Mitschke's Theorem is sharp

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Abstract. A. Mitschke showed that a variety with an m-ary near-unanimity term has Jónsson terms $t_0, \dots, t_{2m-4}$ witnessing congruence distributivity. We show that Mitschke's result is sharp. We also evaluate the best possible number of Day terms witnessing congruence modularity. More generally, we characterize exactly the best bounds for many congruence identities satisfied by varieties with an m-ary near-unanimity term.

1. Introduction

Recall that a term u is a *near-unanimity term* (in some algebra or in some variety) if all the equations of the form

$$u(x, x, \dots, x, y, x, \dots, x, x) = x$$

are satisfied, with just one occurrence of y in any possible position.

Near-unanimity terms have been originally studied in connection with generalizations of the Chinese remainder theorem [2]. More recent research has shown connections with computational complexity, e. g. [3, 4, 5]. Joins of varieties with near-unanimity terms have been studied in [6].

A. Mitschke [18] proved that every variety $\mathcal{V}$ with a near-unanimity term is congruence distributive. A more direct proof, credited to E. Fried, can be found in Kaarli and Pixley [10, Lemma 1.2.12]. Compare also Barto and Kozik [4, Section 5.3.1] and [14, Section 5].

In particular, any variety with a near-unanimity term is congruence modular. The distributivity [9] and modularity [7] levels of varieties with a near-unanimity term have been evaluated.

Theorem 1.1. *Let $m \geq 3$.*

(1) *(Mitschke [18]) A variety with an m-ary near-unanimity term is $2m-4$-distributive.*

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(2) *(Sequeira* [19, Theorem 3.19]*) A variety with an m-ary near-unanimity term is $2m{-}3$-modular.*

In this note we show that Theorem 1.1 gives the best possible evaluations. Section 2 presents our main construction, where we build appropriate subalgebras of certain products. The construction is then iterated in Section 3 in order to get counterexamples showing that Theorem 1.1 cannot be improved. In Section 4 we exemplify our methods by presenting some more concrete examples. Section 4 is largely self-contained. Further remarks are contained in Section 5.

We shall assume familiarity with the basic notions of universal algebras, as presented, e. g., in [16]. The notions we shall use admit equivalent reformulations in terms of congruence identities, as given by the following table.

$$\begin{array}{ll} n\text{-distributive} & \alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha\gamma \circ \overset{n}{\dots} \\ n\text{-alvin} & \alpha(\beta \circ \gamma) \subseteq \alpha\gamma \circ \alpha\beta \circ \overset{n}{\dots} \\ n\text{-modular} & \alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\beta \circ \alpha\gamma \circ \overset{n}{\dots} \\ n\text{-reversed-modular} & \alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\gamma \circ \alpha\beta \circ \overset{n}{\dots} \end{array} \tag{1.1}$$

A notion mentioned on the left holds in some variety $\mathcal{V}$ if and only if $\mathcal{V}$ satisfies the corresponding congruence identity on the right, that is, the identity holds for every algebra $\mathbf{A}$ in $\mathcal{V}$ and all congruences in $\mathbf{A}$. In the above formulae juxtaposition denotes intersection. For ε and δ binary relations, $\varepsilon \circ \delta \circ \overset{k}{\dots}$ denotes the relation $\varepsilon \circ \delta \circ \varepsilon \circ \delta \circ \dots$ with k factors, that is, $k-1$ occurrences of $\circ$. If, say, k is even, then we write $\varepsilon \circ \delta \circ \overset{k}{\dots} \circ \delta$ when we want to make clear that δ is the last factor. Sometimes, for readability or convenience, we might add further factors in the above expressions, as in $\varepsilon \circ \delta \circ \varepsilon \circ \overset{k}{\dots} \circ \varepsilon \circ \delta$. In any case, the number above the dots represents the number of occurrences of $\circ$ minus one.

Usually, the notions introduced in (1.1) are defined in an equivalent way (throughout a variety) by means of the existence of a certain number of terms, called after B. Jónsson, for n-distributive, and after A. Day, for m-modular. The terms for alvin and reversed modularity are obtained from Jónsson and Day terms, respectively, by exchanging the conditions for even and odd indices. As in [15], such "reversed" conditions will be the key to the exact evaluation of the appropriate parameters: dealing only with the more standard notions we could not succeed in proving the exact results. In any case, here we shall not need terms for distributivity and modularity; dealing with congruence identities will prove much simpler. The reasons why in certain cases congruence identities or even relation identities are more convenient than terms are explained in Tschantz [20].

It is immediate from (1.1) that every n-alvin variety is $n{+}1$-distributive, that every n-distributive variety is $n{+}1$-alvin, and corresponding results hold for modularity and reversed modularity.

See [15], in particular, Section 2 therein, for a full discussion of the equivalences presented in table (1.1). See [14, 15] for more results related or similar to Theorem 1.1 and for further comments. In particular in [15], among many other examples, we presented, for every even $n \geq 2$, the example of a locally finite n-distributive variety which is neither $2n-2$-modular, nor $2n-1$-reversed-modular.

2. The main construction

Definition 2.1. Let $m \geq 3$ and $1 \leq k \leq m$.

If some algebra $\mathbf{A}$ has a special element 0, we say that 0 is a *k-absorbing element for a term u* if $u(a_1, \dots, a_m) = 0$, whenever 0 occurs at least k-times in the arguments of u, more formally, whenever $|\{i \mid a_i = 0\}| \geq k$.

A term u is a *k-majority term* in some algebra $\mathbf{A}$ (in some variety $\mathcal{V}$) if every element of $\mathbf{A}$ (of every algebra in $\mathcal{V}$) is k-absorbing for u. In other words, a k-majority term is supposed to satisfy the equation $u(x_1, \dots, x_m) = x$, whenever the variable x occurs at least k-times in the arguments of u. Clearly, $k > \frac{m}{2}$, unless we are in a trivial variety.

An m-ary term u is *idempotent* if it is an m-majority term, namely, if the equation $u(x, x, \dots, x) = x$ is satisfied.

An m-ary term is a *near-unanimity term* if it is an $m-1$-majority term.

An m-ary term u is *symmetrical* in some algebra $\mathbf{A}$ (in some variety $\mathcal{V}$) if all the equations $u(x_1, \dots, x_m) = u(x_{\tau(1)}, \dots, x_{\tau(m)})$ hold in $\mathbf{A}$ (in $\mathcal{V}$), for all permutations τ of $\{1, \dots, m\}$.

In principle, when $k < m-1$, the notion of a k-majority m-ary term has little interest, since it implies the existence of a near-unanimity term of arity $< m$. However, we shall merge different varieties with a k-majority term, for distinct values of k, in such a way that the resulting variety $\mathcal{V}$ has an $m-1$-majority term (namely, a near-unanimity term) and provides all the desired counterexamples.

The next construction and, more generally, all the arguments in the present note share many aspects in common with the constructions we have performed in [15]. However, an important difference should be mentioned. In the constructions in [15], at each inductive step, we have taken the product of some formerly constructed algebra $\mathbf{A}_4$ with three further algebras. One of these additional algebras, the algebra $\mathbf{A}_3$ in [15], is a term-reduct of the two-elements lattice $\mathbf{C}_2$. Thus in [15] at each induction step a reduct of $\mathbf{C}_2$ is added as a new factor. In the present situation, instead, it is necessary to fix the $\mathbf{C}_2$-reduct once and for all at the beginning, hence here the induction steps start with a subalgebra of $\mathbf{A}_3 \times \mathbf{A}_4$. Let us also mention that, for convenience, here we shall shift the third and fourth indices, in comparison with [15]. In particular, the reduct of $\mathbf{C}_2$ here will appear at the fourth place.

We shall frequently consider special elements $0_z \in \mathbf{A}_z$, for $z = 1, 2, 4$. When no confusion is possible, we shall omit the subscripts. The types introduced in the next lemma have been used also in many constructions from [15]. Since, as we mentioned, we are shifting the last two coordinates, the correspondence with [15] is exact only modulo a permutation of the coordinates. This is the reason why the types here are denoted by, say, II^σ, rather than II.

Throughout the present note objects like $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3 \times \mathbf{A}_4$ and, say, $\mathbf{A}_1 \times \mathbf{A}_2 \times (\mathbf{A}_3 \times \mathbf{A}_4)$ shall be always identified, namely, we consider them modulo isomorphism through the natural correspondence.

Lemma 2.2. *Suppose that $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$ are algebras with exactly one m-ary operation u. Suppose that $3 \leq m$, $1 \leq h \leq k$ and $h + k \leq m$. Suppose further that $0_z \in \mathbf{A}_z$, for $z = 1, 2, 4$ and*

(1) 0_z is h-absorbing for u in $\mathbf{A}_z$, for $z = 1, 2$,
(2) u is a k-majority term in $\mathbf{A}_3$, and
(3) 0_4 is 2-absorbing for u in $\mathbf{A}_4$.

Suppose that $a, d \in A_3$, $\mathbf{F}$ is a subalgebra of $\mathbf{A}_3 \times \mathbf{A}_4$ and let $B = B(a, d)$ be the subset of $A_1 \times A_2 \times F$ consisting of the elements which have one of the following types

Type I^σ	*Type II^σ*	*Type III^σ*	*Type IV^σ*
$(-, 0, a, -)$	$(0, 0, -, -)$,	$(0, -, d, -)$	$(-, -, -, 0)$,

where dashed places can be filled with arbitrary elements from the corresponding algebras, and under the provision that each 4-uple actually belongs to $A_1 \times A_2 \times F$, namely, that the couple consisting of the last two coordinates belongs to F. Recall that we are omitting the subscripts relative to the 0's.

