A tight relation between series–parallel graphs and bipartite distance hereditary graphs[*]

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Abstract

Bandelt and Mulder's structural characterization of bipartite distance hereditary graphs asserts that such graphs can be built inductively starting from a single vertex and by repeatedly adding either pendant vertices or twins (i.e., vertices with the same neighborhood as an existing one). Dirac and Duffin's structural characterization of 2–connected series–parallel graphs asserts that such graphs can be built inductively starting from a single edge by adding either edges in series or in parallel. In this paper we give an elementary proof that the two constructions are the same construction when bipartite graphs are viewed as the fundamental graphs of a graphic matroid. We then apply the result to re-prove known results concerning bipartite distance hereditary graphs and series–parallel graphs and to provide a new class of polynomially-solvable instances for the integer multi-commodity flow of maximum value.

Keywords: Series-parallel graphs, bipartite distance hereditary graphs, binary matroids.

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1 Introduction

Distance hereditary graphs are graphs with the *isometric property*, i.e., the distance function of a distance hereditary graph is inherited by its connected induced subgraphs. This important class of graphs was introduced and thoroughly investigated by Howorka in [24, 25]. A bipartite distance hereditary (BDH for short) graph is a distance hereditary graph which is bipartite. Such graphs can be constructed starting from a single vertex by means of the following two operations [6]:

(BDH1) adding a *pendant vertex*, namely a vertex adjacent exactly to an existing one;

(BDH2) adding a *twin* of an existing vertex, namely adding a vertex and making it adjacent to all the neighbors of an existing vertex.

Taken together the two operations above will be referred to as Bandelt and Mulder's construction.

A graph is *series–parallel* [7], if it does not contain the complete graph K_4 as a minor; equivalently, if it does not contain a subdivision of K_4. This is Dirac's [14] and Duffin's [15] characterization by forbidden minors. Since both K_5 and $K_{3,3}$ contain a subdivision of K_4, by Kuratowski's Theorem any series–parallel graph is planar. Like BDH graphs, series–parallel graphs admit a constructive characterization which justifies their name: a connected graph is series–parallel if it can be constructed starting from a single edge by means of the following two operations:

(SP1) adding an edge with the same end-vertices as an existing one (*parallel extension*);

(SP2) subdividing an existing edge by the insertion of a new vertex (*series extension*).

Taken together the two operations above will be referred to as Duffin's construction. Here and throughout the rest of the paper we consider only 2–connected series–parallel graphs which can be therefore obtained by starting from a pair of a parallel edges rather than by starting from a single edge.

The close resemblance between operations (BDH1) and (BDH2) and operations (SP1) and (SP2) is apparent. It becomes even more apparent after our Theorem 3.1, which establishes that the constructions defining BDH and series–parallel graphs, namely, Bandelt and Mulder's construction and Duffin's construction, are the same construction when bipartite graphs are viewed as fundamental graphs of a graphic matroid (Theorem 3.1). Although this fact is fairly well known and short proofs can be given using the deep and refined notions of *branch width* and *tree width* of graphs and matroids[1] (combined with classical results on graph minors), neither an elementary proof nor an explicit statement seem to be at hand.

The intimate relationship between BDH graphs and series–parallel graphs was also already observed by Ellis-Monhagan and Sarmiento in [16]. The authors, motivated by the aim of finding polynomially computable classes of instances for the *vertex–nullity interlace polynomial* introduced by Arratia, Bollobás and Sorkin in [5], under the name of *interlace polynomial*, related the two classes of graphs via a topological construction involving the so called *medial graph* of a planar graph. By further relying on the relationships

[1]In section 5, we give one of such a proof kindly supplied by an anonymous referee of an earlier version of the paper.

between the *Martin polynomial* and the *symmetric Tutte polynomial* of a planar graph, they proved a relation between the the *symmetric Tutte polynomial* of a planar graph H, namely $t(H; x, x)$—recall that the Tutte polynomial is a two variable polynomial–and the interlace polynomial $q(G; x)$ of a graph G derived from the medial graph of G (Theorem 4.1). Such a relation led to the following three interesting consequences:

– the #P–completeness of the interlace polynomial of Arratia, Bollobás and Sorkin [5] in the general case;

– a characterization of BDH graphs via the so-called γ *invariant*, (i.e., the coefficient of the linear term of the interlace polynomial);

– an effective proof that the interlace polynomial is polynomial-time computable within BDH graphs.

In view of a result due to Aigner and van der Holst (Theorem 4.6), the latter two consequences in the list above are straightforward consequences of Theorem 3.1 (see Section 4.1).

Besides the new direct proofs of these results, Theorem 3.1 has some more applications.

– Syslo's characterization's of series–parallel graphs in terms of *Depth First Search* (DFS) *trees*: the characterization asserts that a connected graph H is series–parallel if and only if every spanning tree of H is a DFS-tree of one of its 2–*isomorphic* copies. In other words, up to 2–*isomorphism*, series–parallel graphs have the characteristic property that their spanning trees can be oriented to become *arborescences* so that the corresponding fundamental cycles become directed circuits (cycles whose arcs are oriented in the same way). Recall that an *arborescence* is a directed tree with a single special node distinguished as the *root* such that, for each other vertex, there is a directed path from the root to that vertex.

– New polynomially solvable instances for the problem of finding *integer multi-commodity flow* of maximum value: if the demand graph of a series–parallel graph is a co–tree, then the maximum value of a multi-commodity flow equals the minimum value of a *multi-terminal cut*; furthermore both a maximizing flow and a minimizing cut can be found in strongly polynomial time.

Organization of the paper. The rest of the paper is organized as follows: in Section 2 we give the basic notions used throughout the rest of the paper. In Section 3 we prove our main result (Theorem 3.1) (two more proofs are given in Section 5) and discuss how it fits within *circle graphs* and how it relates with edge-pivoting. Theorem 3.1 is then applied in Section 4: in Section 4.1, we re-prove the previously mentioned couple of results in [16]; in Section 4.2 we re-prove Syslo's characterization of series–parallel graphs and give a sort of hierarchy of characterizations of 2–connected planar graphs by the properties of their spanning trees; finally in Section 4.3, we give an application to multi-commodity flow in series–parallel graphs.

