Multiple positive bound state solutions of a critical Choquard equation

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Abstract

In this paper we consider the problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u + V_\lambda(x)u = (I_\mu * |u|^{2^*_\mu})|u|^{2^*_\mu-2}u \ \text{ in } \ \mathbb{R}^N, \\ u>0 \ \text{ in } \ \mathbb{R}^N, \end{cases}$$

where $V_\lambda = \lambda + V_0$ with $\lambda \geq 0$, $V_0 \in L^{N/2}(\mathbb{R}^N)$, $I_\mu = \frac{1}{|x|^\mu}$ is the Riesz potential with $0<\mu<\min\{N,4\}$ and $2^*_\mu = \frac{2N-\mu}{N-2}$ with $N \geq 3$. Under some smallness assumption on V_0 and λ we prove the existence of two positive solutions of (P_λ). In order to prove the main result, we used variational methods combined with degree theory.

Key Words. Choquard equation, Variational methods, Critical exponents
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1 Introduction

In this paper we will focus our attention on the existence of positive solutions for the following class of Choquard equation

$$(P_\lambda) \qquad \begin{cases} -\Delta u + (\lambda + V_0(x))u = (I_\mu * |u|^{2^*_\mu})|u|^{2^*_\mu-2}u \ \text{ in } \ \mathbb{R}^N, \\ u>0 \ \text{ in } \ \mathbb{R}^N, \end{cases}$$

where $I_\mu = \frac{1}{|x|^\mu}$ is the Riesz potential, $0 < \mu < \min\{N, 4\}$, $2^*_\mu = \frac{(2N-\mu)}{N-2}$ with $N \geq 3$, $\lambda \geq 0$ and $V_0 : \mathbb{R}^N \to \mathbb{R}$ is a positive function satisfying some technical conditions that will be mentioned later on.

The existence of solution for problem (P_λ) ensures the existence of standing waves solutions for a nonlinear Schrödinger equation of the form

$$i\partial_t \Psi = -\Delta\Psi + V_0(x)\Psi - (I_\mu * |\Psi|^{2^*_\mu})|\Psi|^{2^*_\mu - 2}\Psi, \qquad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where V_0 is the external potential and I_μ is the response function, which possesses information on the mutual interaction between the bosons. This type of nonlocal equation appears in a lot of physical applications, for instance in the study of propagation of electromagnetic waves in plasmas [10] and in the theory of Bose-Einstein condensation [15]. We recall that a standing wave solution is a solution of the type

$$\Psi(x,t) = u(x)e^{i\lambda t}, \tag{1.2}$$

which solves (1.1) if, and only if, u solves the equation

$$-\Delta u + (\lambda + V_0(x))u = \left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu - 2}\right)|u|^{2^*_\mu - 2}u \quad \text{in} \quad \mathbb{R}^N, \tag{1.3}$$

which is a Choquard-Pekar equation. In the ansatz (1.2) the frequency λ is related to the energy of the particle, so $\lambda \neq 0$ is a significant case.

In [16], Du and Yang have considered only the case $\lambda = V_0 = 0$, and they showed that any positive solution of (1.3) must be of the form

$$\Psi_{\delta,y}(x) = C\left(\frac{\delta}{\delta^2 + |x-y|^2}\right)^{\frac{N-2}{2}}, \quad x \in \mathbb{R}^N,$$

for some $\delta > 0, y \in \mathbb{R}^N$, and $C > 0$ is a constant that depends only on N. Still related to (1.3), Du, Gao and Yang [17] have studied existence and qualitative properties of solutions of the problem

$$-\Delta u = \frac{1}{|x|^\alpha}\left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu - 2}\right)|u|^{2^*_\mu - 2}u \quad \text{in} \quad \mathbb{R}^N, \tag{1.4}$$

for some values of α and μ. In that paper, the authors has proved an interesting version of the Concentration-Compactness principle due to Lions [28] that can be used for Choquard equations with critical growth, for more details see [17, Lemma 2.5].

In [19], Gao, da Silva, Yang and Zhou showed the existence of solution for (1.3) by supposing $\lambda = 0$ and the following conditions on potential V_0:
(I) $V_0 \in C(\mathbb{R}^N, \mathbb{R})$, $\quad V_0(x) \geq \nu > 0$ in a neighborhood of $\quad 0$.
(II) There are $p_1 < N/2$, $p_2 > N/2$ and for $N = 3$, $p_2 < 3$, such that $V_0 \in L^p(\mathbb{R}^N)$, $\quad \forall p \in [p_1, p_2]$
(III) $|V_0|_{N/2} < C(N,\mu)^{\frac{N-2}{2N-\mu}} S_{H,L}(2^{\frac{N=2-\mu}{2N-\mu}} - 1)$.
where $C(N,\mu)$ and $S_{H,L}$ are as in (2.2) and (2.5), respectively.

A first result fronting problems like (1.3) is due to Benci and Cerami in the seminal paper [8], where the authors studied the existence of solution for the following class of local critical problem

$$-\Delta u + V_0(x)u = |u|^{2^*-2}u \quad \text{in} \quad \mathbb{R}^N, \tag{1.5}$$

with $N \geq 3$, $2^* = \frac{2N}{N-2}$ and the function $V_0 : \mathbb{R}^N \to \mathbb{R}$ satisfies the conditions below

(i) $V_0 \geq 0$ and is strictly positive in an open set.

(ii) $V_0 \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$ with $1 < p_1 < \frac{N}{2} < p_2$, with $p_2 < 3$ if $N = 3$.

(iii) $|V_0|_{L^{N/2}(\mathbb{R}^N)} < S(2^{2/N} - 1)$,
where S denotes the best constant of the immersion $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, that is,

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N), u \neq 0} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{|u|_{2^*}^2}.$$

By using variational methods, the authors were able to prove the existence of a positive solution $u \in D^{1,2}(\mathbb{R}^N)$ with

$$f(u) \in (S \, , \, 2^{2/N} S),$$

where $f : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ is the functional given by

$$f(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_0(x)|u|^2 \right) dx.$$

The main difficulty to prove the existence of solution comes from the fact that the nonlinearity has a critical growth. To overcome this difficult, the authors used Variational methods, Deformation lemma, and the well known Concentration-Compactness principle due to Lions [28]. After the publication of [8], some authors studied problems related to (1.5), see for example, [3], [7], [9], [11], [12], [22], [30], [31], [32], [38] and references therein.

In the present paper, we intend to study the existence and multiplicity of positive solutions for (P_λ) for a new class of potential V_0, and moreover, we also consider the case $\lambda > 0$ that is a novelty for this class of problem. The assumptions on potential V_0 are the following:

(V_1) $V_0(x) \geq 0, \quad \forall x \in \mathbb{R}^N$,

(V_2) $V_0 \in L^{N/2}(\mathbb{R}^N)$,

(V_3) $0 < |V_0|_{L^{N/2}(\mathbb{R}^N)} < \left(2^{\frac{4-\mu}{2N-\mu}} - 1 \right) S$.

Our main results have the following statements:

Theorem 1.1. *Let $\lambda = 0$ and assume that $(V_1) - (V_3)$ hold. Then problem (P_0) has at least a positive solution.*

Theorem 1.2. *Let $\lambda > 0$ and assume that (V_1) and (V_2) hold. Then there exists $\bar{\lambda} > 0$, such that if $\lambda \in (0, \bar{\lambda})$ then problem (P_λ) has at least a positive solutions v_l. If (V_3) also holds, then there exists $\lambda_0 = \lambda_0(V_0) > 0$ such that if $\lambda \in (0, \lambda_0)$ then problem (P_λ) has at least two distinct positive solutions, v_l and v_h.*

Remark 1.3. a) The solutions we find are bound state solutions, indeed assumptions (V_1) and (V_2) imply that problem (P_λ) has no ground state solutions for every $\lambda \geq 0$ (see Proposition 4.1).

b) The solution v_h is an high energy one, while v_l is a low energy one. Namely, as $\lambda \to 0$, the solution v_h converges to the solution provided by Theorem 1.1 while v_l flattens and disappears.

The existence of a solution as in Theorem 1.1 has been suggested in [33, Problem 3] and it is proved in [19, Theorem 1.4] with different conditions on potential V_0 as mentioned above. Hence, the Theorem 1.1 complements the study made [19], in the sense that we find a positive solution for a new class of potential V_0 in the case $\lambda = 0$. Related to the Theorem 1.2, we would like to point out that it is an important contribution of this paper, because it establishes the existence of at least two solutions for $\lambda > 0$ and small enough, which is new for this class of problem. In order to prove the theorems above, we will approach the problem by variational methods. Since the problem lacks of compactness, both for the critical exponent and for the unboundedness of the domain, an analysis of the behaviour of the Palais-Smale sequences needs. We perform it by almost classical techniques developed in the local case, see [8], that in our nonlocal framework requires some careful adjustment (see Theorem 3.2). In particular, the case $\lambda > 0$ presents some more difficulties because in such a case the natural space to work in is $H^1(\mathbb{R}^N)$ while typically the splitting theorem works in $D^{1,2}(\mathbb{R}^N)$. To front these difficulties we follow some ideas from [11] and [23] together with a nonexistence result contained in [35]. Let us remark that in the critical case no extra regularity assumptions need for the nonexistence result, see Theorem 2.2.

Before concluding this introduction, we would like to mention that there is a rich literature associated with Choquard-Pekar equation of the type

$$-\Delta u + V(x)u = K(x)\left(\frac{1}{|x|^{\mu}} * H(u)\right)h(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.6}$$

where $H(t) = \int_0^t h(s)ds$, with $V, K : \mathbb{R}^N \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ being continuous functions verifying some technical conditions. The reader can find some interesting results in [1], [2], [4], [5], [14], [24], [28], [29], [34], [35], [36], [37], [39] and references theirein.

The paper is organized as follows: In Section 2, we prove some results involving the limit problem. In Section 3, we prove a splitting theorem and show some compactness results involving the energy functional associated with (P_λ). In Section 4, we make the proof of some technical lemmas that will be used in Section 5 in the proofs of Theorems 1.1 and 1.2.

2 Variational framework

In this section, we will show some important results involving the limit problem that are crucial in our approach. To begin with, we recall that to apply variational methods, we must have

$$\left|\int_{\mathbb{R}^N} \left(I_\mu * |u|^{2^*_\mu}\right)|u|^{2^*_\mu}\, dx\right| < +\infty, \quad \forall u \in D^{1,2}(\mathbb{R}^N). \tag{2.1}$$

This fact is an immediate consequence of the Hardy-Littlewood-Sobolev inequality, which will be frequently used in this paper.

Proposition 2.1 ([26]). *[Hardy − Littlewood − Sobolev inequality]:*
Let $s, r > 1$ and $0 < \mu < N$ with $1/s + \mu/N + 1/r = 2$. If $g \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C(s, N, \mu, r)$, independent of g, h, such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^{\mu}} \leq C(s, N, \mu, r)|g|_s|h|_r.$$

As a direct consequence of this inequality, we have

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^{\mu}}dxdy\right)^{\frac{1}{2^*{}_\mu}} \leq C(N,\mu)^{\frac{1}{2^*_\mu}}|u|^2_{2^*}, \quad \forall u \in D^{1,2}(\mathbb{R}^N), \tag{2.2}$$

for a suitable positive constant $C(N, \mu)$.

In the sequel, if $\lambda > 0$ we will work with $H = H^1(\mathbb{R}^N)$ endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \lambda \int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{1}{2}}.$$

When $\lambda = 0$, we will consider $H = D^{1,2}(\mathbb{R}^N)$ endowed with the usual norm, that is,

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

Sometimes we use the explicit notations $\|\cdot\|_{D^{1,2}}$ and $\|\cdot\|_{H^1}$.

We say that $u:\mathbb{R}^N\to\mathbb{R}$ is a weak solution of (P_λ) if $u\in H$ is a positive function such that for all $\varphi\in H$ we get

$$\int_{\mathbb{R}^N}\nabla u\nabla\varphi dx+\int_{\mathbb{R}^N}V_\lambda(x)u\varphi dx=\int_{\mathbb{R}^N}(I_\mu*|u|^{2^*_\mu})|u|^{2^*_\mu-2}u\varphi dx,$$

where $V_\lambda:=\lambda+V_0$. It is a standard task to check that the weak solutions of (P_λ) are critical points of the energy functional $E_\lambda:H\to\mathbb{R}$ associated to problem (P_λ) given by

$$E_\lambda(u)=\frac{1}{2}\|u\|^2+\frac{1}{2}\int_{\mathbb{R}^N}V_0(x)u^2dx-\frac{1}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}(I_\mu*|u|^{2^*_\mu})|u|^{2^*_\mu}dx,$$

and that E_λ belongs to $C^1(H,\mathbb{R})$.

In what follows, let us denote by $\mathcal{M}$ the manifold

$$\mathcal{M}=\left\{u\in H\ :\ \int_{\mathbb{R}^N}(I_\mu*|u|^{2^*_\mu})|u|^{2^*_\mu}dx=1\right\}$$

and by J_λ the functional

$$J_\lambda(u)=\|u\|^2+\int_{\mathbb{R}^N}V_0(x)u^2dx,\qquad u\in H.$$

For the case $\lambda=0$, we designate such functionals by E_0 and J_0, respectively.