Then $B = B(a, d)$ is the base set for a subalgebra $\mathbf{B} = \mathbf{B}(a, d)$ of $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{F}$, hence also a subalgebra of $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3 \times \mathbf{A}_4$.

Proof. First notice that B is nonempty, since there exists at least an element of type II^σ. Suppose that $b_1, \dots, b_m \in B$. We have to show that $b = u(b_1, \dots, b_m) \in B$. Since $\mathbf{A}_1$ and $\mathbf{A}_2$ are algebras and $\mathbf{F}$ is a subalgebra of $\mathbf{A}_3 \times \mathbf{A}_4$, if each b_i belongs to $A_1 \times A_2 \times F$, then $b \in A_1 \times A_2 \times F$. Hence it remains to show that b has one of types I^σ - IV^σ.

If at least two b_i's have type IV^σ, then b has type IV^σ, by (3), hence we can suppose that at most one b_i has type IV^σ.

If at least h-many b_i's have type I^σ or II^σ (hence have 0 in the second position) and at least h-many b_i's have type III^σ or II^σ (hence have 0 in the first position), then b has type II^σ, by (1), and we are done in this case, as well.

Otherwise, there are, say, at most $h-1$-many b_i's having type either I^σ or II^σ. Since we have assumed that at most one b_i has type IV^σ, then there are at least $m - (h - 1) - 1 = m - h \geq k$ many b_i's of type III^σ. Then b has type III^σ, by (1), $h \leq k$ and (2). Symmetrically, if there are at most $h - 1$-many

b_i's of type III$^\sigma$ or II$^\sigma$, then there are at least k-many b_i's of type I$^\sigma$, thus b has type I$^\sigma$. $\square$

3. Mitschke's Theorem is sharp

For fixed $m \geq 3$, we now consider lattice terms of the form

$$u_{2,m} = \prod_{i<j<m} (x_i + x_j), \qquad u_{3,m} = \prod_{i<j<k<m} (x_i + x_j + x_k)$$

and so on. We shall combine various lattice reducts defined using the above terms in order to obtain our counterexamples. As shown by the following remarks, it is not enough to consider just one of the above term-reducts.

Example 3.1. For $m \geq 3$, consider the term-reduct $\mathcal{V}_m^d$ of the variety of distributive lattices with the only m-ary operation corresponding to the term $u_{2,m}$ defined above. Trivially the operation in $\mathcal{V}_m$ is an m-ary near-unanimity term, since $u_{2,m}$ is an m-ary near-unanimity term in lattices. Moreover, $\mathcal{V}_m^d$ is locally finite, being a term-reduct of a locally finite variety.

It is easy to see that if $m \geq 4$, then $\mathcal{V}_m^d$ has not an $m{-}1$-ary near-unanimity term, thus, in general, the existence of an m-ary near-unanimity term does not imply an $m{-}1$-ary near-unanimity term, even for locally finite varieties. See [19, Lemma 3.4] for a slightly more involved example (not locally finite).

To check that $\mathcal{V}_m^d$ has not an $m{-}1$-ary near-unanimity term, let $\mathbf{A}$ be the $u_{2,m}$ term-reduct of $\mathbf{C}_2 \times \mathbf{C}_2 \times \cdots \times \mathbf{C}_2$ with $m-1$ factors, where $\mathbf{C}_2$ is the two-elements lattice with universe $\{0, 1\}$. Then $B = A \setminus (1, 1, \dots, 1)$ is the universe for a subalgebra of $\mathbf{A}$, since, for any m-uple of elements of B, we have at least two elements with a 0 at the same component, hence we still get 0 at that component when applying $u_{2,m}$. On the other hand, $\mathcal{V}_m^d$ has not an $m{-}1$-ary near-unanimity term, since, were v such a term, then in $\mathbf{A}$

$$\begin{aligned}
&v((0, 1, 1, \dots, 1), (1, 0, 1, \dots, 1), \dots, (1, 1, 1, \dots, 0)) = \\
&(v(0, 1, 1, \dots, 1), v(1, 0, 1, \dots, 1), \dots, v(1, 1, 1, \dots, 0)) = (1, 1, 1, \dots, 1),
\end{aligned}$$

contradicting the above-proved fact that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, since $(1, 1, 1, \dots, 1) \notin B$.

If $\mathcal{V}_m$ is the corresponding term-reduct of the variety of all lattices, then $\mathcal{V}_m$, too, has an m-ary near-unanimity term but not an $m{-}1$-ary near-unanimity term. In this case, $\mathcal{V}_m$ is not locally finite.

Remark 3.2. In general, we cannot use the above example in order to show that Theorem 1.1 is the best possible result. Indeed, Baker [1] showed that any congruence distributive term reduct of lattices is 4-distributive. While $\mathcal{V}_4$ is actually not 3-distributive and not 4-modular [14], thus $\mathcal{V}_4$ indeed shows that Theorem 1.1 gives the best possible result for $m = 4$, Baker's Theorem prevents $\mathcal{V}_m$ to be a suitable counterexample for larger m.

Henceforth a more involved approach is necessary.

Notice that Baker's result can be generalized to the effect that any congruence distributive term reduct of Boolean algebras is 4-distributive. This statement is immediate from [14, Theorem 6.4(3)]. Hence, for our present purposes, considering Boolean algebras in place of lattices provides no special advantage. In this respect, however, see Proposition 5.3.

Definition 3.3. Suppose that $m \geq 3$ and $1 \leq j \leq m$.

Let $u_{j,m}$ be the following m-ary lattice term

$$u_{j,m}(x_1, \dots, x_m) = \prod_{|J|=j} \sum_{i \in J} x_i \tag{3.1}$$

where J varies on subsets of $\{1, \dots, m\}$.

Of course, strictly speaking, $u_{j,m}$, as a term, is uniquely defined only modulo some fixed arrangement of summands and factors. However, we shall be only interested on $u_{j,m}$ as an operation, hence the actual syntactical definition of the term $u_{j,m}$ shall not be relevant in what follows.

Observation 3.4. Notice that, in every lattice with minimum, the minimum 0 is j-absorbing for $u_{j,m}$. Moreover, in every lattice, $u_{j,m}$ is a p-majority term for $p = \max\{j, m-j+1\}$. In particular, if $j \leq \frac{m+1}{2}$, then $u_{j,m}$ is an $m-j+1$-majority term.

Definition 3.5. Suppose that $m \geq 3$ and $2 \leq j < m$. If $\mathbf{L}$ is a lattice, let $\mathbf{L}^{\mathrm{nu},j,m}$ be the term-reduct of $\mathbf{L}$ with $u_{j,m}$ as the only operation (henceforth always named as u). Let $\mathbf{N}^{j,m} = \mathbf{C}_2^{\mathrm{nu},j,m}$, where $\mathbf{C}_2$ is the two-elements lattice with base set $\{0, 1\}$.

Let $\ell = \frac{m+1}{2}$ if m is odd, and $\ell = \frac{m}{2}$ if m is even. Let $\mathcal{N}_m$ be the variety generated by the algebras

$$\mathbf{N}^{2,m}, \quad \mathbf{N}^{3,m}, \quad \dots, \quad \mathbf{N}^{\ell,m}.$$

The definition is well-posed since the second superscript determines the type of the algebra, in the present case, the arity of the only operation.

Conventionally, we let $\varepsilon \circ \delta \circ \overset{1}{\dots} = \varepsilon$ and $\varepsilon \circ \delta \circ \overset{0}{\dots} = 0$, where 0 is the minimal congruence in the algebra under consideration. If R is a binary relation, R^k denotes $R \circ R \circ \overset{k}{\dots} \circ R$.

Theorem 3.6. *Let $m \geq 3$. The variety $\mathcal{N}_m$ is locally finite and has an m-ary symmetrical near-unanimity term. Moreover*

(1) $\mathcal{N}_m$ is not $2m-4$-alvin, in particular, not $2m-5$-distributive.

(2) More generally, the following congruence identity fails in $\mathcal{N}_m$

$$\alpha(\beta \circ \gamma) \subseteq (\alpha(\gamma \circ \beta))^{m-2}. \tag{3.2}$$

(3) $\mathcal{N}_m$ is not $2m-3$-reversed-modular, in particular, not $2m-4$-modular.

(4) Still more generally, the following congruence identity fails in $\mathcal{N}_m$

$$\alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{\dots} \circ \alpha\beta^\bullet) \circ \gamma^\bullet) \subseteq (\alpha(\gamma \circ \beta \circ \overset{q}{\dots} \circ \beta^\bullet))^{m-2}, \tag{3.3}$$

for every $q \geq 2$, *where* $\beta^\bullet = \beta$, $\gamma^\bullet = \gamma$ *if* q *is even and* $\beta^\bullet = \gamma$, $\gamma^\bullet = \beta$ *if* q *is odd.*

Proof. The variety $\mathcal{N}_m$ has an m-ary near-unanimity term, actually, an m-ary near-unanimity operation, since in each algebra $\mathbf{N}^{j,m}$, for $2 \leq j \leq \ell$, the only operation is a near-unanimity operation. Indeed, by Observation 3.4, the operation of $\mathbf{N}^{j,m}$ is a p-majority term for $p = \max\{j, m-j+1\}$. Since $2 \leq j \leq \ell \leq \frac{m+1}{2}$ and $m \geq 3$, we have $p \leq m-1$, for every j in the interval under consideration. Now notice that if $p \leq p'$, then a p-majority term is a p'-majority term. Hence in each algebra $\mathbf{N}^{j,m}$ the operation is an $m-1$-majority term, that is, a near-unanimity term. The operation is symmetrical, since it is symmetrical on each generating algebra.

Since the variety of distributive lattices is locally finite, each algebra $\mathbf{N}^{j,m}$ generates a locally finite variety, hence $\mathcal{N}_m$ is locally finite, being the join of a finite number of locally finite varieties.

