# SUB-EXPONENTIAL STABILITY FOR THE BEAM EQUATION 

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#### Abstract

We consider a one-parameter family of beam equations with Hamiltonian non-linearity in one space dimension under periodic boundary conditions. In a unified functional framework we study the long time evolution of initial data in two categories of differentiability: (i) a subspace of Sobolev regularity, (ii) a subspace of infinitely many differentiable functions which is strictly contained in the Sobolev space but which strictly contains the Gevrey one. In both cases we prove exponential type times of stability. The result holds for almost all mass parameters and it is obtained by combining normal form techniques with a suitable Diophantine condition weaker than the one proposed by Bourgain. This is the first result of this kind in Sobolev regularity for a degenerate equation, where only one parameter is used to tune the linear frequencies of oscillations.


## Contents

1. Introduction ..... 1
2. Spaces of Hamiltonians and conjugacies ..... 6
3. Small divisors ..... 14
4. Homological equation ..... 17
5. A Birkhoff normal form step ..... 19
6. The iterative scheme ..... 22
7. Times of stability ..... 26
Appendix A. Technical Lemmata ..... 32
References ..... 40

## 1. Introduction

In this paper we consider the one dimensional beam equation

$$
\begin{equation*}
\partial_{t t} \psi+\partial_{x x x x} \psi+m \psi+f(\psi)=0, \tag{1.1}
\end{equation*}
$$

where $\psi=\psi(t, x), x \in \mathbb{T}:=\mathbb{R} / 2 \pi \mathbb{Z}$ and $\mathrm{m} \in[1,2]$. The nonlinearity $f(\psi)$ has the form

$$
\begin{equation*}
f(\psi):=\left(\partial_{\psi} F\right)(\psi) \tag{1.2}
\end{equation*}
$$

for some function $F(y)$ which is real analytic in $y$ in a neighbourhood of $y=0$. We shall assume that $F$ has a zero in $y=0$ and, by analyticity, for some $R>0$ we have

$$
\begin{equation*}
F(y)=\sum_{d=3}^{\infty} F^{(d)} y^{d}, \quad|F|_{R}:=\sum_{d=3}^{\infty}\left|F^{(d)}\right| R^{d}<+\infty . \tag{1.3}
\end{equation*}
$$

We are interested in stability times of initial data, belonging to some Hilbert subspaces $E$ of $L^{2}\left(\mathbb{T}, \mathbb{R}^{2}\right)$. Let $\psi_{0}:=\psi(0)$ and $\psi_{1}:=\partial_{t} \psi(t)_{\mid t=0}$ be the initial conditions of respectively position and velocity of $\psi$ at time $t=0$, by classical local (in time) well-posedness theory we know that given initial data $\left|\left(\psi_{0}, \psi_{1}\right)\right|_{E} \leq \delta$, the solution exists for a certain time $T=T(\delta)>0$ depending only on $\delta$. At least in the case $\delta \ll 1$ we are interested in understanding the optimal time of stability of solutions, i.e. we want to give a lower bound on $T(\delta)$ which is the supremum of times $t$ such that for any $\left|\left(\psi_{0}, \psi_{1}\right)\right|_{E} \leq \delta \ll 1$ one has $\left(\psi(t, x), \partial_{t} \psi(t, x)\right) \in$ $E$ with $\left|\left(\psi(t, \cdot), \partial_{t} \psi(t, \cdot)\right)\right|_{E} \leq 2 \delta$. We refer to this time as stability time. Since the nonlinearity in (1.1) is
quadratic, local theory provides the trivial lower bound $T(\delta) \gtrsim \delta^{-1}$. Moreover, since we are working on a compact domain, no dispersive effects can help to control the behaviour of solutions for longer times. A fruitful approach, in this case, is to reduce the "size" of the non linearity through a convenient normal form analysis.

In this line of thoughts, a fundamental feature of equation 1.1 is that, under a convenient variables' change, we can write it as a Hamiltonian system whose corresponding Hamiltonian has an elliptic fixed point at the origin. Passing to the Fourier side and in appropriate elliptic coordinates $u_{j}$ the Hamiltonian has the form

$$
\begin{equation*}
H=\sum_{j \in \mathbb{Z}} \omega_{j}\left|u_{j}\right|^{2}+O\left(u^{3}\right), \quad \omega_{j}:=\omega_{j}(\mathrm{~m})=\sqrt{j^{4}+\mathrm{m}} \tag{1.4}
\end{equation*}
$$

where $O\left(u^{3}\right)$ denotes a non linearity with a zero at the origin of order at least 3. In this Hamiltonian view point, through suitable symplectic change of coordinates, we shall pull the Hamiltonian $H$ back to Birkhoff Normal Form (BNF)

$$
\widetilde{H}=\sum_{j \in \mathbb{Z}} \omega_{j}\left|u_{j}\right|^{2}+\mathfrak{Z}+\mathfrak{R},
$$

where $\mathfrak{Z}$ depends only on the "actions" $\left|u_{j}\right|^{2}$ (and does not affect the dynamics) while $\mathfrak{R}$ has an high degree of homogeneity $\sim O\left(|u|^{\mathbb{N}+2}\right)$ for some natural $\mathrm{N} \gg 1$. Then, the natural time of stability of the flow of $\widetilde{H}$ becomes $T(\delta) \sim O\left(\delta^{-\mathbb{N}}\right)$.
The crucial difficulties in this approach regard the regularity of the phase space of initial data and interactions among linear frequencies of oscillations $\omega_{j}$.

The problem of long time stability has been widely studied in particular for Sobolev initial data. For instance, we mention the seminal works [Bam03] for the Klein-Gordon equation, and Bambusi-Grébert [BG06], where polynomial bounds for a wide class of tame-modulus PDEs are proved. More precisely, it is shown that for any $\mathrm{N} \gg 1$ there exists $p(\mathrm{~N})$ (tending to infinity as $\mathrm{N} \rightarrow \infty$ ) such that for all $p \geq$ $p(\mathrm{~N})$ and all $\delta$-small initial data in $H^{p}$ one has $T \geq C(\mathrm{~N}, p) \delta^{-\mathrm{N}}$, provided $\delta<\delta_{0}(\mathrm{~N}, p)$. Similar results were also proved for the Klein-Gordon equation on tori and Zoll manifolds in [DS04], [DS06], [BDGS07]. More recently, similar results have been obtained also for nonlinearities containing derivatives, see [YZ14], [Del12], [Del15], [BD18], [FI21]. The above mentioned results are deeply based on the requirement that N -waves interactions are non-resonant. In other words one should impose some diophantine conditions on the vector $\omega=\left(\omega_{j}\right)_{j \in \mathbb{Z}}$ of linear frequencies of oscillations in order to ensure suitable lower bounds on the quantity $\omega \cdot \ell$ with $\ell \in \mathbb{Z}^{\mathbb{Z}},|\ell| \leq \mathrm{N}$. Such quantities arise as the eigenvalues of an appropriate linear operator that must be inverted at each step of the Birkhoff procedue. These arithmetic conditions are typically achieved by exploiting the presence of some "parameter" which modulates the linear frequencies, either "internal", such as the mass parameter $m$ for the Klein-Gordon or Beam equations (see (1.4)), the capillarity of the fluid in the case of water waves, or "external", as convolutions and multiplicative potentials for the Schrödinger equation for example. However, there are cases in which it is not possible to get such a lower bound at any order $|\ell| \leq \mathbb{N}$ for any $\mathrm{N} \in \mathbb{N}$, for instance equations posed in high dimensional generic tori. Nevertheless normal form approach has been successfully applied also to obtain "partial results", i.e. time of stability $T(\delta) \sim O\left(\delta^{-\overline{\widetilde{N}}}\right)$ for some fixed $\overline{\mathrm{N}} \geq 1$. We quote for instance [Del09], [FGI20] on the Klein-Gordon equation, [Ime16], [BFGI21] on the Beam equation in high space dimension, [FM22] on the Schrödinger on generic tori, [HIT16], [IT17], [IP19], [BFP22], [BFF21] on the water waves equation (see also [FIM22] on a different fluid model).

All the results mentioned above regard polynomial stability times in Sobolev spaces. Passing from polynomial estimates to exponential-type ones is not trivial and it is related to the regularity of initial data. Faou and Grébert in [FG13] made a first step forward in this direction, by considering the case of analytic initial data, proving super-exponential bounds of the form $T \geq e^{\ln \left(\frac{1}{\delta}\right)^{1+b}}, b>0$, for classes of NLS equations in $\mathbb{T}^{d}$. See also [CMW20].
The "time-regularity" connection emerges also in finite dimension, where long time behavior of initial data is
carried out through Nekoroshev theory, which gives information over exponentially or superexponentionally long times in the analytic category (see [Nek77, Loc92, BFN20a, GCB16]). In contrast, polynomial stability times are proven in finite differentiability settings, see [Bou11] for $C^{k}, k \in \mathbb{N}$ quasi-convex Hamiltonians and the recent [BMM22], where optimal polynomial stability times are proved for the Hölder steep ones. Also in finite dimension, a sharp BNF theory can be constructed near elliptic equilibria, to get exponential or super-exponential times of stability nearby (see [BFN20b] and references therein).

For the 1-d NLS with convolution potential, a recent achievement is represented by [BMP20a] (see also [BMP19]), where Biasco, Massetti, and Procesi prove exponential-type times of stability, both in Sobolev and Gevrey category, by introducing a suitable functional setting allowing the optimality of time also in finite regularity spaces. It is worth to mentioning that a key ingredient for this result comes from their Diophantine condition, firstly introduced by Bourgain in [Bou05], that allows a control on small divisors which is uniform w.r.t. the dimension of the support of the frequencies. The presence of a convolution potential $V *$, which provides an infinite sequence of parameters $\left(V_{j}\right)_{j \in \mathbb{Z}} \in \ell^{\infty}$ to modulate the frequencies, plays a fundamental role in guaranteeing such arithmetic conditions, and this translates in optimal lower bounds in the divisors, independently of the iterative step. Afterwards, the flexibility of the functional frame of [BMP20a], developed for the stability study, turns out to be pivotal in the study of existence of almost-periodic tori for the same model of NLS [BMP21] in Gevrey regularity. We refer also to the recent result [BMP22] concerning weak and Sobolev almost periodic solutions (see also [BMP20b] for a simple case study).

Following this line of thoughts, in this paper we shall investigate whether it is possible to adapt the functional setting introduced in [BMP20a] to the degenerate case of equation (1.1), where only one physical parameter, the mass m , has to be used for frequency's modulation.

The underlying motivation is to construct a degenerate KAM theory (in the sense of [Rüs01]) for infinite dimensional invariant tori on a model like equation (1.1). A milestone step in this direction is to understand precisely the type of diophantine conditions one is able to impose by moving just $m$ in the frequencies (1.4).

Let us be more precise and introduce our main results. We introduce here the functions spaces we are working on:

$$
\begin{equation*}
H^{s, p}:=\left\{\psi(x)=\sum_{j \in \mathbb{Z}} \psi_{j} e^{\mathrm{i} j x} \in L^{2}(\mathbb{T}, \mathbb{C}):\|\psi\|_{s, p}^{2}:=\sum_{j \in \mathbb{Z}}\left|\psi_{j}\right|^{2}\lfloor j\rfloor^{2 p} e^{2 s \lambda(j)}<+\infty\right\} \tag{1.5}
\end{equation*}
$$

for $s \geq 0, p>1 / 2$ where the weight-function $\lambda: \mathbb{R} \rightarrow \mathbb{R}^{+}$is defined as $\square^{1}$

$$
\begin{align*}
\lambda(y) & :=(\ln (2+\langle y\rangle))^{\mathrm{q}}, & & 1<\mathrm{q} \leq 2 \\
\langle j\rangle & :=\max \{1,|j|\}, & & \lfloor j\rfloor:=\max \{2,|j|\}, \quad j \in \mathbb{Z} \tag{1.6}
\end{align*}
$$

Using the Fourier representation we can identify

$$
\begin{equation*}
\psi(x)=\sum_{j \in \mathbb{Z}} \psi_{j} e^{\mathrm{i} j x} \in L^{2}(\mathbb{T} ; \mathbb{C}) \quad \psi_{j}=\frac{1}{2 \pi} \int_{\mathbb{T}} \psi(x) e^{-\mathrm{i} j x} \mathrm{~d} x \tag{1.7}
\end{equation*}
$$

with its Fourier coefficients, i.e $\underbrace{2}$

$$
\begin{equation*}
L^{2}(\mathbb{T}, \mathbb{C}) \ni \psi(x) \mapsto \psi=\left(\psi_{j}\right)_{j \in \mathbb{Z}} \in \ell^{2}(\mathbb{C}) \tag{1.8}
\end{equation*}
$$

We are interested in understanding the stability times of initial data $\left(\psi_{0}, \psi_{1}\right)$ belonging to some subspace of $H^{2}(\mathbb{T}, \mathbb{R}) \times L^{2}(\mathbb{T}, \mathbb{R})$, which are small w.r.t. an appropriate norm.

[^0]Remark 1.1. Note that, for $s=0$, the space $H^{0, p}$ is the standard Sobolev space $H^{p}(\mathbb{T}, \mathbb{C})$. While for $s>0$, the weight $\lambda$ in 1.6) being logarithmic, the space $H^{s, p}$ is strictly larger than the space of Gevrey functions. However, we are able to guarantee a lifespan for the solutions which is sub-exponential (but super-polynomial) in $1 / \delta$, where $\delta$ is the size of initial conditions, both in the purely Sobolev and subexponential categories $H^{p}$ and $H^{s, p}$ respectively, see Corollary 1.3 and Theorem 1.4 .

In the following we denote by meas : $[1,2] \rightarrow \mathbb{R}^{+}$the Lebesgue probability measure.
Theorem 1.2. (Sobolev stability). Let $s=0, p>1+2^{6}(36)^{2}$ and fix any $0<\gamma<1$. There is a positive measure set $\mathfrak{M}_{\gamma} \subset[1,2]$ with meas $\left([1,2] \backslash \mathfrak{M}_{\gamma}\right)=O(\gamma)$, an absolute constant $\mathrm{c}>0$ such that for any $\mathrm{m} \in \mathfrak{M}$ the following holds. For any

$$
\begin{equation*}
0<\delta \leq \delta_{\mathrm{S}} \gamma^{c p}, \quad \delta_{\mathrm{S}}:=\frac{R}{2^{5}|F|_{R}}, \tag{1.9}
\end{equation*}
$$

and any initial datum $\left(\psi_{0}, \psi_{1}\right) \in H^{0, p+1} \times H^{0, p-1}$ satisfying

$$
\begin{equation*}
\left\|\psi_{0}\right\|_{0, p+1}+\left\|\psi_{1}\right\|_{0, p-1} \leq \frac{\delta}{4} \tag{1.10}
\end{equation*}
$$

the solution $\left(\psi(t), \partial_{t} \psi(t)\right)$ of (1.1) with $\left(\psi(0), \partial_{t} \psi(0)\right)=\left(\psi_{0}, \psi_{1}\right)$ exists and satisfies

$$
\begin{equation*}
\|\psi(t)\|_{0, p+1}+\left\|\partial_{t} \psi(t)\right\|_{0, p-1} \leq 8 \delta, \quad \forall|t| \leq T_{0} \tag{1.11}
\end{equation*}
$$

with

$$
\begin{equation*}
T_{0} \geq \frac{R \gamma^{c p^{2}}}{2|F|_{R} \delta}\left(\frac{\delta_{\mathrm{S}}}{\delta}\right)^{\frac{1}{c}(p-1)^{1 / 3}} . \tag{1.12}
\end{equation*}
$$

As a consequence of the theorem above we get the following.
Corollary 1.3. (Sobolev stability: optimization). Let $\delta_{\mathrm{S}}, \mathrm{c}>0$ be the constants of Theorem 1.2 for any

$$
\delta \leq \bar{\delta}:=\delta_{\mathrm{s}} \gamma^{b}, \quad \mathrm{~b}:=24 \mathrm{c}^{2}\left[2^{6}(36)^{2}\right]^{5 / 3}
$$

and any $\left(\psi_{0}, \psi_{1}\right)$ satisfying

$$
\begin{equation*}
\left\|\psi_{0}\right\|_{0, p+1}+\left\|\psi_{1}\right\|_{0, p-1} \leq \frac{\delta}{4}, \quad p=p(\delta):=1+\left(\frac{1}{24 \mathrm{c}^{2} \ln (1 / \gamma)} \ln \left(\frac{\delta_{\mathrm{S}}}{\delta}\right)\right)^{3 / 5} \tag{1.13}
\end{equation*}
$$

the solution $\left(\psi(t), \partial_{t} \psi(t)\right)$ of (1.1) with $\left(\psi(0), \partial_{t} \psi(0)\right)=\left(\psi_{0}, \psi_{1}\right)$ exists and satisfies

$$
\begin{equation*}
\|\psi(t)\|_{0, p+1}+\left\|\partial_{t} \psi(t)\right\|_{0, p-1} \leq 8 \delta, \quad \forall|t| \leq T_{0} \tag{1.14}
\end{equation*}
$$

with

$$
\begin{equation*}
T_{0} \geq \frac{R}{2|F|_{R} \delta} \exp \left\{\frac{\left.\mathrm{c}(\ln (1 / \gamma))^{-1 / 5}\right)}{\left(24 \mathrm{c}^{2}\right)^{6 / 5}}\left(\ln \left(\delta_{\mathrm{S}} / \delta\right)\right)^{1+\frac{1}{5}}\right\} \tag{1.15}
\end{equation*}
$$

Concerning initial data with sub-exponential decay, belonging to $H^{s, p} s>0$, we prove the following stability result.
Theorem 1.4. (Sub-exponential stability). Let $p>1+1 / 2, s>0,1<\mathrm{q} \leq 2$ and fix any $\gamma>0$. There is a positive measure set $\mathfrak{M}_{\gamma} \subset[1,2]$ such that meas $\left([1,2] \backslash \mathfrak{M}_{\gamma}\right)=O(\gamma)$, an absolute constant $\mathrm{c}>0$ and constants $C_{i}=C_{i}(p, R)>0, i=1,2,3$, such that for any $\mathrm{m} \in \mathfrak{M}$ the following holds. For any

$$
\begin{equation*}
0<\delta \leq \delta_{\mathrm{sE}}:=\min \left\{\frac{1}{C_{1}|F|_{R}} \exp \exp \left(-\left(\frac{\mathrm{c}}{\gamma^{4} s}\right)^{\frac{1}{q-1}}\right), \frac{1}{C_{2}|F|_{R}}\right\}, \tag{1.16}
\end{equation*}
$$

and any $\left(\psi_{0}, \psi_{1}\right) \in H^{s, p+1} \times H^{s, p-1}$ satisfying

$$
\begin{equation*}
\left\|\psi_{0}\right\|_{s, p+1}+\left\|\psi_{1}\right\|_{s, p-1} \leq \frac{\delta}{4} \tag{1.17}
\end{equation*}
$$

the solution $\left(\psi(t), \partial_{t} \psi(t)\right)$ of (1.1) with $\left(\psi(0), \partial_{t} \psi(0)\right)=\left(\psi_{0}, \psi_{1}\right)$ exists and satisfies

$$
\begin{equation*}
\|\psi(t)\|_{s, p+1}+\left\|\partial_{t} \psi(t)\right\|_{s, p-1} \leq 8 \delta, \quad \forall|t| \leq T_{0} \tag{1.18}
\end{equation*}
$$

with

$$
\begin{equation*}
T_{0} \geq C_{3} \frac{\delta_{\mathrm{sE}}}{\delta} \exp \left(\frac{1}{2} \ln \left(\delta_{\mathrm{sE}} / \delta\right)\left(\gamma^{4} s \mathrm{c}^{-1} \ln \ln \left(\delta_{\mathrm{sE}} / \delta\right)\right)^{\frac{\mathrm{q}-1}{2}}\right) \tag{1.19}
\end{equation*}
$$

Remarks on Theorem 1.2, 1.4 and Corollary 1.3. Some remarks are in order.
a) The lack of parameters. As mentioned above, in order to put the Hamiltonian associated to equation (1.1) in a suitable Birkhoff Normal Form, one must require arithmetic conditions on the linear frequencies $\left(\omega_{j}\right)_{j \in \mathbb{Z}}$. The Diophantine condition à la Bourgain reads likg ${ }^{3}$

$$
\mathrm{D}_{\gamma, \mathrm{B}}:=\left\{\omega \in \mathbb{R}^{\mathbb{Z}}:|\omega \cdot \ell| \geq \prod_{n \in \mathbb{Z}} \frac{\gamma}{\left(1+\left|\ell_{n}\right|^{2}\langle n\rangle^{2}\right)^{\tau}}, \forall \ell \in \Lambda: 0<|\ell|<\infty\right\}
$$

where $\langle n\rangle:=\max \{1,|n|\}$ for any $n \in \mathbb{Z}, \gamma, \tau>0$, and $\Lambda$ is a suitable non-resonant sub-lattice of $\mathbb{Z}^{\mathbb{Z}}$ which, in the applications, depends on the frequencies $\omega$. In the case of $\omega_{j} \sim j^{2}+V_{j}$ for an infinite sequence of $\left(V_{j}\right)_{j \in \mathbb{Z}} \in \ell^{\infty}$ as in the model considered in [BMP20a], it has been proved that, for a positive measure set of $V_{j}$, the frequency vector $\omega=\left(j^{2}+V_{j}\right)_{j \in \mathbb{Z}}$ belongs to the Diophantine set $D_{\gamma, \mathrm{B}}$ with the non-resonant sub-lattice $\Lambda:=\mathbb{Z}^{\mathbb{Z}} \backslash\{0\}, \gamma$ and $\tau$ being fixed positive absolute constants. In that case, this translates into uniformity of the scheme w.r.t. the number of frequencies $\omega_{j}$ that are lighted up, and sharp estimates on small divisors. In contrast, in our degenerate case where the frequencies have the form $\omega_{j}=\sqrt{j^{4}+\mathrm{m}}$, the definition of the Diophantine set $D_{\gamma, B}$ is more delicate. The non-resonant sub-lattice $\Lambda$ is strictly contained in $\mathbb{Z}^{\mathbb{Z}} \backslash\{0\}$ and consists of $\ell \in \mathbb{Z}^{\mathbb{Z}}$ such that the function $\omega \cdot \ell$ in not identically zero as function of the mass m . We refer to Section 3 for more details on this, see definition 3.2). In addition to this, the degeneracy of equation (1.1), involving only one parameter for modulating any subset of frequencies, obliges $\tau=\tau(\mathrm{d}(\ell))$ to strongly depend on the "dimension" $\mathrm{d}(\ell)$ of $\ell$ and $\gamma \rightsquigarrow \gamma^{\mathrm{d}(\ell)}, \mathrm{d}(\ell)$ being the number of non zero elements of $\ell \in \mathbb{Z}^{\mathbb{Z}}$ (see Section 3 for the precise definitions). This of course reflects in worst estimates on divisors, which are not uniform in the length $|\ell|$. Remarkably, we are able to control these a priori bad estimates by taking full advantage of the flexibility of our norms, which are extremely effective in keeping sharp track of the constants during the iteration of BNF (see in particular Section 2.4 with Lemmata $2.34 \mid 2.9$. Thus, despite the presence of only one modulation-parameter, Theorem[1.2] entails the same qualitative estimates as the ones in [BMP20a]. As a byproduct a final optimization step can be performed as in [BMP20a], through which we get optimal estimates of exponential type. This is the first result of this kind for a degenerate equation, in Sobolev regularity.
b) Since the norm $\|\cdot\|_{0, p}$ is equivalent to the norm $\|\cdot\|_{H^{p}}:=\|\cdot\|_{L^{2}}+\left\|\partial_{x}^{p} \cdot\right\|_{L^{2}}$ (with equivalence constants independent of $p$ ), we can reformulate condition 1.10) as

$$
\left\|\psi_{0}\right\|_{H^{p+1}}+\left\|\psi_{1}\right\|_{H^{p-1}} \leq \frac{\delta}{8}, \quad\left\|\psi_{0}\right\|_{L^{2}}+\left\|\psi_{1}\right\|_{L^{2}} \leq 2^{-p} \frac{\delta}{8}
$$

We also remark that in [BG06] (see for instance Proposition 1.1 in [BMP20a]) the required smallness condition w.r.t. the classical Sobolev norms reads $\left\|\psi_{0}\right\|_{H^{p+1}}+\left\|\psi_{1}\right\|_{H^{p-1}} \leq \delta \leq \delta_{S} p^{-3 p}$, which is more restrictive with respect to our condition $\delta \leq \delta_{S} \gamma^{p}$. In other words the condition above allows us to work on a slightly bigger ball in $H^{p}$, provided we impose a stronger condition on the $L^{2}$-norm. However our condition is still more restrictive w.r.t. the one in Theorem 1.2 of [BMP20a]. This is again a consequence of our small divisors estimates (see item $a$ ).

