SOME FAMILIES OF BIG AND STABLE BUNDLES ON $K3$ SURFACES AND ON THEIR HILBERT SCHEMES OF POINTS

GILBERTO BINI, SAMUEL BOISSIÈRE, AND FLAMINIO FLAMINI

Abstract. Here we investigate meaningful families of vector bundles on a very general polarized $K3$ surface (X, H) and on the corresponding *Hyper–Kähler variety* given by the Hilbert scheme of points $X^{[k]} := \mathrm{Hilb}^k(X)$, for any integer $k \geqslant 2$. In particular, we prove results concerning bigness and stability of such bundles. First, we give conditions on integers n such that the twist of the tangent bundle of X by the line bundle nH turns out to be big and stable on X; we then prove a similar result for a natural twist of the tangent bundle of $X^{[k]}$. Next, by a careful analysis on Segre classes, we prove bigness and stability results for *tautological bundles* on $X^{[k]}$ arising either from line bundles or from *Mukai-Lazarsfeld* bundles, as well as from *Ulrich* bundles on X.

1. Introduction

Let M be an n-dimensional smooth projective variety over the field of complex numbers. A line bundle L on M is big if the Iitaka dimension $\kappa(M, L)$ is maximal. If L is in particular nef, the bigness of L can be deduced from a numerical criterion, that is, L is big if and only if the top intersection of $c_1(L)$ is a positive integer. If we take into account vector bundles E of rank greater than or equal to 2, there are various notions of bigness: see, for instance, [3] for a comprehensive survey, as well as their base loci. Here we focus on L-bigness which is a natural generalization of ampleness and nefness for vector bundles. More specifically, E is L-big if and only if the tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a big line bundle on the projective bundle $\pi : \mathbb{P}(E) \to M$ of one-dimensional quotients of E. Henceforth, for the sake of notation, by big we mean L-big.

Notably, the numerical criterion for bigness of nef line bundles induces a characterization for bigness of nef vector bundles of higher rank. For these purposes, we recall the definition of the Segre classes of E and give a numerical characterization of bigness in Section 2.2.3, Proposition 2.4. This states that a nef rank r vector bundle E on M is big if and only if the number $(-1)^n \int_M s_n(E)$ is positive, where $s_n(E)$ is the top Segre class of the vector bundle E. In what follows, we will apply this characterization to globally generated vector bundles, which are in fact nef: see Remark 2.5.

In [7], we introduced cohomological criteria on algebraic surfaces and fourfolds in order to verify the numerical characterization mentioned before. What's more, we found out examples of big vector bundles (split and unsplit) on Hirzebruch surfaces and investigated the bigness of some families of Mukai-Lazarsfeld bundles on regular fourfolds. These criteria were also applied in [20] to describe non-big Ulrich bundles on a complex irreducible smooth projective surface. For the sake of completeness, we recall that E is a Ulrich bundle on M if $H^i(M, E(-p))$ vanishes for $i \geqslant 0$ and $1 \leqslant p \leqslant \dim(M)$.

In the present paper, we investigate bigness of vector bundles on other families of varieties, namely *Hyper–Kähler varieties*. Previous results on big and nef line bundles on them, and especially on their base loci, were obtained in [36]. The Beauville-Bogomolov Theorem (see [5], Theorem 1) states that, up to a finite cover, any compact Kähler manifold with trivial first Chern class (in the rational cohomology) can be decomposed as a product of complex tori, (strict) Calabi-Yau varieties and Hyper–Kähler varieties. By definition, the latter have even complex dimension. The first examples are thus given by $K3$ surfaces X. For the purposes of what follows, we shall focus on very general polarized $K3$ surfaces (X, H) where H is an ample divisor on X such that $H^2 = 2g - 2 \geqslant 2$. By general results (see, for instance, [2], Theorem VIII 7.3 on page 366) there exists a smooth, irreducible 19-dimensional moduli space $\mathfrak{F}_g$ which parametrizes (isomorphism classes of) smooth, primitively polarized pairs (X, H) of genus g. A very general point of $\mathfrak{F}_g$ corresponds to a *very general polarized* $K3$ surface (X, H).

Any rank r vector bundle E on X defines a *Mukai vector* $v := v(E)$: see Section 3 for the definition. Correspondingly, one denotes by $M_H(v)^s$ the moduli space of μ_H-stable vector bundles on (X, H) associated with the Mukai vector

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v, where $\mu_H(E) := \frac{\int_X c_1(E)\cdot H}{r}$ is the H-slope of E. Moreover, let us consider the vector bundle $T_X(n) := T_X \otimes H^{\otimes n}$ on (X,H) and denote by $v_{g,n}$ the associated Mukai vector, where g is the genus of (X,H). Then Theorem 3.5. lists pairs of possible integers (g,n) such that $T_X(n)$ is big. Moreover, for any such pair the moduli space $M_H(v_{(g,n)})^s \neq \emptyset$ and its irreducible component $M_H(T_X(n))^s$ is a smooth, quasi-projective variety of dimension 90 whose general element parametrizes a rank 2 vector bundle with Mukai vector $v_{(g,n)}$. The proof follows from explicit calculations by Hirzebruch-Riemann-Roch Theorem and Proposition 2.1, which gives a useful criterion for the bigness of a vector bundle: see [33], Example 6.1.22, for further details. As for the proof of stability of $T_X(n)$ in the cases above, we apply a series of results which are recalled in Proposition 3.1 (see, for instance, [26] for definitions and proofs).

Denote by $X^{[k]} := Hilb^k(X)$ the Hilbert scheme of zero-dimensional subschemes of length k on a polarized $K3$ surface (X,H): see, e.g., [8], [11], [12], [29], [37], [38], [39]. This turns out to be a Hyper–Kähler variety. Similarly to the case of polarized K3 surfaces, we prove some bigness and stability results for the rank $2k$ tangent bundle up to a twist. More precisely, set $Y = X^{[k]}$ and consider the tangent bundle T_Y on Y. The polarization H on the $K3$ surface X gives a big and nef line bundle H_Y as in (4.3). Then Theorem 5.1 lists pairs of positive integers (g,n) such that $T_Y \otimes H_Y^{\otimes n}$ is a μ_{H_Y}-stable, where g is the genus of X.

Besides the tangent bundle, we also focus on other families of vector bundles, first on very general polarized $K3$ surfaces and after that on the Hilbert scheme of points on them. Henceforth, assume the genus of X is greater than or equal to 3. For a primitively polarized $K3$ surface, pick a general curve $C \in |H|$ and a complete linear series $|A| = g^{r-1}_d$ on C, with suitable properties of global generation on A and integrality assumptions on any member of $|H|$: see Section 3.2 for precise statements. Lazarsfeld defines a rank r vector bundle $E := E_{C,A}$ on X, which encodes several properties of Brill-Noether and Petri theory of the scheme parametrizing special linear series on C. The vector bundle E is usually called the *Mukai-Lazarsfeld vector bundle* associated with the pair (C,A). In this setting, Theorem 3.8 proves that for any triple of positive integers $(g, r-1, d)$ such that $d < 2g-2$ and the Brill-Noether number $\rho(g, r-1, d)\geqslant 0$, there exists a vector bundle E which is globally generated and μ_H-stable on X. Moreover, if $\rho(g, r-1, d) = 0$ the moduli space $M_H(v)^s$ consists of a single reduced point, which yields an (isomorphism class) of big bundles. If $\rho(g, r-1, d)$ is positive, the general bundle in the moduli space $M_H(v)^s$ is globally generated and big. Here v is the Mukai vector $v = (r, H, g-1-d+r)$.

Another family of examples comes from Ulrich bundles on very general polarized surfaces of genus $g = h+1$, which are dealt with in Section 3.3. There, Theorem 3.10 proves that for any positive integer $a\geqslant 1$ there exists an $(8a^2 + 2a^2h + 2)$-dimensional family of μ_H-stable Ulrich bundles E on X with Mukai vector $v = (2a, 3aH, 2a(h-1))$. More geometrically, there exists an irreducible component $M^s_H(E)$ of the moduli space $M_H(v)^s$ whose general point corresponds to a μ_H-stable Urlich bundle of rank $2a$, which is also globally generated and big.

If E is a rank r vector bundle, the Hilbert scheme $X^{[k]}$ carries a natural rank rk vector bundle $E^{[k]}$, which is known as the *tautological bundle* associated with E: see Section 5. Then it is natural to consider tautological bundles arising from Mukai-Lazarsfeld bundles and Ulrich bundles, with the same notation and assumptions in Theorem 3.8 and Theorem 3.10, respectively. In order to analyze their bigness and stability, and apply the numerical characterization mentioned before, we need to determine if these tautological bundles are globally generated and if they fulfill the numerical criterion in Proposition 2.4.

To this extent, we proceed as follows. First, set $L_n = H^{\otimes n}$ where H is the polarization on X. As proved by C. Voisin in [42], the tautological bundle $L_1^{[k]}$ is generated by global section for $g > 2k-2$, where g is the genus of the polarized pair (X,H). As for $L_n^{[k]}$, Theorem 5.2 shows that this is a globally generated rank k vector bundle on $X^{[k]}$. The proof is based on the notion of $(k-1)$-very ampleness of L_n. In particular, a vector bundle E is $(k-1)$-very ample on X if and only if the tautological bundle $E^{[k]}$ is globally generated on the Hilbert scheme $X^{[k]}$: see Proposition 4.4, i). Proposition 4.4 ii) proves that if E is globally generated and $L_1^{[k]}$ is globally generated, i.e., L_1 is $(k-1)$-very ample, then $(E \otimes L_1)^{[k]}$ is globally generated. We will apply this proposition to a Mukai-Lazarsfeld bundle or a Ulrich bundle E. To this end, we suitably adapt arguments used in [6] for the stronger notion of k-jet ampleness of vector bundles. As for the bigness behaviour, it remains to check $\int_{X^{[k]}} s_{2k}((E \otimes L_1)^{[k]}) > 0$, where E is either a Mukai-Lazarsfeld bundle or a Ulrich bundle on X.

Therefore, we need a formula for the top Segre class of a tautological bundle $F^{[k]}$ for a rank $r\geqslant 1$ vector bundle F on X. The total Segre class $s(F^{[k]})$ is computed in Proposition 4.2 via a recurrence relation, which is based on a short exact sequence connecting the tautological bundle $F^{[k-1]}$ and the tautological bundle $F^{[k]}$ by pulling them back on the incidence scheme $X^{[k-1,k]}$, which parametrizes triples $(\xi, x, \xi') \in X^{[k-1]} \times X \times X^{[k]}$ such that $\xi \subset \xi'$ with residual subscheme supported at the point x: see Section 4.2 for further details. From the recurrence relations of the total Segre class, we obtain recurrence relations for the d-th Segre class $s_d(F^{[k]})$ in terms lower degree Segre classes for $0 \leqslant d \leqslant 2k$ (cf. 4.8). As an example, we give an explicit formula for $s_2(F^{[2]})$ and $s_3(F^{[3]})$ in Corollary 4.3. This requires the formalism of the Heisenberg algebra (cf., e.g., [19]), which describes the cohomology algebra structure of the Hilbert scheme of points on a surface.

Notation and terminology. Throughout, we work over the field $\mathbb{C}$ of complex numbers. By *variety* we mean an integral algebraic scheme Y. We say that a property holds for a *general* (resp. *very general*) point $y \in Y$ if it holds on a Zariski open nonempty subset of Y (resp. on the complement of the countable union of proper subvarieties of Y). For any vector bundle E on Y, we denote by S^mE the m^{th}–symmetric power of E and by $\mathrm{Sym}(E)$ the symmetric algebra. We denote by T_Y the tangent bundle of Y.

2. Preliminaries

2.1. **Chern and Segre classes.** Let X be a smooth, complex projective variety of dimension $n \geqslant 2$ and E be a rank-r vector bundle on X, $r \geqslant 2$. We set $\mathbb{P}(E) \coloneqq \mathrm{Proj}(\mathrm{Sym}(E))$ the projective bundle parametrizing 1-dimensional quotients of the fibres of E, we denote by $\mathcal{O}_{\mathbb{P}(E)}(1)$ the *tautological line bundle* on $\mathbb{P}(E)$ and by $\mathbb{P}(E) \xrightarrow{\pi} X$ the canonical projection (cf. e.g. [23]). We define the i^{th} *Segre class* of E as:

$$s_i(E) \coloneqq \pi_* \left(c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^{r-1+i} \right) \in H^{2i}(X, \mathbb{Z}). \tag{2.1}$$

The *total Segre class* of E is given by $s(E) \coloneqq 1 + s_1(E) + s_2(E) + \cdots \in H^*(X, \mathbb{Z})$. The *Chern classes* $c_i(E) \in H^{2i}(X, \mathbb{Z})$ of E are defined as the coefficients of the inverse formal series of $s_E(t) \coloneqq \sum_{i=0}^{+\infty} s_i(E)t^i$, i.e. $c_E(t) = s_E(t)^{-1}$ and $c_E(t) \coloneqq \sum_{i=0}^{+\infty} c_i(E)t^i$. We put $c(E) \coloneqq c_E(1)$. In particular, one has (cf. e.g. [18, § 3.2]):

$$c_1(E) = -s_1(E),\ c_2(E) = s_1(E)^2 - s_2(E), \ldots, c_i(E) = -s_1(E)c_{i-1}(E) - s_2(E)c_{i-2}(E) - \cdots - s_i(E),\ \forall\, i \geqslant 3 \tag{2.2}$$

Denoting by $E^\vee$ the dual bundle of E, for any line bundle L on X one has (cf. [18, Rem. 3.2.3 (a), Ex. 3.2.2, Ex. 3.1.1]):

$$c_i(E^\vee) = (-1)^i c_i(E) \ \text{ and } \ c_i(E \otimes L) = \sum_{j=0}^{i} \binom{r-j}{i-j} c_j(E) c_1(L)^{i-j}, \ \ 1 \leqslant i \leqslant r, \tag{2.3}$$

$$s_i(E^\vee) = (-1)^i s_i(E) \ \text{ and } \ s_i(E \otimes L) = \sum_{j=0}^{i} (-1)^{i-j} \binom{r-1+i}{r-1+j} s_j(E) c_1(L)^{i-j}. \tag{2.4}$$

2.2. **Positivity of vector bundles.** We remind some definitions concerning certain *dimension* and *positivity* notions related to vector bundles over X; for more details, we refer the reader to [7, § 2.2] and to references therein. These concepts are first defined for line bundles L on X and then for vector bundles E of rank $r \geqslant 2$ by considering the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$.