Next, let us recall some information involving the problem

$$(P_\infty)\qquad\begin{cases}-\Delta u=(I_\mu*|u|^{2^*_\mu})|u|^{2^*_\mu-2}u\ \text{ in }\ \mathbb{R}^N,\\ u\in D^{1,2}(\mathbb{R}^N).\end{cases}$$

The functions

$$\tilde{U}_{\sigma,z}(x)=\frac{[N(N-2)\sigma]^{\frac{N-2}{4}}}{(\sigma+|x-z|^2)^{\frac{N-2}{2}}},\quad \sigma>0,\quad z\in\mathbb{R}^N,\tag{2.3}$$

are the minimizers for S in $D^{1,2}(\mathbb{R}^N)$ that verify

$$\begin{cases}-\Delta u=|u|^{2^*-2}u\ \text{ in }\ \mathbb{R}^N,\\ u\in D^{1,2}(\mathbb{R}^N)\end{cases}$$

and

$$U_{\sigma,z}(x)=C(N,\mu)^{\frac{2-N}{2(N-\mu+2)}}S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}}\tilde{U}_{\sigma,z}(x),\quad\forall x\in\mathbb{R}^N\tag{2.4}$$

are minimizer for $S_{H,L}$ given by

$$S_{H,L}:=\inf_{u\in D^{1,2}(\mathbb{R}^N),u\neq 0}\frac{\|u\|^2}{\left(\int_{\mathbb{R}^N}(I_\mu*|u|^{2^*_\mu})|u|^{2^*_\mu}dx\right)^{1/2^*_\mu}},\tag{2.5}$$

that satisfy (P_∞) (see [42, 25] and [26, Theorem 4.3]), with

$$\int_{\mathbb{R}^N}|\nabla U_{\sigma,z}|^2dx=\int_{\mathbb{R}^N}(I_\mu*|U_{\sigma,z}|^{2^*_\mu})|U_{\sigma,z}|^{2^*_\mu}dx=S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.\tag{2.6}$$

Moreover, it is possible to show that

$$C(N,\mu)^{\frac{1}{2^*_\mu}}S_{H,L}=S\quad\text{(See [18, \textit{Lemma} 1.2])}$$

and

$$E_\infty(U_{\sigma,z}) = \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}, \tag{2.7}$$

where E_∞ is the energy functional associated to (P_∞) given by

$$E_\infty(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N} \big(I_\mu * |u|^{2_\mu^*}\big)|u|^{2_\mu^*}\,dx.$$

According to the limit problem when $\lambda > 0$, we have the result below that is a key point in our approach.

Theorem 2.2. *If $\lambda > 0$ and $u \in H^1(\mathbb{R}^N)$ is a weak solution of*

$$(P_{\lambda,\infty}) \qquad -\Delta u + \lambda u = (I_\mu * |u|^{2_\mu^*})|u|^{2_\mu^*-2}u,$$

then $u \equiv 0$.

Proof. In what follows, we are proving the assumption of [35, Theorem 2] holds. We have only to verify that in our critical case all the regularity required in the proof is fulfilled. Since here $p = 2_\mu^*$, so $\frac{1}{p} = \frac{N-2}{N+\alpha}$ where $\alpha = N-\mu$, it is sufficient to show that if $u \in H^1(\mathbb{R}^N)$ is a weak solution of $(P_{\infty,\lambda})$ then $u \in H^{2,2}_{\text{loc}}(\mathbb{R}^N)$. The function u solves

$$-\Delta u = (-\lambda + Q(x))u \quad u \in H^1(\mathbb{R}^N),$$

where

$$Q(x) = \Big(I_\mu * |u|^{2_\mu^*}\Big)|u|^{2_\mu^*-2}.$$

By the Hardy-Littlewood-Sobolev inequality, we know that $I_\mu * |u|^{2_\mu^*} \in L^{\frac{2N}{\mu}}(\mathbb{R}^N)$. Since $|u|^{2_\mu^*-2} \in L^{\frac{2N}{4-\mu}}(\mathbb{R}^N)$, we claim that $Q \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Indeed, as

$$|Q(x)|^{\frac{N}{2}} = (I_\mu * |u|^{2_\mu^*})^{\frac{N}{2}}(|u|^{2_\mu^*-2})^{\frac{N}{2}}$$

and

$$(I_\mu * |u|^{2_\mu^*})^{\frac{N}{2}} \in L^{\frac{4}{\mu}}(\mathbb{R}^N), \quad (|u|^{2_\mu^*-2})^{\frac{N}{2}} \in L^{\frac{4}{4-\mu}}(\mathbb{R}^N),$$

we derive from the Hölder inequality that $|Q|^{\frac{N}{2}}$ belongs to $L^1(\mathbb{R}^N)$, that is, $Q \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Now, arguing as in Struwe [41, Lemma B.3],

$$u \in L^q_{loc}(\mathbb{R}^N), \quad \forall q \in [1, +\infty).$$

Now, we claim that $I_\mu * |u|^{2_\mu^*} \in L^\infty(\mathbb{R}^N)$. Indeed, note that,

$$\begin{aligned}
|(I_\mu * |u|^{2_\mu^*})(x)| &= \int_{\mathbb{R}^N} \frac{|u|^{2_\mu^*}}{|x-y|^\mu}dy \\
&\le \int_{|y-x|\le 1} \frac{|u|^{2_\mu^*}}{|x-y|^\mu}dy + \int_{|y-x|\ge 1} \frac{|u|^{2_\mu^*}}{|x-y|^\mu}dy.
\end{aligned}$$

Using the fact that

$$\frac{1}{|z|^\mu} \in L^{\frac{2N}{\mu}}(B_1^c(0)), \quad |u|^{2_\mu^*} \in L^{\frac{2N}{2N-\mu}}(B_1^c(x)), \quad \frac{1}{|z|^\mu} \in L^t(B_1(0)) \quad \text{for} \quad t \in [1, N/\mu)$$

and for every $x \in \mathbb{R}^N$

$$|u|^{2_\mu^*} \in L^{t'}(B_1(x)) \quad \text{for } t' \text{ s.t.} \quad \frac{1}{t} + \frac{1}{t'} = 1,$$

the Hölder inequality ensures that for every compact set $K \subset \mathbb{R}^N$ there is $C = C(K) > 0$ such that

$$|(I_\mu * |u|^{2^*_\mu})(x)| \leq C, \quad \forall x \in K,$$

which implies $I_\mu * |u|^{2^*_\mu} \in L^\infty_{\rm loc}(\mathbb{R}^N)$. Recalling that $|u|^{2^*_\mu - 2} \in L^q_{loc}(\mathbb{R}^N)$ for every $q \in [1, +\infty)$, it follows that

$$Q \in L^q_{loc}(\mathbb{R}^N) \quad \forall q \in [1, +\infty).$$

Thereby, by Calderón-Zygmund inequality, see [21], $u \in W^{2,q}_{loc}(\mathbb{R}^N)$ for all $q \in (1, +\infty)$. This proves the theorem. □

Before concluding this section, we will prove an important estimate involving the nodal solutions of the limit problem, which will be used later on.

Lemma 2.3. *If $u \in D^{1,2}(\mathbb{R}^N)$ is a nodal solution of (P_λ), then*

$$E_\lambda(u) \geq 2^{\frac{4-\mu}{N+2-\mu}} \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

Proof. Arguing as in the proof of [20, Proposition 3.2], for all $t^+, t^- > 0$ we see that

$$E_\lambda(t^+u^+)^{\frac{N+2-\mu}{4-\mu}} + E_\lambda(t^-u^-)^{\frac{N+2-\mu}{4-\mu}} \leq E_\lambda(u)^{\frac{N+2-\mu}{4-\mu}}.$$

Fixing $t^+, t^- > 0$ such that $E'_\lambda(t^\pm u^\pm)(t^\pm u^\pm) = 0$, it follows that

$$E_\lambda(t^\pm u^\pm) \geq \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

The last two inequalities combine to give

$$E_\lambda(u) \geq 2^{\frac{4-\mu}{N+2-\mu}} \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)},$$

finishing the proof.

□

Lemma 2.4. *If $(u_n) \subset D^{1,2}(\mathbb{R}^N)$ is such that*

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} (I_\mu * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu}) dx\right)^{1/2^*_\mu}} \to S_{H,L},$$

then there are $\sigma_n > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\frac{u_n}{|u_n|_{2^*}} = \frac{U_{\sigma_n, y_n}}{|U_{\sigma_n, y_n}|_{2^*}} + o_n(1) \quad in \quad D^{1,2}(\mathbb{R}^N),$$

where $U_{\sigma,y}$ is given in (2.4).

Proof. Note that by (2.2),

$$S_{H,L} + o_n(1) = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} (I_\mu * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu} dx\right)^{1/2^*_\mu}} \geq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{C(N,\mu)^{\frac{1}{2^*_\mu}} |u_n|^2_{2^*}} \geq \frac{S}{C(N,\mu)^{\frac{1}{2^*_\mu}}}.$$

Since $S_{H,L}C(N,\mu)^{\frac{1}{2^*_\mu}} = S$, then

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_n|^{2^*} dx\right)^{2/2^*}} \to S.$$

By Lions [28], Aubin [6] and Talenti [42], there are $\sigma_n > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\frac{u_n}{|u_n|_{2^*}} = \frac{\tilde{U}_{\sigma_n,y_n}}{|\tilde{U}_{\sigma_n,y_n}|_{2^*}} + o_n(1) \quad \text{in} \quad D^{1,2}(\mathbb{R}^N),$$

where $\tilde{U}_{\sigma,y}$ is given in (2.3). This together with (2.4) proves the desired result. □

Next corollary states a first property of Palais-Smale sequences. For short, we shall refer to those sequences as $(\mathrm{PS})_c$ sequences, where c is the energy level, or as (PS) sequences.

Corollary 2.5. *Let (u_n) be a (PS) sequence for the functional E_∞ with*

$$c = \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

Then, there are sequences $(\sigma_n) \subset \mathbb{R}^+$ and $(y_n) \subset \mathbb{R}^N$ such that, up to a subsequence of (u_n), we have

$$\frac{u_n}{|u_n|_{2^*}} = \frac{U_{\sigma_n,y_n}}{|U_{\sigma_n,y_n}|_{2^*}} + o_n(1), \quad \textit{in} \quad D^{1,2}(\mathbb{R}^N). \tag{2.8}$$

Proof. Since (u_n) is a $(PS)_c$ sequence for E_∞ it is bounded and so it is possible to prove that

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^N} (I_\mu * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu}) dx\right)^{1/2^*_\mu}} \to S_{H,L}.$$

Now, we apply Lemma 2.4 and, taking into account that the weak limit of u_n (up to a rescaling) solves (P_∞), we get the desired result. □

Lemma 2.6. *Let (u_n) be a (PS) sequence for E_λ. Then (u_n) is bounded and there exists a weak solution u_0 of (P_λ) such that $u_n \rightharpoonup u_0$ in H, up to a subsequence. The same statement holds for (PS) sequences for the functional E_∞, with u_0 in $D^{1,2}(\mathbb{R}^N)$ a solution of (P_∞).*

Proof. Since $V_0 \geq 0$ on $\mathbb{R}^N$, then

$$\begin{aligned} c + o_n(1)\|u_n\| &= E_\lambda(u_n) - \frac{1}{2 \cdot 2^*_\mu} E'_\lambda(u_n)u_n \\ &\geq \frac{N+2-\mu}{4N-2\mu} \|u_n\|^2, \end{aligned}$$

so we get $(\|u_n\|)$ bounded and the existence of the weak limit u_0. Moreover, clearly u_0 is a weak solution of (P_λ) because (u_n) is a (PS) sequence for E_λ. The same argument works for E_∞.

□

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^N$ a smooth bounded domain. For each $\tau > 0$, there is $M_\tau > 0$ such that*

$$\left[\frac{S}{2^{\frac{2}{N}}} - \tau\right]^{2^*_\mu} \int_\Omega (I_\mu * |v|^{2^*_\mu})|v|^{2^*_\mu} dx \leq C(N,\mu) \left(|\nabla v|^2_{L^2(\Omega)} + M_\tau |v|^2_{L^2(\Omega)}\right)^{2^*_\mu}, \quad \forall v \in H^1(\Omega),$$

where $C(N,\mu)$ is given in (2.2).

Proof. From (2.2),

$$\left[\frac{S}{2^{\frac{2}{N}}}-\tau\right]^{2^*_\mu}\int_\Omega (I_\mu * |v|^{2^*_\mu})|v|^{2^*_\mu}\,dx \le C(N,\mu)\left(\left[\frac{S}{2^{\frac{2}{N}}}-\tau\right]|v|^2_{L^{2^*}(\Omega)}\right)^{2^*_\mu}.$$

Now, we apply the Cherrier's inequality [13] to get the desired result. □

3 A global compactness theorem

In this section we study the Palais-Smale sequences of our functionals First, now prove a technical lemma for E_∞ that will be useful later on.

