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MASSLESS SCALAR FREE FIELD IN 1+1 DIMENSIONS I: WEYL ALGEBRAS PRODUCTS AND SUPERSELECTION SECTORS

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This is the first of two papers on the superselection sectors of the conformal model in the title, in a time zero formulation. A classification of the sectors of the net of observables as restrictions of solitonic (twisted) and non-solitonic (untwisted) sector automorphisms of proper extensions of the observable net is given. All of them are implemented by the elements of a field net in a non-regular vacuum representation and the existence of a global compact Abelian gauge group is proved. A non-trivial center in the fixed-point net of this gauge group appears, but in an unphysical representation and reducing to the identity in the physical one. The completeness of the described superselection structure, to which the second paper is devoted, is shown in terms of Roberts' net cohomology. Some general features of physical field models defined by twisted cross products of Weyl algebras in non-regular representations are also presented.

Keywords: Weyl algebras; massless scalar free field; superselection sectors; conformal models; solitonic sectors; twisted crossed products; non-regular representations.

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Contents

1. Introduction

The general theory of superselection sectors in low dimensional Quantum Field Theory is still lacking, so the study of special models is of great interest. Relevant progress has been achieved in the past years using Algebraic Quantum Field Theory for various classes of models such as loop groups, orbifold models and coset models, see [23, 49] also for a historical review.

In this approach, a complete classification of *rational models* (i.e. with a finite number of sectors) with Virasoro central charge $c < 1$, see [34, 32], and for the local extensions of *compact type* of the Virasoro net on the circle at $c = 1$ has been attained, see [11] for details. In purely massive theories, the triviality of sectors has been proved in [39]. Two notable features of the theory are the presence of topological sectors of solitonic origin, see [35, 42], and the following dichotomy between the rational and non-rational case established in [37]: if in a model all irreducible sectors have conjugates then, it is either rational or has uncountably many different irreducible sectors.

In this paper, we deal with a non-rational $c = 1$ model, the massless scalar free field, also called *the Streater and Wilde model* after its first formulation using the ideas of local nets in [51]. Our main goal is to understand better the interplay between chiral and time zero formulation of the superselection structure in 1+1-dimensional theories, the nature of solitonic and non-solitonic sectors, the relation between DHR sectors and the presence of a quantum global internal symmetry, usually reducing to a global compact gauge group.

The results obtained in this paper, and in its sequel [14], are largely adaptable to other theories based on Weyl algebras and give a strong further application of Roberts' net cohomology for discussing superselection sectors in general spacetime context, see [48] for reference.

The observables of the Streater and Wilde model are obtained from the quantized derivatives of the classical fields by imposing a constraint condition on the solutions of the two-dimensional wave equation. This choice avoids infrared divergences, see [51] and references therein, and for this reason the model is also called *the theory of the potential of the field*.

The existence of a (physical, separable) Hilbert space representation for the observable net is obtained in the usual way by the Fock space second quantization procedure for Weyl algebras. For the same net, the existence of a continuous, i.e. uncountable, family of DHR superselection sectors is known from the original description in [51] (see [26, 48] for general references on the Doplicher, Haag, Roberts theory of superselection sectors). They are labeled by pairs of real numbers, i.e. by the elements of $\mathbb{R}^2_d$ (here the subscript d means discrete topology) and realized as inner automorphisms by some left/right movers (solitons) of a field net extension. Other relevant features of the model were discussed in [29], namely the Tomita–Takesaki modular structure, spacelike and timelike duality.

The usual chiral net formulation of the Streater and Wilde model is treated in [10] and the above cited extension may be classified according to [11, Definition 3.2] as being of compact type.

A relevant step in the description of Weyl algebras models was the introduction of *non-regular representations*, see [55] and [1, 2]: for such a representation π of $\mathcal{W}(V, \sigma_V)$, the Weyl algebra on the symplectic space (V, σ_V), there is a subspace of elements $v \in V$ such that, for $\lambda \in \mathbb{R}$, the map $\lambda \mapsto \pi(W(\lambda v)) \subset \mathcal{B}(\mathcal{H}_\pi)$ is not weakly continuous.

The papers just cited pointed out the utility of such Hilbert space representations in the presence of a theory with uncountable many sectors. This avoids the use of inner product spaces with indefinite metrics, for the (unphysical) representation of charged fields. However, the same papers do not attack the problem of a local net theory in a non-regular representation, nor the full description of its DHR sectors and associated gauge group.

The task of this first paper is to collect the known results on the Streater and Wilde model and reconstruct the DHR sectors theory using a putative field net $\mathfrak{F}$ (that do not satisfy the locality condition) and a compact group $\mathcal{G}$ of gauge symmetries, such that the observable net $\mathfrak{A}$ is the fixed-point net under the action of $\mathcal{G}$, restricted to the representation Hilbert space $\mathcal{H}_a$ of the observables, i.e. $\mathfrak{A} = \mathfrak{F}^{\mathcal{G}} \restriction_{\mathcal{H}_a}$. In the second paper, we legitimate the net $\mathfrak{F}$ as the *complete field net* of $\mathfrak{A}$, and the description of the superselection structure is hence similar to the higher dimensional one of a *field system with gauge symmetry*, see the celebrated [22] for definitions and results, apart from the presence of braid symmetry instead of permutation symmetry.[a]

To construct such a field net, together with the gauge group structure, we use the abstract tool of 2-*cocycle twisted crossed product of Weyl algebras*, i.e. the reinterpretation of the Weyl algebras of fields as an extension of that of the observables by the cocycle twisted action of the charge group. This *simple current extension* is defined by a (generalized) 2-cocycle derived from the symplectic form of the Weyl algebras, in its observable and charge-gauge group components, already partially studied in the physical literature, see for example [28].[b]

[a]It should be said that the superselection theory of the analogous model in 1+3 dimensions is known to be trivial. Here a constrain condition on the symplectic space (not introduced to avoid the infrared divergences as in the 1+1 case) distinguishes the observable net $\mathfrak{A}$ from its Haag dual net $\mathfrak{A}^d \supset \mathfrak{A}$ and the failing of Haag duality for the observable net $\mathfrak{A}$, i.e. $\mathfrak{A}^d \neq \mathfrak{A}$, denotes the presence of a spontaneous breaking of the gauge group. The above mentioned triviality is due to the absence of sector (or solitonic) automorphisms for $\mathfrak{A}^d = \mathfrak{F}$, the dual net equates the field net, and is proved by net cohomology in [9].

[b]More general examples of simple current extensions, derived from loop groups, orbifold models or vertex operator algebras may be considered. See, for example, [33].

The extension of the symplectic spaces in the model has the structure $V_a \subset V_f = V_a \oplus (N \oplus C)$ where the symplectic form splits as $\sigma_f = \sigma_a \oplus \sigma_{N\oplus C}$. Here, the space C is the discrete Abelian charge group. The structure of the corresponding Weyl algebras is then

$$\mathcal{W}(V_f, \sigma_f) = \mathcal{W}(V_a, \sigma_a) \otimes \mathcal{W}(N, \sigma_N) \rtimes_z \mathcal{U}(C). \tag{1.1}$$

Here $\mathcal{U}(C)$ denotes the Abelian group C written multiplicatively. Note that the factor $\mathcal{W}(N, \sigma_N)$ commutes with $\mathcal{W}(V_a, \sigma_a)$, but is acted upon by the charge group through the 2-cocycle z, reflecting the symplectic interaction between N and C.

Such a construction, in a time zero formulation, allows one to classify the sectors labeled by $\mathbb{R}^2_d$ as restrictions of solitonic (twisted) and non-solitonic (untwisted) sector automorphisms of two different simple current extensions of the net of the observables. Hence these sectors accord to the definitions in [42, 35], but this classification reflects a different perspective respect to the equivalent nature of the left/right solitons of the chiral formulation. Moreover, the time zero approach makes evident the presence of a non-trivial center $\mathfrak{F}^{\mathcal{G}} \cap (\mathfrak{F}^{\mathcal{G}})'$ of the fixed-point net under the action of the global compact gauge group $\mathcal{G}$, weakly continuously represented on the unphysical Hilbert space.

Namely, the construction by a simple current extension, gives a *six-term diagram of inclusions* of (localized nets of) symplectic spaces, Weyl algebras and von Neumann algebras with the action of the charge and gauge groups. To introduce the test function spaces used to define the time zero symplectic spaces of the model, we denote by $\mathcal{S}$ the Schwartz space of real valued rapidly decreasing functions on the real line, $\partial\mathcal{S}$ the space of functions that are derivative of functions in $\mathcal{S}$ and by $\partial^{-1}\mathcal{S}$ the C^∞-functions whose derivative is in $\mathcal{S}$.

Referring to the classical theory of the quantum massless field in 1+1 dimensions, if φ denotes the field and $\pi = \dot{\varphi}$ its conjugate momentum field, the currents extension corresponds to two different test function space extensions: the codimension 1 extension from $\partial\mathcal{S}$ to $\mathcal{S}$, which corresponds to lifting the condition that test functions for the massless time-zero field φ should vanish at zero wave number in Fourier space, and the codimension 2 extension from $\mathcal{S}$ to $\partial^{-1}\mathcal{S}$ which corresponds to admitting as test functions for the time-zero conjugate momentum field π both constant functions and odd functions tending to constants at infinity. These extensions of test function spaces, together with the extension of the corresponding symplectic form

$$\sigma_a(F, G) = \int_{\mathbb{R}} (f_0 g_1 - f_1 g_0) dx,$$

where $F = f_0 \oplus f_1$ and $G = g_0 \oplus g_1$ are two different couples of test functions for field and momentum, respectively, give the extension from the algebra of observables to the algebra of fields. We hence denote by C the space of quotient classes $\tilde{f}_0(0)$ for φ and by Q the space of quotient classes $f_1(+\infty) - f_1(-\infty)$ for π. Together, they form a two-dimensional real space of charges $C \oplus Q$, furnished with the discrete topology, i.e. $C \cong Q \cong \mathbb{R}_d$, and playing the role of the charge group of the model, denoted only by C in the more generic formula (1.1).

Omitting some intermediate terms, a reduced version of the cited diagram of inclusions for the (nets of) symplectic spaces and the von Neumann algebras of the observables and putative fields, is

$$\begin{aligned} V_a &:= \partial\mathcal{S} \oplus \mathcal{S} \subset V_b := (\partial\mathcal{S} \oplus \mathcal{S}) \oplus N \subset ((\partial\mathcal{S} \oplus \mathcal{S}) \oplus N) \oplus C \oplus Q \\ &\cong \mathcal{S} \oplus \partial^{-1}\mathcal{S} =: V_f, \\ \mathcal{A} \subset \mathcal{B} &:= \mathcal{A} \otimes \mathcal{Z}_b = \mathcal{F}^{\mathcal{G}} \subset (\mathcal{A} \otimes \mathcal{Z}_b) \rtimes \mathcal{U}(C) \rtimes \mathcal{U}(Q) =: \mathcal{F}. \end{aligned}$$

In these diagrams, $N \cong \mathbb{R}$ denotes the space of constant test functions for π, that plays the same role of N in (1.1). The charge group acts non-trivially only on the non-trivial central tensor factor $\mathcal{Z}_b$, the Abelian von Neumann algebra generated by the representation of N.

A major task, that we postpone to the second paper [14], is the question of the completeness of the $\mathbb{R}^2_d$-labeled superselection theory. A positive answer is given by a careful choice of the index sets defining the nets and by the very effective theory of net cohomology of Roberts. Actually, we determine the sectors for a large class of models given by an extension of Weyl algebras.

In the second paper, the non-trivial center, or relative commutant $\mathfrak{A} \cap \mathfrak{F}'$, is discussed more deeply. This feature is considered for example in [5]. It will also be pointed out that it is related to the $\mathbb{R}$-graded commutation rules of the non local field net $\mathfrak{F}$. Moreover, the relation with the superselection theory in presence of constraints, see [3], and further structural properties of the nets of the Streater and Wilde model, such as duality properties with respect to different index sets and split properties, will also be considered.

The structure of this paper is as follows:

In Sec. 2, we recall known and add new material on Weyl algebras, at the abstract algebraic, C* and von Neumann algebraic level. Particularly, in Sec. 2.1, a definition of the twisted cross product of Weyl algebras is given, and used to describe the observable-charge coupling in a physical model. In Sec. 2.2, we present some useful requirements of independence of the states on the Weyl algebras of a twisted cross product, necessary for physically interesting representations, on a non separable Hilbert space.

A detailed account of non-regular representations of Weyl algebras is presented in Sec. 2.3. The attention is focused on the non-regular representation of the elementary Weyl algebras on the symplectic space $L \cong \mathbb{R} \oplus \mathbb{R}$, presented for example in [55], to be used as a building block for the general twisted product case.[c]

Section 3 is devoted to the twisted cross product formulation of the Streater and Wilde model with initial data on the time zero line. Fixing any charged element

[c]The same algebraic characterization may be used to study the superselection structure of models presenting electromagnetic charges and interaction. In this line, the analysis of the Stückelberg–Kibble QED_2 model will be presented elsewhere.

in the symplectic space, gives a symplectic isomorphism that exponentiates to an isomorphism of Weyl algebras. The defining representations for some intermediate and the larger putative field algebra are also introduced through this isomorphism, in an essentially unique way.

The local net theory of the model is presented in Sec. 4, in the usual approach: the time zero observable net $\mathcal{A}$ on the index set of the open bounded intervals of the time zero line is defined, so that, if I is such an interval and the base of the double cone $\mathcal{O}$, i.e. $I'' = \mathcal{O}$, then $\mathcal{A}(I) = \mathfrak{A}(\mathcal{O})$. Similarly, four more intermediate nets and the putative field net $\mathcal{F}$ are defined. The net $\mathcal{F}$ realizes the cited simple current extensions. In Sec. 4.2, the relation between the chiral and the time zero formulation is discussed. The usual d'Alambert formula gives an isomorphic correspondence between the symplectic spaces and the charges in the two cases. In Sec. 4.3, we present a detailed description of the twisted and untwisted automorphisms describing the sectors. Finally, in Sec. 4.4, the global compact Abelian gauge group $\mathcal{G}$ is derived as the Bohr compactification of a subspace, isomorphic to $\mathbb{R}^2$, of the symplectic space of the fields. Further details on the braided tensor category of the DHR sectors of $\mathfrak{A}$ will be presented in [14].

2. Weyl Algebras

We recall in this section some essential results of the theory of Weyl algebras and fix the general notation, referring mainly to [7, Sec. 5.2] and [38, 50].

For V a separable real (topological) vector space, we denote by σ_V a (continuous) *symplectic form* on it, i.e. a $\mathbb{R}$-bilinear, antisymmetric, real valued, (continuous) form on $V \times V$ and by (V, σ_V) the associated *real symplectic space*. The abstract *-algebra on $\mathbb{C}$ generated by the elements $W(v), v \in V$, with product and involution defined respectively by

$$\begin{aligned} W(v)W(v') &= e^{-\frac{i}{2}\sigma_V(v,v')}W(v+v'),\\ W(v)^* &= W(-v) \end{aligned} \tag{2.1}$$

for $v, v' \in V$ is called the *Weyl algebra of* (V, σ_V) and indicated by $\mathcal{W}(V, \sigma_V)$, or with $\mathcal{W}_V$ if no confusion arises.

We have to note that, as far as we consider abstract non-represented Weyl algebras, it turns out to be needless to specify the topology on V, i.e. we can use the discrete one. Passing to the representations on a Hilbert space, the topology on the (support of the) symplectic space will play its role: Fock representations and, more generally, *regular* representations are typical examples, as we shell better see in the sequel.

The relations (2.1) imply that $W(0) = I$, $W(v)^{-1} = W(-v) = W(v)^*$, i.e. the generators of the Weyl algebra are *formal unitaries* and

$$W(v)W(v') = e^{-i\sigma_V(v,v')}W(v')W(v), \quad v, v' \in V. \tag{2.2}$$

The algebra $\mathcal{W}(V,\sigma_V)$ is a unital, generally non commutative *-algebra, that is simple iff the symplectic form σ_V is *non degenerate*, i.e. if $\sigma_V(v,v')=0$ for all $v\in V$ implies $v'=0$.

In [38, 50], a well-established standard theory associates a unique C*-norm to any Weyl algebra, called the *minimal regular norm*. The symplectic form σ_V is non-degenerate on V iff a unique C*-norm on $\mathcal{W}(V,\sigma_V)$ exists, hence coinciding with the minimal regular one. We denote by $C^*(V,\sigma_V)$ the C*-algebra generated by the Weyl algebra $\mathcal{W}(V,\sigma_V)$ in the minimal regular norm, and call it *the* (*unique*) *C*-algebra* associated with $\mathcal{W}(V,\sigma_V)$. We term

$$N_V := \{v\in V : \forall v'\in V, \sigma_V(v,v')=0\} \tag{2.3}$$

the *degeneracy subspace* of V, so that $\mathcal{W}(N_V,\sigma_V)\cong\mathcal{W}(N_V,0)\subseteq\mathcal{W}(V,\sigma_V)$ is the Abelian *-subalgebra generated by N_V. Its completion in the minimal regular norm on $\mathcal{W}(V,\sigma_V)$, denoted by $C^*(N_V)$, constitutes the center of $C^*(V,\sigma_V)$, i.e.

$$\mathfrak{Z}_V := C^*(V,\sigma_V)\cap C^*(V,\sigma_V)' = C^*(N_V). \tag{2.4}$$

$C^*(V,\sigma_V)$ is simple iff $N_V=\{0\}$ and in the degenerate case, i.e. $N_V\neq\{0\}$, the minimal regular norm on $\mathcal{W}_V$ is not the only C*-norm on $\mathcal{W}_V$.

Clearly, if $V_0:=V/N_V$ and $\sigma_{V_0}:=\sigma_V\restriction V_0$ (we use the notation $\sigma_H:=\sigma_V\restriction H$ for the restriction of the symplectic form to a subspace $H\subset V$), the pair (V_0,σ_{V_0}) is a non-degenerate symplectic space and the C*-algebra $C^*(V_0,\sigma_{V_0})$ it generates is simple. The degenerate case is treated in [38], also when V is replaced by an Abelian topological group.

If σ_V is non-degenerate and V has a *complexification*, i.e. an operator J such that $\sigma_V(\cdot,J\cdot)$ is a positive definite form and

$$\sigma_V(Jv,v')=-\sigma_V(v,Jv'),\quad J^2=-1,\quad v,v'\in V, \tag{2.5}$$

we immediately get a pre-Hilbert space structure for V, whose inner product is defined from the symplectic form by

$$(\cdot,\cdot)_V := \sigma_V(\cdot,J\cdot)+i\sigma_V(\cdot,\cdot).$$

Actually such a correspondence between the pair σ_V, J and $(\cdot,\cdot)_V$ is bijective, up to isomorphism, and the complexification is necessary to obtain a pure quasi-free state and a Fock representation for $\mathcal{W}_V$ (see, e.g., [7]). This is the usual method, necessary in some sense, to obtain a definite metric Hilbert space representation, for the (observable) algebra of a physical model.

2.1. *Isomorphisms and twisted crossed products of Weyl algebras*

We focus in the sequel on two relevant symplectic structures: *isomorphisms* and *twisted compositions* of symplectic spaces; these respectively give rise functorially to isomorphic and twisted crossed products of Weyl and C*-algebras.