2.2.1. *Kodaira–Iitaka dimension, bigness and nefness.* Let L be a line bundle on X; its *Kodaira–Iitaka dimension* $k(L)$ is defined as:

$$k(L) \coloneqq \begin{cases} -\infty & \text{if } h^0(L^{\otimes m}) = 0,\ \forall\, m \in \mathbb{N} \\ \max_{m \in \mathbb{N}} \dim(\varphi_{L^{\otimes m}}(V)), & \text{otherwise,} \end{cases}$$

where $X \overset{\varphi_{L^{\otimes m}}}{\dashrightarrow} \mathbb{P}(H^0(L^{\otimes m})^\vee)$ denotes the rational map given by the linear system $|L^{\otimes m}|$. Then L is said to be *big* if $k(L) = n = \dim(X)$, and L is called *nef* if $L \cdot C \geqslant 0$ for any effective curve $C \subset X$.

Let now E be any rank r vector bundle on X, with $r \geqslant 2$. Similary as above, its *Kodaira–Iitaka dimension* $k(E)$ is defined to be $k(E) \coloneqq k(\mathcal{O}_{\mathbb{P}(E)}(1))$. The vector bundle E is said to be *big* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is big on $\mathbb{P}(E)$. We thus have that E is big if and only if $k(E) = \dim(\mathbb{P}(E)) = n + r - 1$. It is said to be *nef* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle on $\mathbb{P}(E)$ (cf. e.g. [15, Definition 1.9]). We recall for later use the following result (cf. [33, Ex. 6.1.22]):

Proposition 2.1. *Assume that $H^0(X, S^mE) \neq 0$ for some $m \geqslant 1$. Then for any ample line bundle A on X, the vector bundle $E \otimes A$ is big.*

2.2.2. *Numerical dimension.*

Definition 2.2. (cf.[22, II.E, p. 24]) Let L be any *nef* line bundle on X. The *numerical dimension* of L is defined to be the largest integer $n(L)$ such that $c_1(L)^{n(L)} \neq 0$.

By [14] (cf. also [22, (II.E.1), p.24]) one has $k(L) \leqslant n(L)$, and equality holds if $n(L) = \dim(X)$. Let now E be a globally generated vector bundle, of rank $r \geqslant 2$. From Remark 2.5, E is nef, i.e. $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle on $\mathbb{P}(E)$. Therefore, it makes sense to consider its numerical dimension:

Definition 2.3. ([22, § II.E, p.25]) Let E be a globally generated vector bundle of rank r on X. The *numerical dimension* of E is $n(E) := n(\mathcal{O}_{\mathbb{P}(E)}(1))$.

Since $\mathcal{O}_{\mathbb{P}(E)}(1)$ is very ample on the fibres of the projection $\mathbb{P}(E) \xrightarrow{\pi} X$, one has $r-1 \leqslant n(E) \leqslant \dim(\mathbb{P}(E))$. On the other hand, since E is nef, by (2.1) and Definition 2.2, observing that the morphism π_* consists in integrating over the fibers, we have that $n(E)$ is the largest integer with $s_{n(E)-r+1}(E) \neq 0$. We also have $k(E) \leqslant n(E)$, where the equality holds when $n(E) = \dim(\mathbb{P}(E)) = n + r - 1$.

2.2.3. *A numerical characterization of bigness.* The following classical result gives a numerical criterion for the bigness of a nef vector bundle. We briefly recall the proof for the reader's convenience.

Proposition 2.4. *Let E be a nef r vector bundle on a n-dimensional smooth projective variety X. Then E is big if and only if $(-1)^n \int_X s_n(E) > 0$.*

Proof. Collecting all the notions recalled above, since E is nef, its numerical dimension $n(E)$ coincides with the largest integer for which $s_{n(E)-r+1} \neq 0$. Then $n(E)-r+1 = n$ is equivalent to $k(E) = n(E) = \dim(\mathbb{P}(E))$, meaning that E is big, so the bigness of E is equivalent to the non-vanishing of its top Segre class. Applying [15, Theorem 2.5] with $n = k$, $Y = X$ and $a = (1^n)$, this is equivalent to the positivity of the Schur polynomial $P_{(1^n)}(c(E)) = s_n(E^\vee) = (-1)^n s_n(E)$ (cf. [33, 8.3.5]). □

Remark 2.5. If E is globally generated, then E is nef. Indeed, taking $\mathbb{P}(E) \xrightarrow{\pi} X$ the natural projection, global generation of E ensures that π^*E is globally generated. Since $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a quotient of π^*E, the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is globally generated too. Hence, since $|\mathcal{O}_{\mathbb{P}(E)}(1)|$ defines a morphism to a suitable projective space $\mathbb{P}$, then $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef because it is the pull-back via this morphism of the very ample line bundle $\mathcal{O}_{\mathbb{P}}(1)$.

In what follows, we will apply Proposition 2.4 to some globally generated vector bundles.

3. On some big and stable vector bundles on K3 surfaces

Let (X, H) be a smooth polarized $K3$ surface of genus $g \geqslant 2$, where H is an ample divisor on X such that $H^2 = 2g - 2 \geqslant 2$. From [26, Def. 9.1.2, p.169], if E is a rank-r vector bundle on X, its *Mukai vector* is defined as

$$v(E) := (r, c_1(E), \chi(E) - r) = \left(r, c_1(E), \int_X \left(\frac{c_1(E)^2}{2} - c_2(E)\right) + r\right) \in H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) := H^*(X,\mathbb{Z}). \tag{3.1}$$

Moreover, for $\alpha = (\alpha_0, \alpha_2, \alpha_4), \beta = (\beta_0, \beta_2, \beta_4) \in H^*(X,\mathbb{Z})$, the *Mukai pairing* is defined as (cf. [26, Def. 9.4, p.169])

$$\langle \alpha, \beta \rangle := (\alpha_2 \cdot \beta_2) - (\alpha_0 \cdot \beta_4) - (\alpha_4 \cdot \beta_0). \tag{3.2}$$

Any Mukai vector $v := v(E)$ can be uniquely written as $v = mv_0$, where $m \geqslant 0$ is an integer and $v_0 \in H^*(X,\mathbb{Z})$ is indivisible i.e. *primitive*, equivalently the integer m is maximal (cf. [26, § 10.2, p.198]). When $m = 1$, then v itself is a *primitive* Mukai vector. From [26, § 9.3, p. 175], the H-*slope* of a rank-r vector bundle E on X is defined as

$$\mu_H(E) := \frac{\int_X c_1(E) \cdot H}{r}$$

and E is said to be μ_H-*stable* (or *slope-stable*) if for all subsheaves $F \subset E$ one has $\mu_H(F) < \mu_H(E)$. Recall that the μ_H-stability is preserved by taking dual bundles and by tensoring with line bundles.

For a given Mukai vector $v \in H^*(X,\mathbb{Z})$, one denotes by $M_H(v)^s$ the *moduli space* of μ_H-stable vector bundles on (X, H) of given Mukai vector v. Then one has:

Proposition 3.1. *Either $M_H(v)^s$ is empty or it is a smooth, quasi-projective scheme of (equi) dimension $2 + \langle v, v \rangle$ (cf. [26, §;10, Cor. 2.1, p. 196]). Moreover:*

(i) *If* $\langle v, v\rangle = -2$*, then* $M_H(v)^s$ *is either empty or it consists of one reduced point (cf.* [26, § 10, Prop. 3.1, p. 200]*).*
(ii) *If* v *is primitive,* H *is a general polarization, the rank* $r \geqslant 1$ *and* $\langle v, v\rangle \geqslant -2$*, then* $M_H(v)^s$ *is an open dense subset of an irreducible symplectic projective manifold* M *which is deformation equivalent to* $Hilb^{2+\langle v,v\rangle}(X)$*. In particular* $M_H(v)^s$ *is irreducible (cf.* [26, § 10, Thm. 3.10, p. 205]*).*

3.1. **The tangent bundle of a very–general polarized K3.** For any polarized $K3$ surface (X, H) one has (cf. [26, Example 9.1.6, p. 170]):

$$v(T_X) = (2, 0, 2 - e(X)) = (2, 0, -22) \text{ and } \langle v(T_X), v(T_X)\rangle = 88. \tag{3.3}$$

Moreover, the following properties hold:

$$\begin{aligned} &(i)\, h^0(X, T_X) = 0 && \text{(cf. [26, § 1.2.4, p. 13])} \\ &(ii)\, h^0(X, S^mT_X) = 0,\ \forall\ m \geqslant 1 && \text{(cf. [26, Cor. 9.4.13, p. 183])} \\ &(iii)\, T_X \text{ is } \mu_H-\text{stable} && \text{(cf. [26, Prop. 9.4.5, p. 180])} \end{aligned} \tag{3.4}$$

As explained in Introduction, in this paper we are interested in big vector bundles. From (3.4)–(i) and (ii), no m^{th}-symmetric power of T_X can be globally generated, for any $m \geqslant 1$. This means that T_X does not satisfy assumptions as in Proposition 2.1 which, therefore, cannot be applied.

Thus, in what follows, we are concerned in finding sufficient conditions ensuring the existence of a suitable positive integer n_0 for which $T_X \otimes H^{\otimes n}$ turns out to be big, for any integer $n \geqslant n_0$.

To do so, we shall focus on *very–general polarized* $K3$ *surfaces* in the following sense: from [2, Thm.VIII 7.3 and p. 366], there exists a smooth, irreducible moduli space $\mathfrak{F}_g$ of dimension 19 which parametrizes (isomorphism classes of) smooth, primitively polarized $K3$ surfaces (X, H) of genus $g \geqslant 2$. The pair $(X, H) \in \mathfrak{F}_g$ is called *a very general (polarized) K3* when (in the sense of the Introduction) (X, H) corresponds to a very general point of $\mathfrak{F}_g$. Moreover, when (X, H) is very–general, one in particular has $\mathrm{Pic}(X) \cong \mathbb{Z}[H]$. In this case, from (3.4)-(iii), (3.3) and Proposition 3.1, it follows that $M_H((2, 0, -22))^s \neq \emptyset$ and, denoting by $M_H(T_X)^s$ the irreducible component of $M_H((2, 0, -22))^s$ containing the point $[T_X]$, one has moreover that:

$$M_H(T_X)^s \text{ is a smooth, quasi-projective variety of dimension 90.} \tag{3.5}$$

To simplify notation, from now on we will moreover identify multiplicative notation of tensor power of line-bundles and additive notation of Cartier divisors, namely $H^{\otimes n}$ will be simply denoted by nH. Similarly, to denote $T_X \otimes H^{\otimes n}$, for any $n \geqslant 1$, we will simply set $T_X(n)$. Taking into account the isomorphism $T_X \cong \Omega^1_X$, we can reformulate the results in [4, § 5.2] as follows:

Proposition 3.2. *Let* (X, H) *be very–general polarized* $K3$ *surface of genus* $g \geqslant 2$*. Then, one has:*
(a) $h^0(X, T_X(1)) = 0$*, for* $2 \leqslant g \leqslant 9$ *or* $g = 11$*,*
(b) $h^0(X, T_X(1)) = 1$*, for* $g = 10$*,*
(c) $h^1(X, T_X(1)) = 0$*, for* $g = 11$ *or* $g \geqslant 13$*,*
(d) $h^1(X, T_X(1)) \geqslant 1$*, for* $g = 12$*.*

Remark 3.3. For any integer $n \geqslant 0$ one has

$$c_1(T_X(n)) = 2nH \text{ and } \int_X c_2(T_X(n)) = n^2H^2 + \int_X c_2(T_X) = 2n^2(g-1) + 24,$$

as it follows from (2.3) and from the facts that $c_0(T_X) = 1$, $c_1(T_X) = 0$, $\int_X c_2(T_X) = e(X) = 24$, $H^2 = 2(g-1)$. Moreover, $h^2(X, T_X(n)) = 0$; indeed $T_X \cong \Omega^1_X$ so, by Serre duality, $h^2(T_X(n)) = h^0(T_X(-n))$, the latter being zero because $T_X(-n) \subseteq T_X$ and T_X not effective by (3.4)–(i).