Lemma 3.1. *(Main lemma) Let (u_n) be a $(PS)_c$ sequence for the functional E_∞ with $u_n \rightharpoonup 0$ and $u_n \nrightarrow 0$. Then, there are sequences $(\sigma_n) \subset \mathbb{R}^+$ and $(y_n) \subset \mathbb{R}^N$ such that, up to a subsequence of (u_n), we have*

$$v_n(x) := \sigma_n^{(N-2)/2} u_n(\sigma_n x + y_n) \rightharpoonup v_0, \quad in \;\; D^{1,2}(\mathbb{R}^N) \tag{3.1}$$

where v_0 is a non trivial solution of problem (P_∞) and

$$E_\infty(u_n) = E_\infty(v_0) + E_\infty(v_n - v_0) + o_n(1) \tag{3.2}$$

$$\|u_n\|^2 = \|v\|^2 + \|v_n - v\|^2 + o_n(1). \tag{3.3}$$

Proof. Let $(u_n) \subset D^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$ sequence for the functional E_∞, i.e,

$$E_\infty(u_n) \to c \;\text{ and }\; E'_\infty(u_n) \to 0. \tag{3.4}$$

The sequence (u_n) is bounded in $D^{1,2}(\mathbb{R}^N)$, by Lemma 2.6. Then, since $u_n \rightharpoonup 0$ and $u_n \nrightarrow 0$, the (PS) condition and (2.5) imply that

$$c \ge \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

Note that

$$c + o_n(1) = E_\infty(u_n) - \frac{1}{2\cdot 2^*_\mu} E'_\infty(u_n)u_n = \frac{(N+2-\mu)}{4N-2\mu}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx$$

which leads to

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \ge S_{H,L}^{(2N-\mu)/(N+2-\mu)}. \tag{3.5}$$

Let m be a number such that $B_2(0)$ is covered by m balls of radius 1, $k \in \mathbb{N}^*$ to be fixed later, $(\sigma_n) \subset \mathbb{R}^+$, $(y_n) \subset \mathbb{R}^N$ such that

$$\sup_{y\in\mathbb{R}^N}\int_{B_{\sigma_n}(y)} |\nabla u_n|^2 dx = \int_{B_{\sigma_n}(y_n)} |\nabla u_n|^2 dx = \frac{S_{H,L}^{(2N-\mu)/(N+2-\mu)}}{km}$$

and

$$v_n(x) := \sigma_n^{(N-2)/2} u_n\Big(\sigma_n x + y_n\Big).$$

A simple computation gives

$$\|v_n\| = \|u_n\| \quad \text{and} \quad \int_{\mathbb{R}^N} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu} dx = \int_{\mathbb{R}^N} (I_\mu * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu} dx,$$

from where it follows that

$$E_\infty(v_n) = E_\infty(u_n), \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Hereafter, we fix k such way that

$$\frac{S_{H,L}^{(2N-\mu)/(N+2-\mu)}}{k} < \left(\frac{S}{2^{\frac{2}{N}}}\right)^{\frac{2N-\mu}{N+2-\mu}} \left(\frac{1}{C(N,\mu)}\right)^{\frac{N-2}{N+2-\mu}}. \tag{3.7}$$

Using a change of variable, we can prove that

$$\int_{B_1(0)} |\nabla v_n|^2 dx = \frac{S_{H,L}^{(2N-\mu)/(N+2-\mu)}}{km} = \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\nabla v_n|^2 dx. \tag{3.8}$$

Now, for each $\Phi \in D^{1,2}(\mathbb{R}^N)$, we define the function

$$\tilde{\Phi}_n(x) = \frac{1}{\sigma_n^{(N-2)/2}} \Phi\left(\frac{1}{\sigma_n}(x - y_n)\right)$$

that satisfies

$$\int_{\mathbb{R}^N} \nabla u_n \nabla \tilde{\Phi}_n dx = \int_{\mathbb{R}^N} \nabla v_n \nabla \Phi dx \tag{3.9}$$

and

$$\int_{\mathbb{R}^N} (I_\mu * |u_n|^{2^*_\mu})|u_n|^{2^*_\mu - 2} u_n \tilde{\Phi}_n dx = \int_{\mathbb{R}^N} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu - 2} v_n \Phi dx. \tag{3.10}$$

These limits ensure that

$$E'_\infty(v_n) \to 0. \tag{3.11}$$

From (3.6)-(3.11), (v_n) is a $(PS)_c$ sequence for E_∞. Therefore, there exists $v_0 \in D^{1,2}(\mathbb{R}^N)$ such that, up to a subsequence, $v_n \rightharpoonup v_0$ in $D^{1,2}(\mathbb{R}^N)$ and $E'_\infty(v_0) = 0$.

As a consequence of the well known Lions' Lemma [28], we can assume that

$$\int_{\mathbb{R}^N} |v_n|^{2^*} \phi dx \to \int_{\mathbb{R}^N} |v_0|^{2^*} \phi dx + \sum_{j\in J} \phi(x_j)\nu_j, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N) \tag{3.12}$$

and

$$|\nabla v_n|^2 \rightharpoonup \mu \geq |\nabla v_0|^2 + \sum_{j\in J} \delta_{x_j} \mu_j,$$

for some $\{x_j\}_{j\in J} \subset \mathbb{R}^N$, $\{\nu_j\}_{j\in J}$, $\{\mu_j\}_{j\in J} \subset \mathbb{R}^+$ with $S\nu_j^{2/2^*} \leq \mu_j$, where J is at most a countable set.

We are going to show that J is finite. Consider $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\phi(x) = 0$ for all $x \in B_2^c(0)$ and $\phi_\rho(x) = 1$ for all $x \in B_1(0)$. Now fix $x_j \in \mathbb{R}^N$, $j \in J$, and define $\psi_\rho(x) = \phi(\frac{x-x_j}{\rho})$, for each $\rho > 0$. Then $0 \leq \psi_\rho(x) \leq 1$, for all $x \in \mathbb{R}^N$, $\psi_\rho(x) = 0$ for all $x \in B_{2\rho}^c(x_j)$ and $\psi(x) = 1$ for all $x \in B_\rho(x_j)$. We have that $(v_n\psi_\rho)$ is bounded in $D^{1,2}(\mathbb{R}^N)$ and $I'_\infty(v_n)v_n\psi_\rho = o_n(1)$. Hence,

$$\int_{\mathbb{R}^N} \psi_\rho |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} v_n \nabla v_n \nabla \psi_\rho dx = \int_{\mathbb{R}^N} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu} \psi_\rho dx + o_n(1). \tag{3.13}$$

Using Proposition 2.1, and seeing that

$$\lim_{\rho\to 0}\left[\limsup_{n\to\infty}\int_{\mathbb{R}^N} v_n\nabla v_n\nabla\psi_\rho dx\right]=0,$$

we find

$$S\nu_j^{2/2^*}\le\mu_j\le C\nu_j^{\frac{2^*_\mu}{2^*}}.$$

As $2^*_\mu>2$ we deduce that ν_j does not converge to zero and since $\sum_{j\in J}\nu_j^{2/2^*}\le\sum_{j\in J}\mu_j<+\infty$ we have that J is finite. From now on, we denote by $J=\{1,2,...,m\}$ and $\Gamma\subset\mathbb{R}^N$ the set given by

$$\Gamma=\{x_j\in\{x_j\}_{j\in J}\ :\ |x_j|>1\},\quad (x_j\ \text{ given by }\ (3.12)).$$

In the sequel, we are going to show that $v_0\neq 0$. Suppose, by contradiction that $v_0=0$. Thereby, by (3.12),

$$\int_{\mathbb{R}^N}|v_n|^{2^*}\tilde{\varphi}\,dx\to 0,\quad \forall\tilde{\varphi}\in C_0(\mathbb{R}^N\setminus\{x_1,x_2,...,x_m\}). \tag{3.14}$$

Using again Proposition 2.1, for all $\varphi\in C_0(\mathbb{R}^N\setminus\{x_1,x_2,...,x_m\})$ we derive the inequality below

$$\int_{\mathbb{R}^N}(I_\mu*|v_n|^{2^*_\mu})|v_n|^{2^*_\mu}\varphi\,dx\le C|v_n|_{2^*}^{2^*_\mu}\left(\int|v_n|^{2^*}|\varphi|^{\frac{2N}{2N-\mu}}\,dx\right)^{\frac{2N-\mu}{2N}}=o_n(1), \tag{3.15}$$

which leads to

$$\int_{\mathbb{R}^N}|\nabla v_n|^2\varphi=o_n(1). \tag{3.16}$$

Consequently, if $\rho\in\mathbb{R}$ is a number that satisfies $0<\rho<\min\{\text{dist}(\Gamma,\bar{B}_1(0)),1)\}$, it follows that

$$\int_{B_{1+\frac{2\rho}{3}}(0)\setminus B_{1+\frac{\rho}{3}}(0)}|\nabla v_n|^2dx=o_n(1). \tag{3.17}$$

In the sequel, let us consider the sequence (Φ_n) given by $\Phi_n(x)=\Phi(x)v_n(x)$, where $\Phi\in C_0^\infty(\mathbb{R}^N,[0,1])$ satisfies $\Phi(x)=1$ if $x\in B_{1+\rho/3}(0)$ and $\Phi(x)=0$ if $x\in B^c_{1+2\rho/3}(0)$. Note that,

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}}|\nabla\Phi_n|^2dx\le C\left[\int_{B_{1+\frac{2\rho}{3}}(0)\setminus B_{1+\frac{\rho}{3}}}|\nabla v_n|^2dx+\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}|v_n|^2dx\right],$$

that is,

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}}|\nabla\Phi_n|^2dx=o_n(1). \tag{3.18}$$

Since $I'_\infty(v_n)\Phi_n=o_n(1)$, we have

$$\begin{aligned}&\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}\nabla v_n\nabla\Phi_n dx+\int_{B_{1+\frac{\rho}{3}}(0)}\nabla v_n\nabla\Phi_n dx\\ &-\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)}(I_\mu*|v_n|^{2^*_\mu})|v_n|^{2^*_\mu}\Phi\,dx-\int_{B_{1+\frac{\rho}{3}}(0)}(I_\mu*|v_n|^{2^*_\mu})|v_n|^{2^*_\mu}\Phi\,dx=o_n(1),\end{aligned}$$

which implies

$$\begin{aligned} &\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \nabla v_n \nabla \Phi_n dx + \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla v_n|^2 dx \\ &- \int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu} \Phi\, dx - \int_{B_{1+\frac{\rho}{3}}(0)} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu} \Phi\, dx = o_n(1). \end{aligned} \tag{3.19}$$

Note that from Hölder's inequality and (3.17)

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} \nabla v_n \nabla \Phi_n dx \to 0 \quad \text{when} \quad n \to \infty \tag{3.20}$$

and that (3.14) together with Proposition 2.1 gives

$$\int_{B_{1+\rho}(0)\setminus B_{1+\frac{\rho}{3}}(0)} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu} \Phi dx = o_n(1). \tag{3.21}$$

Thereby, from (3.19), (3.20) and (3.21),

$$\int_{B_{1+\frac{\rho}{3}}(0)} |\nabla v_n|^2 dx - \int_{B_{1+\frac{\rho}{3}}(0)} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu} dx = o_n(1). \tag{3.22}$$

The last equality together with the boundedness of (v_n) and (3.8) implies that for some subsequence

$$\lim_{n\to\infty} \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla v_n|^2 dx = \lim_{n\to\infty} \int_{B_{1+\frac{\rho}{3}}(0)} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu} dx = A > 0.$$

These limits combined with the Cherrier's inequality (see Lemma 2.7) give

$$A \geq \left(\frac{S}{2^{\frac{2}{N}}}\right)^{\frac{2N-\mu}{N+2-\mu}} \left(\frac{1}{C(N,\mu)}\right)^{\frac{N-2}{N+2-\mu}}. \tag{3.23}$$

On the other hand, since $B_{1+\frac{\rho}{3}(0)} \subset B_2(0)$ and $B_2(0)$ is covered by m balls of radius 1, we obtain

$$\begin{aligned} \int_{B_{1+\frac{\rho}{3}}(0)} |\nabla v_n|^2 dx &\leq \int_{B_2(0)} |\nabla v_n|^2 dx \\ &\leq \int_{\bigcup_{k=1}^m B_1(\zeta_k)} |\nabla v_n|^2 dx \\ &\leq \sum_{k=1}^m \int_{B_1(\zeta_k)} |\nabla v_n|^2 dx \\ &\leq m \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\nabla v_n|^2 dx \leq \frac{S_{H,L}^{(2N-\mu)/(N+2-\mu)}}{k}. \end{aligned}$$

Then,

$$\int_{B_{1+\frac{\rho}{3}}(0)} |\nabla v_n|^2 dx \leq \frac{S_{H,L}^{(2N-\mu)/(N+2-\mu)}}{k}, \tag{3.24}$$

implying that

$$A \leq \frac{S_{H,L}^{(2N-\mu)/(N+2-\mu)}}{k}.$$

Hence, by (3.7),

$$A < \left(\frac{S}{2^{\frac{2}{N}}}\right)^{\frac{2N-\mu}{N+2-\mu}} \left(\frac{1}{C(N,\mu)}\right)^{\frac{N-2}{N+2-\mu}},$$

which contradicts (3.23), and so, $v_0 \neq 0$.

Finally, using the equalities below,

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \int_{\mathbb{R}^N} |\nabla v_0|^2 dx + \int_{\mathbb{R}^N} |\nabla (v_n - v_0)|^2 dx + o_n(1)$$

and

$$\int_{\mathbb{R}^N} (I_\mu * |v_n|^{2^*_\mu})|v_n|^{2^*_\mu} dx = \int_{\mathbb{R}^N} (I_\mu * |v_0|^{2^*_\mu})|v_0|^{2^*_\mu} dx + \int_{\mathbb{R}^N} (I_\mu * |v_n - v_0|^{2*})|v_n - v_0|^{2^*} dx + o_n(1),$$

(see [18, Lemma 2.2]) it follows that

$$E_\infty(v_n) = E_\infty(v_0) + E_\infty(v_n - v_0) + o_n(1).$$

Since

$$E_\infty(u_n) = E_\infty(v_n),$$

we have

$$E_\infty(u_n) = E_\infty(v_0) + E_\infty(v_n - v_0) + o_n(1),$$

finishing the proof. □

The next result is crucial to study the compactness properties involving the energy functional E_λ. We would like point out that a version of that result for $\lambda = 0$ can be found in [19, Lemma 3.1] by using a different approach.

