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A moment matching method for option pricing under stochastic interest rates

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Summary

In this paper, we present a new and straightforward approximation methodology for pricing a call option in a Black and Scholes market, characterized by stochastic interest rates. The method relies on a Gaussian moment matching technique applied to a conditional Black and Scholes formula, used to disentangle the distributional complexity of the underlying price process. The problem then reduces to exploiting the Gaussian density and the expression of the bond price induced by the interest rate. To check its accuracy and computational time, we implement it for a CIR interest rate model correlated with the underlying, using Monte Carlo simulations as a benchmark. The method performance turns out to be quite remarkable, even when compared with similar results obtained by the affine approximation technique presented in Grzelak and Oosterlee, and by the expansion formula introduced in Kim and Kunimoto. In the last section, we apply the method also to the pricing of Forward-Starting options, to the evaluation of the credit spreads in the Merton structural approach to credit risk, and we outline a possible application to a stochastic volatility model.

KEYWORDS

Cox-Ingersoll-Ross model, moment matching, non-affine models, option pricing, stochastic interest rates

1 | INTRODUCTION

Since the appearance of the seminal Black and Scholes/Merton option pricing formula, there has been an intensive effort to incorporate in the market model additional stochastic factors, such as volatility and interest rates, the latter already discussed by Merton in Reference 1. Over the years, a wide field of research developed, leading to a rich literature on stochastic volatility models, while fewer papers aimed at the inclusion of a dynamic term structure into the valuation of derivatives, for example, Reference 2-8.

As shown in the empirical literature (for instance, see Reference 9,10), adding those risk factors to the models brought a noticeable improvement in the performances of option pricing formulas. Indeed in Reference 11, the author remarked that even including solely stochastic interest rates in the model does affect the pricing formula, especially for longer-dated options.

Including stochastic rates (as any other generalization) implies a higher degree of mathematical complexity. Consequently, the search for efficient pricing techniques to provide accurate evaluations in a short computational time (as opposed to Monte Carlo methods) became relentless. It is even more important nowadays in modern quantitative finance, where an enormous amount of data allows considering strategies that call for real-time model calibration. Computational efficiency has become one of the primary concerns of risk managers, essentially restricting their choice of models to the affine class (see Reference 7).

Indeed, when one models the interest rates by Gaussian processes, as in the popular Hull-White/Vasicek models, it is possible to obtain even analytical prices. Those models are surely appropriate when negative rates occur (as in the current period), but this feature becomes a drawback in periods when rates are usually greater than zero. Perhaps, the most popular affine model used in the literature to ensure the strict positivity of the interest rate is the Cox-Ingersoll-Ross (CIR) one under the Feller condition. The popularity comes from the fact that the CIR falls into the so-called affine models that exploit a very efficient and fast Fourier transform technique to price bonds. Unfortunately, the model affinity is lost when the interest rate correlates with the risky asset dynamics, and searching for accurate approximations of risk-neutral pricing formulas becomes very challenging.

Here, we take a market model where the asset price verifies a linear SDE, correlated with a stochastic risk-free rate, and we present a new and straightforward approximation of the European call option price, based on a moment matching (MM) technique. The problem is a classical one, and the novelty lies in the fact that the proposed method is easy to implement, and it leads to very accurate approximations. First, by conditioning, we provide a representation formula for the claim price in terms of the BS formula, then we build a Gaussian approximation of the involved random variables by matching the first two moments. We also employ a change-of-numeraire technique (introducing the T-forward measure as in Reference 12) to partially disentangle the contributions due to the underlying and to the interest rate. Hence, using the properties of the Normal cumulative distribution function (c.d.f.) (see Lemma 1), we can solve the inner expectation. This procedure leaves to compute a final expectation, expressed only in terms of the interest rate process. When applying the method to the affine models, we can exploit the explicit expressions of the bond price in this framework.

The numerical analysis we run, assuming the interest rate follows a CIR dynamic, shows small relative errors compared to the benchmark Monte Carlo simulated prices. Besides, the performance we reach is comparable with that of other efficient methods proposed in the literature, by Grzelak and Oosterlee in Reference 13, and by Kim and Kunimoto in Reference 6. The same accuracy is transposed to the approximate hedging strategy, as shown by the difference between the payoff Monte Carlo valuation and the approximate replicating strategy final value.

In Section 5, we show we may apply this method to other products also. In particular, we use it to approximate the prices of a Forward-Starting option and a defaultable bond within the Merton structural approach. In the same section, we also outline a possible way to extend the method to include stochastic volatility. Relying on previous work by Ford et al.[14] we consider a stochastic volatility Heston model with correlated CIR rate, and by applying our method to price short maturities contracts we reach a good accuracy compared to Monte-Carlo simulations.

Summarizing the paper is organized as follows. In Section 2, we derive the representation formula for the call option price in a Black and Scholes market with stochastic risk-free short rates, while in Section 3, we describe the MM method in detail. In Section 4, we focus our analysis on affine models, and we apply our technique to a CIR interest rate. In the same section, we briefly introduce the other two approximation techniques, the affine approximation, inspired by Reference 13, and the expansion method proposed in Reference 6, alternative to prices obtained by Monte Carlo simulations. Hence we run a numerical study comparing those methods with ours using Monte Carlo evaluation as a benchmark. In the final section, we apply our method to forward-starting options and defaultable ones in the Merton approach, as well we suggest an extension to stochastic volatility models.

2 | THE PRICE OF A EUROPEAN CALL IN THE BS MODEL WITH STOCHASTIC RATES

In this section, we are concerned with providing an approximation formula for the pricing of a European call option, whose payoff is given by the function $f(x) = (e^x - e^\kappa)^+$ for some $\kappa \in \mathbb{R}$, when stochastic interest rates come into play.

Thus, given a finite time interval $[0, T]$ and a complete probability space $(\Omega, \mathcal{F}, Q)$, with a filtration $\{\mathcal{F}_t\}_{\{t\in[0,T]\}}$ satisfying the "usual hypotheses" (see Reference 15), the market model is defined by the log-price of a risky asset and a risk-free interest (X_t, r_t), whose joint dynamics for any initial condition $(t, x, r) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ and $\forall s \in [t, T]$ is given by

$$\begin{cases} X_s = X_t + \int_t^s (r_v - \frac{\sigma^2}{2})dv + \sigma \left[\rho(B_s^1 - B_t^1) + \sqrt{1-\rho^2}(B_s^2 - B_t^2)\right], \quad X_t = x \\ r_s = r_t + \int_t^s \mu(v, r_v)dv + \int_t^s \eta(v, r_v)dB_v^1, \quad r_t = r, \end{cases} \tag{1}$$

where (B^1, B^2) is a two-dimensional standard Brownian motion, $\rho \in (-1, 1)$, and Q is some risk-neutral probability selected by the market. Finally, the deterministic functions $\mu(\cdot, \cdot)$ and $\eta(\cdot, \cdot)$ are in a class that ensures the existence and uniqueness of a strong solution of (1) (see e.g., Reference 16).

The pair (X_t, r_t) is Markovian, whence the arbitrage-free option price is a deterministic function of the state variables, given by

$$u(t, x, r, T; \rho) = \mathbb{E}\left[e^{-\int_t^T r_s ds}(e^{X_T(\rho)} - e^{\kappa})^+ | X_t = x, r_t = r\right], \tag{2}$$

provided that the coefficients μ and η are such that X_T and $\int_0^T |r_s|ds$ are exponentially integrable. Here, we write $X_T(\rho)$, to stress the dependence of the price on the correlation parameter, and, for shorthand, we denote $\mathbb{E}(\cdot|X_t = x, r_t = r)$ as $\mathbb{E}_t^{x,r}(\cdot)$, $\mathbb{E}(\cdot|r_t = r)$ as $\mathbb{E}_t^r(\cdot)$.

If $u(t, x, r, T; \rho)$ is regular enough in t, x, r, the Feynman-Kaĉ formula implies that it is a classical solution of the following two-dimensional parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}^\rho u = 0 \\ u(T, x, r, T; \rho) = (e^x - e^{\kappa})^+, \end{cases} \tag{3}$$

where $\mathcal{L}^\rho = \mathcal{L}^{\mathbf{0}} + \mathcal{A}$, with

$$\mathcal{L}^{\mathbf{0}} := \left(\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} - r\right) + \left(\frac{\eta^2(t, r)}{2}\frac{\partial^2}{\partial r^2} + \mu(t, r)\frac{\partial}{\partial r}\right) \tag{4}$$

$$\mathcal{A} := \rho\sigma\eta(t, r)\frac{\partial^2}{\partial x \partial r}. \tag{5}$$

By conditioning internally with respect to $\mathcal{F}_T^1 \vee \sigma(X_t, r_t) = \sigma(\{X_t, r_t, B_s^1 : t \le s \le T\})$, we have

$$u(t, x, r, T; \rho) = \mathbb{E}_t^{x,r}\left[e^{-\int_t^T r_s ds}(e^{X_T(\rho)} - e^{\kappa})^+\right] = \mathbb{E}_t^{x,r}\left[e^{-\int_t^T r_s ds}\mathbb{E}_t^{x,r}\left((e^{X_T(\rho)} - e^{\kappa})^+ | \mathcal{F}_T^1\right)\right]. \tag{6}$$

But $X_T(\rho)\Big|\mathcal{F}_T^1 \sim N(M_{t,T}, \Sigma_{t,T})$, where

$$M_{t,T} = x + \int_t^T \left(r_s - \frac{\sigma^2}{2}\right)ds + \sigma\rho(B_T^1 - B_t^1), \quad \text{and} \quad \Sigma_{t,T}^2 = \sigma(1 - \rho^2)(T - t).$$

Hence we obtain

$$\begin{aligned} \mathbb{E}_t^{x,r}\left[(e^{X_T(\rho)} - e^{\kappa})^+ | \mathcal{F}_T^1\right] &= e^{M_{t,T} + \frac{1}{2}\Sigma_{t,T}^2}\mathcal{N}\left(\frac{M_{t,T} - \kappa + \Sigma_{t,T}^2}{\Sigma_{t,T}}\right) - e^{\kappa}\mathcal{N}\left(\frac{M_{t,T} - \kappa}{\Sigma_{t,T}}\right) \\ &= e^{x + \int_t^T (r_s - \frac{\sigma^2}{2})ds + \sigma\rho(B_T^1 - B_t^1) + \frac{1}{2}\sigma^2(1-\rho^2)(T-t)}\mathcal{N}(d_1(\rho)) - e^{\kappa}\mathcal{N}(d_2(\rho)), \end{aligned}$$

where we define

$$d_1(\rho) \equiv d_1(x, t, T, \rho) = \frac{x - \kappa + \int_t^T r_s ds + \sigma\rho(B_T^1 - B_t^1) + \frac{\sigma^2}{2}(T - t) - \sigma^2\rho^2(T - t)}{\sigma\sqrt{1 - \rho^2}\sqrt{T - t}} \tag{7}$$

$$d_2(\rho) \equiv d_1(x, t, T, \rho) = \frac{x - \kappa + \int_t^T r_s ds + \sigma\rho(B_T^1 - B_t^1) - \frac{\sigma^2}{2}(T - t)}{\sigma\sqrt{1 - \rho^2}\sqrt{T - t}}, \tag{8}$$

and $\mathcal{N}$ denotes the cumulative distribution function of the standard Gaussian. It is convenient to introduce the following notations

$$\Lambda_{t,T} = \int_t^T r_s ds, \qquad \Delta B^i_{t,T} = B^i_T - B^i_t,$$
$$\beta(\tau,\rho) = \frac{\rho}{(1-\rho^2)^{1/2}\sqrt{\tau}}, \quad \gamma(\tau,\rho) = \frac{1}{\sigma(1-\rho^2)^{1/2}\sqrt{\tau}}$$
$$\alpha_1(x,\tau,\rho) = \frac{x-\kappa+\frac{\sigma^2}{2}\tau-\sigma^2\rho^2\tau}{\sigma(1-\rho^2)^{1/2}\sqrt{\tau}}, \quad \alpha_2(x,\tau,\rho) = \frac{x-\kappa-\frac{\sigma^2}{2}\tau}{\sigma(1-\rho^2)^{1/2}\sqrt{\tau}},$$

so that we may finally write

$$d_i(x,t,T,\rho) = \alpha_i(x,T-t,\rho) + \beta(T-t,\rho)\Delta B^1_{t,T} + \gamma(T-t,\rho)\Lambda_{t,T}, \quad i=1,2, \tag{9}$$

$$u(t,x,r,T;\rho) = \mathrm{e}^{x-\frac{\sigma^2\rho^2(T-t)}{2}}\mathbb{E}^{x,r}_t\left[\mathrm{e}^{\sigma\rho\Delta B^1_{t,T}}\mathcal{N}(d_1(\rho))\right] - \mathrm{e}^{\kappa}\mathbb{E}^{x,r}_t\left[\mathrm{e}^{-\Lambda_{t,T}}\mathcal{N}(d_2(\rho))\right]. \tag{10}$$