Riemann-Roch-Hirzebruch formula (cf. [18, Corollary 15.2.1]) therefore reads

$$\chi(T_X(n)) = h^0(T_X(n)) - h^1(T_X(n)) = \int_X \frac{c_1(T_X(n))^2}{2} - \int_X c_2(T_X(n)) + 2\,\mathrm{rk}(T_X(n)) =$$

$$= \frac{(2nH)^2}{2} - (2n^2(g-1) + 24) + 4 = \frac{4n^2H^2}{2} - 2n^2(g-1) - 20 = 4n^2(g-1) - 2n^2(g-1) - 20.$$

Thus

$$h^0(T_X(n)) = 2n^2(g-1) - 20 + h^1(T_X(n)) \geqslant 2n^2(g-1) - 20. \tag{3.6}$$

Using the previous computations, we prove the following useful Lemma.

Lemma 3.4. *Let (X,H) be a very–general $K3$ surface of polarization $g \geqslant 2$. Then, for any $g \geqslant 2$, there exists an integer $n_0(g)$, depending on g, for which the vector bundle $T_X(n)$ is effective, for any $n \geqslant n_0(g)$, where the values of $n_0(g)$ according to the genus g are the following:*

g	2	3	4	5	6	7	8	9	10	11	⩾12
$n_0(g)$	4	3	2	2	2	2	2	2	1	2	1

(3.7)

Proof. Notice that, for $g = 10$ and $g \geqslant 12$, Proposition 3.2 gives always $h^0(T_X(1)) \geqslant 1$, i.e. $n_0(g) = 1$ in all these cases. Indeed, for $g = 10$, Proposition 3.2–(b) directly gives $h^0(T_X(1)) = 1$; for $g = 12$, (3.6) applied for $n = 1$ gives $h^0(T_X(1)) = 2 + h^1(T_X(1)) \geqslant 3$, the latter inequality following from Proposition 3.2–(d); at last, for $g \geqslant 13$ Proposition 3.2–(c) gives $h^1(T_X(1)) = 1$ so, by (3.6), we have $h^0(T_X(1)) = 2(g-1) - 20 \geqslant 2(12) - 20 = 4$.

For $g = 11$, differently as above, Proposition 3.2–(a) and (c) give $h^0(T_X(1)) = h^1(T_X(1)) = 0$. On the other hand, formula (3.6) for $n = 2$ gives $h^0(T_X(2)) = 60 + h^1(T_X(2)) \geqslant 60$ so $n_0(11) = 2$.

We are left with the low–genus cases, i.e. $2 \leqslant g \leqslant 9$. From (3.6), for any integer $k \geqslant 0$ we get $h^0(T_X(k)) \geqslant 2k^2(g-1) - 20$, the left–side member being positive as soon as $k^2 > \frac{20}{2(g-1)} = \frac{10}{g-1}$. We therefore set $n_0(g) := \left\lceil \sqrt{\frac{10}{g-1}} \right\rceil$, which gives values as in (3.7). □

Theorem 3.5. *Let (X,H) be a very–general $K3$ surface of polarization $g \geqslant 2$. Then, $T_X(n)$ is a big vector bundle if:*

(1) $n \geqslant 5$, for $g = 2$;
(2) $n \geqslant 4$, for $g = 3$;
(3) $n \geqslant 3$, for $4 \leqslant g \leqslant 9$ or $g = 11$;
(4) $n \geqslant 2$, for $g \geqslant 10$ but $g \neq 11$.

For any pair (g,n) as above, let $v_{(g,n)}$ be the Mukai vector $v(T_X(n))$. Then $M_H(v_{(g,n)})^s \neq \emptyset$. Moreover, denoting by $M_H(T_X(n))^s$ the irreducible component containing the point $[T_X(n)]$, $M_H(T_X(n))^s$ is a smooth, quasi-projective variety of dimension 90 whose general element parametrizes a rank-2 big vector bundle on X of Mukai vector $v_{(g,n)}$.

Proof. We first focus on bigness. From Lemma 3.4, we know that $T_X(n_0(g))$ is effective, where $n_0(g)$ is a positive integer as in (3.7). We can therefore apply Proposition 2.1, with $E := T_X(n_0(g))$, $m = 1$ and $A = kH$, $k \geqslant 1$ any integer, to get that $T_X(n)$ is big for any integer $n := k + 1 \geqslant n_0(g) + 1$.

Concerning stability, from (3.4)-(iii), (3.5) and the fact that μ_H-stability is preserved under tensor product with line bundles, we deduce that, for all pairs (g,n) as in the statement, $M_H(v_{g,n})^s$ is not empty, moreover the irreducible component $M_H(T_X(n))^s$ containing the point $[T_X(n)]$ is a smooth quasi-projective variety of dimension 90. The assertion on $\dim(M_H(T_X(n))^s) = 90$ follows by a direct computation: indeed from Proposition 3.1 one has $\dim M_H(T_X(n))^s = 2 + \langle v_{g,n}, v_{g,n} \rangle$, where

$$v_{g,n} = v(T_X(n)) = \left(2, c_1(T_X(n)), \int_X \left(\frac{c_1(T_X(n))^2}{2} - c_2(T_X(n))\right) + 2\right) = \left(2,\ 2nH,\ 2n^2(g-1) - 22\right),$$

the last equality following from Remark 3.3 above. It is straightforward to compute that $\langle v_{g,n}, v_{g,n} \rangle = 88$ so, by Proposition 3.1, we get $\dim(M_H(T_X(n))^s) = 90$.[1]

At last, since bigness is an open condition and since $T_X(n)$ is big for any $n \geqslant n_0(g) + 1$, the general stable bundle parametrized by $M_H(T_X(n))^s$ is therefore big for any pair (g,n) as above. □

3.2. **Big and stable Mukai-Lazarsfeld vector bundles on $K3$ surfaces.** Smooth curves on polarized $K3$ surfaces, in particular their Brill-Noether theory, play a fundamental role in Algebraic Geometry. Indeed Brill-Noether theory of these curves is deeply connected to the geometry of the surface, to modular properties of curves on it as well as it is fundamental to prove results on smooth curves with general moduli with no use of degeneration techniques (cf. [31]).

Lazarsfeld's approach in [31] uses vector-bundle techniques on X; given (X,H) a primitively polarized $K3$ surface of genus $g \geqslant 3$, a general curve $C \in |H|$ and a complete linear series $|A| = g^{r-1}_d$ on C, with suitable properties of global generations on A and of integrality assumptions on any member of $|H|$, Lazarsfeld associates a rank-r vector bundle $E := E_{C,A}$ on X to the triple (X,C,A), the vector bundle E depending on the choice of $C \in |H|$ and of the line bundle A on C. This vector bundle E encodes several properties of Brill-Noether and Petri's theory of the scheme $W^{r-1}_d(C)$,

[1]The latter equality more intrisically follows from the fact that the operation of tensor product $- \otimes H^{\otimes n}$ establishes, for any integer $n \geqslant 1$, an isomorphism between the irreducible component $M_H(T_X)^s$ of the moduli space $M^s_H((2,0,-22))$ containing the point $[T_X]$ and the irreducible component $M_H(T_X(n))^s$ of the moduli space $M_H(v_{g,n})^s$ containing the point $[T_X(n)]$, the isomorphism sending $[T_X]$ to $[T_X(n)]$. Thus, from (3.5) $M_H(T_X(n))^s$ has the same dimension as $M_H(T_X)^s$ which is of dimension 90

parametrizing *special linear series* on C. Here we will briefly recall Lazarsfeld's approach in [31] as it will allow us to also construct families of stable and big vector bundles of any rank $r \geqslant 2$ on a very–general polarized $K3$ surface.

Let (X, H) be a smooth, polarized, projective $K3$ surface of genus $g \geqslant 3$. Given a curve C and positive integers d and r, consider the *Brill–Noether locus*

$$W_d^{r-1}(C) := \{A \in \operatorname{Pic}^d(C) \mid h^0(C, A) \geqslant r\} \subseteq \operatorname{Pic}^d(C)$$

and its subscheme

$$V_d^{r-1}(C) \subseteq W_d^{r-1}(C)$$

defined to be the non-empty, open subset of $W_d^{r-1}(C)$ consisting of line bundles A on C such that:

(i) $h^0(C, A) = r$, $\deg(A) = d$, and
(ii) both A and $\omega_C \otimes A^\vee$ are globally generated on C (where ω_C denotes the canonical bundle of C).

In this set–up, for any smooth curve $C \in |H|$ and any line bundle $A \in V_d^{r-1}(C)$ one associates to the pair (C, A) a rank-r vector bundle $E := E_{C,A}$ on X as follows: since A is globally generated, we have a canonical surjective map

$$ev_{C,A} : H^0(C, A) \otimes \mathcal{O}_X \twoheadrightarrow A$$

of $\mathcal{O}_X$-modules (thinking A as a sheaf on X); thus, $ker(ev_{C,A})$ is a rank–r vector-bundle on X, therefore, one sets

$$E = E_{C,A} := ker(ev_{C,A})^\vee$$

(for details, cf. [31, § 1]). This gives rise to the exact sequence on X:

$$0 \to E^\vee \to H^0(C, A) \otimes \mathcal{O}_X \to A \to 0. \tag{3.8}$$

Dualizing (3.8), one gets

$$0 \to H^0(C, A)^\vee \otimes \mathcal{O}_X \to E \to \omega_C \otimes A^\vee \to 0, \tag{3.9}$$

since $\mathcal{E}xt^1_{\mathcal{O}_X}(A, \mathcal{O}_X) \cong \omega_C \otimes A^\vee$ (cf. [23, Lemma 7.4, p. 242]). The vector bundle E is called the *Mukai-Lazarsfeld vector bundle* associated to the pair (C, A). If, as it is customary, one considers the *Brill-Noether number*:

$$g - h^0(C, A)h^1(C, A) = g - r(r - 1 + g - d) =: \rho(g, r - 1, d) \tag{3.10}$$

from (3.8), (3.9) and the fact that X is regular with $\omega_X \cong \mathcal{O}_X$, one has (cf. [31, § 1]):

(E1) E is a rank–r, globally generated vector bundle on X,
(E2) $c_1(E) = H$, $\int_X c_2(E) = \deg(A) = d$,
(E3) $h^0(X, E^\vee) = h^2(X, E) = 0$, $h^1(X, E^\vee) = h^1(X, E) = 0$,
(E4) $h^0(X, E) = h^0(C, A) + h^1(C, A) = 2r + g - d - 1$;
(E5) $\chi(X, E \otimes E^\vee) = 2 - 2\rho(g, r - 1, d)$ (cf. [31, (iv), p. 302]). If moreover E is *simple*, more precisely one has $h^0(X, E \otimes E^\vee) = h^2(X, E \otimes E^\vee) = 1$ and $h^1(X, E \otimes E^\vee) = 2\rho(g, r - 1, d)$.

Another fundamental property of the vector bundle E is given by the following:

Lemma 3.6. *(cf.* [31, Lemma 1.3]*) If E has non-trivial endomorphisms, i.e. if $h^0(X, E \otimes E^\vee) \geqslant 2$, the linear system $|H|$ contains a reducible (or multiple) curve.*

Thus, in particular, we have the following:

Proposition 3.7. *Let (X, H) be a primitively polarized $K3$ surface, such that $H^2 > 2$ and that $|H|$ contains neither reducible nor non-reduced curves. Let $C \in |H|$ be any smooth curve and let $A \in V_d^{r-1}(C)$. Then:*
(i) E is a simple bundle on X.
(ii) If moreover (X, H) is assumed to be very–general, then E is a μ_H-stable bundle on X. Thus, setting $v = v(E) := (r, H, g - 1 - d + r)$ as the Mukai vector of E, then the moduli space $M_H^s(v)$ is smooth, irreducible and such that $\dim(M_H^s(v)) = 2\rho(g, r - 1, d)$

Proof. (i) The fact that E is simple directly follows from the assumption on $|H|$, containing neither reducible nor non-reduced curves, and from Lemma 3.6.
(ii) If (X, H) is assumed to be very–general, then in particular $\operatorname{Pic}(X) \cong \mathbb{Z}[H]$. Stability of E is then proved e.g. in [17, Prop. 4.5]. Moreover since $v = v(E) = (r, H, g - 1 - d + r)$ is primitive and since H is the generator of $\operatorname{Pic}(X)$, then Proposition 3.1 (ii) implies irreducibility of $M_H^s(v)$, whose dimension is given by $2 + \langle v, v \rangle = 2\rho(g, r - 1, d)$. □

In this general set–up, we have the following result.

Theorem 3.8. *Let (X, H) be a very–general polarized $K3$ surface of genus $g \geqslant 3$. For any triple $(g, r-1, d)$ of positive integers such that $d < 2g-2$ and $\rho(g, r-1, d) \geqslant 0$, there exists a vector bundle E on X of rank r, with $c_1(E) = H$ and $c_2(E) = d$ which is globally generated and μ_H-stable on X. Setting $v = v(E) = (r, H, g-1-d+r)$ as the Mukai vector of E, then $M_H(v)^s$ is smooth, irreducible of dimension $\dim(M_H(v)^s) = 2\rho(g, r-1, d)$.*

When in particular $\rho(g, r-1, d) = 0$, then $M_H(v)^s$ consists of the single, reduced point $\{E\}$, which is also a big bundle on X. When otherwise $\rho(g, r-1, d) > 0$, the general bundle parametrized by $M_H(v)^s$ is globally generated and big on X.

