Refutation systems: an overview and some applications to philosophical logics

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Abstract

Refutation systems are systems of formal, syntactic derivations, designed to derive the non-valid formulas or logical consequences of a given logic. Here we provide an overview with comprehensive references on the historical development of the theory of refutation systems and discuss some of their applications to philosophical logics.

1 Introduction

The idea of refutation systems goes back to Aristotle's approach to systematic rejection of all invalid syllogisms. Aristotle showed that some syllogisms are invalid by providing suitable examples (in modern terms, counter-models) demonstrating that they can derive false conclusions from true premises. Other syllogisms he then rejected by applying some deductive reasoning, showing that assuming their validity would imply validity of already rejected syllogisms. Essentially, he applied the inference rule of Modus Tollens: "*If B is rejected and $A \rightarrow B$ is accepted (valid), then A must be rejected, too.*". This approach worked quite well, enabling Aristotle to classify most of the invalid syllogisms, though his classification was shown to be incomplete by Jan Łukasiewicz, but only much later, in the 1st half of the 20th century. Łukasiewicz embraced Aristotle's idea and formalised it in the concept of refutation system in [Łukasiewicz, 1939] and applied it to Aristotle's syllogistic in his seminal book [Łukasiewicz, 1951], though a much earlier important pre-cursor of the development of his ideas on refutation systems was [Łukasiewicz, 1921] (see English translation in [Borkowski, 1970]), where he introduced the notion of formal rejection of a false proposition[1]. In [Łukasiewicz, 1951], Łukasiewicz wrote "*...Out of the two intellectual acts, acceptance and rejection of a statement, only the first has been taken into account in modern formal logic. Gottlob Frege introduced into logic the idea of assertion, and the sign of assertion* ($\vdash$)*, accepted afterwards by the authors of Principia Mathematica. The idea of rejection, however, so far as I know, has been neglected up to the present day.*"

Since Łukasiewicz's foundational work in [Łukasiewicz, 1951], the concept of refutation systems has been developing at a slow but steady pace, with various types of refutation systems being designed both for classical logic and for a number of non-classical logics, and several technical and conceptual applications of these have been proposed in the literature.

In this paper we provide a concise, but comprehensively referenced overview of the literature on refutation systems and discuss some of their applications to philosophical logics.

2 Refutation systems: basic concepts

2.1 Semantic vs deductive refutability

Consider a logical system L with a given semantics, defining L-validity, and, more generally, L-logical consequence, both denoted by $\models_{\mathsf{L}}$. Following a common tradition in non-classical logics, we will often identify the logic L with the set of its valid formulas.

An L-formula A is **(semantically) refutable** in L iff it is not valid, i.e. $\not\models A$. That is, there is an L-model falsifying A. Likewise, for logical consequence: $\Phi \not\models A$ means that there is an L-model satisfying all formulas in Φ, but falsifying A.

[1] As noted in [Wybraniec-Skardowska, 2018], for introducing the concept of formal rejection, Łukasiewicz was apparently influenced by Brentano, who was probably the first to consider on a par the two kinds of judgment, viz. *acceptance* and *rejection*, and to use a pair of symbols, $+$ and $-$, to syntactically distinguish between them.

Now, consider also a deductive system $\mathbf{D}$ for L, with a derivation relation $\vdash_{\mathbf{D}}$. Then, for any L-formula A, provability of A in $\mathbf{D}$ is denoted $\vdash_{\mathbf{D}} A$. If $\mathbf{D}$ is sound and complete for L, then $\vdash_{\mathbf{D}} A$ corresponds to validity $\models_{\mathsf{L}} A$.

However, one can argue that, in general, the matching syntactic notion to semantical refutability is *not* non-provability of A, i.e. $\nvdash_{\mathbf{D}} A$. It is rather the notion of "*deductive refutability*", i.e., existence of a formal derivation of A in a suitable *derivation system for* L*-refutable formulas*.

A new symbol is needed for that notion, for which we will use $\dashv_{\mathsf{L}}$, following Łukasiewicz, where

$$\dashv_{\mathsf{L}} A$$

means "A **is deductively refutable in** L", i.e. "*the non-validity of* A *in* L *can be formally derived*", or "*the refutation of* A *in* L *is derivable/derived*". When the logic L is fixed by the context, we will write simply $\dashv$.

The above naturally extends to deductive refutations of logical consequences.

Thus, the notion of "**(deductive) refutation systems**" arises.

2.2 Refutation rules and systems

Pure rule of refutation inference:

$$\frac{\dashv B_1, \ldots, \dashv B_n}{\dashv C}$$

A typical example is Łukasiewicz's rule **Reverse substitution**:

$$\frac{\dashv \sigma(A)}{\dashv A}$$

where σ is a uniform substitution.

The intuitive meaning: if $\sigma(B)$ is derived as non-valid, then B is derived as non-valid, too.

Usually, pure refutation rules do not suffice to capture adequately semantic refutability, so we also consider the following, more general refutation rules.

(Mixed) rule of refutation inference (based on a deductive system $\mathbf{D}$):

$$\frac{\vdash_{\mathbf{D}} A_1, \ldots, \vdash_{\mathbf{D}} A_m,\ \dashv B_1, \ldots, \dashv B_n}{\dashv C}$$

The intuitive interpretation: if each A_i is derived by **D** as valid and each B_j is derived as non-valid, then C is derived as non-valid.

A typical example is Łukasiewicz's rule **Reverse modus ponens**:

$$\frac{\vdash A \to B, \dashv B}{\dashv B}$$

A **refutation system** is a set R of refutation rules. Refutation rules with no premises are called **refutation axioms** and we write them simply as $\dashv A$.

A **refutation derivation in** R, or just an R**-derivation** for a formula A, is a sequence $S_1, ..., S_t$, where S_t is $\dashv A$ and every S_i is either a refutation axiom or has the form $\vdash B$ or is obtained from some preceding formulas by applying a refutation rule from R. Note that when R contains mixed refutation rules, then the notion of R-refutability is relativised to the underlying deductive system **D**. We now say that a formula A is **refutable in** R (or, just R-**refutable**) iff there is a refutation derivation for A in R.

Given a logical system L, we say that a refutation system R is:

- **refutation-sound**, or **Ł-sound, for** L, if only non-valid in L formulae (more generally, logical consequences) are R-refutable.
- **refutation-complete**, or **Ł-complete, for** L, if all non-valid in L formulae (more generally, L logical consequences) are R-refutable.

These are readily relativised to any fixed set Φ of formulas in L.

3 Refutation systems: an overview

3.1 Beginning and early work on refutation systems

The notion of formal rejection was introduced by Łukasiewicz in his early paper [Łukasiewicz, 1921], but his first publication on formal refutation systems was [Łukasiewicz, 1939], where he introduced the concept and proposed a complete refutation system for the classical propositional calculus PL, based on the refutation rules of reverse substitution and Modus Tollens (reverse Modus Ponens).