In the next section, we introduce the MM approximation procedure.

3 | OPTION PRICE APPROXIMATION BY MOMENT MATCHING

The main idea of this section is to replace the random variables $d_i(\rho)$, $i=1,2$, defined by (9), with Gaussian random variables $D_i(\rho)$, matching their first and second moments. We define

$$D_i(\rho) := \alpha_i(x,T-t,\rho) + \hat{\beta}(r,T-t,\rho)\Delta B^1_{t,T} + \gamma(T-t,\rho)\mathbb{E}^{x,r}_t(\Lambda_{t,T}), \quad i=1,2.$$

We impose

$$\mathbb{E}^{x,r}_t(D_i(\rho)) = \alpha_i(x,T-t,\rho) + \gamma(T-t,\rho)\mathbb{E}^r_t(\Lambda_{t,T}) = \mathbb{E}^{x,r}_t(d_i(\rho)), \tag{11}$$

and we fix the new coefficient $\hat{\beta} > 0$ such that for $i=1,2$,

$$\begin{aligned}\mathrm{var}^{x,r}_t(D_i(\rho)) = (T-t)\hat{\beta}^2(r,T-t,\rho) &= \mathrm{var}^{x,r}_t(d_i(\rho))\\ &= \beta^2(T-t,\rho)(T-t) + \gamma^2(T-t,\rho)\mathrm{var}^r_t(\Lambda_{t,T})\\ &\quad + 2\beta(T-t,\rho)\gamma(T-t,\rho)\mathbb{E}^r_t(\Delta B^1_{t,T}\Lambda_{t,T}),\end{aligned} \tag{12}$$

with

$$\mathbb{E}^r_t(\Lambda_{t,T}) = \int_t^T \mathbb{E}^r_t(r_s)ds =: \lambda(t,T) \tag{13}$$

$$\begin{aligned}\mathrm{var}^r_t(\Lambda_{t,T}) &= \mathbb{E}^r_t\left[\Lambda^2_{t,T}\right] - \lambda^2(t,T) = \mathbb{E}^r_t\left[\left(\int_t^T r_s ds\right)^2\right] - \lambda^2(t,T)\\ &= 2\int_t^T\int_t^s \mathbb{E}^r_t(r_s r_v)dsdv - \lambda^2(t,T)\\ &= 2\int_t^T\int_t^s\left[\mathbb{E}^r_t(r_s-r_v)\mathbb{E}^r_t(r_v) + \mathbb{E}^r_t(r_v^2)\right]dsdv - \lambda^2(t,T)\end{aligned} \tag{14}$$

$$\mathbb{E}^r_t(\Delta B^1_{t,T}\Lambda_{t,T}) = \mathbb{E}^r_t\left[(B^1_T - B^1_t)\int_t^T r_s ds\right]. \tag{15}$$

In the last passage in the variance computation, we used the independence of the increments of the process r.

The MM method with Gaussian random variables may be motivated by looking at the empirical distributional properties of the random variables d_i in some well-known rate models: see as examples Figures (1), (2), and (3).

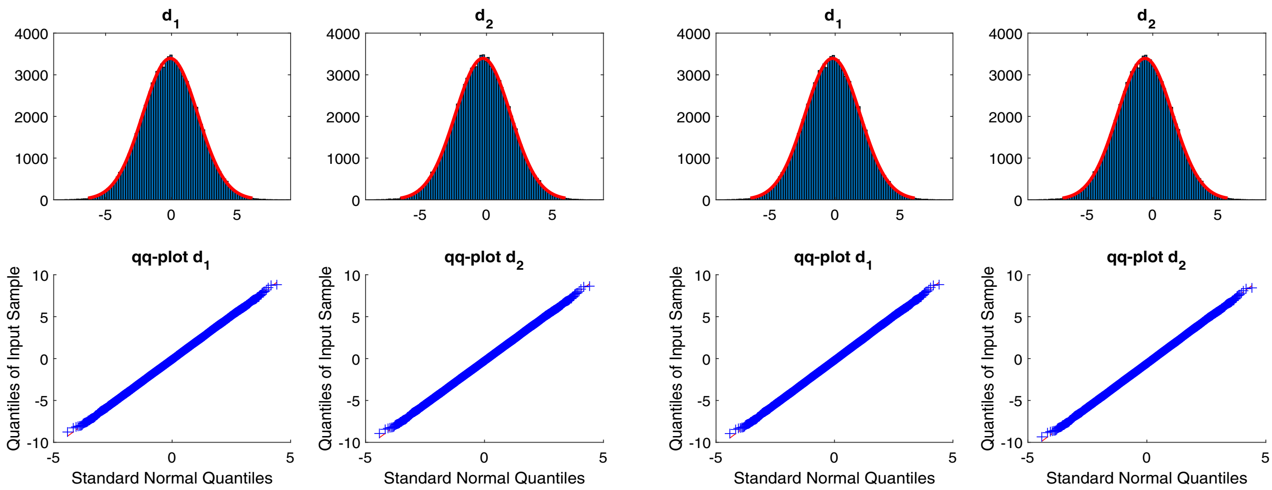

FIGURE 1 The histograms of d_1 and d_2 for $\rho = 0.3$, $T = 1$ (left) and $T = 5$ (right), in comparison with the standard normal law (in red) and related qq-plot, CIR dynamic: $dr_t = \kappa(\theta - r_t)dt + \eta\sqrt{r_t}dB_t^1$. [Color figure can be viewed at wileyonlinelibrary.com]

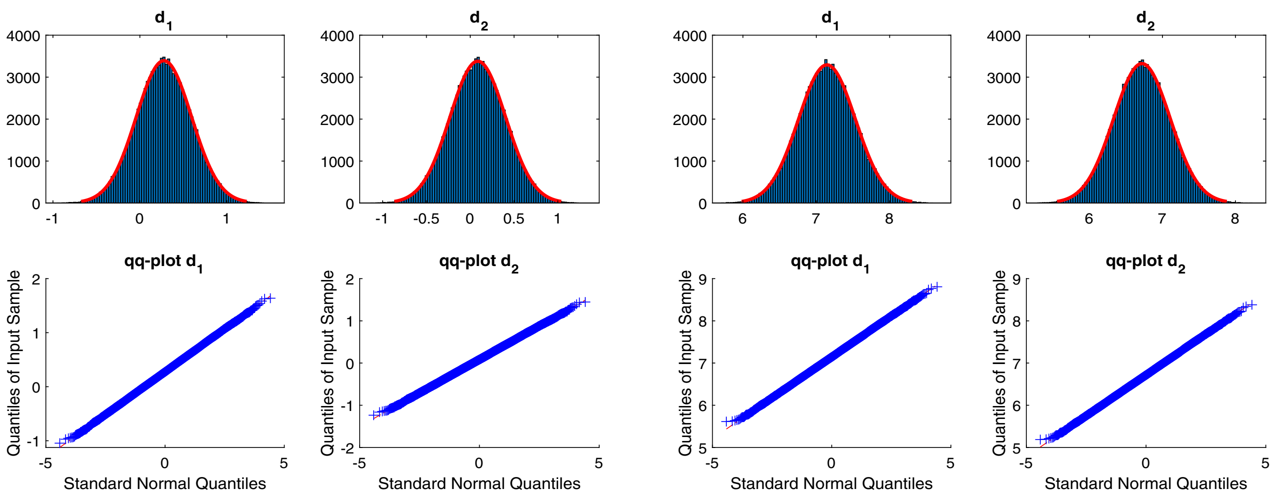

FIGURE 2 The histograms of d_1 and d_2 for $\rho = 0.3$, $T = 1$ (left) and $T = 5$ (right), in comparison with the standard normal law (in red) and related qq-plot, Exponential Vasicek dynamic: $dr_t = r_t(\theta - a\ln(r_t))dt + \eta r_t dB_t^1$. [Color figure can be viewed at wileyonlinelibrary.com]

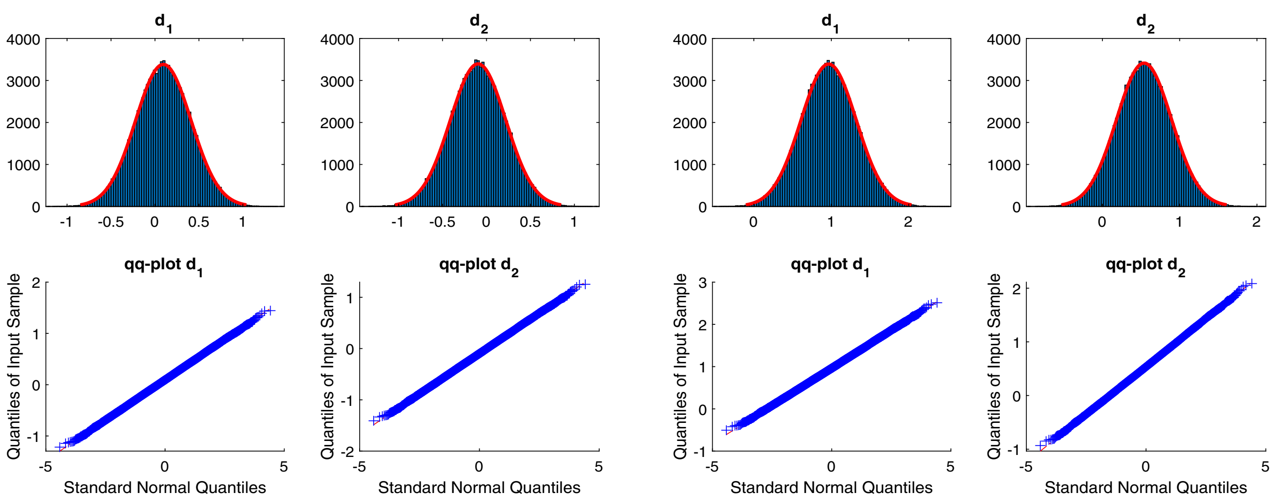

FIGURE 3 The histograms of d_1 and d_2 for $\rho = 0.3$, $T = 1$ (left) and $T = 5$ (right), in comparison with the standard normal law (in red) and related qq-plot, Dothan dynamic: $dr_t = ar_t dt + \eta r_t dB_t^1$. [Color figure can be viewed at wileyonlinelibrary.com]