[^1]c) The function in 1.13 defining $p=p(\delta)$ is explicitly invertible, so one can rephrase the stability time in terms of the regularity, namely, fixing $H^{0, p}$ with $p$ sufficiently large, all the initial data in the ball of radius $\delta=\delta(p)$ remains in the ball of radius $8 \delta$ for times of order
$$
T_{0} \sim \exp \left\{\mathrm{c}(p-1)^{5 / 3}\left(1+(p-1)^{1 / 3}\right) \ln (1 / \gamma)\right\} \sim\left(\frac{1}{\delta(p)}\right)^{1+(p-1)^{1 / 3}}
$$
d) We stress the fact that dealing with functions belonging to the space $H^{s, p}$ with $\exp \left(\ln ^{q}(\cdot)\right)$-decay is more delicate than dealing with the classical sub-exponential Gevrey ones. In particular the monotonic character of the norm together with the estimates on the solution of the Homological equation require a more refined analysis. See Lemmata A. 1 and A. 7 for instance. We restrict here to this more difficult case. One can easily recover the Gevrey regularity case using the corresponding Lemmata in [BMP20a] and following verbatim our proof. In this case we expect a time of stability like (1.19) with only one logarithm at the exponent.
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## 2. Spaces of Hamiltonians and conjugacies

2.1. Hamiltonian structure of the beam equation. We shall deal with the stability problem through a Birkhoff Normal Form apporach, taking advantage of the Hamiltonian structure enjoyed by the beam equation. In fact, by introducing the variable $v=\partial_{t} \psi$, solutions of equation (1.1) correspond to the flow of

$$
X_{\mathcal{B}}:=\left\{\begin{array}{l}
\partial_{t} \psi=v \\
\partial_{t} v=-\omega^{2} \psi-f(\psi)
\end{array}\right.
$$

where $\omega$ is the Fourier multiplier defined by linearity as

$$
\begin{equation*}
\omega e^{\mathrm{i} j \cdot x}=\omega_{j} e^{\mathrm{i} j \cdot x}, \quad \omega_{j}:=\sqrt{|j|^{4}+\mathrm{m}}, \quad \forall j \in \mathbb{Z}, \quad \mathrm{~m} \in[1,2] \tag{2.1}
\end{equation*}
$$

Observe that

$$
X_{\mathcal{B}} \equiv J \nabla H_{\mathbb{R}}(\psi, v)=J\binom{\partial_{\psi} H_{\mathbb{R}}(\psi, v)}{\partial_{v} H_{\mathbb{R}}(\psi, v)}, \quad J=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]
$$

where the Hamiltonian $H_{\mathbb{R}}: H^{2}(\mathbb{T} ; \mathbb{R}) \times L^{2}(\mathbb{T} ; \mathbb{R}) \rightarrow \mathbb{R}$ is defined through

$$
\begin{equation*}
H_{\mathbb{R}}(\psi, v)=\int_{\mathbb{T}}\left(\frac{1}{2} v^{2}+\frac{1}{2}\left(\omega^{2} \psi\right) \psi+F(\psi)\right) \mathrm{d} x \tag{2.2}
\end{equation*}
$$

and $\nabla:=\left(\partial_{\psi}, \partial_{v}\right)$ denotes its $L^{2}$-gradient. Indeed we have

$$
\begin{equation*}
\mathrm{d} H_{\mathbb{R}}(\psi, v) \cdot W=\Omega_{\mathbb{R}}\left(X_{H_{\mathbb{R}}}(\psi, v), W\right) \tag{2.3}
\end{equation*}
$$

for any $W \in H^{2}(\mathbb{T} ; \mathbb{R}) \times L^{2}(\mathbb{T} ; \mathbb{R})$, where $\Omega_{\mathbb{R}}$ is the non-degenerate symplectic form

$$
\Omega_{\mathbb{R}}:=\int_{\mathbb{T}} d \psi \wedge d v d x, \quad \Omega_{\mathbb{R}}\left(W_{1}, W_{2}\right)=\int_{\mathbb{T}} d \psi \wedge d v\left(W_{1}, W_{2}\right) d x=\int_{\mathbb{T}}\left(\psi_{1} v_{2}-v_{1} \psi_{2}\right) d x
$$

for any $W_{1}=\left(\psi_{1}, v_{1}\right), W_{2}=\left(\psi_{2}, v_{2}\right) \in H^{2}(\mathbb{T} ; \mathbb{R}) \times L^{2}(\mathbb{T} ; \mathbb{R})$. The Poisson brackets between two Hamiltonian $H_{\mathbb{R}}, G_{\mathbb{R}}: H^{2}(\mathbb{T} ; \mathbb{R}) \times L^{2}(\mathbb{T} ; \mathbb{R}) \rightarrow \mathbb{R}$ are defined in the classical manner as

$$
\begin{equation*}
\left\{H_{\mathbb{R}}, G_{\mathbb{R}}\right\}:=\Omega_{\mathbb{R}}\left(X_{H_{\mathbb{R}}}, X_{G_{\mathbb{R}}}\right) . \tag{2.4}
\end{equation*}
$$

Let now

$$
\mathcal{R}=\left\{\left(u^{+}, u^{-}\right) \in H^{1}(\mathbb{T} ; \mathbb{C}) \times H^{1}(\mathbb{T}, \mathbb{C}): u^{-}=\bar{u}^{+}\right\}
$$

and let us define the linear isomorphism

$$
\begin{gather*}
\mathcal{C}: H^{2}(\mathbb{T} ; \mathbb{R}) \times L^{2}(\mathbb{T} ; \mathbb{R}) \rightarrow H^{1}(\mathbb{T} ; \mathbb{C}) \times H^{1}(\mathbb{T} ; \mathbb{C}) \cap \mathcal{R}, \\
{\left[\begin{array}{c}
\psi \\
v
\end{array}\right] \mapsto \mathcal{C}\left[\begin{array}{l}
\psi \\
v
\end{array}\right]=\left[\frac{u}{u}\right], \quad \mathcal{C}:=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}
\omega^{\frac{1}{2}} & \mathrm{i} \omega^{-\frac{1}{2}} \\
\omega^{\frac{1}{2}} & -\mathrm{i} \omega^{-\frac{1}{2}}
\end{array}\right),} \tag{2.5}
\end{gather*}
$$

where $\omega$ is the Fourier multiplier defined in 2.1). The vector field $X_{\mathcal{B}}$ is then pushed forward to the new Hamiltonian one

$$
\begin{equation*}
\mathcal{C}_{*} X_{\mathcal{B}}=X_{H}=(\dot{u}, \dot{\bar{u}}), \quad \dot{u}=-\mathrm{i} \omega u-\frac{\mathrm{i}}{\sqrt{2}} \omega^{-1 / 2} f\left(\omega^{-1 / 2}\left(\frac{u+\bar{u}}{\sqrt{2}}\right)\right)=-\mathrm{i} \partial_{\bar{u}} H(u, \bar{u}) \tag{2.6}
\end{equation*}
$$

where $\partial_{\bar{u}}=\left(\partial_{\operatorname{Re} u}+\mathrm{i} \partial_{\operatorname{Im} u}\right) / 2, \partial_{u}=\left(\partial_{\operatorname{Re} u}-\mathrm{i} \partial_{\operatorname{Im} u}\right) / 2$ and

$$
\begin{equation*}
H(u, \bar{u})=H_{\mathbb{R}}\left(\mathcal{C}^{-1}\left[\frac{u}{u}\right]\right)=\int_{\mathbb{T}} \bar{u} \omega u \mathrm{~d} x+\int_{\mathbb{T}} F\left(\frac{\omega^{-1 / 2}(u+\bar{u})}{\sqrt{2}}\right) d x . \tag{2.7}
\end{equation*}
$$

The (complex) induced 2 -form is

$$
\begin{equation*}
\Omega:=\left(\mathcal{C}^{-1}\right)^{*} \Omega_{\mathbb{R}}=\int_{\mathbb{T}} \mathrm{i} d u \wedge d \bar{u} d x \tag{2.8}
\end{equation*}
$$

which yields, for any $w_{1}=(\xi, \bar{\xi}), w_{2}=(\eta, \bar{\eta}) \in H^{1}(\mathbb{T} ; \mathbb{C}) \times H^{1}(\mathbb{T} ; \mathbb{C}) \cap \mathcal{R}$,

$$
\begin{equation*}
\Omega\left(w_{1}, w_{2}\right)=\int_{\mathbb{T}} \xi \bar{\eta}-\bar{\xi} \eta d x \in \mathbb{R} \tag{2.9}
\end{equation*}
$$

and intrinsically defines the Hamiltonian through

$$
\begin{equation*}
\Omega\left(X_{H}(u), w\right)=\mathrm{d} H(u) \cdot w \tag{2.10}
\end{equation*}
$$

for any $w$. Accordingly, we set the (complex) Poisson brackets as

$$
\{H, G\}:=\Omega\left(X_{H}, X_{G}\right)
$$

where $H=H_{\mathbb{R}} \circ \mathcal{C}^{-1}\left(\operatorname{resp} G=G_{\mathbb{R}} \circ \mathcal{C}^{-1}\right)$ and $X_{H}=\mathcal{C}_{*} X_{H_{\mathbb{R}}}\left(\operatorname{resp} X_{G}=\mathcal{C}_{*} X_{G_{\mathbb{R}}}\right)$ which yields $s^{4}$

$$
\begin{equation*}
\{H, G\}=\mathrm{i} \int_{\mathbb{T}}\left(\partial_{u} G \partial_{\bar{u}} H-\partial_{\bar{u}} G \partial_{u} H\right) \mathrm{d} x . \tag{2.11}
\end{equation*}
$$

2.2. Fourier's representation. The stability result will be a consequence of a normalization procedure that amounts in transforming the Hamiltonian into a suitable normal form (the Birkhoff normal form). In order to set the convenient functional setting, we shall rather work in the space of sequences that correspond to the above functional spaces, by systematically identifying $L^{2}(\mathbb{T}, \mathbb{C})$ with the Banach space $\mathcal{F}\left(\ell^{2}(\mathbb{C})\right)$ of $2 \pi$-periodic functions (recall 1.7)) $u(x)=u(x, t)=\sum_{j \in \mathbb{Z}} u_{j} e^{i j x}$ such that their Fourier's coefficients $\left(u_{j}\right)_{j \in \mathbb{Z}} \in \ell^{2}(\mathbb{C})$.
Then, the Hamiltonian in (2.7) reads

$$
\begin{equation*}
H(u, \bar{u})=\sum_{j \in \mathbb{Z}} \omega_{j}\left|u_{j}\right|^{2}+\sum_{p=3}^{\infty} \sum_{\substack{j_{i} \in \mathbb{Z}, \sigma_{i} \in\{ \pm\} \\ \sum_{i=1}^{p} \sigma_{i} j_{i}=0}} F_{j_{1}, \ldots, j_{p}}^{\sigma_{1} \ldots \sigma_{p}} u_{j_{1}}^{\sigma_{1}} \cdots u_{j_{p}}^{\sigma_{p}}=: \sum_{j \in \mathbb{Z}} \omega_{j}\left|u_{j}\right|^{2}+\mathrm{H} \geq 3, \tag{2.12}
\end{equation*}
$$

where we used the analyticity in the neighborhood of the origin of $F$ to expand the second integral in (2.7) for some coefficients $\left|F_{j_{1}, \ldots, j_{p}}^{\sigma_{1} \ldots \sigma_{p}}\right| \lesssim C^{p}$ for some $C>0$, and, rearranging the sum in multi-index notation we set

[^2]$$
\mathrm{H}_{\geq 3}(u, \bar{u})=\sum_{\substack{\alpha, \beta \in \mathbb{N}^{\mathbb{Z}} \\|\alpha \alpha+| \geq 3 \\ \sum_{j} j\left(\alpha_{j}-\beta_{j}\right)=0}} H_{\alpha, \beta} u^{\alpha} \bar{u}^{\beta} .
$$

Accordingly, by defining

$$
\begin{array}{r}
d u_{j}=\frac{1}{\sqrt{2}}\left(d x_{j}+\mathrm{i} d y_{j}\right), \quad d \bar{u}_{j}=\frac{1}{\sqrt{2}}\left(d x_{j}-\mathrm{i} d y_{j}\right), \\
\frac{\partial}{\partial u_{j}}=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_{j}}-\mathrm{i} \frac{\partial}{\partial y_{j}}\right), \quad \frac{\partial}{\partial \bar{u}_{j}}=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_{j}}+\mathrm{i} \frac{\partial}{\partial y_{j}}\right),
\end{array}
$$

the corresponding 2-form and Hamiltonian vector field read

$$
\begin{equation*}
\mathrm{i} \sum_{j \in \mathbb{Z}} d u_{j} \wedge d \bar{u}_{j}, \quad X_{H}^{(j)}=-\mathrm{i} \frac{\partial}{\partial \bar{u}_{j}} H(u) . \tag{2.13}
\end{equation*}
$$

Note that when some real analytic $H$ admits a holomorphic extension $\widehat{H}$ on some ball of radius $r>0$ in $\ell^{2}(\mathbb{C})$ that is

$$
\left(u_{+}, u_{-}\right) \in B_{r}\left(\ell^{2}(\mathbb{C})\right) \times B_{r}\left(\ell^{2}(\mathbb{C})\right) \rightarrow \widehat{H}\left(u_{+}, u_{-}\right): \quad H(u)=\widehat{H}(u, \bar{u}),
$$

then it admits a Taylor expansion

$$
\widehat{H}\left(u_{+}, u_{-}\right)=\sum_{\alpha, \beta \in \mathbb{N}^{Z}}^{*} H_{\alpha, \beta} u_{+}^{\alpha} u_{-}^{\beta},
$$

where we denote by $\sum^{*}$ the sum restricted to those $\alpha, \beta:|\alpha|+|\beta|<\infty$.
One can see that

$$
\frac{\partial}{\partial \bar{u}_{j}} H(u)=\left.\frac{\partial \widehat{H}\left(u_{+}, u_{-}\right)}{\partial u_{-, j}}\right|_{u_{+}=\bar{u}_{-}=u} .
$$

From now on we shall pass to the Fourier side and work on spaces of weighted sequences. Let $\mathrm{w}=\left(\mathrm{w}_{j}\right)_{j \in \mathbb{Z}}$ be the real sequence (recall (1.6)

$$
\begin{align*}
& \mathrm{w}=\mathrm{w}(s, p):=\left(\lfloor j\rfloor^{p} e^{s \lambda(j)}\right)_{j \in \mathbb{Z}}, \quad \text { (sE) Sub-exponential case, }  \tag{2.14}\\
& \mathrm{w}=\mathrm{w}(p):=\mathrm{w}(p, 0)=\left(\lfloor j\rfloor^{p}\right)_{j \in \mathbb{Z}}, \quad \text { (S) Sobolev case , } \tag{2.15}
\end{align*}
$$

and let us set the Hilbert space

$$
\begin{equation*}
\mathrm{h}_{\mathrm{w}}:=\left\{u:=\left(u_{j}\right)_{j \in \mathbb{Z}} \in \ell^{2}(\mathbb{C}):|u|_{\mathrm{w}}^{2}:=\sum_{j \in \mathbb{Z}} \mathrm{w}_{j}^{2}\left|u_{j}\right|^{2}<\infty\right\}, \tag{2.16}
\end{equation*}
$$

endowed with the scalar product

$$
\begin{equation*}
(u, v)_{\mathbf{h}_{\boldsymbol{w}}}:=\sum_{j \in \mathbb{Z}} \mathrm{w}_{j}^{2} u_{j} \bar{v}_{j}, \quad u, v \in \mathrm{~h}_{\mathrm{w}} . \tag{2.17}
\end{equation*}
$$

Moreover, given $r>0$, we denote by $B_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ the closed ball of radius $r$ centred at the origin of $\mathrm{h}_{\mathrm{w}}$.
In the following we shall systematically identify $2 \pi$-periodic functions with their Fourier coefficients, writing $\mathrm{h}_{\mathrm{w}(s, p)}$ instead of $H^{s, p}$. In particular, with abuse of notation, we shall write (recall (1.5p)

$$
\begin{array}{rlrl}
\|u\|_{s, p} & =|u|_{\mathrm{w}(s, p)}, & \forall u \in H^{s, p} \simeq \mathrm{~h}_{\mathrm{w}(s, p)} \\
\|u\|_{p}:=\|u\|_{0, p}=|u|_{\mathrm{w}(p)}, & \forall u \in H^{0, p} \simeq \mathrm{~h}_{\mathrm{w}(p)} . \tag{S}
\end{array}
$$

2.3. Spaces of Hamiltonians. In this section we introduce a suitable graded Poisson algebra of Hamiltonians that we need in order to prove our main abstract result.
Let $w$ as in 2.14 or 2.15 and let $\star: h_{w} \times h_{w} \rightarrow h_{w}$ be the convolution operation defined as

$$
(f, g) \mapsto f \star g:=\left(\sum_{j_{1}+j_{2}=j} f_{j_{1}} g_{j_{2}}\right)_{j \in \mathbb{Z}}
$$

The map $\star:(f, g) \mapsto f \star g$ is continuous in the following sense:
Lemma 2.1. For $p>1 / 2$ we have

$$
\begin{align*}
|f \star g|_{\mathrm{w}(p, s)} & \leq C_{\mathrm{alg}}(p)|f|_{\mathrm{w}(p, s)}|g|_{\mathrm{w}(p, s)}, & \forall f, g \in \mathrm{~h}_{\mathrm{w}(s, p)},  \tag{2.18}\\
|f \star g|_{\mathrm{w}(p)} & \leq C_{\mathrm{alg}, \mathrm{M}}(p)|f|_{\mathrm{w}(p)}|g|_{\mathrm{w}(p)}, & \forall f, g \in \mathrm{~h}_{\mathrm{w}(p)}, \tag{2.19}
\end{align*}
$$

where

$$
\begin{equation*}
C_{\mathrm{alg}}(p):=8^{p}\left(\sum_{i \in \mathbb{Z}}\langle i\rangle^{-p}\right)^{1 / 2}, \quad C_{\mathrm{alg}, \mathrm{M}}(p):=\sqrt{2} \sqrt{2+\frac{2 p+1}{2 p-1}} \tag{2.20}
\end{equation*}
$$

Proof. The proof works verbatim as the one in Lemma 5.5 of $\left[\right.$ BMP20a] where $\langle j\rangle^{p} e^{s\langle j\rangle^{\theta}+a|j|} \rightsquigarrow\lfloor j\rfloor^{p} e^{s \lambda(j)}$ for the $|\cdot|_{p, s, 0} \equiv|\cdot|_{\mathrm{w}_{p, s}}$ norm (just noticing that $\langle j\rangle \leq\lfloor j\rfloor$ and the sublinearity of $\lambda$ ). The case of the Sobolev norm $\|\cdot\|_{p}$ is the same.

By endowing the space $h_{w}$ in 2.16 with the symplectic structure induced by the symplectic form $\Omega$ in (2.13), we introduce the following class of Hamiltonians.

Definition 2.2. (Admissible Hamiltonians). Let $r>0$ and consider a Hamiltonian $H: B_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \rightarrow \mathbb{R}$ such that there exists a pointwise absolutely convergent power series expansion ${ }^{5}$

$$
\begin{equation*}
H(u)=\sum_{\substack{\alpha, \beta \in \mathbb{N}^{\mathbb{Z}}, 2 \leq|\alpha|+|\beta|<\infty}} H_{\alpha, \beta} u^{\alpha} \bar{u}^{\beta}, \quad u^{\alpha}:=\prod_{j \in \mathbb{Z}} u_{j}^{\alpha_{j}} \tag{2.21}
\end{equation*}
$$

We say that $H$ as in 2.21) is admissible if the following properties hold:
(1) Reality condition:

$$
\begin{equation*}
H_{\alpha, \beta}=\bar{H}_{\beta, \alpha}, \quad \forall \alpha, \beta \in \mathbb{N}^{\mathbb{Z}} \tag{2.22}
\end{equation*}
$$

(2) Momentum conservation:

$$
\begin{equation*}
H_{\alpha, \beta} \neq 0 \quad \Rightarrow \quad \pi(\alpha-\beta):=\sum_{j \in \mathbb{Z}} j\left(\alpha_{j}-\beta_{j}\right)=0 \tag{2.23}
\end{equation*}
$$

Finally, given two admissible Hamiltonians $H, G$ the Poisson brackets are given by

$$
\begin{equation*}
\{H, G\}=\mathrm{i} \sum_{j \in \mathbb{Z}}\left(\partial_{u_{j}} G \partial_{\bar{u}_{j}} H-\partial_{\bar{u}_{j}} G \partial_{u_{j}} H\right) \tag{2.24}
\end{equation*}
$$

Let $\mathcal{A}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ be the the space of admissible Hamiltonians such that the majorant

$$
\begin{equation*}
\underline{H}(u):=\sum_{(\alpha, \beta) \in \mathcal{M}}\left|H_{\alpha, \beta}\right| u^{\alpha} \bar{u}^{\beta} \tag{2.25}
\end{equation*}
$$

is point-wise absolutely convergent on $B_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$, where we set

$$
\begin{equation*}
\mathcal{M}:=\left\{(\alpha, \beta) \in \mathbb{N}^{\mathbb{Z}} \times \mathbb{N}^{\mathbb{Z}}: \pi(\alpha-\beta)=0,|\alpha|+|\beta|<\infty\right\} \tag{2.26}
\end{equation*}
$$

and introduce the following class of Hamiltonians.

[^3]Definition 2.3. (Regular Hamiltonians). We denote by $\mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ the subspace of $\mathcal{A}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ of Hamiltonians $H$ such that

$$
\begin{equation*}
|H|_{\mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)}=|H|_{r, \mathrm{w}}:=r^{-1}\left(\sup _{|u|_{\mathrm{w}} \leq r}\left|X_{\underline{H}}\right|_{\mathrm{w}}\right)<\infty . \tag{2.27}
\end{equation*}
$$

Remark 2.4. We remark the following facts:

- Given two positive sequences $\mathrm{w}=\left(\mathrm{w}_{j}\right)_{j \in \mathbb{Z}}, \mathrm{w}^{\prime}=\left(\mathrm{w}_{j}^{\prime}\right)_{j \in \mathbb{Z}}$ we write that $\mathrm{w} \leq \mathrm{w}^{\prime}$ if the inequality holds point wise, namely

$$
\mathrm{w} \leq \mathrm{w}^{\prime} \quad \Longleftrightarrow \quad \mathrm{w}_{j} \leq \mathrm{w}_{j}^{\prime}, \quad \forall j \in \mathbb{Z} .
$$

In this way if $r^{\prime} \leq r$ and $\mathrm{w} \leq \mathrm{w}^{\prime}$ then $B_{r^{\prime}}\left(\mathrm{h}_{\mathrm{w}^{\prime}}\right) \subseteq B_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$.

- If a Hamiltonian $H$ satisfies (2.22), it means that it is real analytic in the real and imaginary part of $u$.
- If a Hamiltonian $H$ satisfies (2.23) then it Poisson commutes with $\sum_{j \in \mathbb{Z}} j\left|u_{j}\right|^{2}$.
- The Hamiltonian functions being defined modulo a constant term, we shall assume without loss of generality that $H(0)=0$.

Finally, let us consider a regular Hamiltonian $S \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathbf{w}}\right)$ and its flow $\Phi_{S, t}$ which is well-defined (see Lemma 2.15 for details), and let

$$
\begin{equation*}
D_{\omega}:=\sum_{j \in \mathbb{Z}} \omega_{j}\left|u_{j}\right|^{2}, \tag{2.28}
\end{equation*}
$$

and its flow $\phi_{\omega, t}$, where $\omega_{j}=\sqrt{j^{4}+\mathrm{m}}$.
Definition 2.5. (i) The Lie derivative of $H$ along the flow of $S$ is given by

$$
\begin{equation*}
L_{S} H=\frac{d}{d t}\left|t=0 ~ \phi_{S, t}^{*} H(u)=\frac{d}{d t}\right|_{t=0} H\left(\phi_{S, t}(u)\right) . \tag{2.29}
\end{equation*}
$$

(ii) Given $H \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ we define the adjoint action of the Hamiltonian $D_{\omega}$ as the Lie derivative operator

$$
\begin{equation*}
L_{\omega} H: \left.=\frac{d}{d t} \right\rvert\, t=0 \text { 就, }{ }^{*} H=\sum_{(\alpha, \beta) \in \mathcal{M}}-\mathrm{i}(\omega \cdot(\alpha-\beta)) H_{\alpha, \beta} u^{\alpha} \bar{u}^{\beta}, \tag{2.30}
\end{equation*}
$$

where $\mathcal{M}$ is the set of indexes defined in (2.26).
Remark 2.6. (Change of variables). Along the paper we shall study how a Hamiltonian $H$ behaves along the flow of a given regular Hamiltonian S. In fact one has

$$
\begin{aligned}
\frac{d}{d t} H\left(\phi_{S, t}(u)\right)=d H\left(\phi_{S, t}(u)\right) \cdot X_{S}\left(\phi_{S, t}(u)\right) & =\Omega\left(X_{H}\left(\phi_{S, t}(u)\right), X_{S}\left(\phi_{S, t}(u)\right)\right) \\
& =\{H, S\} \circ \phi_{S, t}(u) \stackrel{\sqrt{2.29}}{=}\left(L_{S} H\right) \circ \phi_{S, t}(u)
\end{aligned}
$$

Then (2.29) corresponds to $L_{S} H=\{H, S\}$. Moreover, from the formula above, one formally deduces that the well-known "Lie expansion"

$$
H\left(\phi_{S, t}(u)\right)=e^{L_{S}} H=\sum_{k=0}^{\infty} \frac{t^{k}}{k!} L_{S}^{k} H, \quad L_{S}^{k} H:=\left\{L_{S}^{k-1} H, S\right\}, \forall k \geq 1, \quad L_{S}^{0}=\text { Id }
$$

In our work the crucial point is that all the dependence on the parameters $r, \mathrm{w}$ of the norm in (2.27) can be encoded in the coefficients

$$
\begin{equation*}
c_{r, \mathfrak{w}}^{(j)}(\alpha, \beta):=r^{|\alpha|+|\beta|-2} \frac{\mathrm{w}_{j}^{2}}{\mathrm{w}^{\alpha+\beta}}, \quad \mathrm{w}^{\alpha+\beta}=\prod_{j \in \mathbb{Z}} \mathrm{w}_{j}^{\alpha_{j}+\beta_{j}}, \tag{2.31}
\end{equation*}
$$

defined for any $\alpha, \beta \in \mathbb{N}^{\mathbb{Z}}$ and $j \in \mathbb{Z}$. In view of our choices of the weights in (2.14) and (2.15) we have that the coefficients in (2.31) have the following form:

$$
\begin{array}{ll}
\text { sE) case : } & c_{r, \mathbf{w}}^{(j)}(\alpha, \beta)=r^{|\alpha|+|\beta|-2} \frac{\lfloor j\rfloor^{2 p}}{\prod_{i \in \mathbb{Z}}\lfloor j\rfloor^{p\left(\alpha_{i}+\beta_{i}\right)}} e^{s\left(2 \lambda(j)-\sum_{i \in \mathbb{Z}}\left(\alpha_{i}+\beta_{i}\right) \lambda(i)\right)} ; \\
\text { S) case: } & c_{r, \mathbf{w}}^{(j)}(\alpha, \beta)=r^{|\alpha|+|\beta|-2} \frac{\lfloor j\rfloor^{2 p}}{\prod_{i \in \mathbb{Z}}\lfloor i\rfloor^{p\left(\alpha_{i}+\beta_{i}\right)}} . \tag{2.33}
\end{array}
$$

Remark 2.7. (Basic embeddings of spaces of Hamiltonians). Recalling Remark 2.4 one can notice that if $r^{\prime} \leq r$ and $\mathrm{w} \leq \mathrm{w}^{\prime}$ then $\mathcal{A}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \subseteq \mathcal{A}_{r^{\prime}}\left(\mathrm{h}_{\mathrm{w}^{\prime}}\right)$. In the following (see Proposition 2.9) we give conditions on the parameters that $(r, \mathrm{w}),\left(r^{*}, \mathrm{w}^{\prime}\right)$ (with $r^{*} \leq r$ ) which ensure the (not trivial) inclusion $\mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \subseteq \mathcal{H}_{r^{*}}\left(\mathrm{~h}_{\mathrm{w}^{\prime}}\right)$. That condition will be given in terms of the ratio of the coefficients $c_{r, w}^{(j)}(\alpha, \beta), c_{r^{\prime}, w^{\prime}}^{(j)}(\alpha, \beta)$.
2.4. Properties of regular Hamiltonians. We now collect some properties of the norm in (2.27). For any $H \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ we define a map

$$
B_{1}\left(\ell^{2}\right) \rightarrow \ell^{2}, \quad y=\left(y_{j}\right)_{j \in \mathbb{Z}} \mapsto\left(Y_{H}^{(j)}(y ; r, \mathrm{w})\right)_{j \in \mathbb{Z}}
$$

by setting

$$
\begin{equation*}
Y_{H}^{(j)}(y ; r, \mathrm{w}):=\sum_{(\alpha, \beta) \in \mathcal{M}}\left|H_{\alpha, \beta}\right| \frac{\left(\alpha_{j}+\beta_{j}\right)}{2} c_{r, \mathfrak{w}}^{(j)}(\alpha, \beta) y^{\alpha+\beta-e_{j}} \tag{2.34}
\end{equation*}
$$

where $e_{j}$ is the $j$-th basis vector in $\mathbb{N}^{\mathbb{Z}}$, while the coefficient $c_{r, w}^{(j)}(\alpha, \beta)$ is defined right above in 2.31). The following properties give a systematic way for computing the norm of a given Hamiltonian and its relation w.r.t. another one.