A *symplectic morphism* between symplectic spaces is given as a (continuous) map on the spaces, preserving the symplectic forms. An invertible morphism, i.e. an isomorphism, may be defined also in the case of degeneracy as follows

Definition 2.1. Given two symplectic spaces (V_1, σ_1) and (V_2, σ_2), a *symplectic space isomorphism* $\psi : (V_1, \sigma_1) \to (V_2, \sigma_2)$ is a continuous isomorphism between V_1 and V_2 as real topological vector spaces, that preserves the symplectic form, i.e.

$$\sigma_2(\psi(x), \psi(y)) = \sigma_1(x, y), \quad x, y \in V_1.$$

A symplectic isomorphism exponentiates functorially to a Weyl algebras isomorphism between $\mathcal{W}(V_1, \sigma_{V_1})$ and $\mathcal{W}(V_2, \sigma_{V_2})$ and to a center-preserving C*-algebras isomorphism denoted by

$$\Psi : C^*(V_1, \sigma_1) \to C^*(V_2, \sigma_2). \tag{2.6}$$

Consider now

- **Weyl** $:= (W(V, \sigma_V), \varphi)$, the category of all the Weyl algebras as objects and all the (purely algebraic) isomorphisms between them as morphisms.

The above discussion may be formalized saying that there exists a *Weyl exponentiation functor* **W** that realizes an isomorphism of categories between

- **Symp** $:= ((V, \sigma_V), \psi)$, the category of symplectic spaces as objects and symplectic isomorphisms as morphisms;
- **W**(**Symp**), the subcategory of **Weyl**, where the morphisms are only $\mathbf{W}(\psi)$ for ψ a symplectic isomorphism as in Definition 2.1.

Using the C*-closure of Weyl algebras in the uniquely defined minimal regular norm, both of these categories are isomorph to the following one in C*-context, from which **W**(**Symp**) is obtained by a forgetful topology functor:

- $\mathbf{C}^*(\mathbf{W}(\mathbf{Symp}))$, the subcategory of Weyl C*-algebras as objects and the unit preserving isomorphisms $\mathbf{W}(\psi)$ of **W**(**Symp**) as morphisms, extended to the C*-closure.

Notice that the objects in the previously listed isomorphic categories have different algebraic and topological structures, although defined in a natural way starting from **Symp**. A similar natural, physically motivated definition of the representations for the Weyl algebra models is also pursued in the sequel: a typical example is the Fock representation, at least for a Weyl subalgebra and its extension to the whole Weyl algebra.

We may introduce a fourth category, isomorphic to the three above, that is more handy from the point of view of the crossed products theory. The objects of this category are called *Weyl algebra groups* and defined as in the sequel: to any (also degenerate) symplectic space furnished with the discrete topology, a Weyl group $\mathcal{U}(V, \sigma_V)$ is associated such that

$$1 \to \mathbb{T} \to \mathcal{U}(V, \sigma_V) \to \mathcal{U}(V) \to 1 \tag{2.7}$$

is a short exact sequence.[d] This means that the discrete twisted crossed product $\mathcal{U}(V,\sigma_V) := \mathbb{T} \rtimes_{(\iota,y)} \mathcal{U}(V)$ is an extension of the Abelian formal symbols group $\mathcal{U}(V)$ on the symplectic space V, by the torus group $\mathbb{T}$ and the 2-cocycle (see [47])

$$z = (\beta, y) : (\mathcal{U}(V), \mathcal{U}(V) \times \mathcal{U}(V)) \to (\mathrm{Aut}\,\mathbb{T}, \mathbb{T}) \tag{2.10}$$

where the action is trivial, $\beta \equiv \iota$, and the function $y(v,v') := e^{-\frac{i}{2}\sigma_V(v,v')}$ is defined by the symplectic form. Hence, the above announced fourth category is defined by

- $\mathbf{U}(\mathbf{Symp}) := (\mathcal{U}(V,\sigma_V), \Psi)$, the category of the *Weyl algebra groups* as objects and the symplectic derived group isomorphisms as morphisms, i.e. $\Psi = \mathbf{W}(\psi)$ for ψ as in Definition 2.1.

A Weyl algebra is hence simply recovered as a discrete crossed product $\mathcal{W}(V,\sigma_V) = \mathbb{C} \rtimes_{(\iota,y)} \mathcal{U}(V)$. Observe that this is not a semidirect product, eventually defined by a non-trivial action β, but a crossed product twisted by the non-trivial function y.

A useful decomposition in the case of degeneracy is also possible, where Eq. (2.7) is better replaced by an extension making the degeneracy explicit:

$$1 \to \mathbb{T} \times \mathcal{U}(N_V) \to \mathcal{U}(V,\sigma_V) \to \mathcal{U}(V/N_V) \to 1. \tag{2.11}$$

Here $\mathbb{T} \times \mathcal{U}(N_V) \cong Z(\mathcal{U}(V,\sigma_V))$ is the center of the group $\mathcal{U}(V,\sigma_V)$, and we have $\mathcal{U}(V/N_V) \cong \mathcal{U}(V)/\mathcal{U}(N_V)$. This extension may be read as the discrete twisted crossed product

$$\mathcal{U}(V,\sigma_V) = (\mathbb{T} \times \mathcal{U}(N_V)) \rtimes_{(\iota,y)} \mathcal{U}(V/N_V), \tag{2.12}$$

where y take value on the $\mathbb{T}$-part of the normal Abelian subgroup $\mathbb{T} \times \mathcal{U}(N_V)$, and the Weyl algebra is also written as $\mathcal{W}(V,\sigma_V) = \mathcal{W}(N_V) \rtimes_{(\iota,y)} \mathcal{U}(V/N_V)$.

[d] Given a group G an *extension* E of it by another group N is described by the *short exact sequence*

$$1 \to N \to E \to G \to 1 \tag{2.8}$$

where E is the set of pairs $(n,s) \in N \times G$ with multiplication law

$$(n,s)(m,t) := (n\ \beta_s(m)\ y(s,t), st), \quad (n,s),(m,t) \in N \times G$$

for $z = (\beta, y) : (G, G \times G) \to (\mathrm{Aut}\,N, N)$ the non-Abelian 2-cocycle of the extension satisfying the equations

$$\begin{aligned} y(s,t) &\in (\beta_{st}, \beta_s \circ \beta_t), & s,t \in G \\ \beta_r(y(s,t))y(r,st) &= y(r,s)y(rs,t), & r,s,t \in G. \end{aligned} \tag{2.9}$$

The first equation means that y intertwines the action of β_{st} and of $\beta_s \circ \beta_t$, i.e. $y(s,t)\beta_{st}(n) = \beta_s(\beta_t(n))y(s,t)$, for every $s,t \in G$ and $n \in N$; the second relation is a 2-cocycle multiplicative non-Abelian equation. The extensions E are classified, up to isomorphism, by the 2-cohomology of G, with values in 2-category $(Z(N), \mathrm{Aut}(N), N)$, where elements in $Z(N)$, the center of N, implement identity of $\mathrm{Aut}(N)$ of above described cocycles, see [46, 47].

Another simple example of extension is obtained from a *direct sum of symplectic spaces* (H, σ_H) and (L, σ_L) defined by:

$$(V, \sigma_V) := (H \oplus L, \sigma_H + \sigma_L). \tag{2.13}$$

Here we mean that the symplectic form σ_V decomposes according as $\sigma_V = \sigma_H \oplus \sigma_L$, i.e. $\sigma_H = \sigma_V \upharpoonright H$ and $\sigma_L = \sigma_V \upharpoonright L$, such that $(V, \sigma_V) \cong (H, \sigma_H) \oplus (L, \sigma_L)$ is an obvious symplectic isomorphism that at Weyl algebras level gives

$$\mathcal{W}(V, \sigma_V) \cong \mathcal{W}(H, \sigma_H) \otimes \mathcal{W}(L, \sigma_L). \tag{2.14}$$

The definition of the C*-maximal tensor product of two C*-algebras gives the C*-algebra isomorphism

$$\Psi : C^*(V, \sigma_V) \to C^*(H, \sigma_H) \otimes_{\max} C^*(L, \sigma_L). \tag{2.15}$$

This is easy to obtain because denoting by $\|\cdot\|_H$, $\|\cdot\|_L$ and $\|\cdot\|_V$ the minimal regular norms on $\mathcal{W}(H, \sigma_H)$, $\mathcal{W}(L, \sigma_L)$ and $\mathcal{W}(H \oplus L, \sigma_H \oplus \sigma_L)$ respectively, for given $a \in \mathcal{W}(H)$ and $b \in \mathcal{W}(L)$, on a generic elementary tensor $a \otimes b \in \mathcal{W}(H) \otimes \mathcal{W}(L)$ it holds

$$\|a \otimes b\|_{\max} = \|ab\|_V \geq \|\Psi(a)\|_{\max} \|\Psi(b)\|_{\max} = \|a\|_H \|b\|_L$$

being $\|\Psi(a)\|_{\max} = \|a \otimes I\|_{\max} = \|a\|_H$ and similarly for $b \in \mathcal{W}(L)$.

We call such a kind of isomorphism for symplectic spaces, or Weyl and associated C*-algebras, a *splitting isomorphism* and a direct sum as in Eq. (2.13) may also be called *a splitting decomposition of the symplectic space* V.

Remark 2.2. Observe that if both (H, σ_H) and (L, σ_L) are non-degenerate symplectic spaces, the minimal regular norms $\|\cdot\|_H$ and $\|\cdot\|_L$ are unique and the C*-subcross norms on the algebraic tensor product $C^*(H, \sigma_H) \otimes C^*(L, \sigma_L)$ all coincide, so that in this case, it holds (see e.g. [50, 38])

$$C^*(V, \sigma_V) = C^*(H, \sigma_H) \otimes_{\max} C^*(L, \sigma_L) = C^*(H, \sigma_H) \otimes_{\min} C^*(L, \sigma_L). \tag{2.16}$$

The splitting isomorphisms are trivial examples of the following general construction: let (H, σ_H) and (L, σ_L) be two symplectic spaces, with symbol Abelian groups $\mathcal{U}(H)$ and $\mathcal{U}(L)$ and let $\mathcal{U}(H, \sigma_H)$, $\mathcal{U}(L, \sigma_L)$ be their Weyl algebra groups, defined as in the above Eq. (2.7). Consider the 2-cocycle $(\beta, y) : (\mathcal{U}(L), \mathcal{U}(L) \times \mathcal{U}(L)) \to (\mathrm{Aut}\,\mathcal{U}(H, \sigma_H), \mathcal{U}(H, \sigma_H))$, defined for the elements $s = W(l), s' = W(l') \in \mathcal{U}(L)$ and $t = (\zeta, W(h)) \in \mathcal{U}(H, \sigma_H)$ by

$$\beta_s(t) = \beta_s((\zeta, W(h))) = (\zeta e^{-i\alpha(h,l)}, W(h)) \tag{2.17}$$

where the action β is given by a (continuous) real valued, $\mathbb{R}$-bilinear form α, defined on $H \times L$, such that $\alpha(h, 0) = \alpha(0, l) = 0$, and the function y defined by

$$y(s, s') = (e^{-i\sigma_L(l,l')/2}, I_H). \tag{2.18}$$

To the pair of groups $\mathcal{U}(H,\sigma_H)$ and $\mathcal{U}(L)$ is associated the extension

$$e \to \mathcal{U}(H,\sigma_H) \to \mathcal{U}(H\oplus L,\sigma_V) \to \mathcal{U}(L) \to e$$

where σ_V is a symplectic form on $V := H\oplus L$, defined by

$$\sigma_V((h,l),(h',l')) = \sigma_H(h,h') + \sigma_L(l,l') + \alpha(h,l') - \alpha(h',l) \tag{2.19}$$

so that

$$\sigma_{H,L}((h,l),(h',l')) := \alpha(h,l') - \alpha(h',l) \tag{2.20}$$

represents the *interacting content of the non-splitting sum.*

An extension group is defined from the 2-cocycle $z := (\beta, y)$ as above, i.e. in other notation

$$\mathcal{U}(H\oplus L,\sigma_V) = \mathcal{U}(H,\sigma_H) \rtimes_{(\beta,y)} \mathcal{U}(L) = \mathcal{U}(H,\sigma_H) \rtimes_z \mathcal{U}(L). \tag{2.21}$$

Explicitly, for generic elements $t = (\zeta, W(h)), t' = (\zeta', W(h')) \in \mathcal{U}(H,\sigma_H)$ and $s = W(l), s' = W(l') \in \mathcal{U}(L)$, the extension group is defined by the product

$$\begin{aligned}
&((\zeta, W(h)), W(l))((\zeta', W(h')), W(l'))\\
&\quad = ((\zeta, W(h))\beta_s(\zeta', W(h')), W(l)W(l'))\\
&\quad = ((\zeta\zeta' e^{-i\alpha(h',l)} e^{-i\sigma_L(l,l')/2} e^{-i\sigma_H(h,h')/2}, W(h+h')), W(l+l')),
\end{aligned}$$

by the identity $e = ((1, I_H), I_L)$ and the passage to the inverse given by

$$((\zeta, W(h)), W(l))^{-1} = ((\zeta^{-1} e^{-i\alpha(h,l)}, W(-h)), W(-l)).$$

In this generality, we can introduce the following:

Definition 2.3. The algebra on $\mathbb{C}$ associated as group algebra to the extension group $\mathcal{U}(H\oplus L,\sigma_V) = \mathcal{U}(H,\sigma_H) \rtimes_{(\beta,y)} \mathcal{U}(L)$, where the symplectic form σ_V and the 2-cocycle (β, y) are defined as above, is called the *twisted crossed product algebra* of the Weyl algebras $\mathcal{W}(H,\sigma_H)$ and $\mathcal{W}(L,\sigma_L)$. This algebra may also be defined as the Weyl algebra on the symplectic space $(V := H\oplus L, \sigma_V)$.[e]

In particular cases, such a twisted crossed product of Weyl algebras may be derived from a non-splitting decomposition of a symplectic space, as better said in the sequel.

If (V,σ_V) is a (degenerate) symplectic space and H is a real subspace of it, we denote by

$$H' := \{v \in V : \sigma_V(v,h) = 0, h \in H\} \tag{2.22}$$

the *symplectic complement* of H in V and by

$$H^{\perp_{\sigma_V}} := \{S \subset V, \text{linear space} : \sigma_V(s,h) = 0, s\in S, h \in H\} \tag{2.23}$$

the partially ordered set of *the symplectic subspaces of V disjoint to H*. The set $H^{\perp_{\sigma_V}}$ has maximal element H' and obviously contains the (eventually non-trivial)

[e]Such a construction from two symplectic spaces is also called the *semidirect product of Weyl algebras* in the literature, see, e.g., [28].

degeneracy subspace N_V. The decomposition seen in Eq. (2.13) holds iff $\sigma_V(l,h)$ vanishes for all $l \in L$ and $h \in H$, i.e. introducing the symbol $\perp_{\sigma_V}$ called *the symplectic disjunction* in (V, σ_V), iff $L \perp_{\sigma_V} H$.

To construct examples of Weyl algebras products, suppose given a space decomposition $V = B \oplus C$ such that the symplectic form is not splitting, i.e. $\sigma_V \neq \sigma_B + \sigma_C$, and there exists a decomposition of one of the addend as $B = H \oplus N$, with $N \perp_{\sigma_V} H$ and $C \perp_{\sigma_V} H$, i.e. $(N \oplus C) \perp_{\sigma_V} H$. In such a situation we have for the *interacting part of the symplectic form*

$$\sigma_{B,C} = \sigma_{H \oplus N, C} = \sigma_{N,C}$$

and the non-splitting contents of such a decomposition of V is confined in the subspace $L := C \oplus N \cong N \oplus C$. The Weyl algebra associated to (V, σ_V) is isomorphic to a twisted cross product, in accordance with the following

Proposition 2.4. *Given a symplectic space* (V, σ_V) *with decomposition*

$$V = H \oplus N \oplus C, \quad \textit{with} \quad B = H \oplus N \quad \textit{and} \quad L = C \oplus N \perp_{\sigma_V} H$$

there exists a 2*-cocycle*

$$z = (\beta, y) : (\mathcal{U}(C), \mathcal{U}(C) \times \mathcal{U}(C)) \to (\mathrm{Aut}\,(\mathcal{U}(N, \sigma_N), \mathcal{U}(N, \sigma_N)))$$

as in Eq. (2.9), *such that for fixed elements* $s = W(c)$ *and* $s' = W(c')$ *in* $\mathcal{U}(C)$ *an action* $\beta : \mathcal{U}(C) \to \mathrm{Aut}(\mathcal{U}(N, \sigma_N))$, *is defined by*

$$\beta_s(m) = (e^{-i\sigma_L(c,n)}\zeta, W(n)) = \mathrm{ad}\, s(m), \tag{2.24}$$

for the element $m = (\zeta, W(n)) \in \mathcal{U}(N, \sigma_N)$, *and where* $y : \mathcal{U}(C) \times \mathcal{U}(C) \to \mathcal{U}(N, \sigma_N)$ *can be written as*

$$y(s, s') := (e^{-i\sigma_C(c,c')/2}, I) \in \mathcal{U}(N, \sigma_N).$$

Such a 2*-cocycle gives a twisted crossed product decomposition of the Weyl algebra as*

$$\begin{aligned} \mathcal{W}(V, \sigma_V) &= \mathcal{W}(H \oplus N, \sigma_H \oplus \sigma_N) \rtimes_{(\beta,y)} \mathcal{U}(C) \\ &= \mathcal{W}(H, \sigma_H) \otimes \mathcal{W}(N, \sigma_N) \rtimes_{(\beta,y)} \mathcal{U}(C). \end{aligned}$$

Proof. The subspaces H and L in Definition 2.3 have to be, respectively, identified with $H \oplus N$ and C in the case at hand. According to this, in Eq. (2.19) we have to read $\alpha(h,c) = \alpha(n,c) = \sigma_L(n,c)$ for $h \oplus n \in H \oplus N, c \in C$, and the symplectic form σ_V decomposes as $\sigma_V = \sigma_H \oplus \sigma_L$, where $L = C \oplus N$ and $\sigma_L = \sigma_V \upharpoonright L$. □

Observe that the subspace N may be thought, is some sense, as being *in common* between the symplectic subspaces B and L, and that the Weyl elements defined from the subspace C have a non-trivial action on Weyl elements defined from $N \subset B$, by the evaluation of σ_L.

We end this section with some broad ideas about the formalization of physical model by Weyl algebras. In all generality, a *simple current extension* of Weyl algebras is essentially described by a crossed product of Weyl algebras, along the following scheme. The charge carrying fields are defined starting from a symplectic space

$$(V_f, \sigma_f) = (V_a \oplus N \oplus C, \sigma_f). \tag{2.25}$$

Here, we have to read the subspace decomposition $V = B \oplus C = (V_a \oplus N) \oplus C$ as in Proposition 2.4 above, with $H = V_a$. The symplectic form σ_f may not split in the sum $\sigma_B \oplus \sigma_C$, for $\sigma_B = \sigma_f \upharpoonright B$ and $\sigma_C = \sigma_f \upharpoonright C$, but if $N \in V_a^{\perp_{\sigma_f}}$ the field Weyl algebra isomorphically can be written as

$$\mathcal{W}(V_f, \sigma_f) = \mathcal{W}(V_a \oplus N, \sigma_a \oplus \sigma_N) \rtimes_{(\beta,y)} \mathcal{U}(C). \tag{2.26}$$

In these decompositions, V_a has the meaning of the symplectic space for the observables algebra, for which a (regular positive metric) Fock space representation π_a exists. The representation π_f for the field algebra $\mathcal{W}_f$ is in general a non-regular extension of π_a, as it happens in a non-rational model, and $\mathcal{U}(C)$ plays the role of *the charge group* of the theory. Observe that this description is more general that the one treated in Proposition 2.4, where also $C \in V_a^{\perp_{\sigma_f}}$ was assumed.

Hence the Weyl algebra models may be classified on the basis of the different specific properties in the above space decomposition (2.25) and the algebraic ones in Eq. (2.26), such as the dimension of C and N as real linear spaces, the evaluation of σ_V when restricted to C and N, and so on. We will see two different examples below.