2   Preliminaries

For a graph G the edge e with endvertices x and y will be denoted by xy. The graph induced by $U \subseteq V(G)$ is denoted by $G[U]$. If $F \subseteq E(G)$, the graph $G - F$ is the graph $(V(G), E(G) - F)$.

A *digon* is a pair of parallel edges, namely a cycle with two edges. A *hole* in a bipartite graph is an induced subgraph isomorphic to C_n for some $n \geq 6$. A *domino* is a subgraph isomorphic to the graph obtained from C_6 by joining two antipodal vertices by a chord. The domino is denoted by $\boxminus$. A bipartite graph G is a *chordal bipartite graph* if G has no hole. Let $\mathcal{F}$ be a family of graphs. We say that G is $\mathcal{F}$–free if G does not contain any induced copy of a member of $\mathcal{F}$. If G is $\mathcal{F}$–free and $\mathcal{F} = \{G_0\}$, then we say that G is G_0–free.

Graphs dealt with in this paper are, in general, not assumed to be vertex-labeled. However, when needed, vertices are labeled by the first n naturals where n is the order of G. We denote labeled and unlabeled graphs with the same symbol. If u and v are two vertices of G, then a *label swapping* at u and v (or simply uv-swapping) is the labeled graph obtained by interchanging the labels of u and v. For a bipartite graph G with color classes A and B, let $\mathsf{A} \in \{0,1\}^{A \times B}$ be the *reduced adjacency matrix of G*, namely, A is the matrix whose rows are indexed by the vertices of A, whose columns are indexed by the vertices of B and where $\mathsf{A}_{u,v} = 1$ if and only if u and v are adjacent vertices of G. The *incidence graph* of a matrix $\mathsf{A} \in \{0,1\}^{A \times B}$ is the bipartite graph with color classes A and B and where $u \in A$ and $v \in B$ are adjacent if and only $a_{u,v} = 1$.

We review very briefly some basic notions in matroid theory [28, 36, 37]. For a $\{0,1\}$-matrix A the *binary matroid generated by* A, denoted by $M(\mathsf{A})$, is the matroid whose elements are the indices of the columns of A and whose independent sets are those subsets of elements whose corresponding columns are linearly independent over $GF(2)$. A *binary matroid* is a matroid isomorphic to the binary matroid generated by some $\{0,1\}$-matrix A. If T is a basis of a binary matroid M and $f \notin T$, then $T \cup \{f\}$ contains a unique minimal non independent set $C(f,T)$. Thus, if F is a proper subset of $C(f,T)$, then F is an independent set of M. Such a set $C(f,T)$ is the so called *fundamental circuit through f with respect to T* and $C(f,T) - \{f\}$ is the corresponding *fundamental path*. A *partial representation* of a binary matroid M is a $\{0,1\}$-matrix $\widetilde{\mathsf{A}}$ whose columns are the incidence vectors over the elements of a basis of the fundamental circuits with respect to that basis.

A *fundamental graph of a binary matroid M* is simply the incidence bipartite graph of any of its partial representations. Therefore a bipartite graph G is the fundamental graph of a binary matroid M if G is isomorphic to the graph $B_M(T)$ with color classes T and $\overline{T}$ for some basis T and co-basis $\overline{T}$ (i.e., the complement of a basis) of M and where there is an edge of G between $e \in T$ and $f \in \overline{T}$ if $e \in C(f,T)$, where $C(f,T)$ is the fundamental circuit through f with respect to T. If $\widetilde{\mathsf{A}}$ is a partial representation of a binary matroid M, then $M \cong M([\mathsf{I}\,|\,\widetilde{\mathsf{A}}])$, that is M is isomorphic to the matroid generated by $[\mathsf{I}\,|\,\widetilde{\mathsf{A}}]$. Clearly, $\widetilde{\mathsf{A}}$ is a partial representation of M with rows and columns indexed by the elements of the basis T and of the co-basis $\overline{T}$, respectively, if and only if $\widetilde{\mathsf{A}}$ is the reduced adjacency matrix of $B_T(M)$, where the color class T indexes the rows of $\widetilde{\mathsf{A}}$.

The *cycle matroid* (also known as *graphic matroid)* of a graph H, denoted by $M(H)$, is the matroid whose elements are the edges of H and whose independent sets are the forests of H. If H is connected, then the bases of $M(H)$ are precisely the spanning trees of H and its co-bases are precisely the *co-trees*, namely the subgraphs spanned by the complement of the edge–set of a spanning tree. A matroid M is a *cycle matroid* if it is isomorphic to the cycle matroid of some graph H. Cycle matroids are binary: if M is a cycle matroid, then there are a graph H and a spanning forest of H such that $M \cong M([\mathsf{I}\,|\,\widetilde{\mathsf{A}}])$ where $\widetilde{\mathsf{A}}$ is the $\{0,1\}$-matrix whose columns are the incidence vectors over the edges of a spanning forest of the fundamental cycles with respect to that spanning forest.

A *fundamental graph of a graph* H is simply the fundamental graph of its cycle matroid $M(H)$. For a graph H and one of its spanning forests T, we abridge the notation $B_{M(H)}(T)$ into $B_H(T)$ to denote the fundamental graph of H with respect to T (see Figure 1, where $H \cong K_4$). If H is 2–connected, then $B_H(T)$ is connected. Moreover, $B_H(T)$ does not determine H in the sense that non-isomorphic graphs may have isomorphic fundamental graphs. This because, while it is certainly true that isomorphic graphs have isomorphic cycle matroids, the converse is not generally true (see Figure 2). Two graphs having isomorphic cycle matroids are called 2–*isomorphic*.
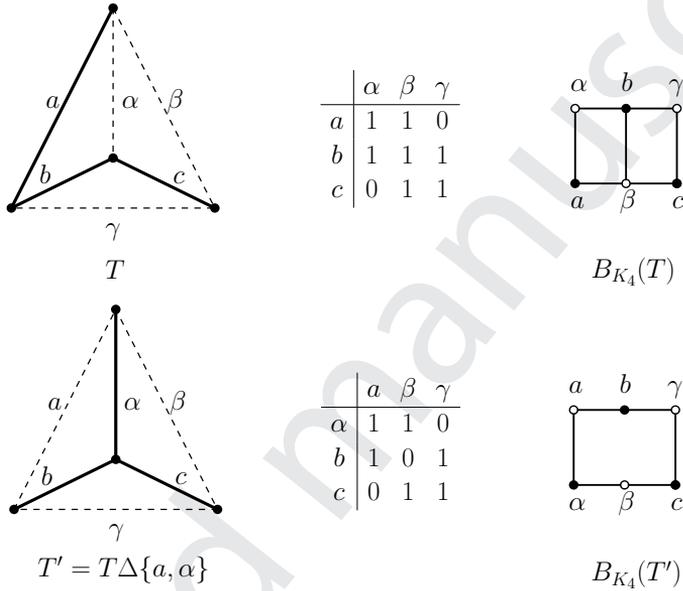