Proof. The existence of E of rank r, with $c_1(E) = H$ and $c_2(E) = d$, the fact that E is globally generated and moreover μ_H-stable are direct consequences of $(E1) - (E2)$ and of Proposition 3.7–(ii) above.

Moreover, by (2.2), one has that the Segre class $\int_X s_2(E) = \int_X (c_1^2(E) - c_2(E)) = H^2 - d = 2g-2-d$ therefore, since $d < 2g-2$ by assumption, $\int_X s_2(E) = (-1)^2 \int_X s_2(E) > 0$. Since E is globally generated, by Proposition 2.4, the bundle E is therefore also big.

The case $\rho(g, r-1, d) = 0$ clearly gives an exceptional Mukai vector $v = v(E)$, i.e. $\langle v, v\rangle = -2$, therefore $M_H(v)^s$ consists only of the single reduced point $[E]$, as it follows from Proposition 3.1–(i), which we have already remarked to be globally generated and big.

When otherwise $\rho(g, r-1, d) > 0$, then $M_H(v)^s$ is smooth, irreducible of positive dimension $2\rho(g, r-1, d)$, as it follows from the facts that $\mathrm{Pic}(X)$ is cyclic generated by H, v is primitive and from Propositions 3.1–(i) and 3.7 above. At last, since global generation and bigness are both open conditions, the general bundle parametrized by $M_H(v)^s$ is therefore globally generated and big on X. □

Remark 3.9. As a very particular case of the previous description, one can consider triples (g, r, d) for which $\rho(g, r-1, d) = 0$, equivalently $\langle v, v\rangle = -2$. In all such cases, by Brill-Noether theory on C and Lazarsfeld's results in [31], on a general member $C \in |H|$ the Brill–Noether locus $W^{r-1}_d(C) = V^{r-1}_d(C)$ consists of finitely many reduced points whereas, from Proposition 3.1 (i), for any associated Mukai vector $v = v(E_{C,A})$, the moduli space $M^s_H(v)$ consists of a single reduced point, i.e. for any general member $C \in |H|$ and any line bundle $A \in V^{r-1}_d(C)$ on C, all Mukai–Lazarsfeld vector bundles $E = E_{C,A}$ as above are isomorphic each other. Cases for which $\rho(g, r-1, d) = 0$ belong to a wider class of μ_H–stable vector bundles F studied in [30, Thm. 2.1] and [10, Prop. 2.1], for which

$$g.c.d.(rk(F), \int_X c_1(F)\cdot H) = 1 \quad \text{and} \quad 2rk(F)\int_X c_2(F) - (rk(F)-1)\int_X c_1(F)^2 = 2(rk(F)^2 - 1).$$

Indeed, if we set $c_1(F) \coloneqq H$ and $\int_X c_2(F) \coloneqq d \geqslant 0$, we have

$$v(F) = (rk(F), H, g-1+rk(F)-d)$$

as $\int_X c_1(F)\cdot H = H^2 = 2(g-1)$ and the condition $g.c.d.(rk(F), \int_X c_1(F)\cdot H) = 1$ implies in particular that $rk(F)$ is certainly odd (moreover not divisible by all prime divisors of $(g-1)$), in particular $rk(F) \geqslant 3$. Furthermore, condition $2rk(F)\int_X c_2(F) - (rk(F)-1)\int_X c_1(F)^2 = 2(rk(F)^2-1)$ above, reads in this case as $rk(F)d = (rk(F)-1)(g+rk(F))$.

Particular triples satisfying these numerical conditions are e.g.

$$(g, r, d) \in \{(6,3,6), (9,3,8), (10,5,12), (12,3,10), \ldots\}.$$

Under these numerical conditions, $v(F)^2 \coloneqq \langle v, v\rangle = -2$ and such a Mukai vector is called *exceptional* (cf. e.g. [10, § 2]). From [30, Thm. 2.1] and [10, Prop. 2.1], for any triples $(rk(F), d, g) = (r, d, g)$ satisfying $c_1(F) = H$ and the previous numerical assumptions, there exists a μ_H-stable (odd) rank–r vector bundle F on X with Mukai vector $v = (r, H, g-1+r-d)$, with $rd = (r-1)(g+r)$, and $M_v(X)^s = \{F\}$ is a single reduced point. Any vector bundle isomorphic to such a F is called *exceptional*, because it is related to an exceptional Mukai vector $v = v(F)$ (cf. [30, 10]). The previous construction shows that bundles F as above arise as Mukai–Lazarsfeld vector bundles $F = E_{C,A}$ associated to pairs (C, A) where $C \in |H|$ is a general member whereas A is a line bundle on C of degree d, with $h^0(C, A) = r$ and such that $\rho(g, r-1, d) = 0$.

3.3. **Big and stable Ulrich bundles on $K3$ surfaces.** Let (X, H) be any primitively polarized K3 surface of genus $g \geqslant 2$, where H globally generated and ample. As in the previous sections, for any vector bundle E on X and any integer n, we will simply denote by $E(n)$ the tensor product $E \otimes H^{\otimes n}$.

A rank–r vector bundle E on X is said to be a *Ulrich bundle* if it satisfies the conditions:

$$H^*(X, E(-1)) = 0 \text{ and } H^*(X, E(-2)) = 0.$$

In particular, $H^i(X, E(-i)) = 0$ for all integers $i > 0$ so, by [32, Def. 1.8.4, p. 100], E is *(Castelnuovo-Mumford) 0–regular with respect to* H and, by [32, Thm. 1.8.5-(i), p. 100], E is therefore globally generated on X and $H^i(X, E) = 0$ for all $i > 0$. Using the Riemann–Roch–Hirzebruch formula [18, Corollary 15.2.1] and equation (2.2) we compute the Euler–Poincaré polynomial of E:

$$\chi(X, E(n)) = r(g-1)n^2 + n\int_X c_1(E)\cdot H + \frac{1}{2}\int_X (c_1(E)^2 - 2c_2(E)) + 2r.$$

By assumption $\chi(X, E(-1)) = \chi(X, E(-2)) = 0$, so $\chi(X, E(n)) = r(g-1)(n+1)(n+2)$. We get by identification:

$$\int_X c_1(E)\cdot H = 3r(g-1), \quad \int_X (c_1(E)^2 - 2c_2(E)) = 4r(g-2), \quad h^0(X, E) = 2r(g-1).$$

If we assume (X, H) to be very–general, in particular $\mathrm{Pic}(X) \cong \mathbb{Z}[H]$, then $c_1(E) = \lambda H$, for some $\lambda \in \mathbb{Z}$, so we get $2\lambda = 3r$ thus necessarily r is even and $c_1(E) = \frac{3r}{2}H$. Assuming this, it follows that $\int_X c_1(E)^2 = \frac{9r^2}{2}(g-1)$, $\int_X c_2(E) = \frac{9r^2}{4}(g-1) - 2r(g-2)$, and finally using (2.1) we get:

$$\int_X s_2(E) = \frac{9r^2}{4}(g-1) + 2r(g-2) > 0 \quad \forall g\geqslant 2. \tag{3.11}$$

From global generation of E and from Proposition 2.4, the positivity of $\int_X s_2(E) = (-1)^2\int_X s_2(E)$ in (3.11) implies that Ulrich bundles E on $K3$ surfaces are therefore big. Taking into account [1, Thm. 0.5], we get:

Theorem 3.10. *For any integer $h \geqslant 2$, let (X, H) be a very–general polarized $K3$ surface, with polarization H of genus $g = h + 1$. Then, for every integer $a \geqslant 1$, there exists an $(8a^2 + 2a^2h + 2)$–dimensional family of μ_H–stable Ulrich bundles E on X such that*

$$rk(E) = 2a, \;\; c_1(E) = 3aH, \;\; \int_X c_2(E) = 9a^2h - 4a(h-1).$$

Setting $v := v(E) = (2a, 3aH, 2a(h-1))$ the Mukai vector of any such bundle E, there exists an irreducible component $M^s_H(E)$ of the corresponding moduli space $M^s_H(v)$, whose general point corresponds to a μ_H–stable Ulrich bundle of rank $r = 2a$ which is also globally generated and big.

Proof. For the proof of the existence of Ulrich bundles as in the statement, we refer the reader for full details to the original paper [1]. Here we briefly remind basic steps for the construction of the aforementioned bundles.

For the case $a = 1$, i.e. for $rk(E) = 2$, Ulrich vector bundles E have been constructed once again via Mukai–Lazarsfeld bundles as in Section 3.2. Namely any rank–2 Ulrich bundle E as in the statement is given by a Mukai–Lazarsfeld vector bundle $E := E_{C,A}$ where the curve $C \in |\mathcal{O}_X(3)|$ is a general cubic section of X (so not anymore a general member in the linear system $|H|$ of the generator of $\mathrm{Pic}(X)$ as in Section 3.2), which is a curve of genus $9h+1$ and of degree $6h$ in $\mathbb{P}^{h+1}$, whereas A is a line bundle on C giving rise to a complete and base–point–free pencil of degree $5h+4$ on C, i.e. $A \in W^1_{5h+4}(C)$. Since the curve C is in this case far from being *Brill-Noether general*, showing that a general cubic section C of X actually carries a pencil $|A| = g^1_{5h+4}$ inducing a Ulrich bundle E on X has been translated by the authors of [1] in a variational problem which has also been used in their proof of Green's conjecture for curves on arbitrary $K3$ surfaces.

For cases $a > 1$, taking direct sums of a–Ulrich bundles as in the rank–two case above, the authors then construct splitting Ulrich bundles of any even rank $r = 2a$ using direct sums of rank–two bundles and then they show that, for a very–general $K3$ surface, these direct sums can be deformed to stable unsplitting Ulrich bundles on X of the same rank $r = 2a$.

With the above set–up, for any $a \geqslant 1$, one has

$$rk(E) = 2a, \;\; c_1(E) = 3aH, \;\; \int_X c_2(E) = 9a^2k - 4a(k-1)$$

and it is a straightforward computation showing that

$$\int_X \left(\frac{c_1(E)^2}{2} - c_2(E)\right) + rk(E) = 2a(k-1)$$

so the Mukai vector of any such E is $v(E) = (2a, 3aH, 2a(k-1))$ as stated. Finally, since the general point of the irreducible component $M^s_H(E)$ is proved to be a Ulrich bundle, it then follows that it is also globally generated and big, as it is 0-regular with $(-1)^2\int_X s_2(E) = \int_X s_2(E) > 0$, as shown in (3.11), and then by applying Proposition 2.4. □

4. Generalities on tautological vector bundles on $Hilb^k$ of a $K3$ surface

In this section we will introduce some generalities on *tautological vector bundles* on varieties $X^{[k]} := Hilb^k(X)$, i.e. the *Hilbert scheme* parametrizing 0–dimensional subschemes of length k on a polarized surface (X, H), for any integer $k \geqslant 2$ (cf. e.g.[8, 11, 12, 13, 29, 37, 38, 39]). These preliminaries will be used in § 5, where we focus on the case of (X, H) a very–general $K3$, so that $X^{[k]}$ turns out to be a *Hyper–Kähler variety*, and where we consider tautological bundles arising from those in § 3.

Let X be any smooth, projective complex surface. Since $X^{[k]}$ is a fine moduli space, it is endowed with a universal family $\Xi_k \subset X \times X^{[k]}$, together with the two natural projections

$$X \overset{\pi_X}{\longleftarrow} \Xi_k \overset{\pi_{X^{[k]}}}{\longrightarrow} X^{[k]},$$

the map $\pi_{X^{[k]}}$ being flat of finite degree k. One can therefore associate to every coherent sheaf F on X the so called *tautological sheaf associated to* F on $X^{[k]}$, which is defined to be

$$F^{[k]} := \pi_{X^{[k]}*}(\pi_X^*(F)). \tag{4.1}$$

If F is locally free of rank m, then the tautological bundle $F^{[k]}$ is locally free of rank km with fibres

$$(F^{[k]})(\xi) \cong H^0(F|_\xi), \ \ \forall \ \xi \in X^{[k]} \tag{4.2}$$

(cf. [29, Rem. 3.6]). Denoting by X^k the k-th cartesian product $X^k = X \times \cdots \times X$ and by $X^{(k)} = X^k/Sym(k)$ the k-th symmetric product of X, it is well–known that $X^{[k]}$ is a resolution of the singularities of $X^{(k)}$ via the *Hilbert–Chow morphism*

$$\mu : X^{[k]} \to X^{(k)}, \ \ \xi \overset{\mu}{\longrightarrow} \sum_{x \in \mathrm{Supp}(\xi)} \mathrm{length}_\xi(x)\, x.$$

Let $\pi_i : X^k \to X$ denote the i-th natural projection. For any $L \in \mathrm{Pic}(X)$ one defines $L^{\boxtimes k} := \otimes_{i=1}^k \pi_i^*(L) \in \mathrm{Pic}(X^k)$. Denoting by $\pi : X^k \to X^{(k)}$ the natural quotient map onto the symmetric product, then $\pi_*(L^{\boxtimes k}) \in \mathrm{Pic}(X^{(k)})$ (cf. [29]). Thus, one can define the natural morphism:

$$D_k : \mathrm{Pic}(X) \to \mathrm{Pic}(X^{[k]}), \quad L \overset{D_k}{\longrightarrow} D_k(L) := \mu^*(\pi_*(L^{\boxtimes k})) \tag{4.3}$$

(cf. e.g. [29]) which is injective and which gives

$$\mathrm{Pic}(X^{[k]}) = D_k(\mathrm{Pic}(X)) \oplus \mathbb{Z}[\Delta], \tag{4.4}$$

where $\Delta := \det(\mathcal{O}_X^{[k]})$ and $c_1(\Delta) = -\frac{1}{2}\mathcal{E}$, where $\mathcal{E}$ denotes the *μ-exceptional divisor* (cf. [16, § 5, p.11]). In this set–up, one has:

$$H^0(X^{[k]}, F^{[k]} \otimes D_k(L)) \cong H^0(X, F \otimes L) \otimes S^{k-1}\left(H^0(X, L)\right) \tag{4.5}$$

(cf. [29, formula (1), p. 2]). When $L \in \mathrm{Pic}(X)$ is an effective and ample line bundle on X, it follows that $D_k(L)$ is a big and nef line bundle on $X^{[k]}$ (cf. [39, p. 5]). Moreover, if $D_k(L)$ is effective, for any effective divisor $D \in |D_k(L)|$, then D is set–theoretically described as $D = \left\{\xi \in X^{[k]} \mid \xi \cap \mathrm{Supp}(D) \neq \emptyset\right\}$ (cf. [39, p. 5]).