Later, in his seminal book [Łukasiewicz, 1951], Łukasiewicz showed that Aristotle's refutation system for syllogisms is "refutationally incomplete", i.e. not deriving all non-valid syllogisms. He then added two more non-valid syllogisms to Aristotle's system, to make it (refutationally) complete for all non-valid syllogisms, but not for all non-valid expressions of the language.

Later, Łukasiewicz's student J. Słupecki showed in [Słupecki, 1955] that no finite set of refutation axioms would suffice for that. In that work, he proposed an additional refutation rule to make Łukasiewicz's system complete. In that work, Słupecki also developed further the concept of refutation system, introduced the notions of rejection consequence relation, Ł-decidability, and Ł-consistency, and solved the problem, posed by Łukasiewicz, of proving decidability of Aristotle's Syllogistics. During the 1970s, Słupecki together with his students Bryll and Wybraniec-Skardowska developed a general systematic theory of rejected propositions, in the style of Tarski's theories of consequence relations and deductive systems, see in [Słupecki et al., 1971] and [Słupecki et al., 1972]. Follow up and related works along that line include [Staszek, 1971], [Staszek, 1972], and the more recent [Wybraniec-Skardowska, 2016], where two notions of axiomatic refutation systems, dual to Tarski's concept of deductive system for PL were constructed and proved to be equivalent.

A recent comprehensive historical overview and discussion of the early work on refutation systems by Łukasiewicz, Słupecki and his school can be found in [Wybraniec-Skardowska, 2018]. For a nice discussion on Łukasiewicz's development of the concept of refutation system see also [Tamminga, 1994].

3.2 Overview of refutation systems in classical logic

In parallel with the fundamental work of Polish school, several other, essentially independent works developing variants of refutation systems for classical logic were published since the 1960s:

- In [Härtig, 1960] Härtig studied Łukasiewicz's refutation system for CPC and established some necessary and sufficient conditions for Ł-completeness of axiomatic refutation systems for the not-provable formulae in the classical propositional calculus CPC.
- In [Hailperin, 1961] Hailperin, apparently unaware of Łukasiewicz's notion of refutation system and related work, developed probably the first explicit axiomatic refutation system for first-order logic FOL, for languages without equality and function symbols. That system was proved

there to be sound and complete for falsifiability in finite FOL models.

- Later, in [Staszek, 1971] Staszek considered Łukasiewicz's refutation system for Aristotle's syllogistic and extends it with rules for the quantifiers.

- In [Stahl, 1958] Stahl introduced the notion of 'opposite system' for axiomatic derivation of all logically false (unsatisfiable) sentences of a given logic and proposed two axiomatizations for such system for CPC. Later, Morgan in [Morgan, 1973] independently presented a finitely axiomatized refutation system for all logically false (unsatisfiable) sentences of CPC.

- Independently from all previous work on refutation systems mentioned so far, Caicedo proposed in [Caicedo, 1978] a finitary formal system for deriving all non-theorems of CPC, consisting only of *pure* refutation rules.

- Later, in [Skura, 1990] Skura proposed a general method for constructing refutation systems with pure refutation rules for sentential logics.

- Another line of relatively independent work started in Japan with Ishimoto's [Ishimoto, 1981], which studied the method of axiomatic rejection in PL. Inoue discussed and followed on Stahl's work in [Inoué, 1989] and Ishimoto's work in [Inoué, 1989]. In [Inoué, 1990], Inoue studied unprovability-preserving translations between formal systems for propositional logic and in [Inoué, 1994] he discussed the atomic formula property of Härtig's refutation calculus in [Härtig, 1960].

- In [Varzi, 1990], Varzi introduced a simplified Ł-complete refutation system for PL, based on the single axiom: $\bot$, and the two rules of inference: reverse substitution and replacement of equivalent subformulas. That system was further refined in [Varzi, 1992].

- In [Kulicki, 2000] and [Kulicki, 2002] Kulicki discusses applications of axiomatic rejection, for refuting the non-valid syllogisms of Aristotle's Syllogistic and, more generally, as syntactic decision procedures.

- [Morgan et al., 2007] presents a sound and complete axiomatic proof system for deriving for the logical contingencies in PL, that is the PL-formulas that are both satisfiable and falsifiable.

Lastly, we note that refutation systems are close in design and purpose to tableaux-based methods, as both can be used to derive non-validity or satisfiability of the input formula. A number of more syntactically presented tableau-based methods for FOL have been developed, of which we only mention here some, which relate more explicitly with the concept of deductive refutation.

- In [Dutkiewicz, 1989] Dutkiewicz developed a tableau-based Ł-complete refutation system for IPL (also mentioned further).
- In [Bry and Torge, 1998] Bry and Torge present a tableaux-based algorithmic deduction method for FOL, for which they prove that it is refutation complete for finite satisfiability in FOL. They also discuss several areas of applications of their methods to computer science, incl. databases, planning, natural language understanding, program verification and theorem proving.
- In [Skura, 2005], Skura introduced 'invertible reduction rules', reversing refutation rules and preserving validity, for IPL.

3.3 Gentzen-style refutation systems

The first constructed refutation systems were of axiomatic type, but gradually the other classical types of deductive systems, such as tableaux, sequent calculi, and systems of natural deduction, were adapted as refutation systems. We mention here those that were constructed for classical logic, while others, specifically designed for non-classical logics, are discussed further in this section.

- In [Tiomkin, 1988], Tiomkin constructed a sequent-style refutation calculus for FOL without function symbols and sketched a proof of its Ł-completeness for the formulas refutable in finite models.

  Independently, a few years later both Bonatti in [Bonatti, 1993][2] and Goranko in [Goranko, 1994] developed an Ł-complete sequent refutation calculi for PL, completing Tiomkin's calculus by including (the

[2] This reference is imported from other sources, albeit none of the authors has seen this technical report and it appears to be currently inacessible in the internet.

same) left- and right-introduction rules for the full propositional language. (See also [Bonatti and Varzi, 1995] containing the rules for implication and negation.) The sequent refutation calculus constructed in [Goranko, 1994] was then extended in the same paper to some important normal modal logics.

- In [Tamminga, 1994] Tamminga developed a system of natural deduction for deriving the non-theorems of PL, which he proved to be (Ł-sound and) Ł-complete. He also presented a subsystem of that system, Ł-complete for the set of all contradictions of PL and illustrated the application of both systems with some examples.

- In [Tiomkin, 2013] Tiomkin constructed an Ł-complete sequent calculus for the contingencies of PL, ie. the propositional formulas that are neither valid nor contradictory, and also proved cut-elimination and the subformula property for that calculus.