We now show that (1) - (3) all follow from (4). Of course, the reader interested only in (1) - (3) might work out the details of the following arguments in the corresponding simplified setting. Cf. also Section 4 below.

To show that (2) follows from (4) observe that (3.2) is the special case $q = 2$ of (3.3). Moreover, (1) is immediate from (2), since $\alpha\beta \circ \alpha\gamma \subseteq \alpha(\beta \circ \gamma)$. Recall the conditions given by (1.1). To show that (4) implies (3), we first establish a condition of independent interest and which, for q odd, is equivalent to (3.3) in every algebra.

Lemma 3.7. *If* $m, q \geq 3$ *and* q *is odd then identity* (3.3) *is equivalent to*

$$\alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{\ldots} \circ \alpha\gamma) \circ \beta) \subseteq \alpha\gamma \circ \big(\alpha(\beta \circ \alpha\gamma \circ \beta \circ \overset{q-2}{\ldots} \circ \alpha\gamma \circ \beta) \circ \alpha\gamma\big)^{m-2} \tag{3.4}$$

in every algebra.

Indeed, by taking $\alpha\gamma$ in place of γ in (3.3) and since q is odd, we get

$$\alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{\ldots} \circ \alpha\gamma) \circ \beta) \subseteq \big(\alpha(\alpha\gamma \circ \beta \circ \alpha\gamma \circ \overset{q}{\ldots} \circ \beta \circ \alpha\gamma)\big)^{m-2} =$$
$$\alpha\gamma \circ \alpha(\beta \circ \alpha\gamma \circ \overset{q-2}{\ldots} \circ \beta) \circ \alpha\gamma \circ \alpha\gamma \circ \alpha(\beta \circ \alpha\gamma \circ \overset{q-2}{\ldots} \circ \beta) \circ \alpha\gamma \ldots$$
$$\ldots \alpha\gamma \circ \alpha(\beta \circ \alpha\gamma \circ \overset{q-2}{\ldots} \circ \beta) \circ \alpha\gamma \circ \alpha\gamma \circ \alpha(\beta \circ \alpha\gamma \circ \overset{q-2}{\ldots} \circ \beta) \circ \alpha\gamma =$$
$$\alpha\gamma \circ \big(\alpha(\beta \circ \alpha\gamma \circ \overset{q-2}{\ldots} \circ \beta) \circ \alpha\gamma\big)^{m-2}$$

since $\alpha(\alpha\gamma \circ \beta \circ \alpha\gamma \circ \overset{q}{\ldots} \circ \beta \circ \alpha\gamma) = \alpha\gamma \circ \alpha(\beta \circ \alpha\gamma \circ \overset{q-2}{\ldots} \circ \beta) \circ \alpha\gamma$ and $\alpha\gamma \circ \alpha\gamma = \alpha\gamma$, both α and $\alpha\gamma$ being equivalence relations. Hence (3.3) implies (3.4).

On the other hand, for all congruences α, β and γ,

$$\alpha\gamma \circ \alpha(\beta \circ \alpha\gamma \circ \beta \circ \overset{q-2}{\ldots} \circ \alpha\gamma \circ \beta) \circ \alpha\gamma \subseteq \alpha(\gamma \circ \beta \circ \gamma \circ \beta \circ \overset{q}{\ldots} \circ \gamma \circ \beta \circ \gamma),$$

thus (3.4) implies (3.3), for q odd. We have proved Lemma 3.7.

If $q = 3$, then $q - 2 = 1$, hence in this case (3.4) becomes exactly the condition (1.1) for $2m-3$-reversed modularity, thus, by Lemma 3.7, the special case $q = 3$ of clause (4) in Theorem 3.6 implies clause (3).

Since we have showed that clause (4) implies all the other clauses in 3.6, it remains to prove (4). The proof shall involve further definitions, notation, claims and shall go through a finite induction divided in three steps, using Lemma 2.2.

Fix $m \geq 3$, $q \geq 2$ and let ℓ be as in Definition 3.5. For every j with $2 \leq j \leq \ell$, let $\mathcal{N}_m^j$ be the variety generated by the algebras

$$\mathbf{N}^{j,m}, \quad \mathbf{N}^{j+1,m}, \quad \dots, \quad \mathbf{N}^{\ell,m}.$$

In particular, $\mathcal{N}_m^2$ is $\mathcal{N}_m$.

Clause (4) of the theorem is immediate from the special case $j = 2$ of the following claim, since $\mathbf{N}^{2,m}$ belongs to $\mathcal{N}_m^2 = \mathcal{N}_m$, and since each $\mathcal{N}_m^j$ is a subvariety of $\mathcal{N}_m$.

Claim. *For every j such that $2 \leq j \leq \ell$, there are an algebra $\mathbf{A}_3^j \in \mathcal{N}_m^j$ and a subalgebra $\mathbf{F}^j$ of $\mathbf{A}_3^j \times \mathbf{N}^{2,m}$ such that the congruence identity*

$$\alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{\dots} \circ \alpha\beta^\bullet) \circ \gamma^\bullet) \subseteq (\alpha(\gamma \circ \beta \circ \overset{q}{\dots} \circ \beta^\bullet))^{m-2j+2} \tag{3.5}$$

fails in $\mathbf{F}^j$.

In order to prove the claim we need to establish some notation.

Notation 3.8. Let $\mathbf{C}_{q+1}$ be the chain with $q+1$ elements $\{0, 1, \dots, q\}$ and the standard lattice operations. Let $\mathbf{N}_{q+1}^{j,m}$ denote $\mathbf{C}_{q+1}^{\mathrm{nu},j,m}$, that is, recalling Definition 3.5, $\mathbf{N}_{q+1}^{j,m}$ is the term-reduct of $\mathbf{C}_{q+1}$ with the only operation given by the term $u_{j,m}$ from Definition 3.3. In particular, $\mathbf{N}^{j,m}$ is $\mathbf{N}_2^{j,m}$.

For every $q \geq 2$, let β_{q+1}^* be the congruence on $\mathbf{C}_{q+1}$ determined by the partition $\{\{q, q-1\}, \{q-2, q-3\}, \dots\}$, where $\{0\}$ is a block of β_{q+1}^* if q is even. Let γ_{q+1}^* be the congruence on $\mathbf{C}_{q+1}$ determined by the partition $\{\{q\}, \{q-1, q-2\}, \{q-3, q-4\}, \dots\}$, where $\{0\}$ is a block of γ_{q+1}^* if q is odd. Notice that β_{q+1}^* and γ_{q+1}^* are congruences on every term-reduct of $\mathbf{C}_{q+1}$.

If $\mathbf{A}$ is an algebra, we let $0_\mathbf{A}$ denote the smallest congruence on $\mathbf{A}$. Similarly, $1_\mathbf{A}$ denotes the largest congruence on $\mathbf{A}$. When there is no risk of ambiguity we shall omit subscripts.

The claim is proved in three steps by induction on decreasing ℓ. During the inductive proof of the claim we shall need some further properties of the constructions witnessing the claim itself. Recall that $\mathbf{N}^{2,m}$ is a reduct of the two-elements lattice with base set $\{0, 1\}$. We shall need the additional properties stated in the following subclaim.

Subclaim. *(*) For every j, the failure of identity* (3.5) *in $\mathbf{F}^j$ can be witnessed by elements of the form $(a^j, 1)$ and $(d^j, 1)$. By the above statement we mean that we can choose a^j and d^j in $\mathbf{A}_3^j$ and congruences α, β and γ of $\mathbf{F}^j$ in such a way that $(a^j, 1), (d^j, 1) \in F^j$ and the pair $((a^j, 1), (d^j, 1))$ belongs to the left-hand side of* (3.5)*, but not to the right-hand side of* (3.5)*.*

(**) We shall also require that the assumption in (*) above that the pair $((a^j, 1), (d^j, 1))$ belongs to $\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{\dots} \circ \alpha\beta^\bullet) \circ \gamma^\bullet$ can be witnessed by

elements of the form $(c^j_i, 0)$*, namely, that there are elements* $c^j_1, \dots, c^j_{q-1}$ *in* A^j_3 *such that* $(a^j, 1)\ \beta\ (c^j_1, 0)\ \alpha\gamma\ (c^j_2, 0)\ \alpha\beta\ (c^j_3, 0)\ \alpha\gamma\ \dots\ \alpha\beta^\bullet\ (c^j_{q-1}, 0)\ \gamma^\bullet\ (d^j, 1)$ *and* $(c^j_1, 0), (c^j_2, 0), \dots \in F^j$.

(***) Finally, we shall prove that we can make (3.5) *fail by taking* α *to be the congruence induced on* $\mathbf{F}^j$ *by the congruence* 1×0 *on* $\mathbf{A}^j_3 \times \mathbf{N}^{2,m}$*. Actually, we shall only need that the second component is* 0*, but the proof shall give the additional result on the first component.*

We now proceed with the proof of the claim, at the same time checking that we can handle the proof in such a way that (*) - (***) in the subclaim are verified.

First step. Consider the case when m is odd and $j = \ell$, thus $m - 2j + 2 = 1$. In this case the claim is almost obvious since if the exponent on the right is 1, then identity (3.5) implies congruence q-permutability (just take $\alpha = 1$, the largest congruence). Lattices are not q-permutable, hence, a fortiori, the term-reduct $\mathcal{N}^\ell_m$ is not q-permutable. It is then enough to take some witness $\mathbf{A}^\ell_3 \in \mathcal{N}^\ell_m$ of the failure of q-permutability and take $\mathbf{F}^\ell = \mathbf{A}^\ell_3 \times \mathbf{N}^{2,m}$.