Theorem 3.2. *(A global compactness result) Let (u_n) be a $(PS)_c$ sequence for E_λ with $u_n \rightharpoonup u_0$ in H. Then, the sequence (u_n) verifies either:*
(a) $u_n \to u_0$ *or*
(b) there exist $k \in \mathbb{N}$ and $u^1, u^2, ..., u^k$ nontrivial solution of (P_∞), such that

$$\|u_n\|^2 = \|u_0\|^2 + \sum_{j=1}^{k} \|u^j\|^2_{D^{1,2}} + o_n(1) \tag{3.25}$$

and

$$E_\lambda(u_n) = E_\lambda(u_0) + \sum_{j=1}^{k} E_\infty(u^j) + o_n(1). \tag{3.26}$$

Proof. Let us first consider the case $\lambda = 0$. By Lemma 2.6, there exists $u_0 \in D^{1,2}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u_0$ in $D^{1,2}$ and u_0 is a critical point of E_λ. Suppose that $u_n \not\to u_0$ in $D^{1,2}(\mathbb{R}^N)$ and let $(z_n^1) \subset D^{1,2}(\mathbb{R}^N)$ be the sequence given by $z_n^1 = u_n - u_0$. Then, $z_n^1 \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^N)$ and $z_n^1 \not\to 0$ in $D^{1,2}(\mathbb{R}^N)$. Arguing as in the proof of Lemma 3.1, we obtain

$$E_\infty(z_n^1) = E_0(u_n) - E_0(u_0) + o_n(1) \tag{3.27}$$

and

$$E'_\infty(z_n^1) = E'_0(u_n) - E'_0(u_0) + o_n(1) \ \text{ in } \ (D^{1,2}(\mathbb{R}^N))'. \tag{3.28}$$

Then, from (3.27) and (3.28), we see that (z_n^1) is a $(PS)_{c_1}$ sequence for E_∞. Hence, by Lemma 3.1, there are sequences $(\sigma_{n,1}) \subset \mathbb{R}$, $(y_{n,1}) \subset \mathbb{R}^N$ and a nontrivial solution $u^1 \in D^{1,2}(\mathbb{R}^N)$ for problem (P_∞) such that

$$v_n^1(x) := \sigma_{n,1}^{(N-2)/2} z_n^1\bigg(\sigma_{n,1}x + y_{n,1}\bigg) \rightharpoonup u_0^1 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N). \tag{3.29}$$

Since

$$\|v_n^1\| = \|u_n^1\| \ \text{ and } \int_{\mathbb{R}^N}(I_\mu * |v_n^1|^{2_\mu^*})|v_n^1|^{2_\mu^*}dx = \int_{\mathbb{R}^N}(I_\mu * |z_n^1|^{2_\mu^*})|z_n^1|^{2_\mu^*}dx, \tag{3.30}$$

it is easy to show that

$$E_\infty(v_n^1) = E_\infty(z_n^1) \quad \text{and} \quad E'_\infty(v_n^1) \to 0 \ \text{ in } \ (D^{1,2}(\mathbb{R}^N))', \tag{3.31}$$

hence (v_n^1) is also a $(PS)_{c_1}$ for E_∞.

Setting $z_n^2 := v_n^1 - u^1$, we derive that

$$E_\infty(v_n^1) = E_\infty(z_n^2) + E_\infty(u^1) + o_n(1)$$

$$E'_\infty(v_n^1) = E'_\infty(z_n^2) + E'_\infty(u^1) + o_n(1) \ \text{ in } \ (D^{1,2}(\mathbb{R}^N))',$$

and

$$\|v_n^1\|^2 = \|z_n^2\|^2 + \|u^1\|^2 + o_n(1).$$

Thus,

$$E_0(u_n) = E_0(u_0) + E_\infty(z_n^1) + o_n(1) = E_0(u_0) + E_\infty(v_n^1) + o_n(1) = E_0(u_0) + E_\infty(u^1) + E_\infty(z_n^2) + o_n(1)$$

and

$$\|u_n\|^2 = \|u_0\|^2 + \|z_n^1\|^2 + o_n(1) = \|u_0\|^2 + \|v_n^1\|^2 + o_n(1) = \|u_0\|^2 + \|u^1\|^2 + \|z_n^2\|^2 + o_n(1).$$

Arguing as above, we have that (z_n^2) is a $(PS)_{c_2}$ sequence for E_∞, that is,

$$E_\infty(z_n^2) \to c_2 \quad \text{and} \quad E'_\infty(z_n^2) \to 0.$$

Therefore, there are sequences $(\sigma_{n,2}) \subset \mathbb{R}$, $(y_{n,2}) \subset \mathbb{R}^N$ and a nontrivial solution $u^2 \in D^{1,2}(\mathbb{R}^N)$ for problem (P_∞) such that

$$v_n^2(x) := \sigma_{n,1}^{(N-2)/2} z_n^1\bigg(\sigma_{n,1}x + y_{n,1}\bigg) \rightharpoonup u^2 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N). \tag{3.32}$$

It is possible to prove that (v_n^2) is a $(PS)_{c_2}$ sequence for E_∞, and fixing the sequence

$$z_n^3(x) := v_n^2(x) - u_0^2(x),$$

we will obtain

$$E_0(u_n) = E_0(u_0) + E_\infty(u^1) + E_\infty(u^2) + E_\infty(z_n^3) + o_n(1)$$

and

$$\|u_n\|^2 = \|u_0\|^2 + \|u^1\|^2 + \|u^2\|^2 + \|z_n^3\|^2 + o_n(1).$$

Arguing as above, we will find $u^1, u^2, ..., u^k$ nontrivial solutions for problem (P_∞) satisfying

$$\|u_n\|^2 = \|u_0\|^2 + \sum_{j=1}^{k} \|u^j\|^2 + \|z_n^{k+1}\|^2 + o_n(1) \tag{3.33}$$

and

$$E_0(u_n) = E_0(u_0) + \sum_{j=1}^{k} E_\infty(u^j) + E_\infty(z_n^{k+1}) + o_n(1) \tag{3.34}$$

for the corresponding sequence (z_n^{k+1}) in $D^{1,2}(\mathbb{R}^N)$.

From the definition of $S_{H,L}$,

$$\left(\int_{\mathbb{R}^N} (I_\mu * |u^j|^{2_\mu^*})|u^j|^{2_\mu^*} dx\right)^{1/2_\mu^*} S_{H,L} \leq \|u^j\|^2, \quad j = 1, 2, ..., k. \tag{3.35}$$

Since u^j is nontrivial solution of (P_∞), for all $j = 1, 2, ..., k$, we have

$$\|u^j\|^2 = \int_{\mathbb{R}^N} (I_\mu * |u^j|^{2_\mu^*})|u^j|^{2_\mu^*} dx.$$

Hence,

$$\|u^j\|^2 \geq S_{H,L}^{(2N-\mu)/(N+2-\mu)}, \quad j = 1, 2, ..., k. \tag{3.36}$$

From (3.33) and (3.36),

$$\|z_n^{k+1}\|^2 = \|u_n\|^2 - \|u_0\|^2 - \sum_{j=1}^{k} \|u^j\|^2 + o_n(1) \leq \|u_n\|^2 - \|u_0\|^2 - k\, S_{H,L}^{(2N-\mu)/(N+2-\mu)} + o_n(1). \tag{3.37}$$

Since (u_n) is bounded in $D^{1,2}(\mathbb{R}^N)$, for k sufficient large, we conclude that $z_n^{k+1} \to 0$ in $D^{1,2}(\mathbb{R}^N)$, this proves the case $\lambda = 0$.

Let us now consider the case $\lambda > 0$. As in the previous case, up to a subsequence $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, where $u_0 \in H^1(\mathbb{R}^N)$ is a solution of (P_λ). We can also assume that $u_n(x) \to u_0(x)$ a.e. in $\mathbb{R}^N$, and $u_n \to u_0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$.

Let us define $z_n^1 := u_n - u_0$. Arguing as in the case $\lambda = 0$, it follows that

$$E_{\lambda,\infty}(z_n^1) = E_\lambda(u_n) - E_\lambda(u_0) + o_n(1) \tag{3.38}$$

and

$$E'_{\lambda,\infty}(z_n^1) = E'_\lambda(u_n) - E'_\lambda(u_0) + o_n(1) \tag{3.39}$$

where

$$E_{\lambda,\infty}(u) := E_\infty(u) + \frac{\lambda}{2}\int_{\mathbb{R}^N} u^2 dx, \qquad u \in H^1(\mathbb{R}^N).$$

Thus, it follows that (z_n^1) is a (PS) sequence for $E_{\lambda,\infty}$ and

$$z_n^1 \to \quad \text{weakly in } H^1(\mathbb{R}^N), \text{ a.e. in } \mathbb{R}^N \text{ and in } L^2_{\text{loc}}(\mathbb{R}^N). \tag{3.40}$$

If $z_n^1 \to 0$ strongly in $H^1(\mathbb{R}^N)$, we are done, so let us assume the existence of $c > 0$ such that

$$\|z_n^1\| \geq c > 0. \tag{3.41}$$

Observe that (3.41) implies the existence of a positive constant $\tilde{c} > 0$ such that

$$|z_n^1|_{2^*}^{2^*} \geq \tilde{c} > 0.$$

Indeed, if this would be not the case, from

$$E'_{\lambda,\infty}(z_n^1)z_n^1 = \int_{\mathbb{R}^N} (|\nabla z_n^1|^2 + \lambda |z_n^1|^2) dx - \int_{\mathbb{R}^N} (I_\mu * |z_n^1|^{2_\mu^*})|z_n^1|^{2_\mu^*} dx = o_n(1), \tag{3.42}$$

and (2.2) we get at once $\|z_n^1\| \to 0$, as $n \to \infty$, contrary to (3.41).

Observe that

$$d_n^1 := \max_{i\in\mathbb{N}} \|z_n^1\|_{L^{2^*}(Q_i)} > C > 0, \tag{3.43}$$

where Q_i, $i \in \mathbb{N}$, are hypercubes with disjoint interior and unitary sides such that $\mathbb{R}^N = \sum_{i\in\mathbb{N}} Q_i$, because

$$\begin{aligned} 0 < \tilde{c} \quad &\le \quad |z_n^1|_{2^*}^{2^*} = \sum_{i=1}^{\infty} |z_n^1|_{L^{2^*}(Q_i)}^{2^*} \le (d_n^1)^{2^*-2} \sum_{i=1}^{\infty} |z_n^1|_{L^{2^*}(Q_i)}^2 \\ &\le \quad (d_n^1)^{2^*-2} c_1 \sum_{i=1}^{\infty} \|z_n^1\|_{H_1(Q_i)}^2 \le c_2 (d_n^1)^{2^*-2}. \end{aligned}$$

For every $n \in \mathbb{N}$, let y_n^1 be the center of an hypercube where d_n is attained and define

$$\tilde{z}_n^1 := z_n^1(\cdot + y_n^1). \tag{3.44}$$

It turns out that $(\tilde{z}_n^1)$ is a (PS) sequence for $E_{\lambda,\infty}$, bounded in $H^1(\mathbb{R}^N)$. Let us call z the weak limit in $H^1(\mathbb{R}^N)$ of $(\tilde{z}_n^1)$, up to a subsequence. Since $|y_n| \to \infty$ by (3.40), z solves $(P_{\lambda,\infty})$, so that by Theorem 2.2 we get $z = 0$.

Now we can argue exactly as in [41, Lemma 3.3], and, taking into account of $|\tilde{z}_n^1|_{L^{2^*}(Q_0)} > C > 0$ and of the other information we get, we find a bounded sequence of points (x_n) in $\mathbb{R}^N$, an infinitesimal sequence (σ_n) in $(0, +\infty)$ and a nontrivial solution u^1 of (P_∞) such that, if we define the sequence (z_n^2) in $H^1(\mathbb{R}^N)$ by

$$z_n^2(x) := \tilde{z}_n^1(x) - \varphi\left(\frac{x - x_n}{\sigma_n^{1/2}}\right) \frac{1}{\sigma^{\frac{N-2}{2}}} u^1\left(\frac{x - x_n}{\sigma_n}\right) \rightharpoonup 0 \qquad \text{in } H^1(\mathbb{R}^N), \tag{3.45}$$

where $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ is a cut-off function such that $\varphi \equiv 1$ on $B_1(0)$, then (z_n^2) is a (PS) sequence for $E_{\lambda,\infty}$, in $H^1(\mathbb{R}^N)$, that verifies

$$z_n^2 \rightharpoonup 0 \qquad \text{in } H^1(\mathbb{R}^N),$$

and

$$E_{\lambda,\infty}(z_n^2) = E_{\lambda,\infty}(\tilde{z}_n^1) - E_\infty(u^1) + o_n(1). \tag{3.46}$$

We observe that in order to get (3.46) it is crucial that $\sigma_n \to 0$, which implies

$$\int_{\mathbb{R}^N} (z_n^2)^2 dx = \int_{\mathbb{R}^N} (\tilde{z}_n^1)^2 dx + o(1) \tag{3.47}$$

by (3.45). By (3.47) we can also write

$$\|u_n\|_{H^1}^2 = \|u_0\|_{H^1}^2 + \|z_n^1\|_{H^1}^2 + o_n(1) = \|u_0\|_{H^1}^2 + \|u^1\|_{D^{1,2}}^2 + \|z_n^2\|_{H^1}^2 + o_n(1). \tag{3.48}$$

By (3.38) and (3.46) we have

$$E_\lambda(u_n) = E_\lambda(u_0) + E_\infty(u^1) + E_{\lambda,\infty}(z_n^2) + o_n(1). \tag{3.49}$$

Moreover, since (z_n^2) is a (PS) sequence for $E_{\lambda,\infty}$, it follows that $E_{\lambda,\infty}(z_n^2) \ge o_n(1)$, so that

$$E_\lambda(u_n) \ge E_\lambda(u_0) + E_\infty(u^1) + o_n(1).$$

If $z_n^2 \to 0$ in $H^1(\mathbb{R}^N)$ we are done, otherwise we can iterate the procedure. Taking into account that at every step k we get

$$E_\lambda(u_n) \ge E_{\lambda,\infty}(u_0) + \sum_{j=1}^{k} E_\infty(u^j) + o_n(1) \ge E_{\lambda,\infty}(u_0) + k\, \frac{(N+2-\mu)}{4N-2\mu}\, S_{H,L}^{(2N-\mu)/(N+2-\mu)} + o_n(1),$$

after a finite number of steps we reach a sequence (z_n^{k+1}) such that $z_n^{k+1} \to 0$ in $H^1(\mathbb{R}^N)$. Hence, we obtain (3.25) and (3.26) by iterating (3.48) and (3.49), that completes the proof.

□

An immediate consequence of the last theorem are the next two corollaries.

