as}}, \Omega^{\mathrm{as}})$ and therefore, by Proposition 2.4 asymptotically complete and non-interacting. To this end, we determine the asymptotic fields in the following proposition:

Proposition 4.1 *For any $A_1 \in \hat{\mathcal{A}}_1(\mathcal{I}_1)$, $A_2 \in \hat{\mathcal{A}}_2(\mathcal{I}_2)$ there holds*

$$\Phi_+^{\mathrm{out/in}}(A_1 \otimes A_2) \ = \ A_1 \otimes (\hat{\Omega}_2 | A_2 \hat{\Omega}_2) I, \tag{48}$$

$$\Phi_-^{\mathrm{out/in}}(A_1 \otimes A_2) \ = \ (\hat{\Omega}_1 | A_1 \hat{\Omega}_1) I \otimes A_2. \tag{49}$$

(In the cases of Φ_+^{out} and Φ_-^{in} it is assumed that $\mathcal{I}_1 \times \mathcal{I}_2 \subset \mathcal{W}$. In the remaining cases $\mathcal{I}_1 \times \mathcal{I}_2 \subset \mathcal{W}'$).

Proof. We consider only Φ_+^{out}, as the remaining cases are analogous. From its definition and formula (46), we obtain

$$\Phi_+^{\mathrm{out}}(A_1 \otimes A_2) = \operatorname*{s-lim}_{T \to \infty} A_1 \otimes \int dt\, h_T(t) \alpha^{(2)}_{\sqrt{2}t}(A_2). \tag{50}$$

We denote $A_2(h_T) := \int dt\, h_T(t) \alpha^{(2)}_{\sqrt{2}t}(A_2)$. This sequence has the following properties:

$$\operatorname*{s-lim}_{T \to \infty} A_2(h_T) \hat{\Omega}_2 \ = \ (\hat{\Omega}_2 | A_2 \hat{\Omega}_2) I, \tag{51}$$

$$\lim_{T \to \infty} \| [A_2(h_T), A] \| \ = \ 0, \text{ for any } A \in \hat{\mathcal{A}}_2(\mathcal{I}), \tag{52}$$

where $\mathcal{I}$ is an arbitrary open, bounded interval. The first identity above follows from the mean ergodic theorem and the fact that $\hat{\Omega}_2$ is the only vector invariant under the action of $\hat{U}_2$. The second equality is a consequence of the locality property (41). Now since $[\hat{\mathcal{A}}_2(\mathcal{I}) \hat{\Omega}_2] = \hat{\mathcal{H}}_2$, we obtain from relations (51), (52)

$$\operatorname*{s-lim}_{T \to \infty} A_2(h_T) = (\hat{\Omega}_2 | A_2 \hat{\Omega}_2) I. \tag{53}$$

This completes the proof. $\square$

Now we can easily prove the main result of this section:

Theorem 4.2 *Any Borchers triple $(\mathcal{R}, U, \Omega)$ associated with a chiral net on $\mathbb{R}^2$ is unitarily equivalent to its asymptotic Borchers triple $(\mathcal{R}^{\mathrm{as}}, U^{\mathrm{as}}, \Omega^{\mathrm{as}})$. More precisely, there exists a unitary map $W : \mathcal{H}^{\mathrm{as}} \to \mathcal{H}$ s.t. $W\mathcal{R}^{\mathrm{as}} = \mathcal{R}W$, $WU^{\mathrm{as}}(x) = U(x)W$ and $W\Omega^{\mathrm{as}} = \Omega$.*

Proof. By cyclicity of Ω under the action of $\mathcal{R}$ and the mean ergodic theorem (cf. formula (5)), there holds

$$\mathcal{H}_{\pm} = [\,\Phi_{\pm}^{\mathrm{out}}(F)\Omega \,|\, F \in \mathcal{R}\,]. \tag{54}$$