- In [Carnielli and Pulcini, 2017], the sequent refutation systems of Tiomkin and Goranko were enriched with two admissible unary cut-rules which come shaped as reverse Weakenings. Then the authors designed a normalization procedure and studied some related computational properties.

- In [Pulcini and Varzi, 2019], the authors propounded a proof-nets system sound and complete with respect to the set of classically invalid (right-sided) sequents. They also provided a normalization procedure that allows for the improvement of the basic computational properties already studied in [Carnielli and Pulcini, 2017].

3.4 Refutation systems for non-validity in the finite

Ł-complete refutation systems can be naturally associated with logics determined by effectively definable classes of finite models, as the non-validities in such logics are effectively enumerable, and therefore possible to capture by deductive refutation systems. Cases of particular importance are first-order logic on the class of all finite structures, hereafter denoted $\mathsf{FOL}^{\mathsf{fin}}$, as well as modal and other non-classical logics with finite model property (FMP). Several such refutation systems have been proposed, including the following.

- We have already mentioned the axiomatic refutation systems in [Hailperin, 1961] and in [Bry and Torge, 1998], as well as the sequent refutation calculus in [Tiomkin, 1988] for FOL without function symbols, each proved to be Ł-sound and Ł-complete for $\mathsf{FOL}^{\mathsf{fin}}$.

- In [Skura, 1991] Skura discussed refutation systems as decision procedures and in [Skura, 1992] he developed a general method for constructing Ł-complete refutation systems for intermediate logics with the FMP, as well as for some logics without the FMP. Further, in [Skura, 1994], he extended that method to modal logics with the FMP.

- In [Skura, 2004b] Skura proposed Scott-style refutation rules for the intermediate logics of finite n-ary trees.

- In [Goranko, 1994] Goranko constructed Ł-complete refutation systems for several modal logics that are complete for classes of finite (intransitive or transitive) trees.

- In [Goranko and Skura, 2018] Goranko and Skura develop generic refutation systems for modal logics with finite model property and for FOL theories determined by classes of finite models. In particular, they construct an Ł-complete refutation system for $\mathsf{FOL}^{\mathsf{fin}}$ in arbitrary FOL languages (incl. function symbols).

3.5 Refutation systems for intuitionistic and modal logics

Research on extending refutation systems to non-classical logics began already with Łukasiewicz, who conjectured in [Łukasiewicz, 1952] that, to derive all intuitionistically non-valid formulae it suffices to add to CPC ($\dashv$) the Disjunction Rule (DR):

$$\frac{\dashv A, \ \dashv B}{\dashv A \vee B}$$

This conjecture was refuted later (in 1957) by Kreisel and Putnam, who showed in [Kreisel and Putnam, 1957] that DR is admissible in a proper extension of the intuitionistic propositional calculus IPC, now known as Kreisel-Putnam's logic, $\mathsf{KP} = \mathsf{IPC} + (\neg A \to (B \vee C)) \to (\neg A \to B) \vee (\neg A \to C)$.

Several complete refutation systems for IPL and its extensions were developed since then, including:

- In [Scott, 1957] D. Scott constructed an Ł-complete refutation system for IPL by introducing an infinite family of non-structural refutation rules (now called Scott-style refutation rules), involving Wajsberg/Mints-style normal forms. These go back to [Wajsberg, 1938] and [Mints, 1990]; see also [Mints, 1992] for further details.
- In [Skura, 1989], Skura improved Scott's system by constructing an Ł-complete refutation system for IPL involving an infinite family of structural rules.
- In [Dutkiewicz, 1989] Dutkiewicz developed another, tableau-based infinitary Ł-complete refutation system for IPL.
- In [Skura, 1992] Skura constructed refutation calculi for some intermediate propositional logics.
- Essential connections between refutations and admissibility of rules have been analysed by Goudsmit in the context of IPL and intermediate logics in [Goudsmit, 2014] and [Goudsmit, 2015], where using some techniques for admissible rules he constructs refutation systems (involving structural refutation rules) for the Gabbay - de Jongh logics.

Refutation systems are very close in purpose to counter-model producing procedures. Of the many and various such methods developed in the literature, we mention just a few that make that relation explicit and make an essential use of it.

- In [Pinto and Dyckhoff, 1995] Pinto and Dyckhoff use the contraction-free sequent calculi LJT for IPL to build (as they say, 'for expository reasons') a sequent 'Calculus for Refutation of Intuitionistic Propositions' CRIP, that is Ł-complete for IPL.
- Related in spirit and purpose is [Goré and Postniece, 2008] where Goré and Postniece present a cut-free sequent calculus for the *bi-intuitionistic logic*, being the union of intuitionistic and dual intuitionistic logic, which also involves refutation calculi for both. See also the related in topic [Pinto and Uustalu, 2009].

- In [Negri, 2014] Negri explores further the duality between proof search and counter-model search or, more generally, refutation search. She presents a method for unifying proof search and countermodel construction that puts together several ideas, incl. sequent calculi with internalized semantics and Tait-Schütte-Takeuti style completeness proofs, combined with procedures for effective finite countermodel construction. The method is applied to a variety of intermediate, modal, and other non-classical logics.

- In [Fiorentini and Ferrari, 2017] Fiorentini and Ferrari employ Maslov's inverse method (a saturation based theorem proving technique using forward proof-search strategy) to develop a 'forward unprovability calculus' FRJ for IPL. They show how to extract a Kripke counter-model for a formula from a derivation of the formula in FRJ.

Other works on refutation systems, specifically for modal logics include:

- The first Ł-complete refutation system for modal logic was the one for S5 constructed by Słupecki and Bryll in [Słupecki and Bryll, 1973].

- In [Goranko, 1994] (see also [Goranko, 1991]) Goranko constructed Ł-complete axiomatic and sequent-style refutation systems for several important modal logics, incl. K, T, K4, GL, S4Grz and others.

- In [Skura, 1995] Skura constructed a refutation system for S4 and proved its Ł-completeness by using algebraic methods. Further, in [Skura, 1996], he presented Scott-style refutation rules for S4. In [Skura, 2002], he also constructed Scott-style refutation rules for K4. Recently, in [Skura, 2019b] Skura explored the relationship between tableaux and refutations in S4. See also his related works [Skura, 1994], [Skura, 1999], and the monograph on refutation systems for modal logics [Skura, 2013], where, refutation systems and constructions of counter-models from syntactic refutations for several specific modal logics are described.

- In [Fiorentini, 2015] Fiorentini introduces a refutation calculus RS4 for the modal logic S4 with the subformula property, by using a contraction-free sequent calculus GS4 for S4 with the same property, and shows how to generate an S4-model of any formula refuted by that calculus.