Figure 1: Two fundamental graphs of K_4 with respect to two spanning trees T and T' along with the corresponding matrices and the respective fundamental graphs. The fundamental graph with respect to T' arises from the one with respect to T by pivoting along edge αa.
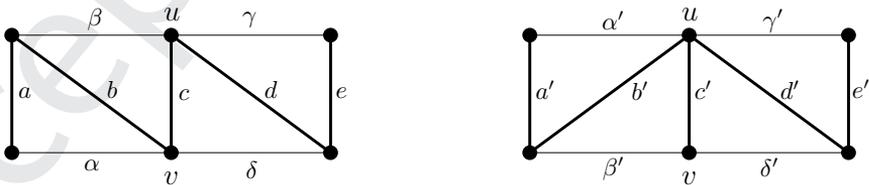

Figure 2: Two 2–isomorphic graphs that are not isomorphic: $x \mapsto x'$ maps bijectively fundamental cycles of the graph on the left to fundamental cycles of the graph on the right.

3   BDH graphs are fundamental graphs of series parallel graphs

In this section we prove our main result.

Theorem 3.1. *A connected bipartite graph G is a bipartite distance hereditary graph if and only if G is a fundamental graph of a 2–connected series–parallel graph.*

Proof. For a bipartite graph G let M^G denote the binary matroid generated by the reduced adjacency matrix of G. Let us examine preliminarily the effect induced on a fundamental graph $B_H(T)$ of a 2–connected graph H by series and parallel extensions and, conversely (and in a sense "dually"), the effect induced on M^G by extending a connected bipartite graph G through the addition of violated vertices and twins. If M^G is a graphic matroid and H is one of the 2–isomorphic graphs whose cycle matroid is isomorphic to M^G, then Table 1 summarizes these effects.

Operation on H		Operation on $B_H(T)$
Parallel extension on edge a of T	$\leftrightarrow$	adding a pendant vertex in color class $\overline{T}$ adjacent to a
Series extension on edge a of T	$\leftrightarrow$	adding a twin of a in color class T
Parallel extension on edge β of $\overline{T}$	$\leftrightarrow$	adding a twin of β in color class $\overline{T}$
Series extension on edge β of $\overline{T}$	$\leftrightarrow$	adding a pendant vertex in color class T adjacent to β.

Table 1: The effects of series and parallel extension on H on its fundamental graph $B_H(T)$.

We can now proceed with the proof. The *only if direction* is proved by induction on the order of G. The assertion is true when G has two vertices because K_2 is a BDH graph and at the same time is also the fundamental graph of a digon. Let now G have $n \geq 3$ vertices and assume that the assertion is true for BDH graphs with $n - 1$ vertices. By Bandelt and Mulder's construction G is obtained from a BDH graph G' either by adding a pendant vertex or a twin. Let H' be a series–parallel graph having G' as fundamental graph with respect to some spanning tree. Since, by Table 1, the last two operations correspond to series or parallel extension of H', the result follows by Duffin's construction of series–parallel graphs. For the *if direction*, let G be the fundamental graph of a series–parallel graph H with respect to some tree T. By Duffin's construction of series–parallel graphs and Table 1, G can be constructed starting from a single edge by either adding twins or pendant vertices. Therefore, G is a BDH graph by Bandelt and Mulder's construction. $\square$

Before going through applications, let us discuss how Theorem 3.1 relates to *circle graphs*, a thoroughly investigated class of graphs which we now briefly describe.

A *double occurrence word* $\mathbf{w}$ over a finite alphabet Σ is a word in which each letter appears exactly twice, where $\mathbf{w}$ is cyclic word, namely, it is the equivalence class of a linear word modulo cyclic shifting and reversal of the orientation. Two distinct symbols of Σ in $\mathbf{w}$ are *interlaced* if one appears precisely once between the two occurrences of the other. By wrapping $\mathbf{w}$ along a circle and by joining the two occurrences of the same symbol of $\mathbf{w}$ by a chord labeled by the same symbols whose occurrences it joins, one obtains a pair $(S, \mathcal{C})$ consisting of a circle S and a set $\mathcal{C}$ of chords of S. In knot theory terminology, such a pair is usually called a *chord diagram* and its intersection graph, namely the graph whose vertex set is $\mathcal{C}$ and where two vertices are adjacent if and only if the corresponding chords

intersects, is called *the interlacement graph of the chord diagram* or the *interlacement graph of the double occurrence word.*

A graph is an *interlacement graph* if it is the interlacement graph of some chord diagram or of some double occurrence words. Interlacement graphs are probably better known as *circle graphs*. The name *interlacement graph* comes historically from the *Gauss Realization Problem of double occurrence words* [13, 31, 34].

Distance hereditary graphs are circle graphs [8]. Thus BDH graphs form a proper subclass of bipartite circle graphs. De Fraysseix [11, 12] proved the following.

Theorem 3.2 ([11, 12])**.** *A bipartite graph is a bipartite circle graph if and only if it is the fundamental graph of a planar graph.*

Therefore Theorem 3.1 specializes de Fraysseix's Theorem to the subclass of series–parallel graphs.

3.1 BDH graphs and edge–pivoting

It follows from Theorem 3.1 that with every 2–isomorphism class of 2–connected series–parallel graphs one can associate all the BDH graphs that are fundamental graphs of each member in the class. Therefore BDH graphs that correspond to the same 2–isomorphism class are graphs in the same "orbit". In this section we make precise the latter sentence and draw the graph-theoretical consequences of this fact.