In detail, recall Notation 3.8, take $\mathbf{A}^\ell_3 = \mathbf{N}^{\ell,m}_{q+1}$, $\mathbf{F}^\ell = \mathbf{A}^\ell_3 \times \mathbf{N}^{2,m}$ and consider the elements $(q, 1), (q-1, 0), (q-2, 0), \dots, (1, 0), (0, 1)$ and the congruences $\beta = \beta^*_{q+1} \times 1$, $\gamma = \gamma^*_{q+1} \times 1$ and $\alpha = 1 \times 0$ on $\mathbf{F}^\ell$ in order to get the failure of (3.5) and at the same time to have (*) - (***) satisfied. Indeed, the pair $((q, 1), (0, 1))$ does belong to $\alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{\dots} \circ \alpha\beta^\bullet) \circ \gamma^\bullet)$, as witnessed by the other elements in the above sequence. On the other hand, suppose by contradiction that $((q, 1), (0, 1))$ belongs to $\alpha(\gamma \circ \beta \circ \overset{q}{\dots} \circ \beta^\bullet)$. Then $(q, 1)\ \gamma\ (e_1, f_1)\ \beta\ (e_2, f_2)\ \gamma \dots \gamma^\bullet\ (e_{q-1}, f_{q-1})\ \beta^\bullet\ (0, 1)$, for certain elements $(e_1, f_1), (e_2, f_2), \dots \in A^\ell_3 \times N^{2,m}$. By γ-equivalence, $e_1 = q$; then, by β-equivalence, $e_2 \geq q - 1$ and, again by γ-equivalence, $e_3 \geq q - 2$. Going on, $e_{q-1} \geq 2$, hence (e_{q-1}, f_{q-1}) is not $\beta^\bullet$-equivalent to $(0, 1)$.

Notice that $\mathbf{N}^{\ell,m}_{q+1}$ belongs to the variety generated by $\mathbf{N}^{\ell,m} = \mathbf{N}^{\ell,m}_2$, since $\mathbf{C}_{q+1}$ belongs to the variety generated by $\mathbf{C}_2$. In particular, $\mathbf{A}^\ell_3 = \mathbf{N}^{\ell,m}_{q+1}$ belongs to $\mathcal{N}^\ell_m$.

Second step. Next, we consider the case $j = \ell$ and m even in the claim. In this case $m - 2j + 2 = 2$. Apply Lemma 2.2 taking $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{N}^{\ell,m}_{q+1}$ and $\mathbf{A}_3$ a one-element algebra with an m-ary operation, say, $A_3 = \{a\}$, with $a = d$. Finally, let $\mathbf{A}_4 = \mathbf{N}^{2,m} = \mathbf{N}^{2,m}_2$ and let $\mathbf{F}$ be the whole of $\mathbf{A}_3 \times \mathbf{A}_4$.

Take $h = k = \ell$ in Lemma 2.2. By Observation 3.4, the element 0 is ℓ-absorbing in $\mathbf{A}_1$ and in $\mathbf{A}_2$ and 0_4 is 2-absorbing in $\mathbf{A}_4$. The operation of $\mathbf{A}_3$ is trivially an ℓ-majority term. By Lemma 2.2 we get a subalgebra $\mathbf{B}$ of $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3 \times \mathbf{A}_4$.

The proof that (3.5) fails in $\mathbf{B}$ for $j = \ell$ presents no significant difference with respect to [14, 15]. We recall the details. Consider the following elements of B.

$$\begin{aligned} c_0 &= (q,0,a,1), & c_q &= (0,q,a,1), \qquad \text{and} \\ c_i &= (q-i,i,a,0), & &\text{for } i=1,\dots,q-1. \end{aligned} \tag{3.6}$$

The above elements are indeed in B, since c_0 has type I^σ, c_q has type III^σ (since $a=d$) and the remaining c_i's have type IV^σ. Recall that in this special case we have taken $\mathbf{F}$ equal to $\mathbf{A}_3 \times \mathbf{A}_4$, hence the above elements automatically belong to $A_1 \times A_2 \times F$.

If q is even, let β and γ be, respectively, the congruences on $\mathbf{B}$ induced by $\beta^*_{q+1} \times \gamma^*_{q+1} \times 1 \times 1$ and $\gamma^*_{q+1} \times \beta^*_{q+1} \times 1 \times 1$. If q is odd, let β and γ be, respectively, the congruences on $\mathbf{B}$ induced by $\beta^*_{q+1} \times \beta^*_{q+1} \times 1 \times 1$ and $\gamma^*_{q+1} \times \gamma^*_{q+1} \times 1 \times 1$. Both in case q even and q odd, let α be the congruence induced by $1 \times 1 \times 1 \times 0$.

We have $c_0 \ \alpha \ c_q$ and $c_0 \ \beta \ c_1 \ \alpha\gamma \ c_2 \ \alpha\beta \ c_3 \dots$, hence $(c_0, c_q) \in \alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{\ldots} \circ \alpha\beta^\bullet) \circ \gamma^\bullet)$. We shall show that $(c_0, c_q) \notin \alpha(\gamma \circ \beta \circ \overset{q}{\ldots} \circ \beta^\bullet) \circ \alpha(\gamma \circ \beta \circ \overset{q}{\ldots} \circ \beta^\bullet)$ in $\mathbf{B}$. Towards a contradiction, suppose the contrary. Then there is some element $f \in B$ such that $(c_0, f) \in \alpha(\gamma \circ \beta \circ \overset{q}{\ldots} \circ \beta^\bullet)$ and $(f, c_q) \in \alpha(\gamma \circ \beta \circ \overset{q}{\ldots} \circ \beta^\bullet)$. Thus $c_0 \ \alpha \ f$ and there are elements $f_0 = c_0, f_1, \dots, f_q = f$ such that $f_0 \ \gamma \ f_1 \ \beta \ f_2 \dots$ Recall that $f_0 = c_0 = (q,0,a,1)$. By γ-equivalence of f_0 and f_1, the first component of f_1 is q. By β-equivalence of f_1 and f_2, the first component of f_2 is $\geq q-1$. Going on, the first component of $f_q = f$ is ≥ 1, in particular, it is not 0. Thus f has neither type II^σ nor III^σ. Moreover, f has not type IV^σ, either, since its fourth component is 1, by α-equivalence of f_0 and f. Since $f \in B$, then f has necessarily type I^σ, thus its second component is 0. However, by performing a symmetric argument, using the assumption $(f, c_q) \in \alpha(\gamma \circ \beta \circ \overset{q}{\ldots} \circ \beta^\bullet)$, we get that the second component of f is ≥ 1, a contradiction. We have showed that (3.5) fails in $\mathbf{B}$ for $j = \ell$.

We are almost done. It is now enough to declare who $\mathbf{A}_3^\ell$ and $\mathbf{F}^\ell$ actually are. Take $\mathbf{A}_3^\ell$ to be $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3$. As in the first step, $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{N}^{\ell,m}_{q+1}$ belong to the variety $\mathcal{N}^\ell_m$, hence $\mathbf{A}_3^\ell$ belongs to $\mathcal{N}^\ell_m$, too. Finally, let $\mathbf{F}^\ell = \mathbf{B}$, thus (3.5) fails in $\mathbf{F}^\ell$. The additional conditions (*) - (***) are verified by construction. Indeed, here we take $a^j = (q,0,a)$ and $d^j = (0,q,a)$, thus $c_0 = (a^j, 1)$ and $c_q = (d^j, 1)$, modulo a standard identification of nested components. Similarly for the other c_i's.

Let us observe that in the present step we could have worked with just three coordinates. However, it is easier to add a dummy third coordinate, rather than state and prove also a three-coordinate (and essentially less general) version of Lemma 2.2. The full four-coordinate version of Lemma 2.2 will be necessary in the proof of the next step.

Third step. Finally, we suppose that we have proved the claim and the subclaim for some j with $2 < j \leq \ell$ and we shall prove the claim for $j-1$. Since we have proved the claim when $j = \ell$, an easy finite induction establishes the claim and the subclaim for all j's, hence the theorem.

The proof of the third step is not really different from the proof of the second step. However, here we shall use a nontrivial $\mathbf{A}_3$ which is given by the inductive hypothesis. Taking into account a nontrivial $\mathbf{A}_3$ involves a bit of further details and, as we mentioned, the full power of Lemma 2.2 will be necessary.

So let $\mathbf{A}_3^j$ and $\mathbf{F}^j \subseteq \mathbf{A}_3^j \times \mathbf{N}^{2,m}$ be given by the case j of the claim and let the failure of (3.5) in $\mathbf{F}^j$ be witnessed by congruences $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$. We can inductively assume that properties (*) - (***) hold, so, by (*), let $((a^j,1),(d^j,1))$ belong to the left-hand side of (3.5), but not to the right-hand side, for certain $a^j, d^j \in \mathbf{A}_3^j$ and where α, β and γ in (3.5) are replaced by $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$.

Apply Lemma 2.2 taking $h = j-1$, $k = m{-}j{+}1$, $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{N}_{q+1}^{j-1,m}$, $\mathbf{A}_3 = \mathbf{A}_3^j$, $\mathbf{A}_4 = \mathbf{N}^{2,m}$, $a = a^j$, $b = b^j$ and $\mathbf{F} = \mathbf{F}^j$. Again, the algebra $\mathbf{N}_{q+1}^{j-1,m} = \mathbf{C}_{q+1}^{\mathrm{nu},j-1,m}$ belongs to the variety generated by $\mathbf{N}^{j-1,m} = \mathbf{C}_2^{\mathrm{nu},j-1,m}$, since $\mathbf{C}_{q+1}$ belongs to the variety generated by $\mathbf{C}_2$. In particular, $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{N}_{q+1}^{j-1,m}$ belong to $\mathcal{N}_m^{j-1}$.