- As already mentioned earlier, refutation systems are very close in purpose to counter-model producing procedures. In addition to the earlier mentioned works exploring that relation, we now also mention [Garg et al., 2012], where Garg et al present a novel countermodel-producing syntactic decision procedure, based on backwards search in labeled sequent calculi, that applies to several intuitionistic and classical multi-modal logics, thus also obtaining new decidability results.

- One topic not explicitly mentioned yet is the application of algebraic semantics and algebraic methods for the construction and analysis of refutation systems for non-classical logics. Such methods have been applied in [Skura, 1989], [Skura, 1992], [Skura, 1994], [Skura, 1995]. and, more recently, in [Citkin, 2013], where Citkin uses Jankov-style formulas to construct refutation rules and systems. These ideas were developed further in [Citkin, 2015b] and [Citkin, 2015a].

3.6 Refutation systems for other non-classical logics

Refutation systems have been developed for several other non-classical logics, too, including the following.

- In [Bonatti, 1996] Bonatti developed sequent refutation calculi for default and autoepistemic logics. Following his approach, Egly and Tompits constructed in [Egly and Tompits, 1997] a similar sequent refutation calculus for the intuitionistic default logic. Later, in [Bonatti and Olivetti, 2002], the authors extended Bonatti's work to sequent refutation calculi for propositional nonmonotonic logics.

- In [Sochacki, 2007] Sochacki studied axiomatic rejection in the implication-negation fragment of Łukasiewicz's many-valued logic. Later, in [Sochacki, 2011], he constructed refutation systems for Finn's 3-valued 'nonsense-logic'. See also [Sochacki, 2010] for a general reference on his related work.

- [Kulicki, 2012], presented a complete axiomatisation of the quantifier-free fragment of Leśniewski's Ontology, called there a "pure calculus of names", by employing the method of axiomatic rejection.

- In [Oetsch and Tompits, 2011] Oetsch and Tompits developed Gentzen-type refutation systems for three-valued logics and proposed applications of them for disproving strong equivalence.

- In [Berger and Tompits, 2013] Berger and Tompits constructed an axiomatic rejection system for the description logic ALC.

- Several refutation calculi and related systems for paraconsistent logics have also been developed: In [Skura, 2017a] Skura used ideas from refutation systems, discussed further here, in Section 5.1.1 to prove that Segerberg's logic $JE'E_1^Q$ is the greatest paraconsistent extension of Johansson's logic.

  In [Pulcini and Varzi, 2018], Pulcini and Varzi observed that any decidable logic which semantically circumscribes a set of truth-functional contingent formulas, classical propositional logic *in primis*, allows for a refutational counterpart which is paraconsistent. The authors stress this observation to draw some philosophical remarks about the notion of paraconsistency.

  In [Trybus, 2018], Trybus developed a method for constructing maximal paraconsistent logics, motivated by ideas from refutation systems.

  Other applications to paraconsistent logics can be found in [Skura, 2004a], also mentioned in Section 3.7 and used further in this paper, in Section 5.1.1.

- In [Skura, 2017b], Skura constructs Scott-style refutation rules for Wansing's nonmonotonic logic and uses them to provide an efficient decision procedure for that logic.

3.7 Systems combining proofs and refutations

Another important line of research on refutation systems is the idea of building systems of deduction combining proofs and refutations on a par. Here are several references exploring that idea.

- The idea of developing combined derivation systems for proofs and refutations was discussed at the end of [Goranko, 1994], where some *hybrid refutation rules* for such systems were suggested. That idea has recently been further developed by designing and exploring combined ('hybrid') proof/refutation systems of natural deduction for classical and intuitionistic logic.

- The idea is also close to the 'complementary systems' for sentential logics proposed in [Bonatti and Varzi, 1995].

- In [Skura, 2004a], Skura studies symmetric inference systems (that is, pairs of inference systems), that can be used for characterizing maximal non-classical logics with certain properties. The method is also applied there to paraconsistent logics.

- In [Wybraniec-Skardowska and Waldmajer, 2011] Wybraniec-Skardowska and Waldmajer explore the general theory of deductive systems employing the two dual consequence operators, the standard logical consequence, inferring validities, and the refutation consequence, inferring non-validities.

- In [Caferra and Peltier, 2008] Caferra and Peltier, motivated by potential applications to automated reasoning, take a unifying perspective on accepting or rejecting propositions by deriving them from other, already accepted or rejected propositions.

- In [Goré and Postniece, 2008] Goré and Postniece combine derivations and refutations to obtain cut-free complete systems for bi-intuitionistic logic.

- In [Negri, 2013], Negri explores the duality of proofs and countermodels in labelled sequent calculi, further developed in the already discussed [Negri, 2014].

- Likewise, in [Fiorentini and Ferrari, 2018], Fiorentini and Ferrari explore the duality between unprovability and provability in forward proof-search for intuitionistic propositional logic, following on their previous, already mentioned work [Fiorentini and Ferrari, 2017].

3.8 Further general references on refutation systems

More recent further general references on the theory of refutation systems include the following.

- A general theory of refutation systems, involving the concept of multiple-conclusion (or Scott-style) consequence relation is presented in [Skura, 2009],
- An introduction to refutation systems, describing and discussing a variety of concepts, methods, and tools connected with them, is provided in [Skura, 2011b].
- A general method for proving completeness with respect to refutation rules involving certain normal forms is presented in [Skura, 2011a].
- The relationship between refutations, tableaux, and counter-models constructions in classical logic is discussed in [Skura, 2016].

4 Refutation systems vs deductive systems

Here we discuss and compare briefly refutation systems with some of the main types of traditional deductive systems, viz. tableaux and sequent calculi. We also outline the closely related idea of 'inverse refutation systems'. For axiomatic refutation systems we refer the reader e.g. to [Skura, 2011b] and for natural deduction style refutation systems – to [Tamminga, 1994].

4.1 Tableaux vs refutation systems

Refutation systems and tableau systems[3] are naturally related, as both are used to establish falsifiability of the input formula. If tableaux are regarded as satisfiability testing procedures then, as noted in [Skura, 2016], tableaux and refutation systems are complementary. Still, there are some clear distinctions, both conceptual and technical, which we discuss here.

Tableau procedures are typically used as *proof-search procedures*: if there is a closed tableau for $\neg A$, then A is proved. When terminating, they can also be used as *validity/satisfiability testing decision procedures*: if there is no closed tableau for $\neg A$, then a (usually finite) counter-model for A can be constructed from open and saturated tableaux for $\neg A$. Note however, that, unlike tableaux (which are derivations in an object language), such model constructions are done on a meta-logical level.

[3] For background on tableau methods for non-classical logics we refer the reader to [Goré, 1999]).

A typical (pure) refutation rule is of the form

$$\frac{\dashv A_1, ..., \dashv A_m}{\dashv A}$$

meaning that if every A_i is (derived as) non-valid, then so is A.