Given a $\{0, 1\}$-matrix A, pivoting A over $GF(2)$ on a nonzero entry (the pivot element) means replacing

$$\mathsf{A} = \left(\begin{array}{cc} 1 & \mathsf{a} \\ \mathsf{b} & \mathsf{D} \end{array} \right) \quad \text{by} \quad \tilde{A} = \left(\begin{array}{cc} 1 & \mathsf{a} \\ \mathsf{b} & \mathsf{D} + \mathsf{ba} \end{array} \right)$$

where a is a row vector, b is a column vector, D is a submatrix of A and the rows and columns of A have been permuted so that the pivot element is $a_{1,1}$ ([10, p. 69], [32, p. 280]). If A is the partial representation of the cycle matroid of a graph H (or more generally a binary matroid), then pivoting on a nonzero entry, $C(e, f)$, say, yields a new tree (basis) with f in the tree (basis) and e in the co-tree (co-basis) and the matrix obtained after pivoting is a new partial representation matrix of the same matroid. Clearly the fundamental graphs associated with the two bases change accordingly so that pivoting on $\{0, 1\}$-matrices induces an operation on bipartite graphs whose concrete interpretation is a change of basis in the associated binary matroid. The latter operation on bipartite graph will be still referred to as *edge–pivoting* or simply to as *pivoting* in analogy to what happens for matrices (see also Figure 1). In the context of circle graphs, the operation of pivoting is a specialization to bipartite graph of the so called *edge–local complementation*. Since any bipartite graph is a fundamental graph of some binary matroid, the operation of *pivoting* can be described more abstractly as follows.

Given a bipartite graph with color classes A and B, *pivoting on edge* $uv \in E(G)$ is the operation that takes G into the graph G^{uv} on the same vertex set of G obtained by complementing the edges between $N_G(u) \setminus \{u\}$ and $N_G(v) \setminus \{v\}$ and then by swapping the labels of u and v (if G is labeled). More formally, if $\ell_G \colon V(G) \to \mathbb{N}$ is a labeling of the vertices of G, then

$$G^{uv} = (V(G), E(G)\Delta((N_G(u) \setminus \{u\}) \times (N_G(v) \setminus \{v\})))$$

and $\ell_{G^{uv}}$ is defined by $\ell_{G^{uv}}(u) = \ell_G(v)$, $\ell_{G^{uv}}(v) = \ell_G(u)$ and $\ell_{G^{uv}}(w) = \ell_G(w)$ if $w \notin \{u, v\}$. If $e \in E(G)$ has endpoints uv, then we use G^e in place of G^{uv}.

We say that a graph $\widetilde{G}$ is *pivot-equivalent* to a graph G, written $\widetilde{G} \sim G$, if for some $k \in \mathbb{N}$, there is a sequence $G_1, \ldots, G_k$ of graphs such that $G_1 \cong G$, $G_k \cong \widetilde{G}$ and, for $i = 1, \ldots, k-1$, $G_{i+1} \cong G_i^{e_i}$, $e_i \in E(G_i)$. The *orbit of G*, denoted by $[G]$, consists of all graphs that pivot-equivalent to G.

For later reference, we state as a lemma the easy though important facts discussed above. Figure 1 illustrates the contents of the lemma.

Lemma 3.3. *Let M be a connected graphic matroid. Then M determines both a class $[G]$ of pivot-equivalent graphs and a class $[H]$ of 2–isomorphic graphs. In particular, any graph in $[G]$ is the fundamental graph of some 2–isomorphic copy of H and the fundamental graph of any graph in $[H]$ is pivot-equivalent to G.*

The operations of pivoting and of taking induced subgraphs commute in (bipartite) graphs.

Lemma 3.4 (see [5])**.** *Let G a bipartite graph, $U \subseteq V(G)$ and e be an edge whose end-vertices are in U. Then $G^e[U] \cong (G[U])^e$.*

The next lemma relates in the natural way minors of a cycle matroid to the induced subgraphs of the fundamental graphs associated with the matroid.

Lemma 3.5. *Let M and N be cycle matroids. Let G be any of the fundamental graphs of M and let K be any of the fundamental graphs of N. Then N is a minor of M if and only if K is an induced subgraph in some bipartite graph in the orbit of G. Equivalently, N is a minor of M if and only if G contains some induced copy of a graph in the orbit of K.*

To get acquainted with pivoting, the reader may check Lemma 3.6 with the help of Figure 3. Refer to Section 2 for the definition of *domino* and *hole*.
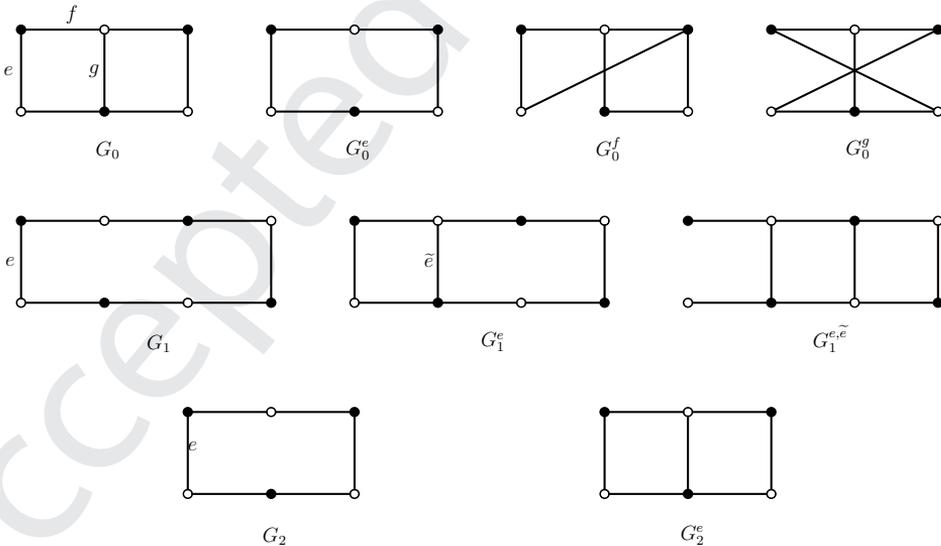

Figure 3: The effect of pivoting a graph G along some of its edges when $G \cong \boxminus, C_8, C_6$.