As a final remark, observe that the purely algebraic constructions above are shown to entails some general functorial features passing to representations, that are also relevant for the nets of von Neumann algebras, defined from localized symplectic subspaces of a given symplectic space.

2.2. *Representations*

We summarize some general results about the representation theory of a Weyl algebra $\mathcal{W}(V, \sigma_V)$ and its associated C*-algebra $C^*(V, \sigma_V)$, see [50, 38] for details:

- every positive linear functional on $\mathcal{W}(V, \sigma_V)$ is continuous with respect to the minimal regular norm and extends to a unique positive, continuous linear functional on $C^*(V, \sigma_V)$;
- every representation π of the Weyl algebra $\mathcal{W}(V, \sigma_V)$ on a Hilbert space $\mathcal{H}_\pi$ extends to a representation of $C^*(V, \sigma_V)$, on the same Hilbert space;
- every *-automorphism on $\mathcal{W}(V, \sigma_V)$ extends uniquely to a *-automorphism of $C^*(V, \sigma_V)$.

In the sequel, we show the relation between the twisted crossed product characterization of Weyl algebras, introduced in the last section, and some factorization properties of their representations. We begin from the simplest situation, the split case of Eq. (2.14) or (2.15), by the following:

Lemma 2.5. *Let $(V = H \oplus L, \sigma_V = \sigma_H \oplus \sigma_L)$ be a direct sum of symplectic spaces with Weyl algebra $\mathcal{W}(V, \sigma_V) \cong \mathcal{W}(H, \sigma_H) \otimes \mathcal{W}(L, \sigma_L)$ as above. Then*

(i) *if $(\pi_{\omega_H}, \mathcal{H}_{\omega_H}, \Omega_H)$ and $(\pi_{\omega_L}, \mathcal{H}_{\omega_L}, \Omega_L)$ are the GNS representations associated to the states ω_H and ω_L on $\mathcal{W}_H$ and $\mathcal{W}_L$ respectively, then the unique product state ω and its GNS representation π_ω is canonically defined for the Weyl algebra $\mathcal{W}_V = \mathcal{W}(H \oplus L, \sigma_H \oplus \sigma_L)$ by the (spatial) tensor product as*

$$(\pi_\omega, \mathcal{H}_\omega, \Omega) = (\pi_{\omega_H}, \mathcal{H}_{\omega_H}, \Omega_H) \otimes (\pi_{\omega_L}, \mathcal{H}_{\omega_L}, \Omega_L);$$

(ii) *if (H, σ_H) and (L, σ_L) are non degenerate symplectic spaces or if (for example) $\sigma_V \upharpoonright L = \sigma_L$ vanish, i.e. if $\mathcal{W}(L, \sigma_L) = \mathcal{W}(L)$ is Abelian, then*

$$(\pi_\omega(C^*(H, \sigma_H) \otimes_{\max} C^*(L, \sigma_L)))'' = \pi_{\omega_H}(C^*(H, \sigma_H))'' \otimes \pi_{\omega_L}(C^*(L, \sigma_L))'',$$

where the latter means the tensor product of the von Neumann algebras $\pi_{\omega_H}(C^*(H, \sigma_H))''$ and $\pi_{\omega_L}(C^*(L, \sigma_L))''$.

Proof.

(i) π_ω is obtained as the GNS representation of the *product state* $\omega := \omega_H \otimes \omega_L$ on the C*-algebra $C^*(H, \sigma_H) \otimes_{\max} C^*(L, \sigma_L)$, i.e. from the state defined by

$$\omega(A \otimes_{\max} B) = \omega_H(A)\omega_L(B), \quad A \in C^*(H, \sigma_H), \quad B \in C^*(L, \sigma_L).$$

Here $\otimes_{\max}$ assures for the product state ω a well behaved passage to the representation π_ω of $C^*(H, \sigma_H) \otimes C^*(L, \sigma_L)$ on the Hilbert space $\mathcal{H}_\omega = \mathcal{H}_{\omega_H} \otimes \mathcal{H}_{\omega_L}$, obtained as the spatial tensor product of the GNS representations π_{ω_H} of $C^*(H, \sigma_H)$ and π_{ω_L} of $C^*(L, \sigma_L)$ (see e.g. [52, Theorem IV.4.9] or [31, Proposition 11.1.1] for details).

(ii) The results follow directly from [52, Theorem IV.4.13] and the identity

$$C^*(H, \sigma_H) \otimes_{\max} C^*(L, \sigma_L) = C^*(H, \sigma_H) \otimes_{\min} C^*(L, \sigma_L).$$

This equality, in the case of non degenerate subspaces is given by Eq. (2.16). In the second case, if $C^*(L, \sigma_L)$ is Abelian, hence nuclear, it is a well known consequence. □

An elementary example of item (ii) in above Lemma 2.5, is given by a splitting isomorphism of symplectic spaces with $L = N_V$, the degenerate subspace of V, and $H = V/N_V$.

Passing to the non-splitting situation, the factorization of representations we are interested in, is described by the following general result, also related to one in [28].

Proposition 2.6. *Let $(V = H \oplus L, \sigma_V = \sigma_H + \sigma_L + \sigma_{H,L})$ be a symplectic space decomposition as in Definition 2.3 and Eq. (2.21), such that the Weyl algebra $\mathcal{W}(V, \sigma_v)$ is not splitting, i.e. the real form α that defines through Eq. (2.20) the interacting part $\sigma_{H,L}$ of the symplectic form σ_V is non-trivial. Then, for given ω_H*

and ω_L two states on $\mathcal{W}(H, \sigma_H)$ and $\mathcal{W}(L, \sigma_L)$ respectively, the linear functional on $\mathcal{W}(V, \sigma_V)$ defined for $v = h \oplus l \in H \oplus L = V$ by

$$\omega(W(v)) := \omega_H(W(h))\omega_L(W(l)), \tag{2.27}$$

is positive, i.e. is a state on $\mathcal{W}(V, \sigma_V)$, if $\omega_L(W(l)) = 0$ for $l \notin H^\perp \cap L$. In particular, if $H^\perp \cap L = \{0\}$, i.e. if α is non-trivial on any subspace of L, such a condition is also necessary, i.e. if $H^\perp \cap L = \{0\}$, ω is a state on $\mathcal{W}(V, \sigma_V)$ iff

$$\omega_L(W(l)) = 0 \quad \text{for } l \neq 0. \tag{2.28}$$

If $H^\perp \cap L = \{0\}$ and the condition (2.28) holds, the state ω is faithful iff so is the state ω_H. Respectively, changing L with H.

Proof. To verify the hermiticity of ω, we may restrict to an element $A = W(l)W(h)$ for $l \in L$ and $h \in H$, so that such a property for ω holds iff

$$\begin{aligned}\omega(A) = \omega_H(W(h))\omega_L(W(l)) &= \overline{\omega(A^*)} = \overline{\omega(W(l)^*\beta_{W(l)}(W(h)^*))} \\ &= \omega_H(\beta_{W(l)}(W(h)))\omega_L(W(l))\end{aligned}$$

i.e. by the definition of the action of β in Eq. (2.24), iff

$$\omega_H(W(h))\omega_L(W(l))(1 - e^{i\alpha(h,l)}) = 0.$$

Hence the hermiticity holds if $\omega_L(W(l)) = 0$ for $l \notin H^\perp \cap L$. In particular, if $H^\perp \cap L = \{0\}$, the hermiticity holds iff the condition (2.28) is satisfied.

To show the positivity, observe that any element $A \in \mathcal{W}_V$ is written for $l_i \in L$, $h_i \in H$ and $a_i \in \mathbb{C}$ with $l_i \neq l_j$ for $i \neq j$, as a finite sum

$$A = \sum_{1 \le i \le n} a_i W(h_i)W(l_i). \tag{2.29}$$

Hence we obtain

$$\begin{aligned}AA^* &= \sum_{1\le i\le n} |a_i|^2 + \sum_{1\le i<j\le n} a_i\overline{a}_j W(h_i)W(l_i)W(l_j)^*W(h_j)^* + adj \\ &= \sum_{1\le i\le n} |a_i|^2 + \sum_{1\le i<j\le n} a_i\overline{a}_j e^{i\alpha_{ij}} W(h_i - h_j)W(l_i - l_j) + adj\end{aligned} \tag{2.30}$$

where $\alpha_{ij} = \frac{1}{2}[\sigma_H(h_i, h_j) + \sigma_L(l_i, l_j) + \sigma_V(h_j, l_i - l_j)]$. Applying the functional ω gives

$$\begin{aligned}\omega(AA^*) &= \sum_{1\le i\le n} |a_i|^2 + \sum_{1\le i<j\le n} 2\Re\{a_i\overline{a}_j e^{i\alpha_{ij}}\omega_H(W(h_i - h_j))\omega_L(W(l_i - l_j))\} \\ &= \sum_{1\le i\le n} |a_i|^2 + \sum_{\substack{1\le i<j\le n \\ l_i - l_j \in H^\perp \cap L}} 2\Re\{a_i\overline{a}_j e^{i\beta_{ij}}\} \\ &\ge \sum_{\substack{1\le i<j\le n \\ l_i - l_j \in H^\perp \cap L}} |a_i|^2 + \sum_{\substack{1\le i<j\le n \\ l_i - l_j \in H^\perp \cap L}} 2\Re\{a_i\overline{a}_j e^{i\beta_{ij}}\} = \sum_{\substack{1\le i<j\le n \\ l_i - l_j \in H^\perp \cap L}} |b_i|^2 \ge 0\end{aligned} \tag{2.31}$$

where we used the condition $\omega_L(l) = 0$ for $l \notin H^\perp \cap L$, redefined β_{ij} from the phases that arise from the evaluation of the states, and the last equality holds because we may call $b_i = a_i e^{i\beta_i}$ for some β_i such that $\beta_{ij} = \beta_i - \beta_j$. If we suppose that the property (2.28) holds true for L, we only have $\omega(AA^*) = \sum_{1\leq i\leq n} |a_i|^2$, so the faithfulness trivially follows in this case. □

Observe that the above defined state ω is a product state on the tensor product algebras $\mathcal{W}_H \otimes \mathcal{W}_L$ in the sense of Lemma 2.5, only if the linear form α trivializes, i.e. if $H^\perp \cap L = L$.

In particular, the Proposition 2.6 applies to the symplectic space decomposition as in Proposition 2.4. For this case, given the two states ω_H on $\mathcal{W}(H, \sigma_H)$ and ω_L on $\mathcal{W}(L, \sigma_L)$, we define in a canonical way as in Lemma 2.5, the product state

$$\omega_p := \omega_H \otimes \omega_L \tag{2.32}$$

on the tensor product Weyl algebra $\mathcal{W}(H, \sigma_H) \otimes \mathcal{W}(L, \sigma_L)$ and denote by $\pi_p := \pi_H \otimes \pi_L$ its GNS representation on the Hilbert space $\mathcal{H}_p = \mathcal{H}_H \otimes \mathcal{H}_L$.

Now let $\omega_{H\oplus N}$ be an extension of the state ω_H from the Weyl algebra $\mathcal{W}(H, \sigma_H)$ to the Weyl algebra $\mathcal{W}(H \oplus N, \sigma_{H\oplus N})$, for an extension $\sigma_{H\oplus N}$ of the symplectic form σ_H of H to $H \oplus N$. For $L = N \oplus C$, let denote by ω_C the restriction of a state ω_L of $\mathcal{W}(L, \sigma_L)$ to the subalgebra $\mathcal{W}(C, \sigma_L \restriction C)$. A result relating the states on a twisted crossed product decomposition (Proposition 2.6) and a splitting Weyl algebra (Lemma 2.5), is obtained confining the interacting part $\sigma_{H,L}$ of the symplectic form σ_V to the subspace N, that is *in common* between H and L.

Hence we give the following result, where indeed, referring to Proposition 2.6, H is replaced by $H \oplus N$ and L by C:

Proposition 2.7. *Consider $V = H \oplus L = H \oplus N \oplus C$ as in notation above, so that the state ω in Eq.* (2.27) *is well defined. Then the following are equivalent*:

(i) $\begin{cases} \omega_{H\oplus N}(W(x)) = \omega_H(W(h)), & x = h \oplus n \in H \oplus N, \\ \omega_L(W(l)) = \omega_C(W(c)), & l = n \oplus c \in N \oplus C = L; \end{cases}$

(ii) *the states ω_p and ω, the first defined as in* (2.32) *with extension and restriction as above, and ω defined as in Proposition* 2.6, *coincide on $\mathcal{W}(V, \sigma_V)$ and on its C^*-algebra $C^*(V, \sigma_V)$.*

Proof. (i) ⇒ (ii) is obtained by a simple calculation from the definitions, for $v = h \oplus l = h \oplus n \oplus c \in H \oplus N \oplus C = V$ and $x = h \oplus n \in H \oplus N$

$$\omega_p(W(v)) := \omega_H(W(h)) \otimes \omega_L(W(l)) = \omega_{H\oplus N}(W(x))\omega_C(W(c)) =: \omega(W(v)).$$

(ii) ⇒ (i) It suffices to use $c = 0$ in the preceding calculation to obtain back the first relation in (i) and analogously, using $h = 0$, for the second. □

Notice that for the element $h = 0$ in H we obtain from the condition (i) in the proposition above that the state $\omega_N := \omega_{H\oplus N} \restriction N$ of $\mathcal{W}_N$ is such that $\omega_N(W(n)) = 1$ for every $n \in N$.

Remark 2.8. (1) Consider a symplectic space decomposition as in Proposition 2.4 and refer to the physical model discussion about Eq. (2.26). If C represents the charge group for V_f and $H \oplus N$ is related to the symplectic space of observables, then the triviality of α, i.e. of $\sigma_{N,C}$, is equivalent to a trivial action of the sector (auto)morphisms on the observables, that means a trivial superselection structure of the model. The condition in Eq. (2.28) instead, defines the state ω_L in a unique way as a non-regular state (see Definition 2.9 below).

(2) The obtained results actually extend the ones of Herdegen in [28], where a crossed product of a Weyl algebra by a CAR algebra is treated: in fact, as observed by Slawny in [50, Sec. 3.10], a CAR algebra may always be written as a Weyl algebra with a non-degenerate symplectic form.

(3) The Proposition 2.7 suggests that, for twisted crossed products as above, it is the non-regular representations of $\mathcal{W}(V, \sigma_V)$ that is of interest, with C the *non-regular subspace* and N the *regular subspace* of V, see Definition 2.9 below. This always leads to non-rational models, with uncountably many superselection sectors, in the sense of [37].

Further general results for twisted crossed products and non-regular representations of CCR algebras of a generic locally compact group, along the lines of [25] are also possible, see [13].

2.3. *Elementary Weyl algebra in non-regular representation*

In the above subsection, we pointed out the utility of non-regular representations when a field algebra model is written as a cross product of Weyl algebras. A regular (Fock space) representation is used instead on the observable part of the model, also called in literature *the physical part*, and a non-regular one on the charge part, see e.g. [1]. We recall hence the following basic:

Definition 2.9. A representation π of a Weyl algebra $\mathcal{W} = \mathcal{W}(V, \sigma_V)$ is said a *regular representation* if for all $v \in V$ the one parameter group

$$\lambda \mapsto \pi(W(\lambda v)), \quad \lambda \in \mathbb{R}, \quad v \in V \tag{2.33}$$

is weakly (and strongly) continuous. If there exist a non-trivial subspace $V_1 \subset V$ such that the map in Eq. (2.33) is not weakly continuous, then the representation π is said to be *non-regular*. The maximal such subspace is said *the non-regular subspace* of π. A state ω whose GNS representation π_ω is regular (non-regular) is also said to be regular (non-regular).

Observe that in general a Weyl algebra is not norm continuous, being $\|W(v_1) - W(v_2)\| = 2$ if $v_1 \neq v_2$ for $v_1, v_2 \in V$. The strong continuity of the regular representations of the Weyl algebras, that is needed for their relevant physical content, is in some sense also necessary to obtain good representation of the associated CCR algebra, see [47, Sec. 3.3] for details.

We now discuss in detail the simplest, but relevant, case of non-regular representation for the symplectic space $L \cong \mathbb{R}_d^2$, in order to use it as a *building block* for the superselection theory of physical models.

Let $C \cong N \cong \mathbb{R}_d$ be copies of the additive group of reals, furnished with the discrete topology; consider the symplectic space (L, σ_L) where $L = C \oplus N \cong \mathbb{R}_d^2$ and the symplectic form is the usual non-degenerate one on $\mathbb{R}_d^2$, i.e. for a pair of elements $l = (c, n), l' = (c', n') \in L$ we have

$$W_L(l)W_L(l') = e^{-\frac{i}{2}\sigma_L(l,l')}W_L(l+l'), \quad \sigma_L(l,l') = (cn' - c'n).$$

The Weyl algebra $\mathcal{W}_L$ associated to (L, σ_L) is hence that of a quantum system with one degree of freedom; its irreducible regular representations, and those of the associated C*-algebra $C^*(L, \sigma_L)$, are well described by the Stone–von Neumann uniqueness theorem, see, e.g., [7].

To consider a particular non-regular representation of $\mathcal{W}_L$, following for example [55, Sec. 2], let ω_L be the functional on $\mathcal{W}_L$ defined by

$$\omega_L(W_L(l)) = \omega_L(W_L((c,n))) = \delta_{c,0}, \quad l = (c,n) \in L. \tag{2.34}$$

This is the unique tracial state on $\mathcal{W}(L, \sigma_L)$, up to the change of C and N, and it turns out to be not faithful because

$$\omega_L((I - W_L((0,n)))(I - W_L^*((0,n)))) = 0, \quad (0,n) \in L.$$

However, if we denote the GNS triple of ω_L by $(\pi_L, \mathcal{H}_L, \Omega_L)$, we observe that π_L is faithful because $\mathcal{W}_L$ is simple, as the degeneracy subspace of L is trivial, i.e. $N_L = \{0\}$. The GNS Hilbert space $\mathcal{H}_L$ is non separable and can be derived from the total set of vectors obtained as

$$|c,n\rangle := \pi_L(W_L((c,n)))\Omega_L, \quad (c,n) \in L.$$

The action of a represented Weyl element $\pi_L(W_L((c,n)))$ on the generic vector $|c',n'\rangle \in \mathcal{H}_L$, for $(c,n),(c',n') \in L$, reads as

$$\pi_L(W_L((c,n)))|c',n'\rangle = e^{-\frac{i}{2}\sigma_L((c,n),(c',n'))}|c+c', n+n'\rangle. \tag{2.35}$$

Moreover, if $|c,n\rangle$ and $|c',n'\rangle$ are two elements of $\mathcal{H}_L$, their scalar product is equal to

$$\langle c,n|c',n'\rangle = \delta_{c,c'}e^{\frac{i}{2}c(n'-n)}, \tag{2.36}$$

so that $\langle 0,n|0,n'\rangle = 1$. It is hence possible to identify the vectors $|c,n\rangle$ for every $n \in N$ with the vector $|c,0\rangle$, up to a phase, giving $|c,n\rangle = e^{\frac{i}{2}cn}|c,0\rangle$ and $\langle c,n|c',n'\rangle = e^{\frac{i}{2}c(n'-n)}\delta_{c,c'}$. Because of this identification, we simply write $|c\rangle := |c,0\rangle$. In particular this is possible for $c = 0$, so that the following relations hold

$$\Omega_L = |0,0\rangle = |0,n\rangle =: |0\rangle, \quad \text{for all } n \in N. \tag{2.37}$$

From the above discussion, the Hilbert space $\mathcal{H}_L$ may also be written as

$$\mathcal{H}_L = \bigoplus_{c\in C} \mathbb{C}|c\rangle \cong \ell^2(C) \cong \ell^2(\mathbb{R}_d), \tag{2.38}$$

$\ell^2(\mathbb{R}_d)$ being the space of square summable maps from $\mathbb{R}_d$ into $\mathbb{C}$, i.e. an element $\xi \in \ell^2(\mathbb{R}_d)$ is a function supported on a countable subset D_ξ of $\mathbb{R}$ such that $\|\xi\| := \sum_{d\in D_\xi} |\xi(d)|^2 < \infty$. According to Eq. (2.36), the inner product on $\ell_2(\mathbb{R}_d)$ is equivalently given by

$$\langle \xi, \eta\rangle = \sum_{d\in D_\xi\cap D_\eta} \overline{\xi(d)}\eta(d), \quad \xi, \eta \in \ell^2(\mathbb{R}_d).$$

Denoting for every $c \in \mathbb{R}$ by $\delta_c \in \ell^2(\mathbb{R}_d)$ the characteristic function of the subset $\{c\} \subset \mathbb{R}$, identifying $|c, 0\rangle$ with δ_c gives the isomorphism in the above Eq. (2.38).