If we turn such a rule around (bottom-up), by contraposition we obtain an **inverse refutation rule**, preserving validity:

$$\frac{\vdash A}{\vdash A_1 \mid ... \mid \vdash A_m}$$

meaning that if A is (derived as) valid, then so is some A_i. (Rules of this kind were introduced in [Skura, 2005, Skura, 2011b].) Using them will facilitate our further discussion.

In a tableau procedure, in order to prove the validity of A, one assumes that A is non-valid, hence $\neg A$ is satisfiable. Starting with $\neg A$ as input, one then applies (satisfiability preserving) tableau rules, the purpose of which is to search systematically for a satisfying truth-assignment (more generally, satisfying model) for the input formula $\neg A$. The aim, however, is to show that this is not possible, by eventually producing a *closed tableau*: a finite tree in which every branch ends with (the unsatisfiable) $\bot$. If such a closed tableau is constructed, it means that $\neg A$ cannot be satisfied. This contradiction with the initial assumption implies that A is valid.

Dually, in a refutation procedure, in order to refute A, we can assume that A is valid. Then, by applying (validity preserving) branching inverse refutation rules as above we aim to obtain a finite tree in which every branch ends with a refutation axiom (which is non-valid). If such a tree is constructed, this is a contradiction with the initial assumption, implying that A is non-valid.

Thus, tableau systems (as proof procedures) and refutation systems (as refutation procedures) can be regarded as mutually dual calculi, and they can be studied by using similar formal tools. We illustrate that in the next subsection.

4.2 Inverse refutation systems

Given finite sets X, Y, Z of (modal) formulae, we define:

$\Box X := \{\Box A : A \in X\}$, $X \longrightarrow Y := \bigwedge X \to \bigvee Y$.

When $Y = \{C\} \cup Z$, we may write simply $X \longrightarrow C, Z$.

Note that an application of an inverse refutation rule to a formula produces $m \geq 1$ immediate successors. These successors can be written just as a set Ψ, representing a multiple-conclusion list $B_1 \mid ... \mid \ B_k$, where $\Psi = \{B_1, ..., \ B_k\}$.

An **inverse refutation tree** for a formula A is a finite tree with root labelled by A, whose nodes have labels containing formulas. That tree is expanded by applying inverse refutation rules to the node labels, and the labels of the leaves (end nodes) are *refutation axioms*, specifically defined for the system. An **inverse refutation system DR** consists of a set of refutation axioms and a set of inverse refutation rules. A formula A is **refutable** in an inverse refutation system **DR** iff there is an inverse refutation tree for A in **DR**. A system **DR** is **Ł-complete** for a logic L iff we have: $A \notin \mathsf{L}$ iff A is refutable in **DR**.

For example, in modal logic, complete inverse refutation systems can be obtained by reformulating the refutation systems involving normal forms in [Skura, 2002, Skura, 2013]. Roughly speaking, such an inverse refutation system for a normal modal logic L contains two rules: MP and a characteristic inverse refutation rule R_{L}. (In fact, usually the whole MP is not necessary, but a few specific cases of MP are sufficient (see [Skura, 2013]). For a concrete example, the characteristic rule for the basic modal logic **K** looks as follows (cf. [Lemmon and Scott, 1977, p.46], [Goranko, 1994], [Skura, 1999]).

(R_K):

$$\frac{\vdash \Box\Phi \longrightarrow C, \Box\Psi}{\{\vdash \Phi \longrightarrow B : B \in \Psi\}}$$

where C is a $\Box$-free formula that is not (valid) in CL.

It seems that the meta-logical constructions of counter-models for **K** from open tableaux (see [Goré, 1999]) and our syntactic refutations (which are formal derivations) are different descriptions of essentially the same procedure. Both terminate because the number of modal connectives is reduced in each step.

However, in other logics, the procedures may generally be different, for the following reasons. The rule R_K (mirroring the modal model construction) is not a sound refutation rule for any proper extension of **K**, as it suffices to refute every formula that is not in **K**. So, new refutation rules must be provided for other modal logics. It seems that, in transitive logics like **S4**, refutation rules must be restricted to Mints-style normal forms (cf.

[Skura, 2013]). But it is in these logics that refutation systems really make a difference because they provide refutations that, at least in some cases, are simpler than the tableau counter-model constructions (cf. [Skura, 2013, p.125]). Also, unlike tableau procedures for transitive logics, our refutation procedures are reduction procedures; in each step the rank of the Mints-style normal form (a specified natural number) is reduced, so our procedures are cycle-free.

Finally, we remark that counter-models can be obtained directly from inverse refutation trees, by removing the nodes to which MP was applied and extracting a suitable valuation from the normal forms involved. For details. see [Skura, 2002, Skura, 2013, Skura, 2017a]. A more detailed analysis of the relationship between refutation systems and tableau systems (illustrated by **S4**) is given in [Skura, 2019b].

4.3 Gentzen-style sequent refutation systems

4.3.1 The antisequent system $\overline{\mathsf{LK}}$ for classical propositional logic

We use capital Greek letters $\Gamma, \Delta, \dots$ to stand for finite multisets of formulas $[A_1, A_2, \dots, A_n]$. To simplify our notation, we write Γ, A and Γ, Δ to mean the multisets $\Gamma \uplus [A]$ and $\Gamma \uplus \Delta$, respectively. If $\Gamma = A_1, A_2 \dots, A_n$, then $\bigwedge \Gamma \equiv A_1 \wedge A_2 \wedge \dots \wedge A_n$ and $\bigvee \Gamma \equiv A_1 \vee A_2 \vee \dots \vee A_n$. With LK we refer to Gentzen's sequent calculus for classical propositional logic, as introduced in [Gentzen, 1935, Szabo, 1969].

Refutational counterparts of ordinary sequent systems are often called in the literature *antisequent* systems. Antisequents are usually written as $\Gamma \dashv \Delta$ and they differ from standard sequents *à la* Gentzen insofar as they are intended to affirm their own invalidity. In semantical terms, affirming $\Gamma \dashv \Delta$ corresponds to say that there is a valuation v such that $v(\bigwedge \Gamma \to \bigvee \Delta) = 0$. Notice that, in spite of the use of the inverted turnstile symbol $\dashv$, the validity of $\Gamma \dashv \Delta$ — i.e. the *in*validity of $\Gamma \vdash \Delta$ and so its unprovability in LK — does not necessarily imply the validity of its mirror image $\Delta \vdash \Gamma$. For a very simple example, the sequent $p \vdash q$ is invalid, therefore the *anti*sequent $p \dashv q$ is valid, but the sequent $q \vdash p$ is classically invalid as well.