By Observation 3.4, 0 is $j-1$-absorbing in $\mathbf{A}_1$ and in $\mathbf{A}_2$. Moreover, the operation of $\mathbf{A}_3$ is an $m{-}j{+}1$-majority term, since $\mathbf{A}_3 = \mathbf{A}_3^j \in \mathcal{N}_m^j$ and each operation on the generators of $\mathcal{N}_m^j$ is an $m{-}j{+}1$-majority term, again by Observation 3.4.

Hence we can apply Lemma 2.2 with $h = j-1$ and $k = m{-}j{+}1$ (notice that $h \le k$, since $j \le \ell$), getting a subalgebra $\mathbf{B}$ of $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{F}$, which is itself a subalgebra of $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3 \times \mathbf{A}_4$.

Recall the definitions of β_{q+1}^* and γ_{q+1}^* from Notation 3.8. If q is even, let β and γ be, respectively, the congruences on $\mathbf{B}$ induced by $\beta_{q+1}^* \times \gamma_{q+1}^* \times \tilde{\beta}$ and $\gamma_{q+1}^* \times \beta_{q+1}^* \times \tilde{\gamma}$. If q is odd, let β and γ be, respectively, the congruences on $\mathbf{B}$ induced by $\beta_{q+1}^* \times \beta_{q+1}^* \times \tilde{\beta}$ and $\gamma_{q+1}^* \times \gamma_{q+1}^* \times \tilde{\gamma}$. In both cases, let α be the congruence induced by $1 \times 1 \times \tilde{\alpha}$. By (**), there are elements $c_1^j, \dots, c_{q-1}^j$ in A_3^j such that $(a^j,1)\ \tilde{\beta}\ (c_1^j,0)\ \tilde{\alpha}\tilde{\gamma}\ (c_2^j,0)\ \tilde{\alpha}\tilde{\beta}\ (c_3^j,0)\dots$ Consider the following elements of B.

$$
\begin{aligned}
c_0 &= (q,0,a^j,1), & c_q &= (0,q,d^j,1), \qquad \text{and}\\
c_i &= (q-i,i,c_i^j,0), & &\text{for } i = 1,\dots,q-1.
\end{aligned}
$$

Notice that c_0 belongs to $A_1 \times A_2 \times F$, since $(a^j,1) \in F$, by (*). Moreover, c_0 has type I^σ, hence c_0 is indeed in B. Recall that we are taking $a = a^j$. Similarly, c_q belongs to $A_1 \times A_2 \times F$ and has type III^σ, hence $c_q \in B$. The remaining c_i's belong to $A_1 \times A_2 \times F$, since $(c_i^j,0) \in F$, by (**). Moreover, each c_i has type IV^σ, hence $c_i \in B$.

One easily checks that $(c_0,c_q) \in \alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{\dots} \circ \alpha\beta^\bullet) \circ \gamma^\bullet)$. We shall show that $(c_0,c_q) \notin (\alpha(\gamma \circ \beta \circ \overset{q}{\dots} \circ \beta^\bullet))^{m-2j+4}$, thus identity (3.5) fails in $\mathbf{B}$ for $j-1$. Suppose the contrary. Then there are elements $f, g \in B$ such that $(c_0,f) \in \alpha(\gamma \circ \beta \circ \overset{q}{\dots} \circ \beta^\bullet)$, $(f,g) \in (\alpha(\gamma \circ \beta \circ \overset{q}{\dots} \circ \beta^\bullet))^{m-2j+2}$ and $(g,c_q) \in \alpha(\gamma \circ \beta \circ \overset{q}{\dots} \circ \beta^\bullet)$. Notice that $m{-}2j{+}2 \ge 1$, since $j \le \ell$. From the first relation we get that $c_0\ \alpha\ f$ and that there are elements $c_0 = f_0, f_1, \dots, f_q = c_q$ in B such that $f_0\ \gamma\ f_1\ \beta\ f_2\ \gamma\ f_3 \dots$ Since $c_0 = f_0 = (q,0,a^j,1)$, then, by γ-equivalence, the first component of f_1 is q. By β-equivalence, the first component of f_2 is $\geq q-1$. Going on, the first component of $f_q = f$ is ≥ 1, thus f has neither type II^σ nor type III^σ. Since $c_0 \ \alpha \ f$, then, by (***), the fourth component of f is 1, hence f has not type IV^σ, either. Since $f \in B$, then f has type I^σ, thus the third component of f is a^j. Symmetrically, the fourth component of g is 1, g has type III^σ, hence the third component of g is d^j. From $(f,g) \in (\alpha(\gamma \circ \beta \circ \overset{q}{\ldots} \circ \beta^\bullet))^{m-2j+2}$, restricting to the third component of $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{F}$, we get $((a,^j, 1), (d^j, 1)) \in (\tilde{\alpha}(\tilde{\gamma} \circ \tilde{\beta} \circ \overset{q}{\ldots} \circ \tilde{\beta}^\bullet))^{m-2j+2}$, contradicting our assumption that the pair $((a^j, 1), (d^j, 1))$ witnesses the failure of (3.5) for $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$.

We have showed that $(c_0, c_q) \in \alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{\ldots} \circ \alpha\beta^\bullet) \circ \gamma^\bullet)$ and $(c_0, c_q) \notin (\alpha(\gamma \circ \beta \circ \overset{q}{\ldots} \circ \beta^\bullet))^{m-2j+4}$. Now it is enough to take $\mathbf{A}_3^{j-1} = \mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{A}_3^j$ and $\mathbf{F}^{j-1} = \mathbf{B} \subseteq \mathbf{A}_3^{j-1} \times \mathbf{N}^{2,m}$ (the inclusion is considered modulo the usual identifications), to get that (3.5) fails in $\mathbf{F}^{j-1}$ for $j-1$. Notice that $\mathbf{A}_3^j$ belongs to $\mathcal{N}_m^j$, by the inductive assumption. As we mentioned, $\mathbf{A}_1$ and $\mathbf{A}_2$ belong to $\mathcal{N}_m^{j-1}$, hence $\mathbf{A}_3^{j-1}$ belongs to $\mathcal{N}_m^{j-1}$, too, since $\mathcal{N}_m^j \subseteq \mathcal{N}_m^{j-1}$. As in the second step, (*) - (***) are verified by construction. $\square$

Theorem 3.6 is optimal. It is immediate from Theorem 1.1 that clauses (1) and (3) in 3.6 are optimal. Clause (4), too, is the best possible result, as shown in the next remark.

Remark 3.9. In [14, Proposition 5.1] we have showed that if $m \geq 3$ and some variety $\mathcal{V}$ has an m-ary near-unanimity term, then, for every $q \geq 2$, $\mathcal{V}$ satisfies

$$\alpha(\beta \circ \gamma \circ \overset{q}{\ldots} \circ \gamma) \subseteq \alpha\beta \circ \alpha\gamma \circ \overset{(m-2)q}{\ldots} \circ \alpha\gamma, \qquad \text{if } q \text{ is even}, \tag{3.7}$$

$$\alpha(\beta \circ \gamma \circ \overset{q}{\ldots} \circ \beta) \subseteq \alpha\beta \circ \alpha\gamma \circ \overset{1+(m-2)(q-1)}{\ldots} \circ \alpha\beta, \qquad \text{if } q \text{ is odd}. \tag{3.8}$$

Clause (4) in Theorem 3.6 shows that the above result is best possible. Indeed, $\mathcal{N}_m$ has an m-ary near-unanimity term. If q is even and, by contradiction, (3.7) can be improved by considering $(m-2)q-1$ factors on the right-hand side, then

$$\begin{aligned}\alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{q-2}{\ldots} \circ \alpha\beta) \circ \gamma) &\subseteq \alpha(\beta \circ \gamma \circ \beta \circ \overset{q}{\ldots} \circ \beta \circ \gamma) \\ \subseteq \alpha\beta \circ \alpha\gamma \circ \overset{(m-2)q-1}{\ldots} \circ \alpha\beta &\subseteq \alpha\gamma \circ \alpha\beta \circ \alpha\gamma \circ \overset{(m-2)q}{\ldots} \circ \alpha\beta \\ &\subseteq (\alpha(\gamma \circ \beta \circ \overset{q}{\ldots} \circ \beta))^{m-2},\end{aligned}$$

contradicting clause (4) in Theorem 3.6.

The case q odd is similar, using Lemma 3.7 and identity (3.4). In fact, the arguments show that if $m \geq 3$, then $\mathcal{N}_m$ is a variety with a symmetric m-ary near-unanimity term for which the following identities fail.

$$\alpha(\beta \circ \gamma \circ \overset{q}{\ldots} \circ \gamma) \subseteq \alpha\gamma \circ \alpha\beta \circ \overset{(m-2)q}{\ldots} \circ \alpha\beta, \qquad \text{if } q \text{ is even},$$

$$\alpha(\beta \circ \gamma \circ \overset{q}{\ldots} \circ \beta) \subseteq \alpha\gamma \circ \alpha\beta \circ \overset{1+(m-2)(q-1)}{\ldots} \circ \alpha\gamma, \qquad \text{if } q \text{ is odd}.$$

Notice that here $\alpha\beta$ and $\alpha\gamma$ are exchanged on the right-hand side, in comparison with (3.7).

4. An explicit example

Following the proof of Theorem 3.6 we shall present explicit examples of algebras in $\mathcal{N}_m$ for which identity (3.2) fails. Then we hint to the details for a full counterexample to (3.3). In particular, such counterexamples show in a more direct way that Theorem 1.1 cannot be improved.