The representation π_L turns out to be irreducible and non-regular, not being strongly continuous in first coordinate $c \in C$ of elements $W_L(c, r)$ but only in the second. The von Neumann algebras $\mathcal{L} := \pi_L(C^*(L, \sigma_L))''$ is irreducibly represented on $\mathcal{H}_L$ by π_L.

Moreover, we may consider the Weyl elements $W_L(c, 0), c \in C$ and $W_L(0, n)$, $n \in N$, and the Abelian C*-subalgebra of $C^*(L, \sigma_L)$ they generate, respectively $C^*(C, 0) =: C^*(C)$ and $C^*(N, 0) =: C^*(N)$; finally if $\mathcal{C} := \pi_L(C^*(C))''$ and $\mathcal{N} := \pi_L(C^*(N))''$ are the associated Abelian von Neumann subalgebras of $\mathcal{L}$ they define, the properties of these unitaries and algebras are collected in the following:

Proposition 2.10. *Under the above definitions we have*:

(i) *if* $c \neq 0$ *and* $n \neq 0$ *then the spectrum of* $W_L((c, 0))$ *is* $\mathbb{T}$ *and equals both the spectrum of* $W_L((0, n))$ *and the discrete spectrum of* $\pi_L(W_L(0, n))$;

(ii) *there is a self-adjoint operator* Φ_N *on* $\mathcal{H}_L$ *such that for any* $n \in N$ *we have* $\pi_L(W_L(0, n)) = \exp\{\frac{i}{2}n\Phi_N\}$, *i.e.* Φ_N *is the generator of the one parameter group associated to* N; *there is no self-adjoint operator* Φ_C *on* $\mathcal{H}_L$, *generating a one-parameter group associated to* C. *In particular, for all* $c \in C$ *we have*

$$\begin{aligned}\Phi_N|c, 0\rangle &= -i \lim_{n\to 0} n^{-1}(\pi_L(W(0, n)) - I_L)|c, 0\rangle \\ &= -i \lim_{n\to 0} n^{-1}(e^{inc} - 1)|c, 0\rangle = c|c, 0\rangle;\end{aligned}$$

(iii) $C^*(C) \cong C(bC) \cong C(b\mathbb{R}) \cong C(bN) \cong C^*(N)$, *where* $C(b\mathbb{R})$ *indicates the* C^*-*algebra of the continuous functions on the Bohr compactification* $b\mathbb{R}$ *of* $\mathbb{R}$, *i.e. is the algebra of the almost periodic functions on* $\mathbb{R}$, *i.e. the group* C^*-*algebra of the compact group* $b\mathbb{R}$;

(iv) $C^*(L, \sigma_L) = C^*(N) \rtimes_{(\iota,y)} \mathcal{U}(C)$, *i.e.* $C^*(L, \sigma_L)$ *is the twisted crossed product of* $C^*(N)$ *by the discrete additive group* C, *with* 2-*cocycle* (ι, y), *defined from the symplectic form* σ_L;

(v) *every non zero vector in $\mathcal{H}_L$ is cyclic for the von Neumann subalgebra $\mathcal{C}$; $\mathcal{N}$ is a spectrally multiplicity free von Neumann subalgebra of $\mathcal{L}$. In particular, $\mathcal{C}$ and $\mathcal{N}$ are maximal Abelian in $\mathcal{B}(\mathcal{H}_L)$;*

(vi) $\mathcal{L} := \pi_L(C^*(L,\sigma_L))''$ *is a type II_1 factor.*

Proof. (i)–(iii) are standard results, see, for example, [50] and also [27].
For (iv), see the above Proposition 2.4.

(v) Considering the total set of vectors $|c\rangle, c \in C$ and the action of $\mathcal{C}$ on them as read off from Eq. (2.35), cyclicity for $\mathcal{C}$ is trivial, and the maximality is a classical result. For the subalgebra $\mathcal{N}$, consider for every $c \in C$, the spectral measures μ_c defined on $\widehat{C^*(N)} \cong bN$, the Gelfand spectrum of $C^*(N)$, and the Baire set $X_c = bN \setminus c$. These data satisfy the hypothesis of [12, Corollary 5.2] that characterizes π_L as a *spectrally multiplicity-free representation* of $C^*(L,\sigma_L)$, a stronger condition for $\mathcal{N}$ to be maximally Abelian in $\mathcal{B}(\mathcal{H}_L)$.

(vi) The von Neumann algebras $\mathcal{L}$ is equal to the represented discrete crossed product algebra $\mathcal{N} \rtimes_\alpha \Gamma$, where $\Gamma := \pi_L(\mathcal{U}(C))$ denotes the discrete group $\mathcal{U}(C)$ represented on the Hilbert space $\mathcal{H}_L$. The automorphic action α defined by $\alpha_c := \mathrm{ad}\, W(c), c \in C$ on the maximal Abelian *-algebra $\mathcal{N}$ is given by the 2-cocycle (ι, y), i.e. through the symplectic form σ_L. It easy to see that α acts ergodically on $\mathcal{N}$, so that $\mathcal{L}$ is a factor. Moreover, the state ω_L is a tracial state on $\mathcal{L}$ so that the algebra is of finite type, and is of type II_1 being not finite dimensional. □

Remark 2.11. (1) The elements $\pi(W((c,0)))$ for $c \in C$, may be considered as operators carrying the charge c. The elements $\pi(W((0,n)))$ for $n \in N$ act as different phases, on the distinct charge subspaces, in the charge decomposition of $\mathcal{H}_L$ given by Eq. (2.38).

(2) The physical idea is to profit of a symplectic space $L = C \oplus N$ with $C \cong \mathbb{R}^n_d \cong N$ for some natural n, such that C represents the additive group of the charges and its associated dual group $\widehat{C} \cong b\mathbb{R} \cong bN$ the internal (gauge) symmetry group of the model. The non-regular representation of $\mathcal{W}_L$ with C the non-regular subspace of L and obtained as the nth tensor product of the representation π_L above, may be used to manage the twisted crossed product of Weyl algebras in a defining state satisfying the requirements of Propositions 2.6 and 2.7.

(3) From a mathematical point of view, a similar construction may also be given for the Weyl algebras $\mathcal{W}(L,\sigma_L)$ where now $L = C \oplus N$, for C an Abelian discrete group and N the locally compact group such that its Bohr compactification bN is equal to $\widehat{C}$, the dual group of C, i.e. the space of *all* arbitrary characters of C, see [38, 50, 25, 13].

3. The Streater and Wilde Model

We begin in this section to analyze the scalar massless free Bosonic quantum field, on the 1+1-dimensional Minkowski spacetime, along the lines of the twisted crossed products described in the previous Sec. 2. General references are [51,29] and a recent more physical introduction is [15].

With respect to the original formulation on chiral left/right light lines of the Streater and Wilde paper, we prefer a time zero formulation, i.e. giving Cauchy data on the time zero real line, because the classification of the superselection sectors, e.g. their different (solitonic) origin, is more clear in this approach, as it will be shown in Sec. 4.3.

In contrast to the same model in higher dimension it is well known that in the 1+1-dimensional case infrared infinities occur, so that the observable algebra may be defined in a physical Fock space representation only after furnishing some supplementary constrain conditions reducing the algebra of non observable fields.

If we write $\mathcal{S} := \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ for the Schwartz space of real valued rapidly decreasing functions on the real line, the cited constrains are realized in the Weyl formulation as a restriction of the test function space $\mathcal{S}$ to a subspace of functions with vanishing Fourier transform at zero momentum, i.e. $\tilde{f}(0) := \int f(x)dx = 0$, for $f \in \mathcal{S}$.[f] This condition is also written for short as $f = \partial g$, for some $g \in \mathcal{S}$, and is required for smearing the functions of the observable fields at time zero, not for their Lagrangian associated time zero momentum. For this reason, the observable theory is also called in the literature *the theory of the potential of the field*.

3.1. *Weyl algebras for the Streater and Wilde model*

The symplectic spaces of the model are introduced by the following restriction and extensions of the function space $\mathcal{S}$:

$$\begin{aligned}
\partial\mathcal{S} &:= \{f \in \mathcal{S} : f = \partial g,\ g \in \mathcal{S}\},\\
\partial^{-1}\mathcal{S} &:= \{f \in C^\infty(\mathbb{R}) : \partial f \in \mathcal{S}\},\\
\partial_0^{-1}\mathcal{S} &:= \left\{f \in \partial^{-1}\mathcal{S} : \lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x)\right\},\\
\partial_q^{-1}\mathcal{S} &:= \left\{f \in \partial^{-1}\mathcal{S} : \lim_{x\to-\infty} f(x) = -\lim_{x\to+\infty} f(x)\right\}.
\end{aligned} \tag{3.1}$$

Denoting by $\mathbb{R}_d$ the constant functions on the time zero line, we have the following additive group quotients

$$\mathcal{S} = \partial_0^{-1}\mathcal{S}/\mathbb{R}_d \quad \text{and} \quad \partial_q^{-1}\mathcal{S} = \partial^{-1}\mathcal{S}/\mathbb{R}_d, \tag{3.2}$$

and the inclusions

$$\partial\mathcal{S} \subset \mathcal{S}, \quad \mathcal{S} \subset \partial_0^{-1}\mathcal{S} \subset \partial^{-1}\mathcal{S} \quad \text{and} \quad \mathcal{S} \subset \partial_q^{-1}\mathcal{S} \subset \partial^{-1}\mathcal{S}. \tag{3.3}$$

[f]Considering localized subspaces of test function space as defining local algebras, we may also replace $\mathcal{S}$ by the space of smooth functions with compact support.

From the linear spaces in Eqs. (3.1) and because of the inclusions in (3.3) we define properly included (symplectic) spaces as follows:

$$\begin{array}{ccccc} V_a := \partial\mathcal{S} \oplus \mathcal{S} & \subset V_b := \partial\mathcal{S} \oplus \partial_0^{-1}\mathcal{S} \subset & V_c := \mathcal{S} \oplus \partial_0^{-1}\mathcal{S} \\ \cap & \cap & \cap \\ V_q := \partial\mathcal{S} \oplus \partial_q^{-1}\mathcal{S} \subset & V_e := \partial\mathcal{S} \oplus \partial^{-1}\mathcal{S} \subset & V_f := \mathcal{S} \oplus \partial^{-1}\mathcal{S}. \end{array} \tag{3.4}$$

Moreover, because of the quotient maps in Eq. (3.2) we have, for $N \cong \mathbb{R}_d$

$$V_b = V_a \oplus N \quad \text{and} \quad V_e = V_q \oplus N. \tag{3.5}$$

The symplectic spaces associated to the above support spaces are defined as follows: for $F = f_0 \oplus f_1$ and $G = g_0 \oplus g_1$ elements in V_a, a (non-degenerate) symplectic form σ_a is defined by

$$\sigma_a(F,G) = \int_{\mathbb{R}} (f_0 g_1 - f_1 g_0) dx. \tag{3.6}$$

The following standard procedure gives a pre-Hilbert space from the symplectic space V_a, defining it on the mass shell (i.e. the positive light cone in momentum space), and leads to a Fock space representation of the associated observable algebras. Let T_a be the map defined by

$$\begin{aligned} T_a : V_a &\to \mathfrak{H}_a := L^2(\mathbb{R}, dp) \\ F = f_0 \oplus f_1 &\mapsto \omega^{-1/2}\tilde{f}_0 + i\omega^{1/2}\tilde{f}_1, \quad \text{for } \omega := |p|, \end{aligned} \tag{3.7}$$

from the real space V_a to the real subspace $T_a(V_a)$ of $\mathfrak{H}_a$, whose closure in $\mathfrak{H}_a$ constitutes the real subspace $H_a := \overline{T_a(V_a)}$ of $\mathfrak{H}_a$, i.e. the *one particle Hilbert space* of the observables of the model, see [51, 29] for details.[g] The scalar product on H_a is explicitly given, for two elements obtained from $F, G \in V_a$, by

$$(T_a F, T_a G)_{H_a} := \int (\omega^{-1}\bar{\tilde{f}}_0 \tilde{g}_0 + \omega \bar{\tilde{f}}_1 \tilde{g}_1) dp + i \int (f_0 g_1 - f_1 g_0) dx. \tag{3.8}$$

For $F = f_0 \oplus f_1 \in V_a$, a quasi-free state on the algebra $\mathcal{W}(V_a, \sigma_a)$ is given by

$$\begin{aligned} \omega_a(W_a(F)) &:= \exp\left\{-\frac{1}{4}\|T_a(F)\|^2_{H_a}\right\} \\ &= \exp\left\{-\frac{1}{4}\int (\omega^{-1}|\tilde{f}_0|^2 + \omega|\tilde{f}_1|^2) dp\right\}. \end{aligned} \tag{3.9}$$

The GNS representation of ω_a, whose triple we denote by $(\pi_a, \mathcal{H}_a, \Omega_a)$, is the Fock (vacuum) representation of the algebra $\mathcal{W}(V_a, \sigma_a)$.

This Fock space construction is possible only for the symplectic space (V_a, σ_a), see, e.g., [1], but construction with a canonical non-regular representation exists instead for all the Weyl algebras associated to the symplectic spaces in diagram (3.4) as illustrated below.

[g] Because the map T_a is an isomorphism of symplectic spaces, the pair $(T_a(V_a), \sigma_a)$ itself is often regarded as the symplectic space of the observables of the model.

Indicated the extension of σ_a to the space V_k by σ_k, for V_k each of the real spaces in (3.4), the pairs (V_k, σ_k) are real symplectic space, properly included as in the same diagram. Only the symplectic spaces (V_b, σ_b) and (V_e, σ_e) are degenerate, with the same degenerate subspace

$$N := \{F = f_0 \oplus f_1 \in V_b : f_0 \oplus f_1 = 0 \oplus n, n \in \mathbb{R}\} \cong \mathbb{R}_d, \tag{3.10}$$

and because of the results on degenerate symplectic forms as in Eq. (2.4), there exist a common non-trivial center of the C*-algebras $C^*(V_b, \sigma_b)$ and $C^*(V_e, \sigma_e)$ given by

$$C^*(N) = \mathfrak{Z}_b := C^*(V_b, \sigma_b) \cap C^*(V_b, \sigma_b)' = C^*(V_e, \sigma_e) \cap C^*(V_e, \sigma_e)' =: \mathfrak{Z}_e. \tag{3.11}$$

We introduce now some symplectic isomorphisms to discuss the model in terms of twisted crossed products of Weyl algebras as in Sec. 2.1. To any function $f_1 \in \partial^{-1}\mathcal{S}$, we associate three real values

$$F_+ := \lim_{x\to+\infty} f_1(x), \quad F_- := \lim_{x\to-\infty} f_1(x), \quad F_\infty := \frac{1}{2}(F_+ + F_-). \tag{3.12}$$

Observe that subtracting the real value F_∞ from the second coordinate function f_1 of an element $F = f_0 \oplus f_1 \in V_b, V_e$ realizes the quotient maps in Eq. (3.2). Hence we state the following:

Lemma 3.1. *For given an element $F = f_0 \oplus f_1$ in V_b (or V_e), we denote by $[f_1] := f_1 - F_\infty$ the element in the quotient group $\mathcal{S} = \partial_0^{-1}\mathcal{S}/\mathbb{R}_d$ (or $\partial_q^{-1}\mathcal{S} = \partial^{-1}\mathcal{S}/\mathbb{R}_d$) as in the quotient maps of Eq.* (3.2)*. Then*

(i) *there exists a splitting isomorphism of degenerate symplectic spaces ψ_∞ from (V_b, σ_b) to $(V_a, \sigma_a) \oplus (N, \sigma_N)$, with $\sigma_N \equiv 0$, defined as*

$$\begin{aligned} \psi_\infty : V_b &\to V_a \oplus N = (\partial\mathcal{S} \oplus \mathcal{S}) \oplus N \\ F &\mapsto (f_0 \oplus [f_1]) \oplus F_\infty. \end{aligned} \tag{3.13}$$

The Weyl functor gives an isomorphism of C-algebras as*

$$\Psi_b : C^*(V_b, \sigma_b) \to C^*(V_a, \sigma_a) \otimes_{\max} C^*(N) = C^*(V_a, \sigma_a) \otimes_{\min} C^*(N).$$

(ii) *there exists a splitting isomorphism of degenerate symplectic spaces, from (V_e, σ_e) to $(V_q, \sigma_q) \oplus (N, \sigma_N)$ with $\sigma_N \equiv 0$, also indicated by ψ_∞ and defined by*

$$\begin{aligned} \psi_\infty : V_e &\to V_q \oplus N = (\partial\mathcal{S} \oplus \partial_q^{-1}\mathcal{S}) \oplus N \\ F &\mapsto (f_0 \oplus [f_1]) \oplus F_\infty. \end{aligned} \tag{3.14}$$

The corresponding isomorphism of C-algebras given by the Weyl functor is*

$$\Psi_e : C^*(V_e, \sigma_e) \to C^*(V_q, \sigma_q) \otimes_{\max} C^*(N) = C^*(V_q, \sigma_q) \otimes_{\min} C^*(N).$$

Proof. ψ_∞ is an isomorphism between the real discrete vector spaces V_b and $V_a \oplus N$, so it trivially preserves the symplectic form and the image of the degeneracy subspace N of V_b. The last equality about tensor product holds because $C^*(N)$ Abelian. The same for the V_e case. □

A more general symplectic space isomorphism extending ψ_∞ is also possible for the larger symplectic space (V_f, σ_f) itself, once we have picked an element in it. This isomorphism turns out to be essentially unique. We begin this construction by defining *the charges* of a generic element $F = f_0 \oplus f_1 \in V_f$. These are the two real quantities canonically associated to F by

$$F_c := \int_{\mathbb{R}} f_0 dx, \quad F_q := \int_{\mathbb{R}} \partial f_1 dx = F_+ - F_-. \tag{3.15}$$

A complete useful re-parametrization of the symplectic spaces in diagram (3.4) is obtained by the choice of a *regularizing element* $T \in V_f$ with non-vanishing charges, i.e. such that $T_c \neq 0 \neq T_q$. A useful choice for such an element of V_f is the following: given a function $t \in \partial^{-1}\mathcal{S}$ such that

$$\lim_{x\to+\infty} t(x) = -\lim_{x\to-\infty} t(x) = \frac{1}{2} \tag{3.16}$$

and denoted by $\partial t \in \mathcal{S}$ its derivative, we choice $T = \partial t \oplus t \in \mathcal{S} \oplus \partial^{-1}\mathcal{S}$. For example, we may take $t(x) = \frac{1}{\pi}\arctan x$, or a half-line or bounded smooth localization obtained from a composition of it with a C^∞ function of compact support. Note that $\int_{\mathbb{R}} t(x)\partial t(x)dx = 0$ and the charges associated to T both equate 1, in fact

$$T_c = \int_{\mathbb{R}} \partial t(x)dx = \lim_{x\to+\infty} t(x) - \lim_{x\to-\infty} t(x) = T_q = 1.$$

Now, because of the fixing of the regularizing element, we may associate to every element $F \in V_f$ two more real numbers

$$F_n := \int_{\mathbb{R}} f_1 \partial t dx, \quad F_r := \int_{\mathbb{R}} f_0 t dx, \tag{3.17}$$

and functions

$$f_{0\partial t} := f_0 - F_c \partial t \in \partial\mathcal{S}, \quad f_{1t} := f_1 - F_q t - F_\infty \in \mathcal{S}. \tag{3.18}$$

In this way, denoting by $(L = N \oplus C, \sigma_L)$ and $(M = R \oplus Q, \sigma_M)$ two copies of the elementary symplectic space as in Sec. 2.3, i.e. $N \cong C \cong R \cong Q \cong \mathbb{R}_d$, and setting

$$F_t = f_{0\partial t} \oplus f_{1t} \in V_a, \quad F_l = F_c \oplus F_n \in L, \quad F_m = F_r \oplus F_q \in M, \tag{3.19}$$

it is possible to state the following

Proposition 3.2. *For every choice of an element $T \in V_f$ as above, for example as in* (3.16):

(i) *there exists a non-splitting isomorphism of symplectic spaces between (V_f, σ_f) and $(\psi_T(V_f), \sigma_f) \subsetneq (V_a \oplus L \oplus M, \sigma_a \oplus \sigma_L \oplus \sigma_M)$, defined by*

$$\begin{aligned} \psi_T : V_f &\to \psi_T(V_f) \subsetneq V_a \oplus L \oplus M \\ F &\mapsto F_t \oplus F_l \oplus F_m. \end{aligned} \tag{3.20}$$

ψ_T extends the splitting isomorphisms ψ_∞ of Lemma 3.1 *for V_b and V_e and reduces to the isomorphisms $(V_k, \sigma_k) \to (\psi_T(V_k), \sigma_k)$ of symplectic spaces when*

applied to V_k, $k = c, q$ *respectively, i.e. such that*:

- ψ_T *is a* $\mathbb{R}$*-linear continuous bijection on its image, preserving symplectic forms and degeneracy subspaces*;
- *for elements* $F = f_0 \oplus f_1$ *and* $G = g_0 \oplus g_1$ *in* V_f *we have*

$$\sigma_f(F, G) = \int (f_0 g_1 - f_1 g_0) dx = \sigma_a(F_t, G_t) + \sigma_L(F_l, G_l) + \sigma_M(F_m, G_m);$$

- ψ_T *maps* V_c *into elements with* $F_q = 0$, V_q *into elements with* $F_c = F_n = 0$ *and* V_e *into elements with* $F_c = 0$.