We introduce the call price approximation

$$
\begin{aligned}
\tilde{u}(t,x,r,T;\rho) &:= \mathrm{e}^{x}\mathrm{e}^{-\frac{\sigma^2\rho^2}{2}(T-t)}\mathbb{E}_t^{x,r}\left[\mathrm{e}^{\sigma\rho\Delta B^1_{t,T}}\mathcal{N}\left(D_1(\rho)\right)\right] - \mathrm{e}^{\kappa}\mathbb{E}_t^{x,r}\left[\mathrm{e}^{-\Lambda_{t,T}}\mathcal{N}\left(D_2(\rho)\right)\right] \\
&=: \mathrm{e}^{x}\mathrm{e}^{-\frac{\sigma^2\rho^2}{2}(T-t)}\,F(\rho) - \mathrm{e}^{\kappa}G(\rho).
\end{aligned}
\tag{16}
$$

The function F can be evaluated in closed form by the following

Lemma 1. *Let $p \in \mathbb{R}$ and $X \sim N(\mu, \nu^2)$, $(\mu,\nu) \in \mathbb{R}\times\mathbb{R}^+$, then*

$$
\mathbb{E}(\mathrm{e}^{pX}\mathcal{N}(X)) = \mathrm{e}^{p\mu+\frac{(p\nu)^2}{2}}\mathcal{N}\left(\frac{\mu+p\nu^2}{\sqrt{1+\nu^2}}\right).
$$

Proof. See Reference 17 for $p=0$, the general case follows by a "completing the squares" argument. ∎

Since

$$
\Delta B^1_{t,T} = [D_1(\rho) - \alpha_1(x,T-t,\rho) - \gamma(T-t,\rho)\lambda(t,T)]\,\hat{\beta}(r,T-t,\rho)^{-1},
$$

applying the above Lemma, we may rewrite F as

$$
\begin{aligned}
F(\rho) &= \mathbb{E}_t^{x,r}\left[\mathrm{e}^{\sigma\rho(D_1(\rho)-\alpha_1(x,T-t,\rho)-\gamma(T-t,\rho)\lambda(t,T)\hat{\beta}(r,T-t,\rho)^{-1}}\mathcal{N}(D_1(\rho))\right] \\
&= \mathrm{e}^{-\sigma\rho[\alpha_1(x,T-t,\rho)+\gamma(T-t,\rho)\lambda(t,T)]\hat{\beta}(r,T-t,\rho)^{-1}}\mathbb{E}_t^{x,r}\left[\mathrm{e}^{\sigma\rho D_1(\rho)\hat{\beta}(r,T-t,\rho)^{-1}}\mathcal{N}(D_1(\rho))\right] \\
&= \mathrm{e}^{-\frac{\sigma\rho}{\hat{\beta}(r,T-t,\rho)}[\alpha_1(x,T-t,\rho)+\gamma(T-t,\rho)\lambda(t,T)]} \\
&\quad\times \mathrm{e}^{\frac{\sigma\rho\mathbb{E}_t^{x,r}(D_1(\rho))}{\hat{\beta}(r,T-t,\rho)}+\frac{\sigma^2\rho^2\mathrm{var}_t^{x,r}(D_1(\rho))}{2\hat{\beta}^2(r,T-t,\rho)}}\mathcal{N}\left(\frac{\mathbb{E}_t^{x,r}(D_1(\rho))+\sigma\rho\mathrm{var}_t^{x,r}(D_1(\rho))\hat{\beta}(r,T-t,\rho)^{-1}}{\sqrt{1+\mathrm{var}_t^{x,r}(D_1(\rho))}}\right).
\end{aligned}
$$

From (11) and (12), we may conclude

$$
F(\rho) = \mathrm{e}^{\frac{\sigma^2\rho^2(T-t)}{2}}\mathcal{N}\left(\frac{\alpha_1(x,T-t,\rho)+\sigma\rho\hat{\beta}(r,T-t,\rho)(T-t)+\gamma(T-t,\rho)\mathbb{E}_t^r(\Lambda_{t,T})}{\sqrt{1+\hat{\beta}^2(r,T-t,\rho)(T-t)}}\right).
\tag{17}
$$

If $\lambda(t,T) = \mathbb{E}_t^r(\Lambda_{t,T})$ and (14), (15) can be computed analytically, then F is totally explicit.

On the contrary, the function G cannot be evaluated in such a straightforward manner, as it involves a detailed knowledge of the joint distribution of $\Lambda_{t,T}$ and $B^1_T - B^1_t$ and not only of their moments and covariance. To represent G, we suggest employing a change-of-numeraire technique to exploit the bond pricing theory.

Let us define

$$
P(s,T) := \mathbb{E}\left(\mathrm{e}^{-\int_s^T r_v dv}|\mathcal{F}_s\right),
\tag{18}
$$

the Zero Coupon Bond price. Again, since $r.$ is a Markov process, $P(s,T)$ is a deterministic function of the state variable, say $g(s,r_s)$, where $g(t,r) = \mathbb{E}_s^r\left(\mathrm{e}^{-\int_s^T r_v dv}\right)$, which we assume to be $C^{1,2}([0,T]\times\mathbb{R}^+)$. For $0 \le t \le s \le T$, we define the martingale (we remark it is a true martingale thanks to the exponential integrability of $\Lambda_{t,T}$)

$$
L_{t,s} = \frac{\mathbb{E}(\mathrm{e}^{-\int_t^T r_v dv}|\mathcal{F}_s)}{P(t,T)} = \mathrm{e}^{-\Lambda_{t,s}}\frac{P(s,T)}{P(t,T)} = \mathrm{e}^{-\Lambda_{t,s}}\frac{g(s,r_s)}{g(t,r)}, \quad r_t = r.
\tag{19}
$$

By applying the Itô formula, we have the dynamic of L

$$
\begin{aligned}
dL_{t,s} &= \frac{e^{-\Lambda_{t,s}}}{g(t,r)}\left[\frac{\partial g}{\partial t}(s,r_s)+\frac{1}{2}\eta^2(s,r_s)\frac{\partial^2 g}{\partial r^2}(s,r_s)+\mu(s,r_s)\frac{\partial g}{\partial r}(s,r_s)-r_s g(s,r_s)\right]ds \\
&+\frac{e^{-\Lambda_{t,s}}}{g(t,r)}\eta(s,r_s)\frac{\partial g}{\partial r}(s,r_s)dB^1_s \\
&=\frac{e^{-\Lambda_{t,s}}}{g(t,r)}\eta(s,r_s)\frac{\partial g}{\partial r}(s,r_s)dB^1_s = L_{t,s}\left(\frac{\eta}{g}\frac{\partial g}{\partial r}\right)(s,r_s)dB^1_s, \quad L_{t,t}=1,
\end{aligned}
$$

and we may define the T-forward measure $\mathcal{F}$ by $Q^T(A) := \mathbb{E}_0(L_{0,T}1_A), \forall\, A \in \mathcal{F}_T$, (see Reference 18 for the method and Reference 19 for a similar application). Under Q^T, we get

$$
G(\rho) = \mathbb{E}^{x,r}_t\left[e^{-\Lambda_{t,T}}\mathcal{N}(D_2(\rho))\right] = P(t,T)\mathbb{E}^{Q^T}_t\left[\mathcal{N}(D_2(\rho))\right], \tag{20}
$$

and by the Girsanov theorem, setting

$$
\xi_s := \int_0^s \frac{\eta(v,r_v)}{g(v,r_v)}\frac{\partial g}{\partial r}(v,r_v)dv,
$$

we have that the process $\tilde{B}^1_s := B^1_s - \xi_s$ is a Q^T–Brownian motion.

When choosing an interest rate model that provides an explicit expression of the bond price, $\mathbb{E}^{Q^T}_t\left[\mathcal{N}(D_2(\rho))\right]$ remains the final quantity to compute. Under Q^T, $D_2(\rho)$ has the following expression

$$
D_2(\rho) = \alpha_2(x,T-t,\rho) + (\xi_T-\xi_t)\hat{\beta}(r,T-t,\rho) + \hat{\beta}(r,T-t,\rho)\Delta\tilde{B}^1_{t,T} + \gamma(T-t,\rho)\lambda(t,T),
$$

whence its distribution is no longer known. To circumvent this difficulty, we replace $D_2(\rho)$ with the r.v.

$$
\overline{D}_2(\rho) := \alpha_2(x,T-t,\rho) + \mathbb{E}^r_t(\xi_T-\xi_t)\hat{\beta}(r,T-t,\rho) + \hat{\beta}(r,T-t,\rho)\Delta\tilde{B}^1_{t,T} + \gamma(T-t,\rho)\lambda(t,T),
$$

where we are taking the expectation $\mathbb{E}^{x,r}_t(\xi_T-\xi_t) = \mathbb{E}^r_t(\xi_T-\xi_t)$, under the probability Q. From now on, we denote such expectation by $\epsilon(r,t,T)$, and we have that $\overline{D}_2(\rho)$ is a Gaussian r.v., whence we may apply Lemma 1 once again, obtaining

$$
\mathbb{E}^{Q^T}_t(\mathcal{N}(\overline{D}_2(\rho))) = \mathcal{N}\left(\frac{\mathbb{E}^{Q^T}_t(\overline{D}_2(\rho))}{\sqrt{1+\mathrm{var}^{Q^T}_t(\overline{D}_2(\rho))}}\right),
$$

with

$$
\begin{aligned}
\mathbb{E}^{Q^T}_t\left(\overline{D}_2(\rho)\right) &= \alpha_2(x,T-t,\rho) + \epsilon(r,t,T)\hat{\beta}(r,T-t,\rho) + \gamma(T-t,\rho)\lambda(t,T), \\
\mathrm{var}^{Q^T}_t\left(\overline{D}_2(\rho)\right) &= \hat{\beta}^2(r,T-t,\rho)(T-t).
\end{aligned}
$$

Hence we denote by

$$
\overline{G}(\rho) := P(t,T)\mathbb{E}^{Q^T}_t\left[\mathcal{N}(\overline{D}_2(\rho))\right]
$$

the approximation of $G(\rho)$, and we may define the final approximation of the call option price $u(t,x,r,T;\rho)$ as

$$
\begin{aligned}
\bar{u}(t,x,r,T;\rho) &:= e^{x-\frac{1}{2}\sigma^2\rho^2(T-t)}F(\rho) - e^{\kappa}\overline{G}(\rho) \\
&= e^x\mathcal{N}\left(\frac{\alpha_1(x,T-t,\rho)+\sigma\rho\hat{\beta}(r,T-t,\rho)(T-t)+\gamma(T-t,\rho)\lambda(t,T)}{\sqrt{1+\hat{\beta}^2(r,T-t,\rho)(T-t)}}\right) \\
&- e^{\kappa}P(t,T)\mathcal{N}\left(\frac{\alpha_2(x,T-t,\rho)+\epsilon(r,t,T)\hat{\beta}(r,T-t,\rho)+\gamma(T-t,\rho)\lambda(t,T)}{\sqrt{1+\hat{\beta}^2(r,T-t,\rho)(T-t)}}\right).
\end{aligned} \tag{21}
$$

We conclude this section with some final remarks. First, we summarize the requirements to make the approximation (21) explicitly computable and hopefully efficient:

1. the distributions of $d_i(\rho), i = 1, 2$ should be close to a Gaussian distribution;
2. the bond price $P(t, T)$ should be theoretically computable;
3. the quantities $\mathbb{E}_t^r(\Lambda_{t,T})$, $\text{var}_t^r(\Lambda_{t,T})$ and $\mathbb{E}_t^r(\Delta B^1_{t,T}\Lambda_{t,T})$ and/or their approximations, should be easy to compute;
4. it should be possible to apply the change of numeraire technique (the Girsanov theorem).