Corollary 3.3. *Let (u_n) be a $(PS)_c$ sequence for E_λ with $c \in \left(0, \dfrac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}\right)$. Then, up to a subsequence, (u_n) strongly converges in H.*

Proof. Since (u_n) is a (PS) sequence, (u_n) is bounded in H, and so, for some subsequence, it follows that $u_n \rightharpoonup u_0$ in H and $E'_\lambda(u_0) = 0$ for some $u_0 \in H$. Suppose, by contradiction, that

$$u_n \nrightarrow u_0 \quad \text{in} \quad H.$$

From Theorem 3.2, there are $k \in \mathbb{N}$ and nontrivial solutions $z_0^1, z_0^2, ..., z_0^k$ of problem (P_∞) such that,

$$\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{k} \|z_0^j\|^2$$

and

$$E_\lambda(u_n) \to E_\lambda(u_0) + \sum_{j=1}^{k} E_\infty(z_0^j).$$

Note that

$$\begin{aligned} E_\lambda(u_0) &= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_\lambda(x) u_0^2 dx - \frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N} (I_\mu * |u_0|^{2_\mu^*})|u_0|^{2_\mu^*} dx \\ &= \left(\frac{1}{2} - \frac{1}{2\cdot 2_\mu^*}\right)\int_{\mathbb{R}^N} (I_\mu * |u_0|^{2_\mu^*})|u_0|^{2_\mu^*} dx \geq 0. \end{aligned}$$

Then,

$$\begin{aligned} c &= E_\lambda(u_0) + \sum_{j=1}^{k} E_\infty(z_0^j) \geq \sum_{j=1}^{k} E_\infty(z_0^j) \\ &\geq k\frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)} \geq \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}, \end{aligned}$$

which is a contradiction with $c \in \left(0, \dfrac{N+2-\mu}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}\right)$. □

Corollary 3.4. *The functional $E_\lambda : H \to \mathbb{R}$ satisfies the Palais-Smale condition in the range $\left(\dfrac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}, 2^{\frac{4-\mu}{N+2-\mu}} \dfrac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}\right)$.*

Proof. Let (u_n) be a sequence in H that satisfies

$$E_\lambda(u_n) \to c \quad \text{and} \quad E'_\lambda(u_n) \to 0.$$

Since (u_n) is bounded, up to a subsequence, we have $u_n \rightharpoonup u_0$ in H, moreover $E_\lambda(u_0) \geq 0$. Suppose by contradiction that

$$u_n \nrightarrow u_0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N).$$

From Theorem 3.2, there are $k \in \mathbb{N}$ and nontrivial solutions $z_0^1, z_0^2, ..., z_0^k$ of problem (P_∞) such that

$$\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{k} \|z_0^j\|^2$$

and

$$E_\lambda(u_n) \to c = E_\lambda(u_0) + \sum_{j=1}^{k} E_\infty(z_0^j).$$

The above information ensures that $u_0 \neq 0$. Since $E_\lambda(u_0) \geq 0$, then $k = 1$ and z_0^1 cannot change of sign, because otherwise, by Lemma 2.3,

$$E_\infty(z_0^j) \geq 2^{\frac{4-\mu}{N+2-\mu}} \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)},$$

which leads to a contradiction. Thereby, as z_0^1 has definite sign, $z_0^1 = U_{\sigma,z}$ for suitable $\sigma > 0$ and $z \in \mathbb{R}^N$ and, by (2.7),

$$E_\infty(z_0^j) = \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

On the other hand, by a direct computation,

$$E_\lambda(u_0) \geq \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

Hence,

$$c = E_\lambda(u_0) + E_\infty(z_0^1) \geq 2\frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)} > 2^{\frac{4-\mu}{N+2-\mu}} \frac{(N+2-\mu)}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)},$$

obtaining again a contradiction. This proves the result. □

The next results provide us the (PS) condition for the functional J_λ. The first one is a direct computation and we omit its proof, the second one is an immediate consequence of the study made above.

Lemma 3.5. *Let $(u_n) \subset \mathcal{M}$ be a sequence that satisfies*

$$J_\lambda(u_n) \to c \quad \textit{and} \quad J'_\lambda|_{\mathcal{M}}(u_n) \to 0.$$

Then, the sequence $v_n = c^{(N-2)/(2N-2\mu+4)} u_n$ satisfies the following limits.

$$E_\lambda(v_n) \to \frac{(N+2-\mu)}{4N-2\mu} c^{(2N-\mu)/(N+2-\mu)} \quad \textit{and} \quad E'_\lambda(v_n) \to 0.$$

Corollary 3.6. *Suppose that there are a sequence $(u_n) \subset \mathcal{M}$ and*

$$c \in (S_{H,L}, 2^{\frac{4-\mu}{2N-\mu}} S_{H,L})$$

such that

$$J_\lambda(u_n) \to c \quad \textit{and} \quad J'_\lambda|_{\mathcal{M}}(u_n) \to 0.$$

Then

a) *there exists $u_0 \in \mathcal{M}$ such that, up to a subsequence, $u_n \to u_0$ in $D^{1,2}(\mathbb{R}^N)$ and u_0 is a critical point for J_λ constrained on $\mathcal{M}$;*

b) *E_λ has a critical point $v_0 \in H$ with $E_\lambda(v_0) = \dfrac{(N+2-\mu)}{4N-2\mu} c^{(2N-\mu)/(N+2-\mu)}$.*

4 Main tools and basic estimates

We are looking for solutions of problem (P_λ) as critical points of the functional J_λ constrained on $\mathcal{M}$, up to a multiplier. Next proposition shows that the problem cannot be solved by minimization, so no ground state solution exists.

Proposition 4.1. *Set*

$$m := \inf\{J_\lambda(u) : u \in \mathcal{M}\}. \tag{4.1}$$

Then

$$m = S_{H,L}$$

and the minimization problem (4.1) has no solution.

Proof. Let $u \in \mathcal{M}$ be arbitrarily chosen. Then, by (V_1) we get

$$J_\lambda(u) \geq S_{H,L},$$

which implies

$$m \geq S_{H,L}.$$

In order to show the opposite inequality, let us consider the sequence

$$\tilde{\Psi}_n(x) = \xi(|x|)U_{\frac{1}{n},0}(x),$$

where $\xi \in C_0^\infty((0,+\infty),[0,1])$ is such that $\xi(s) = 1$ is $s \in [0,1/2]$ and $\xi(s) = 0$ is $s \geq 1$. Using (2.6) together with the definition of $\tilde{\Psi}_n$, we have

$$\int_{\mathbb{R}^N} |\nabla \tilde{\Psi}_n|^2 dx = \int_{\mathbb{R}^N} |\nabla U_{\frac{1}{n},0}|^2 dx + o_n(1), \tag{4.2}$$

$$\int_{\mathbb{R}^N} (I_\mu * |\tilde{\Psi}_n|^{2^*_\mu})|\tilde{\Psi}_n|^{2^*_\mu} dx = \int_{\mathbb{R}^N} (I_\mu * |U_{\frac{1}{n},0}|^{2^*_\mu})U_{\frac{1}{n},0}|^{2^*_\mu} dx + o_n(1) \tag{4.3}$$

and

$$\lambda \int_{\mathbb{R}^N} |\tilde{\Psi}_n|^2 dx = o_n(1). \tag{4.4}$$

On the other hand, for all $\rho > 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} V_0(x)|\tilde{\Psi}_n|^2 dx &= \int_{B_\rho(0)} V_0(x)|\tilde{\Psi}_n|^2 dx + \int_{\mathbb{R}^N\setminus B_\rho(0)} V_0(x)|\tilde{\Psi}_n|^2 dx \\
&\leq |\tilde{\Psi}_n|^2_{L^{2^*}(\mathbb{R}^N)} \left(\int_{B_\rho(0)} |V_0(x)|^{N/2} dx\right)^{2/N} \\
&+ |V_0|_{L^{N/2}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N\setminus B_\rho(0)} |\tilde{\Psi}_n(x)|^{2^*} dx\right)^{2/2^*}.
\end{aligned}$$

Now, recalling that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N\setminus B_\rho(0)} |\tilde{\Psi}_n(x)|^{2^*} dx = 0,$$

$$\sup_{n\in\mathbb{N}} |\tilde{\Psi}_n|_{2^*} < +\infty$$

and

$$\lim_{\rho\to 0}\int_{B_\rho(0)} |V_0(x)|^{N/2}dx = 0,$$

we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V_0(x)|\tilde{\Psi}_n(x)|^2 dx = 0. \tag{4.5}$$

Now, if we define

$$\hat{\Psi}_n(x) = \frac{1}{\int_{\mathbb{R}^N}(I_\mu * |\tilde{\Psi}_n|^{2^*_\mu})|\tilde{\Psi}_n|^{2^*_\mu}dx}\,\tilde{\Psi}_n(x)$$

then $\hat{\Psi}_n \in \mathcal{M}$, $\forall n \in \widetilde{N}$, and from (4.2) – (4.5),

$$\lim_{n\to\infty} J_\lambda(\hat{\Psi}_n(x)) = S_{H,L},$$

which concludes the first part of the proof. Now suppose that the minimization problem (4.1) has a solution u^*. Then

$$S_{H,L} \le \frac{\displaystyle\int_{\mathbb{R}^N}|\nabla u^*|^2dx}{\left(\displaystyle\int_{\mathbb{R}^N}(I_\mu * |u|^{2^*_\mu})|u|^{2^*_\mu}dx\right)^{1/2^*_\mu}} \le \frac{\displaystyle\int_{\mathbb{R}^N}[|\nabla u^*|^2 + V_\lambda(x)|u^*|^2]dx}{\left(\displaystyle\int_{\mathbb{R}^N}(I_\mu * |u^*|^{2^*_\mu})|u^*|^{2^*_\mu}dx\right)^{1/2^*_\mu}} = S_{H,L}.$$

The above relation implies that $\int_{\mathbb{R}^N} V_\lambda(x)|u^*|^2dx = 0$ and $u^* = U_{\sigma,z}$ for some $\sigma > 0$ and $z \in \mathbb{R}^N$. Thus, using the assumptions on V_λ and the fact that $U_{\sigma,z} > 0$ for all $x \in \mathbb{R}^N$, we deduce

$$0 = \int_{\mathbb{R}^N} V_\lambda(x)|u^*|^2dx = \int_{\mathbb{R}^N} V_\lambda(x)|U_{\sigma,z}|^2dx > 0,$$

which is impossible.

□

In view of the previous proposition, the main goal of this section will be to introduce some tools and to establish some basic estimates in oder to find bound state solutions in the next section. To begin with, let us introduce a barycenter type map $\beta : H \setminus \{0\} \to \mathbb{R}^N$ given by

$$\beta(u) = \frac{1}{|u|^{2^*}_{2^*}}\int_{\mathbb{R}^N}\frac{x}{1+|x|}|u|^{2^*}dx$$

and a kind of inertial momentum $\gamma : H \setminus \{0\} \to \mathbb{R}$ given by

$$\gamma(u) = \frac{1}{|u|^{2^*}_{2^*}}\int_{\mathbb{R}^N}\left|\frac{x}{1+|x|} - \beta(u)\right||u|^{2^*}dx.$$

It is readily seen that the maps β and γ are continuous and, moreover, $\beta(tu) = \beta(u)$ and $\gamma(tu) = \gamma(u)$, for all $t \in \mathbb{R}$ and for all $u \in H \setminus \{0\}$.

Lemma 4.2. *Let $\lambda \ge 0$ and define*

$$\mathcal{B}_{V_\lambda} = \inf\left\{J_\lambda(u) : u \in \mathcal{M}, \beta(u) = 0, \gamma(u) = \frac{1}{2}\right\}.$$

Then the following inequalities hold

$$\mathcal{B}_{V_\lambda} > S_{H,L}, \quad \text{for all} \;\; \lambda \ge 0.$$

Proof. By Proposition 4.1,

$$\inf\left\{J_\lambda(u) : u \in \mathcal{M}, \beta(u) = 0, \gamma(u) = \frac{1}{2}\right\} \geq S_{H,L}.$$

Now suppose, by contradiction, that the equality is true. Then, there exists a sequence $(u_n) \subset H$ such that

$$\begin{cases} (a) \quad u_n \in \mathcal{M}, \quad \beta(u_n) = 0, \quad \gamma(u_n) = \dfrac{1}{2}; \\ (b) \quad \lim\limits_{n\to+\infty} J_\lambda(u_n) = S_{H,L}. \end{cases} \tag{4.6}$$

Note that

$$S_{H,L} = J_\lambda(u_n) + o_n(1) \geq \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + o_n(1) \geq S_{H,L} + o_n(1),$$

then

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + o_n(1) = S_{H,L}. \tag{4.7}$$

By Lemma 2.4 and [8, Theorem 2.5], we get

$$u_n(x) = \Theta\, U_{\sigma_n,y_n}(x) + \epsilon_n(x) \tag{4.8}$$

for some positive constant Θ with U_{σ_n,y_n} as in (2.4), $\sigma_n > 0$, $y_n \in \mathbb{R}^N$ and $\epsilon_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$. In order to get the constant $\Theta > 0$, we recall that $|U_{\sigma_n,y_n}|_{2^*} = |U_{1,0}|_{2^*}$ and that by (4.7) the sequence (u_n) is bounded in $D^{1,2}(\mathbb{R}^N)$, and so, we can assume that for some subsequence $\lim\limits_{n\to+\infty} |u_n|_{2^*} = L_1$. Moreover, since $u_n \in \mathcal{M}$, we must have $L_1 > 0$.