(ii) *the subspace* $N \subset V_b$ *is invariant under the action of* ψ_T, *i.e. for any* $F = (0, F_\infty) \in N$ *it holds* $F \mapsto (0,0) \oplus (0, F_\infty) \oplus (0,0)$, *and the non-trivial center of* $C^*(V_b, \sigma_b)$, *denoted* $\mathfrak{Z}_b$, *equals its relative commutant in* $C^*(V_f, \sigma_f)$ *and* $C^*(V_c, \sigma_c)$, *i.e. for* $k = c, f$ *we have*

$$\mathfrak{Z}_b = C^*(V_b, \sigma_b)^c := C^*(V_b, \sigma_b)' \cap C^*(V_k, \sigma_k).$$

Similarly for $C^*(V_e, \sigma_e)$ *instead of* $C^*(V_b, \sigma_b)$;

(iii) *for a different choice of an element* $T' \in V_f$ *that defines the symplectic isomorphism* $\psi_{T'}$, *there exists a symplectic isomorphism* $\varphi_{TT'}$ *between* $\psi_T(V_f)$ *and* $\psi_{T'}(V_f)$ *such that* $\psi_{T'} = \varphi_{TT'} \circ \psi_T$. *The isomorphism* $\varphi_{TT'}$ *reduces to the identity on* $C \oplus Q$ *iff* $T - T' \in V_b$, *i.e. iff* T *and* T' *have the same charge content. Moreover, it exponentiates to a Weyl algebra and C*-algebra isomorphism.*

Proof. The existence of the isomorphisms with the properties described in (i) and (ii) follows from easy calculations on the above definitions. The essential uniqueness up to the symplectic isomorphism in (iii) follows because $\varphi_{TT'}$ is defined from the function $T' - T \in V_f$, so that the property of invariance of the charges holds iff $T' - T$ has zero charge content. □

We will refer to the additive group $C \oplus Q$ as the *charge group* of the theory.

Remark 3.3. (1) ψ_T being non-splitting, its exponentiation Ψ_T gives a Weyl algebra $\mathbf{W}(\psi_T(V_f)) = \Psi_T(\mathcal{W}_f) \cong \mathcal{W}_f$ that is only properly contained in the algebraic tensor product $\mathcal{W}_a \otimes \mathcal{W}_L \otimes \mathcal{W}_M$. The associated C*-algebra is isomorphic to $C^*(V_f, \sigma_f)$ and only properly contained in the respective (maximal) tensor product as well. Hence, because of the results in Sec. 2, we have

$$C^*(V_f, \sigma_f) \cong ((C^*(V_a, \sigma_a) \otimes \mathfrak{Z}_b) \rtimes_{\sigma_L} \mathcal{U}(C)) \rtimes_{\sigma_M} \mathcal{U}(Q). \tag{3.21}$$

(2) Observe that the isomorphism ψ_∞ is introduced to quotient out from an element in V_b or V_e its degenerate part in N. Instead, the isomorphism ψ_T is tailored to extract from an element in V_f its charge content, i.e. to fix its equivalence class in the additive group $C \oplus Q$ of the charges, dividing out the remaining components in V_a, N and R, according to the choice of $T \in V_f$.

3.2. *Defining representations for the Streater and Wilde model*

We define a reference representation for each Weyl and C*-algebra that functorially derives from the six term diagram (3.4). These extend the Fock representation of the observable algebra and are non-regular representations that satisfy the properties of factorization indicated in Proposition 2.6. The twisted crossed product in (3.21), explains the observable and charge content of each of the algebras associated to the symplectic spaces in diagram (3.4).

We begin by noticing that Eq. (3.21) gives an interpretation of the non-regular representations introduced for different models of this kind in [1,2]. In these papers, the representation of $\mathcal{W}(V_f, \sigma_f)$ is obtained as the GNS representation associated to the non-regular state

$$\omega(W_f(F)) = \exp\left\{-\frac{1}{4}q(F)\right\}, \quad F \in V_f, \tag{3.22}$$

where $q(F)$ is a so called *generalized quadratic form*, defined by

$$q(F) = \begin{cases} \int (\omega^{-1}|\tilde{f}_0|^2 + \omega|\tilde{f}_1|^2)dp & \text{if } F = f_0 \oplus f_1 \in V_a, \\ +\infty & \text{if } F \notin V_a. \end{cases} \tag{3.23}$$

If ω_a is the Fock state on $\mathcal{W}_a$ and for two symplectic spaces with supports $L \cong M \cong \mathbb{R}^2_d$ we take two copies ω_L and ω_M of the non-regular elementary state introduced in Sec. 2.3, respectively, on the algebras $\mathcal{W}_L$ and $\mathcal{W}_M$, similarly to Eq. (3.22) we give the following:

Definition 3.4. For given $T \in V_f$ a charge regularizing element as in Sec. 3.1 and notation as in the previous Proposition 3.2, and for δ being the Kronecker delta, we define a state on the Weyl algebra $\mathcal{W}_f$ by

$$\begin{aligned} \omega_f(W_f(F)) &:= \exp\left\{-\frac{1}{4}\|T_a(F_t)\|^2_{H_a}\right\}\delta_{F_c,0}\delta_{F_q,0} \\ &= \omega_a(W_a(F_t))\delta_{F_c,0}\delta_{F_q,0} \\ &= \omega_a(W_a(F_t))\omega_L(W_L(F_L))\omega_M(W_M(F_M)). \end{aligned}$$

Observe that the state in Definition 3.4 differs from the one in Eq. (3.22) only in restriction to elements $W_b(F) \in \mathcal{W}_b$, with $F \in V_b$ and $F_\infty \neq 0$: the GNS representation of the state ω_f, unlike that of ω, maps them in elements different from the identity, as seen in Sec. 2.3 for the representation π_L. This means that ω_f, unlike ω, results to be faithful on the Weyl elements obtained from the subspace N, hence it is preferred in the sequel for obtaining the physical content also for this subspace.

For different regularizing elements, the above definition gives different states, whose GNS representations are actually related by the following

Proposition 3.5. (i) *The GNS representations π_f, associated to the states ω_f for different choices $T_1, T_2 \in V_f$ of the non vanishing charge regularizing element in Definition* 3.4, *are unitary equivalent. The unitary operator on $\mathcal{H}_f$ giving this equivalence by adjoint action is $\pi_f(W(F))$ for $F = T_1 - T_2$; moreover, such a $F \in V_b$ if the charges of T_1 and T_2 are the same*;

(ii) *the representation π_f is unitary equivalent to the representations $\pi_a \otimes \pi_L \otimes \pi_M \restriction \mathcal{W}_f$ and, denoting by $\mathcal{H}_{(c,q)} \cong \mathcal{H}_a$ the Hilbert space at fixed charges $(c, q) \in C \oplus Q$ and $\pi_{(c,q)} \cong \pi_a$ the relative representation, also to the representation $\oplus_{(c,q)\in C\oplus Q}\pi_{(c,q)}$. π_f acts on the Hilbert space*

$$\mathcal{H}_f \cong \mathcal{H}_a \otimes \mathcal{H}_L \otimes \mathcal{H}_M \cong \mathcal{H}_a \otimes \ell^2(C) \otimes \ell^2(Q) \cong \bigoplus_{(c,q)\in C\oplus Q} \mathcal{H}_{(c,q)}. \tag{3.24}$$

Equivalently, we have $\pi_b = \pi_c \cong \pi_a \otimes \pi_L$, $\pi_q \cong \pi_a \otimes \pi_M$ and $\pi_e \cong \pi_f$, where it is meant $\pi_b \cong (\pi_a \otimes \pi_L)\restriction_{\mathcal{W}_b}$ an similar. The representations π_a and π_b are the only regular ones;

(iii) *the GNS vector $\Omega_f \in \mathcal{H}_f$ of the state ω_f is the vector $|0, 0\rangle \in \mathcal{H}_{(0,0)} \subset \mathcal{H}_f$. It is cyclic for $\pi_f(\mathcal{W}_f)$ and is identified by the isomorphisms in Eq.* (3.24) *with the vectors $\Omega_a \otimes \Omega_L \otimes \Omega_M$ and $\Omega_a \otimes \delta_0 \otimes \delta_0$ respectively. Similarly, in the other cases, Ω_b and Ω_e not being cyclic for $\mathcal{W}_b$ and $\mathcal{W}_e$, respectively.*

Proof. (i) The equivalence for different T is a consequence of Proposition 3.2, because of the isomorphism of the Weyl and C*-algebras and because the states ω_f satisfies the factorization condition (2.28) in Proposition 2.6.

(ii) It suffices to observe that the state ω_f reduces to the state ω_a on $\mathcal{W}_a$ and that if $F \in V_f$ with at least a non vanishing charge, then $\omega_f(W(F)) = 0$ because of the Kronecker deltas.

(iii) it is trivial. □

4. Local Theory, DHR Sectors and Gauge Group

In this section we treat the net approach of AQFT to the Streater and Wilde model. As general reference we use [26, 48], and for the low dimensional theories also [34].

Let M be the 1+1-dimensional Minkowski spacetime and consider $\mathbb{R}$ as its time zero axis. We recall some notation and fix new concerning Open (M) and Open $(\mathbb{R})$, the set of open non-void subsets of M and $\mathbb{R}$, respectively, partially ordered by inclusion.

A causal disjointness relation $\perp$ is induced on Open (M) from the Minkowski metric. A triple consisting of a subset $\mathcal{P} \subset$ Open $(\mathbb{R})$, the inclusion partial order $\subset$ and the disjointness relation $\perp$ will be called an *index set* and shortly indicated as $(\mathcal{P}, \subset, \perp)$. If $P \in \mathcal{P}$, we denote by $P^\perp$ the *causally disjoint set* of P, i.e. the set of the elements of $\mathcal{P}$ disjointed from P.

On the Minkowski spacetime the preferred index set is the set of causally complete bounded connected regions, indicated by $\mathcal{K}$ and called the set of *open double* *cones on* M and defined as follows: for given $x, y \in M$ with $y \in V_x^+ \subset M$, the *open future cone of* x and for $V_y^- \subset M$ the *open past cone* of y, a *double cone* $\mathcal{O} \in \mathcal{K}$ is defined by the intersection of $V_x^+ \cap V_y^-$.

The relation $\perp$ on M reduces to the set theoretic disjointness relation on the time zero line $\mathbb{R}$, i.e. for $A, B \in \mathrm{Open}\,(\mathbb{R}) A \perp B \Leftrightarrow A \cap B = \emptyset$. The index set we work with in this paper, will only be the following:

$$\mathcal{I} := \{\text{non-empty, open, bounded intervals of } \mathbb{R}\}. \tag{4.1}$$

A generic abstract *net* $\mathbf{N}$ *on the index set* $\mathcal{P}$ is defined by an inclusion preserving map

$$\mathbf{N} : P \mapsto \mathbf{N}(P), \quad P \in \mathcal{P}. \tag{4.2}$$

A net hence takes its image elements in the objects of a generic category, also furnished with the structures of partial order and disjointness. To distinguish nets defined on different index set and with the same image category, we use the notation $\mathbf{N}_{\mathcal{P}}$ and define *a maximal element* of the net $\mathbf{N}_{\mathcal{P}}$ by $\mathbf{N}^{\mathcal{P}}(M) := \bigvee_{P \in \mathcal{P}} \mathbf{N}_{\mathcal{P}}(P)$, where the symbol $\bigvee$ has to be properly understood relative to the category of the image elements.

A relevant property of a net is *locality*, i.e. $\mathbf{N}(P_1) \perp \mathbf{N}(P_2)$ for $P_1, P_2 \in \mathcal{P}$ and $P_1 \perp P_2$; it distinguishes Bosonic (i.e. local) from non Bosonic nets.

For given a symplectic space (V, σ_V) and an appropriate notion of *localization* of an element of V in an element of an index set $\mathcal{P}$, it remains naturally defined a category of symplectic subspaces with order and disjointness relation $(\mathrm{Sub}(V(\mathcal{P})), \subset, \perp_{\sigma_V})$. The partial order is inherited from the one of $\mathcal{P}$, by the defined localization, and the disjointness of two symplectic subspaces V_1 and V_2 in this category means that $V_2 \in V_1^{\perp_{\sigma_V}}$, i.e. $\sigma(v_1, v_2) = 0$ for any element $v_1 \in V_1$ and $v_2 \in V_2$. Hence $(\mathrm{Sub}(V(\mathcal{P})), \subset, \perp_{\sigma_V})$ is a first example of the image category of a net that, accordingly to (4.2), we indicate by $V_{\mathcal{P}}$ and call the net of the symplectic subspaces of the symplectic space (V, σ_V) on the index set $(\mathcal{P}, \perp)$. Notice that the notion of localization of the elements of V on $\mathcal{P}$, also accounts for the locality of $V_{\mathcal{P}}$.

The Weyl functor preserves the net structure on $\mathcal{P}$, so that a net of Weyl algebra $\mathcal{W}_{\mathcal{P}} : P \mapsto \mathcal{W}(V(P), \sigma_V)$ and a net of C*-algebra $\mathfrak{N}_{\mathcal{P}}$ are also defined, by

$$\mathfrak{N}_{\mathcal{P}} : P \mapsto \mathfrak{N}(P) := \mathcal{W}((V(P), \sigma))^-, \quad P \in \mathcal{P}. \tag{4.3}$$

Here the completion may be intended with respect to the minimal regular norm.

Once fixed a defining representation π_n, a net $\mathcal{N}_{\mathcal{P}}$ of von Neumann algebras is canonically associated to a Weyl algebra net, by

$$\mathcal{N}_{\mathcal{P}} : P \mapsto \mathcal{N}(P) := \pi_n(\mathcal{W}(V(P), \sigma))'', \quad P \in \mathcal{P}. \tag{4.4}$$

If $A \in \mathrm{Open}(\mathbb{R})$ then $\mathfrak{N}^{\mathcal{P}}(A)$ is the C*-algebra generated by the von Neumann algebras $\mathcal{N}_{\mathcal{P}}(P), P \subset A, P \in \mathcal{P}$, so that we may define $\mathcal{N}^{\mathcal{P}}(A) := \mathfrak{N}^{\mathcal{P}}(A)''$.

Hence, $\mathfrak{N}^{\mathcal{P}}(A)$ and $\mathcal{N}^{\mathcal{P}}(A)$ are the C* and von Neumann algebras generated by additivity on the index set $\mathcal{P}$.[h]

In particular, $\mathfrak{N}^{\mathcal{I}}(\mathbb{R})$ is the C*-algebra generated by all the local von Neumann algebras $\mathcal{N}_{\mathcal{I}}(I)$, for $I \in \mathcal{I}$. It is called the *C*-algebra of quasi local elements* of the net $\mathcal{N}_{\mathcal{I}}$ and is the C*-algebra referred to in studying the DHR superselection sectors of the net, with respect to the index set $\mathcal{I}$. The net and this C*-algebra are usually both indicated by the same symbol $\mathcal{N}$.

4.1. *Nets for the Streater and Wilde model*

To define useful nets for the study of the Streater and Wilde model, a proper definition of localization is now in order: for $j = a, b, c, q, e, f$ we define the localization of a generic element $F = f_0 \oplus f_1 \in V_j$ as

$$\operatorname{loc} F := \operatorname{supp} f_0 \cup \operatorname{supp} \partial f_1, \quad f_0 \in \mathcal{S}, \quad f_1 \in \partial^{-1}\mathcal{S}. \tag{4.5}$$

According to this definition, the various nets of symplectic subspaces $V_{j,\mathcal{P}} \subseteq V_{f,\mathcal{P}}$ with range category $(\mathrm{Sub}(V_j(\mathcal{P})), \subset, \perp_{\sigma_j})$ are defined: explicitly $V_{j,\mathcal{I}}$ is given by

$$V_{j\mathcal{I}} : I \in \mathcal{I} \mapsto V_{j\mathcal{I}}(I) := \{F \in V_j : \operatorname{loc} F \subset I\}, \quad I \in \mathcal{I}.$$

Observe that the elements $F = 0 \oplus n \in V_f$, with $n \in N \cong \mathbb{R}_d$, have vanishing localization Weyl unitaries according to the definition in Eq. (4.5), so that they may be thought as localized in any interval of the time zero line.

A possible first result for the model, is a complete *simple current extension* characterization of the nets of von Neumann algebras, derived from the nets of symplectic subspaces in the six term inclusion of symplectic spaces in diagram (3.4), in the representations defined in Sec. 3. To obtain such a result, we show the existence of *local* symplectic isomorphisms obtained from a natural local choice of the regularizing element $T \in V_f$. Such a characterization will be used in Sec. 4.4 to discuss the existence of the relative gauge simmetry groups.

The following result collects the local properties of the symplectic isomorphism ψ_T, according to the relative position of $\operatorname{loc} T$ and of the localized algebras. Observe that for $T = \partial t \oplus t$ we have $\operatorname{loc} T = \operatorname{supp} \partial t$. We omit for brevity the reference to the symplectic form of the symplectic subspaces involved, and always suppose that the nets are defined on a fixed index set $\mathcal{P}$.