We also observe that one might plug the market bond price for $P(t, T)$ into (19) and use it for calibration purposes.

Eventually, to validate this approximation technique, one has to compare results with Monte Carlo simulated prices, as well as with other methods present in the literature, as we are going to do in the next section.

We finally remark that the same methodology may provide an approximation of the hedging strategy. Indeed, given formula (16), differentiating $\bar{u}(t, x, r, T; \rho)$ for $z = e^x$, we obtain

$$\partial_z \bar{u}(x, r, t, T; \rho) = e^{-\frac{1}{2}\sigma^2\rho^2(T-t)} F(\rho) + e^x e^{-\frac{1}{2}\sigma^2\rho^2(T-t)} \partial_z F(\rho) - e^\kappa \partial_z \overline{G}(\rho), \tag{22}$$

where

$$\partial_z F(\rho) = \mathcal{N}'\left(\frac{\alpha_1(x, T-t, \rho) + \sigma\rho\hat{\beta}(r, T-t, \rho)(T-t) + \gamma(T-t, \rho)\lambda(t, T)}{\sqrt{1 + \hat{\beta}^2(r, T-t, \rho)(T-t)}}\right)$$
$$\frac{\partial_x \alpha_1(x, T-t, \rho)}{e^x\sqrt{1 + \hat{\beta}^2(r, T-t, \rho)(T-t)}},$$
$$\partial_z \overline{G}(\rho) = P(t, T)\mathcal{N}'\left(\frac{\alpha_2(x, T-t, \rho) + \epsilon(r, t, T)\hat{\beta}(r, T-t, \rho) + \gamma(T-t, \rho)\lambda(t, T)}{\sqrt{1 + \hat{\beta}^2(r, T-t, \rho)(T-t)}}\right),$$
$$\frac{\partial_x \alpha_2(x, T-t, \rho)}{e^x\sqrt{1 + \hat{\beta}^2(r, T-t, \rho)(T-t)}},$$
$$\partial_x \alpha_i(x, T-t, \rho) = \frac{1}{\sigma\sqrt{(1-\rho^2)(T-t)}}, \quad i = 1, 2.$$

All the quantities appearing in the above formulas have already been obtained during the price computation, making the classical hedging approach feasible.

4 | NUMERICS AND COMPARISON WITH OTHER METHODOLOGIES

In this section, we employ an affine model for the interest rate, which provides an explicit expression for the Zero Coupon Bond price (18). Our market model is then given by

$$X_s = X_t + \int_t^s \left(r_v - \frac{\sigma^2}{2}\right) dv + \sigma\left[\rho(B^1_s - B^1_t) + \sqrt{1-\rho^2}(B^2_s - B^2_t)\right], \quad X_t = x$$
$$r_s = r_t + \int_t^s [a(v)r_v + b(v)]dv + \int_t^s [c(v)r_v + d(v)]^{1/2} dB^1_v, \quad r_t = r, \tag{23}$$

where $a, b, c, d : [0, T] \to \mathbb{R}$ are bounded functions. In this framework, the ZCB price has the following expression as function of the state variable $r_t = r$

$$P(t, T) = g(t, r) = A(t, T)e^{-rB(t,T)},$$

for suitable deterministic functions $A(\cdot, T)$ and $B(\cdot, T)$. Two very classical models fall into this setting

$$\begin{array}{lllll} \text{(Vasicek)} & a(v) = -\gamma, & b(v) = \gamma\theta, & c(v) = 0, & d(v) = \eta^2 \\ \text{(CIR)} & a(v) = -\gamma, & b(v) = \gamma\theta, & c(v) = \eta^2, & d(v) = 0, \end{array} \quad \gamma, \theta, \eta > 0.$$

In both cases, the functions $A(t, T)$ and $B(t, T)$, characterized as the solution of a Riccati system of ODEs, are known explicitly,[12] the same being true also for their time-dependent extensions (Hull-White/Vasicek, and Hull-White/CIR models). Unfortunately, when X and r are correlated, the pair might be no longer jointly affine since its diffusion matrix

$$\sigma(v, x, r)\sigma(v, x, r)^T = \begin{pmatrix} \sigma^2 & \rho\sigma[c(v)r + d(v)]^{1/2} \\ \rho\sigma[c(v)r + d(v)]^{1/2} & c(v)r + d(v) \end{pmatrix}, \tag{24}$$

may have non-linear entries in the state variables (this happens for the CIR model), and the above approach does not apply.

Considering the correlation in the market model (23) is not irrelevant. Indeed, it has a noticeable impact on the term structure of the implied volatility determined by at-the-money options, strengthening the sensitivity of the model to the parameters (in particular to the pair r, θ) see Figure (4). Depending on the choice of r and θ, correlated rates may create upward or downward sloping term structures, which add more flexibility to the pricing model. This effect grows as maturity increases: while it is below few basis points within the year, it rises to hundreds of basis points for five years.

In this context, we apply the approximation procedure presented in the previous section. In this case (see e.g., Reference 12), setting $\delta = \sqrt{\gamma^2 + 2\eta^2}$, we have

$$A(t, T) = e^{\frac{2\gamma\theta}{\eta^2}} \frac{2\delta e^{\gamma+\delta(T-t)}}{\delta - \gamma + (\delta + \gamma)e^{\delta(T-t)}}, \quad B(0, T) = \frac{2(e^{\delta(T-t)} - 1)}{\delta - \gamma + (\delta + \gamma)e^{\delta(T-t)}},$$

and we need to calculate $\mathbb{E}_t^r(\Lambda_{t,T})$, $\text{var}_t^r(\Lambda_{t,T})$ and $\mathbb{E}_t^r\left(\Lambda_{t,T}(B_T^1 - B_t^1)\right)$.

1. **Computation of** $\mathbb{E}_t^r(\Lambda_{t,T})$. It is straightforward to see

$$\mathbb{E}_t^r(\Lambda_{t,T}) = \int_t^T \mathbb{E}_t^r(r_s)ds = \int_t^T \left[(r - \theta)e^{-\gamma(s-t)} + \theta\right] ds = \theta(T - t) + (r - \theta)\frac{1 - e^{-\gamma(T-t)}}{\gamma}.$$

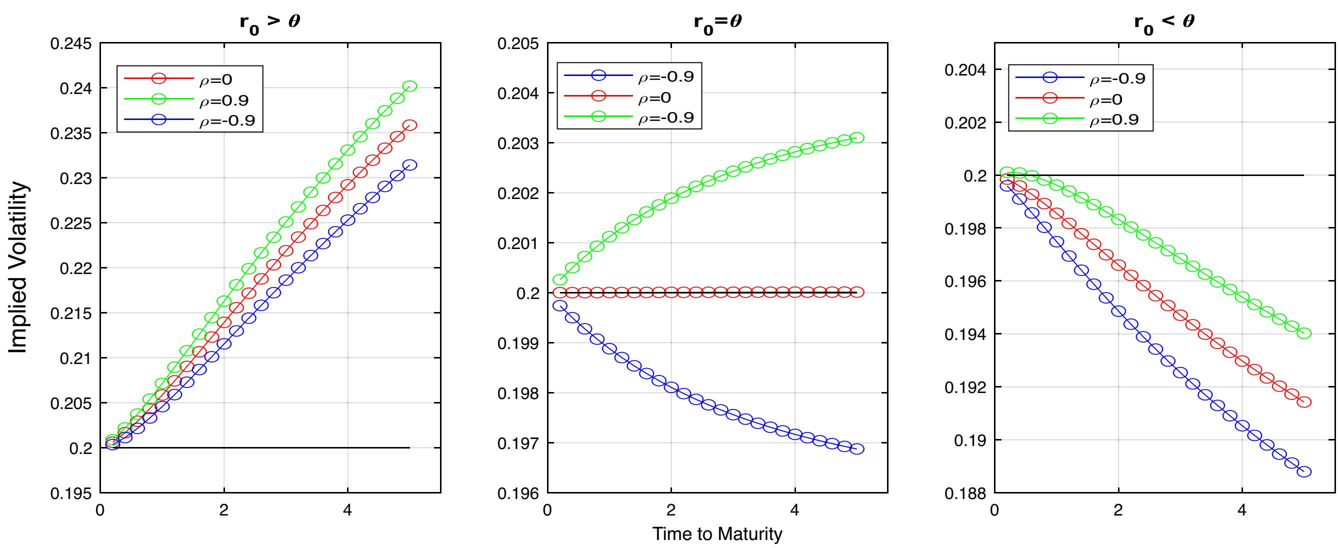

FIGURE 4 Impact of model correlation on at-the-money implied volatilities. The rate parameters of CIR are $r_0 = 0.01$, $\gamma = 0.58$, $\eta = 0.03$ and $\theta \in \{0.005, 0.01, 0.03\}$. The variations with respect to the constant level implied volatility (i.e., the constant rate model) range from few percentage points up to 20%. [Color figure can be viewed at wileyonlinelibrary.com]

2. **Computation of** $\mathrm{var}_t^r(\Lambda_{t,T})$. From (14), the problem reduces to computing $\mathbb{E}_t^r(r_v^2)$ for $v \geq t$. By applying the Itô formula and taking expectations, we arrive at the following equation

$$\mathbb{E}_t^r(r_v^2) = r^2 + \int_t^v \left[(2\gamma\theta + \eta^2)\mathbb{E}_t^r(r_w) - 2\gamma\mathbb{E}_t^r(r_w^2)\right] dw$$

that has explicit solution

$$\mathbb{E}_t^r(r_v^2) = \left[(r-\theta)^2 - \frac{\eta^2}{\gamma}\left(r - \frac{\theta}{2}\right)\right] \mathrm{e}^{-2\gamma(v-t)} + \left(2\theta + \frac{\eta^2}{\gamma}\right)(r-\theta)\mathrm{e}^{-\gamma(v-t)} + \theta\left(\theta + \frac{\eta^2}{2\gamma}\right).$$

3. **Computation of** $\mathbb{E}_t^r(\Delta B_{t,T}^1 \Lambda_{t,T})$. First, we use the Ito integration by parts formula to write

$$\mathbb{E}_t^r(\Delta B_{t,T}^1 \Lambda_{t,T}) = \int_t^T \mathbb{E}_t^r(\Delta B_{t,s}^1 r_s) ds,$$

then applying the integration by parts again, we have

$$\Delta B_{t,s}^1 r_s = \int_t^s \left[\Delta B_{t,v}^1 \gamma(\theta - r_v) + \eta\sqrt{r_v}\right] dv + \eta \int_t^s \left[\Delta B_{t,v}^1 \sqrt{r_v} + r_v\right] dB_v^1,$$

and taking expectations, one gets the equation

$$\mathbb{E}_t^r(\Delta B_{t,s}^1 r_s) = -\gamma \int_t^s \mathbb{E}_t^r(\Delta B_{t,v}^1 r_v) dv + \eta \int_t^s \mathbb{E}_t^r(\sqrt{r_v}) dv,$$

with explicit solution

$$\mathbb{E}_t^r(\Delta B_{t,s}^1 r_s) = \eta \int_0^s \mathrm{e}^{-\gamma(s-v)} \mathbb{E}_t^r(\sqrt{r_v}) dv$$

$$\Rightarrow \quad \mathbb{E}_t^r(\Delta B_{t,s}^1 \Lambda_{t,T}) = \eta \int_t^T \int_t^s \mathrm{e}^{-\gamma(s-v)} \mathbb{E}_t^r(\sqrt{r_v}) dv ds.$$

Thus the crucial point is computing $\mathbb{E}_t^r(\sqrt{r_v})$, which is quite delicate (see Reference 20). Indeed, this expectation is not computable explicitly, and we exploit the approximation proposed by Grzelak and Oosterlee in Reference 14, where they write it as

$$\mathbb{E}_t^r(\sqrt{r_v}) \approx a + b\mathrm{e}^{-c(v-t)}, \tag{25}$$

for appropriate numerically efficient parameters a, b and c.