Recall that $\mathbf{C}_{q+1}$ is the chain with $q+1$ elements $\{0,1,\dots,q\}$, considered as a lattice. Moreover, $\mathbf{N}^{j,m}_{q+1}$ is the term-reduct of $\mathbf{C}_{q+1}$ endowed with the m-ary operation induced by the lattice term $u_{j,m}(x_1,\dots,x_m)=\prod_{|J|=j}\sum_{i\in J}x_i$, where J varies on subsets of $\{1,\dots,m\}$. We have set $\mathbf{N}^{j,m}=\mathbf{N}^{j,m}_2$. Compare Definitions 3.3, 3.5 and Notation 3.8.

Fix some $q\geq 2$, $m\geq 3$ and let $\ell=\frac{m}{2}$ if m is even and $\ell=\frac{m+1}{2}$ if m is odd. Consider the following product $\mathbf{P}=\mathbf{P}(m,q)$ in the cases, respectively, m even and m odd.

$$\begin{gathered}(\mathbf{N}^{2,m}_{q+1}\times\mathbf{N}^{2,m}_{q+1})\times(\mathbf{N}^{3,m}_{q+1}\times\mathbf{N}^{3,m}_{q+1})\times\dots\times(\mathbf{N}^{\ell,m}_{q+1}\times\mathbf{N}^{\ell,m}_{q+1})\times\mathbf{N}^{2,m},\\ (\mathbf{N}^{2,m}_{q+1}\times\mathbf{N}^{2,m}_{q+1})\times(\mathbf{N}^{3,m}_{q+1}\times\mathbf{N}^{3,m}_{q+1})\times\dots\times(\mathbf{N}^{\ell-1,m}_{q+1}\times\mathbf{N}^{\ell-1,m}_{q+1})\times\mathbf{N}^{\ell,m}_{q+1}\times\mathbf{N}^{2,m},\end{gathered}\tag{P}$$

where the grouping of the factors is only for notational convenience. In any case, $\mathbf{P}(m,q)$ has $m-1$ factors; for example, $\mathbf{P}(3,q)=\mathbf{N}^{2,3}_{q+1}\times\mathbf{N}^{2,3}$, $\mathbf{P}(4,q)=(\mathbf{N}^{2,4}_{q+1}\times\mathbf{N}^{2,4}_{q+1})\times\mathbf{N}^{2,4}$ and $\mathbf{P}(5,q)=(\mathbf{N}^{2,5}_{q+1}\times\mathbf{N}^{2,5}_{q+1})\times\mathbf{N}^{3,5}_{q+1}\times\mathbf{N}^{2,5}$. By Observation 3.4, the operation of $\mathbf{P}$ is an m-ary near-unanimity term.

A member p of P is *good* if either (a) its last component is 0, or (b) its last component is 1 and, disregarding the last component,

(b1) p begins with a (possibly empty, possibly covering all pairs) sequence of null pairs $(0,0)$,

(b2) the first (if any) pair of p which is not null has either the form $(\text{-},0)$, or the form $(0,\text{-})$, and

(b3) all the subsequent pairs, if any, have, correspondingly, the form $(q,0)$ or $(0,q)$.

If m is odd, we follow the same rules, considering the penultimate component as a "half pair" and applying the above rules only to the first component of the pair. Typical good elements are given by the sequences

$$\begin{array}{ccccccccc}
(\text{-},\text{-}) & (\text{-},\text{-}) & \dots & (\text{-},\text{-}) & (\text{-},\text{-}) & (\text{-},\text{-}) & \dots & (\text{-},\text{-}) & \text{-} & 0\\
(\text{-},0) & (q,0) & \dots & (q,0) & (q,0) & (q,0) & \dots & (q,0) & q & 1\\
(0,0) & (0,0) & \dots & (0,0) & (\text{-},0) & (q,0) & \dots & (q,0) & q & 1\\
(0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (\text{-},0) & q & 1\\
(0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (0,0) & \text{-} & 1\\
(0,\text{-}) & (0,q) & \dots & (0,q) & (0,q) & (0,q) & \dots & (0,q) & 0 & 1\\
(0,0) & (0,0) & \dots & (0,0) & (0,\text{-}) & (0,q) & \dots & (0,q) & 0 & 1\\
(0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (0,\text{-}) & 0 & 1
\end{array}$$

in the case m odd, while we get typical elements in the case m even simply discarding the penultimate column in the above table.

The set of good elements of P is the universe for a subalgebra $\mathbf{B} = \mathbf{B}(m, q)$ of $\mathbf{P}$. This can be checked directly using arguments similar to those used in the proof of Lemma 2.2. Roughly, suppose that $b_1, \dots, b_m \in B$ and $b = u(b_1, \dots, b_m)$. If at least two b_i's have 0 as the last component, then this applies to b, as well, hence $b \in B$, by (a). Otherwise, there are enough 0's in the components of the b_i's in order to make 0 at least one element of each pair of b, using Observation 3.4. Then the rules describing the elements of B, together with Observation 3.4 again, show that b has a sufficient number of 0's and q's in the appropriate places. Alternatively, in order to show that B is the universe for a subalgebra of $\mathbf{P}$, work out the proof of Theorem 3.6, going in the backward direction.

Now suppose, say, that q is even and consider the congruences β, γ and α on $\mathbf{B}$ induced, respectively, by the congruences

$$\begin{aligned} \beta^* &= (\beta^*_{q+1} \times \gamma^*_{q+1}) \times (\beta^*_{q+1} \times \gamma^*_{q+1}) \times \dots \times (\beta^*_{q+1} \times \gamma^*_{q+1}) \times \beta^*_{q+1} \times 1, \\ \gamma^* &= (\gamma^*_{q+1} \times \beta^*_{q+1}) \times (\gamma^*_{q+1} \times \beta^*_{q+1}) \times \dots \times (\gamma^*_{q+1} \times \beta^*_{q+1}) \times \gamma^*_{q+1} \times 1, \\ \alpha^* &= (1 \times 1) \times (1 \times 1) \times \dots \times (1 \times 1) \times 1 \times 0, \end{aligned} \tag{4.1}$$

where, as usual by now, in each line the penultimate congruence appears only if m is odd. Recall that β^*_{q+1} and γ^*_{q+1} are the congruences determined, respectively, by the partitions $\{\{q, q-1\}, \{q-2, q-3\}, \dots\}$ and $\{\{q\}, \{q-1, q-2\}, \{q-3, q-4\}, \dots\}$.

For simplicity, let $q = 2$. Indeed, this is the case showing that Theorem 1.1(1) cannot be improved. Consider the following elements of P

$$\begin{array}{ccccccccccc} a & = & (2,0) & (2,0) & \dots & (2,0) & (2,0) & (2,0) & \dots & (2,0) & 2 & 1 \\ c & = & (1,1) & (1,1) & \dots & (1,1) & (1,1) & (1,1) & \dots & (1,1) & 1 & 0 \\ d & = & (0,2) & (0,2) & \dots & (0,2) & (0,2) & (0,2) & \dots & (0,2) & 0 & 1 \end{array}$$

The element c witnesses that $(a, d) \in \beta \circ \gamma$. Moreover, $a \mathrel{\alpha} d$, hence $(a, d) \in \alpha(\beta \circ \gamma)$. On the other hand, the only other element $\alpha\beta$-connected to a is

$$\begin{array}{ccccccccccc} f_1 & = & (1,0) & (2,0) & \dots & (2,0) & (2,0) & (2,0) & \dots & (2,0) & 2 & 1 \end{array}$$

due to the rule (b3) in the formation of P. Due to the definition of γ, the only other element γ-connected to f_1 and with last component 1 is

$$\begin{array}{ccccccccccc} f_2 & = & (0,0) & (2,0) & \dots & (2,0) & (2,0) & (2,0) & \dots & (2,0) & 2 & 1 \end{array}$$

Continuing this way, the only possibility to go from a to d through an $\alpha\beta$-or-$\alpha\gamma$-chain is to consider *all* the elements

$$\begin{array}{rcllllllllll}
a & = & (2,0) & (2,0) & \dots & (2,0) & (2,0) & (2,0) & \dots & (2,0) & 2 & 1 \\
f_1 & = & (1,0) & (2,0) & \dots & (2,0) & (2,0) & (2,0) & \dots & (2,0) & 2 & 1 \\
f_2 & = & (0,0) & (2,0) & \dots & (2,0) & (2,0) & (2,0) & \dots & (2,0) & 2 & 1 \\
f_3 & = & (0,0) & (1,0) & \dots & (2,0) & (2,0) & (2,0) & \dots & (2,0) & 2 & 1 \\
 & & & & & & \dots & & & & & \\
f_{m-4} & = & (0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (1,0) & 2 & 1 \\
f_{m-3} & = & (0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (0,0) & 2 & 1 \\
f_{m-2} & = & (0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (0,0) & 1 & 1 \\
f_{m-1} & = & (0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (0,0) & 0 & 1 \\
f_m & = & (0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (0,1) & 0 & 1 \\
 & & & & & & \dots & & & & & \\
f_{2m-7} & = & (0,0) & (0,1) & \dots & (0,2) & (0,2) & (0,2) & \dots & (0,2) & 0 & 1 \\
f_{2m-6} & = & (0,0) & (0,2) & \dots & (0,2) & (0,2) & (0,2) & \dots & (0,2) & 0 & 1 \\
f_{2m-5} & = & (0,1) & (0,2) & \dots & (0,2) & (0,2) & (0,2) & \dots & (0,2) & 0 & 1 \\
d & = & (0,2) & (0,2) & \dots & (0,2) & (0,2) & (0,2) & \dots & (0,2) & 0 & 1
\end{array}$$

in the case m odd, while in the case m even the penultimate column should be deleted and the above "middle" block is replaced by

$$\begin{array}{rclllllllll}
 & & & & & & \dots & & & & \\
f_{m-3} & = & (0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (1,0) & 1 \\
f_{m-2} & = & (0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (0,0) & 1 \\
f_{m-1} & = & (0,0) & (0,0) & \dots & (0,0) & (0,0) & (0,0) & \dots & (0,1) & 1 \\
 & & & & & & \dots & & & &
\end{array}$$

Since we need to consider all the above elements, we get $(a,d) \notin \alpha\beta \circ \alpha\gamma \circ \stackrel{2m-5}{\dots}$, hence $\alpha(\beta \circ \gamma) \nsubseteq \alpha\beta \circ \alpha\gamma \circ \stackrel{2m-5}{\dots}$, that is, $\mathbf{B}$ does not belong to a $2m{-}5$-distributive variety. Recall that $\alpha\beta \circ \alpha\gamma \circ \stackrel{k}{\dots}$ denotes the relation $\alpha\beta \circ \alpha\gamma \circ \alpha\beta \circ \dots$ with $k-1$ occurrences of $\circ$ and that R^k is $R \circ R \circ \stackrel{k}{\dots}$.