4.1. Tautological bundles and stability.

The notion of slope–(semi)stability can be formally extended to big and nef line bundles (cf. [41, p. 437]). Indeed, using notation and terminology as above, one has the following:

Theorem 4.1. *(cf. [39, Thm. 1.4]) Let (X, H) be a smooth, irreducible polarized surface, where H a globally generated and ample line bundle on X. Let $k \geqslant 2$ be any integer. Let F be a rank-r bundle on X, where $r \geqslant 1$.*

If $F \neq \mathcal{O}_S$ and if moreover F is μ_H-stable on X, then the tautological bundle $F^{[k]}$ is $\mu_{D_k(H)}$-stable on $X^{[k]}$.

4.2. Segre classes of tautological bundles.

Consider the incidence variety $X^{[k-1,k]} \subset X^{[k-1]} \times X \times X^{[k]}$ parametrizing triples $(\xi, x, \xi') \in X^{[k-1]} \times X \times X^{[k]}$ such that $\xi \subset \xi'$ with residual subscheme supported at the point x. This variety has dimension $2k$. We denote the projections as follows:

$$\begin{array}{ccccc} & & X^{[k-1,k]} & \overset{\rho}{\longrightarrow} & X \\ & \swarrow^{\varphi} & & \searrow^{\psi} & \\ X^{[k-1]} & & & & X^{[k]} \end{array}$$

Given a triple $(\xi, x, \xi') \in X^{[k-1,k]}$, either x is already in the support of ξ, meaning that ξ' is obtained by thickening ξ at the point x, or not. We denote by $\mathfrak{E}$ the set of those triples such that x is in the support of ξ.

If $\mathcal{I}_{\Xi_k}$ denotes the ideal sheaf of the universal family $\Xi_k \subset X \times X^{[k]}$, we will denote by $\mathbb{P}(\mathcal{I}_{\Xi_k})$ the *blowing–up of* $X \times X^{[k]}$ *w.r.t. the coherent sheaf of ideals* $\mathcal{I}_{\Xi_k}$ in the sense of [23, Def., Cap. 7, p.163]. Then, there are isomorphisms

$$X^{[k-1,k]} \cong \mathbb{P}(\mathcal{I}_{\Xi_k}) \cong \mathrm{Bl}_{\Xi_k}(X \times X^{[k]})$$

(cf. [19, § 1.2]). From these isomorphisms, we see that the $\mathfrak{E}$ turns out to be the exceptional divisor of the blow–up on the right hand side.

Let $\mathbb{H}_k := \bigoplus_{j=0}^{4k} H^j(X^{[k]}, \mathbb{Q})$ and $\mathbb{H} := \bigoplus_{k \geqslant 0} \mathbb{H}_k$. For any $\alpha \in H^*(X, \mathbb{Q})$ and any $i \geqslant 1$, the Nakajima operator $q_i(\alpha) \in \mathrm{End}(\mathbb{H})$ is defined by

$$q_i(\alpha)(\beta) := \psi_*(\varphi^*\beta \cdot \rho^*\alpha), \quad \forall \beta \in \mathbb{H}.$$

Following always [19], we denote by $\partial \in \mathrm{End}(\mathbb{H})$ the operator acting on $\mathbb{H}_k$ by cup-product with the class $c_1(\mathcal{O}_X^{[k]})$ and we define the *derived Nakajima operators* by:

$$q_i'(\alpha) := [\partial, q_i(\alpha)] = \partial \circ q_i(\alpha) - q_i(\alpha) \circ \partial.$$

(Since the K3 surface X has no odd cohomology, there is no sign in the definition of the Lie bracket).

Starting from $s(F^{[0]}) = 1$, the total Segre classes can be computed recursively using the following formula which, in particular, extends either to non–primitive line bundles or to higher–rank vector bundles computations on Segre classes in [42], holding for $H^{[k]}$:

Proposition 4.2. *Let F be a rank $r \geqslant 1$ vector bundle on a K3 surface X. Then for any $k \geqslant 1$ one has:*

$$s(F^{[k]}) = \frac{1}{k}\left(\sum_{i=0}^{2k}\sum_{j=0}^{\min\{i,2\}}(-1)^{i-j}\binom{r-1+i}{r-1+j} q_1^{(i-j)}(s_j(F))\right)\left(s(F^{[k-1]})\right).$$

Proof. Following [19, equation (11)] (see also [16, Lemma 2.1]), for any rank r vector bundle F on X, we have an exact sequence relating the tautological bundles associated to F on $X^{[k-1]}$ and $X^{[k]}$:

$$0 \longrightarrow \rho^*F \otimes \mathcal{O}_{X^{[k-1,k]}}(-\mathfrak{E}) \longrightarrow \psi^*F^{[k]} \longrightarrow \varphi^*F^{[k-1]} \longrightarrow 0.$$

The basic properties of the total Segre class give:

$$\psi^*s(F^{[k]}) = \varphi^*s(F^{[k-1]}) \cdot s\left(\rho^*F \otimes \mathcal{O}_{X^{[k-1,k]}}(-E)\right). \tag{4.6}$$

We put $\lambda := c_1(\mathcal{O}_{X^{[k-1,k]}}(-\mathfrak{E}))$. Using formula (2.4) we get:

$$s\left(\rho^*F \otimes \mathcal{O}_{X^{[k-1,k]}}(-E)\right) = \sum_{i=0}^{2k}\sum_{j=0}^{\min\{i,2\}}(-1)^{i-j}\binom{r-1+i}{r-1+j}\rho^*s_j(F)\lambda^{i-j}. \tag{4.7}$$

For any k, we denote by $\mathbf{s}(F^{[k]}) \in \mathrm{End}(\mathbb{H}_k)$ the operator acting on $\mathbb{H}_k$ by cup-product with the total Segre class $s(F^{[k]})$. For any $\beta \in \mathbb{H}_{k-1}$, following the same lines as the proof of [19, Theorem 4.2] we compute, using (4.6) and (4.7):

$$\begin{aligned}
\mathbf{s}(F^{[k]}) \circ q_1(1) \circ \mathbf{s}(F^{[k-1]})^{-1}(\beta) &= s(F^{[k]}) \cdot \psi_*\left(\varphi^*\left(\mathbf{s}(F^{[k-1]})^{-1}(\beta)\right)\right)\\
&= \psi_*\left(\psi^*s(F^{[k]}) \cdot \varphi^*\left(s(F^{[k-1]})^{-1}\right) \cdot \varphi^*(\beta)\right)\\
&= \psi_*\left(\sum_{i=0}^{2k}\sum_{j=0}^{\min\{i,2\}}(-1)^{i-j}\binom{r-1+i}{r-1+j}\rho^*s_j(F)\lambda^{i-j}\varphi^*(\beta)\right).
\end{aligned}$$

By [19, Lemma 3.9], for any $\beta \in \mathbb{H}$ and any $\nu \geqslant 0$, we have $q_1^{(\nu)}(\alpha)(\beta) = \psi_*(\lambda^\nu \cdot \varphi^*\beta \cdot \rho^*\alpha)$, so we obtain:

$$\mathbf{s}(F^{[k]}) \circ q_1(1) \circ \mathbf{s}(F^{[k-1]})^{-1}(\beta) = \sum_{i=0}^{2k}\sum_{j=0}^{\min\{i,2\}}(-1)^{i-j}\binom{r-1+i}{r-1+j} q_1^{(i-j)}(s_j(F) \cdot \beta).$$

We denote by $\mathbf{1}$ the unit in the ring $\mathbb{H}_0$. Recall that $\frac{1}{k!}q_1(1)^k\mathbf{1}$ is the unit in the ring $\mathbb{H}_k$ for any k. the above formula with $\beta = \frac{1}{(k-1)!}q_1(1)^{k-1}\mathbf{1}$ we get the expected recursion relation:

$$s(F^{[k]}) = \frac{1}{k}\left(\sum_{i=0}^{2k}\sum_{j=0}^{\min\{i,2\}}(-1)^{i-j}\binom{r-1+i}{r-1+j} q_1^{(i-j)}(s_j(F))\right)(s(F^{[k-1]})).$$

□

In particular, since each operator $q_n^{(\nu)}(\alpha)$ has cohomological degree $2\nu+2(n-1)+\deg(\alpha)$, we deduce a recursion formula for the d-th Segre class $s_d(F^{[k]})\in\mathbb{H}_{2d}$ for any $0\leqslant d\leqslant 2k$:

$$s_d(F^{[k]})=\frac{1}{k}\left(\sum_{i\geqslant\max\{0,d-2(k-1)\}}^{d}\sum_{j=0}^{\min\{i,2\}}(-1)^{i-j}\binom{r-1+i}{r-1+j}q_1^{(i-j)}(s_j(F))\right)\left(s_{d-i}(F^{[k-1]})\right). \tag{4.8}$$

Corollary 4.3. *Let F be a rank $r\geqslant 1$ vector bundle on a K3 surface X. Then:*

$$\int_X s_4(F^{[2]})=12\binom{r+3}{4}-\frac{1}{2}\binom{r+2}{2}\cdot\int_X s_1(F)^2-\left(\frac{r^2+3r+3}{2}\right)\cdot\int_X s_2(F)+\frac{1}{2}\cdot\left(\int_X s_2(F)\right)^2;$$

$$\begin{aligned}\int_X s_6(F^{[3]})&=-2\left(4r^4+21r^2+35r+20\right)\binom{r+2}{3}+\frac{1}{3}\binom{r+2}{2}\left(3r^2+8r+6\right)\cdot\int_X s_1(F)^2\\&+\frac{1}{6}\left(6r^4+35r^3+72r^2+61r+20\right)\cdot\int_X s_2(F)-\frac{1}{2}\binom{r+2}{2}\cdot\int_X s_1(F)^2\cdot\int_X s_2(F)\\&-\left(\frac{r^2+3r+3}{2}\right)\cdot\left(\int_X s_2(F)\right)^2+\frac{1}{6}\cdot\left(\int_X s_2(F)\right)^3.\end{aligned}$$

Proof. We apply (4.8), starting from $s(F)=1+s_1(F)+s_2(F)=q_1(1)\mathbf{1}+q_1(s_1(F))\mathbf{1}+q_1(s_2(F))\mathbf{1}$:

$$\begin{aligned}2s_4(F^{[2]})=&\binom{r+1}{r-1}q_1^{(2)}(1)q_1(s_2(F))\mathbf{1}-(r+1)q_1'(s_1(F))q_1(s_2(F))\mathbf{1}+q_1(s_2(F))q_1(s_2(F))\mathbf{1}\\&-\binom{r+2}{r-1}q_1^{(3)}(1)q_1(s_1(F))\mathbf{1}+\binom{r+2}{r}q_1^{(2)}(s_1(F))q_1(s_1(F))\mathbf{1}-(r+2)q_1'(s_2(F))q_1(s_1(F))\mathbf{1}\\&+\binom{r+3}{r-1}q_1^{(4)}(1)q_1(1)\mathbf{1}-\binom{r+3}{r}q_1^{(3)}(s_1(F))q_1(1)\mathbf{1}+\binom{r+3}{r+1}q_1^{(2)}(s_2(F))q_1(1)\mathbf{1}.\end{aligned}$$

Let us explain in details the computation of the first term of the sum:

$$\begin{aligned}q_1^{(2)}(1)q_1(s_2(F))\mathbf{1}&=(\partial q_1'(1)-q_1'(1)\partial)q_1(s_2(F))\mathbf{1}\\&=\partial q_1'(1)q_1(s_2(F))\mathbf{1}-q_1'(1)\partial q_1(s_2(F))\mathbf{1}.\end{aligned}$$

Since $q_1(s_2(F))\mathbf{1}\in H^4(X)$, we have $\partial q_1(s_2(F))\mathbf{1}=0$, so:

$$q_1^{(2)}(1)q_1(s_2(F))\mathbf{1}=\partial[q_1'(1),q_1(s_2(F))]\mathbf{1}+\partial q_1(s_2(F))q_1'(1)\mathbf{1}.$$