Finally, employing the above three points, we may compute $\hat{\beta}(r, T-t, \rho)$ from (12), and we may also approximate $\mathbb{E}_t^r(\xi_T - \xi_t)$ as

$$\mathbb{E}_t^r(\xi_T - \xi_t) = -\eta \int_t^T B(s,T)\mathbb{E}_t^r\left(\sqrt{r_s}\right) ds \approx -\eta \int_t^T B(s,T)(a + b\mathrm{e}^{-c(s-t)}) ds.$$

For completeness in the next two subsections, we briefly describe the two approximation techniques, in respectively References 14 and 6, that we are going to use for comparison.

4.1 | The Grzelak-Oosterlee (GO) approximation

The GO approximation consists simply of modifying the operator $\mathcal{A}$ given in (5), replacing the state variable in the coefficient with a deterministic function. Namely, the new operator is

$$\mathcal{A}^{GO}u(t,x,r) := \rho\sigma\mathbb{E}(\eta(t,r_t))\frac{\partial^2 u}{\partial x \partial r}(t,x,r).$$

In the case of the CIR model, η is time-homogeneous, and this operator becomes

$$\mathcal{A}^{GO}u(t,x,r) := \rho\sigma\eta\mathbb{E}(\sqrt{r_t})\frac{\partial^2 u}{\partial x\partial r}(s,x,r) \approx \rho\sigma\eta(a+b\mathrm{e}^{-ct})\frac{\partial^2 u}{\partial x\partial r}(t,x,r).$$

Once this replacement has taken place, the modified PDE presents only linear coefficients (in the state variables), and one can employ Fourier transform methods to compute the solution. We shall denote this approximation by $u^{GO}(t,x,r,T;\rho)$.

The discounted transform, for $\zeta \in \mathbb{C}$, (see Reference 7) for the affine approximation is

$$\phi(\zeta,t,x,r,T) := \mathbb{E}_t^{x,r}\left(\mathrm{e}^{-\int_t^T r_s ds}\mathrm{e}^{\zeta X_T}\right) = \mathrm{e}^{A(\zeta,t,T)+B(\zeta,t,T)x+C(\zeta,t,T)r},$$

where the functions A, B, C satisfy a system of solvable ODE's, and they are

$$B(\zeta,t,T) = \zeta,$$

$$C(\zeta,t,T) = \frac{1-\mathrm{e}^{-d(T-t)}}{\eta^2(1-g\mathrm{e}^{-d(T-t)})},\quad d=\sqrt{\gamma^2+2\eta^2(1-\zeta)},\quad g=\frac{\gamma-d}{\gamma+d},$$

$$A(\zeta,t,T) = -\frac{\sigma^2}{2}(T-t)\zeta(1+\zeta)+\frac{\gamma-d}{\eta}\int_t^T\left[\frac{\gamma\theta}{\eta}+\rho\sigma\zeta(a+b\mathrm{e}^{-cs})\right]\frac{1-\mathrm{e}^{-d(T-s)}}{1-g\mathrm{e}^{-d(T-s)}}ds.$$

Finally, by Lévy inversion formula as in Reference 7, or Fourier inversion as in Reference 21, one gets an integral representation for the price function: in our implementation, we use the Fourier inversion

$$u^{GO}(t,x,r,T;\rho) = \frac{\mathrm{e}^{\nu\gamma}}{\pi}\int_0^{+\infty}\mathcal{R}\left(\frac{\mathrm{e}^{-\mathrm{i}\zeta\gamma}}{\nu^2-\nu-\zeta^2+\mathrm{i}\zeta(1-2\nu)}\phi(t,\zeta,x,r,T)\right)d\zeta, \tag{26}$$

where $\nu < 0$ is a dumping factor, and $\mathcal{R}(z)$ is the real part for $z \in \mathbb{C}$.

4.2 | The Kim–Kunimoto (KK) approximation

Kim and Kunimoto, in Reference 6, consider a Taylor expansion of the process r_s in powers of η around $\eta = 0$. When looking at the first-order polynomial, they obtain

$$r_s = \theta+(r-\theta)\mathrm{e}^{-\gamma(s-t)}+\eta\int_t^s \mathrm{e}^{-\gamma(s-v)}\sqrt{\theta+(r-\theta)\mathrm{e}^{-\gamma(v-t)}}(\rho dB_v^1+\sqrt{1-\rho^2}dB_v^2)+o(\eta). \tag{27}$$

Denoting by $\varphi(t,s) = \theta+(r-\theta)\mathrm{e}^{-\gamma(s-t)}$, and inserting the approximation (27) in the evaluation formula for the call option, after some manipulations, one can approximate the option price as

$$\begin{aligned} u^{KK}(t,x,r,T;\rho) &= \mathrm{e}^x\mathcal{N}(d_1)-\mathrm{e}^{\kappa-\int_t^T\varphi(t,s)ds}\mathcal{N}(d_2) \\ &\quad+\eta C_1\left[d_2\mathrm{e}^x\mathcal{N}'(d_1)-d_1\mathrm{e}^{\kappa-\int_t^T\varphi(t,s)ds}\mathcal{N}'(d_2)\right] \end{aligned} \tag{28}$$

where

$$C_1 = -\frac{\rho}{\sigma(T-t)}\frac{2\sqrt{\theta}\left[(1+2\mathrm{e}^{\gamma(T-t)})\sqrt{r}-3\gamma_K\right]+\left[r-\theta(1+2\mathrm{e}^{\gamma(T-t)})\right]\lambda_K}{2\mathrm{e}^{\gamma(T-t)}\gamma^2\sqrt{\theta}},$$

$$d_1 = \frac{x-\kappa+\theta(T-t)+(r-\theta)(1-\mathrm{e}^{-\gamma(T-t)})/\gamma+\sigma^2(T-t)/2}{\sqrt{\sigma^2(T-t)}},\quad d_2 = d_1-\sigma\sqrt{(T-t)},$$

being $\gamma_K = \mathrm{e}^{\gamma(T-t)/2}\sqrt{r-\theta(1-\mathrm{e}^{\gamma(T-t)})}$, and $\lambda_K = \log\left(\frac{(\sqrt{r}+\sqrt{\theta})^2}{r-\theta(1-2\mathrm{e}^{\gamma(T-t)})+2\gamma_K\sqrt{\theta})}\right)$.

TABLE 1 Results of the approximations for the parameters $\gamma = 0.6$, $\theta = 0.02$, $\eta = 0.1$, $r_0 = 0.001$, and σ_L. The time to maturity is $T = 1$ and $K = 100$. In parenthesis the confidence interval of the Monte Carlo (MC) estimates. The error is defined as the difference between the MC price and the related approximation.

ρ	−0.9	−0.6	−0.3	0.0	0.3	0.6	0.9
Prices							
MC	8.1543	8.1799	8.2055	8.2314	8.2574	8.2832	8.3085
	(0.0225)	(0.0137)	(0.0064)	(0.0003)	(0.0069)	(0.0142)	(0.0230)
GO	8.1192	8.1568	8.1943	8.2315	8.2686	8.3055	8.3423
KK	8.1361	8.1677	8.1993	8.2309	8.2625	8.2941	8.3258
MM	8.146	8.1745	8.2029	8.2313	8.2595	8.2877	8.3157
Errors							
GO	0.0351	0.0231	0.0113	−0.0001	−0.0113	−0.0223	−0.0338
KK	0.0182	0.0121	0.0062	0.0005	−0.0052	−0.0109	−0.0172
MM	0.0083	0.0053	0.0026	0.0001	−0.0022	−0.0045	−0.0072
Rel. Err.							
GO	0.0043	0.0028	0.0013	1.4e-05	0.0014	0.0027	0.0041
KK	0.0022	0.0015	0.0008	5.9e-05	0.0006	0.0013	0.0021
MM	0.0010	0.00065	0.0003	1.7e-05	0.0003	0.0005	0.0009

4.3 | Numerical results

We ran all the numerical experiments for $t = 0$, comparing the results of the different approximations with the benchmark Monte Carlo method, applied to the price (10). Thus we had to simulate only the interest rate process to get samples from $d_1(\rho)$ and $d_2(\rho)$, which was done by Euler discretization with a full truncation algorithm (see Reference 22). In our numerical experiments, we generated $M = 10^6$ sample paths with a time step discretization equal to 10^{-3} for all the maturities. We implemented the algorithms in MatLab (R2019b) on an Intel Core i7 2.40GHZ with 8GB RAM, by using the available building-in functions, in particular for the computation of all the integrals involved. The average time to compute one price was (in secs) 32.1 (MC), 0.055 (GO), 0.005 (KK) and 0.009 (MM).

We chose different sets of parameters (κ, θ, η) and volatility scenarios: a low volatility $\sigma_L = 0.2$ and a high volatility $\sigma_H = 0.4$, and for each set, we varied the correlation ρ, the rate volatility η and the maturity of the contract T, for at-the-money options at level 100. Tables 1, 2, 3, 4, 5, 6, 7 and 8 summarize the numerical results. At least in the CIR model, these show that the MM method produces the best approximations in most scenarios when compared with the benchmark Monte Carlo evaluation.

4.3.1 | Δ-Hedging

We verified the quality of the Delta approximation (22) by implementing the classical Δ-hedging strategy over a set of simulated sample paths, according to the market model (23) with CIR rates. We ran the computations for an at-the-money call with 1000 scenarios for three different levels of correlation with daily rebalancing, and we report them in Figure 5. For values of $\rho = -0.3, 0, 0.3$, the mean absolute hedging errors (the absolute difference between the replicating portfolio final value and the observed call payoff) was respectively 0.0572 (standard deviation 0.3621), 0.0720 (st. dev. 0.3653), and 0.0776 (st. dev. 0.3605). The obtained results are comparable with the mean absolute hedging error, equal to 0.0140 (st. dev. 0.3874), made in the Black-Scholes framework with risk-free rate r_0, using the analytical values of the price and the Δ.