Proposition 4.1. *Let $P \subset \mathbb{R}$ be a generic non-void open subset of the time zero real line and ψ_T be a symplectic isomorphism defined as in Proposition* 3.2 *for $T = \partial t \oplus t \in V_f(\operatorname{loc} T)$. Letting $v = F \oplus l \oplus m = (f_0 \oplus f_1) \oplus (c \oplus n) \oplus (r \oplus q)$ be the*

[h] For example, the additivity property for the von Neumann algebras net $\mathcal{N}_{\mathcal{P}}$ means that $\mathcal{N}(A) = \vee_{P \subset A} \mathcal{N}_{\mathcal{P}}(P)$. For the general properties of abstract nets we refer to [48], and to [14] for further discussions.

generic element in $V_a \oplus L \oplus M$, *we have*:

(i) *if* $\operatorname{loc} T \perp P$ *then* $\psi_T(V_a(P)) = V_a(P) \oplus 0 \oplus 0$; *for a generic* P *and* $\operatorname{loc} T$ *it holds*

$$\psi_T(V_b(P)) = \psi_T(V_a(P)) \oplus N,$$
$$\psi_T(V_e(P)) = \psi_T(V_q(P)) \oplus N;$$

(ii) *if* $\operatorname{loc} T \subset P$ *then*

$$\begin{aligned}
\psi_T(V_c(P)) &= \psi_T(V_b(P)) \oplus C = \psi_T(V_a(P)) \oplus (C \oplus N),\\
\psi_T(V_q(P)) &= \psi_T(V_a(P)) \oplus Q,\\
\psi_T(V_e(P)) &= \psi_T(V_b(P)) \oplus Q = \psi_T(V_q(P)) \oplus N\\
&= \psi_T(V_a(P)) \oplus N \oplus Q,\\
\psi_T(V_f(P)) &= \psi_T(V_c(P)) \oplus Q = \psi_T(V_e(P)) \oplus C\\
&= \psi_T(V_a(P)) \oplus (C \oplus N) \oplus Q.
\end{aligned}$$

Proof. (i) The case $\operatorname{loc} T \perp P$ is trivial. If $\operatorname{loc} T$ and P are not disjoint, then

$$\psi_T(V_a(P)) = \left\{ (f_0, f_1) \oplus \left(0, \int_{\mathbb{R}} f_1 \partial t dx\right) \oplus \left(\int_{\mathbb{R}} f_0 t dx, 0\right) \in V_a \oplus L \oplus M : \right.$$
$$\left. f_0 \in \partial \mathcal{S}(P), f_1 \in \mathcal{S}(P) \right\}$$

is a proper subset of $V_a \oplus N \oplus R$, that does not split into a symplectic sum. Moreover

$$\begin{aligned}
\psi_T(V_b(P)) &= \left\{ (f_0, f_1) \oplus (0, n) \oplus \left(0, \int_{\mathbb{R}} f_0 t dx\right) \in V_a \oplus L \oplus M : \right.\\
&\qquad \left. f_0 \in \partial \mathcal{S}(P), f_1 \in \mathcal{S}(P) \right\}\\
&= \psi_T(V_a(P)) \oplus N.
\end{aligned}$$

The decompositions for $\psi_T(V_q(P))$ and $\psi_T(V_e(P))$ are in the proof of (ii) below.

(ii) The definition of ψ_T as in Proposition 3.2 for generic $\operatorname{loc} T$ and domain P gives

$$\begin{aligned}
\psi_T(V_f(P)) = \Big\{ & (f_0, f_1) \oplus (c, n) \oplus \left(\int_{\mathbb{R}} f_0 t dx, q\right) \in V_a \oplus L \oplus M :\\
& f_0 \in \partial \mathcal{S}(P \cup \operatorname{loc} T), f_0 + c\partial t \in \mathcal{S}(P),\\
& f_1 \in \mathcal{S}(P \cup \operatorname{loc} T), f_1 + qt \in \partial^{-1} \mathcal{S}(P) \Big\}
\end{aligned}$$

and similarly for the other local symplectic subspaces. Taking $\operatorname{loc} T \subset P$, the results follow. □

Observe that whatever $\operatorname{loc} T$ and P are, with $\operatorname{loc} T \perp P$, the local algebra $\mathcal{A}(V_a(P))$ defined as in Eq. (4.4) satisfies

$$\mathcal{A}(V_a(P)) \cong \pi_a \otimes \pi_L \otimes \pi_M(\mathbf{W}(\psi_T(V_a(P))))'' = \mathcal{A}(V_a(P)) \otimes I_L \otimes I_M. \tag{4.6}$$

Similarly we define from the symplectic local subspaces of V_b, V_c, V_q, V_e and V_f the nets $\mathcal{B}, \mathcal{C}, \mathcal{Q}, \mathcal{E}$ and $\mathcal{F}$, respectively. Remembering that we also indicated by C and Q the additive groups of the charges, we denote without ambiguity by $\mathcal{U}(C)$ and $\mathcal{U}(Q)$ their unitary Weyl representations on $\mathcal{H}_L$ and $\mathcal{H}_M$, respectively.

Recalling that in Remark 3.3, for chosen $T \in V_f$ we denoted by Ψ_T the isomorphism at the Weyl algebra level obtained from the symplectic space isomorphism ψ_T by the Weyl functor, we use the same symbol Ψ_T for its extension to the von Neumann algebras, so that from the representation π_f to the representation $\pi_a \otimes \pi_L \otimes \pi_M$, it holds $\mathcal{A}(V_a(P)) \otimes I_L \otimes I_M = \Psi_T(\mathcal{A}(V_a(P)))$, for every $T \in V_f$.[i]

Hence, denoting the Abelian algebra of the symplectic subspace N in representation π_b by

$$\mathcal{Z}_b := \pi_b(\mathcal{W}_b(N))'' \cong \pi_L(\mathcal{W}_L(N))'', \tag{4.7}$$

the above proposition and the isomorphism Ψ_T, give the following characterization of the nets involved, as simple current extensions from the net of observables:

Proposition 4.2. *For given* $P \subseteq \mathbb{R}$ *a generic non-void subset of the line and for notations as above we have*:

(i) *for generic* P *and* $\operatorname{loc} T$:

$$\mathcal{B}(V_b(P)) = \mathcal{A}(V_a(P)) \otimes \mathcal{Z}_b,$$
$$\mathcal{E}(V_e(P)) = \mathcal{Q}(V_q(P)) \otimes \mathcal{Z}_b;$$

(ii) *if* $\operatorname{loc} T \subset I \in \mathcal{I}$ *then*

$$\begin{array}{ccccc}
 & & \mathcal{B}(I) & & \mathcal{C}(I) \\
 & & \| & & \| \\
\mathcal{A}(I) & \subset & \mathcal{A}(I) \otimes \mathcal{Z}_b & \subset & (\mathcal{A}(I) \otimes \mathcal{Z}_b) \rtimes_{\sigma_L} \mathcal{U}(C) \\
\cap & & \cap & & \cap \\
\mathcal{A}(I) \rtimes_{\sigma_M} \mathcal{U}(Q) & \subset & (\mathcal{A}(I) \otimes \mathcal{Z}_b) \rtimes_{\sigma_M} \mathcal{U}(Q) & \subset & (\mathcal{A}(I) \otimes \mathcal{Z}_b) \rtimes_{\sigma_M} \mathcal{U}(Q) \rtimes_{\sigma_L} \mathcal{U}(C) \\
\| & & \| & & \| \\
\mathcal{Q}(I) & & \mathcal{E}(I) & & \mathcal{F}(I).
\end{array} \tag{4.8}$$

Proof. From the Proposition 4.1, by functoriality and the splitting of the representation. □

[i]Such an isomorphism Ψ_T turns out to be weakly continuous only for the closed linear subspaces of the von Neumann algebras with fixed charge, see [14] for detail.

4.2. *Chiral versus time zero formulation*

We introduce in the sequel the chiral formulation of the scalar massless free field nets: this allows one to relate the time zero approach to the Streater and Wilde original one and to the conformal chiral one given by Buchholz, Mack and Todorov in [10].

To fix notation, let $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_{\mathcal{K}}$ be the observable net of the Streater and Wilde model on the index set of the double cones $\mathcal{K}$ of the 1+1-dimensional Minkowski spacetime M.

If $I \in \mathcal{I}$ is a time zero open interval and $\mathcal{O} = I'' \in \mathcal{K}$ is the open double cone it generates by causal completion, i.e. the interior of the causal closure of the interval I, the net $\widetilde{\mathcal{A}}$ has time zero generating net $\mathcal{A}_{\mathcal{I}}$, i.e. it holds $\widetilde{\mathcal{A}}(\mathcal{O}) = \mathcal{A}_{\mathcal{I}}(I)$.

A wherever based double cone, is described as $\mathcal{O} = I_+ \times I_-$, for $I_\pm \in \mathcal{I}_\pm$, the index set of the open bounded intervals on the right/left light ray lines, see e.g. [44].

The Streater and Wilde original model in [51] is formulated in the left/right chiral fields formalism and, for every chiral component, it gives a family labeled by $\mathbb{R}_d$ of DHR automorphisms for the net $\widetilde{\mathcal{A}}$. Disliking this, we preferred to write the symplectic spaces and nets in a time zero formulation and then, using the symplectic isomorphism ψ_T defined in Proposition 3.2, we extracted the charge content. To relate the two approaches, and to discuss the geometric covariance properties, we may use a second symplectic isomorphism constructed as in the sequel.

In the chiral approach, the symplectic spaces are given by the smearing test functions of the left/right mover solutions $\theta_\pm$ of the classical wave equation, i.e. $(\partial_t \pm \partial_x)\theta_\pm = 0$.

To consider the charge carrying fields, we have to restrict to the functions $\theta_\pm(x) \in C^\infty_{\mathbb{R}}(\mathbb{R})$ with $\partial\theta_\pm \in \mathcal{S}$ and a general solution of the wave equation can be written as[j]

$$\Theta(x,t) := \theta_-(x-t) + \theta_+(x+t).$$

To give a symplectic isomorphism between the two formulation, we start from a direct application of the d'Alambert formula: if $F = f_0 \oplus f_1 \in V_f$, we have

$$\Theta(x,t) = \frac{1}{2}\left[f_1(x+t) + f_1(x-t) + \int_{x-t}^{x+t} f_0(y)dy\right]. \tag{4.9}$$

Denoted by $V := \{\theta \in C^\infty_{\mathbb{R}}(\mathbb{R}) : \partial\theta \in \mathcal{S}\}$ a chiral symplectic space support, and by

$$\sigma_\pm(\theta,\varphi) = \pm\int_{\mathbb{R}} (\varphi\partial\theta - \theta\partial\varphi)dx$$

two symplectic forms on V, we can hence define the *left/right chiral symplectic spaces of the fields* by $V_+ := (V, \sigma_+)$ and $V_- := (V, \sigma_-)$.

[j]Actually in Streater's and Wilde's original formulation the unnecessary requirement $\lim_{y\to-\infty}\theta_\pm(y) = 0$ is made. We omit such a choice that does not reveal the presence of $\mathcal{Z}_b$.

Consider now the isomorphism ψ of real linear spaces defined by

$$\begin{aligned} \psi : V_f &\to \psi(V_f) = V \oplus V \\ F = f_0 \oplus f_1 &\mapsto \psi(F) := \Theta^F := \theta_+^F \oplus \theta_-^F, \end{aligned} \tag{4.10}$$

where the direct and inverse transformations are explicitly given by

$$\begin{cases} \theta_+^F(x) = \dfrac{1}{2}\left\{ f_1(x) + \displaystyle\int_{-\infty}^x f_0(y)dy \right\} \\ \theta_-^F(x) = \dfrac{1}{2}\left\{ f_1(x) - \displaystyle\int_{-\infty}^x f_0(y)dy \right\} \end{cases}, \quad \begin{cases} f_0(x) = \partial(\theta_+^F(x) - \theta_-^F(x)) \\ f_1(x) = \theta_+^F(x) + \theta_-^F(x) \end{cases}. \tag{4.11}$$

Define moreover for every $\Theta = (\theta_+, \theta_-) \in V \oplus V$ the two real constants

$$\Theta(+\infty) := (\theta_+(+\infty), \theta_-(+\infty)) \in \mathbb{R}_d^2$$

where

$$\theta_\pm(+\infty) := \lim_{x\to+\infty} \theta_\pm(x). \tag{4.12}$$

Observe that for $F \in V_f$ and $\Theta = \psi(F)$, this pair of constants is given by

$$\theta_\pm^F(+\infty) = \lim_{x\to+\infty} \frac{1}{2}\left[f_1(x) \pm \int_{-\infty}^x f_0(y)dy \right]. \tag{4.13}$$

Introducing the (continuous) real valued $\mathbb{R}$-linear form $\alpha : \mathbb{R}^2 \to \mathbb{R}$ such that $\alpha(a,b) = ab$, we may denote by σ_∞ the usual symplectic form on $\mathbb{R}^2$, defined by

$$\sigma_\infty((a_1,b_1),(a_2,b_2)) = \alpha(a_1b_2) - \alpha(b_1a_2) = a_1b_2 - b_1a_2, \tag{4.14}$$

and finally state the following

Lemma 4.3. *There exists a non-splitting symplectic isomorphism ψ_∞, equating the isomorphism ψ of Eq.* (4.10) *as a real linear space isomorphism, defined by*

$$\begin{aligned} \psi_\infty : (V_f, \sigma_f) &\to \psi_\infty((V_f,\sigma_f)) = (V_+ \oplus V_-, \sigma_+ + \sigma_- + \sigma_\infty) \\ F = f_0 \oplus f_1 &\mapsto \Theta^F := (\theta_-^F, \theta_-^F), \end{aligned} \tag{4.15}$$

such that for the symplectic forms it holds

$$\sigma_f(F,G) = \sigma_+(\theta_+^F, \theta_+^G) + \sigma_-(\theta_-^F, \theta_-^G) + \sigma_\infty(\Theta^F(+\infty), \Theta^G(+\infty)), \quad F, G \in V_f.$$

The corresponding Weyl algebras isomorphism is

$$\Psi_\infty : \mathcal{W}(V_f, \sigma_f) \to \Psi_\infty(\mathcal{W}(V_f,\sigma_f)) = \mathcal{W}(V_+, \sigma_+) \rtimes_{z_-} \mathcal{U}(V_-) \tag{4.16}$$

where in the 2*-cocycle $z_- = (\beta, y)$, the action β is defined by the α appearing in Eq.* (4.14) *and the $\mathbb{T}$-valued function y is defined by the symplectic form σ_-.*

Proof. The proof it trivial, it is enough to reefer to the general case treated in Sec. 2.1, in particular Eqs. (2.17) and (2.18) for the definition of z_-. Observe that the part σ_∞ in the symplectic form, corresponds to the interacting part of σ_f (actually denoted by σ_{V_+,V_-} according to the notation of Sec. 2.1). $\square$

Various observations are now in order:

- *Picking a point at infinity* $\infty \in \mathbb{R} \cup \{\pm\infty\}$ in the limits of the integrals appearing in Eq. (4.11) is necessary in order to define ψ_∞ and σ_∞. The choice $\infty = -\infty$ gives $\theta_-^F(-\infty) = \theta_+^F(-\infty) = \frac{1}{2} f_1(-\infty)$, so that the *interacting part* σ_∞ of the symplectic form depends only on the limit value of the functions at $+\infty$.
- The space $\psi_\infty(V_f)$ is not a direct sum because ψ_∞ is not splitting. As a consequence, the Weyl algebra $\Psi_\infty(\mathcal{W}(V_f, \sigma_f))$ is not a tensor product but a twisted crossed product of Weyl algebras. Changing the role of V_+ and V_- in Eq. (4.16) we also have $\mathcal{W}(V_f, \sigma_f) \cong \mathcal{W}(V_-, \sigma_-) \rtimes_{z_+} \mathcal{U}(V_+)$, where now the 2-cocycle z_+ is given in terms of α and σ_+. To make evident the symmetry behind this construction, and its physical significance, we use the following notation
$$\mathcal{W}(V_f, \sigma_f) \cong \mathcal{W}(V_+, \sigma_+) \underset{\infty}{\bowtie} \mathcal{W}(V_-, \sigma_-), \tag{4.17}$$
where the symbol $\underset{\infty}{\bowtie}$ accounts for the *twisted interaction* operated by σ_∞ between the two chiral field algebras at the chosen point ∞.
- For $\mathcal{S}$, the Schwartz space of functions on the chiral line and $V_a \cong \mathcal{S}$, we define the two *chiral symplectic space of observables* as $V_{a_\pm} := (V_a, \sigma_\pm) \subset (V_\pm, \sigma_\pm)$. Choosing $\infty = -\infty$ we have $\Theta(+\infty) = (0,0) \in \mathbb{R}_d$, for any $\Theta \in V_{a+} \oplus V_{a-}$. Hence σ_∞ vanishes, independently of the choice of ∞ we made, when at least one of its arguments is in V_{a_+} or V_{a_-}. We may interpret this as and independence relation, given by the vanishing of the commutator at ∞ between any field of one chirality and any observable of the other chirality.
- The isomorphism ψ_∞ splits when restricted to the observable symplectic subspace, i.e. it hold $\psi_\infty(V_a, \sigma_a) = (V_{a_+}, \sigma_+) \oplus (V_{a_-}, \sigma_-)$, and we have
$$\mathcal{W}(V_a, \sigma_a) \cong \mathcal{W}(V_{a_+}, \sigma_+) \otimes \mathcal{W}(V_{a_-}, \sigma_-).$$

The (vacuum) states of the algebras $\mathcal{W}(V_{a_\pm}, \sigma_\pm)$, i.e. the quasi-free state on the chiral observables algebras $\mathcal{W}(V_{a_\pm}, \sigma_\pm)$, are defined for $\theta \in V_{a_\pm}$ and $T_{a_\pm}(\theta)(p) := |p|^{\frac{1}{2}}\tilde{\theta}(p)$ by
$$\omega_{a_\pm}(W(\theta)) := \exp\left\{-\frac{1}{2}\|T_{a_\pm}(\theta)\|^2_{H_{a_\pm}}\right\} = \exp\left\{-\frac{1}{2}\int_{\mathbb{R}} |p||\tilde{\theta}|^2 dp\right\}. \tag{4.18}$$

Similarly to the time zero formulation, the GNS construction from these states gives Fock space representation on the Hilbert spaces $\mathcal{H}_{a_\pm}$, with chiral one particle spaces $H_{a_\pm} := T_{a_\pm}(V_{a_\pm})^-$, and GNS (vacuum) vector $\Omega_{a_\pm}$, see, for example, [10] for details. From these representations of the chiral observable algebras, we can obtain the non-regular representations of the chiral field algebras, using the regularizing elements tool of Sec. 3.1 as follows.

Initially we observe that a simple relation between the charges carried by a field in the two formulations is also obtained from the d'Alambert formula (4.9). The charges carried by $\Theta = (\theta_+, \theta_-) \in V_+ \oplus V_-$ are the pair $(c_+, c_-) \in C_+ \oplus C_- \cong \mathbb{R}_d \oplus \mathbb{R}_d$ where

$$c_\pm := \lim_{y\to+\infty} \theta_\pm(y) - \lim_{y\to-\infty} \theta_\pm(y). \tag{4.19}$$

If $F \in V_f$ and if $c_\pm$ are the charges of $\psi_\infty(F) = \Theta^F$, we have

$$F_c = c_+ - c_- \quad \text{and} \quad F_q = c_+ + c_-. \tag{4.20}$$

In particular we have that $(F_c, F_q) = (0,0)$ iff $(c_+, c_-) = (0,0)$.