Lemma 3.6. *Let $k \geq 6$ be an even integer.*

- *If either $H \cong \boxminus$ or $H \cong C_k$, then for each $uv \in E(H)$ there exists an induced subgraph H' of H^{uv} such that either $H' \cong \boxminus$ or $H' \cong C_k$.*

- *If $G \cong C_k$, then there is a graph $\widetilde{G}$ in the orbit of H such that $\widetilde{G}$ contains an induced copy of either $\boxminus$ or C_6.*

Proof. By inspecting Figure 3 one checks that if $G \cong \boxminus$, then either $G^e \cong \boxminus$ or $G^e \cong C_6$. If $G \cong C_6$, then $G^e \cong \boxminus$ for every $e \in E(G)$. If $G \cong C_k$, $k > 6$, then by pivoting on $uv \in E(G)$ and deleting u and v results in a graph $G' \cong C_{k-2}$. In particular, by repeatedly pivoting on new formed edges (like edge $\widetilde{e}$ of graph G_1^e in Figure 3), one obtains a graph in the orbit of G which contains an induced copy of either $\boxminus$ or C_6. The second part of the proof is left to the reader.   □

We are ready to extract the graph-theoretical consequence of Theorem 3.1. To this end let us recall that besides their constructive characterization, Bandelt and Mulder characterized the class of BDH graphs also by forbidden induced subgraphs as follows.

Theorem 3.7 ([6, Corollaries 3 and 4])**.** *Let G be a connected bipartite graph. Then G is BDH if and only if G contains neither holes nor induced dominoes.*

The following two corollaries follow straightforwardly from Theorem 3.1 after Theorem 3.7 and assert that the class of BDH graphs–that is, of $\{hole, domino\}$–free graphs–is closed under pivoting, namely, that the orbit of a bipartite $\{hole, domino\}$–free graph consists of $\{hole, domino\}$–free graphs.

Corollary 3.8. *The following statement about a chordal bipartite graph G are equivalent:*

(i) G does not contain any induced domino;

(ii) any graph in the orbit of G is a chordal bipartite graph.

Corollary 3.9. *Let G be a bipartite domino-free graph. If G is chordal, then so is any other graph in its orbit.*

4   Applications

4.1   BDH graphs and the interlace polynomial

As already mentioned, Ellis-Monaghan and Sarmiento related series–parallel graphs and BDH graphs topologically, via the medial graph. Let H be a plane graph (or even a 2-cell embedded graph in an oriented surface). For our purposes, we can assume that H is 2–connected. The medial graph $m(H)$ of H is the graph obtained as follows: first place a vertex v_e into the interior of every edge e of H. Then, for each face F of H, join v_e to v_f by an edge lying in F if and only if the edges e and f are consecutive on the boundary of F. Notice that if F is bounded by a digon $\{e, e'\}$ or if e and e' share a degree-2 endpoint in H, then vertices v_e and $v_{e'}$ are joined by two parallel edges. Let $m(H)$ be the plane (2-cell embedded) graph obtained in this way. The graph underlying $m(H)$ is the medial graph of H. The medial graph is clearly 4-regular, as each face creates two adjacencies for each edge on its boundary. Moreover, it can be oriented so that each vertex is entered by 2 arcs and left by 2 arcs. Given a 4-regular labeled graph N and one of its Eulerian circuits

C, we can associate with N a double occurrence word **w** which is the word consisting of the labels of the vertices of C cyclically met during the tour on C. The circle graph formed from C and chords between repeated pairs of letters of **w** is called the *the circle graph of* N. Ellis-Monaghan and Sarmiento, building also on the relations between the *Martin polynomial* and the symmetric *Tutte Polynomial*, proved the following relation between the symmetric Tutte polynomial $t(H; x, x)$ of a planar graph H and the vertex nullity interlace polynomial $q(G; x)$ of a graph G derived, as described in the theorem below, from the medial graph of any of its plane embedding.

Theorem 4.1 ([16])**.** *If H is a plane embedding of a planar graph and G is the circle graph of some Eulerian circuit of the medial graph of H, then $q(G; x) = t(H; x, x)$.*

The results were then specialized so as to give the following characterization of BDH graphs.

Theorem 4.2 ([16])**.** *G is a BDH graph with at least two vertices if and only if it is the circle graph of an Euler circuit in the medial graph of a plane embedding of a series–parallel graph H.*

Using Theorem 4.1 and Theorem 4.2, the authors deduced the following consequences stated below as Corollary 4.3, Corollary 4.4 and Corollary 4.5.

Corollary 4.3. *The vertex-nullity interlace polynomial is #P-hard in general.*

Corollary 4.4. *If G is a BDH graph, then $q(G; x)$ is polynomial-time computable.*

Corollary 4.4 follows because the Tutte polynomial is polynomial-time computable for series–parallel graphs [29].

Corollary 4.5. *A connected graph G is a BDH graph if and only if the coefficient of the linear term of $q(G; x)$ equals 2.*

The latter coefficient referred to in Corollary 4.5, denoted by $\gamma(G)$, is called the γ-invariant of G in analogy with the Crapo invariant $\beta(G)$ which is the common value of the coefficients of the linear terms of $t(G; x, y)$ where G has at least two edges. By a result due to Brylawski [9] (in the more general context of matroids) series–parallel graphs can be characterized by the value of the Crapo invariant as follows: a graph G is a series–parallel graph if and only if $\beta(G) = 1$. Both the corollaries above can be deduced directly by Theorem 3.1 after the following result due to Aigner and van der Holst [1].

Theorem 4.6 ([1])**.** *If G is a bipartite graph, then*

$$q(G; x) = t(M^G; x, x)$$

where M^G is the binary matroid generated by the reduced adjacency matrix of G and $t(M^G; x, x)$ is the Tutte polynomial of M^G.

Theorem 3.1 and Theorem 4.6 have the following straightforward consequence which re-proves directly Corollary 4.4 and Corollary 4.5.