The first antisequent system for classical logic was constructed by Tiomkin in [Tiomkin, 1988]. The propositional part of Tiomkin's calculus consists in one refutation axiom scheme together with left- and right-introduction rules for negation and disjunction. The refutation axiom scheme allows the user to introduce any antisequent $\Gamma \dashv \Delta$ provided that Γ and Δ are *two disjoint multisets of atoms* (example: $p, q \dashv t$). Thus, the empty antisequent '$\dashv$' is readily provable in the system as a limit case of the refutation axiom wherein $\Gamma = \Delta = \varnothing$. As noted in Section 3.3, antisequent calculi for the full language of PL were independently constructed a few years later in [Goranko, 1994] and (reportedly) in [Bonatti, 1993].

From now on we will write $\overline{\mathsf{LK}}$ to refer to the full antisequent system, as formulated in [Goranko, 1994] (cf. Figure 1). Notice that $\overline{\mathsf{LK}}$'s logical rules are all unary so that $\overline{\mathsf{LK}}$-proofs turn out to be simple chains of antisequents. From semantic viewpoint, logical rules are designed in a way that the very valuation that falsifies the premise, also falsifies the conclusion. In Figure 2 we give an example of a $\overline{\mathsf{LK}}$-derivation ending in the truth-functional contingency $(p \to q) \to (\neg p \to q)$.

In general we have that:

- $\overline{\mathsf{LK}}$ proves $\dashv A$ if and only if A is *not* a tautology.
- $\overline{\mathsf{LK}}$ proves $A \dashv$ if and only if A is *not* a contradiction.

Therefore, by combining the two previous points, we can conclude that $\overline{\mathsf{LK}}$ proves both $\dashv A$ and $A \dashv$ if and only if A is a truth-functional contingency [Tiomkin, 2013].

In [Carnielli and Pulcini, 2017], the authors observed how $\overline{\mathsf{LK}}$ logical rules can be algorithmically generated by a bottom-up reading of the tableau rules for classical logic once: (*i*) formulas labelled as true are placed on the left-hand side of the sequent symbol, (*ii*) formulas labelled as false are placed on the right-hand side, and (*iii*) context variables are added on both sides of the sequents. The guiding idea is that branching tableau rules generate two distinct $\overline{\mathsf{LK}}$ logical rules, whereas non-branching rules generate a single $\overline{\mathsf{LK}}$ logical rule. This procedure is similar to the one employed in [D'Agostino, 1999] to derive the rules of sequent systems for classical logic dealing with implicit structural rules (cfr. Kleene's system G4 or its variant G3 [Kleene, 1967, Negri et al., 2008]) from tableau rules.

As an illustrative example, consider the tableau rules for classical conjunction. First, consider the case in which the formula $A \wedge B$ is true, namely:

Axiom:

$$\frac{}{\Gamma \dashv \Delta}\ \overline{ax}. \quad \text{where } \Gamma \text{ and } \Delta \text{ are two finite multisets of atoms s.t. } \Gamma \cap \Delta = \varnothing$$

Logical rules:

$$\frac{\Gamma, A, B \dashv \Delta}{\Gamma, A \wedge B \dashv \Delta}\ \wedge\dashv \qquad \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A \wedge B}\ \dashv\wedge_{\mathcal{R}} \qquad \frac{\Gamma \dashv \Delta, B}{\Gamma \dashv \Delta, A \wedge B}\ \dashv\wedge_{\mathcal{L}}$$

$$\frac{\Gamma, A \dashv \Delta}{\Gamma, A \vee B \dashv \Delta}\ \vee_{\mathcal{R}}\dashv \qquad \frac{\Gamma, B \dashv \Delta}{\Gamma, A \vee B \dashv \Delta}\ \vee_{\mathcal{L}}\dashv \qquad \frac{\Gamma \dashv A, B, \Delta}{\Gamma \dashv A \vee B, \Delta}\ \dashv\vee$$

$$\frac{\Gamma \dashv A, \Delta}{\Gamma, A \rightarrow B \dashv \Delta}\ \rightarrow\dashv_{\mathcal{R}} \qquad \frac{\Gamma, B \dashv \Delta}{\Gamma, A \rightarrow B \dashv \Delta}\ \rightarrow\dashv_{\mathcal{L}} \qquad \frac{\Gamma, A \dashv B, \Delta}{\Gamma \dashv A \rightarrow B, \Delta}\ \dashv\rightarrow$$

$$\frac{\Gamma \dashv A, \Delta}{\Gamma, \neg A \dashv \Delta}\ \neg\dashv \qquad \frac{\Gamma, A \dashv \Delta}{\Gamma \dashv \neg A, \Delta}\ \dashv\neg$$

Figure 1: The $\overline{\mathsf{LK}}$ sequent calculus.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\dashv q, p, p}\ \overline{ax}.}{\neg p \dashv q, p}\ \neg\dashv}{p \rightarrow q, \neg p \dashv q}\ \rightarrow\dashv_{\mathcal{R}}}{p \rightarrow q \dashv \neg p \rightarrow q}\ \dashv\rightarrow}{\dashv (p \rightarrow q) \rightarrow (\neg p \rightarrow q)}\ \dashv\rightarrow$$

Figure 2: An example of a $\overline{\mathsf{LK}}$-proof.

$$\begin{array}{c}\mathtt{T} : A \wedge B \\ | \\ \mathtt{T} : A \\ | \\ \mathtt{T} : B\end{array}$$

In this case we proceed by putting all the true formulas on the left-hand side of the antisequent symbol, distinguishing the conclusions from the premises as follows: $\dfrac{A, B \dashv}{A \wedge B \dashv}$. Then we add context variables on both sides so as to finally achieve the following $\overline{\mathsf{LK}}$ rule:

$$\frac{\Gamma, A, B \dashv \Delta}{\Gamma, A \wedge B \dashv \Delta} \wedge \dashv.$$

When we focus, instead, on the case in which $A \wedge B$ is taken to be false, we have to consider the following branching rule

$$\begin{array}{c}\mathtt{F} : A \wedge B \\ \swarrow \quad \searrow \\ \mathtt{F} : A \qquad \mathtt{F} : B\end{array}$$

which is expected to produce two different logical rules, one for each branch. Consider, for instance, the left-most branch, the one claiming the falsity of the first conjunct. In this case we rewrite the rule as follows: $\dfrac{\dashv A}{\dashv A \wedge B}$. Then we add contexts variables on both sides so as to get the following logical rule:

$$\frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta, A \wedge B} \dashv \wedge_{\mathcal{R}}.$$

Each of the other logical rules can be obtained by applying the procedure in the same way.