Consider now two copies of the elementary symplectic space, defined in Sec. 2.3, that we denote by $(L_\pm, \sigma_L)$, with $L_\pm = C_\pm \oplus N_\pm \cong \mathbb{R}_d^2$. Let moreover $\omega_{L_\pm}$ be the non-regular states on the Weyl algebras $\mathcal{W}(L_\pm, \sigma_L)$ respectively, defined as in Eq. (2.34) by $\omega_{L_\pm}(W((c_\pm, n_\pm)) = \delta_{c_\pm,0}$, for $(c_\pm, n_\pm) \in L_\pm$ and δ the Kronecker delta.

A regularizing element for the left field movers (similarly for the right movers case), is a non vanishing charge element S_+, i.e. $S \in V_+/V_{a_+}$. We can choose it such that $\int_\mathbb{R} S_+ \partial S_+ dx = 0$ for simplicity. Then, as in Proposition 3.2, there exist a regularizing isomorphism

$$\begin{aligned} \psi_{S_+} : (V_+, \sigma_+) \to \psi_{S_+}((V_+, \sigma_+)) \subsetneq (V_{a+}, \sigma_+) \oplus (L_+, \sigma_L) \\ \theta \mapsto \theta - c_+ S_+ \oplus (c_+, n_+) \end{aligned} \tag{4.21}$$

where the charge $c_+ = \lim_{y\to+\infty} \theta(y) - \lim_{y\to-\infty} \theta(y)$ and $n_+ := \int_\mathbb{R} (\theta \partial S_+ - S_+ \partial\theta) dx$. Observe that $c_+ = 0$ if $\theta \in V_{a_+} \subset V_+$. Similarly, we obtain $(c_-, n_-) \in L_-$ and, as in Proposition 2.6, it is easy to see that two non-regular states $\omega_\pm$ on $\mathcal{W}(V_\pm, \sigma_\pm)$ respectively, are defined by

$$\omega_\pm(W(\theta)) = \omega_{a_\pm}(W(\theta - c_\pm S_\pm))\omega_{L_\pm}((c_\pm, n_\pm)), \quad \theta \in V_\pm, \tag{4.22}$$

with non-separable GNS representation spaces $\mathcal{H}_\pm \cong \mathcal{H}_{a_\pm} \otimes \mathcal{H}_{L_\pm}$, respectively.

A state for the algebra $\mathcal{W}(V_+, \sigma_+) \underset{\infty}{\bowtie} \mathcal{W}(V_-, \sigma_-)$ is defined by

$$\omega_\infty(W(\Theta)) := \omega_{a_+}(W(\theta_+ - c_+ S_+))\omega_{a_-}(W(\theta_- - c_- S_-))\delta_{c_+,0}\delta_{c_-,0}, \tag{4.23}$$

for the element $W(\Theta) \in \mathcal{W}(V_+, \sigma_+) \underset{\infty}{\bowtie} \mathcal{W}(V_+, \sigma_+)$ associated to the pair $\Theta = \theta_+ \oplus \theta_- \in V_+ \oplus V_-$, with charges $(c_+, c_-) \in C_+ \oplus C_-$. Observe that the definition of the state ω_∞ above is well posed, in fact once we write the obvious algebraic isomorphism

$$\mathcal{W}(V_+, \sigma_+) \underset{\infty}{\bowtie} \mathcal{W}(V_-, \sigma_-) \cong \mathcal{W}(V_{a_+} \oplus N_+, \sigma_+) \otimes \mathcal{W}(V_{a_-} \oplus N_-, \sigma_-) \rtimes \mathcal{U}(C_+ \oplus C_-),$$

the condition (2.28) of Proposition 2.6 is satisfied because for the state $\omega_C = \omega_{L_+ \oplus L_-} \restriction_{C_+ \oplus C_-}$ it holds $\omega_C(W(c_+, c_-)) = \delta_{(0,0),(c_+,c_-)}$.

Moreover, the choice of the point at infinity ∞ (note that the values of the functions at the point at infinity are not involved in the definition of ω_∞) and of the regularizing elements $S_\pm \in V_\pm$, defines the state ω_∞ uniquely, up to isomorphism, as in point (i) of Proposition 3.5.

From the GNS representations $\pi_\pm := (\pi_{a_\pm} \otimes \pi_{L_\pm}) \circ \psi_{S_\pm}$ of the Weyl algebras $\mathcal{W}(V_\pm, \sigma_\pm)$, respectively, we obtain for the non-regular GNS representation of the state ω_∞:

$$\pi_\infty \cong \pi_+ \otimes \pi_- := ((\pi_{a_+} \otimes \pi_{L_+}) \circ \psi_{S_+}) \otimes ((\pi_{a_-} \otimes \pi_{L_-}) \circ \psi_{S_-}).$$

For the Hilbert spaces $\mathcal{H}_\infty$ of this representation, it holds $\mathcal{H}_\infty \cong \mathcal{H}_+ \otimes \mathcal{H}_- \cong \mathcal{H}_f$, i.e. π_∞ is unitarily equivalent to the representation π_f. In fact, for $F = (f_0 \oplus f_1) \in V_a$ we have

$$\begin{aligned}\|T_a(F)\|^2_{H_a} &= \int_{\mathbb{R}} (|p|^{-1}|\tilde{f}_0|^2 + |p||\tilde{f}_1|^2)dp = \int_{\mathbb{R}} 2|p||\widetilde{\theta^F_+}|^2 + 2|p||\widetilde{\theta^F_-}|^2 dp \\ &= 2\|T_{a_+}(\theta^F_+)\|^2_{H_{a_+}} + 2\|T_{a_-}(\theta^F_-)\|^2_{H_{a_-}}. \end{aligned} \tag{4.24}$$

We pass now to define the chiral and 1+1-dimensional nets of the fields and their relation with the time zero one.

For a time zero based double cone $\mathcal{O} = I'' = I_+ \times I_-$, with $I \in \mathcal{I}$ and $I_\pm \in \mathcal{I}_\pm$, we have:

$$\widetilde{\mathcal{F}}(\mathcal{O}) = \mathcal{F}(I) \cong \mathcal{F}_+(I_+) \underset{\infty}{\bowtie} \mathcal{F}_-(I_-). \tag{4.25}$$

Here $\underset{\infty}{\bowtie}$ denotes the twisted crossed product for the point at infinity obtained in Eq. (4.17) and $\mathcal{F}_\pm : I_\pm \mapsto \pi_\pm(\mathcal{W}(V_\pm(I_\pm))''$ are the chiral field nets on the non separable Hilbert spaces $\mathcal{H}_\pm$ defined, for $I_\pm \in \mathcal{I}_\pm$ the light ray intervals, from the localized symplectic space $V_\pm(I_\pm) = \{\theta \in V_\pm : \operatorname{supp} \partial V \subset I_\pm\}$.

The observable chiral nets are defined on the separable Hilbert Fock spaces $\mathcal{H}_{a_\pm}$ by

$$\mathcal{A}_\pm := I_\pm \mapsto \pi_{a_\pm}(\mathcal{W}(V_{a_\pm}(I_\pm))'', \quad I_\pm \in \mathcal{I}_\pm,$$

and the following equality holds for $\mathcal{O}$ as above

$$\widetilde{\mathcal{A}}(\mathcal{O}) = \mathcal{A}(I) \cong \mathcal{A}_+(I_+) \otimes \mathcal{A}_-(I_-). \tag{4.26}$$

As a last observation, notice that if $N = \{0 \oplus n, n \in \mathbb{R}\} \subset V_f$ is the time zero subspace defined in above Sec. 3.1, than $\psi_\infty(N) = N \oplus N \subset V_+ \oplus V_-$, because $\psi_\infty(n) = \frac{n}{2}$. Hence $\psi_\infty(N)$ generates the diagonal constant elements in the field, and it holds

$$\mathcal{Z}_b \cong \pi_+(\mathcal{W}(\psi_\infty(N)))'' \otimes \pi_-(\mathcal{W}(\psi_\infty(N)))'' \cong \cap_{I_\pm \in \mathcal{I}_\pm} \mathcal{F}_\pm(I_\pm),$$

i.e. the common Abelian von Neumann subalgebra of the local left and right movers is isomorphic to the non-trivial center $\mathcal{Z}_b$ of the time zero net $\mathcal{B}$.

It is well known that the 1+1-dimensional observable net $\widetilde{\mathcal{A}}$, in the vacuum representation $\pi_a \cong \pi_{a_+} \otimes \pi_{a_-}$, satisfies (see [51, 29, 8] for this model and [32] for a general formulation):

- covariance under the action of $\overline{PSL}(2,\mathbb{R}) \times \overline{PSL}(2,\mathbb{R})$, the universal covering group of the Möbius group on M, implemented by a continuous, unitary, positive energy representation that we denote by $U_a = U_{a_+} \otimes U_{a_-}$;

- Reeh–Schlieder property for $\widetilde{\mathcal{A}}$ relative to the cyclic and separating vector $\Omega_a \in \mathcal{H}_a$ (and also for the net $\mathcal{A}$, relative to the same vector, and for $\mathcal{A}_\pm$ relative to $\Omega_{a_\pm} \in \mathcal{H}_{a_\pm}$); and
- the local algebras $\widetilde{\mathcal{A}}(\mathcal{O})$ ($\mathcal{A}_\mathcal{I}(I)$ and $\mathcal{A}_\pm(I_\pm)$) appear as type III_1 factors.

Moreover, as a consequence of the modular theory for the local algebras and of the Möbius covariance, it is also shown in [29] that we have

- Haag duality for the net $\widetilde{\mathcal{A}}$, with respect to the double cone index set $\mathcal{K}$ (respectively, of $\mathcal{A}_\mathcal{I}$ with respect to the index set $\mathcal{I}$ and nets $\mathcal{A}_\pm$ with respect to the index sets $\mathcal{I}_\pm$);
- timelike duality for the net $\widetilde{\mathcal{A}}$, with respect to the index set $\mathcal{K}$, and
- unitary equivalence of the local algebras $\widetilde{\mathcal{A}}(\mathcal{O})$ with any local algebras defined by additivity on bounded simply connected regions or wedges in M (respectively, of the algebra $\mathcal{A}_\mathcal{I}(I)$ and the algebra defined by additivity on the half lines, in time zero formulation).

4.3. *DHR sectors for the observable net $\mathcal{A}_\mathcal{I}$*

We begin this section recalling some basic facts about DHR superselection theory, see, e.g., [48].

Given a directed index set $\mathcal{P}$ and a von Neumann algebras net $\mathcal{N}_\mathcal{P}$ in representation π_n, it holds $\mathcal{N}_\mathcal{P} \subset \mathcal{N}^\mathcal{P} := (\pi_n, \pi_n)'$. The DHR superselection sectors of $\mathcal{N}_\mathcal{P}$ are described by the W^*-category $\mathrm{Rep}^\perp \mathcal{N}_\mathcal{P}$ defined as follows:

- objects: the representations π of $\mathcal{N}_\mathcal{P}$ such that

— π is a representation *in* $\mathcal{N}^\mathcal{P}$, i.e. $\pi(\mathcal{N}_\mathcal{P}(P)) \subset \mathcal{N}^\mathcal{P}$ for $P \in \mathcal{P}$;
— π satisfies the DHR *superselection criterion*, i.e.

$$\pi\restriction P^\perp \cong \pi_n\restriction P^\perp, \quad P \in \mathcal{P}; \quad \text{and} \tag{4.27}$$

- arrows: the intertwiners between these representations, i.e. operators $T \in \mathcal{N}^\mathcal{P}$ such that for any $P \in \mathcal{P}$ and $\pi_1, \pi_2 \in \mathrm{Rep}^\perp \mathcal{N}_\mathcal{P}$ we have

$$T\pi_1(A) = \pi_2(A)T, \quad A \in \mathcal{N}_\mathcal{P}(P). \tag{4.28}$$

By [18, 20], it is possible to analyze $\mathrm{Rep}^\perp \mathcal{N}_\mathcal{P}$ in terms of the tensor W^*-category $\mathcal{T}_t$ of *localized transportable endomorphisms* of the net $\mathcal{N}_\mathcal{P}$ (if $\mathcal{P}$ is directed this is actually an equivalence of W^*-categories). The correspondence functor is given on the objects of $\mathcal{T}_t$ by $\pi = \pi_n \circ \rho$, for $\rho \in \mathcal{T}_t$ and $\pi \in \mathrm{Rep}^\perp \mathcal{N}_\mathcal{P}$.[k] This functor allows

[k]The endomorphism ρ is said to be *localized* in $P \in \mathcal{P}$, where P is the element appearing in Eq. (4.27); moreover it is said to be *transportable* if there exists a field of endomorphisms $\mathcal{P} \ni a \mapsto \rho^a$ such that $\rho^a = \rho$ if $a = P$, the localization region of ρ and if for every $b \in \mathcal{P}$ such that $a, P \subset b$, there exists a unitary operator $u(b) \in \mathcal{U}(\mathcal{H}_\pi)$ such that $\rho^a = \mathrm{ad}\, u(b) \circ \rho$. For a general approach, also on non-directed set, see [48].

one to reduce the study of all the representations to the Hilbert space $\mathcal{H}_n$, with intertwiners defined as in Eq. (4.28).

In our case, using the above recalled identification of $\mathrm{Rep}^\perp \mathcal{A}_\mathcal{I}$ and $\mathcal{T}_t$, we define for any $F \in V_f(I)$ that carries the charges $(F_c, F_q) \in \widehat{\mathcal{G}}$ a DHR representation for the nets $\mathcal{A}_\mathcal{I}$ and $\widetilde{\mathcal{A}}$, by the adjoint action automorphism ρ_F^I obtained by F and localized in I, such that

$$\pi_a \circ \mathrm{ad}\, \pi_f(W_f(F)) := \pi_a \circ \rho_F^I. \tag{4.29}$$

For the representations in $\mathrm{Rep}^\perp \mathcal{N}_\mathcal{P}$, the covariance property under the action of a geometrical symmetry group of the spacetime is a standard requirement.

In our model, in order to define and discuss such a covariance and the positivity of the energy for the net $\mathcal{A}_\mathcal{I}$ and for the introduced subsidiary nets, we have to pass to the 1+1-dimensional theory. In fact it is well known that if $\widetilde{\mathcal{N}}_\mathcal{K}$ is a (Poincaré or Möbius[1]) translation covariant net on the index set $\mathcal{K}$ of the double cones in a 1+1-dimensional Minkowski spacetime M, its time zero restriction net $\mathcal{N}_\mathcal{I}$ is only a space-translation covariant net, without spectrum condition, see, for example, [34].

A representation π of a net $\mathcal{N}_\mathcal{K}$ is said to be *Möbius covariant* if there exist a unitary representation U_π of the group $\overline{PSL}(2,\mathbb{R}) \times \overline{PSL}(2,\mathbb{R})$ on the Hilbert space $\mathcal{H}_\pi$ such that, for any double cone $\mathcal{O} = I_+ \times I_-$ and element $g = (g_+, g_-) \in \mathcal{U} \subset \overline{PSL}(2,\mathbb{R}) \times \overline{PSL}(2,\mathbb{R})$ in the connected neighborhood $\mathcal{U}$ of the identity element of the covering of the Möbius group, we have

$$\mathrm{ad}\, U_\pi(g)(\pi(\mathcal{N}(I_+ \times I_-))) = \pi(\mathcal{N}(g_+ I_+ \times g_- I_-)). \tag{4.30}$$

Returning to the Streater and Wilde model, if we take $F \in V_f(I)$ implementing the sector automorphism ρ_F^I and the representation $\pi_F^I := \pi_a \circ \rho_F^I$ as in Eq. (4.29) labeled by the charges $(F_c, F_q) \in \widehat{\mathcal{G}}$, we may pass to the chiral formulation using the symplectic isomorphism ψ_∞ of Lemma 4.3. For given the chiral charges $(c_+^F, c_-^F) = (\frac{F_q+F_c}{2}, \frac{F_q-F_c}{2})$, see Eq. (4.20), we have a unitary representation of $\overline{PSL}(2,\mathbb{R}) \times \overline{PSL}(2,\mathbb{R})$ on the subspace

$$\mathcal{H}_{c_+^F} \otimes \mathcal{H}_{c_-^F} \subset \mathcal{H}_+ \otimes \mathcal{H}_- \cong (\oplus_{c_+ \in C_+} \mathcal{H}_{c_+}) \otimes (\oplus_{c_- \in C_-} \mathcal{H}_{c_-}).$$

For every chiral charge $c_\pm \in C_\pm$, and for $\mathcal{H}_{a_\pm}$ the observables separable Hilbert space, we have the Hilbert space equivalence $\mathcal{H}_{c_\pm} \cong \mathcal{H}_{a_\pm}$, but carrying different representations. If this equivalence is given by the unitaries $Y_\pm : \mathcal{H}_{a_\pm} \to \mathcal{H}_{c_\pm}$ and if $U_{a_\pm}$ are the representations of the Möbius group on $\mathcal{H}_{a_\pm}$, respectively, then we may define by $U_{c_\pm} := \mathrm{ad}\, Y_\pm(U_{a_\pm})$ the unitary representations of the Möbius group on the Hilbert spaces $\mathcal{H}_{c_\pm}$. The representations of the same group on $\mathcal{H}_{c_+} \otimes \mathcal{H}_{c_-}$ are defined by $U_{c_+,c_-} := U_{c_+} \otimes U_{c_-}$.

Denote now by $\xi_\pm \in I_\pm$ the coordinates on ray light lines, and take an element $\Theta = (\theta_+, \theta_-) \in V_+(I_+) \oplus V_-(I_-)$, i.e. with $\mathrm{supp}\,\Theta = I_+ \times I_- = \mathcal{O}$, and charges equal

[1]The name *conformal covariance* is better reserved for covariance under the diffeomorphisms group, that will not appear in this paper.

to (c_+, c_-). For a given DHR representation $\pi_{c_+} \otimes \pi_{c_-}$, and for any $g = (g_+, g_-) \in \mathcal{U} \subset \overline{PSL}(2,\mathbb{R}) \times \overline{PSL}(2,\mathbb{R})$, the symmetry group representation U_{c_+,c_-} acts as

$$\begin{aligned}
&\operatorname{ad} U_{c_+,c_-}(g)(\pi_{c_+}(W(\theta_+(\xi_+))) \otimes \pi_{c_-}(W(\theta_-(\xi_-)))) \\
&\quad = \operatorname{ad} U_{c_+}(g_+)(\pi_{c_+}(W(\theta_+(\xi_+)))) \otimes \operatorname{ad} U_{c_-}(g_-)(\pi_{c_-}(W(\theta_-(\xi_-)))) \\
&\quad = \pi_{c_+}(W(\theta_+(g_+^{-1}\xi_+))) \otimes \pi_{c_-}(W(\theta_-(g_-^{-1}\xi_-))). \qquad (4.31)
\end{aligned}$$

Observe that it is not necessary to define the action of the Möbius group on the two left/right chirality points at infinity $\infty_\pm$, that not even explicitly appear in the action of the representation U_{c_+,c_-} for no $(c_+, c_-) \in C_+ \oplus C_-$. In fact it is possible to choose ∞_+ not contained in the left supports I_+ of Θ, and take the open connected neighborhood $\mathcal{U}$ such to remain with $\infty_+ \notin g_+ I_+$, for every $(g_+, g_-) \in \mathcal{U}$, and similarly for ∞_-.

Moreover, the defined representation U_{c_+,c_-} may also be used in restriction to the elements of the chiral observable net $\widetilde{\mathcal{A}}_{\mathcal{K}} = \mathcal{A}_+ \otimes \mathcal{A}_-$ in any representation $\pi_F^I \cong \pi_{c_+} \otimes \pi_{c_-}$.