Thus, applying Proposition 4.1, and exploiting the cyclicity of $\hat{\Omega}_1$, $\hat{\Omega}_2$ under the action of the respective local algebras, we obtain

$$\mathcal{H}_+ = \hat{\mathcal{H}}_1 \otimes [c\hat{\Omega}_2], \tag{55}$$

$$\mathcal{H}_- = [c\hat{\Omega}_1] \otimes \hat{\mathcal{H}}_2. \tag{56}$$

Recalling that $\mathcal{H}^{\mathrm{as}} = \mathcal{H}_+ \otimes \mathcal{H}_-$ and $\mathcal{H} = \hat{\mathcal{H}}_1 \otimes \hat{\mathcal{H}}_2$, we define a unitary map $W : \mathcal{H}^{\mathrm{as}} \to \mathcal{H}$, extending by linearity the relation

$$W((\Psi_1 \otimes \hat{\Omega}_2) \otimes (\hat{\Omega}_1 \otimes \Psi_2)) = \Psi_1 \otimes \Psi_2, \quad \Psi_1 \in \hat{\mathcal{H}}_1, \Psi_2 \in \hat{\mathcal{H}}_2. \tag{57}$$

It is readily verified that

$$WU^{\mathrm{as}}(x) = U(x)W, \tag{58}$$

$$W\Omega^{\mathrm{as}} = \Omega, \tag{59}$$

$$\begin{aligned} &W\{\Phi_+^{\mathrm{out}}(A_1 \otimes A_2)|_{\mathcal{H}_+} \otimes \Phi_-^{\mathrm{in}}(B_1 \otimes B_2)|_{\mathcal{H}_-}\} \\ &\qquad = (\hat{\Omega}_1|B_1\hat{\Omega}_1)(\hat{\Omega}_2|A_2\hat{\Omega}_2)\{A_1 \otimes B_2\}W, \end{aligned} \tag{60}$$

where $A_1 \otimes A_2$, $B_1 \otimes B_2$ comply with the assumptions of Proposition 4.1. By definition (47), the elements in the curly bracket on the r.h.s. of (60) generate $\mathcal{R}$. Making use of this fact and of the identities $\Phi_+^{\mathrm{out}}(F)|_{\mathcal{H}_+} = P_+F|_{\mathcal{H}_+}$, $\Phi_-^{\mathrm{in}}(F)|_{\mathcal{H}_-} = P_-F|_{\mathcal{H}_-}$, where $F \in \mathcal{R}$, we obtain that the double commutant of the set of elements in the curly bracket on the l.h.s. of (60) coincides with $\mathcal{R}^{\mathrm{as}}$. Hence $W\mathcal{R}^{\mathrm{as}} = \mathcal{R}W$, which concludes the proof. □

In view of the above theorem, we obtain from Proposition 2.4:

Corollary 4.3 *Any Borchers triple $(\mathcal{R}, U, \Omega)$, associated with a chiral net, gives rise to an asymptotically complete, non-interacting theory. Moreover, $\mathrm{sp}\, U = V_+$.*

Hence, by Corollary 3.3, deformations of such Borchers triples give rise to asymptotically complete, interacting theories.

5 Concluding remarks

In this paper we applied the deformation method, developed in [BLS10], to two-dimensional massless theories. We have shown that the deformation procedure not only introduces interaction, as expected from the massive case [GL08, BS08], but also preserves the property of asymptotic completeness.

By deforming chiral conformal field theories, we obtained a large class of wedge-local theories which are both interacting and asymptotically complete. As the resulting scattering matrices are Lorentz invariant, one can hope for the existence of local observables in these models. We recall that negative results, concerning this issue, have so far been established only in spacetimes of dimension larger than two [BLS10].

A large part of our investigation was devoted to scattering theory for wedge-local theories of massless particles in two-dimensional spacetime. It turned out that the collision theory developed in [Bu75] for local nets of observables generalizes naturally to the wedge-local framework: To construct the two-body scattering matrix, it suffices to know the Borchers triple. It is an interesting open problem, if this fact remains true for scattering of massless particles in spacetimes of higher dimension. We recall that for local nets of observables collision theory of massless excitations is well understood in spacetimes of even dimension [Bu77].

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