Corollary 4.7. *If G is BDH graph, then*

$$q(G; x) = t(H; x, x)$$

for some series–parallel graph H having G as fundamental graph and where $t(H; x, x)$ is the Tutte polynomial of H, namely the Tutte polynomial of the cycle matroid of H.

4.2   Characterizing series–parallel graphs by DFS-trees

As credited by Syslo [35], Shinoda, Chen, Yasuda, Kajitani, and Mayeda, proved that series–parallel graphs can be completely characterized as in Theorem 4.8 by a property of their spanning trees, and Syslo himself gave a constructive algorithmic proof of the result [35].

Theorem 4.8 (S. Shinoda et al., 1981; Syslo, 1984)**.** *Every spanning tree of a connected graph H is a DFS-tree of one of its 2–isomorphic copies if and only if H is a series–parallel graph.*

When H is assumed to be 2–connected (an assumption that guarantees the connectedness of its fundamental graphs), Theorem 4.8 will be equivalently stated as statement (1) below.

Let $\mathcal{T}$ be a family of trees (or a family of oriented trees) and let G be a bipartite graph with color classes A and B. We say that G is a *path/ $\mathcal{T}$ bipartite graph on A* if there exists a member T of $\mathcal{T}$ and a bijection $\xi\colon A \to E(T)$ such that, for each $v \in B$, $\{\xi w \mid w \in N_G(v)\}$ is the edge–set (arc–set if T is oriented) of a simple cycle (directed circuit if T is oriented) in the (oriented) graph $(V(T), A \cup B)$. *Path/ $\mathcal{T}$ bipartite graphs on B* are defined similarly. G is a *path/ $\mathcal{T}$ bipartite graph* if it is a path/ $\mathcal{T}$ bipartite graph on A or on B. G is a *self–dual path/ $\mathcal{T}$* bipartite graph if it is a path/ $\mathcal{T}$ bipartite graph on both A and B. In any case T will be referred to as a *supporting tree* for G. For instance, if $G \cong K_{1,3}$ and G has color classes $A = \{a\}$ and $B = \{\alpha, \beta, \gamma\}$ and if $\mathcal{T}$ is any family of paths containing paths of order 2 and order 4, then G is a path/ $\mathcal{T}$ bipartite graph: G is supported on A by a path of order 2 whose unique edge is labeled a and G is supported on B by a path of order 4 with three edges labeled α, β and γ.

Recall that an *arborescence* is a directed tree with a single special node distinguished as the *root* such that, for each other vertex, there is a directed path from the root to that vertex. A *DFS tree* for a connected graph H (in the sense of [35]), is a pair (T, ϕ) consisting of a spanning tree and an orientation ϕ of H, such that ϕT is a spanning arborescence of ϕH and for each $f \in E(H) \setminus E(T)$, $\phi C(f, T)$ is a directed circuit in ϕH (i.e, all arcs of $\phi C(f, T)$ are oriented in the same way). By choosing for $\mathcal{T}$ the class **arborescence** of arborescences, one can reformulate Theorem 4.8 in the following way

(1) H is series–parallel if and only if for each spanning tree T of H the fundamental graph $B_T(H)$ is a self–dual path/**arborescence** bipartite graph.

Indeed, if (T, ϕ) is a DFS-tree in a 2–isomorphic copy H' of H, then T is a spanning tree of graph H' whose cycle matroid is $M(H)$; hence $B_H(T) \cong B_{H'}(T)$ and ϕT is a supporting arborescence for $B_H(T)$. Conversely, suppose that G is a fundamental graph of H and that G is a path/**arborescence** bipartite graph. Let G have color classes A and B. Since G is a path/**arborescence** bipartite graph, then there is a supporting arborescence $\overrightarrow{T}$ for G that induces an orientation ϕ of the graph $H' = (V(T), A \cup B)$, T being the underlying undirected graph of $\overrightarrow{T}$. Clearly (T, ϕ) is a DFS tree in H' which in turn is 2–isomorphic to H because G is one of its fundamental graphs (i.e., H and H' have the same cycle matroid).

Statement (1) is now a rather straightforward consequence of Corollary 3.8 and the fact that BDH graphs are self–dual path/**arborescence** bipartite graphs as shown by the following result proved in [4].

Theorem 4.9 ([4])**.** *Every connected BDH graph is a self–dual path/**arborescence** bipartite graph.*

Proof of (1). Let H be a 2–connected series–parallel graph. Then, by Theorem 3.1 $B_H(T)$ is BDH for each spanning tree T of H. Hence, for every spanning tree T of H, $B_H(T)$ is a self–dual path/**arborescence** bipartite graph by Theorem 4.9. Conversely, suppose that for every spanning tree T of a 2–connected graph H, the fundamental graph $B_H(T)$ is a path/**arborescence** bipartite graph. Thus $B_H(T)$ is chordal (see, e.g., [8]). Moreover, since if T' is any other spanning tree of H, then $B_H(T')$ is in the orbit of $B_H(T)$, we conclude that each bipartite graph in the orbit of $B_H(T)$ is a chordal bipartite graph. Therefore $B_H(T)$ is a BDH graph by Corollary 3.8 and, consequently, H is a series–parallel graph. □

It is worth observing that, in the same way as Theorem 3.1 specializes de Fraysseix's Theorem 3.2, Statement (1) specializes the following statement (see also [12]):

(2) a bipartite graph is a bipartite circle graph if and only if it is a self–dual path/**tree** bipartite graph, **tree** being the class of trees.

Proof. By Whitney's planarity criterion [38] a graph is planar if and only if its cycle matroid is also *co-graphic*, namely, it is the dual matroid of another cycle matroid. Let now G be the fundamental graph of a 2–connected graph H with respect to some spanning tree T of H. Let A be the reduced adjacency matrix of G with rows indexed by the edges of T and columns indexed by the edges of $\overline{T}$. Then, while $[\mathsf{I}\,|\,\mathsf{A}]$ generates $M(H)$, $[\mathsf{I}\,|\,\mathsf{A}^t]$ generates $M^*(H)$, the dual of $M(H)$. Hence, when H is planar, by Whitney's planarity criterion, $M^*(H)$ is the cycle matroid of a 2–isomorphic copy of a plane dual H^* of H. Therefore the neighbors of each vertex in the color class T spans a path in the co-tree $\overline{T}$ which is in turn a spanning tree of a 2–isomorphic copy of plane dual H^* of H. □

In view of such a discussion it is reasonable to wonder whether there is a class of self dual path/ $\mathfrak{T}_0$ bipartite graphs closed under edge–pivoting, where $\mathfrak{T}_0$ is a family of trees sandwiched between **trees** and **arborescences**. The next result gives a negative answer in a sense. In what follows **di-tree** is the class of oriented trees.