We claim that $\lim\limits_{n\to\infty} \sigma_n = \overline{\sigma} > 0$ and $\lim\limits_{n\to\infty} y_n = \overline{y}$ in $\mathbb{R}^N$. Let us first show that (σ_n) is bounded. In fact, if for some subsequence, still denoted by (σ_n), $\lim\limits_{n\to+\infty} \sigma_n = +\infty$ occurs, then for all $\rho > 0$, we have

$$\lim_{n\to+\infty} \int_{B_\rho(0)} |u_n|^{2^*} dx \quad = \quad \Theta^{2^*} \lim_{n\to+\infty} \int_{B_\rho(0)} |U_{\sigma_n,y_n}|^{2^*} dx = 0.$$

Since $\beta(u_n) = 0$, for all $\rho > 0$,

$$\begin{aligned} \gamma(u_n) &= \frac{1}{|u_n|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \frac{|x|}{1+|x|} |u_n|^{2^*} dx \\ &= \frac{1}{|u_n|_{2^*}^{2^*}} \int_{\mathbb{R}^N \setminus B_\rho(0)} \frac{|x|}{1+|x|} |u_n|^{2^*} dx \\ &\quad + \frac{1}{|u_n|_{2^*}^{2^*}} \int_{B_\rho(0)} \frac{|x|}{1+|x|} |u_n|^{2^*} dx \\ &\geq \frac{\rho}{1+\rho} + o_n(1), \end{aligned}$$

so

$$\liminf_{n\to+\infty} \gamma(u_n) \geq \frac{\rho}{1+\rho}, \quad \forall \rho > 0,$$

and then

$$\liminf_{n\to+\infty} \gamma(u_n) \geq 1, \tag{4.9}$$

obtaining, therefore, a contradiction. Thus, (σ_n) is bounded and we can assume that

$$\lim_{n\to+\infty} \sigma_n = \overline{\sigma} \ \text{ with } \ \overline{\sigma} \geq 0.$$

We claim that $\overline{\sigma}$ is positive. In fact, if $\overline{\sigma} = 0$, for all $\rho > 0$ we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N\setminus B_\rho(y_n)} |u_n|^{2^*} dx \quad = \quad \Theta^{2^*} \lim_{n\to+\infty} \int_{\mathbb{R}^N\setminus B_\rho(y_n)} |U_{\sigma_n,y_n}|^{2^*} dx = 0.$$

As $\beta(u_n) = 0$, we get

$$\begin{aligned} \frac{|y_n|}{1+|y_n|} &= \left|\frac{y_n}{1+|y_n|} - \beta(u_n)\right| = \frac{1}{|u_n|_{2^*}^{2^*}} \left|\int_{\mathbb{R}^N} \left(\frac{y_n}{1+|y_n|} - \frac{x}{1+|x|}\right) |u_n|^{2^*} dx\right| \\ &\leq \frac{1}{|u_n|_{2^*}^{2^*}} \left|\int_{B_\rho(y_n)} \left(\frac{y_n}{1+|y_n|} - \frac{x}{1+|x|}\right) |u_n|^{2^*} dx\right| \\ &\quad + \frac{1}{|u_n|_{2^*}^{2^*}} \left|\int_{\mathbb{R}^N\setminus B_\rho(y_n)} \left(\frac{y_n}{1+|y_n|} - \frac{x}{1+|x|}\right) |u_n|^{2^*} dx\right| \leq \rho + o_n(1). \end{aligned} \tag{4.10}$$

Hence,

$$\limsup_{n\to+\infty} \frac{|y_n|}{1+|y_n|} \leq \rho, \quad \forall \rho > 0,$$

from which it follows

$$\lim_{n\to+\infty} |y_n| = 0.$$

On the other hand, by the same calculus performed in (4.10),

$$\begin{aligned} \lim_{n\to+\infty} \gamma(u_n) &= \lim_{n\to+\infty} \frac{1}{|u_n|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \left|\frac{x}{1+|x|} - \beta(u_n)\right| |u_n|^{2^*} dx \\ &= \lim_{n\to+\infty} \frac{1}{|u_n|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \left|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\right| |u_n|^{2^*} dx = 0, \end{aligned}$$

which is a contradiction.

Now, we are able to prove that (y_n) is bounded. For this, suppose by contradiction, that there is a subsequence, still denoted by (y_n), such that

$$\lim_{n\to+\infty} |y_n| = +\infty.$$

Then, for all $\epsilon > 0$, there is $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$|x - y_n| < R \Rightarrow \left|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\right| < \epsilon, \quad \forall n \geq n_0 \tag{4.11}$$

and

$$\int_{\mathbb{R}^N\setminus B_R(y_n)} |u_n|^{2^*} dx = \Theta^{2^*} \int_{\mathbb{R}^N\setminus B_R(y_n)} |U_{\sigma_n,y_n}|^{2^*} dx + o_n(1) < \epsilon. \tag{4.12}$$

From (4.11) and (4.12),

$$\begin{aligned} \left|\beta(u_n) - \frac{y_n}{1+|y_n|}\right| &\leq \frac{1}{|u_n|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \left|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\right| |u_n|^{2^*} dx \\ &= \frac{1}{|u_n|_{2^*}^{2^*}} \int_{B_R(y_n)} \left|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\right| |u_n|^{2^*} dx \\ &\quad + \frac{1}{|u_n|_{2^*}^{2^*}} \int_{\mathbb{R}^N\setminus B_R(y_n)} \left|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\right| |u_n|^{2^*} dx \\ &\leq \epsilon + 2\,\frac{\epsilon}{|u_n|_{2^*}^{2^*}} + o_n(1) \end{aligned}$$

which implies

$$\lim_{n\to+\infty}|\beta(u_n)|=1,$$

which again leads us to a contradiction. Therefore, (y_n) is bounded and we can assume that

$$\lim_{n\to+\infty} y_n=\overline{y}.$$

Then

$$\begin{aligned} S_{H,L} &= \lim_{n\to\infty}\left[\int_{\mathbb{R}^N}(|\nabla u_n|^2+V_\lambda(x)u_n^2)dx\right] \\ &= \Theta^2\left[\int_{\mathbb{R}^N}(|\nabla U_{\overline{\sigma},\overline{y}}|^2+V_\lambda(x)|U_{\overline{\sigma},\overline{y}}|^2)dx\right] \\ &> \Theta^2\int_{\mathbb{R}^N}|\nabla U_{\overline{\sigma},\overline{y}}|^2dx=S_{H,L}, \end{aligned}$$

which is an absurd. □

Lemma 4.3. *If $\lambda>0$, then*

$$\Upsilon_\lambda:=\inf\{J_\lambda(u):u\in\mathcal{M},\beta(u)=0,\gamma(u)\geq 1/2\}>S_{H,L}. \tag{4.13}$$

Proof. We start observing that

$$\Upsilon_\lambda=\inf\{J_\lambda(u):u\in\mathcal{M},\beta(u)=0,\gamma(u)\geq 1/2\}\geq S_{H,L}.$$

Now suppose, by contradiction, that the equality is true. Then, there exists a sequence (u_n) such that

$$\begin{cases}(a) & u_n\in\mathcal{M},\quad \beta(u_n)=0,\quad \gamma(u_n)\geq 1/2;\\ (b) & \lim\limits_{n\to+\infty}J_\lambda(u_n)=S_{H,L}.\end{cases} \tag{4.14}$$

Then, the same computations made in Lemma 4.2 allow to assert that

$$u_n(x)=\Theta\, U_{\sigma_n,y_n}(x)+\epsilon_n(x) \tag{4.15}$$

with $\Theta>0$, $\sigma_n>0$, $y_n\in\mathbb{R}^N$, $\epsilon_n\to 0$ in $D^{1,2}(\mathbb{R}^N)$ verifying $\lim\limits_{n\to\infty}\sigma_n=\overline{\sigma}\in(0,+\infty]$ and $\lim\limits_{n\to\infty}y_n=\overline{y}$ in $\mathbb{R}^N$. Let us show that $\overline{\sigma}=+\infty$ cannot occur. If this would be the case, then

$$\begin{aligned} S_{H,L} &\geq \liminf_{n\to+\infty}\left[\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\lambda\int_{B_{\sqrt{\sigma_n}}(y_n)}|u_n|^2dx\right] \\ &\geq \left[S_{H,L}+\lambda\Theta^2\liminf_{n\to+\infty}\sigma_n\int_{B_1(0)}|U_{1,0}|^2dx\right] \\ &= +\infty \end{aligned}$$

that is a contradiction. So, we can assume that $\overline{\sigma}\in(0,\infty)$, and then

$$\begin{aligned} S_{H,L} &\geq \lim_{n\to\infty}\left[\int_{\mathbb{R}^N}(|\nabla u_n|^2+\lambda|u_n|^2)dx\right] \\ &\geq \Theta^2\left[\int_{\mathbb{R}^N}|\nabla U_{\overline{\sigma},\overline{y}}|^2\,dx+\lambda\overline{\sigma}\int_{B_{\overline{\sigma}}(\overline{y})}|U_{\overline{\sigma},\overline{y}}|^2dx\right] \\ &> \Theta^2\int_{\mathbb{R}^N}|\nabla U_{\overline{\sigma},\overline{y}}|^2dx=S_{H,L}, \end{aligned}$$

which is again a contradiction. □

Remark 4.4. *Testing the functional J_0 by the functions $\frac{1}{N}U_{n,0}$, $n \in \mathbb{N}$, it is readily seen that*

$$\inf\{J_0(u) : u \in \mathcal{M}, \beta(u) = 0, \gamma(u) \geq 1/2\} = S_{H,L}.$$

Let $a \in (0,1)$ be such that

$$|V_0|_{L^{N/2}(\mathbb{R}^N)} = S\left(2^{a\,\frac{4-\mu}{2N-\mu}} - 1\right) \tag{4.16}$$

and let us fix a number $\overline{c}$ such that

$$S_{H,L} < \overline{c} < \min\left(\frac{\mathcal{B}_{V_0} + S_{H,L}}{2}, 2^{(1-a)\,\frac{4-\mu}{2N-\mu}} S_{HL}\right). \tag{4.17}$$

Note that this interval is not empty by Lemma 4.2.

In the sequel, φ is a function that belongs to $C_0^\infty(B_1(0))$ and satisfyies the following properties:

$$\begin{cases} \varphi \in C_0^\infty(B_1(0)), \quad \varphi(x) > 0 \quad \forall x \in B_1(0), \\ \varphi \ \text{ is symmetric and } |x_1| < |x_2| \Rightarrow \varphi(x_1) > \varphi(x_2), \\ \varphi \in \mathcal{M} \ \text{ and } \displaystyle\int_{\mathbb{R}^N} |\nabla\varphi|^2 dx = \Sigma \in (S_{H,L}, \overline{c}). \end{cases} \tag{4.18}$$

For every $\sigma > 0$ and $y \in \mathbb{R}^N$, we set

$$\varphi_{\sigma,y}(x) = \begin{cases} \sigma^{-\frac{N-2}{2}}\varphi\left(\dfrac{x-y}{\sigma}\right), & x \in B_\sigma(y), \\ 0, & x \notin B_\sigma(y). \end{cases} \tag{4.19}$$

We remark that by the definition of $\varphi_{\sigma,y}$ and by variable change, it follows that for every $\sigma > 0$ and $y \in \mathbb{R}^N$

$$\int_{\mathbb{R}^N} |\nabla\varphi_{\sigma,y}|^2 dx = \int_{B_\sigma(y)} |\nabla\varphi_{\sigma,y}|^2 dx = \int_{B_1(0)} |\nabla\varphi|^2 dx, \tag{4.20}$$

$$\int_{\mathbb{R}^N} |\varphi_{\sigma,y}|^{2^*} dx = \int_{B_\sigma(y)} |\varphi_{\sigma,y}|^{2^*} dx = \int_{B_1(0)} |\varphi|^{2^*} dx \tag{4.21}$$

and

$$\int_{\mathbb{R}^N} (I_\mu * |\varphi_{\sigma,y}|^{2^*_\mu})\, |\varphi_{\sigma,y}|^{2^*_\mu} dx = \int_{\mathbb{R}^N} (I_\mu * \varphi^{2^*_\mu})\, \varphi^{2^*_\mu} dx = 1$$

so that, in particular,

$$\varphi_{\sigma,y} \in \mathcal{M} \quad \text{ and } \quad \int_{\mathbb{R}^N} |\nabla\varphi_{\sigma,y}|^2 dx = \Sigma \in (S_{H,L}, \overline{c}) \qquad \forall \sigma > 0 \ \text{ and } \ \forall y \in \mathbb{R}^N. \tag{4.22}$$

Lemma 4.5. *The following relations hold:*

$$\begin{cases} (a) \quad \displaystyle\lim_{\sigma\to 0} \sup\left\{\int_{\mathbb{R}^N} V_0(x)|\varphi_{\sigma,y}|^2 dx; \quad y \in \mathbb{R}^N\right\} = 0; \\ (b) \quad \displaystyle\lim_{\sigma\to+\infty} \sup\left\{\int_{\mathbb{R}^N} V_0(x)|\varphi_{\sigma,y}|^2\, dx; \quad y \in \mathbb{R}^N\right\} = 0; \\ (c) \quad \displaystyle\lim_{r\to+\infty} \sup\left\{\int_{\mathbb{R}^N} V_0(x)|\varphi_{\sigma,y}|^2\, dx; \quad |y| = r, \quad \sigma > 0, \quad y \in \mathbb{R}^N\right\} = 0. \end{cases} \tag{4.23}$$

Proof. Let $y \in \mathbb{R}^N$ be chosen arbitrarily. Then, by the Hölder inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} V_0(x)|\varphi_{\sigma,y}|^2\,dx &= \int_{B_\sigma(y)} V_0(x)|\varphi_{\sigma,y}|^2\,dx \le |V_0|_{L^{N/2}(B_\sigma(y))}|\varphi_{\sigma,y}|^2_{L^{2^*}(B_\sigma(y))}\\ &= |V_0|_{L^{N/2}(B_\sigma(y))}|\varphi|^2_{2^*}, \qquad \forall\sigma>0,\end{aligned}$$

hence

$$\sup\left\{\int_{\mathbb{R}^N} V_0(x)|\varphi_{\sigma,y}|^2\,dx;\quad y\in\mathbb{R}^N\right\} \le |\varphi|^2_{L^{2^*}(B_1(0))}\sup\left\{|V_0|_{L^{N/2}(B_\sigma(y))};\quad y\in\mathbb{R}^N\right\}. \tag{4.24}$$

Since

$$\lim_{\sigma\to 0}\sup_{y\in\mathbb{R}^N}|V_0|_{L^{N/2}(B_\sigma(y))}=0,$$

so (a) follows from (4.24).