Two interesting facts are worth observing at this point of our discussion. The first is that $\overline{\mathsf{LK}}$ is paraconsistent, in the sense that, for any truth-functional contingency A, $\overline{\mathsf{LK}}$ proves both the antisequents $\dashv A$ and $\dashv \neg A$. This peculiarity of the calculus has been emphasised in [Pulcini and Varzi, 2018] to draw some philosophical remarks and reflect light on the logical nature of the notion of *paraconsistency*, cf. [Priest, 2002, Carnielli and Coniglio, 2016]. The other fact is that $\overline{\mathsf{LK}}$ can be classified as a substructural logic insofar as it can be proved sound and complete with respect to the set of invalid sequents without resorting to the structural rules Weakening and Contraction, neither explicitly nor implicitly.

4.3.2 Cut-elimination and related computational properties.

As is well-known, there are two ways to formulate the cut-rule. The context-mixing version is termed *multiplicative* cut:

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,; \tag{MC}$$

whereas the context-sharing rule is called *additive* cut:

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\,. \tag{AC}$$

In presence of the structural rules Weakening and Contraction, the two versions MC and AC can be easily proved to be equivalent. However, as already observed, the $\overline{\mathsf{LK}}$ calculus is a completely substructural system and so this equivalence cannot be imported from the affirmative to the refutational side. As a matter of fact, whereas the additive version proves admissible in $\overline{\mathsf{LK}}$ (namely, adding AC to the $\overline{\mathsf{LK}}$ sequent calculus does not entail new provable antisequents), its multiplicative counterpart is not. The following is an easy counterexample to the admissibility of MC in $\overline{\mathsf{LK}}$.

$$\frac{p \dashv q \qquad q \dashv p}{p \vdash p}\,\text{MC}$$

Tiomkin proposes the two following 'hybrid' structural rules that he calls "cuts for the unprovability" [Tiomkin, 1988].

$$\frac{\Gamma \dashv \Delta \qquad \Gamma \vdash \Delta, A}{\Gamma, A \dashv \Delta} \qquad\qquad \frac{\Gamma \dashv \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \dashv \Delta, A}$$

The problem with these rules is that they do not violate the subformula property so, from a strict proof-theoretic point of view, they can be hardly defined as ‘cuts’ for all intent and purposes: where is the cut-formula? what is to be cut?

As shown in [Carnielli and Coniglio, 2016], the additive cut, though admissible, is unnecessarily general for $\overline{\mathsf{LK}}$, in the sense that it can be replaced by the two following simpler unary cuts.

$$\frac{\Gamma, A \dashv \Delta}{\Gamma \dashv \Delta} \dashv cut \qquad\qquad \frac{\Gamma \dashv \Delta, A}{\Gamma \dashv \Delta} \, cut \dashv$$

Once these rules are added to the system, AC turns out not only admissible, but also trivially derivable.

In [Carnielli and Pulcini, 2017], the authors provide a fully syntactical proof of cut-elimination, i.e., any antisequent provable by means of a proof that resorts to cut applications is also provable by means of a cut-free proof. Such a result is, of course, implicit in the completeness proof provided, for instance, in [Tiomkin, 1988]. However, the fact of focussing on the dynamics of a normalization procedure instead of dealing with the static concept of cut-redundancy provides the full setting for the study of the computational properties of $\overline{\mathsf{LK}}$. In particular, the normalization procedure outlined in [Carnielli and Pulcini, 2017] has been shown to enjoy the weak normalization property as well as the uniqueness of the normal form. Computational properties related to normalization in $\overline{\mathsf{LK}}$ have been further optimized in [Pulcini and Varzi, 2019] by resorting to the proof-nets technique.

In the traditional terminology of proof theory, a proof is said to be *analytic* in case any formula displayed in the proof is a subformula of some formula among those displayed in the end-sequent. To take an example, the proof presented in Figure 2 is analytic. In both LK and $\overline{\mathsf{LK}}$, cut-elimination implies the fact that any provable (anti)sequent is provable analytically, i.e., by means of an analytic proof. However, in LK, analyticity does not necessarily imply simplicity since cut-elimination may increase the size of LK-proofs. On the contrary, the implementation of cut-elimination in the complementary system $\overline{\mathsf{LK}}$ always returns one of the simplest proofs (intended as the shortest) for any provable antisequent [Carnielli and Pulcini, 2017].

5 Some applications to non-classical logics

5.1 Defining non-classical logics via refutability

5.1.1 Positive and non-negative ways of defining a logic

Propositional logics are usually defined in a positive way, as the least sets of formulas containing certain given formulas (axioms, deemed a priori acceptable/valid) and closed under given inference rules (usually, *uniform substitution* SUB, MP, and possibly others). However, non-classical logics are often motivated by negative requirements saying that certain principles are not acceptable, hence any formulas that formalise or imply them should be rejected. For example, Intuitionistic Logic can be obtained from some systems of Classical Logic CL by rejecting the law of excluded middle (LEM) $p \vee \neg p$. Or, paraconsistent logics (see [Skura, 2004a]) can be obtained from CL by rejecting the law of explosion:

$$(\mathsf{XP}) \qquad \neg p \to (p \to q)$$

A positive way to define such a logic would be to:

1. Specify a set POS of formulas (axioms) that we regard as acceptable.
2. Ensure that the rejected law is not derivable from POS by the inference rules.
3. Define the new logic as the least set of formulas containing POS and closed under the inference rules.

In [Skura, 2004a], a new, 'non-negative' method of defining a logic was introduced. It can be described as follows.

- In addition to POS, specify a set NEG of rejected formulas.
- Make sure that no $A \in \mathrm{NEG}$ is derivable from POS.
- Define the set $\mathsf{Ref}(\mathrm{POS}, \mathrm{NEG})$ (of *refutable formulas*) as follows:

  $A \in \mathsf{Ref}(\mathrm{POS}, \mathrm{NEG})$ iff some $B \in \mathrm{NEG}$ is derivable from A by using POS and the inference rules.

- Define the new logic to be the set of non-refutable formulas (that is, the complement $-\mathsf{Ref}(\mathrm{POS}, \mathrm{NEG})$ of $\mathsf{Ref}(\mathrm{POS}, \mathrm{NEG})$.

Metaphorically speaking, in the positive approach we want to keep in what is good, and in the 'non-negative' approach we want to keep out what is bad.

We now briefly discuss two examples of non-classical logics obtained from CL in the non-negative way.

5.1.2 Paraconsistent logics

Let $\mathrm{FOR}(\to, \wedge, \vee, \bot)$ be the set of all formulas generated from the set VAR of propositional variables by the binary connectives $\to, \wedge, \vee$ and the constant $\bot$ (falsum). We define $\neg A := A \to \bot$.

We take as POS the axioms of Positive Logic, i.e. the fragment of the intuitionistic logic IPL with the connectives $\to, \wedge, \vee$ (cf. e.g. [Odintsov, 2008]). As NEG we take $\{\mathsf{XP}\}$.