By [19, Theorem 3.10], we have $[q_1'(1),q_1(s_2(F))]=-q_2(s_2(F))$, and we observe that $q_1'(1)\mathbf{1}=\partial q_1(1)\mathbf{1}=c_1(\mathcal{O}_X)=0$. The relation between the derived Nakajima operators and the Virasoro operators given again in [19, Theorem 3.10] gives, using $\partial\mathbf{1}=0$:

$$\partial q_2(s_2(F))\mathbf{1}=q_2'(s_2(F))\mathbf{1}=(q_1(1)q_1(1))\delta_*(s_2(F))\mathbf{1}=\int_X s_2(F)\cdot q_1([x])q_1([x])\mathbf{1},$$

where $\delta_*\colon H^*(X)\to H^*(X)\otimes H^*(X)$ is the push-forward map associated to the diagonal embedding and $[x]\in H^4(X)$ is the class of a point. Using similar computations, that can be performed efficiently with the help of a computer, we get:

$$\begin{aligned}q_1^{(2)}(1)q_1(s_2(F))\mathbf{1}&=-\int_X s_2(F)\cdot q_1([x])q_1([x])\mathbf{1}\\q_1'(s_1(F))q_1(s_2(F))\mathbf{1}&=0\\q_1(s_2(F))q_1(s_2(F))\mathbf{1}&=\left(\int_X s_2(F)\right)^2\cdot q_1([x])q_1([x])\mathbf{1}\\q_1^{(3)}(1)q_1(s_1(F))\mathbf{1}&=0\\q_1^{(2)}(s_1(F))q_1(s_1(F))\mathbf{1}&=-\int_X s_1(F)^2\cdot q_1([x])q_1([x])\mathbf{1}\\q_1'(s_2(F)q_1(s_1(F))\mathbf{1}&=0\\q_1^{(4)}(1)q_1(1)\mathbf{1}&=24\cdot q_1([x])q_1([x])\mathbf{1}\end{aligned}$$

$$q_1^{(3)}(s_1(F))q_1(1)\mathbf{1} = 0$$
$$q_1^{(2)}(s_2(F))q_1(1)\mathbf{1} = -\int_X s_2(F)\cdot q_1([x])q_1([x])\mathbf{1}.$$

Since $\int_{X^{[2]}} q_1([x])q_1([x])\mathbf{1} = 1$, the value of $\int_X s_4(F^{[2]})$ follows directly. Similar computations give the formula for $\int_X s_6(F^{[3]})$. □

4.3. **Global generation of tautological bundles.** This section will be focused on finding sufficient conditions for global generation of tautological vector bundles on $X^{[k]}$, for any integer $k \geqslant 2$. To do so, we first remind some useful terminology (cf. e.g. [28] for line bundles and [40] for the more general set–up of vector bundles).

Let X be any smooth, irreducible projective surface and let G be a rank–r vector bundle on X, for $r \geqslant 1$. Let $k \geqslant 1$ be an integer. One says that G is $(k-1)$*–very ample* on X if, for any 0–dimensional subscheme ξ of X of length $h^0(\xi, \mathcal{O}_\xi) = k$, the natural evaluation map

$$H^0(X,G) \xrightarrow{ev_{G,\xi}} H^0(X, G\otimes\mathcal{O}_\xi)$$

is surjective. Notice that G is 0-very ample if and only if it is globally generated.

In this set–up, we prove the following result which will be used later on.

Proposition 4.4. *Let X be a smooth, irreducible projective surface and let G be a rank r vector bundle on X, with $r \geqslant 1$. Let $k \geqslant 2$ be any integer.*

(i) G is $(k-1)$-very ample on X if and only if the tautological bundle $G^{[k]}$ is globally generated (i.e. 0–very ample) on $X^{[k]}$.

(ii) If G is globally generated on X and if L is a $(k-1)$–very ample line bundle on X, then $G\otimes L$ is $(k-1)$–very ample on X, equivalently $(G\otimes L)^{[k]}$ is globally generated (i.e. 0–very ample) on $X^{[k]}$.

Proof. (i) (⇒) If G is $(k-1)$–very ample, then for any 0–dimensional subscheme ξ of X of length $h^0(\mathcal{O}_\xi) = k$, the evaluation map

$$H^0(X,G) \xrightarrow{ev_{G,\xi}} H^0(X, G\otimes\mathcal{O}_\xi)$$

is surjective. By (4.2), we have that $H^0(X, G\otimes\mathcal{O}_\xi) \cong (G^{[k]})(\xi)$. Moreover, by (4.5), one has $H^0(X^{[k]}, G^{[k]}) \cong H^0(X,G)$ therefore the surjectivity of $ev_{G,\xi}$, for any $\xi \in X^{[k]}$, implies that $G^{[k]}$ is globally generated on $X^{[k]}$.

(⇐) Conversely, assume that $G^{[k]}$ is globally generated on $X^{[k]}$. Thus, there exists a surjective map

$$H^0(X^{[k]}, G^{[k]})\otimes\mathcal{O}_{X^{[k]}} \to G^{[k]} \to 0$$

so, for every $\xi \in X^{[k]}$, we have a surjective map

$$H^0(X^{[k]}, G^{[k]})\otimes\mathcal{O}_{X^{[k]},\xi} \to G^{[k]}(\xi) \to 0.$$

As above, by (4.5), we have $H^0(X^{[k]}, G^{[k]}) \simeq H^0(X,G)$ moreover, by (4.2), we have $G^{[k]}(\xi) \simeq H^0(X, G\otimes\mathcal{O}_\xi)$. This implies that, for every $\xi \in X^{[k]}$, the exact sequence $0 \to \mathcal{I}_{\xi/X}\otimes G \to G \to G\otimes\mathcal{O}_\xi \to 0$ on X gives rise in cohomology to the surjective map $H^0(X,G) \to H^0(X, G\otimes\mathcal{O}_\xi) \to 0$, i.e., G is $(k-1)$-very ample on X.

(ii) The proof is inspired by that in [6, Lemma 2.2]. Let ξ be any 0-dimensional subscheme of X of length $h^0(\mathcal{O}_\xi) = k$; the $(k-1)$-very ampleness of L ensures that, for any such ξ, the evaluation map

$$H^0(X,L) \xrightarrow{ev_{L,\xi}} H^0(X, L\otimes\mathcal{O}_\xi)$$

is surjective.

Let $\mathrm{Supp}(\xi) = \{x_1, x_2, \ldots, x_s\}$ be the support of ξ, where $1 \leqslant s \leqslant k$ is an integer; notice that the equality $s = k$ holds if and only if ξ consists of k distinct reduced points of X whereas $s = 1$ holds if and only if ξ is a 0-dimensional subscheme of X concentrated at just one point and of length k. The proof below is given for $s \geqslant 2$. The case $s = 1$ can be dealt with similarly, and it will not be specified any further.

For any integer $1 \leqslant i \leqslant s$, denote by $\xi_i \subset \xi$ the maximal subscheme of ξ whose support is the point x_i, i.e. denoting by $\eta_i := \xi \setminus \xi_i$ the *residual* 0-dimensional subscheme of ξ_i in ξ, one has $\mathrm{Supp}(\eta_i) = \{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s\}$. Set $k_i := h^0(\mathcal{O}_{\xi_i}) = \mathrm{length}(\xi_i) \geqslant 1$, so that $\sum_{i=1}^s k_i = k$ and $h^0(\mathcal{O}_{\eta_i}) = \mathrm{length}(\eta_i) = k - k_i$.

Consider the vector subspace

$$V_i := H^0(X, L\otimes\mathcal{I}_{\eta_i/X}) \subset H^0(X,L),$$

where $\mathcal{I}_{\eta_i/X} \subset \mathcal{O}_X$ denotes the ideal sheaf of η_i in X, $1 \leqslant i \leqslant s$. The $(k-1)$–ampleness of L ensures that $V_i \subsetneq H^0(X, L)$, that $V_i \neq V_j$ for $1 \leqslant i \neq j \leqslant s$ and, moreover, that the restriction of $ev_{L,\xi}$ to V_i induces a surjective map

$$V_i \stackrel{\alpha_i}{\twoheadrightarrow} H^0(X, L \otimes \mathcal{I}_{\eta_i/X} \otimes \mathcal{O}_{\xi_i}) \cong H^0(L \otimes \mathcal{O}_{\xi_i}) \cong \mathbb{C}^{k_i}_{(x_i)},$$

$1 \leqslant i \leqslant s$, where the first isomorphism on the right follows from the fact that $\mathrm{Supp}(\eta_i) \cap \mathrm{Supp}(\xi_i) = \emptyset$ whereas the second isomorphism holds by the definition of $k_i = \mathrm{length}(\xi_i)$ and $x_i = \mathrm{Supp}\{\xi_i\}$. Here $\mathbb{C}^{k_i}_{(x_i)}$ is the stalk at the point x_i.

Similarly, global generation of G ensures that, for any $1 \leqslant i \leqslant s$, the evaluation map

$$H^0(X, G) \stackrel{ev_{G,x_i}}{\longrightarrow} H^0(X, G \otimes \mathcal{O}_{x_i})$$

is surjective. To ease notation, set $\beta_i := ev_{G,x_i}$, so that we have epimorphisms

$$H^0(X, G) \stackrel{\beta_i}{\twoheadrightarrow} H^0(X, G \otimes \mathcal{O}_{x_i}) \cong \mathbb{C}^{r}_{(x_i)},$$

$1 \leqslant i \leqslant s$, where the isomorphism on the right follows from the fact that G has rank r.

Consider $H^0(X, G) \otimes V_i \subset H^0(X, G) \otimes H^0(X, L)$, $1 \leqslant i \leqslant s$. If we denote by $\mu := \mu_{G,L}$ the natural multiplication map among global sections $H^0(X, G) \otimes H^0(X, L) \stackrel{\mu}{\longrightarrow} H^0(X, G \otimes L)$, set

$$\mathcal{E}_i := \mu(H^0(X, G) \otimes V_i) \subset H^0(X, G \otimes L).$$

By definition of V_i, notice that $\mathcal{E}_i \subseteq H^0(X, G \otimes L \otimes \mathcal{I}_{\eta_i/X})$. If we consider the evaluation map

$$ev_{G\otimes L,\xi} : H^0(X, G \otimes L) \longrightarrow H^0(X, G \otimes L \otimes \mathcal{O}_{\xi}),$$

set $\rho_i := ev_{G\otimes L,\xi_{|\mathcal{E}_i}}$, $1 \leqslant i \leqslant s$. By definition of V_i, one has that

$$\mathcal{E}_i \stackrel{\rho_i}{\longrightarrow} H^0(G \otimes L \otimes \mathcal{O}_{\xi_i}), \quad 1 \leqslant i \leqslant s.$$

Claim 4.5. *The map ρ_i is surjective, for any $1 \leqslant i \leqslant s$.*

Proof of Claim 4.5. For any $1 \leqslant i \leqslant s$ we have a commutative diagram

$$\begin{array}{ccc} H^0(X,G) \otimes V_i & \stackrel{\beta_i \otimes \alpha_i}{\longrightarrow} & \mathbb{C}^s_{(x_i)} \otimes \mathbb{C}^{k_i}_{(x_i)} \\ \downarrow^{\mu_i} & & \downarrow^{\varphi_i} \\ \mathcal{E}_i & \stackrel{\rho_i}{\longrightarrow} & \mathbb{C}^{k_i\, s}_{(x_i)} \end{array}$$

where $\mu_i := \mu_{|_{H^0(X,G)\otimes V_i}}$ is surjective, by the definition of $\mathcal{E}_i$, and φ_i is an isomorphism. Since $\varphi_i \circ (\beta_i \otimes \alpha_i) = \rho_i \circ \mu_i$, to prove the surjectivity of ρ_i it suffices to show that $\beta_i \otimes \alpha_i$ is surjective. In turn, this directly follows from the fact that both α_i and β_i are surjective, as observed above, and from the properties of tensor product (or *Kronecker product*) of linear maps, i.e. $\mathrm{rk}(\beta_i \otimes \alpha_i) = \mathrm{rk}(\beta_i)\mathrm{rk}(\alpha_i)$ (cf. e.g.[25, Ex. 4.2.1]). Therefore ρ_i is surjective, for any $1 \leqslant i \leqslant s$. □

By definition of $\mathcal{E}_i$ and by Claim 4.5, global sections in $\mathcal{E}_i$ vanish at η_i but generate the stalk $G \otimes L \otimes \mathcal{O}_{\xi_i}$, $1 \leqslant i \leqslant s$. Since $\mathcal{E}_i \subset H^0(X, G \otimes L)$, for any $1 \leqslant i \leqslant s$, and since $\rho_i = ev_{G\otimes L,\xi_{|\mathcal{E}_i}}$, this implies that global sections of $H^0(X, G \otimes L)$ separate the scheme ξ via the evaluation map $ev_{G\otimes L,\xi}$, i.e. $ev_{G\otimes L,\xi}$ is surjective. Since ξ is arbitrary, previous arguments imply that $G \otimes L$ is $(k-1)$-very ample. Last part of (ii) directly follows from part (i). □

5. On some big and stable tautological bundles on $Hilb^k$ of a $K3$ surface

In this section we extend results proved in Section 3 for $K3$'s to Hyper–Kähler varieties given by $X^{[k]} := Hilb^k(X)$, the *Hilbert scheme* parametrizing 0–dimensional subschemes of X of length k, where $k \geqslant 2$ is an integer and where (X, H) is a very–general, primitively polarized $K3$ surface.