By Lemma 3.1 in [BMP20a] (see also Lemmata 3.3, 3.4 and $A .1$ in [PS22]) we have the following.
Lemma 2.8. Let $r, r^{\prime}>0, \mathrm{w}, \mathrm{w}^{\prime} \in \mathbb{R}_{+}^{\mathbb{Z}}$. The following properties hold.
(1) The norm of $H$ can be expressed as

$$
\begin{equation*}
|H|_{r, w}=\sup _{|y|_{\ell^{2}} \leq 1}\left|Y_{H}(y ; r, \mathrm{w})\right|_{\ell^{2}} . \tag{2.35}
\end{equation*}
$$

(2) Given $H^{(1)} \in \mathcal{H}_{r^{\prime}, w^{\prime}}$ and $H^{(2)} \in \mathcal{H}_{r, w}$, such that $\forall \alpha, \beta \in \mathbb{N}^{\mathbb{Z}}$ and $j \in \mathbb{Z}$ with $\alpha_{j}+\beta_{j} \neq 0$ one has

$$
\begin{equation*}
\left|H_{\alpha, \beta}^{(1)}\right| c_{r^{\prime}, w^{\prime}}^{(j)}(\alpha, \beta) \leq c\left|H_{\alpha, \beta}^{(2)}\right| c_{r, \mathfrak{w}}^{(j)}(\alpha, \beta), \tag{2.36}
\end{equation*}
$$

for some $c>0$, then

$$
\left|H^{(1)}\right|_{r^{\prime}, w^{\prime}} \leq c\left|H^{(2)}\right|_{r, \mathbf{w}} .
$$

The following proposition gathers the immersion properties of the norm $|\cdot|_{r, w(p, s)}$ with respect to the parameters $p, s$.
Proposition 2.9. (Monotonicity). For any $p>1, s>0$ the norm $|\cdot|_{r, w}$ is monotone increasing in $r$. Moreover, letting $r>0$ the following holds.
sE) Consider w as in (2.14). For any $\sigma, s>0$ we have

$$
\begin{equation*}
|H|_{r, w(s+\sigma, p)} \leq|H|_{r, w(s, p)} . \tag{2.37}
\end{equation*}
$$

S) Consider w as in 2.15. For any $p^{\prime}>0, p>1$, we have

$$
\begin{equation*}
|H|_{r, \mathbf{w}\left(p+p^{\prime}\right)} \leq|H|_{r, \mathbf{w}(p)} \tag{2.38}
\end{equation*}
$$

For the moment we omit the proof of the proposition above and we refer the reader to Appendix A
By Proposition 2.1 in [BMP20a] and Lemma 3.5 in [PS22] we have that the scale $\left\{\mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)\right\}_{r>0}$ is a BanachPoisson algebra in the following sense.

Proposition 2.10. (Poisson Brackets). For $0<\rho \leq r$ we have

$$
\begin{equation*}
|\{F, G\}|_{r, \mathrm{w}} \leq 4\left(1+\frac{r}{\rho}\right)|F|_{r+\rho, \mathrm{w}}|G|_{r+\rho, \mathrm{w}} \tag{2.39}
\end{equation*}
$$

2.5. Graded Poisson structure and conjugations. We start by defining a degree decomposition which endows $\mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ with a graded Poisson algebra structure.

Definition 2.11. (Scaling degree). Given $\mathrm{d} \in \mathbb{N}$, let $\mathcal{H}^{(\mathrm{d})}$ be the vector space of homogeneous polynomials of degree $\mathrm{d}+2$, that is admissible Hamiltonians of the form

$$
\sum_{\substack{(\alpha, \beta) \in \mathcal{M} \\|\alpha|+|\beta|=\mathrm{d}+2}} H_{\alpha, \beta} u^{\alpha} \bar{u}^{\beta}
$$

We shall say that a Hamiltonian $H$ has scaling degree $\geq \mathrm{d}=\mathrm{d}(H)$ if

$$
H \in \mathcal{H}^{(\geq \mathrm{d})}=\mathcal{H}^{(\mathrm{d})} \oplus_{h>\mathrm{d}} \mathcal{H}^{(h)}
$$

Accordingly, we shall define projections associated with this direct sum decomposition and write

$$
\begin{equation*}
\Pi^{(\mathrm{d})} H=\sum_{\substack{(\alpha, \beta) \in \mathcal{M} \\|\alpha|+|\beta|=\mathrm{d}+2}} H_{\alpha, \beta} u^{\alpha} \bar{u}^{\beta}, \quad \Pi^{(>\mathrm{d})} H=\sum_{\substack{(\alpha, \beta) \in \mathcal{M} \\|\alpha|+|\beta|>\mathrm{d}+2}} H_{\alpha, \beta} u^{\alpha} \bar{u}^{\beta} \tag{2.40}
\end{equation*}
$$

We say that $\mathrm{d}(0)=+\infty$.
Remark 2.12. With this definitions, quadratic Hamiltonians have scaling degree 0. Essentially $H$ has scaling degree d if and only if it has a zero of order $\mathrm{d}+2$ at zero.

Definition 2.11 produces a graded Poisson algebra structure. Moreover one has the following result.
Lemma 2.13. The projection operators are continuos. In particular, the following hold.
(i) If $H \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ with $\mathrm{d}(H)=\mathrm{d}$, then one has

$$
\begin{equation*}
\left|\Pi^{(\mathrm{d})} H\right|_{r, \mathrm{w}} \leq|H|_{r, \mathrm{w}}, \quad\left|\Pi^{(>\mathrm{d})} H\right|_{r, \mathrm{w}} \leq|H|_{r, \mathrm{w}} \tag{2.41}
\end{equation*}
$$

(ii) If $H \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ with $\mathrm{d}(H) \geq \mathrm{d}$, then for all $r^{\prime} \leq r$ one has

$$
|H|_{r^{\prime}, \mathrm{w}} \leq\left(\frac{r^{\prime}}{r}\right)^{\mathrm{d}}|H|_{r, \mathrm{w}}
$$

Proof. (i) We only prove the first in 2.41 . The estimate for $\Pi^{(>d)}$ follows similarly.
By absolute convergence of $H$, we can rearrange the terms and write $H=\sum_{\mathrm{d} \geq 0} \Pi^{(\mathrm{d})} H$ where each term reads as in 2.40 . In general, for any Hamiltonian $H$ by definition of majorant norm, we have that

$$
X_{\underline{H}}^{(j)}(u)=-\mathrm{i} \sum_{(\alpha, \beta) \in \mathcal{M}}\left|H_{\alpha, \beta}\right| \beta_{j} u^{\alpha} \bar{u}^{\beta-e_{j}}
$$

which trivially yields $\left|X_{\underline{H}}(u)\right|_{\mathrm{w}} \leq\left|X_{\underline{H}}(\underline{u})\right|_{\mathrm{w}}$, where $\underline{u}=\left(\left|u_{j}\right|\right)_{j \in \mathbb{Z}}$. Observing that when evaluating the supremum of $X_{\underline{H}}$ over $|u|_{\mathrm{w}} \leq r$ we can restrict to the case in which $u=\left(u_{j}\right)_{j \in \mathbb{Z}}$ has positive real components, we get

$$
|H|_{r, \mathrm{w}}=r^{-1} \sup _{|u|_{\mathrm{w}} \leq r}\left|\left(W_{H}^{j}(u)\right)_{j \in \mathbb{Z}}\right|_{\mathrm{w}}, \quad W_{H}^{j}(u)=\sum_{(\alpha, \beta) \in \mathcal{M}}\left|H_{\alpha, \beta}\right| \boldsymbol{\beta}_{j}|u|^{\alpha+\beta-e_{j}}
$$

Then inequality 2.41 follows trivially from $W_{\Pi^{(\mathrm{d})} H}^{j}(u) \leq W_{H}^{j}(u)$.
(ii) By item (2) in Lemma 2.8, it suffices to observe that

$$
\frac{c_{r^{\prime}, \mathrm{w}(\alpha, \beta)}^{j}}{c_{r, \mathrm{w}}(\alpha, \beta)}=\left(\frac{r^{\prime}}{r}\right)^{|\alpha|+|\beta|-2} \leq\left(\frac{r^{\prime}}{r}\right)^{\mathrm{d}+2-2}
$$

This concludes the proof.
Remark 2.14. If $F$ and $G$ are Hamiltonians in $\mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ with scaling degree $\mathrm{d}_{1}, \mathrm{~d}_{2}$ respectively, then the Poisson $\{F, G\}$ has scaling degree equal to $\mathrm{d}_{1}+\mathrm{d}_{2}$. In general, if the scaling degrees are $\geq \mathrm{d}_{1}, \mathrm{~d}_{2}$, then the scaling degree of $\{F, G\}$ is $\geq \mathrm{d}_{1}+\mathrm{d}_{2}$.

The following Lemma guarantees that the flow of a regular Hamiltonian is well-posed on $\mathrm{h}_{\mathrm{w}}$. Moreover it shows how regular Hamiltonians changes under conjugation through flows.

Lemma 2.15. (Hamiltonian flow). Let $0<\rho<r$, and $S \in \mathcal{H}_{r+\rho}\left(\mathrm{h}_{\mathbf{w}}\right)$ with

$$
\begin{equation*}
|S|_{r+\rho, w} \leq \delta:=\frac{\rho}{8 e(r+\rho)}, \tag{2.42}
\end{equation*}
$$

Then the time 1-Hamiltonian flow $\Phi_{S}^{1}: B_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \rightarrow B_{r+\rho}\left(\mathrm{h}_{\mathrm{w}}\right)$ is well defined, analytic, symplectic with

$$
\begin{equation*}
\sup _{u \in B_{r}\left(\mathbf{h}_{\mathbf{w}}\right)}\left|\Phi_{S}^{1}(u)-u\right|_{\mathbf{h}_{\mathbf{w}}} \leq(r+\rho)|S|_{r+\rho, w} \leq \frac{\rho}{8 e} \tag{2.43}
\end{equation*}
$$

Moreover, for any $H \in \mathcal{H}_{r+\rho}\left(\mathrm{h}_{\mathrm{w}}\right)$ we have that $H \circ \Phi_{S}^{1}=e^{L_{S}} H \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ and

$$
\begin{align*}
&\left|e^{L_{S}} H\right|_{r, w} \leq 2|H|_{r+\rho, \mathrm{w}},  \tag{2.44}\\
&\left|\left(e^{L_{S}}-\mathrm{id}\right) H\right|_{r, \mathrm{w}} \leq \delta^{-1}|S|_{r+\rho, \mathrm{w}}|H|_{r+\rho, \mathrm{w}}  \tag{2.45}\\
&\left|\left(e^{L_{S}}-\mathrm{id}-\{S, \cdot\}\right) H\right|_{r, \mathrm{w}} \leq \frac{1}{2} \delta^{-2}|S|_{r+\rho, \mathrm{w}}^{2}|H|_{r+\rho, \mathrm{w}} \tag{2.46}
\end{align*}
$$

More generally for any $h \in \mathbb{N}$ and any sequence $\left(c_{k}\right)_{k \in \mathbb{N}}$ with $\left|c_{k}\right| \leq 1 / k!$, we have

$$
\begin{equation*}
\left|\sum_{k \geq h} c_{k} L_{S}^{k}(H)\right|_{r, \mathrm{w}} \leq 2|H|_{r+\rho, \mathrm{w}}\left(|S|_{r+\rho, \mathrm{w}} / 2 \delta\right)^{h} \tag{2.47}
\end{equation*}
$$

Proof. Follows verbatim by Lemma 2.1 in [BMP20a] with $\eta=0$ and ad ${ }_{S} \rightsquigarrow L_{S}$.
The following classical Lemma gives a priori estimates on the time of definition of flows generated by a wider class of Hamiltonians.

Lemma 2.16. Let $\mathcal{N} \in \mathcal{A}_{r}(\mathrm{w})$ and $R \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ (recall Def. 2.3) for some $r>0$. Assume that

$$
\begin{equation*}
\operatorname{Re}\left(X_{\mathcal{N}}(v), v\right)_{\mathrm{h}_{\mathrm{w}}}=0, \quad \forall v \in \mathrm{~h}_{\mathrm{w}} . \tag{2.48}
\end{equation*}
$$

Consider the dynamical system

$$
\dot{v}=X_{\mathcal{N}}(v)+X_{R}(v), \quad v(0)=v_{0}, \quad\left|v_{0}\right|_{\mathrm{w}} \leq \frac{3}{4} r .
$$

Then one has

$$
\left||v(t)|_{\mathrm{w}}-\left|v_{0}\right|_{\mathrm{w}}\right| \leq \frac{r}{8}, \quad \forall|t| \leq \frac{1}{8|R|_{r, w}} .
$$

Proof. See Lemma 5.4 in [BMP20a].

Resonant Hamiltonians. We define the resonant subset of $\mathcal{M}$ (see (2.26) as

$$
\begin{equation*}
\mathrm{R}=\left\{(\alpha, \beta) \in \mathcal{M}: \alpha_{j}=\beta_{j} \vee \alpha_{j}=\beta_{-j} \forall j \in \mathbb{Z}\right\} \tag{2.49}
\end{equation*}
$$

and we denote by $\mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ the subset of resonant Hamiltonians, i.e.

$$
\begin{equation*}
\mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)=\left\{H \in \mathcal{H}_{r}\left(h_{\mathrm{w}}\right): \sum_{(\alpha, \beta) \in \mathrm{R}} H_{\alpha, \beta} u^{\alpha} \bar{u}^{\beta}\right\} . \tag{2.50}
\end{equation*}
$$

Remark 2.17. Let $(\alpha, \beta) \in \mathrm{R}$ and $\ell=\alpha-\beta$. The condition in (2.49) implies that

$$
\ell_{j}+\ell_{-j}=\alpha_{j}-\beta_{j}+\alpha_{-j}-\beta_{-j} \equiv 0, \quad \forall j \in \mathbb{Z} .
$$

The following results regards a fundamental properties of resonant Hamiltonians.
Lemma 2.18. (Flows of Kernel Hamiltonians). Let

$$
\mathrm{f}: \mathrm{h}_{\mathrm{w}} \rightarrow \mathrm{~h}_{\mathrm{w}}, \quad u \mapsto \mathrm{f}(u)=\left(\mathrm{f}_{j} u_{j}\right)_{j \in \mathbb{Z}}, \quad \mathrm{f}_{j}=\mathrm{f}_{-j}, \quad \forall j \in \mathbb{Z}
$$

Then any $H \in \mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ poisson commutes with $|\mathrm{f}(u)|_{\mathrm{w}}^{2}$.
Proof. Let $H \in \mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ (see (2.50)). Using (2.24) one has that

$$
\left\{H, \sum_{j \in \mathbb{Z}} \mathrm{w}_{j}^{2} \mathrm{f}_{j}^{2}\left|u_{j}\right|^{2}\right\}=\sum_{\alpha, \beta \in \mathbb{R}} H_{\alpha, \beta}\left\{u^{\alpha} \bar{u}^{\beta}, \sum_{j \in \mathbb{Z}} \mathrm{w}_{j}^{2} \mathrm{f}_{j}^{2}\left|u_{j}\right|^{2}\right\}=\sum_{\alpha, \beta \in \mathbb{R}} H_{\alpha, \beta} \sum_{\ell \in \mathbb{Z}}\left(\alpha_{\ell}-\beta_{\ell}\right) \mathrm{w}_{\ell}^{2} \mathrm{f}_{\ell}^{2} u^{\alpha} \bar{u}^{\beta} .
$$

Since since f is even in $\ell$ (as well as w) we have that the right hand side of the equation above reads

$$
\sum_{\alpha, \beta \in \mathrm{R}} H_{\alpha, \beta} \sum_{\ell>0}\left(\alpha_{\ell}+\alpha_{-\ell}-\beta_{-\ell}-\beta_{\ell}\right) \mathbf{f}(\ell)^{2} u^{\alpha} \bar{u}^{\beta}=0
$$

where in the last inequality we used Remark 2.17. This concludes the proof.
Remark 2.19. Let $\mathcal{N} \in \mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$. We have that

$$
2 \operatorname{Re}\left(X_{\mathcal{N}}(v), v\right)_{\mathbf{h}_{\mathfrak{w}}}=\left\{\mathcal{N}(v),|v|_{\mathbf{w}}^{2}\right\}=0,
$$

by Lemma 2.18 Hence we deduce that the vector field $X_{\mathcal{N}}$ satisfies the condition 2.48) in Lemma 2.16
Remark 2.20. Let $H \in \mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ and assume $H=\Pi^{(\mathrm{d})} H$ for some $\mathrm{d} \geq 1$. By using 2.49, 2.50) one can check that $H \equiv 0$ if d is odd.

## 3. Small divisors

Recalling that

$$
\begin{align*}
\omega & :=\omega(\mathrm{m}):=\left(\omega_{j}\right)_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}, \\
\omega_{j} & :=\omega_{j}(\mathrm{~m}):=\sqrt{|j|^{4}+\mathrm{m}}, \quad j \in \mathbb{Z}, \quad \mathrm{~m} \in[1,2], \tag{3.1}
\end{align*}
$$

and the resonant set

$$
\mathrm{R}=\left\{(\alpha, \beta) \in \mathcal{M}: \alpha_{j}=\beta_{j} \vee \alpha_{j}=\beta_{-j} \forall j \in \mathbb{Z}\right\},
$$

we now give arithmetic conditions on non- resonant indexes belonging to the following set:

$$
\begin{equation*}
\Lambda:=\left\{\ell \in \mathbb{Z}^{\mathbb{Z}}: \ell:=\alpha-\beta, \forall(\alpha, \beta) \in \mathbb{R}^{c}\right\} . \tag{3.2}
\end{equation*}
$$

Finally, given a vector $\ell:=\left(\ell_{i}\right)_{i \in \mathbb{Z}} \in \Lambda$ consider the $\operatorname{set} \mathcal{A}(\ell):=\left\{i \in \mathbb{Z}: \ell_{i} \neq 0\right\}$. We define the map

$$
\begin{equation*}
\ell \mapsto \mathrm{d}:=\mathrm{d}(\ell) \in \mathbb{N} \tag{3.3}
\end{equation*}
$$

where $\mathrm{d}(\ell):=\# \mathcal{A}(\ell)$. We call $\mathrm{d}(\ell)$ the cardinality of $\ell$, i.e. the number of components of $\ell$ which are different form zero.

Definition 3.1. (Diophantine frequencies). Given $\gamma>0$ we denote by $\mathrm{D}_{\gamma}$ the set of diophantine frequencies

$$
\begin{equation*}
\mathrm{D}_{\gamma}:=\left\{\omega \in \mathbb{R}^{\mathbb{Z}}:|\omega \cdot \ell| \geq \prod_{n \in \mathbb{Z}} \frac{\gamma^{\mathrm{d}(\ell)}}{\left(1+\left|\ell_{n}\right|^{2}\langle n\rangle^{2}\right)^{\tau}}, \tau:=\mathrm{d}(\ell)(\mathrm{d}(\ell)+2), \quad \forall \ell \in \Lambda\right\} \tag{3.4}
\end{equation*}
$$

where $\langle n\rangle:=\max \{1,|n|\}$ for any $n \in \mathbb{Z}$.
The key proposition of this section guarantees that, for "almost all" choices of the parameter $m$ the frequency vector $\omega$ in (3.1) belongs to the diophantine set in (3.4). Our aim is to prove the following result.

Proposition 3.2. (Measure estimates). There exists a positive measure set $\mathfrak{M} \subseteq[1,2]$ such that for any $\mathrm{m} \in \mathfrak{M}$, the vector $\omega(\mathrm{m})$ belongs to the diophantine set of frequencies $\mathrm{D}_{\gamma}$ in (3.4). Moreover, there exists $a$ positive constant C such that

$$
\operatorname{meas}([1,2] \backslash \mathfrak{M}) \leq \mathrm{C} \gamma
$$

The proof of the proposition above involves several argument which will be discussed below. First of all let us define the quantity (see 2.26 )

$$
\begin{equation*}
\psi(\mathrm{m}, \ell):=\omega \cdot \ell, \quad \forall \ell \in \mathcal{M} \tag{3.5}
\end{equation*}
$$

and recall that we shall provide lower bounds on $\psi(\omega, \ell)$ only for $\ell$ belonging to the set $\Lambda$ in (3.2). Moreover, according to the notation (3.3), we can write the function in (3.5) as

$$
\begin{equation*}
\psi(\mathrm{m}, \ell)=\sum_{i=1}^{\mathrm{d}} \ell_{j_{i}} \omega_{j_{i}}, \quad j_{i} \in \mathbb{Z} \tag{3.6}
\end{equation*}
$$

Estimates of a single "bad set". We have the following.
Lemma 3.3. For any $\ell \in \Lambda$ there exists $0 \leq k \leq \mathrm{d}(\ell)-1$ such that

$$
\begin{equation*}
\left|\partial_{\mathrm{m}}^{k} \psi(\mathrm{~m}, \ell)\right| \geq \prod_{i=1}^{\mathrm{d}} \frac{1}{\left(1+\left|\ell_{j_{i}}\right|^{2}\left\langle j_{i}\right\rangle^{2}\right)^{\mathrm{d}(\ell)}} \tag{3.7}
\end{equation*}
$$

Proof. To lighten the notation we shall write d instead of $\mathrm{d}(\ell)$.
Given $\ell \in \Lambda$, after a reordering of the indexes we can write $\ell=(\bar{\ell}, 0)$, where $\bar{\ell}=\left(\ell_{j_{1}}, \ldots, \ell_{j_{\mathrm{d}}}\right)$. Without loss of generality, we can always assume that the vector $\bar{\ell}$ satisfies

$$
\begin{equation*}
j_{i} \neq-j_{k}, \quad \forall j, k=1, \ldots, \mathrm{~d} . \tag{3.8}
\end{equation*}
$$

Indeed, the d-pla $\left(j_{1}, \ldots, j_{\mathrm{d}}\right)$ can be written as

$$
\left(k_{1}, \ldots, k_{p}, q_{1},-q_{1}, q_{2},-q_{2} \ldots, q_{r},-q_{r}\right), \quad 0 \leq p \leq \mathrm{d}
$$

for some $0 \leq p \leq \mathrm{d}$ and $p+2 r=\mathrm{d}$, where $k_{i}, i=1, \ldots, p$ satisfy (3.8). The small divisors has the form

$$
\omega \cdot \bar{\ell}=\sum_{i=1}^{\mathrm{d}} \omega_{j_{i}} \ell_{j_{i}}=\sum_{i=1}^{p} \omega_{k_{i}} \ell_{k_{i}}+\sum_{i=1}^{r} \omega_{q_{i}}\left(\ell_{q_{i}}+\ell_{-q_{i}}\right)
$$

Hence we can define

$$
\tilde{\ell}=\left(\tilde{\ell}_{k_{1}}, \ldots, \tilde{\ell}_{k_{p}}, \tilde{\ell}_{q_{1}}, \ldots, \tilde{\ell}_{q_{r}}\right), \quad \text { where } \quad\left\{\begin{array}{l}
\tilde{\ell}_{k_{i}}=\ell_{k_{i}}, \quad i=1, \ldots p \\
\tilde{\ell}_{q_{i}}=\ell_{q_{i}}+\ell_{-q_{i}}, \quad i=1, \ldots, r
\end{array}\right.
$$

Since $\ell \in \Lambda$ it is not possible that at the same time $p=0$ and $\ell_{q_{i}}+\ell_{-q_{i}}=0$ for any $i=1, \ldots, r$. Otherwise $\ell$ is a resonant vector (recall (2.49) and Remark 2.17). As a consequence up to reducing the length of $\tilde{\ell}$ to $\tilde{\mathrm{d}}=\mathrm{d}(\tilde{\ell}) \leq \mathrm{d}(\ell)$ (by eliminating the components for which $\ell_{q_{i}}+\ell_{-q_{i}}=0$ ), we have obtained a vector satisfying condition 3.8 with $\tilde{\mathrm{d}} \leq \mathrm{d}$.