Johansson's logic J (cf. [Odintsov, 2008] is the least set of formulas of $\mathrm{FOR}(\to, \wedge, \vee, \bot)$ containing POS and closed under SUB and MP. We note that adding XP to **J** results in IPL.

For any $A \in \mathrm{FOR}$, the symbol $\mathbf{J}A$ denotes the extension of **J** with the additional axiom A, i.e. the least set of formulas containing $\mathbf{J} \cup \{A\}$ and closed under uniform substitution and MP. For example, the logic $\mathbf{J}E$, where

(E) $\quad p \vee (p \to q) \quad$ (the extended law of excluded middle)

is called "*the logic of classical refutability*" (see [Odintsov, 2008]).

Then, by using some results in [Skura, 2004a, Skura, 2017a] and in [Odintsov, 2008], it can be shown that

$$-\mathsf{Ref}(\mathrm{POS}, \mathrm{NEG}) = \mathbf{J}E.$$

As an interesting corollary, we obtain the following. Let L be an (axiomatic) extension of **J**. We say that L is a **paraconsistent analogue of CL** iff $\mathsf{L} \subseteq \mathsf{CL}$ and $\mathsf{XP} \notin \mathsf{L}$. (Hence, $\mathsf{L} \subset \mathsf{CL}$.) It turns out that $-\mathsf{Ref}(\mathrm{POS}, \mathrm{NEG})$ is the greatest paraconsistent analogue of CL (see [Skura, 2004a, Corollary 3.2]).

5.1.3 Implicational relevance logics

A logic L is said to have the **variable-sharing property** (VSP) if

$A \to B$ is non-valid in L, whenever A and B share no variables.

For instance, *relevance logics* are logics with the VSP (see [Anderson and Belnap, 1975]).

The VSP is readily presented as a negative property. It turns out (see [Skura, 2019a]) that, in a large class of *implicational logics* (i.e. logics with $\to$ as the only logical connective), the VSP is equivalent to the simpler property that the following formula PP, regarded as unacceptable in relevance logics, is non-valid:

$$(\mathsf{PP}) \qquad p \to (q \to p) \qquad \text{(Positive Paradox)}$$

We now take the axioms of the implicational fragment $\mathbf{R}_{\to}$ of the relevance logic $\mathbf{R}$ as POS and $\{\mathsf{PP}\}$ (or all formulas $A \to B$, where A and B share no variables) as NEG.

Characterizing the set $-\mathsf{Ref}(\mathrm{POS}, \mathrm{NEG})$ turns out to be a hard technical problem (see [Skura, 2019a] for some open problems). However, extending the basic logic to $\mathbf{RMO}^{\top}_{\to}$, defined below, makes the task feasible.

Let $\mathrm{FOR}(\to, \top)$ be the set of all formulas generated from VAR by the connective $\to$ and the constant $\top$. The logic $\mathbf{RMO}^{\top}_{\to}$ is defined as the least set of formulas in $\mathrm{FOR}(\to, \top)$ closed under the rules SUB and MP, and containing the following axioms (cf. [Avron, 1984, Avron, 1990], [Dunn and Restall, 2002]).

- $p \to \top$
- $p \to (p \to p)$
- $(p \to q) \to ((q \to r) \to (p \to r))$
- $(p \to (q \to r)) \to (q \to (p \to r))$
- $(p \to (p \to q)) \to (p \to q)$

Consider the logical matrix (see [Skura, 2019a] for details)

$$\mathbf{3} := (\{-1, 0, 1\}, \{0, 1\}, \to),$$

where the set of designated values is $\{0, 1\}$ and $x \to y = max(-x, y)$ if $x \leq y$; else $x \to y = min(-x, y)$. A **valuation** in the matrix $\mathbf{3}$ is a function v from VAR to $\{-1, 0, 1\}$ extended with $v(\top) = 1$ and $v(A \to B) = v(A) \to v(B)$.

We say that a formula A is **valid in 3**, denoted $A \in Val(\mathbf{3})$, iff $v(A) \in \{0, 1\}$ for every valuation v. We remark that $\mathbf{RMO}^{\top}_{\to} \subseteq Val(\mathbf{3})$, and $\mathsf{PP} \notin Val(\mathbf{3})$.

Let us now take as POS the set of axioms of $\mathbf{RMO}_{\rightarrow}^{\top}$ listed above, and as NEG we take $\{\mathsf{PP}\}$. Then, as shown in [Skura, 2019a], we have:

$$-\mathsf{Ref}(\mathrm{POS}, \mathrm{NEG}) = Val(\mathbf{3}).$$

Also, let L be an axiomatic extension of $\mathbf{RMO}_{\rightarrow}^{\top}$. We say that L is a **relevant analogue** of CL iff $\mathsf{L} \subseteq \mathsf{CL}$ and $\mathsf{PP} \notin \mathsf{L}$. (Thus, $\mathsf{L} \subset \mathsf{CL}$.) It follows that $-\mathsf{Ref}(\mathrm{POS}, \mathrm{NEG})$ is the greatest relevant analogue of CL (see [Skura, 2004a, Corollary 3.2]).

6 Concluding remarks

Refutation systems are not yet explored nearly as deep as traditional logical deductive systems, and there is an abundance of related open questions and potential applications, in addition to those discussed in the present paper. We only briefly mention here one such application, viz. to provide a recursive axiomatization of the non-validities of a logic that may not be (known to be) recursive axiomatizatizable, but the non-validities of which are known to be effectively enumerable. A prime example is $\mathsf{FOL}^{\mathsf{fin}}$, for which several Ł-complete refutation systems proposed in the literature (referenced in Section 3.4) illustrate the idea. More generally, recursive axiomatizatizable Ł-complete refutation systems exist for any logic or theory that is defined in terms of an effectively enumerable class of finite models. For some generic results of that type see [Goranko and Skura, 2018].

A well-known example of a logic with no known (and, possibly not existing) recursive axiomatization is *Medvedev's logic of finite problems*, for which a surprisingly simple Ł-complete refutation system (though, employing deduction in Kreisel-Putnam's logic KP) was designed in [Skura, 1992]. There are, however, many less-known such logical systems waiting for Ł-complete refutation systems to be designed for them; just one example is the interval temporal logic of reflexive subinterval relation over the integers, proved to be (surprisingly) undecidable in [Marcinkowski and Michaliszyn, 2011], while it has the finite model property by definition, and therefore an effectively enumerable set of non-validities.

In conclusion, the inherent importance and potential of the idea of refutation systems is yet to be unravelled. That idea is still waiting for new, sufficiently convincing applications that would attract the attention of the wider logic community and would eventually properly address Łukasiewicz's concern about the disparity between deductive acceptance and rejection of a logical statement, raised in [Łukasiewicz, 1951]. Discovering such applications is the so far main challenge in that field.

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