To ease notation, we will set $Y := X^{[k]}$ and $H_Y := D_k(H)$, according to (4.3). As already observed, since H is globally generated, ample and effective, then H_Y is a big and nef line bundle on Y. Moreover, for any locally free sheaf $\mathcal{F}$ on Y and any positive integer n we will simply set

$$\mathcal{F}(n) := \mathcal{F} \otimes H_Y^{\otimes n}.$$

We follow section sub–division as in Section 3. Therefore, we will start with tangent bundles.

5.1. **The tangent bundle of $Hilb^k$ of a $K3$.** From § 3.1, when (X, H) is a very–general $K3$ surface with polarization of genus $g \geqslant 2$, Theorem 3.5 gives sufficient conditions for $T_X(n)$ to be big and μ_H-stable. We will make use of Theorem 3.5 to prove the main result of this section, namely Theorem 5.1 below.

Theorem 5.1. *Let (X, H) be a very–general $K3$ surface of genus $g \geqslant 2$. Let $Y = X^{[k]}$ be the Hilbert scheme parametrizing 0–dimensional subschemes of X of length k and let $H_Y = D_k(H)$ be the big and nef line bundle defined as in (4.3). Let T_Y denote the tangent bundle of Y.*

Then, the vector bundles $(T_X)^{[k]}(n)$ (cf. (4.1)) and $T_Y(n)$ are μ_{H_Y}-stable, of rank $2k$ on Y, for any integer n. Furthermore, $(T_X)^{[k]}(n)$ and $T_Y(n)$ are also big if

(1) *$n \geqslant 5$, for $g = 2$*
(2) *$n \geqslant 4$, for $g = 3$*
(3) *$n \geqslant 3$, for $4 \leqslant g \leqslant 9$ or $g = 11$*
(4) *$n \geqslant 2$, for $g \geqslant 10$ but $g \neq 11$.*

Proof. From [39, Theorem B], one has an exact sequence

$$0 \to (T_X)^{[k]} \to T_Y \to Q \to 0,$$

where Q is a torsion sheaf on Y; this implies that $(T_X)^{[k]}$ and T_Y are vector bundles of the same rank $2k$ on Y. The same conclusion holds for $(T_X)^{[k]}(n)$ and $T_Y(n)$, for any integer n.

Focusing on μ_{H_Y}–stability, from (3.4)–(iii) we know that T_X is μ_H–stable on X so, from Theorem 4.1, $(T_X)^{[k]}$ is μ_{H_Y}-stable on Y. Since slope–stability is preserved under tensor product via line–bundles, one deduces that $(T_X)^{[k]}(n)$ is μ_{H_Y}-stable, for any integer n.

As for μ_{H_Y}–stability of T_Y notice that, since Y is an irreducible holomorphic symplectic manifold (IHS, for short), in particular it is smooth and one can observe that Y satisfies in particular assumptions as in [21, Definition (8.16.2)]: indeed, in [24, beginning of §2] it is observed that when Y is smooth (as it occurs in our case), by the purity of the branch locus, any *quasi–étale* morphism $f : Y' \to Y$ (i.e. f étale in codimension one, using same terminology as in [21, Definition (8.16.2)]) is actually étale. On the other hand, from [21, Rem. 8.19], when Y is smooth satisfying assumptions as in [21, Definition (8.16.2)], then Y is simply connected (as it occurs in our case since $Y = X^{[k]}$ where X a $K3$); thus any étale $f : Y' \to Y$ is actually an isomorphism. Thus, the global generation assumption on exterior algebra of forms is intrinsically satisfied by the fact that $Y = X^{[k]}$, where X a $K3$, is Hyper–Kähler. One can therefore apply [21, Prop. 8.20] to $Y = X^{[k]}$ to get that T_Y is *strongly stable* (in the sense of [21, Def. 7.2]). This implies in particular that T_Y is μ_A-stable w.r.t. any ample line bundle $A \in \mathrm{Pic}(Y)$. Being μ_A–stable for any (ample) polarization A, then in particular T_Y is *simple*, i.e. $\mathrm{End}(T_Y) \cong \mathbb{C}$ (cf. [27, Corollary 1.2.8]). Since H_Y is a big and nef line bundle, then H_Y belongs to the boundary of the ample cone of Y; thus, passing to limit the numerical condition encoding stability, one deduces that T_Y is μ_{H_Y}–semistable. On the other hand, since T_Y is a simple bundle, then it cannot be μ_{H_Y}–strictly semistable. Therefore one concludes that T_Y is μ_{H_Y}–stable. As above, since slope–stability is preserved under tensor product via line–bundles, one deduces that $T_Y(n)$ is μ_{H_Y}-stable, for any integer n.

The rest of the proof will be devoted to the "bigness part" of the statement. From Lemma 3.4, we know that $T_X(n_0(g)) := T_X \otimes H^{\otimes n_0(g)}$ is an effective vector bundle, the integer $n_0(g)$ depending on g defined as follows:

g	2	3	4	5	6	7	8	9	10	11	$\geqslant 12$
$n_0(g)$	4	3	2	2	2	2	2	2	1	2	1

Applying (4.5) with $F = T_X$ and $L = H^{\otimes n_0(g)} = n_0(g)H$ (recall we interchangebly identify divisors and line bundles and use additive notation for divisor equivalently to tensor products of line bundles) one has

$$H^0(X^{[k]}, (T_X)^{[k]} \otimes D_k(n_0(g)H)) \cong H^0(X, T_X(n_0(g))) \otimes S^{k-1}\left(H^0(X, n_0(g)H)\right)$$

which shows that the vector bundle

$$E := (T_X)^{[k]} \otimes D_k(n_0(g)H)) = (T_X)^{[k]}(n_0(g)H_Y)$$

is an effective vector bundle on Y.

Considering the projective bundle $\mathbb{P}(E) \xrightarrow{\pi} Y$, then $\xi := c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ is an effective line bundle on Y. Taking into account that H_Y is big and nef (since H is very–ample on X) then, from [32, Corollary 2.2.7, p. 141], it follows that for any ample line bundle A_Y on Y there exist a positive integer $m_Y := m_{A_Y}$ and an effective line bundle $N_Y := N_{A_Y}$ such that

$$(1) \quad m_Y H_Y \sim A_Y + N_Y,$$

where $\sim$ denote linear equivalence of divisors on Y. On the other hand, since ξ is π-ample then, from [32, Proposition 1.7.10, p. 97], it follows that

$$(2) \quad \xi + \pi^*(mA_Y)$$

is ample, for any integer $m >> 0$. Notice that, from (1) above, one has that for any integer $m >> 0$

$$m(m_Y H_Y) \sim mA_Y + mN_Y.$$

Therefore, using (2), for any $m >> 0$ one has

$$m\, m_Y(\xi + \pi^*(H_Y)) = m\, m_Y \xi + m\pi^*(m_Y H_Y) \sim m\, m_Y \xi + \pi^*(mA_Y) + \pi^*(mN_Y).$$

Notice that

$$m\, m_Y \xi + \pi^*(mA_Y) + \pi^*(mN_Y) = (\xi + \pi^*(mA_Y)) + ((m\, m_Y - 1)\xi + \pi^*(mN_Y))$$

where the first summand on the right-side of the equality is ample by (2) whereas the second summand is effective since $m\, m_Y - 1 > 0$ and since ξ and $\pi^*(mN_Y)$ are both effective. Thus, from [32, Corollary 2.2.7 (iii), p. 141], it follows that $\xi + \pi^*(H_Y)$ is a big line bundle on $\mathbb{P}(E)$ and so

$$E \otimes H_Y = \Big((T_X)^{[k]} \otimes (n_0(g) H_Y)\Big) \otimes H_Y = (T_X)^{[k]} \otimes \mathcal{O}_Y((n_0(g)+1)H_Y)$$

is a big vector bundle on Y. Since H_Y is big and nef, then $(T_X)^{[k]} \otimes \mathcal{O}_Y(nH_Y)$ is a big vector bundle, for any $n \geqslant n_0(g)+1$.

Finally, consider the exact sequence

$$(3) \quad 0 \to (T_X)^{[k]} \to T_Y \to Q \to 0$$

from [39, Theorem B], where $(T_X)^{[k]}$ and T_Y are vector bundles on Y of the same rank $2k$ whereas Q is a torsion sheaf on Y. Using the fact that $(T_X)^{[k]} \otimes \mathcal{O}_Y(nH_Y)$ is big for any $n \geqslant n_0(g)+1$, the exact sequence above, the fact that $(T_X)^{[k]}$ and T_Y have the same rank and finally the characterization of bigness in terms of global sections of the corresponding tautological divisors on $\mathbb{P}(E)$ as in [32, Lemma 2.2.3, p. 139], it follows that $T_Y(nH_Y)$ is big for any $n \geqslant n_0(g)+1$, which completes the proof of the statement. □

5.2. **Big and stable tautological bundles on $X^{[k]}$ arising from line bundles on X a very–general $K3$.** Examples of further tautological bundles on $Y = X^{[k]}$, which are big and $\mu_{D_k(H)}$–stable, when X is a very–general primitively polarized $K3$ surface of genus g can be easily obtained as follows.

Theorem 5.2. *Let $k \geqslant 2$ be any integer and let (X, H) be a very–general primitively polarized $K3$ surface of genus $g > 2k-2$. Let $L_n := H^{\otimes n} \in \operatorname{Pic}(X)$, where $n \geqslant 1$ any integer. Set $Y := X^{[k]}$.*

Then, the rank–k tautological vector bundle $(L_n)^{[k]}$ on Y is globally generated and $\mu_{D_k(H)}$–stable. If moreover $\int_Y s_{2k}((L_n)^{[k]}) > 0$ then $(L_n)^{[k]}$ is also a big vector bundle on $X^{[k]}$.

Proof. For $n = 1$, i.e. $L_1 = H$, C. Voisin [42, Lemma 2.2] proves that $H^{[k]}$ is generated by global sections when $g > 2k-2$.

If $n > 1$, we use [28, Theorem 1.1], which gives necessary and sufficient conditions for the line bundle L_n to be $(k-1)$–very ample. Indeed it is a straightforward computation to showing that, if $k \geqslant 2$ and $g > 2k-2$, then $L_n^2 \geqslant 4(k-1)$ certainly holds and moreover that there are no effective divisors D on X such that

$$2D^2 \leqslant L_n \cdot D \leqslant D^2 + k \leqslant 2k.$$

Therefore, since condition (iii) in [28, Theorem 1.1] holds true, it follows that under the numerical assumptions $k \geqslant 2$ and $g > 2k-2$, L_n is $(k-1)$–very ample for any $n \geqslant 1$. Thus, from Proposition 4.4–(i), it follows that $(L_n)^{[k]}$ is globally generated on $X^{[k]}$.

Since any line bundle is μ_H–stable on X, then the rank–k vector bundle $L_n^{[k]}$ is certainly $\mu_{D_k(H)}$-stable on Y, for any $k \geqslant 2$ and any $n \geqslant 1$, as it follows from Theorem 4.1.

Finally, since $k \geqslant 2$ and $g > 2k-2$ imply that $(L_n)^{[k]}$ is globally generated on $X^{[k]}$, from Proposition 2.4 we know that $(-1)^{2k} \int_Y s_{2k}((L_n)^{[k]}) = \int_Y s_{2k}((L_n)^{[k]}) > 0$ implies that $(L_n)^{[k]}$ is big. □

As a direct consequence of the previous result we have the following:

Corollary 5.3. *Let $k \in \{2,\ 3\}$ be an integer and let (X, H) be a very–general primitively polarized $K3$ surface of genus $g > 2k-2$. Let $L_n := H^{\otimes n} \in \mathrm{Pic}(X)$, for any integer $n \geqslant 1$. Set $Y := X^{[k]}$. Then, the rank–k tautological vector bundle $(L_n)^{[k]}$ is globally generated and $\mu_{D_k(H)}$–stable on Y. If moreover one has $n \geqslant 2$, then $(L_n)^{[k]}$ is also big.*

Proof. Global generation and $\mu_{D_k(H)}$–stability of $(L_n)^{[k]}$ hold true as in the proof of Theorem 5.2.