Finally, the covariance of the vacuum representation $\pi_\infty \cong \pi_+ \otimes \pi_-$ of the field net $\widetilde{\mathcal{F}}_{\mathcal{K}}$ is also easily obtained, defining the Möbius group representation by

$$U := \oplus_{(c_+,c_-)\in C_+\oplus C_-} U_{c_+,c_-} = (\oplus_{c_+\in C_+} U_{c_+}) \otimes (\oplus_{c_-\in C_-} U_{c_-}), \qquad (4.32)$$

under the proper notion of convergence, i.e. summing up on finite sets of charges $\Lambda \subset C_+ \oplus C_-$. The same result holds for the time zero formulation, under the isomorphisms seen above.

After this excursion on the chiral formulation and the representation of the Möbius group, the reason why we choose time zero description is going to be clear in a moment.

Solitonic sectors appeared as part of AQFT in [45], for a general formulation see also [24, 43]. In the Streater and Wilde model, this notion emerge for the sectors of the intermediate net $\mathcal{C}_{\mathcal{I}}$, that turn out to be similar to a class studied in [40] for massive and conformal field theories.

For $I \in \mathcal{I}$, we define its *left and right causally disjoint sets* $I_l^\perp, I_r^\perp \subset \mathcal{I}$, by $I_l^\perp \cup I_r^\perp = I^\perp$ and $I_l^\perp \cap I_r^\perp = \emptyset$, and use the following

Definition 4.4. Let $\mathcal{N}_{\mathcal{I}}$ be a net in the defining representation π_n, and let K be a group with a local action α on the net $\mathcal{N}_{\mathcal{I}}$. A translation covariant representation π of the net $\mathcal{N}_{\mathcal{I}}$ is said to be a *K-solitonic representation with support* $I \in \mathcal{I}$ if there exist $h, k \in K$ such that for any $I_1 \in I_l^\perp$ and $I_2 \in I_r^\perp$ we have

$$\pi\restriction\mathcal{N}(I_1) = \pi_n \circ \alpha_h(\mathcal{N}(I_1)), \quad \pi\restriction\mathcal{N}(I_2) = \pi_n \circ \alpha_k(\mathcal{N}(I_2)).$$

A *K-solitonic automorphism* $\rho_{k,h}^I$ of the net $\mathcal{N}_{\mathcal{I}}$ is a net automorphism such that $\pi = \pi_n \circ \rho_{k,h}^I$ is a solitonic representation with support $I \in \mathcal{I}$, i.e. $\rho_{k,h}^I\restriction\mathcal{N}_{\mathcal{I}}(I_1) = \alpha_k$ and $\rho_{k,h}^I\restriction\mathcal{N}_{\mathcal{I}}(I_2) = \alpha_h$.

We remember other notions, presented e.g. in [41,42], also useful for the case at hand.

If K is a (finite) compact group and $\mathcal{R}_\mathcal{I} := \mathcal{N}_\mathcal{I}^K$ is the fixed-point net of $\mathcal{N}_\mathcal{I}$ under the action of K so that the DHR sectors of the net $\mathcal{R}_\mathcal{I}$ are described by K-solitonic automorphisms of net $\mathcal{N}_\mathcal{I}$, we speak of an *orbifold model*; moreover, such a model is said to be *holomorphic* if the net $\mathcal{N}_\mathcal{I}$ has only trivial superselection sectors. Starting from a 1+1-dimensional net $\widetilde{\mathcal{N}}_\mathcal{K}$, with time zero restriction $\mathcal{N}_\mathcal{I}$, being the solitonic automorphisms not locally normal at infinity, it is only as representations of the fixed-point subnet $\widetilde{\mathcal{R}} = \widetilde{\mathcal{N}}_\mathcal{K}^K$, with time zero restriction $\mathcal{R}_\mathcal{I}$, that they represent true positive energy DHR sectors.

In the Streater and Wilde model, the properties of the sector automorphisms of the observable net $\mathcal{A}_\mathcal{I}$ (and $\widetilde{\mathcal{A}}$) implemented by the Weyl elements of the auxiliary nets, are collected in the following result, where geometric covariance properties are also discussed

Proposition 4.5. *In the above notation, we have*

(i) *for $F = f_0 \oplus f_1 \in V_e(I)$, the adjoint action of the represented Weyl element $\pi_f(W_f(F))$ implements an N-solitonic transportable automorphism ρ_F^I of the net $\mathcal{C}_\mathcal{I}$, localized in $I \in \mathcal{I}$, i.e. such that for $G \in V_c$ we have*

$$\rho_F^I(\pi_f(W_f(G))) := \mathrm{ad}\,\pi_f(W_f(F))(\pi_f(W_f(G))) = e^{i\sigma_f(F,G)}\pi_f(W_f(G)). \tag{4.33}$$

Moreover, if $\lim_{x\to\pm\infty} f_1(x) = F_\pm \in N$ and if $\mathrm{loc}\, G \in I_r^\perp$, we have

$$\rho_F^I\pi_f((W_f(G))) = \alpha_{F_+}((W_f(G)))) = e^{-iF_+G_c}\pi_f(W_f(G)),$$

or if $\mathrm{loc}\, G \in I_l^\perp$ we have

$$\rho_F^I\pi_f((W_f(G))) = \alpha_{F_-}((W_f(G)))) = e^{-iF_-G_c}\pi_f(W_f(G)).$$

The elements in $\mathcal{C}_\mathcal{I}$ intertwine solitonic automorphisms with the same values of $F_q = F_+ - F_-$, preserving supports, if localized with the same support as the automorphisms. Such automorphisms turn out to be DHR sector automorphisms in restriction to the nets $\mathcal{B}_\mathcal{I}$ and $\mathcal{A}_\mathcal{I}$, localized in I, with charge $F_q \in Q$ and intertwiners in $\mathcal{B}_\mathcal{I}$ and $\mathcal{A}_\mathcal{I}$, respectively;

(ii) *for $F \in V_c(I)$ the automorphism $\rho_F^I := \mathrm{ad}\,\pi_f(W_f(F))$ is a DHR automorphism of the net $\mathcal{E}_\mathcal{I}$, localized in I, i.e. such that for $G \in V_e$ equality* (4.33) *holds and $\rho_F^I \restriction \mathcal{E}(I_1) = \iota$ if $I_1 \perp I$. The elements in $\mathcal{E}_\mathcal{I}$ intertwine automorphisms with same charge, preserving supports if localized as the automorphisms are. The restriction to the nets $\mathcal{Q}_\mathcal{I}$ and $\mathcal{A}_\mathcal{I}$ gives DHR sector automorphisms for these nets as well, with intertwiners in $\mathcal{Q}_\mathcal{I}$ and $\mathcal{A}_\mathcal{I}$, respectively;*

(iii) *for any $F \in V_f(I)$ the automorphism ρ_F^I defined as in equality* (4.33) *is equivalent to a positive energy, Möbius covariant DHR sector automorphism $\rho_F^\mathcal{O}$ for the* 1+1*-dimensional net $\widetilde{\mathcal{A}}$, with charges $(F_c, F_q) \in \widehat{\mathcal{G}}$, localized in $\mathcal{O} = I''$. The intertwiners of two such automorphisms are given by elements in $\widetilde{\mathcal{A}}$.*

Proof. (i) The transportability is a result of a simple calculation, see, for example, [51], and using the correspondence of the given automorphism sectors on the time zero axes. Moreover $\pi := \pi_c \circ \rho_F^I$ satisfies Definition 4.4 for the group N, since

$$F_\pm \mapsto I_a \otimes \pi_L(W_L((0, F_\pm))) \otimes I_M \in \mathcal{U}(\mathcal{H}_f)$$

is a strongly continuous unitary representation of N on the Hilbert space $\mathcal{H}_f$, acting locally on the net $\mathcal{C}_\mathcal{I}$.

(ii) and (iii) It is easy to check the DHR superselection criterion requirements for such automorphisms, relatively to the indicated nets. In particular, to show the equivalence in (iii) it is enough to observe that the map

$$\rho_F^I := \mathrm{ad}(\pi_f(W(F))) \mapsto \mathrm{ad}(\pi_+(W(\psi_\infty(F))) \otimes \pi_-(W(\psi_\infty(F)))) =: \rho_F^\mathcal{O}$$

defines an automorphism for the net $\widetilde{\mathcal{A}}$ with the required properties. Positivity of the energy and Poincaré covariance of $\rho_F^\mathcal{O}$ is proved in [51]. The Möbius covariance is proved using the chiral formulation and the representation of the Möbius group given in Eq. (4.31). □

Remark 4.6. (1) Notice that the time zero formulation of Proposition 4.5 distinguishes between the different nature of the two families of DHR automorphisms obtained for the net $\mathcal{A}$: the ones in (i) are the restriction of solitonic automorphisms of the bigger net $\mathcal{C}$; the ones in (ii) instead are the restriction of DHR automorphisms of the bigger net $\mathcal{Q}$. Similarly for the net $\widetilde{\mathcal{A}}$ because of (iii). In the following Sec. 4.4, the net $\mathcal{A}$ is characterized as the fixed-point subnet of $\mathcal{Q}$ under the action of a compact gauge group $\mathcal{G}_q$, i.e. $\mathcal{A} = \mathcal{Q}^{\mathcal{G}_q}$. Using the discussion and terminology in [35], we say that the automorphisms in (i) give *twisted* representations of $\mathcal{A}$ and the ones in (ii) *untwisted*, relatively to the representation of the net $\mathcal{Q}$. In contrast, the chiral approach describes the sectors as indistinguished restrictions of solitonic automorphisms on the light lines.

(2) In the general situation treated in [40], the split property for the net is required in order to construct *disorder operators* that implement the solitonic automorphisms; in the Streater and Wilde case such operators are directly given as Weyl elements in the larger algebras.

4.4. *Gauge symmetry group*

We consider in the sequel the gauge automorphisms defined on net $\mathcal{F}_\mathcal{I}$.

The general theory in [18,19], the construction of the Weyl algebras sector automorphisms and the structure of simple current extensions in (ii) of Proposition 4.2, suggest looking for gauge symmetries in the dual of the charge group $\widehat{\mathcal{G}}$. We also recall the results (iii) and (iv) in Proposition 2.10, where the properties of the elementary Weyl algebra on the symplectic space $L = C \oplus N \cong \mathbb{R}^2_d$, in non-regular representation, are displayed. Of particular relevance is the Bohr compactification of N.

Consider the introduced two Abelian groups: the charge group $\widehat{\mathcal{G}} := C \oplus Q \cong \mathbb{R}^2_d$ furnished with the discrete topology and the additive group $\mathcal{G}_0 := N \oplus R$ as defined in the symplectic isomorphism in Sec. 3.1 and furnished with the usual topology of $\mathbb{R}^2$.

Consider moreover the Abelian group of *all* characters on $\widehat{\mathcal{G}}$, i.e. the group of all the functions $\chi : \widehat{\mathcal{G}} \to \mathbb{T}$ such that $\chi(s+s') = \chi(s)\chi(s')$ for $s, s' \in \widehat{\mathcal{G}}$, with the identity function as neutral element and complex conjugation as inverse. If we furnish this group with the compact-open topology, that coincides with the pointwise convergence topology, $\widehat{\mathcal{G}}$ being discrete, we obtain an Abelian compact group we denote by $\mathcal{G}$, because of the Pontryagin duality theorem.

Moreover, the group $\mathcal{G}$ is isomorphic to the *Bohr compactification of* $\mathcal{G}_0$, i.e. the closure of the group $\mathcal{G}_0$ in the compact-open topology defined above.

Observe that the group $\mathcal{G}_0$ is identified with the subgroup of the characters of $\widehat{\mathcal{G}}$ that correspond to strongly continuous one-dimensional representation of $\widehat{\mathcal{G}}$, with the usual topology. This means that under the groups morphism

$$\begin{aligned} \chi : \mathcal{G}_0 &\to \mathcal{G} \\ (n, r) &\mapsto \chi_{(n,r)}, \end{aligned}$$

the elements of $\mathcal{G}_0$ are the elements $\chi_{(n,r)}$ of $\mathcal{G}$ with the property that for $\lambda \in \mathbb{R}$ the map $\chi_{(n,r)}(\lambda(c,q)) = e^{i\lambda(nc+rq)}$, $(c,q) \in \widehat{\mathcal{G}}$ is continuous. The properties collected in the following result, authorize to call $\mathcal{G}$ *the gauge group of the net* $\mathcal{A}$, at least relatively to the net $\mathcal{F}$.

Proposition 4.7. *The map defined by*

$$\begin{aligned} V : \mathcal{G}_0 &\to \mathcal{U}(\mathcal{H}_f) \\ (n, r) &\mapsto V(n,r) := I_a \otimes \pi_L(W_L((0,n))) \otimes \pi_M(W_M((r,0))) \end{aligned}$$

satisfies:

(i) V *is a strongly continuous unitary representation of* $\mathcal{G}_0$ *on* $\mathcal{H}_f$;

(ii) *the invariant Hilbert subspace of* $V(\mathcal{G}_0)$ *is* $\mathcal{H}_a$. *Moreover, the representation* V *leaves invariant* $\Omega_f = \Omega_a \otimes \Omega_L \otimes \Omega_M$, *the vacuum vector of* $\mathcal{F}_{\mathcal{I}}$;

(iii) *on the net* $\mathcal{F}_{\mathcal{I}}$ *the adjoint action of* V *implements an automorphism group we indicate by the same symbol* $\mathcal{G}_0$, *such that for* $(n,r) \in \mathcal{G}_0$ *and* $F \in V_f$ *we have*

$$\alpha_{(n,r)}(\pi_f(W_f(F))) = \operatorname{ad} V(n,r)(\pi_f(W_f(F))) = e^{-i(nF_c + rF_q)}\pi_f(W_f(F)).$$

This automorphism group acts strictly locally on the net $\mathcal{F}_{\mathcal{I}}$ *and on the* 1+1-*dimensional net* $\widetilde{\mathcal{F}} = \mathcal{F}_+ \underset{\infty}{\bowtie} \mathcal{F}_-$, *commuting with the action of the Möbius group on the net* $\widetilde{\mathcal{F}}$, *represented on* $\mathcal{H}_f \cong \mathcal{H}_+ \otimes \mathcal{H}_-$ *as in Eq.* (4.32) *and acting on the subspace of fixed charges* (c_+, c_-) *as in Eq.* (4.31);

Moreover, for the Abelian compact group $\mathcal{G}$ *defined above, we have*:

(iv) $\mathcal{G} \cong \mathcal{G}_c \times \mathcal{G}_q$, *where* $\mathcal{G}_c$ *and* $\mathcal{G}_q$ *are the Bohr compactifications of* N *and* R *respectively. The group* $\mathcal{G}$ *is strong continuously represented on* $\mathcal{H}_f$, *extend-*

ing the representation V, and preserving the properties in (ii) *and* (iii) *of the subgroup $\mathcal{G}_0$;*

(v) *the dual group of $\mathcal{G}$ is $\widehat{\mathcal{G}}$, the charge group*;

(vi) *it holds $\mathcal{F}_{\mathcal{I}}^{\mathcal{G}_q} = \mathcal{C}_{\mathcal{I}}$, $\mathcal{F}_{\mathcal{I}}^{\mathcal{G}_c} = \mathcal{E}_{\mathcal{I}}$ and $\mathcal{F}_{\mathcal{I}}^{\mathcal{G}} = \mathcal{B}_{\mathcal{I}}$.*

Proof. Strong continuity in (i) and (iv) holds because the group $\mathcal{G}_0$ is represented by V using only the regular subspace $N \oplus R \subset L \oplus M$ of the representation $\pi_L \otimes \pi_M$ and, by density, extends to elements in $\mathcal{G}$ because of the universal property of Bohr compactification, see [16, Chap. 16].

(ii) The invariance of the subspace $\mathcal{H}_a$ and of the vector Ω_f hold because of the factorization property of the state $\omega_f = \omega_a \otimes \omega_L \otimes \omega_M$, and of its GNS representation, as illustrated in Proposition 2.6.

(iii) For the elements in $\mathcal{G}_0$ this is trivial, because the action is implemented by Weyl elements. The commutation of the action with the action of the Möbius group derives from the invariance of charges and the $\mathbb{T}$-valued action of the latter group.

(iv) results from the discussion above. In particular, the locality of the action for elements of $\mathcal{G}$ not in $\mathcal{G}_0$ holds, because it is $\mathbb{T}$-valued.

(v) by the Pontryagin's duality theorem.

(vi) follows from the general case in [18], averaging over the compact group $\mathcal{G}$, or the indicated subgroups, with respect to the Haar measure, in the one-dimensional representation fixed by specifying the charges of the Weyl generators. □

Remark 4.8. Observe that the previous introduction of a non-splitting isomorphism, i.e. of a regularizing element $T \in V_f$, has been also useful to simplify the proofs of the Propositions 4.5 and 4.7. Actually this isomomorphism corresponds to a section $\widehat{\mathcal{G}} \ni (c,q) \to \Delta_c/\mathrm{Inn}\,\mathcal{A}$, where Δ_c is the group of the covariant DHR automorphisms and $\mathrm{Inn}\,\mathcal{A}$ the one of the inner automorphisms of $\mathcal{A}$. This section gives a group homomorphism, as said in [19, Sec. IV] and a similar choice is also made in a non-trivial center situation, as illustrated in [4].

Having established the existence of the superselection automorphisms and their relative gauge groups, the following diagram concerns the full meaning of the simple current extension diagram seen in (4.8):

$$\begin{array}{ccccc}
 & & \mathcal{A} \otimes \mathcal{Z}_b & & \mathcal{B} \rtimes \mathcal{U}(C) \\
 & & \| & & \| \\
\mathcal{A} = \mathcal{Q}^{\mathcal{G}_q} & \subset & \mathcal{B} = \mathcal{E}^{\mathcal{G}_q} = \mathcal{C}^{\mathcal{G}_c} = \mathcal{F}^{\mathcal{G}} & \subset & \mathcal{C} = \mathcal{F}^{\mathcal{G}_q} \\
\cap & & \cap & & \cap \\
\mathcal{Q} & \subset & \mathcal{E} = \mathcal{F}^{\mathcal{G}_c} & \subset & \mathcal{F} \\
\| & & \| & & \| \\
\mathcal{A} \rtimes \mathcal{U}(Q) & & \mathcal{B} \rtimes \mathcal{U}(Q) & & \mathcal{B} \rtimes \mathcal{U}(Q) \rtimes \mathcal{U}(C)
\end{array} \tag{4.34}$$

Notice that

- the vertical lines describe fixed-point restrictions under the action of the compact subgroup $\mathcal{G}_q$; dually they account for (twisted) DHR sectors of $\mathcal{A}_\mathcal{I}$ and $\mathcal{B}_\mathcal{I}$ and solitonic (automorphism) sectors for $\mathcal{C}_\mathcal{I}$;
- the horizontal lines describe fixed-point restrictions under the action of the compact subgroup $\mathcal{G}_c$; dually they account for (untwisted) DHR sectors of nets $\mathcal{A}_\mathcal{I}, \mathcal{B}_\mathcal{I}, \mathcal{Q}_\mathcal{I}$ and $\mathcal{E}_\mathcal{I}$.

To conclude, it worth recalling that such a kind of diagrams is present in other situation where the sectors of a net are obtained along two different extension procedures, through *partial field nets*, see e.g. the *square of nets* in [43, 40]. For a general explanation of these features in terms of *braided crossed G-categories*, for G a finite gauge group, see [42]. Regarding the last cited paper, it is to observe that in the Streater and Wilde model only the discrete subgroup $N \cong \mathbb{R}_d$ is present in the description of solitonic sectors, and not the entire gauge group.

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