To prove (b), we fix arbitrarily $y\in\mathbb{R}^N$ and note that by the Hölder inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} V_0|\varphi_{\sigma,y}|^2\,dx &= \int_{B_\rho(0)} V_0(x)|\varphi_{\sigma,y}|^2\,dx + \int_{\mathbb{R}^N\setminus B_\rho(0)} V_0(x)|\varphi_{\sigma,y}|^2\,dx\\ &\le |V_0|_{L^{N/2}(B_\rho(0))}|\varphi_{\sigma,y}|^2_{L^{2^*}(B_\rho(0))} + |V_0|_{L^{N/2}(\mathbb{R}^N\setminus B_\rho(0))}|\varphi_{\sigma,y}|^2_{L^{2^*}(\mathbb{R}^N\setminus B_\rho(0))}\\ &\le |V_0|_{N/2}\sup_{y\in\mathbb{R}^N}|\varphi_{\sigma,y}|^2_{L^{2^*}(B_\rho(0))} + |\varphi|^2_{2^*}|V_0|_{L^{N/2}(\mathbb{R}^N\setminus B_\rho(0))}, \qquad \forall\rho,\sigma>0.\end{aligned}$$

Using the fact that

$$\lim_{\sigma\to+\infty}\sup_{y\in\mathbb{R}^N}|\varphi_{\sigma,y}|_{L^{2^*}(B_\rho(0))}=0,$$

we get

$$\lim_{\sigma\to+\infty}\sup\left\{\int_{\mathbb{R}^N} V_0(x)|\varphi_{\sigma,y}|^2\,dx;\quad y\in\mathbb{R}^N\right\}\le|\varphi|^2_{2^*}\cdot|V_0|_{L^{N/2}(\mathbb{R}^N\setminus B_\rho(0))}.$$

Passing the limit of $\rho\to+\infty$ in the last inequality, we obtain (b).

To prove (c), we will assume by contradiction that there are sequences $(y_n)\subset\mathbb{R}^N$ and a sequence $(\sigma_n)\subset(0,+\infty)$ such that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} V_0(x)\varphi^2_{\sigma_n,y_n}dx = L>0 \quad\text{and}\quad |y_n|\to+\infty. \tag{4.25}$$

From (a) and (b), we can suppose that

$$\lim_{n\to+\infty}\sigma_n=\overline{\sigma}>0.$$

Using the hypotheses that $|y_n|\to+\infty$ and $V_0\in L^{N/2}(\mathbb{R}^N)$, the Lebesgue's Theorem leads to

$$\lim_{n\to+\infty}|V_0|_{L^{N/2}(B_{\sigma_n}(y_n))}=0,$$

from where it follows that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} V_0(x)\varphi^2_{\sigma_n,y_n}dx\le\lim_{n\to+\infty}|V_0|_{L^{N/2}(B_{\sigma_n}(y_n))}=0,$$

which contradicts (4.25). Therefore (c) occurs. □

Lemma 4.6. *The following relations hold:*

$$\begin{cases}(a)\quad \lim_{\sigma\to 0}\sup\left\{\gamma(\varphi_{\sigma,y});\quad y\in\mathbb{R}^N\right\}=0;\\ (b)\quad \lim_{\sigma\to+\infty}\inf\left\{\gamma(\varphi_{\sigma,y});\quad y\in\mathbb{R}^N,\quad |y|\le r\right\}=1,\quad\forall r>0;\\ (c)\quad (\beta(\varphi_{\sigma,y})|y)_{\mathbb{R}^N}>0;\quad\forall y\in\mathbb{R}^N\setminus\{0\},\quad\forall\sigma>0.\end{cases} \tag{4.26}$$

Here $(x|y)$ denotes the usual inner product in $\mathbb{R}^N$ of the vectors $x,y\in\mathbb{R}^N$.

Proof. Let $y \in \mathbb{R}^N$ be chosen arbitrarily. For any $\sigma > 0$, using the fact that $\varphi_{\sigma,y} \in \mathcal{M}$ and the definitions of β, γ, we find

$$\begin{aligned} 0 \quad &\leq \quad \gamma(\varphi_{\sigma,y}) = \frac{1}{|\varphi_{\sigma,y}|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \left| \frac{x}{1+|x|} - \beta(\varphi_{\sigma,y}) \right| |\varphi_{\sigma,y}|^{2^*} dx \\ &\leq \quad \frac{1}{|\varphi_{\sigma,y}|_{2^*}^{2^*}} \int_{B_\sigma(y)} \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| |\varphi_{\sigma,y}|^{2^*} dx + \left| \frac{y}{1+|y|} - \beta(\varphi_{\sigma,y}) \right|. \end{aligned} \tag{4.27}$$

Now, from (4.18) and (4.19),

$$\begin{aligned} \left| \frac{y}{1+|y|} - \beta(\varphi_{\sigma,y}) \right| \quad &= \quad \frac{1}{|\varphi_{\sigma,y}|_{2^*}^{2^*}} \left| \int_{\mathbb{R}^N} \left(\frac{y}{1+|y|} - \frac{x}{1+|x|} \right) |\varphi_{\sigma,y}|^{2^*} dx \right| \\ &\leq \quad \frac{1}{|\varphi_{\sigma,y}|_{2^*}^{2^*}} \int_{B_\sigma(y)} \left| \frac{y}{1+|y|} - \frac{x}{1+|x|} \right| |\varphi_{\sigma,y}|^{2^*} dx. \end{aligned} \tag{4.28}$$

Combining (4.27) with (4.28) we derive that

$$0 \leq \gamma(\varphi_{\sigma,y}) \quad \leq \quad \frac{2}{|\varphi_{\sigma,y}|_{2^*}^{2^*}} \int_{B_\sigma(y)} \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| |\varphi_{\sigma,y}|^{2^*} dx \leq 2\sigma.$$

Hence

$$0 \leq \sup \left\{ \gamma(\varphi_{\sigma,y}); \quad y \in \mathbb{R}^N \right\} \leq 2\sigma,$$

which gives (a) letting $\sigma \to 0$.

To prove (b), let us first show that for all $y \in \mathbb{R}^N$,

$$\lim_{\sigma \to +\infty} \sup_{|y| \leq r} |\beta(\varphi_{\sigma,y})| = 0. \tag{4.29}$$

Since $\varphi_{\sigma,0}$ is a symmetric function, we have $\beta(\varphi_{\sigma,0}) = 0$. This combined with the limit below

$$\lim_{\sigma \to +\infty} \sup_{|y| \leq r} |\varphi_{1,y/\sigma} - \varphi_{1,0}|_{2^*} = 0,$$

and the definition of β gives (4.29).

Now, fix $r > 0$ arbitrarily and let $y \in \mathbb{R}^N$ such that $|y| \leq r$. For any $\sigma > 0$, we see that

$$\begin{aligned} \gamma(\varphi_{\sigma,y}) \quad &= \quad \frac{1}{|\varphi_{\sigma,y}|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \left| \frac{x}{1+|x|} - \beta(\varphi_{\sigma,y}) \right| |\varphi_{\sigma,y}|^{2^*} dx \\ &\leq \quad \frac{1}{|\varphi_{\sigma,y}|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \frac{|x|}{1+|x|} |\varphi_{\sigma,y}|^{2^*} dx + |\beta(\varphi_{\sigma,y})| \\ &\leq \quad 1 + |\beta(\varphi_{\sigma,y})|, \end{aligned}$$

which together with (4.29) leads us to

$$\limsup_{\sigma \to +\infty} \left[\inf \left\{ \gamma(\varphi_{\sigma,y}); \quad y \in \mathbb{R}^N, \quad |y| \leq r \right\} \right] \leq 1. \tag{4.30}$$

If

$$\limsup_{\sigma \to +\infty} \left[\inf \left\{ \gamma(\varphi_{\sigma,y}); \quad y \in \mathbb{R}^N, \quad |y| \leq r \right\} \right] < 1,$$

there are sequences $(\sigma_n) \subset (0, +\infty)$ and $(y_n) \subset \mathbb{R}^N$ such that $\sigma_n \to +\infty$, $|y_n| \leq r$ and

$$\lim_{n \to +\infty} \gamma(\varphi_{\sigma_n, y_n}) < 1. \tag{4.31}$$

On the other hand, considering (4.29), for all $\rho > 0$ we deduce that

$$\begin{aligned} \gamma(\varphi_{\sigma_n,y_n}) &= \frac{1}{|\varphi_{\sigma_n,y_n}|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \left| \frac{x}{1+|x|} - \beta(\varphi_{\sigma_n,y_n}) \right| |\varphi_{\sigma_n,y_n}|^{2^*} dx \\ &\geq \frac{1}{|\varphi_{\sigma_n,y_n}|_{2^*}^{2^*}} \int_{\mathbb{R}^N} \frac{|x|}{1+|x|} |\varphi_{\sigma_n,y_n}|^{2^*} dx - |\beta(\varphi_{\sigma_n,y_n})| \\ &\geq \frac{1}{|\varphi_{\sigma_n,y_n}|_{2^*}^{2^*}} \int_{\mathbb{R}^N \setminus B_\rho(0)} \frac{|x|}{1+|x|} |\varphi_{\sigma_n,y_n}|^{2^*} dx - o_n(1) \\ &\geq \frac{\rho}{1+\rho} \frac{1}{|\varphi_{\sigma_n,y_n}|_{2^*}^{2^*}} \int_{\mathbb{R}^N \setminus B_\rho(0)} |\varphi_{\sigma_n,y_n}|^{2^*} dx - o_n(1) \\ &\geq \frac{\rho}{1+\rho} \frac{1}{|\varphi_{1,0}|_{2^*}^{2^*}} \int_{\mathbb{R}^N \setminus B_{\frac{\rho}{\sigma_n}}(-y_n/\sigma_n)} |\varphi_{1,0}|^{2^*} dx - o_n(1), \end{aligned}$$

hence

$$\lim_{n\to+\infty} \gamma(\varphi_{\sigma_n,y_n}) \geq \frac{\rho}{1+\rho}, \quad \forall \rho > 0.$$

From this, since $\rho > 0$ is arbitrarily,

$$\lim_{n\to+\infty} \gamma(\varphi_{\sigma_n,y_n}) \geq 1,$$

which contradicts (4.31). Thus, the equality in (4.30) holds and the proof of (b) is finished.

Now, we will prove (c). We note that if $0 \notin B_\sigma(y)$, we have $(x|y) > 0$ $\forall x \in B_\sigma(y)$ and thus

$$(\beta(\varphi_{\sigma,y})|y) = \int_{\mathbb{R}^N} \frac{(x|y)}{1+|x|} |\varphi_{\sigma,y}|^{2^*} dx > 0.$$

If $0 \in B_\sigma(y)$, for each $x \in B_\sigma(y)$ such that $(x|y) < 0$, the point $-x$ belongs to $B_\sigma(y)$, $(-x|y) > 0$ and $\varphi_{\sigma,y}(-x) > \varphi_{\sigma,y}(x)$, which is enough to prove that $(\beta(\varphi_{\sigma,y})|y) > 0$, as desired. □

Corollary 4.7. *There exist $r > 0$ and $0 < \sigma_1 < \frac{1}{2} < \sigma_2$ such that*

(a) $\gamma(\varphi_{\sigma_1,y}) < \dfrac{1}{2}, \quad \forall y \in \mathbb{R}^N,$

(b) $\gamma(\varphi_{\sigma_2,y}) > \dfrac{1}{2}, \quad \forall y \in \mathbb{R}^N$ *with* $|y| \leq r,$

and

$$\sup\{J_0(\varphi_{\sigma_1,y}));\quad (\sigma,y) \in \partial\mathcal{H}\} < \overline{c}, \tag{4.32}$$

where

$$\mathcal{H} = [\sigma_1, \sigma_2] \times B_r(0). \tag{4.33}$$

Points (a) and (b) follow from points (a) and (b) of Lemma 4.6, respectively, while (4.32) is a consequence of (4.22) and Lemma 4.5.

Lemma 4.8. *Let $\mathcal{H}$ be the set defined in* (4.33). *Then, there exist $(\tilde{\sigma}, \tilde{y}) \in \partial\mathcal{H}$ and $(\overline{\sigma}, \overline{y}) \in int\,\mathcal{H}$ satisfying*

$$\beta(\varphi_{\tilde{\sigma},\tilde{y}}) = 0 \ \text{ and } \ \gamma(\varphi_{\tilde{\sigma},\tilde{y}}) > \frac{1}{2} \tag{4.34}$$

and

$$\beta(\varphi_{\overline{\sigma},\overline{y}}) = 0 \ \text{ and } \ \gamma(\varphi_{\overline{\sigma},\overline{y}}) = \frac{1}{2}. \tag{4.35}$$

Proof. First of all, note that by the symmetry of φ, we have $\beta(\varphi_{\sigma,0}) = 0$, $\forall \sigma > 0$. Then $(\tilde{\sigma}, \tilde{y}) := (\sigma_2, 0)$ verifies (4.34), by Corollary 4.7.

In order to get (4.35) it is sufficient to consider also that $\sigma \mapsto \gamma(\varphi_{\sigma,0})$ is a continuous map and that $\sigma \mapsto \gamma(\varphi_{\sigma_1,0}) < \frac{1}{2}$ while $\sigma \mapsto \gamma(\varphi_{\sigma_2,0}) > \frac{1}{2}$ by Corollary 4.7.