As implicit in the proof of 3.6, we see that a is $\alpha\gamma$-connected only to itself, hence we also get $\alpha(\beta \circ \gamma) \nsubseteq \alpha\gamma \circ \alpha\beta \circ \stackrel{2m-4}{\dots}$, that is, $\mathbf{B}$ does not belong to a $2m{-}4$-alvin variety.

In the above arguments we have considered identities involving $\alpha\gamma \circ \alpha\beta \circ \dots$ on the right only for simplicity. While a is β-connected to further elements of B, since we can consider elements with 0 as the last coordinate, on the other hand, f_1 is the only other element such that $(a, f_1) \in \alpha(\gamma \circ \beta)$. Continuing the same way, the only elements h such that $(f_1, h) \in \alpha(\gamma \circ \beta)$ are a, f_2 and f_3. Of course, it is no use to turn back to a, and the "fastest way to d" uses f_3. Going on, we see that $(a,d) \notin (\alpha(\gamma \circ \beta))^{m-2}$, hence $\alpha(\beta \circ \gamma) \nsubseteq (\alpha(\gamma \circ \beta))^{m-2}$, that is, clause (4) in Theorem 3.6.

Dealing with larger even q presents no significant difference, while if q is odd it is enough to modify the definitions displayed in (4.1): all the pairs in the definitions of β and γ should be, respectively, $(\beta^*_{q+1} \times \beta^*_{q+1})$ and $(\gamma^*_{q+1} \times \gamma^*_{q+1})$. Then it is more convenient to deal with identity (3.4) in Lemma 3.7. For example, the first elements in the shortest chain from a to d in the case $q = 3$ are

$$\begin{array}{lcccccccccccc}
a & = & (3,0) & (3,0) & \dots & (3,0) & (3,0) & (3,0) & \dots & (3,0) & 3 & 1 \\
f_1 & = & (2,0) & (3,0) & \dots & (3,0) & (3,0) & (3,0) & \dots & (3,0) & 3 & 1 \\
f_2 & = & (1,0) & (3,0) & \dots & (3,0) & (3,0) & (3,0) & \dots & (3,0) & 3 & 1 \\
f_3 & = & (0,0) & (2,0) & \dots & (3,0) & (3,0) & (3,0) & \dots & (3,0) & 3 & 1 \\
 & & & & & & \dots & & & & &
\end{array}$$

Notice that, as in the case q even, $a \ \beta \ f_1$, but there is no other element $\alpha\gamma$-connected to a. On the other hand, in the case q odd we are sometimes able to move two components at a time, as is the case for f_2 and f_3 above.

In conclusion, the above arguments show that the following proposition holds. Recall that the definitions of $\mathbf{P} = \mathbf{P}(m, q)$ and $\mathbf{B} = \mathbf{B}(m, q)$ depend on m and q, though sometimes we have not explicitly indicated the dependence in the above arguments.

Proposition 4.1. *For every $m \geq 3$ and $q \geq 2$, the algebra $\mathbf{B}(m, q)$, as constructed above, has an m-ary near-unanimity term. Identity* (3.3) *fails in* $\mathbf{B}(m, q)$.

5. Further remarks

It is well-known that, for every $m \geq 4$, there is a variety with an m-ary near-unanimity term and without an $m-1$-ary near-unanimity term. See, e. g., [19, Lemma 3.4] or Example 3.1 here. The variety $\mathcal{N}_m$ introduced in Definition 3.5 furnishes another counterexample, as we shall show in the next corollary. In addition, the counterexamples presented here have a symmetric m-ary near-unanimity term and are locally finite. Notice that the variety denoted by $\mathcal{N}_m$ in [19] is distinct from the variety denoted by $\mathcal{N}_m$ here. Also, the indices are shifted by 1 in most definitions, with respect to [19].

For every $n \geq 2$ there are known examples of n-distributive not $n-1$-distributive varieties, e. g. [8, 15] and further references there. For n even, the variety $\mathcal{N}_m$ provides another example (with $m = \frac{n+4}{2}$). Corresponding examples appear in [15] regarding n-modularity; again, $\mathcal{N}_m$ provides further counterexamples.

Corollary 5.1. *If $m \geq 4$, then $\mathcal{N}_m$ has a symmetric m-ary near-unanimity term but no $m-1$-ary near-unanimity term (symmetric or not).*

The variety $\mathcal{N}_m$ is $2m-4$-distributive but not $2m-5$-distributive, $2m-3$-modular but not $2m-4$-modular.

Proof. We have proved in Theorem 3.6 that $\mathcal{N}_m$ has an m-ary symmetric near-unanimity term. The arguments in Example 3.1 show that the variety generated by the algebra $\mathbf{N}^{2,m}$ from Definition 3.5 has not an $m-1$-ary near-unanimity term. Since $\mathbf{N}^{2,m}$ is one among the generators of $\mathcal{N}_m$, then $\mathcal{N}_m$ has not an $m-1$-ary near-unanimity term.

The second statement is immediate from Theorems 1.1 and 3.6(1)(3).

From Theorems 1.1 and 3.6(1)(3) we can also obtain another proof that $\mathcal{N}_m$ has not an $m-1$-ary near-unanimity term. If, by contradiction, such a term exists, then $\mathcal{N}_m$ is $2m-6$-distributive. However, $\mathcal{N}_m$ is not even $2m-5$-distributive, absurd. A similar proof is obtained by dealing with the modularity levels. Remark 3.9 could be used to get still another proof. □

Remark 5.2. Recall the definition of $\mathbf{L}^{\mathrm{nu},j,m}$ from Definition 3.5.

For any given $m \geq 3$, let $\ell = \frac{m+1}{2}$ if m is odd, $\ell = \frac{m}{2}$ if m is even and let $\mathcal{N}'_m$ be the variety generated by all the lattice reducts $\mathbf{L}^{\mathrm{nu},j,m}$, with $\mathbf{L}$ an arbitrary lattice, $2 \leq j \leq \ell$.

Theorem 3.6 and Corollary 5.1 hold for $\mathcal{N}'_m$, as well, with the only exception that $\mathcal{N}'_m$ is not locally finite.

In another classical paper Mitschke [17] showed that the variety $\mathcal{I}$ of implication algebras is congruence 3-distributive, 3-permutable, not 2-distributive and not permutable. Then in [18] she proved that for no m the variety $\mathcal{I}$ has an m-ary near-unanimity term. Another proof can be found in [14, Remarks 2.2(a)(b)].

In [13, Section 5] we expanded $\mathcal{I}$ by adding a 4-ary near-unanimity term in such a way that the distributive and permutable levels remain unchanged. Combining the arguments from [17, 18, 13] and from Example 3.1, we get a variety sharing the same levels of $\mathcal{I}$, with an $m+1$-ary near-unanimity term but without an m-ary near-unanimity term. See the next proposition.

This shows that 3-permutability has no effect on the integers m for which an m-ary near-unanimity can exist. Notice that, on the other hand, it is immediate from the characterizations in (1.1) that a congruence distributive (modular) n-permutable variety is n-distributive (n-modular). Notice also that a congruence permutable variety with a near-unanimity term is congruence distributive, hence, by permutability, 2-distributive, and this means the existence of a majority term, namely, a 3-ary near-unanimity term. Thus, in contrast with 3-permutability, the stronger notion of permutability does trivialize the sets of integers m for which an m-ary near-unanimity exists.

The operations of a Boolean algebra shall be denoted by $+$, $\cdot$ and $'$. The variety $\mathcal{I}$ of (dual) *implication algebras* is the variety generated by term-reducts of Boolean algebras in which $i(x,y) = xy'$ is the only basic operation. Let $f(x,y,z)$ be the Boolean term $x(y'+z)$. The variety $\mathcal{I}^-$ is the variety generated by reducts of Boolean algebras having f as the only basic operation. If $m \geq 3$, we let $\mathcal{I}_m$, resp. $\mathcal{I}^-_m$, be the varieties generated by reducts of Boolean algebra with two basic operations corresponding to i and $u_{2,m}$, resp., f and $u_{2,m}$. Recall the definition of $u_{2,m}$ from Definition 3.3.

Proposition 5.3. *If $m \geq 4$, then both $\mathcal{I}_m$ and $\mathcal{I}^-_m$ are 3-distributive, congruence 3-permutable, not congruence permutable, not 2-distributive, have an m-ary near-unanimity term but not an $m-1$-ary near-unanimity term.*

Proof. Since $\mathcal{I}$ and $\mathcal{I}^-$ are 3-distributive and congruence 3-permutable [17], then so are their expansions $\mathcal{I}_m$ and $\mathcal{I}_m^-$. To show that 2-distributivity and permutability fail, consider the reduct $\mathbf{A}$ of $\mathbf{2}^3$, where $\mathbf{2}$ is the two-elements Boolean algebra. Since $m \geq 4$, then $2^3 \setminus (1,1,1)$ is closed under $u_{2,m}$, compare an argument in Example 3.1. Moreover, $2^3 \setminus (1,1,1)$ is closed also under i (or f), thus it is the universe for a subalgebra of $\mathbf{A}$. The original argument in [17] (credited in that form to the referee) now shows that 2-distributivity fails, hence also permutability fails. The argument is recalled also in the proof of [13, Proposition 5.1].