Hence from now on we consider $\ell \in \Lambda$ with $\mathrm{d}(\ell)=\mathrm{d}$ and satisfying (3.8). Notice that, for any $k \geq 1$,

$$
\begin{equation*}
\partial_{\mathrm{m}}^{k} \psi(\mathrm{~m}, \ell)=\sum_{i=1}^{\mathrm{d}} \ell_{j_{i}} \partial_{\mathrm{m}}^{k} \omega_{j_{i}}=\Gamma(k) \sum_{i=1}^{\mathrm{d}} \ell_{j_{i}}\left(\omega_{j_{i}}\right)^{1-2 k}, \quad \Gamma(k):=\frac{(-1)^{k+1}}{2^{k}}(2 k-3)!! \tag{3.9}
\end{equation*}
$$

Let us define $\mathrm{a}:=\left(\mathrm{a}_{i}\right)_{i=0, \ldots, \mathrm{~d}} \in \mathbb{R}^{\mathrm{d}}$ as

$$
\begin{equation*}
\partial_{\mathrm{m}}^{k} \psi(\mathrm{~m}, \ell)=\mathrm{a}_{k+1}, \quad k=0, \ldots, d-1 \tag{3.10}
\end{equation*}
$$

Our aim is to prove that there is at least one component of the vector a satisfying the bound 3.7). In view of 3.9 we rewrite 3.10 as

$$
\begin{equation*}
\Gamma M O \ell=\mathrm{a} \tag{3.11}
\end{equation*}
$$

where

$$
\begin{align*}
& \Gamma:=\left(\begin{array}{cccc}
1 & \ldots & \ldots & 0 \\
0 & \Gamma(1) & \ldots & \vdots \\
\vdots & \ldots & \ddots & \vdots \\
0 & \ldots & \ldots & \Gamma(\mathrm{~d}-1)
\end{array}\right), \quad O:=\left(\begin{array}{cccc}
\omega_{j_{1}} & \ldots & \ldots & 0 \\
0 & \omega_{j_{2}} & \ldots & \vdots \\
\vdots & \ldots & \ddots & \vdots \\
0 & \ldots & \ldots & \omega_{j_{\mathrm{d}}}
\end{array}\right),  \tag{3.12}\\
& M:=\left(\begin{array}{cccc}
1 & \ldots & \ldots & 1 \\
\omega_{j_{1}}^{-2} & \ldots & \cdots & \omega_{j_{\mathrm{d}}}^{-2} \\
\vdots & \ldots & \ldots & \vdots \\
\omega_{j_{1}}^{-2(\mathrm{~d}-1)} & \ldots & \ldots & \omega_{j_{\mathrm{d}}}^{-2(\mathrm{~d}-1)}
\end{array}\right) .
\end{align*}
$$

Notice that the matrix $M$ is a Vandermonde matrix. Moreover using that $\ell \in \Lambda$ and that 3.8 holds, its determinant is given by

$$
\operatorname{det}(M)=\prod_{i \neq k}\left(\omega_{j_{i}}^{-2}-\omega_{j_{k}}^{-2}\right) \neq 0
$$

so that the matrix $M$ is invertible. It is also easy to check that

$$
\max _{i, k=1, \ldots, \mathrm{~d}}\left|\left(M^{-1}\right)_{i}^{k}\right| \leq(\mathrm{d}-1)!\prod_{i \neq k} \frac{\omega_{j_{i}}^{2} \omega_{j_{k}}^{2}}{\omega_{j_{i}}^{2}-\omega_{j_{k}}^{2}} \lesssim 2^{-\mathrm{d}} \mathrm{~d}^{-1}\left(\prod_{i=1}^{\mathrm{d}} \omega_{j_{i}}\right)^{\mathrm{d}} \sim 2^{-\mathrm{d}} \mathrm{~d}^{-1}\left(\prod_{i=1}^{\mathrm{d}}\left|j_{i}\right|^{2}\right)^{\mathrm{d}}
$$

Recalling (3.12) we note

$$
\max _{i=1, \ldots, \mathrm{~d}}\left|\left(\Gamma^{-1}\right)_{i}^{i}\right| \leq 2^{\mathrm{d}}, \quad \max _{i=1, \ldots, \mathrm{~d}}\left|\left(O^{-1}\right)_{i}^{i}\right| \leq 1
$$

Therefore

$$
\begin{equation*}
\max _{i, k=1, \ldots, \mathrm{~d}}\left|\left((\Gamma M O)^{-1}\right)_{i}^{k}\right| \lesssim \mathrm{d}^{-1}\left(\prod_{i=1}^{\mathrm{d}}\left|j_{i}\right|^{2}\right)^{\mathrm{d}} \lesssim \mathrm{~d}^{-1}\left(\prod_{i=1}^{\mathrm{d}} \frac{1}{\left(1+\left|\ell_{j_{i}}\right|^{2}\left\langle j_{i}\right\rangle^{2}\right)}\right)^{-\mathrm{d}} \tag{3.13}
\end{equation*}
$$

Since by (3.11), we have $\ell=(\Gamma M O)^{-1}$ a, we deduce

$$
1 \leq|\ell| \lesssim \mathrm{d}_{i, k=1, \ldots, \mathrm{~d}}\left|\left((\Gamma M O)^{-1}\right)_{i}^{k}\right|\|\mathrm{a}\|_{\ell \infty}
$$

which, together with (3.13), implies the bound (3.7).
Now we need the following result (see for example Lemma B. 1 [Eli02] ):
Lemma 3.4. Let $\mathfrak{g}(x)$ be a $C^{n+1}$-smooth function on the segment $[1,2]$ such that

$$
\left|\mathfrak{g}^{\prime}\right|_{C^{n}}=\beta \quad \text { and } \quad \max _{1 \leq k \leq n} \min _{x}\left|\partial^{k} \mathfrak{g}(x)\right|=\sigma .
$$

Then one has

$$
\operatorname{meas}\left(\{x||\mathfrak{g}(x)| \leq \rho\}) \leq C_{n}\left(\beta \sigma^{-1}+1\right)\left(\rho \sigma^{-1}\right)^{1 / n}\right.
$$

Now, for any fixed $\ell \in \Lambda$ and $\eta>0$, we define the "bad set" of parameters

$$
\begin{equation*}
\mathcal{B}(\ell):=\left\{m \in[1,2]:|\omega \cdot \ell| \leq \prod_{n \in \mathbb{Z}} \frac{\gamma^{\mathrm{d}}}{\left(1+\left|\ell_{n}\right|^{2}\langle n\rangle^{2}\right)^{\tau}}\right\} \tag{3.14}
\end{equation*}
$$

with $\tau$ as in (3.4). Thanks to Lemma 3.3 we shall apply Lemma 3.4 with $n=\mathrm{d}-1$ and

$$
\sigma \geq \prod_{i=1}^{\mathrm{d}} \frac{1}{\left(1+\left|\ell_{j_{i}}\right|^{2}\left\langle j_{i}\right\rangle^{2}\right)^{\mathrm{d}}}, \quad \rho=\prod_{n \in \mathbb{Z}} \frac{\gamma^{\mathrm{d}}}{\left(1+\left|\ell_{n}\right|^{2}\langle n\rangle^{2}\right)^{\tau}}, \quad \beta \leq \mathrm{d}!\lesssim \prod_{i=1}^{\mathrm{d}}\left(1+\left|\ell_{j_{i}}\right|^{2}\left\langle j_{i}\right\rangle^{2}\right)
$$

Therefore we obtain

$$
\begin{equation*}
\operatorname{meas}(\mathcal{B}(\ell)) \lesssim \gamma^{\frac{\mathrm{d}}{\mathrm{~d}-1}}\left(\prod_{i=1}^{\mathrm{d}} \frac{1}{1+\left|\ell_{j_{i}}\right|^{2}\left\langle j_{i}\right\rangle^{2}}\right)^{\frac{\tau}{\mathrm{d}}-\mathrm{d}-1} \tag{3.15}
\end{equation*}
$$

Proof of Proposition 3.2. We define (see (3.4) and (3.14))

$$
\mathcal{B}:=\bigcup_{\ell \in \Lambda} \mathcal{B}(\ell)
$$

and we set $\mathfrak{M}=\mathcal{B}^{c}$. Then the thesis follows by using the sub-additivity of the Lebesgue measure, the bound (3.15) and by reasoning as in the proof of Lemma 4.1 in [BMP20a].

Remark 3.5. By Remark 2.17 for any $(\alpha, \beta) \in \mathrm{R}$ and $\ell=\alpha-\beta$ one has that $\omega \cdot \ell \equiv 0$ is identically zero for $\mathrm{m} \in[1,2]$. On the other hand, by Proposition 3.2 for any $\omega \in \mathrm{D}_{\gamma}$ one has $\omega \cdot \ell \neq 0$ for any $\ell \in \Lambda$.

## 4. Homological equation

Given a diophantine vector $\omega \in \mathrm{D}_{\gamma}$, in view of Remark 3.5 and by formula 2.30) we deduce that

$$
L_{\omega} H=0 \quad \Leftrightarrow \quad H \in \mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)
$$

Hence the operator $L_{\omega}$ is formally invertible when acting on the subspace

$$
\begin{equation*}
\mathcal{R}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)=\mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)^{\perp}:=\left\{H \in \mathcal{H}_{r}\left(h_{\mathrm{w}}\right): \sum_{(\alpha, \beta) \in \mathrm{R}^{c}} H_{\alpha, \beta} u^{\alpha} \bar{u}^{\beta}\right\} \tag{4.1}
\end{equation*}
$$

containing those Hamiltonians supported on monomials $u^{\alpha} \bar{u}^{\beta}$ with $(\alpha, \beta) \in \mathrm{R}^{c}$. We decompose the space of regular Hamiltonians $\mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ as

$$
\mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)=\mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \oplus \mathcal{R}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right),
$$

and we denote by $\Pi_{\mathcal{K}}$ and $\Pi_{\mathcal{R}}$ the continuous projections on the subspaces $\mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right), \mathcal{R}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$. One can note

$$
\begin{equation*}
\left|\Pi_{\mathcal{K}} H\right|_{r, \mathrm{w}},\left|\Pi_{\mathcal{R}} H\right|_{r, \mathrm{w}} \leq|H|_{r, \mathrm{w}} \tag{4.2}
\end{equation*}
$$

Obviously, for diophantine frequency, $\mathcal{R}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ and $\mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right)$ represent the range and kernel of $L_{\omega}$ respectively.
Proposition 4.1. (Inverse of the adjoint action). Fix $\mathrm{N} \in \mathbb{N}, r>0, p>1$ and $s>0$. Consider $\mathrm{w}(s, p)$ (resp. $\mathrm{w}(p)$ ) and a Hamiltonian function $f \in \mathcal{R}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \cap \mathcal{H}^{(\mathrm{N})}$ (see Def. 2.11 and recall 4.1). For any $\omega \in \mathrm{D}_{\gamma}$ the following holds.
(case (sE)) There exists an absolute constant $\mathrm{C}>0$ (independent of N ) such that for any $0<\sigma \ll 1$ one has that

$$
\left|L_{\omega}^{-1} f\right|_{r, \mathrm{w}(p, s+\sigma)} \leq J_{0}^{\mathrm{sE}}|f|_{r, \mathrm{w}(p, s)}
$$

where $L_{\omega}$ is in 2.30 and

$$
\begin{equation*}
J_{0}^{\mathrm{sE}}:=J_{0}^{\mathrm{sE}}(\sigma, \mathrm{~N}):=\gamma^{-4 \mathrm{~N}} \exp \exp \left(\left(\frac{\mathrm{~N}^{2}}{\sigma} \mathrm{C}\right)^{\frac{1}{\mathrm{q}-1}}\right) \tag{4.3}
\end{equation*}
$$

(case (S)) Fix $\zeta \geq(36 \mathrm{~N})^{2}$. There exists an absolute constant $\mathrm{C}>0$ such that

$$
\left|L_{\omega}^{-1} f\right|_{r, \mathrm{w}(p+\zeta)} \leq J_{0}^{\mathrm{S}}|f|_{r, \mathrm{w}(p)}
$$

where

$$
\begin{equation*}
J_{0}^{\mathrm{S}}:=J_{0}^{\mathrm{S}}(\zeta, \mathrm{~N}):=\gamma^{-4 \mathrm{~N}} e^{\mathrm{C} \zeta} \tag{4.4}
\end{equation*}
$$

Proof. Case (sE). Since, by hypothesis, $f$ belongs to the range of the operator $L_{\omega}$, the Hamiltonian $L_{\omega}^{-1} f$ is well-defined with coefficients given by

$$
\left(L_{\omega}^{-1} f\right)_{\alpha, \beta}=\frac{f_{\alpha, \beta}}{-\mathrm{i} \omega \cdot(\alpha-\beta)}, \quad \forall(\alpha, \beta) \in \mathrm{R}^{c}
$$

Recall the coefficients in 2.32). In view of property (2.36) with $\mathrm{w}^{\prime}=\mathrm{w}(p, s+\sigma)$ and $\mathrm{w}=\mathrm{w}(p, s)$ and formula (2.30) , in order to get the result it is sufficient to estimate the quantity

$$
\begin{equation*}
J_{0}:=\sup _{\substack{j \in \mathbb{Z},(\alpha, \beta) \in \Lambda \\ \alpha_{j}+\beta_{j} \neq 0 \\|\alpha-\beta| \leq \mathrm{N}+2}} \frac{c_{r, \mathrm{w}(s+\sigma, p)}^{(j)}(\alpha, \beta)}{c_{r, \mathrm{w}(s, p)}^{(j)}(\alpha, \beta)|\omega \cdot(\alpha-\beta)|} \tag{4.5}
\end{equation*}
$$

By an explicit computation using (2.32) we get

$$
J_{0}=\sup _{\substack{j \in \mathbb{Z},(\alpha, \beta) \in \Lambda \\ \alpha_{j}+\beta_{j} \neq 0 \\|\alpha-\beta-\beta| \leq N+2}} \frac{e^{-\sigma\left(\sum_{i} \lambda(i)\left(\alpha_{i}+\beta_{i}\right)-2 \lambda(j)\right)}}{|\omega \cdot(\alpha-\beta)|} .
$$

By Lemma A.7, we just have to study the case in which A.17) holds true. Let $\omega \in \mathrm{D}_{\gamma}$ (recall (3.3)-(3.4). Since $\ell=\alpha-\beta,|\ell| \leq \mathrm{N}+2$ we notice that $\mathrm{d}=\mathrm{d}(\ell) \leq 4 \mathrm{~N}$ and $\tau=\tau(\ell) \leq 36 \mathrm{~N}^{2}$. Therefore we have

$$
\begin{aligned}
J_{0} & \leq \gamma^{-\mathrm{d}} \exp \left(-\sigma\left(\sum_{i} \lambda(i)\left(\alpha_{i}+\beta_{i}\right)-2 \lambda(j)\right)+\sum_{i \in \mathbb{Z}} \tau \ln \left(1+\left(\alpha_{i}-\beta_{i}\right)\langle i\rangle^{2}\right)\right) \\
& \stackrel{\text { A.18 }}{\leq} \gamma^{-4 \mathrm{~N}} \exp \left(\sum_{i}\left[-\frac{\sigma \kappa}{63}\left|\alpha_{i}-\beta_{i}\right| \lambda(\sqrt{\langle i\rangle})\right]+36 \mathrm{~N}^{2} \ln \left(1+\left(\alpha_{i}-\beta_{i}\right)^{2}\langle i\rangle^{2}\right)\right) \\
& \leq \gamma^{-4 \mathrm{~N}} \exp \left(-144 \mathrm{~N}^{2} \sum_{i} H_{i}\left(\left|\alpha_{i}-\beta_{i}\right|\right)\right),
\end{aligned}
$$

where for $0<\sigma \leq 1, i \in \mathbb{Z}$, we defined

$$
H_{i}(x):=\frac{\sigma \kappa}{63 \times 36 \mathrm{~N}^{2}} x \lambda(\sqrt{\langle i\rangle})-\ln (1+\sqrt{x\langle i\rangle})
$$

where $x:=\left|\alpha_{i}-\beta_{i}\right| \geq 1$. By definition of $\lambda$ (recall (1.6)), by denoting

$$
\begin{equation*}
\alpha=\frac{\sigma \kappa}{63 \times 36 \mathrm{~N}^{2}}, \tag{4.6}
\end{equation*}
$$

we observe that there exists $X(\alpha)$ such that the following inequalities hold:

$$
\alpha x \lambda(\sqrt{\langle i\rangle})-\ln (1+\sqrt{x\langle i\rangle}) \geq \alpha \lambda(\sqrt{x\langle i\rangle})-\ln (1+\sqrt{x\langle i\rangle}) \geq 0, \quad \text { if } \quad\langle i\rangle \geq X^{2}(\alpha) .
$$

By an explicit computation one can check that

$$
\begin{equation*}
X(\alpha)=\exp \left\{\left(\frac{2 \cdot 63 \cdot 36 \mathrm{~N}^{2}}{\sigma \kappa}\right)^{\frac{1}{q-1}}\right\} \leq e^{\left(\frac{2}{\alpha}\right)^{\frac{1}{q}-1}} \tag{4.7}
\end{equation*}
$$

## Consequently

$$
\begin{equation*}
J_{0} \leq \gamma^{-4 \mathrm{~N}} \exp \left(-144 \mathrm{~N}^{2} \inf _{x \geq 1} \sum_{i:\langle i\rangle \leq X^{2}(\alpha)} H_{i}(x)\right) \tag{4.8}
\end{equation*}
$$

Let us compute $\inf _{x \geq 1} H_{i}(x)$. We have

$$
H_{i}(x) \geq \hat{H}_{i}(x):=\alpha x \lambda(\sqrt{\langle i\rangle})-\ln (1+\sqrt{x})-\ln (1+\sqrt{\langle i\rangle}) .
$$

Then, since

$$
\hat{H}_{i}^{\prime}(x)=\alpha \lambda(\sqrt{\langle i\rangle})-\frac{1}{2 \sqrt{x}(1+\sqrt{x})}=0 \quad \Leftrightarrow \quad \sqrt{x}=\frac{1}{2}\left(-1+\sqrt{1+\frac{2}{\alpha \lambda(\sqrt{\langle i\rangle})}}\right),
$$

we deduce

$$
\begin{aligned}
H_{i}(x) & \geq \lambda(\sqrt{i})\left(-\frac{\alpha}{2}+\frac{\alpha}{2} \sqrt{1+\frac{2}{\alpha(\sqrt{\langle i\rangle})}}\right)^{2}-\ln \left(1+\sqrt{1+\frac{2}{\alpha \lambda(\sqrt{\langle i\rangle})}}\right)-\ln (1+\sqrt{\langle i\rangle}) \\
& \geq-\ln \left(1+\sqrt{1+\frac{2}{\alpha \lambda(1)}}\right)-\ln (1+X(\alpha)) .
\end{aligned}
$$

The latter bound, together with (4.8), implies

$$
\begin{aligned}
J_{0} & \leq \gamma^{-4 \mathrm{~N}}\left(\left(1+\sqrt{1+\frac{2}{\alpha \lambda(1)}}\right)(1+X(\alpha))\right)^{144 \mathrm{~N}^{2} X^{2}(\alpha)} \\
& \leq \gamma^{-4 \mathrm{~N}}\left(\left(1+\sqrt{1+\frac{2}{\alpha \ln ^{q} 2}}\right)(1+X(\alpha))\right)^{144 \mathrm{~N}^{2} X^{2}(\alpha)}
\end{aligned}
$$

By (4.6) and (4.7) it follows the desired bound (4.3) choosing a suitable constant $\mathrm{C}>0$ large enough.
case (S). We proceed as in the case ( sE ). By definition of the coefficients in the Sobolev case (2.33) we have

$$
\begin{equation*}
J_{0}:=\sup _{\substack{j \in \mathbb{Z},(\alpha, \beta) \in \Lambda \\ \alpha_{j}+\beta_{j} \neq 0 \\|\alpha-\beta| \leq \mathrm{N}+2}} \frac{c_{r, \mathbf{w}(p+\delta)}^{(j)}(\alpha, \beta)}{c_{r, w}^{(j)}(\alpha, \beta)|\omega \cdot(\alpha-\beta)|}=\sup _{\substack{j \in \mathbb{Z},(\alpha, \beta) \in \Lambda \\ \alpha_{j}+\beta, \beta \neq 0 \\|\alpha-\beta| \leq \mathrm{N}+2}} \frac{\lfloor j\rfloor^{2 \delta}}{|\omega \cdot(\alpha-\beta)|} \prod_{i \in \mathbb{Z}}\lfloor i\rfloor^{-\delta\left(\alpha_{i}+\beta_{i}\right)} . \tag{4.9}
\end{equation*}
$$

By the diophantine condition (3.4) we have

$$
J_{0} \leq \gamma^{-4 \mathrm{~N}} \sup _{\substack{j \in \mathbb{Z},(\alpha, \beta) \in \Lambda \\ \alpha_{j}+\beta_{j} \neq 0 \\|\alpha-\beta| \leq \mathrm{N}+2}}\left(\frac{\lfloor j\rfloor^{2}}{\prod_{i \in \mathbb{Z}}\lfloor i\rfloor^{\alpha_{i}+\beta_{i}}}\right)^{\delta} \prod_{i \in \mathbb{Z}}\left(\left(1+\left|\alpha_{i}-\beta_{i}\right|^{2}\right)\langle i\rangle^{2}\right)^{\tau} .
$$

By Lemma A. 7 we only have to consider the case in which A.17) holds. Recalling that $|\alpha|+|\beta|=\mathrm{N}+2$, $\mathrm{d} \leq 4 \mathrm{~N}, \zeta \geq(36 \mathrm{~N})^{2}$, we can apply Lemma A.8. The bound (A.22) implies the estimate (4.4).

## 5. A Birkhoff normal form step

Notations. Let $r>r^{\prime}>0, \sigma, s>0, \zeta, p>0, \mathrm{~K} \gg 1$ and $1 \leq \mathrm{N} \leq \mathrm{K}-1$. In the following we shall write

$$
\begin{equation*}
\mathrm{w}=\mathrm{w}(s, p) \quad(\text { resp. } \mathrm{w}=\mathrm{w}(p)) \quad \text { and } \quad \mathrm{w}^{\prime}=\mathrm{w}(s+\sigma, p) \quad\left(\text { resp. } \mathrm{w}^{\prime}=\mathrm{w}^{\prime}(p+\zeta)\right) . \tag{5.1}
\end{equation*}
$$

We consider an Hamiltonian function of the form

$$
\begin{equation*}
H=D_{\omega}+\sum_{\mathrm{d}=1}^{\mathrm{N}-1} Z^{(\mathrm{d})}+\sum_{\mathrm{d}=\mathrm{N}}^{\mathrm{K}} R^{(\mathrm{d})}+R^{(\geq \mathrm{K}+1)}, \tag{5.2}
\end{equation*}
$$

where $D_{\omega}$ is in 2.28 and

$$
\begin{aligned}
& Z^{(\mathrm{d})} \in \mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \cap \mathcal{H}^{(\mathrm{d})}, \quad 1 \leq \mathrm{d} \leq \mathrm{N}-1 \\
& R^{(\mathrm{d})} \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \cap \mathcal{H}^{(\mathrm{d})}, \quad \mathrm{N} \leq \mathrm{d} \leq \mathrm{K} \\
& R^{(\geq \mathrm{K}+1)} \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \cap \mathcal{H}^{(\geq \mathrm{K}+1)}
\end{aligned}
$$

In the case $\mathrm{N}=1$ we assume $Z^{(\mathrm{d})} \equiv 0$. We set

$$
\begin{equation*}
\epsilon_{\mathrm{d}}:=\left|R^{(\mathrm{d})}\right|_{r, \mathrm{w}}, \quad \mathrm{~N} \leq \mathrm{d} \leq \mathrm{K}, \quad \epsilon_{\mathrm{K}+1}:=\left|R^{(\geq \mathrm{K}+1)}\right|_{r, \mathrm{w}} \tag{5.3}
\end{equation*}
$$

Lemma 5.1. (Birkhoff normal form step). Consider the Hamiltonian $H$ in 5.2 and fix $\omega \in \mathrm{D}_{\gamma}$. Assume that

$$
\begin{equation*}
J_{0}^{\star}\left(\sum_{\mathrm{d}=\mathrm{N}}^{\mathrm{K}} \epsilon_{\mathrm{d}}+\epsilon_{\mathrm{K}+1}\right) \leq \delta \quad \text { with } \quad \delta:=\frac{r-r^{\prime}}{16 e r} \tag{5.4}
\end{equation*}
$$

where $J_{0}^{\star}=J_{0}^{\mathrm{sE}}(\sigma, \mathrm{N})$ in (4.3) (respectively $J_{0}^{\star}=J_{0}^{\mathrm{S}}(\zeta, \mathrm{N})$ in (4.4).
Then there exists a change of variables

$$
\begin{equation*}
\Phi: B_{r^{\prime}}\left(\mathrm{h}_{\mathrm{w}^{\prime}}\right) \rightarrow B_{r}\left(\mathrm{~h}_{\mathrm{w}^{\prime}}\right), \tag{5.5}
\end{equation*}
$$

such that

$$
\begin{equation*}
H \circ \Phi=D_{\omega}+\sum_{\mathrm{d}=1}^{\mathrm{N}} Z_{+}^{(\mathrm{d})}+\sum_{\mathrm{d}=\mathrm{N}+1}^{\mathrm{K}} R_{+}^{(\mathrm{d})}+R_{+}^{(\geq \mathrm{K}+1)} \tag{5.6}
\end{equation*}
$$

where

$$
\begin{aligned}
& Z_{+}^{(\mathrm{d})} \in \mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \cap \mathcal{H}^{(\mathrm{d})}, \quad 1 \leq \mathrm{d} \leq \mathrm{N} \\
& R_{+}^{(\mathrm{d})} \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \cap \mathcal{H}^{(\mathrm{d})}, \quad \mathrm{N}+1 \leq \mathrm{d} \leq \mathrm{K} \\
& R_{+}^{(\geq \mathrm{K}+1)} \in \mathcal{H}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \cap \mathcal{H}^{(\geq \mathrm{K}+1)}
\end{aligned}
$$

Moreover the following estimates hold

$$
\begin{gather*}
Z_{+}^{(\mathrm{d})}:=Z^{(\mathrm{d})}, \quad 1 \leq \mathrm{d} \leq \mathrm{N}-1, \quad\left|Z_{+}^{(\mathrm{N})}\right|_{r^{\prime}, \mathrm{w}^{\prime}} \leq \epsilon_{\mathrm{N}}  \tag{5.7}\\
\left|R_{+}^{(p)}\right|_{r^{\prime}, \mathrm{w}^{\prime}} \leq \epsilon_{p}+\sum_{\substack{j \geq 2 \\
(j-1) \mathrm{N}+\mathrm{N}=p}} \frac{\epsilon_{\mathrm{N}}}{j!}\left(\frac{\epsilon_{\mathrm{N}} J_{0}^{\star}}{2 \delta}\right)^{j-1}+\sum_{\substack{1 \leq j \leq \mathrm{d} \leq \mathrm{K} \\
j \mathrm{~N}+\mathrm{d}=p}} \frac{1}{j!}\left(\frac{\epsilon_{\mathrm{N}} J_{0}^{\star}}{2 \delta}\right)^{j}\left|Z^{(\mathrm{d})}\right|_{r, \mathrm{w}}  \tag{5.8}\\
+\sum_{\substack{1 \leq j \leq \mathrm{K} \\
\mathrm{~N} \leq \mathrm{d} \leq \mathrm{K} \\
j \mathrm{~N}+\mathrm{d}=p}} \frac{\epsilon_{\mathrm{d}}}{j!}\left(\frac{\epsilon_{\mathrm{N}} J_{0}^{\star}}{2 \delta}\right)^{j}, \quad \mathrm{~N}+1 \leq \mathrm{d} \leq \mathrm{K} \\
\left|R_{+}^{(\geq \mathrm{K}+1)}\right|_{r^{\prime}, \mathrm{w}^{\prime}} \leq \epsilon_{\mathrm{K}+1}+2 \sum_{\mathrm{d}=\mathrm{N}}\left(\frac{\epsilon_{\mathrm{N}} J_{0}^{\star}}{2 \delta}\right)^{\left[\frac{\mathrm{K}+1-\mathrm{d}}{\mathrm{~N}}\right]} \epsilon_{\mathrm{d}}+2\left(\frac{\epsilon_{\mathrm{N}} J_{0}^{\star}}{2 \delta}\right)^{\mathrm{K}+1}\left(|Z|_{r, \mathrm{w}}+\epsilon_{\mathrm{N}}\right) \tag{5.9}
\end{gather*}
$$

Finally, for any $\sigma^{\sharp} \geq 0, \zeta^{\sharp} \geq 0$, setting $\mathrm{w}^{\sharp}:=\mathrm{w}\left(s+\sigma+\sigma^{\sharp}, p\right)\left(\right.$ resp. $\mathrm{w}^{\sharp}:=\mathrm{w}\left(p+\zeta+\zeta^{\sharp}\right)$ ) assume the further conditions

$$
\begin{equation*}
\widetilde{J}_{0}^{\star}\left(\sum_{\mathrm{d}=\mathrm{N}}^{\mathrm{K}} \epsilon_{\mathrm{d}}+\epsilon_{\mathrm{K}+1}\right) \leq \delta, \tag{5.10}
\end{equation*}
$$

where $\widetilde{J}_{0}^{\star}=J_{0}^{\mathrm{SE}}\left(\sigma^{\sharp}, \mathrm{N}\right)$ in 4.3$)$ (respectively $\widetilde{J}_{0}^{\star}=J_{0}^{\mathrm{S}}\left(\zeta^{\sharp}, \mathrm{N}\right)$ in (4.4). Then

$$
\begin{array}{r}
\left.\Phi\right|_{B_{r^{\prime}}\left(\mathrm{h}_{\mathrm{w}^{\sharp}}\right)}: B_{r^{\prime}}\left(\mathrm{h}_{\mathrm{w}^{\sharp}}\right) \rightarrow B_{r}\left(\mathrm{~h}_{\mathrm{w}^{\sharp}}\right), \\
\sup _{u \in B_{r^{\prime}}\left(\mathrm{h}_{\mathrm{w}^{\sharp}}\right)}|\Phi(u)-u|_{\mathrm{h}_{\mathrm{w}^{\sharp}}} \leq r \widetilde{J}_{0}^{\star}\left|R^{(\mathrm{N})}\right|_{r, \mathrm{w}} . \tag{5.11}
\end{array}
$$