5 | FURTHER APPLICATIONS

Once established the performance of our method in the case of a call option, we present three other feasible applications that could be useful for operative desks.

TABLE 2 Results of the approximations for the parameters $\gamma = 0.6$, $\theta = 0.02$, $\eta = 0.1$, $r_0 = 0.001$, and σ_L. The time to maturity is $T = 5$ and $K = 100$. In parenthesis the confidence interval of the Monte Carlo (MC) estimates. The error is defined as the difference between the MC price and the related approximation.

ρ	**−0.9**	**−0.6**	**−0.3**	**0.0**	**0.3**	**0.6**	**0.9**
Prices							
MC	19.8443	20.1287	20.4125	20.6936	20.9705	21.2425	21.5086
	(0.0595)	(0.0351)	(0.0153)	(0.0026)	(0.0202)	(0.0404)	(0.0649)
GO	19.6375	19.9974	20.3492	20.6936	21.0308	21.3614	21.6856
KK	19.7487	20.0582	20.3678	20.6773	20.9869	21.2964	21.606
MM	19.7747	20.085	20.3892	20.6875	20.981	21.269	21.5522
Errors							
GO	0.2067	0.1313	0.0632	5.2e-07	−0.0603	−0.1189	−0.1769
KK	0.0956	0.0705	0.0447	0.0162	−0.0164	−0.0540	−0.0973
MM	0.0695	0.0436	0.0232	0.0061	−0.0104	−0.0265	−0.0435
Rel. Err.							
GO	0.0104	0.0065	0.0031	2.5e-08	0.0029	0.0056	0.0082
KK	0.0048	0.0035	0.0022	0.0008	0.0008	0.0025	0.0045
MM	0.0035	0.0022	0.0011	0.0003	0.0005	0.0012	0.0020

TABLE 3 Results of the approximations for the parameters $\gamma = 0.6$, $\theta = 0.02$, $\eta = 0.1$, $r_0 = 0.001$, and σ_H. The time to maturity is $T = 1$ and $K = 100$. In parenthesis the confidence interval of the Monte Carlo (MC) estimates. The error is defined as the difference between the MC price and the related approximation.

ρ	**−0.9**	**−0.6**	**−0.3**	**0.0**	**0.3**	**0.6**	**0.9**
Prices							
MC	16.0337	16.0533	16.073	16.0933	16.1141	16.1351	16.156
	(0.0504)	(0.0301)	(0.0139)	(0.0002)	(0.0144)	(0.0306)	(0.0509)
GO	15.9831	16.0199	16.0567	16.0934	16.1300	16.1665	16.2030
KK	15.9997	16.0309	16.062	16.0932	16.1243	16.1555	16.1866
MM	16.0094	16.0374	16.0654	16.0933	16.1211	16.1489	16.1767
Errors							
GO	0.0506	0.0333	0.0162	−0.0001	−0.0159	−0.0314	−0.0469
KK	0.0339	0.0224	0.0109	0.0001	−0.0102	−0.0203	−0.0306
MM	0.0242	0.0159	0.0076	1.8e-05	−0.0071	−0.0138	−0.0207
Rel. Err.							
GO	0.0032	0.0021	0.0010	7.0e-06	0.0010	0.0019	0.0029
KK	0.0021	0.0013	0.0007	6.9e-06	0.0006	0.0012	0.0019
MM	0.0015	0.0009	0.0005	1.1e-06	0.0004	0.0009	0.0013

5.1 The pricing of forward-starting options

Forward-starting options are financial contracts characterized by the payoff

$$(e^{X_T} - \alpha e^{X_s})^+,$$

TABLE 4 Results of the approximations for the parameters $\gamma = 0.6$, $\theta = 0.02$, $\eta = 0.1$, $r_0 = 0.001$, and σ_H. The time to maturity is $T = 5$ and $K = 100$. In parenthesis the confidence interval of the Monte Carlo (MC) estimates. The error is defined as the difference between the MC price and the related approximation.

ρ	**−0.9**	**−0.6**	**−0.3**	**0.0**	**0.3**	**0.6**	**0.9**
Prices							
MC	36.1379	36.3566	36.5912	36.8358	37.0875	37.3439	37.6015
	(0.0405)	(0.0102)	(0.0141)	(0.0008)	(0.0163)	(0.0324)	(0.0571)
GO	35.8574	36.1877	36.5138	36.8358	37.154	37.4683	37.7789
KK	35.9641	36.2539	36.5437	36.8335	37.1233	37.4132	37.703
MM	35.9876	36.2725	36.5543	36.8329	37.1089	37.3819	37.6520
Errors							
GO	0.2805	0.1690	0.0774	2.2e-06	−0.0664	−0.1244	−0.1773
KK	0.1739	0.1028	0.04748	0.0023	−0.0358	−0.0693	−0.1014
MM	0.1504	0.0842	0.0368	0.0029	−0.0214	−0.0379	−0.0504
Rel. Err.							
GO	0.0078	0.0045	0.0021	6.1e-08	0.0018	0.0033	0.0047
KK	0.0048	0.0028	0.0013	6.2e-05	0.0010	0.0018	0.0027
MM	0.0041	0.0023	0.0010	7.8e-05	0.0006	0.0010	0.0013

TABLE 5 Results of the approximations for the parameters $\gamma = 0.58$, $\theta = 0.0345$, $\rho = 0.2$, $r_0 = 0.01$, and σ_L. The time to maturity is $T = 1$ and $K = 100$. In parenthesis the confidence interval of the Monte Carlo (MC) estimates. The error is defined as the difference between the MC price and the related approximation.

η	**0.005**	**0.02**	**0.04**	**0.06**	**0.09**	**0.12**	**0.15**
Prices							
MC	8.7198	8.7257	8.7337	8.7418	8.754	8.7662	8.7782
	(0.0043)	(0.0045)	(0.0044)	(0.0045)	(0.0046)	(0.0047)	(0.0046)
GO	8.7130	8.7195	8.7283	8.7372	8.7504	8.7635	8.7763
KK	8.7187	8.7246	8.7323	8.7401	8.7518	8.7634	8.7751
MM	8.7187	8.7246	8.7324	8.7404	8.7523	8.7641	8.7756
Errors							
GO	0.0068	0.0062	0.0054	0.0046	0.0036	0.0027	0.0019
KK	0.0011	0.0012	0.0014	0.0017	0.0022	0.0027	0.0031
MM	0.0011	0.0012	0.0013	0.0014	0.0017	0.0021	0.0026
Rel. Err.							
GO	0.0008	0.0007	0.0006	0.0005	0.0004	0.0003	0.0002
KK	0.0001	0.0001	0.0002	0.0002	0.0003	0.0003	0.0004
MM	0.0001	0.0001	0.0001	0.0002	0.0002	0.0002	0.0003

where $\alpha \in [0, 1]$ is a pre-specified percentage. The intermediate time s is contractually fixed, and it is known as *strike determination time*, so they are path-dependent options. Forward-starting contracts are the fundamental components of the so-called *cliquet options*, which are equivalent to a series of forward-starting at-the-money options, activated over a series of future dates. They mainly serve as protection against downside risk, though preserving an upside potential, furthermore they are also used as employee stock options. As financial contracts that incorporate the forward-starting feature, they get stipulated with long-dated maturities, and therefore they are sensitive to the interest rate fluctuations during their lifetime. In an affine setting, one can consider their pricing even under stochastic volatility but keeping the

TABLE 6 Results of the approximations for the parameters $\gamma = 0.58$, $\theta = 0.0345$, $\rho = 0.2$, $r_0 = 0.01$, and σ_L. The time to maturity is $T = 5$ and $K = 100$. In parenthesis the confidence interval of the Monte Carlo (MC) estimates. The error is defined as the difference between the MC price and the related approximation.

η	0.001	0.02	0.04	0.06	0.08	0.1	0.12
Prices							
MC	23.5335	23.5767	23.6354	23.6949	23.7837	23.8699	23.9509
	(0.0125)	(0.0131)	(0.0139)	(0.0147)	(0.0159)	(0.0171)	(0.0183)
GO	23.5221	23.5694	23.6337	23.6986	23.7955	23.8894	23.9770
KK	23.5284	23.5707	23.6270	23.6834	23.7679	23.8524	23.9370
MM	23.5289	23.5731	23.6331	23.6935	23.7828	23.8672	23.9423
Errors							
GO	0.0113	0.0072	0.0017	−0.0037	−0.0118	−0.0195	−0.0261
KK	0.0050	0.0060	0.0084	0.0115	0.0158	0.0175	0.0139
MM	0.0046	0.0035	0.0023	0.0014	0.0010	0.0027	0.0086
Rel. Err.							
GO	0.0005	0.0003	0.0001	0.0002	0.0005	0.0008	0.0011
KK	0.0002	0.0003	0.0004	0.0005	0.0007	0.0007	0.0006
MM	0.0002	0.0002	0.0001	0.0001	4.0e-05	0.0001	0.0004

TABLE 7 Results of the approximations for the parameters $\gamma = 0.58$, $\theta = 0.0345$, $\rho = 0.2$, $r_0 = 0.01$, and σ_H. The time to maturity is $T = 1$ and $K = 100$. In parenthesis the confidence interval of the Monte Carlo (MC) estimates. The error is defined as the difference between the MC price and the related approximation.

η	0.001	0.02	0.04	0.06	0.09	0.12	0.15
Prices							
MC	16.5324	16.5381	16.5459	16.5536	16.5652	16.5765	16.5874
	(0.0095)	(0.0095)	(0.0096)	(0.0097)	(0.0098)	(0.0100)	(0.0101)
GO	16.5238	16.5301	16.5386	16.5471	16.5596	16.5718	16.5833
KK	16.5302	16.536	16.5436	16.5513	16.5628	16.5743	16.5859
MM	16.5303	16.5362	16.544	16.5519	16.5636	16.575	16.586
Errors							
GO	0.0086	0.0080	0.0072	0.0065	0.0055	0.0047	0.0041
KK	0.0021	0.0022	0.0022	0.0023	0.0023	0.0021	0.0016
MM	0.0021	0.0020	0.0019	0.0017	0.0016	0.0015	0.0014
Rel. Err.							
GO	0.0005	0.0005	0.0004	0.0004	0.0003	0.0003	0.0002
KK	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	9.5e-05
MM	0.0001	0.0001	0.0001	0.0001	9.0e-05	8.9e-05	8.6e-05

processes' reciprocal independence to preserve the affine property. The novelty we introduce here is the computation of an approximated price under correlation with the interest rate. To this aim, we notice that the Markov property still holds, and the risk-neutral price of a forward- starting option is

$$c_{FS}(t, x, r, T; \rho) = \mathbb{E}_t^{x,r}\left[e^{-\int_t^T r_v dv}(e^{X_T} - \alpha e^{X_s})^+\right], \quad t < s < T. \tag{29}$$

TABLE 8 Results of the approximations for the parameters $\gamma = 0.58$, $\theta = 0.0345$, $\rho = 0.2$, $r_0 = 0.01$, and σ_H. The time to maturity is $T = 5$ and $K = 100$. In parenthesis the confidence interval of the Monte Carlo (MC) estimates. The error is defined as the difference between the MC price and the related approximation.