Theorem 4.10. *If G is a connected bipartite graph whose orbit consists of self–dual path/**di-tree** bipartite graphs, then the orbit of G consists of path/**arborescence** bipartite graphs.*

Proof. Path/**di-tree** bipartite graphs are *balanced* (see [2]). Recall that a bipartite graph Γ is *balanced* if its reduced adjacency matrix does not contain the vertex-edge adjacency matrix of a chordless cycle of odd order. Equivalently, Γ is *balanced* if each hole of Γ has order congruent to zero modulo 4. Hence, since G and any other graph in its orbit, is a self–dual path/**di-tree** bipartite graph, then G, and any other graph in its orbit must be balanced as well. Let $\widetilde{G}$ be any member of $[G]$ and suppose that $\widetilde{G}$ contains a hole C. Let $e \in E(C)$. The order t of C is at least eight, because $\widetilde{G}$ is balanced. Nevertheless $\widetilde{G}^e$ contains a hole of order $t-2$ by Lemma 3.6. Since $t-2 \equiv 2 \pmod 4$ we conclude that any graph in the orbit of G must be hole-free. Therefore G is BDH by Corollary 3.9, and, by Theorem 3.1, it is the fundamental graph of a series–parallel graph. The thesis now follows by Statement (1). □

Remark 4.11. It is worth observing that by the proof above, if **A** is a class of balanced matrices closed under pivoting over $GF(2)$, then **A** consists of totally balanced matrices, namely those matrices whose bipartite incidence graph is hole-free. Actually, and more sharply, in view of Corollary 3.9, every member of **A** is the incidence matrix of a γ-acyclic hypergraph [3].

4.3   Packing paths and multi-commodity flows in series–parallel graphs

In this section we give an application of Theorem 3.1 in Combinatorial Optimization. We show that a notoriously hard problem contains polynomially solvable instances when restricted to series–parallel graphs. Let $H = (V, E)$ be a graph and let $F \subseteq E$ be a set of prescribed edges of H called the *nets* of H. Following [19] a path P of H will be called *F-admissible* if it connects two vertices s, t of V with $st \in F$ and $E(P) \subseteq E - F$. Let U be the set of end-vertices of the nets. In the context of network-flow, vertices of U are thought of as terminals to be connected by a flow of some commodity (the nets are in fact also known as *commodities*). Let $\mathcal{P}_F$ denote the family of all F-admissible paths of G and let $\mathcal{P}_{F,f} \subseteq \mathcal{P}_F$ be the family of those F-admissible paths connecting the endpoints s, t of net f. An F-*multiflow* (see e.g. [33]), is a function $\lambda \colon \mathcal{P}_F \to \mathbb{R}_+$, $P \mapsto \lambda_P$. The multiflow is integer if λ is integer valued. The value of the F-*multiflow on the net f* is $\phi_f = \sum_{P \in \mathcal{P}_{F,f}} \lambda_P$. The total value of the multiflow is the number $\phi = \sum_{f \in F} \phi_f$. Let $w \colon E - F \to \mathbb{Z}_+$ be a function to be though of as a capacity function. An F-*multiflow subject to w* in H is an F-multiflow such that,

$$\sum_{P \in \mathcal{P}_F : E(P) \ni e} \lambda_P \leq w(e), \;\; \forall e \in E - F \tag{4.1}$$

When $w(e) = 1$ for all $e \in E - F$, an integer multiflow is simply a collection of edge–disjoint F-admissible paths of H. The F-*Max- Multiflow Problem* is the problem of finding, for a given capacity function w, an F-multiflow subject to w of maximum total value. An F-*multicut* of H is a subset of B edges of $E - F$ that intersects the edge–set of each F-admissible path. The name F-multicut is due to the fact that the removal of the edges of B from H leaves a graph with no F-admissible path: in the graph $H - B$ it is not possible to connect the terminals of any net. The *capacity of the F-multicut B* is the number $\sum_{e \in B} w(e)$.

Multiflow Problems are very difficult problems (see [18], [19] and Vol. C, Chapter 70 in [33]). In [20] it has been shown that the Max-Multiflow Problem is NP-hard even for trees and even for $\{1, 2\}$-valued capacity functions. The problem though is shown to be polynomial time solvable for constant capacity functions by a dynamic programming approach. However, even for constant functions, the linear programming problem of maximizing the value of the multiflow over the system of linear inequalities (4.1) has not even, in general, $\frac{1}{2}\mathbb{Z}$-valued optimal solutions. In [26], the NP-completeness of the Edge–Disjoint–Multi commodity Path Problem for series–parallel graph (and partial 2–trees) has been established while, previously in [39], the polynomial time solvability of the same problem for partial 2–trees was proved under some restriction either on the number of the commodities (required to be a logarithmic function of the order of the graph) or on the location of the nets.

Theorem 4.12. *Let $H = (V, E)$ be a 2–connected series–parallel graph and let F be the edge–set of any of its spanning co-trees. Then the maximum total value of an F-multiflow*

equals the minimum capacity of an F-multicut. Furthermore, both a maximizing multiflow and a minimizing multicut can be found in strongly polynomial time.