Proof. Recalling (5.2) we define

$$
\begin{equation*}
Z_{+}:=\sum_{\mathrm{d}=1}^{\mathrm{N}} Z_{+}^{(\mathrm{d})}, \quad Z_{+}^{(\mathrm{d})}:=Z^{(\mathrm{d})}, \quad 1 \leq \mathrm{d} \leq \mathrm{N}-1, \quad Z_{+}^{(\mathrm{N})}:=\Pi_{\mathcal{K}} R^{(\mathrm{N})} . \tag{5.12}
\end{equation*}
$$

By (5.12), (5.3), (4.2) and (2.41) we deduce that $Z_{+} \in \mathcal{K}_{r}\left(\mathrm{~h}_{\mathrm{w}}\right) \cap \mathcal{H}^{(\leq \mathrm{N})}$ and satisfies the bound (5.7). Let

$$
\begin{equation*}
S:=L_{\omega}^{-1}\left(\Pi_{\mathcal{R}} R^{(\mathrm{N})}\right) \tag{5.13}
\end{equation*}
$$

be the unique solution of the homological equation $L_{\omega} S=\left\{S, D_{\omega}\right\}=\Pi_{\mathcal{R}} R^{(\mathbb{N})}$. By Proposition 4.1 we have that $S \in \mathcal{R}_{r}\left(\mathrm{~h}_{\mathrm{w}^{\prime}}\right) \cap \mathcal{H}^{(\mathrm{N})}$ and satisfies the estimate

$$
\begin{equation*}
|S|_{r, w^{\prime}} \leq J_{0}^{\star}\left|R^{(\mathbb{N})}\right|_{r, w} \stackrel{\sqrt{5.3 \mid}}{\leq} J_{0}^{\star} \epsilon_{\mathrm{N}}, \tag{5.14}
\end{equation*}
$$

where $J_{0}^{\star}=J_{0}^{\mathrm{sE}}(\sigma, \mathrm{N})$ in (4.3) (respectively $J_{0}^{\star}=J_{0}^{\mathrm{S}}(\zeta, \mathrm{N})$ in (4.4). We now apply Lemma 2.15 with $(r, \mathrm{w}) \rightsquigarrow\left(r^{\prime}, \mathrm{w}^{\prime}\right)$ and $\rho:=r-r^{\prime}$. Note that (5.4) and (5.14) imply (2.42). Setting $\Phi:=\Phi_{S}^{1}$ we have that the conjugated Hamiltonian reads

$$
\begin{aligned}
H \circ \Phi & =D_{\omega}+\sum_{\mathrm{d}=1}^{\mathrm{N}-1} Z^{(\mathrm{d})}+\Pi_{\mathcal{K}} R^{(\mathrm{N})}+\left\{D_{\omega}, S\right\}+\Pi_{\mathcal{R}} R^{(\mathrm{N})}+\left(e^{L_{S}}-\mathrm{id}-\{\cdot, S\}\right) D_{\omega} \\
& +\left(e^{L_{S}}-\mathrm{id}\right)\left(\sum_{\mathrm{d}=1}^{\mathrm{N}-1} Z^{(\mathrm{d})}+R^{(\mathrm{N})}\right)+e^{L_{S}}\left(\sum_{\mathrm{d}=\mathrm{N}+1}^{\mathrm{K}} R^{(\mathrm{d})}+R^{(\geq \mathrm{K}+1)}\right) \\
& \stackrel{[5.13],[5.12]}{=} D_{\omega}+Z_{+}+R_{+},
\end{aligned}
$$

where

$$
\begin{aligned}
R_{+} & :=-\sum_{j=2}^{\infty} \frac{\left(L_{S}\right)^{j-1}}{j!} \Pi_{\mathcal{R}} R^{(\mathrm{N})} \\
& +\left(e^{L_{S}}-\mathrm{id}\right)\left(\sum_{\mathrm{d}=1}^{\mathrm{N}-1} Z^{(\mathrm{d})}+\sum_{\mathrm{d}=\mathrm{N}}^{\mathrm{K}} R^{(\mathrm{d})}+R^{(\geq \mathrm{K}+1)}\right)+\sum_{\mathrm{d}=\mathrm{N}+1}^{\mathrm{K}} R^{(\mathrm{d})}+R^{(\geq \mathrm{K}+1)} .
\end{aligned}
$$

Therefore we have

$$
\begin{align*}
R_{+} & :=\sum_{p=\mathrm{N}+1}^{\mathrm{K}} R_{+}^{(p)}+R_{+}^{(\geq \mathrm{K}+1)}, \\
R_{+}^{(p)} & :=R^{(p)}+\sum_{\substack{j \geq 2 \\
j \mathrm{~N}+\mathrm{N}=p}} \frac{\left(L_{S}\right)^{j-1}}{j!} \Pi_{\mathcal{R}} R^{(\mathrm{N})}+\sum_{\substack{1 \leq j, \mathrm{~d} \leq \mathrm{K} \\
j \mathrm{~N}+\mathrm{d}=p}} \frac{\left(L_{S}\right)^{j}}{j!} Z^{(\mathrm{d})}+\sum_{\substack{1 \leq j \leq \mathrm{K} \\
\mathrm{~N} \leq \mathrm{d} \leq \mathrm{K} \\
j \mathrm{~N}+\mathrm{d}=p}} \frac{\left(L_{S}\right)^{j}}{j!} R^{(\mathrm{d})} \tag{5.15}
\end{align*}
$$

and $R_{+}^{(\geq \mathrm{K}+1)}$ defined by difference. Moreover by, the explicit formulæ (5.15), Lemma 2.15, bounds (5.14), (2.41), the smallness assumption (5.4), Remark 2.14 and the monotonicity property (see Proposition 2.9) we get $R_{+} \in \mathcal{R}_{r^{\prime}}\left(\mathrm{h}_{\mathrm{w}^{\prime}}\right) \cap \mathcal{H}^{(>\mathrm{N})}$ which satisfies (5.8)-(5.9).
Finally, let us assume (5.10). By Proposition 4.1 let $S^{\sharp}=L_{\omega}^{-1} \Pi_{\mathcal{R}} R^{(\mathrm{N})}$ in $\mathcal{R}_{r}\left(\mathrm{~h}_{\mathrm{w}^{\sharp}}\right)$ be the solution of the homological equation $L_{\omega} S^{\sharp}=\Pi_{\mathcal{R}} R^{(\mathrm{N})}$ on $B_{r}\left(\mathrm{~h}_{\mathrm{w}^{\sharp}}\right) \subseteq B_{r}\left(\mathrm{~h}_{\mathrm{w}^{\prime}}\right)$ for any $\mathrm{w}^{\sharp} \geq \mathrm{w}^{\prime}$. Since $S$ and $S^{\sharp}$ solve the same linear equation on $B_{r}\left(\mathrm{~h}_{\mathrm{w}^{\sharp}}\right)$, we have that

$$
S^{\sharp}=S_{\left.\right|_{B_{r}\left(\mathbf{h}_{w^{\sharp}}\right)}} .
$$

By Proposition 4.1] we get

$$
\begin{equation*}
|S|_{r, w^{\sharp}} \leq \widetilde{J}_{0}^{\star}\left|R^{(\mathrm{N})}\right|_{r, w} . \tag{5.16}
\end{equation*}
$$

We now apply Lemma 2.15 with $(r, \mathrm{w}) \rightsquigarrow\left(r, \mathrm{w}^{\sharp}\right)$ and $\rho:=r-r^{\prime}$. Note that (5.10) and (5.16) imply (2.42). Then (5.11) follows by (2.43) and (5.16).

## 6. The iterative scheme

Here we apply repeatedly Lemma 5.1. Let $\bar{r}, s_{0}, p>0,0<\gamma<1$, fix a natural number $\mathrm{K} \geq 1$ and define $\mathrm{w}_{0}:=\mathrm{w}\left(s_{0}, p\right)$ (resp. $\mathrm{w}_{0}:=\mathrm{w}(p)$ ). Consider a Hamiltonian $H$ such that $H-D_{\omega} \in \mathcal{H}_{\bar{r}}\left(\mathrm{~h}_{\mathrm{w}_{0}}\right)$ satisfying

$$
\begin{align*}
& H:=D_{\omega}+R_{0}, \quad R_{0}=\sum_{\mathrm{d}=1}^{\mathrm{K}} R_{0}^{(\mathrm{d})}+R_{0}^{(\geq \mathrm{K}+1)},  \tag{6.1}\\
& R_{0}^{(\mathrm{d})} \in \mathcal{H}_{\bar{r}}\left(\mathrm{~h}_{\mathrm{w}_{0}}\right) \cap \mathcal{H}^{(\mathrm{d})}, \quad R_{0}^{(\geq \mathrm{K}+1)} \in \mathcal{H}_{\bar{r}}\left(\mathrm{~h}_{\mathrm{w}_{0}}\right) \cap \mathcal{H}^{(\geq \mathrm{K}+1)},
\end{align*}
$$

where $D_{\omega}$ is in (2.28), $\omega \in D_{\gamma}$ (see (3.4). Consider the constant C $>0$ provided by Proposition 4.1 and define

$$
\begin{gather*}
r_{0}^{\star}:=\min \left\{\bar{r},\left(\frac{4^{\mathrm{K}+3}\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{0}}}{\bar{r}} J_{\mathrm{K}}^{\star} 32 e \mathrm{~K}\right)^{-1}\right\}, \quad \star \in\{\mathrm{sE}, \mathrm{~S}\},  \tag{6.2}\\
J_{\mathrm{K}}^{\mathrm{sE}}:=\gamma^{-4 \mathrm{~K}} \exp \left(e^{\frac{\mathrm{k}^{2} \mathrm{c}}{s_{0}}}\right), \quad J_{\mathrm{K}}^{\mathrm{S}}:=\gamma^{-4 \mathrm{~K}} \exp \left(\mathrm{C} 2^{12} \mathrm{~K}^{3}\right) .
\end{gather*}
$$

Remark 6.1. Without loss of generality we can always assume that $\left|R_{0}\right|_{\bar{r}, w_{0}} \leq 1$. Indeed if $\left|R_{0}\right|_{\bar{r}, w_{0}}>1$ one can choose $\widetilde{r}<\bar{r}$ such that (recall Lemma 2.13)

$$
\left|R_{0}\right|_{\tilde{r}, w_{0}} \leq\left(\frac{\widetilde{r}}{\bar{r}}\right)\left|R_{0}\right|_{\bar{r}, w_{0}} \leq 1
$$

The main result of this section is the following.
Theorem 6.2. (Birkhoff normal form). Consider $H$ in 6.1). Then, for any $0<r_{0} \leq r_{0}^{\star}$ with $\star \in\{\mathrm{sE}, \mathrm{S}\}$, there exists a symplectic map

$$
\begin{align*}
& \Phi: B_{\frac{r_{0}}{2}}\left(\mathrm{~h}_{\mathrm{w}_{f}}\right) \rightarrow B_{r_{0}}\left(\mathrm{~h}_{\mathrm{w}_{f}}\right) \\
& \sup _{u \in B_{\frac{r_{0}}{2}}\left(\mathrm{~h}_{\mathrm{w}_{f}}\right)}|\Phi(u)-u|_{\mathrm{h}_{\mathrm{w}_{f}}} \leq \mathrm{C}_{1}^{\star} r_{0}^{2} \leq \frac{r_{0}}{8}, \quad \mathrm{C}_{1}^{\star}:=\frac{\mid R_{0}{\overline{\bar{r}}, \mathrm{w}_{0}}^{\bar{r}}}{\bar{r}} J_{\mathrm{K}}^{\star}, \tag{6.3}
\end{align*}
$$

where $\mathrm{w}_{f}:=\mathrm{w}\left(\frac{3}{2} s_{0}, p\right)$ (resp. $\mathrm{w}_{f}=\mathrm{w}(p+\zeta)$ with $\left.\zeta=36^{2} \sum_{i=1}^{\mathrm{K}} i^{2}\right)$ such that the following holds. The Hamiltonian

$$
\begin{align*}
& H_{f}:=H \circ \boldsymbol{\Phi}:=D_{\omega}+\mathfrak{Z}+\mathfrak{R}, \\
& \mathfrak{Z} \in \mathcal{K}_{\frac{r_{0}}{2}}\left(\mathrm{~h}_{\mathrm{w}_{f}}\right) \cap \mathcal{H}^{(\leq \mathrm{K})}, \quad \Re \in \mathcal{H}_{\frac{r_{0}}{2}}\left(\mathrm{~h}_{\mathrm{w}_{f}}\right) \cap \mathcal{H}^{(\geq \mathrm{K}+1)} \tag{6.4}
\end{align*}
$$

satisfies

$$
\begin{equation*}
|\mathfrak{Z}|_{\frac{r_{0}}{2}, w_{f}} \leq \mathrm{C}_{2}^{\star} r_{0}^{2}, \quad|\mathfrak{R}|_{\frac{r_{0}}{2}, w_{f}} \leq \mathrm{C}_{3}^{\star} r_{0}^{\mathrm{K}+1} \tag{6.5}
\end{equation*}
$$

with

$$
\mathrm{C}_{2}^{\star}:=\frac{16 e \mathrm{~K}\left|R_{0}\right|_{\bar{r}, \mathrm{~W}} 4^{\mathrm{K}+1}}{\bar{r}^{2}} J_{\mathrm{K}}^{\star}, \quad \mathrm{C}_{3}^{\star}:=\frac{\left|R_{0}\right|_{\bar{r}, \mathrm{~W}}\left(16 e \mathrm{~K} 4^{\mathrm{K}+2}\right)^{\mathrm{K}}}{\bar{r}^{\mathrm{K}+1}}\left(J_{\mathrm{K}}^{\star}\right)^{\mathrm{K}} .
$$

The proof of Theorem 6.2 is based on the following iterative scheme.
Setting of parameters. For any $0 \leq k \leq K$, let us recursively define:

$$
\begin{align*}
& r_{k}=r_{0}\left(1-\frac{k}{2 \mathrm{~K}}\right), \quad \delta_{k}=\frac{r_{k}-r_{k+1}}{16 e r_{k}}, \quad s_{k}=s_{0}\left(1+\frac{k}{2 \mathrm{~K}}\right), \quad \zeta_{k}:=(36 k)^{2} \\
& \sigma_{k}:=s_{0} \frac{k}{2 \mathrm{~K}}, \quad \mathrm{w}_{k}:=\mathrm{w}\left(s_{k}, p\right), \quad\left(\text { resp. } \mathrm{w}_{k}:=\mathrm{w}\left(p+\sum_{i=1}^{k} \zeta_{i}\right)\right) . \tag{6.6}
\end{align*}
$$

Moreover, let us define

$$
\begin{equation*}
\varepsilon:=\left(\frac{r_{0}}{\bar{r}}\right), \quad \mathrm{R}_{0}:=\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{0}} \tag{6.7}
\end{equation*}
$$

By Lemma 2.13 and Remark 6.1 we have that

$$
\begin{equation*}
\left|R_{0}^{(\mathrm{d})}\right|_{r_{0}, \mathrm{w}_{0}} \leq \varepsilon^{\mathrm{d}} \mathrm{R}_{0}, \quad 1 \leq \mathrm{d} \leq \mathrm{K}, \quad\left|R_{0}^{(\geq \mathrm{K}+1)}\right|_{r_{0}, \mathrm{w}_{0}} \leq \varepsilon^{\mathrm{K}+1} \mathrm{R}_{0} \tag{6.8}
\end{equation*}
$$

Let us introduce

$$
\begin{equation*}
J_{k}:=J_{0}^{\mathrm{sE}}\left(\sigma_{k}, k\right), \quad\left(\text { resp. } J_{k}:=J_{0}^{\mathrm{S}}\left(\sum_{i=1}^{k} \zeta_{i}, k\right)\right) \quad 0 \leq k \leq \mathrm{K} \tag{6.9}
\end{equation*}
$$

where $J_{0}^{\mathrm{sE}}, J_{0}^{\mathrm{S}}$ are introduced in (4.3), (4.4) respectively, and assume the following smallness condition:

$$
\begin{equation*}
\mathrm{R}_{0} 4^{\mathrm{K}+3} J_{\mathrm{K}} \varepsilon \leq \delta_{0} \tag{6.10}
\end{equation*}
$$

We now prove the following.
Lemma 6.3. (Iteration lemma). The following holds true for any $0 \leq k \leq \mathrm{K}$ :
$(\mathbf{S 1})_{k}$ there are Hamiltonians $H_{k}$ of the form

$$
\begin{align*}
& H_{k}=D_{\omega}+Z_{k}+R_{k} \\
& Z_{k}:=\sum_{\substack{1 \leq \mathrm{d} \leq k \\
\mathrm{~d} \text { even }}} Z_{k}^{(\mathrm{d})}, \quad R_{k}:=\sum_{\mathrm{d}=k+1}^{\mathrm{K}} R_{k}^{(\mathrm{d})}+R_{k}^{(\geq \mathrm{K}+1)} \tag{6.11}
\end{align*}
$$

where $Z_{0} \equiv 0$, and

$$
\begin{array}{rlrl}
Z_{k}^{(\mathrm{d})} & \in \mathcal{K}_{r_{k}}\left(\mathrm{~h}_{\mathrm{w}_{k}}\right) \cap \mathcal{H}^{(\mathrm{d})}, & & 1 \leq \mathrm{d} \leq k \\
R_{k}^{(\mathrm{d})} & \in \mathcal{H}_{r_{k}}\left(\mathrm{~h}_{\mathrm{w}_{k}}\right) \cap \mathcal{H}^{(\mathrm{d})}, & & k+1 \leq \mathrm{d} \leq \mathrm{K}  \tag{6.12}\\
R_{k}^{(\geq \mathrm{K}+1)} & \in \mathcal{H}_{r_{k}}\left(\mathrm{~h}_{\mathrm{w}_{k}}\right) \cap \mathcal{H}^{(\geq \mathrm{K}+1)} ; &
\end{array}
$$

(S2) ${ }_{k}$ one has, for $1 \leq k \leq K$,

$$
\begin{equation*}
J_{k}\left(\sum_{\mathrm{d}=k}^{\mathrm{K}}\left|R_{k-1}^{(\mathrm{d})}\right|_{r_{k-1}, \mathrm{w}_{k-1}}+\left|R_{k-1}^{(\geq \mathrm{K}+1)}\right|_{r_{k-1}, \mathrm{w}_{k-1}}\right) \leq \delta_{k-1} \tag{6.13}
\end{equation*}
$$

$(\mathbf{S 3})_{k}$ one has, for $1 \leq k \leq K$,

$$
\begin{align*}
& \quad\left|Z_{k}^{(\mathrm{d})}\right|_{r_{k}, \mathrm{w}_{k}} \leq \varepsilon^{\mathrm{d}} \mathrm{R}_{0}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{\mathrm{d}-1} 2^{\mathrm{d}-1}, \quad 1 \leq \mathrm{d} \leq k  \tag{6.14}\\
& \quad\left|R_{k}^{(\mathrm{d})}\right|_{r_{k}, \mathrm{w}_{k}} \leq \varepsilon^{\mathrm{d}} \mathrm{R}_{0}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{\mathrm{d}-1} 2^{k-1}, \quad k+1 \leq \mathrm{d} \leq \mathrm{K}  \tag{6.15}\\
& \left|R_{k}^{(\geq \mathrm{K}+1)}\right|_{r_{k}, \mathrm{w}_{k}} \leq \varepsilon^{\mathrm{K}+1} \mathrm{R}_{0}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{\mathrm{K}} 2^{k} \tag{6.16}
\end{align*}
$$

$(\mathbf{S} 4)_{k}$ for any $1 \leq k \leq \mathrm{K}$ there are maps $\Phi_{k}: B_{r_{k}}\left(\mathrm{~h}_{\mathrm{w}_{k}}\right) \rightarrow B_{r_{k-1}}\left(\mathrm{~h}_{\mathrm{w}_{k}}\right)$ such that

$$
\begin{equation*}
H_{k}=H_{k-1} \circ \Phi_{k} \tag{6.17}
\end{equation*}
$$

Moreover, for any $k \leq n \leq \mathrm{K}$, one has

$$
\begin{equation*}
\Phi_{k}: B_{r_{k}}\left(\mathrm{~h}_{\mathrm{w}_{n}}\right) \rightarrow B_{r_{k-1}}\left(\mathrm{~h}_{\mathrm{w}_{n}}\right) \tag{6.18}
\end{equation*}
$$

with

$$
\begin{equation*}
\sup _{u \in B_{r_{k}}\left(\mathrm{~h}_{\mathrm{w}_{n}}\right)}\left|\Phi_{k}(u)-u\right|_{\mathrm{w}_{n}} \leq r_{k-1} \mathrm{R}_{0} \frac{1}{2^{k}} J_{\mathrm{K}} \varepsilon \tag{6.19}
\end{equation*}
$$

Proof. We reason inductively and apply iteratively Lemma 5.1. Assume that $(\mathbf{S i})_{k}, \mathbf{i}=1, \ldots, 4$, hold for $0 \leq k \leq \mathrm{K}-2$. We claim that

$$
\begin{equation*}
J_{k+1}\left(\sum_{\mathrm{d}=k+1}^{\mathrm{K}}\left|R_{k}^{(\mathrm{d})}\right|_{r_{k}, \mathrm{w}_{k}}+\left|R_{k}^{(\geq \mathrm{K}+1)}\right|_{r_{k}, \mathrm{w}_{k}}\right) \leq \delta_{k} \tag{6.20}
\end{equation*}
$$

which is condition $(\mathbf{S 2})_{k+1}(\operatorname{see}(6.13)$ with $k \rightsquigarrow k+1)$. First of all notice that (recall 4.3), 6.6)

$$
\begin{equation*}
J_{k+1} \leq J_{\mathrm{K}}, \quad \forall 0 \leq k \leq \mathrm{K}-1 \tag{6.21}
\end{equation*}
$$

By the inductive assumption $(\mathbf{S 3})_{k}-6.15$ (see 6.11) one can note that

$$
\begin{aligned}
& \sum_{\mathrm{d}=k+1}^{\mathrm{K}}\left|R_{k}^{(\mathrm{d})}\right|_{r_{k}, \mathrm{w}_{k}}+\left|R_{k}^{(\geq \mathrm{K}+1)}\right|_{r_{k}, \mathrm{w}_{k}} \\
& \quad \leq \sum_{\mathrm{d}=k+1}^{\mathrm{K}} \mathrm{R}_{0} \varepsilon^{\mathrm{d}}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{\mathrm{d}-1} 2^{k-1}+\mathrm{R}_{0} \varepsilon^{\mathrm{K}+1}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{\mathrm{K}} 2^{k} \\
& \quad \leq 2^{k-1} \mathrm{R}_{0} \varepsilon^{k+1}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{k}\left[\sum_{j=0}^{\mathrm{K}-(k+1)} \varepsilon^{j}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{j}+\varepsilon^{\mathrm{K}-k}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{\mathrm{K}-k} 2\right] \\
& \\
& \quad \frac{6.10}{\leq} \mathrm{R}_{0} \varepsilon^{k+1}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{k} 2^{k}
\end{aligned}
$$

Recalling 6.9, (4.3), we deduce that

$$
\begin{aligned}
J_{k+1}\left(\sum_{\mathrm{d}=k+1}^{\mathrm{K}}\left|R_{k}^{(\mathrm{d})}\right|_{r_{k}, \mathrm{w}_{k}}\right. & \left.+\left|R_{k}^{(\geq \mathrm{K}+1)}\right|_{r_{k}, \mathrm{w}_{k}}\right) \leq \mathrm{R}_{0} \varepsilon^{k+1}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{k} 2^{k} J_{k+1} \\
& \leq \mathrm{R}_{0}\left(\varepsilon 4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{k+1} 2^{k} 4^{-k} \delta_{0} \stackrel{\sqrt[6.10]{\leq}}{\leq} \delta_{0} \stackrel{6.6}{\leq} \delta_{k}
\end{aligned}
$$

which proves the claim. Condition 6.20 implies the smallness assumption (5.4) by setting

$$
\epsilon_{\mathrm{d}}:=\left|R_{k}^{(\mathrm{d})}\right|_{r_{k}, \mathrm{w}_{k}}, \quad \epsilon_{\mathrm{K}+1}:=\left|R_{k}^{(\geq \mathrm{K}+1)}\right|_{r_{k}, \mathrm{w}_{k}}
$$

Therefore Lemma 5.1 applies with

$$
\begin{equation*}
\mathrm{N}, \sigma, r, r^{\prime}, \mathrm{w}, \mathrm{w}^{\prime} \rightsquigarrow k+1, \sigma_{k}, r_{k}, r_{k+1}, \mathrm{w}_{k}, \mathrm{w}_{k+1}, \quad Z^{(\mathrm{d})}, R^{(\mathrm{d})}, R^{(\geq \mathrm{K}+1)} \rightsquigarrow Z_{k}^{(\mathrm{d})}, R_{k}^{(\mathrm{d})}, R_{k}^{(\geq \mathrm{K}+1)} \tag{6.22}
\end{equation*}
$$

Then we obtain a map $\Phi_{k+1}$ satisfying (6.17) with $k \rightsquigarrow k+1$ (see (5.5)-(5.6). In particular the new hamiltonian $H_{k+1}$ has the form (6.11) (with $k \rightsquigarrow k+1$ ) for some Hamiltonians $Z_{k+1}, R_{k+1}$. This proves $(\mathbf{S 1})_{k+1}$. Let us check the $(\mathbf{S 3})_{k+1}$, i.e. the bounds (6.14)-6.16) with $k \rightsquigarrow k+1$. By (5.7) we have

$$
\begin{equation*}
Z_{k+1}=\sum_{\mathrm{d}=1}^{k+1} Z_{k+1}^{(\mathrm{d})}, \quad Z_{k+1}^{(\mathrm{d})}:=Z_{k}^{(\mathrm{d})}, \quad 1 \leq \mathrm{d} \leq k, \quad Z_{k+1}^{(k+1)}:=\Pi_{\mathcal{K}} R_{k}^{(k+1)} \tag{6.23}
\end{equation*}
$$