For the rest of the statement, if $k = 2$, Corollary 4.3 gives that

$$\int_Y s_4((L_n)^{[k]}) = 2(n^4(g-1)^2 - 5n^2(g-1) + 6).$$

Thus, $n^2 > \frac{3}{g-1}$ implies that $\int_Y s_4((L_n)^{[k]})$ is certainly positive. Since $k = 2$ and $g > 2k-2 = 2$, notice that $\frac{3}{g-1} < 3$. Therefore, if $n \geqslant 2$, $\int_Y s_4((L_n)^{[k]}) > 0$ holds true. If otherwise $k = 3$, by Corollary 4.3 we get

$$\int_Y s_6((L_n)^{[k]}) = \frac{1}{3}\left(4n^6(g-1)^3 + 3n^4(g-1)^2 + 684n^2(g-1) - 480\right).$$

This equals $\frac{4}{3}(g-1)^3\left(n^2 - \frac{4}{g-1}\right)\left(n^2 - \frac{5}{g-1}\right)\left(n^2 - \frac{6}{g-1}\right)$, which is positive for $g > 4$ and $n \geqslant 2$. □

5.3. **Big and stable bundles on $X^{[k]}$ arising from Mukai–Lazarsfeld bundles on X a very–general $K3$.** Taking into account what proved in § 3.2, here we have the following:

Theorem 5.4. *Let $k \geqslant 2$ be a positive integer. Let (X, H) be a very–general polarized $K3$ surface of genus $g > 2k-2$. Let $(g, r-1, d)$ be any triple of positive integers such that $d < 2g-2$ and $\rho(g, r-1, d) \geqslant 0$, where $\rho(g, r-1, d)$ the* Brill–Noether number *as in (3.10). Set $Y := X^{[k]}$.*

Then, for any Mukai–Lazarsfeld rank–r vector bundle E on X as in Theorem 3.8, the tautological rank–kr vector bundle $(E \otimes H)^{[k]}$ is globally generated and $\mu_{D_k(H)}$–stable on Y.

If moreover $\int_Y s_{2k}((E \otimes H)^{[k]}) > 0$, then $(E \otimes H)^{[k]}$ is also big.

Proof. From Theorem 3.8, any Mukai–Lazarsfeld vector bundle $E = E_{C,A}$ constructed therein is μ_H–stable; so it is $E \otimes H$. Therefore, the rank–rk vector bundle $(E \otimes H)^{[k]}$ is certainly $\mu_{D_k(H)}$-stable on Y, for any $k \geqslant 2$, as it follows from Theorem 4.1.

Any such $E = E_{C,A}$ is also globally generated; since, by assumption, we have $g > 2k-2$ then, from [42, Lemma 2.2] or even from what discussed in the proof of Theorem 5.2, the line bundle H is $(k-1)$–very ample on X. Therefore, from Proposition 4.4–(ii), $E \otimes H$ is $(k-1)$-very ample on X so, by Proposition 4.4–(i), $(E \otimes H)^{[k]}$ is globally generated on Y. Thus, from Proposition 2.4, $(-1)^{2k}\int_Y s_{2k}((E \otimes H)^{[k]}) = \int_Y s_{2k}((E \otimes H)^{[k]}) > 0$ implies that $(E \otimes H)^{[k]}$ is big. □

Recall that Mukai-Lazarsfeld bundles $E = E_{C,A}$ as above are such that

$$\mathrm{rk}(E) = r, \ \ c_1(E) = H, \ \ \int_X c_2(E) = d$$

therefore, from (2.3), we have

$$rk(E \otimes H) = r, \ \ c_1(E \otimes H) = rH + c_1(E) = (r+1)H, \ \ \int_X c_2(E \otimes H) = 2\left(\binom{r}{2} + (r-1)\right)(g-1) + d.$$

Morevover, from (2.4), we have

$$s_1(E \otimes H) = -c_1(E \otimes H), \ \ s_2(E \otimes H) = c_1(E \otimes H)^2 - c_2(E \otimes H).$$

Using these expressions, one has:

Corollary 5.5. *Let $k \in \{2,\ 3\}$ be an integer and let (X, H) be a very–general primitively polarized $K3$ surface of genus $g > 2k-2$. Let $(g, r-1, d)$ be any triple of positive integers such that $d < 2g-2$ and $\rho(g, r-1, d) \geqslant 0$, where $\rho(g, r-1, d)$ the* Brill–Noether number *as in (3.10). Then the rank–rk vector bundle $(E \otimes H)^{[k]}$ is globally generated, $\mu_{D_k(H)}$–stable and big on $X^{[k]}$ for $k = 2, 3$.*

Proof. This is a direct consequence of Theorem 5.4 and of Corollary 4.3, namely one needs to show that, for $k = 2$, $\int_Y s_4((E \otimes H)^{[k]}) > 0$ (respectively, $\int_Y s_6((E \otimes H)^{[k]}) > 0$ for $k = 3$). In order to prove bigness, the strategy is similar for $k = 2$ and $k = 3$. In both cases we used a MAPLE script. Here we illustrate the case $k = 2$. The other case can be dealt with analogously. More details are available from the authors upon request.

The numerical condition coming from the positivity of the Segre classes, i.e., $\int_{X^{[2]}} s_4\left((E\otimes H)^{[2]}\right) > 0$, is given by

$$29r-23g+\frac{11}{2}d+8g^2-3rgd-4gd+\frac{1}{2}d^2+\frac{23}{2}r^3+2r^4+\frac{3}{2}r^2d+$$
$$+\frac{9}{2}rd+\frac{1}{2}r^4g^2+3r^3g^2+\frac{17}{2}r^2g^2+12rg^2+\frac{53}{2}r^2-\frac{59}{2}r^2g-38rg-2r^4g-\frac{23}{2}r^3g-r^2gd+15.$$

This polynomial expression $p(d,r,g)$ in the variables d,r,g has degree 2 in d and in g. We look at the locus $\{(d,g): p(d,r,g)=0\}$ as a plane conic in the real plane (g,d) whose coefficients depend on r. By direct inspection, the conic is a parabola for any value of r. In fact, if we perform the coordinate change

$$\begin{cases} g = G + (-r^2-3r-4)D, \\ d = (r^2+3r+4)G + D, \end{cases}$$

we obtain the locus in the plane (G,D) which is defined by the vanishing of the following polynomial, whose coefficients depend on r:

$$\left(\frac{1}{2}r^4+\frac{5}{2}r^3+\frac{9}{2}r^2+\frac{7}{2}r+1\right)G = \left(\frac{1}{2}r^8+6r^7+35r^6+126r^5+\frac{611}{2}r^4+510r^3+577r^2+408r+\frac{289}{2}\right)D^2+$$
$$+\left(2r^6+\frac{35}{2}r^5+72r^4+\frac{345}{2}r^3+\frac{513}{2}r^2+\frac{451}{2}r+\frac{195}{2}\right)D+\left(2r^4+\frac{23}{2}r^3+\frac{53}{2}r^2+29r+15\right).$$

In order to show that the top Segre class is always positive under the assumption $d<2g-2$, we prove that the parabola is contained in the half-plane $d>2g-2$. For these purposes, we first show that the conic takes positive values along the line $d=2g-2$. In other words, the parabola does not intersect this line. By connectedness, either the parabola is contained in the half-plane $d<2g-2$, or it is contained in the half-plane $d>2g-2$. By continuity of the real parameter r and the canonical form of the parabola, if the parabola is contained in one of the half-planes for one value of r, then it is contained in the same half-plane for every value of r. Therefore, it suffices to check the sign at one point of the parabola for one value of r. We find a point for $r=3$ that is contained in the half-plane $d>2g-2$. Hence the parabola are always contained in this half-plane, so the Segre number above is always positive. Thus the claim follows.

□

5.4. **Big and stable bundles on $X^{[k]}$ arising from Ulrich bundles on X a very general $K3$.** Taking into account what proved in § 3.3, here we have the following:

Theorem 5.6. *Let $k\geqslant 2$ and $h>2k-3$ be positive integers. Let (X,H) be a very–general polarized $K3$ surface, with polarization H of genus $g=h+1$. Set $Y:=X^{[k]}$.*

For every integer $a\geqslant 1$, consider any μ_H–stable Ulrich bundle E of rank-$2a$ on X as in Theorem 3.10. Thus, the tautological vector bundle $(E\otimes H)^{[k]}$ of rank $2ka$ is globally generated and $\mu_{D_k(H)}$–stable on Y.

If moreover $\int_Y s_{2k}(E\otimes H)^{[k]}>0$, then $(E\otimes H)^{[k]}$ is also big.

Proof. From Theorem 3.10, any Ulrich bundle E considered therein is μ_H–stable; so it is $E\otimes H$. Thus, the rank–$2ak$ vector bundle $(E\otimes H)^{[k]}$ is certainly $\mu_{D_k(H)}$-stable on Y, for any $k\geqslant 2$, as it follows from Theorem 4.1.

Since E is an Ulrich bundle on X so in particular it is globally generated. From the assumption $h>2k-3$, it follows that $g=h+1>2k-2$ therefore, from [42, Lemma 2.2] (or even from what discussed in the proof of Theorem 5.2), H is $(k-1)$–very ample on X. Thus, from Proposition 4.4–(ii), $E\otimes H$ is $(k-1)$-very ample on X and so, by Proposition 4.4–(i), the tautological bundle $(E\otimes H)^{[k]}$ is globally generated on Y. Thus, from Proposition 2.4, $(-1)^{2k}\int_Y s_{2k}(E\otimes H)^{[k]} = \int_Y s_{2k}(E\otimes H)^{[k]}>0$ implies that $(E\otimes H)^{[k]}$ is big on Y. □

Similarly as for Mukai–Lazarsfeld vector bundles, any Ulrich bundle E as above is such that

$$\mathrm{rk}(E)=2a,\quad c_1(E)=3aH,\quad \int_X c_2(E)=9a^2h-4a(h-1)$$

therefore, from (2.3), we have

$$\mathrm{rk}(E\otimes H)=2a,\quad c_1(E\otimes H)=2aH+c_1(E)=5aH,\quad \int_X c_2(E\otimes H)=2(2a-1)(a+1)(g-1)+9a^2h-4a(h-1).$$

Morevover, from (2.4), we have

$$s_1(E\otimes H)=-c_1(E\otimes H),\quad s_2(E\otimes H)=c_1(E\otimes H)^2-c_2(E\otimes H).$$

Using these expressions, one has:

Corollary 5.7. *Let $k \in \{2,\ 3\}$ and $h > 2k-3$ be integers. Let (X, H) be a very–general polarized $K3$ surface, with polarization H of genus $g = h+1$. For every integer $a \geqslant 1$, consider any globally generated and μ_H–stable Ulrich bundles E of rank-$2a$ on X as in Theorem 3.10. Set $Y := X^{[k]}$.*

- (i) *The vector bundle $(E \otimes H)^{[2]}$ is globally generated, $\mu_{D_2(H)}$–stable and big on $X^{[2]}$ for $g > 2$, without any restriction on the rank a.*
- (ii) *There exists a rational number $g(a)$, depending on the integer a, such that for $g > g(a)$ the vector bundle $(E \otimes H)^{[3]}$ is globally generated, $\mu_{D_3(H)}$–stable and big on $X^{[3]}$.*

Proof. This is a direct consequence of Theorem 5.6 and the numerical conditions on the top Segre classes. In both cases we used a MAPLE script. More details are available from the authors upon request. As for (i), the positivity of the Segre number $\int_{X^{[4]}} s_4((E \otimes H)^{[2]}) > 0$ translates into the following inequality, namely:

$$\frac{841}{2}a^4 + 606a^3 + \frac{521}{2}a^2 + 30a - \left(725a^4 + 874a^3 + \frac{581}{2}a^2 + 18a\right)g - \left(\frac{625}{2}a^4 + 300a^3 + 72a^2\right)g^2 > 0.$$

The left–hand–side member of the previous inequality is a degree–2 polynomial in the indeterminate g, with coefficients depending on a. An elementary numerical study of the real maximal root of this polynomial - as a function of a - shows that the maximal root is always smaller than 2, under the assumption $g > 2$. As for (ii), the condition $\int_{X^{[6]}} s_6((E \otimes H)^{[k]}) > 0$ is equivalent to the following inequality:

$$a_0 + a_1 g + a_2 g^2 + a_3 g^3 = \left(\frac{512}{3}a^7 - \frac{36961}{6}a^6 - \frac{42046}{3}a^5 - \frac{75263}{6}a^4 - 5339a^3 - 1066a^2 - 80a\right) +$$
$$+\left(\frac{27225}{2}a^6 + 27942a^5 + 21153a^4 + 7139a^3 + \frac{3010}{3}a^2 + 40a\right)g +$$
$$+\left(-\frac{20625}{2}a^6 - -18050a^5 - \frac{22301}{2}a^4 - 2784a^3 - 216a^2\right)g^2 + \left(\frac{15625}{6}a^6 + 3750a^5 + 1800a^4 + 288a^3\right)g^3 > 0.$$

The rational number $g(a)$ can be computed by means of the formula for resolving third degree equations. If we set $\delta_0 = a_2^2 - 3a_3a_1$, $\delta_1 = 2a_2^3 - 9a_3a_2a_1 + 27a_3^2a_0$, and $C = -\left|\delta_1 - \sqrt[3]{|\frac{|\delta_1^2 - 4\delta_0^3|}{2}}\right|$, then we have $g(a) = -\frac{a_2 + C + \frac{\delta_0}{C}}{3a_3}$. ☐

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Gilberto Bini, Dipartimento di Matematica ed Informatica, Università degli Studi di Palermo, Via Archirafi 34, 90133 Palermo – Italy

Email address: gilberto.bini@unipa.it

Samuel Boissière, Laboratoire de Mathématiques et Applications, Université de Poitiers, UMR 7348 du CNRS, Site du Futuroscope, TSA 61125, 11 bd Marie et Pierre Curie, 86073 Poitiers Cedex 9, France

Email address: samuel.boissiere@univ-poitiers.fr

Flaminio Flamini, Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Viale della Ricerca Scientifica 1, 00133 Roma – Italy

Email address: flamini@mat.uniroma2.it