Proof. Let A be a $\{0,1\}^{m \times n}$−valued matrix and $b \in \mathbb{Z}_+^m$ be a vector. Let 1_d be the all ones vector in $\mathbb{R}^d$. Consider the linear programming problem

$$\max_{x \in \mathbb{R}_+^n} \left\{ 1_n^T x \mid Ax \leq b \right\} \tag{4.2}$$

and its dual

$$\min_{y \in \mathbb{R}_+^m} \left\{ b^T y \mid A^T y \geq 1_n \right\}. \tag{4.3}$$

By the results of Hoffman, Kolen and Sakarovitch [23] and Farber [17], if A is a totally balanced matrix (i.e., A is the reduced adjacency matrix of a bipartite chordal graph), then both the linear programming problems above have integral optimal solutions and, by linear programming duality, the two problems have the same optimum value. Furthermore, an integral optimal solution x^* to the maximization problem in (4.2) satisfying the additional constraint

$$x^* \leq 1_n \tag{4.4}$$

and an integral optimal solution y^* to the minimization problem in (4.3) satisfying the additional constraint

$$y^* \leq 1_n \tag{4.5}$$

can be found in strongly polynomial time.

Let now H be a 2–connected graph and let F be the edge–set of a co–tree $\overline{T}$ of some spanning T tree of H. By giving a total order on the edge–set of T, one can define a vector b whose entries are the values of the capacity function $w \colon E(H) - F \to \mathbb{Z}_+$. If A is the the incidence matrix of $\mathcal{P}_F$, namely the matrix whose columns are the incidence vectors of the F-admissible paths of H, then A is a partial representation of $M(H)$. Moreover, if H is series–parallel, then A is totally balanced: by Theorem 3.1, A is the reduced adjacency matrix of a BDH graph which is chordal being hole-free (by Theorem 3.7). On the other hand, integral solutions to the problem in (4.2) satisfying constraint (4.4) and to the problem in (4.3) satisfying constraint (4.5) are incidence vectors of F-multiflows and F-multicuts, respectively. Hence, both an F-multiflow of maximum value and an F-multicut of minimum capacity can be found in strongly polynomial-time by solving the linear programming problems above. Moreover, linear programming duality implies that the maximum value of an F-multiflow and the minimum capacity of an F-multicut coincide. $\qquad\square$

5   Two more proofs of Theorem 3.1

In this section, we give two more proofs of Theorem 3.1: one is due to an anonymous referee of an earlier version of the paper and it relies on the deep and refined notion of *branch-* and *rank-width* of a matroid (for the undefined terms given in the proof we address the reader to the references therein); the other fits the theory of double occurrences words and relies on a result in [5].

Second proof of Theorem 3.1. Suppose that a connected bipartite graph G is a fundamental graph of a 2-connected series parallel graph H. Since 2-connected graphs of branch-width at most 2 are exactly 2-connected series parallel graphs ([30]), the branch-width of H is

at most 2. As proved in [22], the branch-width of a graph equals that of its cycle matroid. Hence, the branch-width of H equals the branch-width of $M(H)$. By a result in [27], the branch-width of a binary matroid (in particular of a cycle matroid) equals the rank-width of any of its fundamental graphs plus 1. By definition, G is a fundamental graph of $M(H)$ and thus $\text{rw}(G) + 1 = \text{bw}(M(H)) = \text{bw}(H) \leq 2$, where $\text{rw}(\cdot)$ and $\text{bw}(\cdot)$ denote the rank-width and branch-width parameters, respectively. Hence the rank-width of G is at most 1 and we conclude that G is bipartite distance hereditary because, still by a result in [27], distance hereditary graphs are precisely the graphs of rank-width at most 1.

For the other direction, suppose that a connected bipartite graph G is distance-hereditary. Let M^G be the binary matroid generated by the reduced adjacency matrix of G. By the same reasons (and the same notation) given above, it holds that $\text{bw}(M^G) = \text{rw}(G) + 1 \leq 2$. By a result in [21], M^G is a *series parallel matroid* (see [36] for the definition) and any such a matroid is the cycle matroid of a series parallel graph (see Lemma 4.2.12 in [36]). Hence $M^G = M(H)$ for some series parallel graph H. Furthermore, H is 2-connected, otherwise, G is disconnected. $\qquad\square$

The third proof will require a result in [5]. Let C be an Eulerian cycle in a 4-regular labeled graph H and let $\mathbf{w}$ be the double occurrence word it induces (Section 3, following the first proof of Theorem 3.1). Recall that two vertices, say labeled a and b, are interlaced in $\mathbf{w}$ if $\mathbf{w} = \mathbf{u}ax b y a z$ for some (possibly empty) intervals $\mathbf{u}$, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of $\mathbf{w}$. For two vertices u and v, labeled a and b, respectively, the uv-transposition of $\mathbf{w}$ is the word $\mathbf{w}^{uv} = \mathbf{u}a y b x a z$ [5]. Thus a uv-transposition of $\mathbf{w}$ amounts to replace one of the the subpaths of C connecting u and v with he other one. The relation between uv-transposition and uv pivoting is given in the next lemma which specializes a more general result in [5] (see also [13]).

Lemma 5.1. *Let H be a 4-regular graph and let $\mathbf{w}$ be any of the double occurrence words it induces. Further, let $G(H, \mathbf{w})$ denote the interlacement graph of $\mathbf{w}$. Suppose that $G(H, \mathbf{w})$ is a bipartite graph. Then, for any edge uv of $G(H, \mathbf{w})$ of H, one has $G(H, \mathbf{w})^{uv} = G(H, \mathbf{w}^{uv})$.*

Third proof of Theorem 3.1. If G is a fundamental graph of a series–parallel graph, then M^G is a binary matroid with no $M(K_4)$ minor by Dirac and Duffin's characterization. Dominoes are fundamental graphs of K_4 and holes can be pivoted to either dominoes or C_6 (recall Lemma 3.6)—notice that C_6 is a fundamental graph of K_4 as well (Figure 1)–it follows that G is BDH-free by Lemma 3.3. Conversely, if G is BDH, then by Theorem 4.2 (in the language of Lemma 5.1), $G \cong G(m(H), \mathbf{w})$ for some series–parallel graph H (observe that $m(H)$ is a 4-regular graph) and some code $\mathbf{w}$. By Lemma 5.1, pivoting on edges G affects neither H nor $m(H)$. Consequently, every graph in $[G]$ is a BDH. Therefore M^G has no $M(K_4)$ minor by Lemma 3.3 and Lemma 3.5 and G is a fundamental graph of such a matroid and therefore the fundamental graph of a series–parallel graph. $\qquad\square$

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