η	0.001	0.02	0.04	0.06	0.08	0.1	0.12
Prices							
MC	38.945	38.9857	39.0397	39.0931	39.1707	39.2439	39.3112
	(0.0257)	(0.0262)	(0.0267)	(0.0272)	(0.0278)	(0.0283)	(0.0289)
GO	38.9249	38.9694	39.0283	39.0863	39.1702	39.2486	39.3192
KK	38.9377	38.9782	39.0323	39.0863	39.1673	39.2484	39.3294
MM	38.9382	38.9802	39.0356	39.0901	39.1682	39.2397	39.3016
Errors							
GO	0.0201	0.0163	0.0114	0.0067	0.0005	−0.0047	−0.0080
KK	0.0073	0.0074	0.0074	0.0068	0.0033	−0.0045	−0.0182
MM	0.0068	0.0055	0.0041	0.0030	0.0025	0.0043	0.0097
Rel. Err.							
GO	0.0005	0.0004	0.0003	0.0002	1.2e-05	0.0001	0.0002
KK	0.0002	0.0002	0.0002	0.0002	8.5e-05	0.0001	0.0005
MM	0.0002	0.0001	0.0001	7.7e-05	6.5e-05	0.0001	0.0002

By conditioning internally w.r.t. $\mathcal{F}_s$, we get

$$c_{FS}(t,x,r,T;\rho) = \mathbb{E}_t^{x,r}\left[\mathrm{e}^{-\int_t^s r_v dv}\mathbb{E}\left(\mathrm{e}^{-\int_s^T r_v dv}(\mathrm{e}^{X_T} - \alpha \mathrm{e}^{X_s})^+ | \mathcal{F}_s\right)\right].$$

The inner conditional expectation is the price $u(s, X_s, r_s, T; \rho)$, at time s, of a call option with fixed log-strike $\kappa = \log(\alpha) + X_s$ hence we may write

$$c_{FS}(t,x,r,T;\rho) = \mathbb{E}_t^{x,r}\left[\mathrm{e}^{-\int_t^s r_v dv} u(s, X_s, r_s, T; \rho)\right], \tag{30}$$

where, from (10), we have

$$u(s, X_s, r_s, T; \rho) = \mathrm{e}^{X_s - \frac{\sigma^2\rho^2}{2}(T-s)}\mathbb{E}_s^{r_s}\left[\mathrm{e}^{\sigma\rho\Delta B^1_{s,T}}\mathcal{N}\left(d_1(\rho)\right)\right] - \alpha \mathrm{e}^{X_s}\mathbb{E}_s^{r_s}\left[\mathrm{e}^{-\Lambda_{s,T}}\mathcal{N}\left(d_2(\rho)\right)\right], \tag{31}$$

with $d_i(\rho) = \alpha_i(T-s,\rho) + \beta(T-s,\rho)\Delta B^1_{s,T} + \gamma(T-s,\rho)\Lambda_{s,T}$, for $i = 1, 2$, and $\alpha_i(T-s,\rho)$ independent of the underlying value X_s.

By using the same technique which leads to the MM approximation (21), we obtain

$$\bar{u}(s, X_s, r_s, T; \rho) = \mathrm{e}^{X_s}\left\{\mathcal{N}\left(\mathcal{D}_1(\rho)\right) - \alpha g(s, r_s)\mathcal{N}\left(\overline{\mathcal{D}}_2(\rho)\right)\right\},$$

with

$$\mathcal{D}_1(\rho) = \frac{\alpha_1(T-s,\rho) + \sigma\rho\hat{\beta}(r, T-s, \rho)(T-s) + \gamma(T-s,\rho)\mathbb{E}_t^r(\Lambda_{s,T})}{\sqrt{1+\hat{\beta}^2(r, T-s, \rho)(T-s)}}$$

$$\overline{\mathcal{D}}_2(\rho) = \frac{\alpha_2(T-s,\rho) + \epsilon(r, s, T)\hat{\beta}(r, T-s, \rho) + \gamma(T-s,\rho)\mathbb{E}_t^r(\Lambda_{s,T})}{\sqrt{1+\hat{\beta}^2(r_s, T-s, \rho)(T-s)}},$$

to replace u. By inserting $\bar{u}$ in (30), we have the following approximation:

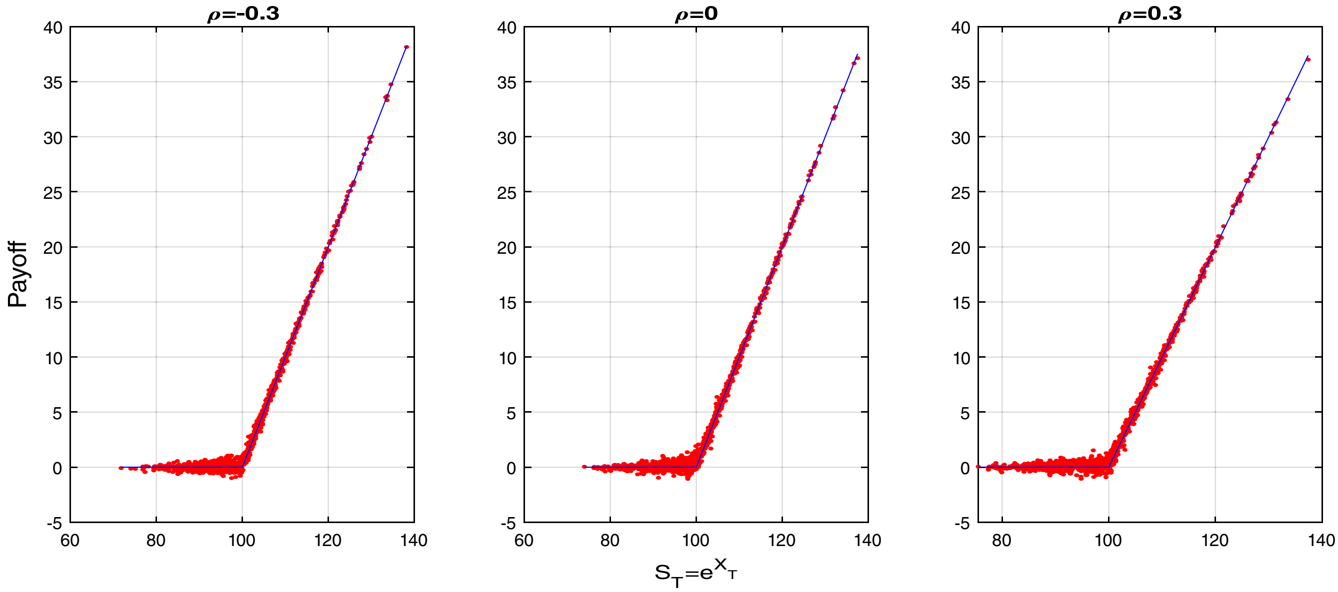

FIGURE 5 Simulation results for the Δ-hedging strategy. Red dots are the values at maturity of the replicating portfolios, built by using the approximate Δ, compared to the realized call payoffs. The call option has strike $K = 100$ and maturity $T = 3$ months. The CIR parameters are $r_0 = 0.01$, $\theta = 0.0345$, $\gamma = 0.58$ and $\eta = 0.06$. The underlying value at time $t = 0$ is $e^x = 100$ with $\sigma = 0.2$. [Color figure can be viewed at wileyonlinelibrary.com]

$$\begin{aligned}\bar{c}_{FS}(t,x,r,T;\rho) &= \mathbb{E}_t^{x,r}\left[e^{-\int_t^s r_v dv} e^{X_s}\left\{\mathcal{N}\left(\mathcal{D}_1(\rho)\right) - \alpha g(s,r_s)\mathcal{N}\left(\overline{\mathcal{D}}_2(\rho)\right)\right\}\right] \\ &= e^x \mathcal{N}\left(\mathcal{D}_1(\rho)\right) - \alpha \mathbb{E}_t^{x,r}\left[e^{-\int_t^s r_v dv} e^{X_s} g(s,r_s)\right]\mathcal{N}\left(\overline{\mathcal{D}}_2(\rho)\right).\end{aligned}$$

It remains to compute $\mathbb{E}_t\left[e^{-\int_t^s r_v dv} e^{X_s} g(s,r_s)\right]$. By using the affine property for $g(s,r_s)$, we conclude

$$\begin{aligned}\mathbb{E}_t\left[e^{-\int_t^s r_v dv} e^{X_s} g(s,r_s)\right] &= \mathbb{E}_t\left[e^{-\int_t^s r_s dv} e^{X_s} e^{A(s,T)+B(s,T)r_s}\right] \\ &= e^{X_t-\sigma^2/2(s-t)+A(s,T)}\mathbb{E}_t\left[e^{\sigma\sqrt{1-\rho^2}(B_s^2-B_t^2)}\right]\mathbb{E}_t\left[e^{\sigma\rho(B_s^1-B_t^1)+B(s,T)r_s}\right] \\ &= e^{X_t+A(s,T)}\mathbb{E}_t\left[e^{B(s,T)r_s}e^{\sigma\rho(B_s^1-B_t^1)-\sigma^2\rho^2(s-t)/2}\right].\end{aligned}$$

To evaluate the last expectation, we use a further change-of-numeraire by taking a new measure $Q^\Gamma(A) = \mathbb{E}(\Gamma_T 1_A)$ with $\Gamma_s = e^{-\sigma^2\rho^2 s/2+\sigma\rho B_s^1}$. Consequently

$$\mathbb{E}_t\left(e^{-\int_t^s r_v dv} e^{X_s} g(s,r_s)\right) = e^{X_t+A(s,T)}\mathbb{E}_t^{Q^\Gamma}\left(e^{B(s,T)r_s}\right).$$

Girsanov theorem implies that $W_t^1 = B_t^1 - \sigma\rho t$ is a Q^Γ-Brownian motion, and the rate's dynamic becomes

$$dr_s = [\gamma(\theta - r_s) + \sigma\eta\rho\sqrt{r_s}]ds + \eta\sqrt{r_s}dW_s^1,$$

which is no more a CIR process. As a final step, we approximate r_s under Q^Γ by

$$d\tilde{r}_s = \gamma(\tilde{\theta} - \tilde{r}_s)ds + \eta\sqrt{\tilde{r}_s}dW_s^1,$$

where $\tilde{\theta} = \theta + \sigma\eta\rho\sqrt{r_0}/\gamma$. Since the conditional density $\tilde{r}_u|\tilde{r}_t$ is known, we can evaluate

$$H(\rho) = \mathbb{E}_t^{Q^\Gamma}\left(e^{B(s,T)\tilde{r}_s}\right)$$

in closed form. We arrive at the final handy formula, approximating the value of a forward-starting call option

$$\tilde{c}_{FS}(x, r, t, T; \rho) = e^{x}\left[\mathcal{N}\left(\mathcal{D}_1(\rho)\right) - \alpha e^{A(s,T)} H(\rho)\mathcal{N}\left(\overline{\mathcal{D}}_2(\rho)\right)\right]. \tag{32}$$

The approximation (32) proved to be very effective when compared with the Monte Carlo evaluation, as outlined in Table 9.

TABLE 9 Results of the MM approximations of the Forward Starting option price for the rate parameters $\gamma = 0.58$, $\theta = 0.0345$, $\eta = 0.1$, $r_0 = 0.001$ and the two price volatility scenarios, σ_L, σ_H. The time to maturity is $T = 1$, the determination time is $u = 0.25$ and $\alpha = 1$. The error is defined as the difference between the MC price and the related approximation.