For $1 \leq \mathrm{d} \leq k$ one has

$$
\left|Z_{k+1}^{(\mathrm{d})}\right|_{r_{k+1}, \mathrm{w}_{k+1}} \stackrel{\boxed{6.23}}{=}\left|Z_{k}^{(\mathrm{d})}\right|_{r_{k+1}, \mathrm{w}_{k+1}} \stackrel{\sqrt{6.14}}{\leq} \mathrm{R}_{0} \varepsilon^{\mathrm{d}}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{\mathrm{d}-1} 2^{\mathrm{d}-1} \leq \mathrm{R}_{0} \varepsilon^{\mathrm{d}}\left(4^{k+1} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{\mathrm{d}-1} 2^{\mathrm{d}-1}
$$

where the last inequality is trivial. Moreover for $\mathrm{d}=k+1$ one has

This proves the bound (6.14) with $k \rightsquigarrow k+1$ on the Hamiltonians $Z_{k+1}^{(\mathrm{d})}$ with $1 \leq \mathrm{d} \leq k+1$. We notice that $Z_{k+1} \equiv 0$ when $k+1$ is odd (see Remark 2.20. We now prove the estimate (6.15) with $k \rightsquigarrow k+1$ on
$R_{k+1}$. By (5.8), (5.3), (6.22), (6.9), (6.21), (6.6), and using (6.14), (6.15) and the fact that $\mathrm{R}_{0} \leq 1$, we get, for any $k+2 \leq p \leq \mathrm{K}$,

$$
\begin{aligned}
\left|R_{k+1}^{(p)}\right|_{r_{k+1}, \mathrm{w}_{k+1}} & \leq \mathrm{R}_{0} \varepsilon^{p}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{p-1} 2^{k-1} \\
& +\mathrm{R}_{0} \sum_{\substack{(j-1)(k+1)+(k+1)=p}} \frac{\varepsilon^{k+1}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{k} 2^{k-1}}{j!}\left(\frac{\varepsilon^{k+1}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{k} 2^{k-1} J_{\mathrm{K}}}{2 \delta_{0}}\right)^{j-1} \\
& +\mathrm{R}_{0} \sum_{\substack{1 \leq j, \mathrm{~d} \leq \mathrm{K} \\
j \mathrm{~N}+\mathrm{d}=p}} \frac{\varepsilon^{\mathrm{d}}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{\mathrm{d}-1} 2^{k-1}}{j!}\left(\frac{\varepsilon^{k+1}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{k} 2^{k-1} J_{\mathrm{K}}}{2 \delta_{0}}\right)^{j} \\
& +\mathrm{R}_{0} \sum_{\substack{1 \leq j \leq \mathrm{K} \\
\mathrm{~N} \leq \mathrm{d} \leq \mathrm{K}}} \frac{\varepsilon^{\mathrm{d}}\left(2^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{\mathrm{d}-1} 2^{k-1}}{j!}\left(\frac{\varepsilon^{k+1}\left(2^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{k} 2^{k-1} J_{\mathrm{K}}}{2 \delta_{0}}\right)^{j} \\
& \leq \mathrm{R}_{0} \varepsilon^{p}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{p-1} 2^{k-1} \\
& +\mathrm{R}_{0} \varepsilon^{p}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{p-1} 2^{k-1} \sum_{j \geq 2} \frac{1}{j!} \frac{2^{(k-1)(j-1)}}{2^{(j-1)} 4^{k(j-1)}} \\
& +\mathrm{R}_{0} \varepsilon^{p}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{p-1} 2^{k-1} 2 \cdot \sum_{j \geq 1} \frac{1}{j!} \frac{2^{(k-1) j}}{2^{j} 4^{k j}} \\
& \leq \mathrm{R}_{0} \varepsilon^{p}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{p-1} 2^{k-1}\left(1+\sum_{j \geq 2} \frac{1}{j!}\left(\frac{2^{k-1}}{2 \cdot 4^{k}}\right)^{j-1}+\sum_{j \geq 1} \frac{1}{j!}\left(\frac{2^{k}}{2 \cdot 4^{k}}\right)^{j}\right) \\
& \leq \mathrm{R}_{0} \varepsilon^{p}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{p-1} 2^{k-1}\left(\sum_{j \geq 0} \frac{1}{j!}\left(\frac{1}{2^{k}}\right)^{j}\right) \\
& \leq \mathrm{R}_{0} \varepsilon^{p}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{p-1} 2^{k-1} \sqrt{e} \leq \\
& \leq \mathrm{R}_{0} \varepsilon^{p}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{p-1} 2^{k} \leq \mathrm{R}_{0} \varepsilon^{p}\left(4^{k+1} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{p-1} 2^{k} \\
&
\end{aligned}
$$

which is the 6.15 with $k \rightsquigarrow k+1$. The estimate for the term $R_{k+1}^{(\geq \mathrm{K}+1)}$ follows similarly.
It remains to show the $(\mathbf{S 4})_{k+1}$, i.e. the bound (6.19) with $k \rightsquigarrow k+1$. By estimate (5.11) (recalling again (6.22)) and taking $k \leq n \leq \mathrm{K}$ we get

$$
\begin{aligned}
\sup _{u \in B_{r_{k+1}}\left(\mathrm{~h}_{\mathrm{w}_{n}}\right)}\left|\Phi_{k+1}(u)-u\right|_{\mathrm{w}_{n}} & \leq r_{k} J_{k+1}\left|R_{k}^{(k+1)}\right|_{r_{k}, \mathrm{w}_{k}} \\
& \stackrel{6.15}{\leq} r_{k} J_{\mathrm{K}} \varepsilon^{k+1} \mathrm{R}_{0}\left(4^{k} J_{\mathrm{K}} \delta_{0}^{-1}\right)^{k} 2^{k-1}
\end{aligned}
$$

which implies the thesis 6.19 using the smallness condition 6.10).
Proof of Theorem 6.2, The condition 6.2 and the choice of $\varepsilon$ in 6.7 imply the smallness condition 6.10, so the iterative lemma 6.3 applies. By (5.11) we have (recall 6.6)

$$
\Phi_{k}: B_{r_{k}}\left(\mathrm{~h}_{\mathrm{w}_{\mathrm{K}}}\right) \rightarrow B_{r_{k-1}}\left(\mathrm{~h}_{\mathrm{WK}}\right)
$$

with

$$
\sup _{u \in B_{r_{k}}\left(\mathrm{~h}_{\mathrm{w}_{\mathrm{K}}}\right)}\left|\Phi_{k}(u)-u\right|_{\mathrm{w}_{\mathrm{K}}} \leq r_{k-1} \mathrm{R}_{0} \frac{1}{2^{k}} J_{\mathrm{K}}^{\star} \varepsilon
$$

Then, defining

$$
\mathbf{\Phi}:=\Phi_{1} \circ \cdots \circ \Phi_{\mathrm{K}}
$$

we immediately get

$$
\boldsymbol{\Phi}: B_{\frac{r_{0}}{2}}\left(\mathrm{~h}_{\mathrm{w}_{\mathrm{K}}}\right) \rightarrow B_{r_{0}}\left(\mathrm{~h}_{\mathrm{wK}_{\mathrm{K}}}\right)
$$

which, adding and subtracting id to each $\Phi_{i}$ with $i=1, \ldots \mathrm{~K}$ entails

$$
\boldsymbol{\Phi}-\mathrm{id}=\left(\Phi_{1}-\mathrm{id}\right) \circ \Phi_{2} \circ \cdots \circ \Phi_{\mathrm{K}}+\left(\Phi_{2}-\mathrm{id}\right) \circ \Phi_{3} \circ \cdots \circ \Phi_{\mathrm{K}}+\cdots+\Phi_{\mathrm{K}}-\mathrm{id} .
$$

Hence we get the estimate
which implies 6.3) by using the definitions of $J_{\mathrm{K}}, \varepsilon$ and $\mathrm{R}_{0}$, the smallness assumption (6.2) on $r_{0}$ and setting $\mathrm{w}_{f}=\mathrm{w}_{\mathrm{K}}$.

The new Hamiltonian $H \circ \boldsymbol{\Phi}$ is equal to $H_{\mathrm{K}}$ given in (6.11) with $k \rightsquigarrow \mathrm{~K}$. We then set $\mathfrak{Z}:=Z_{\mathrm{K}}$ and $\mathfrak{R}:=R_{\mathrm{K}}$. The (6.4) follows. The bounds (6.5) follow by (6.14)-(6.16) recalling (6.9), (6.8), (6.7).

## 7. Times of Stability

In this section we provide the proof of our main stability results for the solutions of the equation (1.1). The proof basically rely on a combination of the Birkhoff normal form result of Theorem 6.2 and Lemma 2.16. In view of section 2.1 we have that equation (1.1) is equivalent to (2.6). In particular, recalling (2.5), for any $\left(\psi_{0}, \psi_{1}\right) \in H^{s, p+1} \times H^{s, p-1}$ we define

$$
\begin{equation*}
u_{0}:=\frac{1}{\sqrt{2}}\left(\omega^{\frac{1}{2}} \psi_{0}+\mathrm{i} \omega^{-\frac{1}{2}} \psi_{0}\right) . \tag{7.1}
\end{equation*}
$$

One can notice that condition (1.17) (resp. (1.10) implies that the function $u_{0}$ in (7.1) satisfies

$$
\begin{equation*}
\left\|u_{0}\right\|_{s, p} \leq \delta \tag{7.2}
\end{equation*}
$$

Assume now that $u(t)$ is a solution of (2.6), with initial condition $u(0)=u_{0}$, satisfying

$$
\begin{equation*}
\sup _{t \in\left[0, T_{0}\right]}\|u(t)\|_{s, p} \leq 2 \delta \quad \text { for some } \quad T_{0}>0 \tag{7.3}
\end{equation*}
$$

Hence the solution $\left(\psi(t), \partial_{t} \psi(t)\right)$ of (1.1) with initial conditions $\left(\psi_{0}, \psi_{1}\right)$ satisfies the a priori bound

$$
\|\psi(t)\|_{s, p+1}+\left\|\partial_{t} \psi(t)\right\|_{s, p-1} \leq 4\|u(t)\|_{s, p} \leq 8 \delta, \quad \forall t \in\left[0, T_{0}\right]
$$

which implies (1.18) (resp. 1.11)).
The discussion above implies that in order to proof Theorems $1.4,1.2$ we just have to prove the claim (7.3) on solutions of (2.6) with initial conditions satisfying (7.2) and we shall provide suitable lower bounds on the lifespan $T_{0}>0$.

In order to apply our abstract Birkhoff normal form result we need some preliminary results. More precisely we shall prove that the the Hamiltonian function $H$ in (2.7) (see also (2.12)) can be written in the form (6.1) with

$$
\begin{equation*}
R_{0}:=\mathrm{H}_{\geq 3}:=\int_{\mathbb{T}} F\left(\frac{\omega^{-1 / 2}(u+\bar{u})}{\sqrt{2}}\right) d x \tag{7.4}
\end{equation*}
$$

where $F$ is the analytic function in (1.3). This is the content of the following lemma.
Lemma 7.1. Let $R>0$ as in (1.3) and consider the Hamiltonian $R_{0}$ in (7.4).
Case (sE). Fix s, p>0 as in Theorem 1.4 and let (recall (1.6)

$$
\begin{equation*}
s_{0}:=\frac{2}{3} s, \quad \mathrm{w}_{0}:=\mathrm{w}\left(s_{0}, p\right) . \tag{7.5}
\end{equation*}
$$

For any $\bar{r}>0$ satisfying (see 2.20)

$$
\begin{equation*}
2 \mathrm{C}_{\mathrm{alg}}(p) \bar{r}<R, \tag{7.6}
\end{equation*}
$$

one has that the Hamiltonian $R_{0}$ in (7.4) belongs to the space $\mathcal{H}_{\bar{r}}\left(\mathrm{~h}_{\mathrm{w}_{0}}\right)$ of regular Hamiltonians (see Def. 2.3), and

$$
\begin{equation*}
\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{0}} \leq \mathrm{C}(p, R)|F|_{R} \bar{r}<+\infty, \quad \mathrm{C}(p, R):=\frac{8 C_{\mathrm{alg}}(p)}{R^{3}} \tag{7.7}
\end{equation*}
$$

Case (S). Fix $s=0$ and let

$$
\begin{equation*}
\mathrm{w}_{0}:=\mathrm{w}\left(p_{0}\right), \tag{7.8}
\end{equation*}
$$

For any $p_{0}>1$ and for any $\bar{r}>0$ satisfying (recall (2.20))

$$
\mathrm{C}_{\mathrm{alg}, \mathrm{M}}\left(p_{0}\right) \bar{r}<R,
$$

one has that the Hamiltonian $R_{0}$ in (7.4) belongs to the space $\mathcal{H}_{\bar{r}}\left(\mathrm{~h}_{\mathrm{w}_{0}}\right)$ of regular Hamiltonians and

$$
\begin{equation*}
\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{0}} \leq \frac{\mathrm{C}_{\mathrm{alg}, \mathrm{M}}\left(p_{0}\right)}{R}|F|_{R} \bar{r}<+\infty . \tag{7.9}
\end{equation*}
$$

Proof. Let us consider the case (sE). It follows using the analyticity of the function $F$ in (1.3) and Lemma 2.1. and reasoning as in Proposition 5.2 in [BMP20a].

By analyticity we have that

$$
\begin{aligned}
R_{0}=\int_{\mathbb{T}} F\left(\frac{\omega^{-1 / 2}(u+\bar{u})}{\sqrt{2}}\right) d x & =\sum_{d=3}^{\infty} F^{(d)} \int_{\mathbb{T}}\left(\frac{\omega^{-1 / 2}(u+\bar{u})}{\sqrt{2}}\right)^{d} d x \\
& =\sum_{d=3}^{\infty} \frac{F^{(d)}}{2^{d / 2}} \sum_{k=0}^{d}\binom{d}{k} \int_{\mathbb{T}}\left(\omega^{-\frac{1}{2}} u\right)^{k}\left(\omega^{-\frac{1}{2}} \bar{u}\right)^{d-k} d x
\end{aligned}
$$

hence, passing to the Fourier basis, we get

$$
R_{0}=\sum_{d=3}^{\infty} \frac{F^{(d)}}{2^{d / 2}} \sum_{k=0}^{d}\binom{d}{k} \sum_{\sum_{i=1}^{k} j_{i}=\sum_{p=k+1}^{d} j_{p}} C^{(d)}\left(j_{1}, \ldots, j_{d}\right) u_{j_{1}} \cdots u_{j_{k}} \bar{u}_{j_{k+1}} \cdots \bar{u}_{j_{d}}
$$

where the coefficients $C^{(d)}\left(j_{1}, \ldots, j_{d}\right)$ are symmetric in $j_{i}, i=1, \ldots, d$, and have the form

$$
\begin{equation*}
C^{(d)}\left(j_{1}, \ldots, j_{d}\right):=\prod_{i=1}^{d} \omega^{-\frac{1}{2}}\left(j_{i}\right) \stackrel{\sqrt[2.1 \mid]{=}}{=} \prod_{i=1}^{d} \frac{1}{\sqrt[4]{\left|j_{i}\right|^{4}+\mathrm{m}}} \leq 1 \tag{7.10}
\end{equation*}
$$

In view of (2.6) we now compute the first component of the vector field $X_{R_{0}}$. We have

$$
\begin{aligned}
X_{R_{0}}^{(j)} & :=-\mathrm{i} \partial_{\bar{u}_{j}} R_{0}=-\mathrm{i} \sum_{d=3}^{\infty} \frac{F^{(d)}}{2^{d / 2}} \sum_{k=0}^{d}\binom{d}{k}(d-k) \mathrm{a}^{(d, k)}(j), \\
\mathrm{a}^{(d, k)}(j) & :=\sum_{\sum_{i=1}^{k} j_{i}-\sum_{p=k+1}^{d-1} j_{p}=j} C^{(d)}\left(j_{1}, \ldots, j\right) u_{j_{1}} \cdots u_{j_{k}} \bar{u}_{j_{k+1}} \cdots \bar{u}_{j_{d-1}} .
\end{aligned}
$$

where we used the symmetry of the coefficients $C^{(d)}\left(j_{1}, \ldots, j\right)$. Notice that

$$
\left|X_{\underline{R_{0}}}(u)\right|_{\mathfrak{w}(s, p)} \leq\left|X_{\underline{R_{0}}}(\underline{u})\right|_{\mathrm{w}(s, p)} \leq \sum_{d=0}^{\infty} \frac{\left|F^{(d)}\right|}{2^{d / 2}} \sum_{k=0}^{d}\binom{d}{k}(d-k)\left|\mathrm{a}^{(d, k)}\right|_{\mathrm{w}(s, p)}
$$

where $\underline{u}=\left(\left|u_{j}\right|\right)_{j \in \mathbb{Z}}$ and for any $0 \leq k \leq d, d \geq 3$ we set $\mathbf{a}^{(d, k)}:=\left(\mathbf{a}^{(d, k)}(j)\right)_{j \in \mathbb{Z}}$. Notice moreover that

$$
\left|\mathrm{a}^{(d, k)}(j)\right| \leq|(\underbrace{u \star \ldots u}_{k} \star \underbrace{\bar{u} \star \ldots \star \bar{u}}_{d-k-1})_{j}|, \quad \forall j \in \mathbb{Z} .
$$

Therefore, using estimate 2.18 in Lemma 2.1, we obtain

$$
\begin{aligned}
\left|X_{\underline{R_{0}}}(u)\right|_{\mathrm{w}(s, p)} & \leq \sum_{d=3}^{\infty} \frac{\mid F^{(d) \mid}}{2^{d / 2}} \sum_{k=0}^{d}\binom{d}{k}(d-k)\left(C_{\mathrm{alg}}(p)\right)^{d-2}|u|_{\mathrm{w}(s, p)}^{d-1} \\
& \leq \sum_{d=3}^{\infty} d 2^{\frac{d}{2}-1}\left|F^{(d)}\right|\left(C_{\mathrm{alg}}(p)\right)^{d-2}|u|_{\mathrm{w}(s, p)}^{d-1} \leq 4 \sum_{d=3}^{\infty}\left|F^{(d)}\right|\left(2 C_{\mathrm{alg}}(p)\right)^{d-2}|u|_{\mathrm{w}(s, p)}^{d-1}
\end{aligned}
$$

Then

$$
\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{0}} \leq \sum_{d=3}^{\infty}\left|F^{(d)}\right|\left(2 C_{\mathrm{alg}}(p) \bar{r}\right)^{d-2}
$$

which implies the bound (7.7) using (7.6) and the hypothesis (1.3). This proves item $(i)$.
The case $(\mathrm{S})$ in item $(i i)$ can be proved by following almost word by word the proof of item $(i)$ using the estimate 2.19 instead of 2.18 in Lemma 2.1 .
7.1. Sub-exponential stability and proof of Theorem 1.4. For future convenience we set

$$
\begin{equation*}
\bar{r}:=\frac{R}{\mathrm{C}_{\mathrm{alg}}(p) \mathrm{C}(p, R)|F|_{R}} \tag{7.11}
\end{equation*}
$$

where $\mathrm{C}(p, R)$ is given in 7.7). Therefore by case sE in Lemma 7.1 we have that the Hamiltonian $H$ in (2.7) can be written, for any $\mathrm{K} \geq 1$, in the form (6.1) with $s_{0}, \mathrm{w}_{0}$ as in 7.5 . Our aim is to apply Theorem 6.2 to the Hamiltonian $H=D_{\omega}+R_{0}$ with $R_{0}$ in (7.5). Recalling the parameters in (6.6)-(6.7) we fix

$$
\begin{equation*}
r_{0}=2 \delta \tag{7.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{K}=\mathrm{K}\left(r_{0}\right):=\left[\left(\frac{\gamma^{4} s_{0}}{2^{10} \mathrm{C}}\right)^{\frac{1}{2}}\left(\frac{1}{2^{8}} \ln \ln \frac{\bar{r}}{r_{0}}\right)^{\frac{\mathrm{q}-1}{2}}\right] \tag{7.13}
\end{equation*}
$$

where $[\cdot]$ is the integer part and where $C>0$ is the absolute constant given by Proposition 4.1 .
We claim that 7.12 and 1.16 implies that $r_{0}$ satisfies the smallness condition 6.2 with $\mathrm{K}\left(r_{0}\right)$ in 7.13 ) and $J_{\mathrm{K}}^{*}=J_{\mathrm{K}}^{\mathrm{sE}}$. In other words we prove that

$$
\mathrm{C}_{0}:=32 e 4^{3}\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{0}} \mathrm{~K}\left(\frac{4}{\gamma^{4}}\right)^{\mathrm{K}} \exp \exp \left(\left(\frac{\mathrm{~K}^{2} \mathrm{C}}{s_{0}}\right)^{\frac{1}{\mathrm{q}-1}}\right) \frac{r_{0}}{\bar{r}} \leq 1
$$

In fact, with that choice of $K$, we can easily check that conditions (7.12) and (1.16) actually implies a much stronger bound:

$$
\begin{aligned}
& \mathrm{C}_{0} \leq 2^{16}\left|R_{0}\right|_{\bar{r}, w_{0}} \exp \left\{\exp \left(\left(\mathrm{~K}^{2} \frac{2^{4} \mathrm{C}}{\gamma^{4} s_{0}}\right)^{\frac{1}{\mathrm{q}-1}}\right)-\ln \frac{\bar{r}}{r_{0}}\right\} \\
& \quad \stackrel{\overline{7.13]}}{\leq} 2^{16}\left|R_{0}\right|_{\bar{r}, w_{0}} \exp \left\{\exp \left(2^{-14} \ln \ln \frac{\bar{r}}{r_{0}}\right)-\ln \frac{\bar{r}}{r_{0}}\right\} \\
& \quad \leq 2^{[7.7]} \leq 2^{16} \mathrm{C}(p, R)|F|_{R} \bar{r} \exp \left\{\left(\ln \frac{\bar{r}}{r_{0}}\right)^{\frac{1}{2^{14}}}-\ln \frac{\bar{r}}{r_{0}}\right\} \\
&=\left.\right|_{R} \bar{r} \exp \left\{-\frac{1}{2} \ln \frac{\bar{r}}{r_{0}}\right\} \leq 1
\end{aligned}
$$

where the last bound holds provided that

$$
\begin{equation*}
r_{0} / \bar{r} \leq \exp 2^{\frac{2^{14}}{1-2^{14}}} \tag{7.14}
\end{equation*}
$$

and

$$
\begin{equation*}
r_{0} \leq \frac{\bar{r}}{2^{32}\left(\mathrm{C}(p, R)|F|_{R} \bar{r}\right)^{2}} \stackrel{\sqrt{7.11]}}{=} \frac{\mathrm{C}_{\text {alg }}(p)}{2^{28} C(p, R)|F|_{R} R} \tag{7.15}
\end{equation*}
$$

are satisfied. Since $r_{0}=2 \delta$ and $\delta \leq \delta_{\mathrm{sE}}$, we have that condition (1.16) implies 7.15) if one requires

$$
\begin{equation*}
C_{2}=C_{2}(p, R) \geq \frac{2^{29} C(p, R) R}{\mathrm{C}_{\mathrm{alg}}(p)} . \tag{7.16}
\end{equation*}
$$

Using (7.12) we notice that (1.16) implies (7.14) if one requires

$$
\begin{equation*}
C_{2}=C_{2}(p, R) \geq \frac{8 \mathrm{C}_{\mathrm{alg}}(p) C(p, R)}{R} . \tag{7.17}
\end{equation*}
$$

Theorem 6.2, together with Lemma 2.16 (recall also Remark 2.19 ), implies that the solution $u(t)$ of (2.6) evolving from initial data satisfying (7.2) remains in the ball of radius $2 \delta$ for time $t \in\left[0, T_{0}\right]$ (i.e. the bound (7.3) is satisfied) with (see the bound (6.5))

$$
T_{0}:=\frac{\bar{r}^{\mathrm{K}+1}}{8 r_{0}^{\mathrm{K}+1}\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{f}}} \frac{\left(16 e \mathrm{~K} 4^{\mathrm{K}+2}\right)^{-\mathrm{K}}}{\left(\gamma^{-4 \mathrm{~K}} \exp \exp \mathrm{~K}^{2} c / s_{0}\right)^{\mathrm{K}}} \geq \frac{1}{8\left|R_{0}\right|_{\bar{r}, w_{f}}} \frac{\bar{r}}{r_{0}} \frac{\left(\frac{\bar{r}}{r_{0}}\right)^{\mathrm{K}}}{\left(\left(\frac{2^{9}}{\gamma^{4}}\right)^{\mathrm{K}} \mathrm{~K} \exp \exp \mathrm{~K}^{2} c / s_{0}\right)^{\mathrm{K}}} .
$$

We have to show that, thanks to (7.13) and (7.11) and the smallness of $r_{0}$, the time $T_{0}$ above satisfies the bound (1.19). Actually we prove the slightly better bound

$$
T_{0} \geq \frac{1}{8\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{f}}} \frac{\bar{r}}{r_{0}} \exp \left(\frac{1}{2} \ln \left(\frac{\bar{r}}{r_{0}}\right)\left(\frac{\gamma^{4} s_{0}}{2^{16} \mathrm{C}} \ln \ln \frac{\bar{r}}{r_{0}}\right)^{\frac{\mathrm{q}-1}{2}}\right)
$$

Indeed $\delta_{\mathrm{sE}} \leq \bar{r} / 2$ using (7.11) and (1.16) and taking $C_{2}(p, R)$ as in (7.17). Let us observe that

$$
\begin{aligned}
\exp \left(\mathrm{K} \ln \frac{r_{0}}{\bar{r}}+\mathrm{K}^{2} \ln \left(\frac{2^{9}}{\gamma^{4}}\right)\right)\left(\mathrm{K} \exp \left\{e^{\mathrm{K}^{2} \mathrm{C} / s_{0}}\right\}\right)^{\mathrm{K}} & \leq \exp \left(\mathrm{K} \ln \frac{r_{0}}{\bar{r}}\right) \exp \left(\mathrm{K}^{2} \ln \left(\frac{2^{9}}{\gamma^{4}}\right)+\mathrm{K}^{2} e^{\mathrm{K}^{2} \mathrm{C} / s_{0}}\right) \\
& \leq \exp \left(\mathrm{K} \ln \frac{r_{0}}{\bar{r}}\right) \exp \exp \left(\frac{2^{10} \mathrm{C}}{\gamma^{4} s_{0}} \mathrm{~K}^{2}\right),
\end{aligned}
$$

hence

$$
\begin{align*}
T_{0} & \geq \frac{1}{8\left|R_{0}\right|_{\bar{r}, W_{f}}} \frac{\bar{r}}{r_{0}} \exp \left(\mathrm{~K} \ln \frac{\bar{r}}{r_{0}}-\exp \left(\frac{2^{10} \mathrm{C}}{\gamma^{4} s_{0}} \mathrm{~K}^{2}\right)\right) \\
& \stackrel{(7.13}{\geq} \frac{1}{8\left|R_{0}\right|_{\bar{r}, w_{f}}} \frac{\bar{r}}{r_{0}} \exp \left(\left[\left(\frac{\gamma^{4} s_{0}}{2^{10} \mathrm{C}}\right)^{\frac{1}{2}}\left(\frac{1}{2^{8}} \ln \ln \frac{\bar{r}}{r_{0}}\right)^{\frac{q}{2}-1}\right] \ln \frac{\bar{r}}{r_{0}}-\exp \left(\left(\frac{1}{2^{8}} \ln \ln \frac{\bar{r}}{r_{0}}\right)^{\mathrm{q}-1}\right)\right) \\
& \geq \frac{1}{8\left|R_{0}\right|_{\bar{r}, w_{f}}} \frac{\bar{r}}{r_{0}} \exp \left(\left[\left(\frac{\gamma^{4} s_{0}}{2^{10} \mathrm{C}}\right)^{\frac{1}{2}}\left(\frac{1}{2^{8}} \ln \ln \frac{\bar{r}}{r_{0}}\right)^{\frac{\mathrm{q}-1}{2}}\right] \ln \frac{\bar{r}}{r_{0}}-\exp \left(\left(\ln \ln ^{1 / 2^{8}} \frac{\bar{r}}{r_{0}}\right)^{\mathrm{q}-1}\right)\right)  \tag{7.18}\\
& \geq \frac{1}{8\left|R_{0}\right|_{\bar{r}, w_{f}}} \frac{\bar{r}}{r_{0}} \exp \left(\left[\left(\frac{\gamma^{4} s_{0}}{2^{10} \mathrm{C}}\right)^{\frac{1}{2}}\left(\frac{1}{2^{8}} \ln \ln \frac{\bar{r}}{r_{0}}\right)^{\frac{q-1}{2}}\right] \ln \frac{\bar{r}}{r_{0}}-\exp \left(\left(\ln \ln ^{1 / 2^{8}} \frac{\bar{r}}{r_{0}}\right)\right)\right) \\
& \geq \frac{1}{8\left|R_{0}\right|_{\bar{r}, W_{f}}} \frac{\bar{r}}{r_{0}} \exp \left(\ln \frac{\bar{r}}{r_{0}}\left(\left(\frac{\gamma^{4} s_{0}}{2^{10} \mathrm{C}}\right)^{\frac{1}{2}}\left(\frac{1}{2^{8}} \ln \ln \frac{\bar{r}}{r_{0}}\right)^{\frac{\mathrm{q}-1}{2}}-2\right)\right) \\
& \geq \frac{1}{8\left|R_{0}\right|_{\bar{r}, W_{f}}} \frac{\bar{r}}{r_{0}} \exp \left(\frac{1}{2} \ln \left(\frac{\bar{r}}{r_{0}}\right)\left(\left(\frac{\gamma^{4} s_{0}}{2^{18} \mathrm{C}}\right)^{1 / 2}\left(\ln \ln \frac{\bar{r}}{r_{0}}\right)^{\frac{\mathrm{q}-1}{2}}\right)\right)
\end{align*}
$$

where we required that

$$
\begin{align*}
& r_{0} \leq \frac{\bar{r}}{\exp \exp 2^{8}}  \tag{7.19}\\
& r_{0} \leq \bar{r} \exp \exp \left(-\left(\frac{2^{22} \mathrm{C}}{\gamma^{4} s_{0}}\right)^{\frac{1}{q-1}}\right) \tag{7.20}
\end{align*}
$$