To show that neither $\mathcal{I}_m$ nor $\mathcal{I}_m^-$ have an $m-1$-ary near-unanimity term argue as in Example 3.1, considering the reduct of $\mathbf{2}^{m-1} \setminus (1,1,\dots,1)$. ☐

Remark 5.4. For $m \geq 3$, an m-ary near-unanimity term implies the existence of a sequence $t_1, \dots, t_{m-2}$ of *directed Jónsson terms*, i. e., terms satisfying

$$
\begin{aligned}
x &= t_1(x,x,z), && t_{m-2}(x,z,z) = z, \\
t_i(x,z,z) &= t_{i+1}(x,x,z), && \text{for } 1 \leq i < m-2, \text{ and} \\
x &= t_i(x,y,x), && \text{for } 1 \leq i \leq m-2.
\end{aligned}
$$

See Barto and Kozik [4, Section 5.3.1]. Directed Jónsson terms provide another characterization of congruence distributivity [11].

The mentioned observation from [4, Section 5.3.1] is optimal: the variety $\mathcal{N}_m$ fails to have $t_1, \dots, t_{m-3}$ directed Jónsson terms, since otherwise $\mathcal{N}_m$ would be $2m-6$-distributive, by [11, Observation 1.2], thus contradicting Theorem 3.6(1). Notice that the counting conventions in [4, 11] are sometimes different from the conventions adopted in the present note.

Remark 5.5. Theorem 1.1 suggests that, for every $m \geq 3$, there should be some condition C_m strictly between the strength of an m-ary and of an $m+1$-ary near-unanimity term and such that C_m implies $2m-3$-distributivity but does not imply $2m-4$-distributivity. Of course,

($\diamondsuit_m$) there is an $m+1$-ary near-unanimity term + $2m-3$-distributivity
is possibly such a condition, but it looks quite artificial. One should check that ($\diamondsuit_m$) does not imply $2m-4$-distributivity, a fact which can be probably obtained by combining the present methods with [15]. For $m = 3$, a 3-distributive not 2-distributive variety with a 4-ary near-unanimity term does indeed exist, see [13, Proposition 5.1] or Proposition 5.3 above in the case $m = 4$.

A possibly more natural condition is presented in the next definition.

Definition 5.6. If $m \geq 3$, an $m\frac{1}{2}$*-near-unanimity term* is an $m+2$-ary term u such that the following equations hold.

$$u(z,z,x,x,\dots,x) = x, \tag{5.1}$$

$$u(x,\dots,x,\underset{i}{z},x,\dots,x) = x, \qquad \text{for } 2 \leq i \leq m+2, \tag{5.2}$$

$$u(x,x,x,z,z,\dots,z) = u(x,z,z,z,z,\dots,z). \tag{5.3}$$

The terminology comes from the fact that if u is an $m\frac{1}{2}$-near-unanimity term, then the $m+1$-ary term v defined by

$$v(x_1, x_2, x_3, \dots, x_{m+1}) = u(x_1, x_1, x_2, x_3, \dots, x_{m+1})$$

is a near-unanimity term, by (5.1) and (5.2). On the other hand, if w is an m-ary near-unanimity term, then by adding two initial dummy varables, w becomes an $m\frac{1}{2}$-near-unanimity term.

Proposition 5.7. *Let $m \geq 3$.*

(1) If some variety $\mathcal{V}$ has an $m\frac{1}{2}$-near-unanimity term, then $\mathcal{V}$ is $2m-3$-distributive.
(2) There is a variety $\mathcal{V}$ with an $m+1$-ary near-unanimity term but without an $m\frac{1}{2}$-near-unanimity term.
(3) There is a variety $\mathcal{V}$ with an $m\frac{1}{2}$-near-unanimity term but without an m-ary near-unanimity term.

Details for the proof of Proposition 5.7 shall be presented elsewhere. We just notice that, granted clause (1) in Proposition 5.7, then, by Theorem 3.6(1), the variety $\mathcal{N}_{m+1}$ furnishes an example for 5.7(2).

Problem 5.8. Study "dissent" terms in the following sense. A term u of arity ≥ 3 is a *lone-dissent term* if all the equations of the form

$$u(x, x, \dots, x, y, x, \dots, x, x) = y$$

are satisfied, with just one occurrence of y in any possible position.

A ternary lone-dissent term is a *minority term*, see [12] for a detailed study of minority terms. It is easy to see that a variety of abelian groups has an $m+1$-ary lone-dissent term if and only if its exponent divides m. In particular, contrary to the case of near-unanimity terms, the existence of an m-ary lone-dissent term does not imply an $m+1$-ary lone-dissent term. However, there are some positive results. Some simple facts are stated in the next proposition.

It is probably also interesting to study dissent-unanimity terms in the following sense. If $m \geq 3$, a $2m$-ary term u is a *dissent-unanimity term* if all the equations of the form

$$u(x, x, \dots, x, \underset{i}{y}, x, \dots, x, x; y, y, \dots, y, \underset{m+i}{z}, y, \dots, y, y) = y$$

are satisfied for all i, $1 \leq i \leq m$, where the semicolon separates the first m arguments of u with the last m arguments. The case $m = 3$ has been dealt with in [12].

Proposition 5.9. *Let $m, n \geq 2$.*

(1) If some variety $\mathcal{V}$ has an $m+1$-ary lone-dissent term, then, for all $k \geq 1$, $\mathcal{V}$ has a $km+1$-ary lone-dissent term.
(2) More generally, if some variety $\mathcal{V}$ has both an $m+1$-ary and an $n+1$-ary lone-dissent term, then $\mathcal{V}$ has an $m+n+1$-ary lone-dissent term.

(3) *If some variety $\mathcal{V}$ has an $m+1$-ary lone-dissent term, then $\mathcal{V}$ has a Maltsev term, hence $\mathcal{V}$ is congruence permutable.*

(4) *If some variety $\mathcal{V}$ has both an $m+1$-ary and an $m+2$-ary lone-dissent term, then $\mathcal{V}$ is an arithmetical variety.*

Proof. (1) If d is an $m+1$-ary lone-dissent term, then $d(d(-,-,\dots),-,\dots)$ is a $2m+1$-ary lone-dissent term, $d(d(d(-,-,\dots),-,\dots),-,\dots)$ is a $3m+1$-ary lone-dissent term and so on.

(2) If d, e are, respectively, an $m+1$-ary and an $n+1$-ary lone-dissent term, then $d(e(-,-,\dots),-,\dots)$ is an $m+n+1$-ary lone-dissent term.

(3) If d is an $m+1$-ary lone-dissent term, then $t(x,y,z) = d(x,y,y,\dots,y,z)$ is a Maltsev term.

(4) If d, e are an $m+1$-ary and an $m+2$-ary lone-dissent term, then

$$\begin{aligned}
&t(x_1,\dots,x_{m+2}) = \\
&e(d(x_1,x_2,\dots,x_m,x_{m+1}), d(x_1,x_2,\dots,x_m,x_{m+2}), \\
&\quad d(x_1,x_2,\dots,x_{m-1},x_{m+1},x_{m+2}),\dots, d(x_1,x_2,x_4,\dots,x_{m+1},x_{m+2}), \\
&\quad d(x_1,x_3,\dots,x_{m+1},x_{m+2}), d(x_2,x_3,\dots,x_{m+1},x_{m+2}))
\end{aligned}$$

is an $m+1$-ary near-unanimity term, hence $\mathcal{V}$ is congruence distributive. By (3) $\mathcal{V}$ is congruence permutable, hence $\mathcal{V}$ is arithmetical. $\square$

Corollary 5.10. *If $m, n \geq 2$, m and n are coprime and some variety $\mathcal{V}$ has both an $m+1$-ary and an $n+1$-ary lone-dissent term, then $\mathcal{V}$ is arithmetical.*

Proof. Since m and n are coprime, then the Diophantine equation $km+hn=1$ has a solution with $k,h \in \mathbb{Z}$. Changing a sign, we have either $km = hn+1$ or $hn = km+1$ with $k,h \in \mathbb{N} \setminus \{0\}$. Applying 5.9(1) twice, $\mathcal{V}$ has both a $km+1$- and a $hn+1$-lone-dissent term. Since hn and km differ by 1, then $\mathcal{V}$ is arithmetical by 5.9(4). $\square$

As a final remark, we notice that, for $m \geq 3$, the existence of an m-ary lone-dissent term implies the existence of a sequence $t_1,\dots,t_{m-2}$ of *directed minority terms*, i. e., terms satisfying

$$\begin{aligned}
y &= t_1(x,x,y), & t_{m-2}(x,y,y) &= x, \\
t_i(x,y,y) &= t_{i+1}(x,x,y), & &\text{for } 1 \leq i < m-2, \text{ and} \\
y &= t_i(x,y,x), & &\text{for } 1 \leq i \leq m-2.
\end{aligned}$$

The proof presents no variation with respect to [4, Section 5.3.1]. It is probably interesting to study this and similar conditions. See also [15, Remark 8.19] for further comments.

The author considers highly inappropriate and strongly discourages the use of indicators extracted from the following list (even in aggregate forms in combination with similar lists) in decisions about individuals, attributions of funds, selections or evaluations of research projects.

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