	σ_L						
ρ	**−0.9**	**−0.6**	**−0.3**	**0.0**	**0.3**	**0.6**	**0.9**
MC Prices	7.4795	7.4920	7.5064	7.5216	7.5366	7.5510	7.5652
95% c.i.	(0.0072)	(0.0072)	(0.0071)	(0.0073)	(0.0072)	(0.0073)	(0.0073)
MM Prices	7.4771	7.4918	7.5065	7.5211	7.5358	7.5504	7.5649
err.	(0.0023)	(0.0002)	(−0.0001)	(0.0004)	(0.0008)	(0.0006)	(0.0002)
rel err.	(3.1e-04)	(2.8e-05)	(−8.7e-06)	(5.5e-05)	(1.0e-04)	(8.2e-05)	(3.1e-05)
	σ_H						
ρ	**−0.9**	**−0.6**	**−0.3**	**0.0**	**0.3**	**0.6**	**0.9**
MC Prices	14.2742	14.2856	14.3012	14.3183	14.3352	14.3501	14.3648
95% c.i.	(0.0160)	(0.0161)	(0.0161)	(0.0159)	(0.0160)	(0.0161)	(0.0162)
MM Prices	14.2670	14.2856	14.3010	14.3170	14.3328	14.3486	14.3644
err.	(0.0049)	(0.0004)	(0.0001)	(0.0015)	(0.0025)	(0.0015)	(0.0004)
rel. err.	(3.4e-04)	(2.8e-05)	(1.0e-05)	(1.0e-04)	(1.7e-04)	(1.0e-04)	(2.6e-05)

TABLE 10 Results of the MM approximation of the call price in the stochastic volatility Heston model with CIR rates. The volatility parameters are $\gamma_v = 0.5$, $\theta_v = 0.04$, $\eta_v = 0.1$ and $\nu = -0.34$, and $y = 0.034$; the rate parameters are $\gamma = 0.6$, $\theta = 0.02$, $\eta = 0.1$, and $r = 0.01$. The call has strike price $K = 105$ and $S_0 = 100$. The error is defined as the difference between the MC price and the related approximation.

	$T = 0.1$						
ρ	**−0.9**	**−0.6**	**−0.3**	**0.0**	**0.3**	**0.6**	**0.9**
MC Prices	0.6714	0.6727	0.6741	0.6754	0.6768	0.6783	0.6798
95% c.i.	(0.0037)	(0.0037)	(0.0038)	(0.0037)	(0.0038)	(0.0038)	(0.0039)
MM Prices	0.6710	0.6724	0.6737	0.6751	0.6765	0.6778	0.6792
err.	(3.9e-04)	(3.3e-04)	(3.1e-04)	(3.2e-04)	(3.7e-04)	(4.6e-04)	(5.8e-04)
rel err.	(5.8e-04)	(4.9e-04)	(4.6e-04)	(4.7e-04)	(5.5e-04)	(6.7e-04)	(8.5e-04)
	$T = 0.25$						
ρ	**−0.9**	**−0.6**	**−0.3**	**0.0**	**0.3**	**0.6**	**0.9**
MC Prices	1.8146	1.8209	1.8273	1.8338	1.8406	1.8474	1.8544
95% c.i.	(0.0075)	(0.0076)	(0.0075)	(0.0077)	(0.0076)	(0.0077)	(0.0078)
MM Prices	1.8111	1.8176	1.8241	1.8307	1.8372	1.8437	1.8502
err.	(0.0035)	(0.0032)	(0.0031)	(0.0031)	(0.0034)	(0.0038)	(0.0043)
rel. err.	(0.0019)	(0.0018)	(0.0017)	(0.0017)	(0.0018)	(0.0020)	(0.0023)

5.2 | The pricing of defaultable bonds in the structural approach

Merton structural approach to the price $D(t,T)$ of a defaultable bond issued by a firm leads to the following formula

$$D(t,T) = e^{x} - u(t,x,r,T;\rho), \tag{33}$$

where the process $\{X_s\}, X_t = x$ is the logarithm of the firm value and u, given by (10), is the price of a European call written on the firm value with strike-price equal to the bond value, that is, $e^{\kappa} = D(T,T) := K$.

In the Merton model, default can take place only at maturity, and the option represents the shareholders' equity value of the firm. Shimko et al. in Reference 4, sect. 3, gave an explicit formula for $D(t,T)$ when the interest rate follows a Vasicek dynamic. A direct consequence of our results is to provide a way to evaluate (33) in an approximate efficient manner, when the interest rate follows a CIR dynamic. Hence the approximated value is

$$\overline{D}(t,T) := e^{x} - \bar{u}(t,x,r,T;\rho) = \overline{D}(t,T) = e^{x}(1 - e^{-\frac{1}{2}\sigma^2\rho^2(T-t)}F(\rho)) - K\overline{G}(\rho), \tag{34}$$

with $\bar{u}$ given by (21), and $\overline{G}(\rho) := P(t,T)\mathbb{E}_t^{Q^T}\left[\mathcal{N}\left(\overline{D}_2(\rho)\right)\right]$. From this last formula, we deduce an efficient estimate $\overline{\Delta}$ of the credit spread which is

$$\overline{\Delta} = -\frac{1}{T-t}\log\left(\frac{\overline{D}(t,T)}{K}\right) + \frac{1}{T-t}\log(P(t,T)) = \frac{1}{T-t}\log\left(\frac{KP(t,T)}{\overline{D}(t,T)}\right),$$

where $P(t,T)$ is the price of a default-free zero-coupon bond with the same maturity (Figure 6).

5.3 | Towards stochastic volatility models

As a final application, we discuss the extension of our approximation technique to market models that include stochastic volatility. In particular, we consider a stochastic volatility Heston model with correlated CIR rates

$$\begin{cases} X_s = X_t + \int_t^s (r_v - \frac{Y_v}{2})dv + \int_t^s \sqrt{Y_v}\left[\rho dB_s^1 + \nu dB_s^2 + \sqrt{1-\rho^2-\nu^2}dB_s^3\right], \quad X_t = x \\ Y_s = Y_t + \int_t^s \gamma_{\mathrm{v}}(\theta_{\mathrm{v}} - Y_v)dv + \int_t^s \eta_{\mathrm{v}}\sqrt{Y_v}dB_v^2, \quad Y_t = y, \\ r_s = r_t + \int_t^s \gamma(\theta - r_v)dv + \int_t^s \eta\sqrt{r_v}dB_v^1, \quad r_t = r, \end{cases} \tag{35}$$

where (B^1, B^2, B^3) is a standard three-dimensional Brownian motion and $\rho^2 + \nu^2 \leq 1$. We assume that the parameters satisfy the Feller condition for both r and Y. Model (35) was considered in Reference 14. The system is still Markovian, therefore the price of a call option is given by

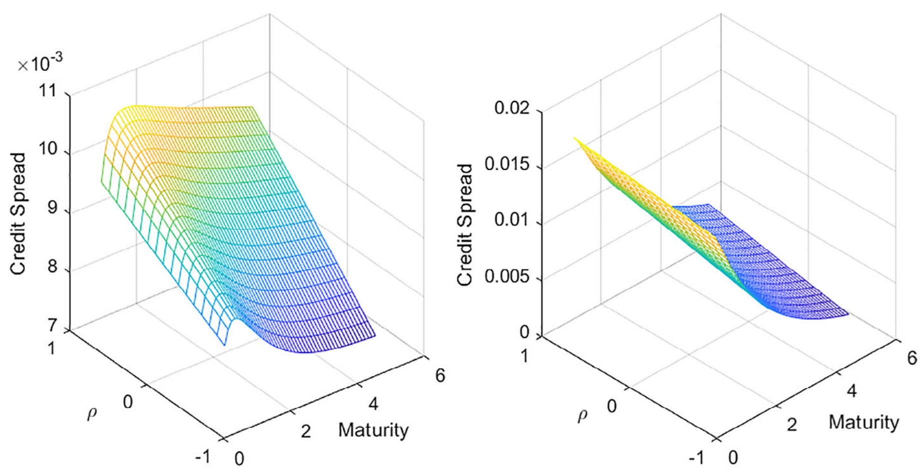

FIGURE 6 Credit spread in the CIR rate model. On the left plot, the CIR parameters are $r_0 = 0.05$ and $\theta = 0.001$ with $F = 93$; on the right $r_0 = 0.03$, $\theta = 0.05$ and $F = 95$. The other parameters are $\gamma = 0.58$ and $\eta = 0.06$. The underlying value at time $t = 0$ is $e^x = 100$ with $\sigma = 0.1$. [Color figure can be viewed at wileyonlinelibrary.com]

$$u(t,x,y,r,T;\rho,\nu)=\mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds}(\mathrm{e}^{X_T(\rho,\nu)}-\mathrm{e}^{\kappa})^+|X_t=x,Y_t=y,r_t=r\right]. \tag{36}$$

To apply our technique, we consider the implied volatility approximation developed in Reference 13 that we present here for $t=0$ without loss of generalization.

If we denote by $c_{BS}(x,\kappa,T,\sigma)$ the Black and Scholes price of a call option with strike price e^{κ}, maturity T, and volatility σ, by definition of implied volatility, we have

$$c_{Hes}(x,\kappa,T)=c_{BS}(x,\kappa,T,\sigma_{Hes}(x-\kappa,T)),$$

where c_{Hes} is given by (36) for $\gamma=\eta=0$. In Reference 13, the authors provide an asymptotic small-time expansion for the implied volatility of the Heston model

$$\sigma_{Hes}^2(m,T)=\sigma_0^2(m,T)+a_1(m,T)+o(T),\quad \forall m\in\mathbb{R}\setminus\{0\}, \tag{37}$$

where $m=x-\kappa$, and the first two terms $\sigma_0(m,T), a_1(m,T)$ are explicit (see Reference 13, Corollary 4.3), giving numerical evidence of the accuracy of the formula induced by the approximate implied volatility defined as

$$\hat{\sigma}_{Hes}(m,T)=\sqrt{\sigma_0^2(m,T)+a_1(m,T)}.$$

We exploit this approach by replacing the dynamic (35) with the market model (1) where we take $\sigma=\hat{\sigma}_{Hes}(x-\kappa,T)$, so that we may consider the following approximation for the call price under the Heston stochastic volatility model

$$u(0,x,y,r,T;\rho,\nu)\approx\bar{u}(0,x,r,T,\hat{\sigma}_{Hes}(x-\kappa,T);\rho). \tag{38}$$

where we highlighted the dependence on the volatility σ. We remark that the price's dependence on the correlation ν gets embedded into $\hat{\sigma}_{Hes}$.

We checked the numerical performance of this approach for short maturities, as required by the expansion. In Table (10), we report the results for two different scenarios when comparing the prices computed according to formula (38) with Monte Carlo prices obtained by simulating the system (35) with $M=1000,000$ sample paths generated by the Euler discretization with a full truncation algorithm. At least for those choices of parameters, we achieve a high degree of efficiency, as the relative errors show. We finally notice that large-time asymptotic could be used as well (see Reference 23) for large maturities.

A full treatment of stochastic volatility/interest rate models via MM, if possible, would require careful handling of the singularities introduced in the pricing formulas by the stochastic volatility and of the system correlations. It is a complex task that we hope to address in future work.

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