We have that conditions 1.16 and 7.11 imply 7.19 if one requires

$$
\begin{equation*}
C_{2}=C_{2}(p, R) \geq \frac{2 \exp \exp \left\{2^{8}\right\} \mathrm{C}_{\mathrm{alg}}(p) C(p, R)}{R} \tag{7.21}
\end{equation*}
$$

The bound 7.20 is implied by the 1.16 setting $\mathrm{c}=3 \cdot 2{ }^{15} \mathrm{C}$ and

$$
\begin{equation*}
C_{1}=C_{1}(p, R) \geq \frac{R}{\mathrm{C}_{\mathrm{alg}}(p) C(p, R)} \tag{7.22}
\end{equation*}
$$

The bound 7.18 , together with (7.7), implies the lower bound 1.19 by setting

$$
\begin{equation*}
C_{3}=C_{3}(p, R) \geq 16 C(p, R) \tag{7.23}
\end{equation*}
$$

Theorem 1.4 follows by the discussion above choosing the constant $C_{1}$ as in (7.22), $C_{3}$ as in 7.23), $C_{2}$ satisfying (7.16, 7.17), 7.21) where $\mathrm{C}_{\mathrm{alg}}(p)$ is given in 2.20) and $C(p, R)$ in 7.7).
7.2. Sobolev stability and proof of Theorem 1.2 . In order to prove Theorem 1.2 we reason as done in Section 7.1 and we assume that the initial condition $u_{0}$ satisfies (see 1.5 )

$$
\begin{equation*}
\left\|u_{0}\right\|_{p}:=\left\|u_{0}\right\|_{0, p} \leq \delta \tag{7.24}
\end{equation*}
$$

Fix

$$
\begin{equation*}
\mathrm{K}:=\mathrm{K}(p):=\left[\left(\frac{p-1}{2^{4} 3^{4}}\right)^{\frac{1}{3}}\right]-1, \quad p_{0}:=p-\zeta:=p-2^{4} 3^{4} \sum_{i=1}^{\mathrm{K}} i^{2} \tag{7.25}
\end{equation*}
$$

Recall that in Theorem 1.2 we required $p>2^{6}(36)^{2}+1$. Then one can check that

$$
\begin{equation*}
1 \leq \widetilde{c}(p-1)^{\frac{1}{3}} \leq \mathrm{K}(p) \leq p^{1 / 3}, \quad \widetilde{c}:=\frac{1}{2(36)^{2 / 3}}, \quad p_{0}>1 \tag{7.26}
\end{equation*}
$$

For future convenience we set

$$
\begin{equation*}
\bar{r}:=\frac{R}{\mathrm{C}_{\mathrm{alg}, \mathrm{M}}\left(p_{0}\right)|F|_{R}} \tag{7.27}
\end{equation*}
$$

Therefore by item (ii) in Lemma 7.1 we have that the Hamiltonian $H$ in 2.7) can be written, for any $\mathrm{K} \geq 1$, in the form (6.1) with $w_{0}$ as in (7.8). Here again we apply Theorem 6.2 to the Hamiltonian $H=D_{\omega}+R_{0}$ with $R_{0}$ in (7.5. Recalling the parameters in 6.6)-6.7) we fix

$$
\begin{equation*}
r_{0}=2 \delta \tag{7.28}
\end{equation*}
$$

We claim that (7.28) and (1.16) imply that $r_{0}$ satisfies the smallness condition (6.2) with $\mathrm{K}(p)$ in (7.25) and $J_{\mathrm{K}}^{*}=J_{\mathrm{K}}^{\mathrm{S}}$. That is, we show that

$$
\mathrm{C}_{0}:=32 e 4^{3}\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{0}} \mathrm{~K}\left(\frac{4}{\gamma^{4}}\right)^{\mathrm{K}} \exp \left(2^{12} \mathrm{CK}^{3}\right) \frac{r_{0}}{\bar{r}} \leq 1
$$

First, we notice that

$$
\begin{equation*}
p_{0}>1 \quad \Rightarrow \quad 1 \leq \mathrm{C}_{\mathrm{alg}, \mathrm{M}}\left(p_{0}\right):=\sqrt{2} \sqrt{2+\frac{2 p_{0}+1}{2 p_{0}-1}} \leq 2^{3} \tag{7.29}
\end{equation*}
$$

Hence

$$
\begin{aligned}
& \mathrm{C}_{0} \leq\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{0}} \exp \left\{2^{19} \mathrm{C} \ln (1 / \gamma) \mathrm{K}^{3}\right\} \frac{r_{0}}{\bar{r}} \\
& \quad \frac{7.9,,\left(7.27,, \frac{7.25}{\leq} \frac{|F|_{R}}{R} \exp \left\{2^{21} \mathrm{C} \ln (1 / \gamma) p\right\} r_{0} \leq 1\right.}{} \quad \text {, }
\end{aligned}
$$

provided that

$$
r_{0} \leq \frac{R}{|F|_{R}} \exp \left\{-2^{21} \mathrm{C} \ln (1 / \gamma) p\right\}
$$

The last inequality follows from $(7.28$ and $(1.9)$, taking

$$
\begin{equation*}
\mathrm{c} \geq 2^{21} \mathrm{C} \tag{7.30}
\end{equation*}
$$

In view of Theorem 6.2 and Lemma 2.16 we have that the solution $u(t)$ of 2.6) evolving from initial data satisfying (7.24) remains in the ball of radius $2 \delta$ for time $t \in\left[0, T_{0}\right]$ with (recall the estimate 6.5 ) and $J_{\mathrm{K}}^{\mathrm{S}}$ in (6.2))

$$
T_{0}:=\frac{\bar{r}^{\mathrm{K}+1}}{8 r_{0}^{\mathrm{K}+1}\left|R_{0}\right|_{\bar{r}, \mathrm{w}_{f}}} \frac{\left(16 e \mathrm{~K} 4^{\mathrm{K}+2}\right)^{-\mathrm{K}}}{\left(\gamma^{-4 \mathrm{~K}} \exp \left(2^{12} \mathrm{CK}^{3}\right)\right)^{\mathrm{K}}}
$$

Observe that

$$
\begin{aligned}
T_{0} & \geq \frac{1}{8\left|R_{0}\right|_{\bar{r}, \mathrm{w}}} \frac{\bar{r}}{r_{0}}\left(\frac{\bar{r}}{r_{0}}\right)^{\mathrm{K}} \frac{1}{\left[\exp \left(2^{14} \mathrm{C} \ln (1 / \gamma) \mathrm{K}^{3}\right)\right]^{\mathrm{K}}} \\
& \stackrel{(7.9), \boxed{77.27}}{\geq} \frac{R}{8 \mathrm{C}_{\mathrm{alg}}\left(p_{0}\right)|F|_{R}} \frac{1}{r_{0}}\left(\frac{\bar{r}}{r_{0}}\right)^{\mathrm{K}} \frac{1}{\left[\exp \left(2^{14} \mathrm{C} \ln (1 / \gamma) \mathrm{K}^{3}\right)\right]^{\mathrm{K}}} \\
& \geq \frac{R}{|F|_{R}} \frac{1}{r_{0}}\left(\frac{\bar{r}}{r_{0}}\right)^{\mathrm{K}} \frac{1}{2^{6}\left[\exp \left(2^{14} \mathrm{C} \ln (1 / \gamma) \mathrm{K}^{3}\right)\right]^{\mathrm{K}}}
\end{aligned}
$$

since $p \geq p_{0}$ and (7.29) holds. We also remark that condition (1.9) with (7.27)-7.28) implies that $\delta_{\mathrm{S}} \leq \bar{r} / 4$. Finally, by (7.26) one has

$$
\begin{aligned}
T_{0} & \geq \frac{R}{|F|_{R}} \frac{1}{r_{0}}\left(\frac{\delta_{\mathrm{S}}}{\delta}\right)^{\widetilde{c}(p-1)^{1 / 3}} \frac{1}{2^{6}\left[\exp \left(p 2^{14} \mathrm{C} \ln (1 / \gamma)\right)\right]^{p}} \\
& \geq \frac{R}{|F|_{R}} \frac{1}{r_{0}}\left(\frac{\delta_{\mathrm{S}}}{\delta}\right)^{\widetilde{c}(p-1)^{1 / 3}} \frac{1}{\left[\exp \left(p 2^{15} \mathrm{C} \ln (1 / \gamma)\right)\right]^{p}}
\end{aligned}
$$

Setting $c=\max \left\{2(36)^{2 / 3}, 2^{21} \mathbf{C}\right\}$, we get the thesis.
7.3. Sobolev stability optimization and proof of Corollary 1.3. Let us fix $\delta$ such that

$$
\begin{equation*}
0<\delta \leq \bar{\delta}:=\delta_{\mathrm{S}} \exp \{-\mathrm{b} \ln (1 / \gamma)\} \tag{7.31}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{b}:=\max \left\{24 \mathrm{c}^{2}\left(\frac{1}{48 \mathrm{c}}\right)^{\frac{10}{9}}, 24 \mathrm{c}^{2}\left[2^{6}(36)^{2}\right]^{5 / 3}\right\} \tag{7.32}
\end{equation*}
$$

and let us consider

$$
\begin{equation*}
p=p(\delta)=1+\left(\frac{1}{24 \mathrm{c}^{2} \ln (1 / \gamma)} \ln \left(\frac{\delta_{\mathrm{S}}}{\delta}\right)\right)^{3 / 5} \tag{7.33}
\end{equation*}
$$

where $\mathrm{c}>0$ is the absolute constant given by Theorem 1.2 and $\delta_{\mathrm{S}}$ given in (1.9).
In order to prove Corollary 1.3 we reason as above and we assume that that

$$
\left\|u_{0}\right\|_{p}:=\left\|u_{0}\right\|_{0, p} \leq \delta
$$

Our aim is to apply Theorem 1.2. We shall verify that condition (1.9) holds for $\delta$ small enough. First notice that (7.33) implies

$$
\begin{equation*}
\delta=\delta_{\mathrm{S}} \exp \left\{-24 \mathrm{c}^{2}(p-1)^{\frac{5}{3}} \ln (1 / \gamma)\right\} \tag{7.34}
\end{equation*}
$$

Then smallness condition (1.9) translates in proving that

$$
\exp \left\{\mathrm{c} \ln (1 / \gamma)\left[p-24 \mathrm{c}(p-1)^{5 / 3}\right]\right\} \leq 1 \quad \Leftrightarrow \quad p-24 \mathrm{c}(p-1)^{5 / 3} \leq 0
$$

recalling that $0<\gamma<1$, which is true as long as

$$
\begin{aligned}
& p=1+\left(\frac{1}{24 \mathrm{c}^{2} \ln (1 / \gamma)} \ln \left(\frac{\delta_{\mathrm{S}}}{\delta}\right)\right)^{3 / 5} \geq 1+\left(\frac{1}{48 \mathrm{c}}\right)^{\frac{2}{3}}, \quad \Leftrightarrow \\
& \ln \left(\delta_{\mathrm{S}} / \delta\right) \geq 24 \mathrm{c}^{2}\left(\frac{1}{48 \mathrm{c}}\right)^{\frac{10}{9}} \ln (1 / \gamma)
\end{aligned}
$$

This follows by $(7.31)-(\sqrt{7.32})$. With similar computations we can check that (7.31)-(7.32), together with (7.33), yield $p=p(\delta)>1+2^{6}(36)^{2}$. Hence,Theorem 1.2 applies, guaranteeing time of stability of the form

$$
\begin{aligned}
T_{0} & \geq \frac{R}{2|F|_{R} \delta}\left(\frac{\delta_{\mathrm{S}}}{\delta}\right)^{\frac{1}{\mathrm{c}}(p-1)^{1 / 3}} \exp \left\{-p^{2} \mathrm{c} \ln (1 / \gamma)\right\} \\
& \stackrel{7.34 \mid}{=} \frac{R}{2|F|_{R} \delta} \exp \left\{24 \mathrm{c}(p-1)^{2} \ln (1 / \gamma)-p^{2} \mathrm{c} \ln (1 / \gamma)\right\} \\
& \geq \frac{R}{2|F|_{R} \delta} \exp \left\{\mathrm{c} \ln (1 / \gamma)\left(24(p-1)^{2}-p^{2}\right)\right\} \\
& \geq \frac{R}{2|F|_{R} \delta} \exp \left\{\mathrm{c}(p-1)^{2} \ln (1 / \gamma)\right\} \\
& \stackrel{7.33)}{\geq} \frac{R}{2|F|_{R} \delta} \exp \left\{\frac{\left.\mathrm{c}(\ln (1 / \gamma))^{-1 / 5}\right)}{\left(24 \mathrm{c}^{2}\right)^{6 / 5}}\left(\ln \left(\delta_{\mathrm{S}} / \delta\right)\right)^{1+\frac{1}{5}}\right\},
\end{aligned}
$$

which is the stated bound (1.15).

## Appendix A. Technical Lemmata

We collect some technical lemmata.

## Lemma A.1. Consider the function

$$
\begin{equation*}
\lambda(x):=(\log (2+x))^{\mathrm{q}}, \quad 1<\mathrm{q} \leq 2, \quad x>0 . \tag{A.1}
\end{equation*}
$$

Then, there exists a constant $1 \leq \kappa \leq 5 / 4$ such that for any integer $N \geq 4$ and $x_{2} \geq x_{3} \geq \cdots \geq x_{N} \geq 1$,

$$
\begin{equation*}
\sum_{\ell=2}^{N} \lambda\left(x_{\ell}\right) \geq \lambda\left(\sum_{\ell=2}^{N} x_{\ell}\right)+c \sum_{\ell=3}^{N} \lambda\left(x_{\ell}\right), \quad \text { with } c:=1-\frac{1}{\kappa} \geq 0 \tag{A.2}
\end{equation*}
$$

Proof. First of all we notice that the function $\lambda$ is sub-linear i.e.

$$
\begin{equation*}
\lambda(x+y) \leq \lambda(x)+\lambda(y), \quad \forall x, y \in \mathbb{N}_{+} . \tag{A.3}
\end{equation*}
$$

Recall (1.6) and that $\langle x+y\rangle \leq\langle x\rangle+\langle y\rangle$ and set

$$
F(x, y):=(\ln (2+x+y))^{\mathrm{q}}-(\ln (2+x))^{\mathrm{q}}, \quad \forall x, y \in \mathbb{R}, \quad x, y \geq 0 .
$$

To obtain (A.3) it is sufficient to show that, for any $y \in \mathbb{R}, y \geq 1$, one has

$$
\begin{equation*}
F(x, y) \leq(\ln (2+y))^{\mathrm{q}}, \quad \forall x \in \mathbb{R}, \quad x \geq 1 . \tag{A.4}
\end{equation*}
$$

We claim that for fixed $y \geq 1$ the function $[1,+\infty] \ni x \mapsto F(x, y)$ is decreasing. Then A.4] follows since $F(0, y) \leq(\ln (2+y))^{\mathrm{q}}$. We have

$$
\partial_{x} F(x, y)=\mathrm{q}\left\{\frac{(\ln (2+x+y))^{\mathrm{q}-1}}{2+x+y}-\frac{(\ln (2+x))^{\mathrm{q}-1}}{2+x}\right\},
$$

hence $\partial_{x} F(x, y)=0$ for some $x \geq 1$ if and only if

$$
\begin{equation*}
g(x, y)=\frac{(\ln (2+x))^{\mathrm{q}-1}}{2+x} \tag{A.5}
\end{equation*}
$$

where

$$
g(x, y):=\frac{(\ln (2+x+y))^{\mathbf{q}-1}}{2+x+y} .
$$

The right hand side of the equation (A.5) does not depend in $y \in \mathbb{R}$. One can check (using also that $1<\mathrm{q} \leq 2$ ) that for fixed $x$, the function $y \mapsto g(x, y)$ is decreasing if $y \geq 1$. Hence A.5) cannot be satisfied, and, on the other hand, one has

$$
g(x, y)<\frac{(\ln (2+x))^{\mathrm{q}-1}}{2+x}, \quad x, y \geq 1,
$$

which implies $\partial_{x} F(x, y)<0$ and hence the claim. Then A.4) holds true. We claim that there is a constant $1 \leq \kappa \leq 5 / 4$ such that

$$
\begin{align*}
x \lambda(1) & \geq \kappa \lambda(x), & \forall x \geq 1  \tag{A.6}\\
\lambda^{\prime}(x) & \geq \kappa \lambda^{\prime}(2 x), & \forall x \geq 1  \tag{A.7}\\
x \lambda(2) & \geq \kappa \lambda(2 x), & \forall x \geq 1 . \tag{A.8}
\end{align*}
$$

By using (A.1) one has that conditions A.6, (A.7), A.8) are equivalent to

$$
\begin{equation*}
\kappa \leq \min _{i=1,2,3}\left\{\inf _{x \geq 1} h_{1}(x), \inf _{x \geq 1} h_{2}(x), \inf _{x \geq 1} h_{3}(x)\right\}, \tag{A.9}
\end{equation*}
$$

setting

$$
h_{1}(x):=x\left(\frac{\log (3)}{\log (2+x)}\right)^{\mathrm{q}}, \quad h_{2}(x):=\frac{2+2 x}{2+x}\left(\frac{\log (2+x)}{\log (2+2 x)}\right)^{\mathrm{q}-1}, \quad h_{3}(x):=x\left(\frac{\log (4)}{\log (2+2 x)}\right)^{\mathrm{q}} .
$$

Notice that

$$
\begin{aligned}
& h_{1}^{\prime}=\left(\frac{\log (3)}{\log (2+x)}\right)^{\mathrm{q}}\left[1-\frac{\mathrm{q} x}{2+x} \frac{1}{\log (2+x)}\right], \\
& h_{3}^{\prime}=\left(\frac{\log (4)}{\log (2+2 x)}\right)^{\mathrm{q}}\left[1-\frac{2 \mathrm{q} x}{2+2 x} \frac{1}{\log (2+2 x)}\right], \\
& h_{2}^{\prime}=\frac{1}{(2+x)^{2}}\left(\frac{\log (2+x)}{\log (2+2 x)}\right)^{\mathrm{q}-1}\left[2-(\mathrm{q}-1)\left(\frac{4+2 x}{\ln (2+2 x)}-\frac{2+2 x}{\ln (2+x)}\right)\right] .
\end{aligned}
$$

Consider the function $h_{1}(x)$. One can note that

$$
h_{1}^{\prime}(x) \geq 0 \quad \Leftrightarrow \quad \mathrm{q} x \leq(2+x) \ln (2+x) \quad \Leftrightarrow \quad 1 \leq \frac{1}{\mathrm{q}} \ln (2+x)+\frac{2}{2+x},
$$

which is true for $x \geq 1$, since $1<\mathrm{q} \leq 2$. Hence $\min _{h \geq 1} h_{1}(x) \geq 1$. With similar computations one can check that the constant $\kappa$ in A.9) can be fixed as $\kappa=1$. Then A.6, A.7), A.8) hold.
We are now in position to prove A.2). Let $1 \leq N_{0} \leq N$ such that

$$
x_{N_{0}+1}=\ldots=x_{N}=1, \quad x_{N_{0}} \geq 2 .
$$

Let $N_{1}:=N-N_{0} \geq 0$. Define

$$
g\left(x_{2}, x_{3}, \ldots, x_{N_{0}}\right):=\sum_{\ell=2}^{N_{0}} \lambda\left(x_{\ell}\right)-\lambda\left(N_{1}+\sum_{\ell=2}^{N_{0}} x_{\ell}\right)-c \sum_{\ell=3}^{N_{0}} \lambda\left(x_{\ell}\right)+(1-c) N_{1} \lambda(1) .
$$

In order to prove $\widehat{A .2}$ we have to show that $g \geq 0$. If $N_{0}=1$ we have that

$$
g=-\lambda(N-1)+(1-c)(N-1) \lambda(1) \stackrel{[A .6]}{\geq}\left[\left(1-c-\frac{1}{\kappa}\right)(N-1)\right] \lambda(1) \geq 0 .
$$

Assume now that $N_{0} \geq 2$. We have

$$
\begin{equation*}
\partial_{x_{2}} g=\lambda^{\prime}\left(x_{2}\right)-\lambda^{\prime}\left(N_{1}+\sum_{\ell=2}^{N_{0}} x_{\ell}\right) \geq 0 \tag{A.10}
\end{equation*}
$$

since $\lambda^{\prime}(x)$ is decreasing for $x \geq 2$. If $N_{0}=2$ we have that

$$
\begin{aligned}
g\left(x_{2}\right) \geq g(2) & =\lambda(2)-f(N)+(1-c)(N-2) \lambda(1) \\
& \stackrel{A .3}{\geq}-\lambda(N-2)+(1-c)(N-2) \lambda(1) \stackrel{A \cdot(6)}{\geq}\left(1-c-\frac{1}{\kappa}\right)(N-2) \lambda(1) \geq 0 .
\end{aligned}
$$

Assume now that $N_{0} \geq 3$. Then, since $x_{2} \geq x_{3}$, by A.10 we get

$$
\begin{aligned}
& g\left(x_{2}, x_{3}, \ldots, x_{N_{0}}\right) \geq g\left(x_{3}, x_{3}, \ldots, x_{N_{0}}\right) \\
& \quad=(2-c) \lambda\left(x_{3}\right)-\lambda\left(N_{1}+2 x_{3}+\sum_{\ell=4}^{N_{0}} x_{\ell}\right)+(1-c) \sum_{\ell=4}^{N_{0}} \lambda\left(x_{\ell}\right)+(1-c) N_{1} \lambda(1) \\
& \quad=: g_{3}\left(x_{3}, \ldots, x_{N_{0}}\right) .
\end{aligned}
$$

Proceeding analogously, since $\lambda^{\prime}$ is decreasing and positive, we get

$$
\begin{align*}
\partial_{x_{3}} g_{3}\left(x_{3}, \ldots, x_{N_{0}}\right) & =2\left(\left(1-\frac{c}{2}\right) \lambda^{\prime}\left(x_{3}\right)-\lambda^{\prime}\left(N_{1}+2 x_{3}+\sum_{\ell=4}^{N_{0}} x_{\ell}\right)\right)  \tag{A.11}\\
& \geq 2\left(\left(1-\frac{c}{2}\right) \lambda^{\prime}\left(x_{3}\right)-\lambda^{\prime}\left(2 x_{3}\right)\right) \stackrel{\stackrel{\rightharpoonup A .77}{\geq}}{\geq} 2\left(1-\frac{c}{2}-\frac{1}{\kappa}\right) \lambda^{\prime}\left(x_{3}\right) \geq 0 .
\end{align*}
$$

Let us note that by A.3 and A.6 we have

$$
\lambda(N+1) \leq \lambda(5)+\left(1+\frac{N-5}{\kappa}\right) \lambda(1), \quad \forall N \geq 4 .
$$

Then, if $N_{0}=3$, and recalling that $\kappa \leq 5 / 4$ we get

$$
\begin{aligned}
g_{3}\left(x_{3}\right) & \stackrel{A .11}{\geq} g_{3}(2) \\
& =(2-c) \lambda(2)-\lambda(N+1)+(1-c)(N-3) \lambda(1) \\
& \geq(2-c) \lambda(2)-\lambda(5)-\left(1+\frac{N-5}{\kappa}\right) \lambda(1)+(1-c)(N-3) \lambda(1) \\
& \stackrel{A .8}{\geq}\left(2-c-\frac{5}{2 \kappa}\right) \lambda(2)+\left((1-c)(N-3)-1-\frac{N-5}{\kappa}\right) \lambda(1) \\
& \stackrel{A .6}{\geq}\left(\frac{4-2 c}{\kappa}-\frac{5}{\kappa^{2}}+(1-c)(N-3)-1-\frac{N-5}{\kappa}\right) \lambda(1) \geq 0,
\end{aligned}
$$

by definition of $c$. Otherwise, for $N_{0} \geq 4$ we get, since $x_{3} \geq x_{4}$,

$$
\begin{aligned}
& g_{3}\left(x_{3}, x_{4}, \ldots, x_{N_{0}}\right) \stackrel{\text { A..11] }}{\geq} g_{3}\left(x_{4}, x_{4}, \ldots, x_{N_{0}}\right) \\
& \quad=(3-2 c) \lambda\left(x_{4}\right)-\lambda\left(N_{1}+3 x_{4}+\sum_{\ell=5}^{N_{0}} x_{\ell}\right)+(1-c) \sum_{\ell=5}^{N_{0}} \lambda\left(x_{\ell}\right)+(1-c) N_{1} \lambda(1) \\
& \quad=: g_{4}\left(x_{4}, \ldots, x_{N_{0}}\right)
\end{aligned}
$$

Proceeding in this way we obtain by induction, for $3 \leq n \leq N_{0}$ functions

$$
\begin{aligned}
g_{n} & =g_{n}\left(x_{n}, \ldots, x_{N_{0}}\right) \\
& :=((n-1)-(n-2) c) \lambda\left(x_{n}\right)-\lambda\left(N_{1}+(n-1) x_{n}+\sum_{\ell=n+1}^{N_{0}} x_{\ell}\right) \\
& +(1-c) \sum_{\ell=n+1}^{N_{0}} \lambda\left(x_{\ell}\right)+(1-c) N_{1} \lambda(1),
\end{aligned}
$$

with

$$
g_{n-1}\left(x_{n-1}, \ldots, x_{N_{0}}\right) \geq g_{n}\left(x_{n}, \ldots, x_{N_{0}}\right), \quad \forall x_{n-1} \geq x_{n} .
$$

Indeed, since

$$
\begin{aligned}
& \partial_{x_{n}} g_{n}\left(x_{n}, \ldots, x_{N_{0}}\right)= \\
& \quad=((n-1)-(n-2) c) \lambda^{\prime}\left(x_{n}\right)-(n-1) \lambda^{\prime}\left(N_{1}+(n-1) x_{n}+\sum_{\ell=n+1}^{N_{0}} x_{\ell}\right) \\
& \quad \geq(n-1)\left[\left(1-\frac{n-2}{n-1} c\right) \lambda^{\prime}\left(x_{n}\right)-\lambda^{\prime}\left(2 x_{n}\right)\right] \\
& \quad \stackrel{\text { A..7. }}{\geq}(n-1)\left(1-\frac{n-2}{n-1} c-\frac{1}{\kappa}\right) \lambda^{\prime}\left(x_{n}\right) \geq 0,
\end{aligned}
$$

we have that for every $x_{n} \geq x_{n+1}$

$$
g_{n}\left(x_{n}, x_{n+1}, \ldots, x_{N_{0}}\right) \geq g_{n}\left(x_{n+1}, x_{n+1}, \ldots, x_{N_{0}}\right)=g_{n+1}\left(x_{n+1}, \ldots, x_{N_{0}}\right) .
$$

In conclusion we get

$$
\begin{aligned}
& g\left(x_{2}, x_{3}, \ldots, x_{N_{0}}\right) \geq g_{N_{0}}\left(x_{N_{0}}\right) \\
& \quad=\left(\left(N_{0}-1\right)-\left(N_{0}-2\right) c\right) \lambda\left(x_{N_{0}}\right)-\lambda\left(N-N_{0}+\left(N_{0}-1\right) x_{N_{0}}\right) \\
& \quad+(1-c)\left(N-N_{0}\right) \lambda(1)
\end{aligned}
$$

with $\partial_{x_{N_{0}}} g_{N}\left(x_{N_{0}}\right) \geq 0$. Then, recalling that we are in the case $N_{0} \geq 4$, we get

$$
\begin{aligned}
g_{N_{0}}\left(x_{N_{0}}\right) & \geq g_{N_{0}}(2)= \\
& =\left(\left(N_{0}-1\right)-\left(N_{0}-2\right) c\right) \lambda(2)-\lambda\left(N+N_{0}-2\right)+(1-c)\left(N-N_{0}\right) \lambda(1) \\
& \stackrel{A .3 \sqrt{A}}{\geq}\left(N_{0}-1\right) \lambda(2)-c\left(N_{0}-2\right) \lambda(2)-\lambda\left(2 N_{0}-2\right)-\lambda\left(N-N_{0}\right)+(1-c)\left(N-N_{0}\right) \lambda(1) \\
& \stackrel{A .8),(A .6]}{\geq}\left[\left(1-\frac{1}{\kappa}\right)\left(N_{0}-1\right)-c\left(N_{0}-2\right)\right] \lambda(2)+\left(1-c-\frac{1}{\kappa}\right)\left(N-N_{0}\right) \lambda(1) \\
& =c \lambda(2) \geq 0 .
\end{aligned}
$$

This complete the proof of A.2).