□

Lemma 4.9. *Let $g : \mathcal{H} \to \mathbb{R} \times \mathbb{R}^N$ the function defined by*

$$g(\sigma, y) = (\gamma(\varphi_{\sigma,y}), \beta(\varphi_{\sigma,y})). \tag{4.36}$$

Then,

$$deg(g, int(\mathcal{H}), (0, 1/2))) = 1.$$

Proof. Let us consider the homotopy $G : [0, 1] \times \partial\mathcal{H} \to \mathbb{R} \times \mathbb{R}^N$ given by

$$G(t, \sigma, y) = (1 - t)(\sigma, y) + t(\gamma(\varphi_{\sigma,y}), \beta(\varphi_{\sigma,y})). \tag{4.37}$$

We remark G is continuous and that

$$G(0, \sigma, y) = (\sigma, y)$$

and

$$G(1, \sigma, y) = (\gamma(\varphi_{\sigma,y}), \beta(\varphi_{\sigma,y})) = g(\sigma, y).$$

So, it remains to show that

$$\left(\frac{1}{2}, 0\right) \notin G(t, \partial\mathcal{H}) \quad \forall t \in [0, 1] \tag{4.38}$$

or, equivalently,

$$G(t, \sigma, y) \neq \left(\frac{1}{2}, 0\right) \quad \forall (\sigma, y) \in \partial\mathcal{H} \;\text{ and }\; \forall t \in [0, 1].$$

In fact, set $\partial\mathcal{H} = K_1 \cup K_2 \cup K_3$ with

$$\begin{cases} K_1 = \{(\sigma, y); \quad |y| \leq r, \quad \sigma = \sigma_1\}, \\ K_2 = \{(\sigma, y); \quad |y| \leq r, \quad \sigma = \sigma_2\}, \\ K_3 = \{(\sigma, y); \quad |y| = r, \quad \sigma \in [\sigma_1, \sigma_2]\}. \end{cases}$$

If $(\sigma, y) \in K_1$, then $\sigma = \sigma_1$ and by the Corollary 4.7 (a)

$$(1 - t)\sigma_1 + t\gamma(\varphi_{\sigma_1,y}) < (1 - t)\frac{1}{2} + t\frac{1}{2} = \frac{1}{2}, \quad \forall t \in [0, 1].$$

Analogously, if $(\sigma, y) \in K_2$, then $\sigma = \sigma_2$ and again by the Corollary 4.7 (b)

$$(1 - t)\sigma_2 + t\gamma(\varphi_{\sigma_2,y}) > (1 - t)\frac{1}{2} + t\frac{1}{2} = \frac{1}{2}, \quad \forall t \in [0, 1].$$

If $(\sigma, y) \in K_3$, then $|y| = r$ and $0 < \sigma_1 \leq \sigma \leq \sigma_2$, so using Lemma 4.6 (c), we obtain

$$((1 - t)y + t\beta(\gamma_{\sigma,y})|y) = (1 - t)|y|^2 + t(\beta(\varphi_{\sigma,y})|y) > 0 \qquad \forall t \in [0, 1].$$

Now, the results follows by employing the proprieties of the Brouwer's Topological degree. □

Lemma 4.10. *Let $\mathcal{H}$ be the set defined in* (4.33)*, and assume that* (V_3) *holds, then*

$$L = \max\{J_0(\varphi_{\sigma,y}) : (\sigma, y) \in \mathcal{H}\} < 2^{\frac{4-\mu}{2N-\mu}} S_{H,L}.$$

Proof. Using (4.21), we have for all (σ, y) that

$$\begin{aligned} J_0(\varphi_{\sigma,y}) &= \left[\|\varphi_{\sigma,y}\|^2 + \int_{\mathbb{R}^N} V_0(x)\varphi_{\sigma,y}^2 dx\right] \leq \|\varphi\|^2 + |V_0|_{L^{N/2}}|\varphi|_{2^*}^2 \\ &\leq \|\varphi\|^2 + \frac{|V_0|_{L^{N/2}}}{S}\|\varphi\|^2 \leq \left[1 + \frac{|V_0|_{L^{N/2}}}{S}\right]\Sigma \leq \left[1 + \frac{|V_0|_{L^{N/2}}}{S}\right]\overline{c}. \end{aligned}$$

The result follows by (V_3), (4.16) and (4.17).

□

5 Proof of Theorems

Finally, with the help of the previous lemmas we are ready to prove our main results. For $c \in \mathbb{R}$, let us fix the set

$$J_\lambda^c = \{u \in \mathcal{M} : \quad J_\lambda(u) \leq c\}.$$

Proof of Theorem 1.1**.** Combining the definition of $\overline{c}$ in (4.17), and Lemma 4.8, we have

$$S_{H,L} < \overline{c} < \mathcal{B}_{V_0} \leq J_0(\varphi_{\bar{\sigma},\bar{y}}) \leq L < 2^{\frac{4-\mu}{2N-\mu}} S_{H,L}.$$

We will prove that functional J_0 constrained to $\mathcal{M}$ has a critical level in the interval $(S_{H,L}, 2^{\frac{4-\mu}{2N-\mu}} S_{H,L})$. Suppose, by contradiction, that is not true. From Corollary 3.6, J_0 satisfies the Palais-Smale condition in interval $(S_{H,L}, 2^{\frac{4-\mu}{2N-\mu}} S_{H,L})$. Thus, using a variant of the Deformation Lemma (see [40]) we can find a $\delta > 0$ such that $\mathcal{B}_{V_0} - \delta > \overline{c}$, $L + \delta < 2^{\frac{4-\mu}{2N-\mu}} S_{H,L}$ and a continuous map $\eta : J_0^{L+\delta} \to J_0^{\mathcal{B}_{V_0}-\delta}$ such that

$$\eta(u) = u, \quad \forall u \in J_0^{\mathcal{B}_{V_0}-\delta}.$$

Then, the map $\eta(\varphi_{\sigma,y})$, $(\sigma, y) \in \mathcal{H}$, is well defined and we remark that

$$J_0(\eta(\varphi_{\sigma,y})) < \mathcal{B}_{V_0} - \delta, \quad \forall (y, \sigma) \in \mathcal{H},$$

which implies

$$\Theta(\varphi_{\sigma,y}) := (\gamma(\eta(\varphi_{\sigma,y})), \beta(\eta(\varphi_{\sigma,y}))) \neq \left(\frac{1}{2}, 0\right). \tag{5.1}$$

On the other hand, by Corollary 4.7,

$$J_0(\varphi_{\sigma,y}) < \overline{c} < \mathcal{B}_{V_0} - \delta, \quad \forall (y, \sigma) \in \partial\mathcal{H}, \tag{5.2}$$

which implies $\eta(\varphi_{\sigma,y}) = \varphi_{\sigma,y}$, from which

$$\Theta(\varphi_{\sigma,y}) = g(\sigma, y) = (\gamma(\varphi_{\sigma,y}), \beta(\varphi_{\sigma,y})), \quad \forall (y, \sigma) \in \partial\mathcal{H},$$

where g is the map introduced in (4.36).

Therefore, by the homotopy invariance of topological degree, taking into account Lemma 4.9, we deduce

$$1 = d(g, \mathcal{H}, (1/2, 0)) = d(\Theta, \mathcal{H}, (1/2, 0)),$$

that implies the existence of (σ, y) such that $\Theta(\sigma, y) = (1/2, 0)$, contradicting (5.1). Therefore, the functional J_0 constrained on $\mathcal{M}$ has at least one critical point $u \in \mathcal{M}$ such that $\overline{c} < J(u) < 2^{\frac{4-\mu}{2N-\mu}} S_{H,L}$. Moreover, by Lemma 2.3, we also have $u > 0$, finishing the proof.

Proof of Theorem 1.2**.**

The proof of this results follows as in [11], however for the reader's convenience we will write its proof.

Lemma 5.1. *Let $\mathcal{H}$ be the set defined in* (4.33)*. Then, there exists $\bar{\lambda} > 0$ such that, for all $0 < \lambda < \bar{\lambda}$, we have*

$$l_\lambda := \max\{J_\lambda(\varphi_{\sigma,y}) : (\sigma, y) \in \partial\mathcal{H}\} < \overline{c}.$$

Moreover, $\gamma(\varphi_{\sigma_1,y}) < \frac{1}{2}$, $\gamma(\varphi_{\sigma_2,y}) > \frac{1}{2}$, for all $y \in \mathbb{R}^N$, $|y| < r$.

Proof. Note that

$$\lambda \int_{\mathbb{R}^N} |\varphi_{\sigma,y}|^2 dx = \lambda\sigma^2 \int_{B_1(0)} |\varphi|^2 dx,$$

and so,

$$J_\lambda(\varphi_{\sigma,y}) = J_0(\varphi_{\sigma,y}) + \lambda\sigma^2 \int_{B_1(0)} |\varphi|^2 dx.$$

Now, the result follows from (4.32). □

Combining the definition of $\overline{c}$ and Lemma 5.1, for every $\lambda \in (0, \bar{\lambda})$ we have

$$S_{H,L} < \Upsilon_\lambda \leq J_\lambda(\varphi_{\tilde{\sigma},\tilde{y}}) \leq l_\lambda < \overline{c} < \mathcal{B}_{V_0}, \tag{5.3}$$

where Υ_λ has been defined in (4.13) and $(\tilde{\sigma}, \tilde{y})$ has been introduced in Lemma 4.8.

We will prove that functional J_λ constrained to $\mathcal{M}$ has a critical level in the interval $(\Upsilon_\lambda, l_\lambda)$. Suppose, by contradiction, that is not true. From Corollary 3.6, J_λ satisfies the Palais-Smale condition in interval $(\Upsilon_\lambda, l_\lambda)$. Thus, using a variant of the Deformation Lemma (see [40]) we can find a positive number $\delta_1 > 0$ such that $\Upsilon_\lambda - \delta_1 > S_{H,L}$, $l_\lambda + \delta_1 < \overline{c}$ and a continuous function

$$\eta : [0, 1] \times J_\lambda^{l_\lambda+\delta_1} \longrightarrow J_\lambda^{l_\lambda+\delta_1}$$

such that

$$\begin{aligned} \eta(0, u) &= u && \forall u \in J_\lambda^{l_\lambda+\delta_1} \\ \eta(s, u) &= u && \forall u \in J_\lambda^{\Upsilon_\lambda-\delta_1},\ \forall s \in [0, 1] \\ J_\lambda(\eta\,(s, u)) &\leq J_\lambda(u) && \forall u \in J_\lambda^{\Upsilon_\lambda+\delta_1},\ \forall s \in [0, 1] && (5.4) \\ \eta(1, J_\lambda^{l_\lambda+\delta_1}) &\subset J_\lambda^{\Upsilon_\lambda-\delta_1}. && && (5.5) \end{aligned}$$

Therefore, definition of l_λ and (5.5) give

$$(\sigma, y) \in \partial\mathcal{H} \ \Rightarrow\ J_\lambda(\varphi_{\sigma,y}) \leq l_\lambda \ \Rightarrow\ J_\lambda(\eta(1, \varphi_{\sigma,y}) \leq \Upsilon - \delta_1. \tag{5.6}$$

Let us consider $\forall s \in [0, 1]$, $\forall(\sigma, y) \in \partial\mathcal{H}$

$$\Gamma(\sigma, y, s) \ = \ \begin{cases} G(\sigma, y, 2s), & s \in [0, 1/2] \\ (\gamma \circ \eta\,(2s - 1), \varphi_{\sigma,y}), \beta \circ \eta\,(2s - 1, \varphi_{\sigma,y})) & s \in [1/2, 1], \end{cases}$$

where G is the map defined in (4.37). As already shown in Lemma 4.9,

$$\forall s \in [0, 1/2],\ \forall(\sigma, y) \in \partial\mathcal{H},\ G(\sigma, y, s) \neq \left(\frac{1}{2}, 0\right). \tag{5.7}$$

Furthermore, from (5.3) and (5.4), we deduce $\forall s \in [1/2, 1]$ $\forall(\sigma, y) \in \partial\mathcal{H}$

$$J_\lambda(\eta(2s - 1, \varphi_{\sigma,y})) \leq J_\lambda(\varphi_{\sigma,y}) \leq l_\lambda < \bar{c} < \mathcal{B}_{V_0} \leq \mathcal{B}_{V_\lambda}, \quad \forall \lambda > 0,$$

which gives

$$\forall s \in [1/2, 0],\ \forall(\sigma, y) \in \partial\mathcal{H},\ G(\sigma, y, s) \neq \left(\frac{1}{2}, 0\right). \tag{5.8}$$

By (5.7), (5.8) and the continuity of Γ, we obtain the existence of $(\breve{\sigma}, \breve{y}) \in \partial\mathcal{H}$ such that

$$\gamma \circ \eta\,(1, \varphi_{\breve{\sigma},\breve{y}}) \geq \frac{1}{2}, \quad \beta \circ \eta\,(1, \varphi_{\breve{\sigma},\breve{y}}) = 0.$$

Then

$$J_\lambda(\eta(1, \varphi_{\breve{\sigma},\breve{y}})) \geq \Upsilon_\lambda,$$

which contradicts (5.6). Therefore, the functional J_λ constrained on $\mathcal{M}$ has at least one critical point $u_l \in \mathcal{M}$, such that $S_{H,L} < J(u_l) < \overline{c}$, $\forall\lambda \in (0, \bar{\lambda})$. Moreover, by the Lemma 2.3, we deduce $u_l > 0$, concluding the proof of the first part of the theorem.

Now, let us suppose that (V_3) holds. Then the existence of an high energy positive solution, i.e. of a critical point for J_λ constrained on $\mathcal{M}$ such that $\bar{c} \leq J_\lambda(u_h) < 2S_{H,L}$, can be proved for small λ arguing exactly as in the proof of Theorem 1.1, taking into account that

$$\lim_{\lambda\to 0} \sup\{J_\lambda(\varphi_{\sigma,y}) : (\sigma, y) \in \mathcal{H}\} = \sup\{J_0(\varphi_{\sigma,y}) : (\sigma, y) \in \mathcal{H}\}$$

and

$$\lim_{\lambda\to 0} \sup\{J_\lambda(\varphi_{\sigma,y}) : (\sigma, y) \in \partial\mathcal{H}\} = \sup\{J_0(\varphi_{\sigma,y}) : (\sigma, y) \in \partial\mathcal{H}\}.$$

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