Remark A.2. From the proof of Lemma A.1 actually we can deduce that, if $\lambda(x)$ is defined as in A.1, one can choose the constant $\kappa=1$ in A.6)-(A.9). This means that the constant $c$ in A.2) can be chosen to be 0. However, we state it in this more general form to make it transparent also for a different choice of $\lambda(x)$, provided it is sublinear and satisfies conditions (A.6)-A.9.
Lemma A. 3 (Lemma C. 1 in BMP20a]). For $p, \beta>0$ and $x_{0} \geq 0$ we have that

$$
\max _{x \geq x_{0}} x^{p} e^{-\beta x}= \begin{cases}(p / \beta)^{p} e^{-p} & \text { if } x_{0} \leq p / \beta \\ x_{0}^{p} e^{-\beta x_{0}} & \text { if } x_{0}>p / \beta\end{cases}
$$

Lemma A.4. Let $x_{1} \geq x_{2} \geq \ldots \geq x_{N} \geq 2$. Then

$$
\frac{\sum_{1 \leq \ell \leq N} x_{\ell}}{\prod_{1 \leq \ell \leq N} \sqrt{x_{\ell}}} \leq \sqrt{x_{1}}+\frac{4}{\sqrt{x_{1}}} .
$$

Proof. By induction over $N$. The result is obvious for $N=1$. Let assume it for $N$ and show it for $N+1$. We have

$$
\begin{aligned}
& \frac{\sum_{1 \leq \ell \leq N+1} x_{\ell}}{\prod_{1 \leq \ell \leq N+1} \sqrt{x_{\ell}}} \leq \frac{\left(\sum_{1 \leq \ell \leq N} x_{\ell}\right)+x_{N+1}}{\left(\prod_{1 \leq \ell \leq N} \sqrt{x_{\ell}}\right) \sqrt{x_{N+1}}} \\
\leq & \left(\sqrt{x_{1}}+\frac{4}{\sqrt{x_{1}}}\right) \frac{1}{\sqrt{x_{N+1}}}+\frac{\sqrt{x_{N+1}}}{\sqrt{2}} .
\end{aligned}
$$

since $x_{1} \geq x_{2} \geq \ldots \geq x_{N} \geq 2$ implies $\prod_{1 \leq \ell \leq N} \sqrt{x_{\ell}} \geq \sqrt{2}$. It remains to prove that

$$
\left(\sqrt{x_{1}}+\frac{4}{\sqrt{x_{1}}}\right) \frac{1}{\sqrt{x_{N+1}}}+\frac{\sqrt{x_{N+1}}}{\sqrt{2}} \leq \sqrt{x_{1}}+\frac{4}{\sqrt{x_{1}}} .
$$

Denoting $t:=\sqrt{x_{1}}$ and $s:=\sqrt{x_{N+1}}$, the above inequality is equivalent to

$$
f(t, s):=2 t^{2} s-\sqrt{2} t s^{2}+8 s-2 t^{2}-8 \geq 0
$$

for $\sqrt{2} \leq s \leq t$. Since $f$ is a concave function of $s$ we have that

$$
f(t, s) \geq \min \{f(t, \sqrt{2}), f(t, t)\}
$$

It is immediate to see that

$$
\min _{t \geq \sqrt{2}} f(t, \sqrt{2})=7 \sqrt{2}-9>0, \quad \min _{t \geq \sqrt{2}} f(t, t)=12 \sqrt{2}-16>0,
$$

showing that $f(t, s)>0$ for $\sqrt{2} \leq s \leq t$ and concluding the proof.
In the following it will be convenient to use the following way of reordering of the indexes $j \in \mathbb{Z}$ appearing in the Hamiltonian (2.21).
Definition A.5. Consider a vector $v=\left(v_{i}\right)_{i \in \mathbb{Z}} v_{i} \in \mathbb{N},|v|<\infty$.
(i) We denote by $\widehat{n}=\widehat{n}(v)$ the vector $\left(\widehat{n}_{l}\right)_{l \in I}$ (where $I \subset \mathbb{N}$ is finite) which is the decreasing rearrangement of

$$
\left\{\mathbb{N} \ni h>1 \text { repeated } v_{h}+v_{-h} \text { times }\right\} \cup\left\{1 \text { repeated } v_{1}+v_{-1}+v_{0} \text { times }\right\}
$$

(ii) Define the vector $m=m(v)$ as the reordering of the elements of the set

$$
\left\{j \neq 0, \quad \text { repeated } \quad\left|u_{j}\right| \text { times }\right\},
$$

where $D<\infty$ is its cardinality, such that $\left|m_{1}\right| \geq\left|m_{2}\right| \geq \cdots \geq\left|m_{D}\right| \geq 1$.

Remark A.6. We observe that the number $N:=|\alpha|+|\beta|$ is the cardinality of $\widehat{n}$ and that, by momentum conservation, there exists a choice of $\sigma_{i}= \pm 1,0$ such that

$$
\begin{equation*}
\sum_{l} \sigma_{l} \widehat{n}_{l}=0 \tag{A.12}
\end{equation*}
$$

with $\sigma_{l} \neq 0$ if $\widehat{n}_{l} \neq 1$. Hence,

$$
\begin{equation*}
\widehat{n}_{1} \leq \sum_{l \geq 2} \widehat{n}_{l} \tag{A.13}
\end{equation*}
$$

Indeed, if $\sigma_{1}= \pm 1$, the inequality follows directly from A.12; if $\sigma_{1}=0$, then $\widehat{n}_{1}=1$, hence $\widehat{n}_{l}=1 \forall l$.

Given $\alpha \neq \beta \in \mathbb{N}^{\mathbb{Z}}$, with $|\alpha|+|\beta|<\infty$ we consider $m=m(\alpha-\beta)$ and $\widehat{n}=\widehat{n}(\alpha+\beta)$. If we denote by $D$ the cardinality of $m$ and $N$ the one of $\widehat{n}$ we have

$$
\begin{align*}
D+\alpha_{0}+\beta_{0} & \leq N  \tag{A.14}\\
(\left|m_{1}\right|, \ldots,\left|m_{D}\right|, \underbrace{1, \ldots, 1}_{N-D \text { times }}) & \leq\left(\widehat{n}_{1}, \ldots \widehat{n}_{N}\right) \tag{A.15}
\end{align*}
$$

Set $\sigma_{l}=\operatorname{sign}\left(\alpha_{m_{l}}-\beta_{m_{l}}\right)$. For every function $g$ defined on $\mathbb{Z}$ we have that

$$
\begin{align*}
\sum_{i \in \mathbb{Z}} g(i)\left|\alpha_{i}-\beta_{i}\right| & =g(0)\left|\alpha_{0}-\beta_{0}\right|+\sum_{l \geq 1} g\left(m_{l}\right) \\
\sum_{i \in \mathbb{Z}} g(i)\left(\alpha_{i}-\beta_{i}\right) & =g(0)\left(\alpha_{0}-\beta_{0}\right)+\sum_{l \geq 1} \sigma_{l} g\left(m_{l}\right) \tag{A.16}
\end{align*}
$$

Lemma A.7. For all $(\alpha, \beta) \in \mathcal{M}$ (see 2.26) the following holds.
(i) If

$$
\begin{equation*}
\sum_{i}\left(\alpha_{i}-\beta_{i}\right)|i|^{2} \leq 10 \sum_{i}\left|\alpha_{i}-\beta_{i}\right| \tag{A.17}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\sum_{i}\left|\alpha_{i}-\beta_{i}\right| \lambda(\sqrt{\langle i\rangle}) \leq 63 \sum_{l \geq 3} \lambda\left(\widehat{n}_{l}\right) \leq \frac{63}{\kappa}\left(\sum_{i}\left(\alpha_{i}+\beta_{i}\right) \lambda(\langle i\rangle)-2 \lambda(\langle j\rangle)\right) \tag{A.18}
\end{equation*}
$$

where $\lambda$ is in (1.6) and

$$
\begin{equation*}
\prod_{i}\left(1+\left|\alpha_{i}-\beta_{i}\right|\langle i\rangle\right) \leq e^{27} N^{6} \prod_{l=3}^{N} \widehat{n}_{l}^{\tau_{0}} \tag{A.19}
\end{equation*}
$$

where $N=|\alpha|+|\beta|$.
(ii) If on the contrary A.17) does not hold then

$$
\begin{equation*}
|\omega \cdot(\alpha-\beta)| \geq 1 \tag{A.20}
\end{equation*}
$$

where $\omega$ is given in (3.1).
Proof. Let us prove item (i). Inequality A.19 is contained in BMP20a, Lemma 7.1] so we send the reader there for the related proof. For inequality A.18) we proceed as follows. We claim that, given $g$ defined on $\mathbb{Z}$ and non negative, even and not decreasing on $\mathbb{N}$, if $\alpha \neq \beta$ one has

$$
\begin{equation*}
\sum_{i \in \mathbb{Z}} g(i)\left|\alpha_{i}-\beta_{i}\right| \leq 2 g\left(m_{1}\right)+\sum_{l \geq 3} g\left(\widehat{n}_{l}\right) \tag{A.21}
\end{equation*}
$$

Indeed, by A.16 we note that

$$
\begin{aligned}
\sum_{i \in \mathbb{Z}} g(i)\left|\alpha_{i}-\beta_{i}\right| & =g(0)\left|\alpha_{0}-\beta_{0}\right|+\sum_{l \geq 1} g\left(m_{l}\right) \\
& \leq g(1)\left(\alpha_{0}+\beta_{0}\right)+2 g\left(m_{1}\right)+\sum_{l \geq 3} g\left(m_{l}\right) .
\end{aligned}
$$

Hence A.21) follows by A.14 and A.15). Applying A.21) with $g(x)=f(\sqrt{x})$, we have

$$
\begin{aligned}
\sum_{i}\left|\alpha_{i}-\beta_{i}\right| g(\langle i\rangle) & \leq 2 g\left(m_{1}\right)+\sum_{l \geq 3} g\left(\widehat{n}_{l}\right) \leq 2 g\left(31 \sum_{l \geq 3} \widehat{n}_{l}^{2}\right)+\sum_{l \geq 3} g\left(\widehat{n}_{l}\right) \\
& \leq 62 \sum_{l \geq 3} g\left(\widehat{n}_{l}^{2}\right)+\sum_{l \geq 3} g\left(\widehat{n}_{l}\right) \leq 63 \sum_{l \geq 3} \lambda\left(\widehat{n}_{l}\right),
\end{aligned}
$$

which implies the first inequality in A.18). The second one follows by A.2.
We now show item (ii). Notice that

$$
\left|\sqrt{i^{4}+m}-i^{2}\right| \leq \frac{m}{2 i^{2}} \leq 1, \quad m \in[1,2], \quad i \in \mathbb{Z}
$$

Then, by triangular inequality, we have

$$
\begin{aligned}
\left|\sum_{i \in \mathbb{Z}}\left(\alpha_{i}-\beta_{i}\right) \sqrt{i^{4}+m}\right| & \geq\left|\sum_{i \in \mathbb{Z}}\left(\alpha_{i}-\beta_{i}\right) i^{2}\right|-\left|\sum_{i \in \mathbb{Z}}\left(\alpha_{i}-\beta_{i}\right)\left(\sqrt{i^{4}+m}-i^{2}\right)\right| \\
& \geq 10 \sum_{i}\left|\alpha_{i}-\beta_{i}\right|-\sum_{i}\left|\alpha_{i}-\beta_{i}\right| \geq 1
\end{aligned}
$$

which is A.20.
Lemma A.8. Fix $\mathrm{N} \geq 1, \delta \geq(36 \mathrm{~N})^{2}, \tau \leq 36 \mathrm{~N}^{2}$ and $\mathrm{d} \geq 4 \mathrm{~N}$. Then one has

$$
\begin{equation*}
\mathrm{J}:=\sup _{j \in \mathbb{Z},(\alpha, \beta) \in \mathcal{A}}\left(\frac{\lfloor j\rfloor^{2}}{\prod_{i \in \mathbb{Z}}\lfloor i\rfloor^{\alpha_{i}+\beta_{i}}}\right)^{\delta} \prod_{i \in \mathbb{Z}}\left(\left(1+\left|\alpha_{i}-\beta_{i}\right|^{2}\right)\langle i\rangle^{2}\right)^{\tau} \leq 2^{\delta-1}\left(4^{6} e^{27}\right)^{72 \mathbb{N}^{2}} 6^{\delta} . \tag{A.22}
\end{equation*}
$$

where $\mathcal{A} \subseteq \Lambda$ is the set of indexes $(\alpha, \beta)$ such that A .17 ) holds and

$$
|\alpha|+|\beta|=\mathrm{N}+2, \quad \alpha_{j}+\beta_{j} \neq 0, \quad|\alpha-\beta| \leq \mathrm{N}+2 .
$$

Proof. By A.19) (recall Definition A.5) toghether with $|\alpha|+|\beta|=\mathrm{N}+2$ and $\mathrm{d} \leq 4 \mathrm{~N}$ we get

$$
\mathrm{J} \leq \sup _{\substack{\alpha_{j}+\beta_{j} \neq 0 \\|\alpha-\beta| \leq \mathrm{N}+2}}\left(\frac{\lfloor j\rfloor^{2}}{\prod_{i \in \mathbb{Z}}\lfloor i\rfloor^{\alpha_{i}+\beta_{i}}}\right)^{\delta}\left(e^{27 \tau}(\mathrm{~N}+2)^{6 \tau}\right)^{2}\left(\prod_{\ell=3}^{\mathrm{N}+2} \widehat{n}_{\ell}^{\frac{15}{2} \tau}\right)^{2},
$$

with $\widehat{n}=\widehat{n}(\alpha+\beta)$. We claim that

$$
\begin{equation*}
\mathrm{N}+2 \leq 4 \prod_{l=3}^{\mathrm{N}+2}\left\lfloor\widehat{n}_{l}\right\rfloor^{\frac{1}{4 \ln 2}} . \tag{A.23}
\end{equation*}
$$

Indeed if $N=0$, the inequality is trivial. The case $N \geq 1$ follows by Lemma A.4. Recalling Def. A. 5 we have

$$
\begin{equation*}
\prod_{i}\lfloor i\rfloor^{\alpha_{i}+\beta_{i}}=\prod_{l \geq 1}\left\lfloor\widehat{n}_{l}\right\rfloor . \tag{A.24}
\end{equation*}
$$

Then

$$
\sup _{\substack{j, \alpha, \beta \\ \alpha_{j}+\beta_{j} \geq 1}} \frac{\lfloor j\rfloor^{2}}{\prod_{i}\lfloor i\rfloor^{\alpha_{i}+\beta_{i}}} \leq \frac{\left\lfloor\hat{n}_{1}\right\rfloor^{2}}{\prod_{l \geq 1}\left\lfloor\widehat{n}_{l}\right\rfloor}=\frac{\left\lfloor\hat{n}_{1}\right\rfloor}{\prod_{l \geq 2}\left\lfloor\widehat{n}_{l}\right\rfloor} \leq \frac{\sum_{l \geq 2}\left\lfloor\widehat{n}_{l}\right\rfloor}{\prod_{l \geq 2}\left\lfloor\widehat{n}_{l}\right\rfloor}=\frac{1}{\prod_{l \geq 3}\left\lfloor\widehat{n}_{l}\right\rfloor}+\frac{\sum_{l \geq 3}\left\lfloor\widehat{n}_{l}\right\rfloor}{\prod_{l \geq 2}\left\lfloor\widehat{n}_{l}\right\rfloor},
$$

where the last inequality holds by momentum conservation. Recall that $\tau \leq 36 \mathrm{~N}^{2}$ and $\delta \geq(36 \mathrm{~N})^{2}$. Then, by A.23) and by the fact that $(a+b)^{\delta} \leq 2^{\delta-1}\left(a^{\delta}+b^{\delta}\right)$ for $a, b \geq 0, \delta \geq 1$, one has

$$
\begin{aligned}
\mathrm{J} & \leq 2^{\delta-1}\left(\frac{1}{\prod_{l \geq 3}\left\lfloor\widehat{n}_{l}\right\rfloor^{\delta}}+\frac{\left(\sum_{l \geq 3}\left\lfloor\widehat{n}_{l}\right\rfloor\right)^{\delta}}{\prod_{l \geq 2}\left\lfloor\widehat{n}_{l}\right\rfloor^{\delta}}\right)\left(4^{6} e^{27}\right)^{72 \mathrm{~N}^{2}} \prod_{l \geq 3}\left\lfloor\widehat{n}_{l}\right\rfloor^{\delta / 2} \\
& \leq 2^{\delta-1}\left(4^{6} e^{27}\right)^{72 \mathrm{~N}^{2}}\left(1+\frac{\left(\sum_{l \geq 3}\left\lfloor\widehat{n}_{l}\right\rfloor\right)^{\delta}}{\left\lfloor\widehat{n}_{2}\right\rfloor^{\delta} \prod_{l \geq 3}\left\lfloor\widehat{n}_{l}\right\rfloor^{\delta / 2}}\right) \\
& \leq 2^{\delta-1}\left(4^{6} e^{27}\right)^{72 \mathrm{~N}^{2}}\left(1+\frac{\left(\left\lfloor\widehat{n}_{3}\right\rfloor^{1 / 2}+4\right)^{\delta}}{\left\lfloor\widehat{n}_{2}\right\rfloor^{\delta}}\right)
\end{aligned}
$$

where we used Lemma A. 4 Then the thesis follows.
Proof of Proposition 2.9. In all that follows we shall use systematically the fact that our Hamiltonians preserve the momentum and are zero at the origin. These facts imply that $|\alpha|+|\beta| \geq 1$.
Case sE ) Let us start by proving the bound (2.37). It follows by (2.36) in Lemma 2.8 provided that (recall Remark 2.7)

$$
C_{0}:=\sup _{\substack{j \in \mathbb{Z}, \alpha, \beta \in \mathbb{N}^{Z}, \alpha_{j}+\beta_{j} \neq 0, \sum_{i} i\left(\alpha_{i}-\beta_{i}\right)=0}} \frac{c_{r, w^{\prime}}^{(j)}(\alpha, \beta)}{c_{r, \mathbf{w}}^{(j)}(\alpha, \beta)}<\infty,
$$

where $\mathrm{w}^{\prime}=\mathrm{w}(p, s+\sigma)$ and where $\mathrm{w}(p, s)$ is the weight defined in (2.14). We actually show that $C_{0}$ is equal to 1 . By direct computation

$$
\begin{equation*}
\frac{c_{r, w^{\prime}}^{(j)}(\alpha, \beta)}{c_{r, w}^{j j}(\alpha, \beta)}=\exp \left(-\sigma\left(\sum_{i} \lambda(\langle i\rangle)\left(\alpha_{i}+\beta_{i}\right)-2 \lambda(\langle j\rangle)\right)\right) . \tag{A.25}
\end{equation*}
$$

By momentum conservation, inequality (A.13) holds, which together with the sub-linearity of $\lambda$ and the definition of $\widehat{n}$ gives the following chain of inequalities, for all $\alpha, \beta$ in $\sum_{*}$ such that $\alpha_{j}+\beta_{j} \neq 0$ :

$$
\begin{aligned}
\sum_{i} \lambda(\langle i\rangle)\left(\alpha_{i}+\beta_{i}\right)-2 \lambda(\langle j\rangle) & \geq \sum_{i} \lambda(\langle i\rangle)\left(\alpha_{i}+\beta_{i}\right)-2 \lambda\left(\widehat{n}_{1}\right) \\
& \geq \sum_{l \geq 1} \lambda\left(\widehat{n}_{l}\right)-\lambda\left(\widehat{n}_{1}\right)-\lambda\left(\sum_{l \geq 2} \widehat{n}_{l}\right) \\
& \geq \sum_{l \geq 2} \lambda\left(\widehat{n}_{l}\right)-\lambda\left(\sum_{l \geq 2} \widehat{n}_{l}\right) \geq c \sum_{l \geq 3} \lambda\left(\widehat{n}_{l}\right) \geq 0,
\end{aligned}
$$

where the last inequality follows from $(\mathrm{A} .2$. This concludes the proof.
Case $S$ ) In order to prove the bound (2.38) we follows the ideas in the proof of Proposition 6.3 in [BMP20a] where the norm $|\cdot|_{r, w(p)}$ (see Def. 2.3) with w $(p)$ in (2.15) is denoted by $\|\cdot\|_{r, p}$. Again in view of (2.36) in Lemma 2.8 we only have to prove that (recall (2.33))

$$
\begin{equation*}
\sup _{\substack{j \in \mathbb{Z}, \alpha, \beta \in \mathbb{N}^{\mathbb{Z}}, \alpha_{j}+\beta_{j} \neq 0, \sum_{i} i\left(\alpha_{i}-\beta_{i}\right)=0}} \frac{c_{r, w^{\prime}}^{(j)}(\alpha, \beta)}{c_{r, w}^{(j)}(\alpha, \beta)}=\sup _{\substack{j \in \mathbb{Z}, \alpha, \beta \in \mathbb{N}^{\mathbb{Z}}, \alpha_{j}+\beta_{j} \neq 0, \sum_{i} i\left(\alpha_{i}-\beta_{i}\right)=0}}\left(\frac{\lfloor j\rfloor^{2}}{\prod_{i \in \mathbb{Z}}\lfloor i\rfloor^{\alpha_{i}+\beta_{i}}}\right)^{p^{\prime}} 1, \tag{A.26}
\end{equation*}
$$

where $\mathrm{w}^{\prime}=\mathrm{w}\left(p+p^{\prime}\right)$ and where $\mathrm{w}(p)$ is the weight defined in (2.15).
We first show that the inequality holds in the case $j=0, \pm 1$. Indeed we have

$$
\prod_{i}\lfloor i\rfloor^{\alpha_{i}+\beta_{i}} \geq \prod_{i} 2^{\alpha_{i}+\beta_{i}}=2^{\sum_{i} \alpha_{i}+\beta_{i}} \geq 4
$$

since $\sum_{i} \alpha_{i}+\beta_{i} \geq 2$.
Consider now the case $|j|=\lfloor j\rfloor \geq 2$. Since $\alpha_{j}+\beta_{j} \geq 1$, the inequality A.26) follows by showing that

$$
\begin{equation*}
\sup _{j, \alpha, \beta} \frac{|j|}{\prod_{i \neq j}\lfloor i\rfloor^{\alpha_{i}+\beta_{i}}} \leq 1 \tag{A.27}
\end{equation*}
$$

By momentum conservation we have

$$
|j| \leq \sum_{i \neq j}|i|\left(\alpha_{i}+\beta_{i}\right) \leq \sum_{i \neq j}\lfloor i\rfloor\left(\alpha_{i}+\beta_{i}\right)
$$

Then A.27) follows by showing that

$$
\sup _{j, \alpha, \beta} \frac{\sum_{i \neq j}\lfloor i\rfloor\left(\alpha_{i}+\beta_{i}\right)}{\prod_{i \neq j}\lfloor i\rfloor^{\alpha_{i}+\beta_{i}}} \leq 1
$$

where we can restrict the sum and the product to the indexes $i$ such that $\alpha_{i}+\beta_{i} \geq 1$. The latter bound follows by the fact that, given $x_{k} \geq 1$,

$$
\frac{\sum_{2 \leq k \leq n} k x_{k}}{\prod_{2 \leq k \leq n} k^{x_{k}}} \leq 1
$$

as one can prove by induction oven $n$ and recalling that $n^{x} \geq n x$ for $n \geq 2, x \geq 1$.
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[^0]:    ${ }^{1}$ note that the weight is sub-linear in the sense $\lambda\left(y_{1}+y_{2}\right) \leq \lambda\left(y_{1}\right)+\lambda\left(y_{2}\right)$ for any $y_{1}, y_{2} \in \mathbb{R}$.
    ${ }^{2}$ We denote by $\ell^{2}(\mathbb{R})$ the subspace of $\ell^{2}(\mathbb{C})$ made of sequences $\left(\psi_{j}\right)_{j \in \mathbb{Z}}$ such that $\psi_{j}=\bar{\psi}_{-j}$.

[^1]:    ${ }^{3}$ Here as usual for integer vectors we denote $|\ell|:=\sum_{i \in \mathbb{Z}}\left|\ell_{i}\right|$.

[^2]:    ${ }^{4}$ Note that the naturality of the Poisson brackets holds $\left\{H_{\mathbb{R}}, G_{\mathbb{R}}\right\} \circ \mathcal{C}^{-1}=\{H, G\}$.

[^3]:    ${ }^{5}$ As usual given a vector $k \in \mathbb{Z}^{\mathbb{Z}},|k|:=\sum_{j \in \mathbb{Z}}\left|k_{j}\right|$.

