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A note on operator semigroups associated to chaotic flows

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Abstract. The transfer operator associated to a flow (continuous time dynamical system) is a one-parameter operator semigroup. We consider the operator-valued Laplace transform of this one-parameter semigroup. Estimates on the Laplace transform have been used in various settings in order to show the rate at which the flow mixes. Here we consider the case of exponential mixing and the case of rapid mixing (superpolynomial). We develop the operator theory framework amenable to this setting and show that the same estimates may be used to produce results, in terms of the operators, which go beyond the results for the rate of mixing.

1. *Introduction*

Flows are important dynamical systems, arguably the origin of much of the research in the area of dynamical systems. It has proved significantly more difficult to study strong statistical properties of flows compared to corresponding questions for discrete time systems. Of particular importance is proving the rate of mixing of a given system or family of systems. Substantial initial progress was made by studying the Laplace transform of the correlation function [**10**, **23**]. A certain estimate (the oscillatory cancellation estimate pioneered by Dolgopyat [**10**]) can then be translated into an exponential mixing estimate for the flow. These ideas were developed by Liverani for the closely related question of studying the resolvent operator of the infinitesimal generator of the semigroup of transfer operators [**19**]. An identical argument is used by Baladi and Liverani [**3**] and by Giuletti *et al* [**14**]. We further develop these ideas, extending the idea of considering the operator-valued Laplace transform of the transfer operator [**4**] and show that one may squeeze a little more information from this line of thinking. The cases for exponentially mixing flows and rapid mixing flows are presented side-by-side in the same language and so are easily comparable.

The improved operator-theoretic result is of interest in several ways. Beyond the rate of mixing there are many other statistical properties that can often be deduced from the spectral results [**16**, §9] and cannot be deduced directly from the rate of mixing. Another important use of the functional analysis is for studying how statistical properties behave under perturbations of the dynamical system [**17**]. Such perturbations could be deterministic or random. Moreover, the same ideas (as interpreted in [**18**]) can be used for the physically important question of understanding coupled dynamical systems.

The improvements are as follows. The calculation involved is completely streamlined. This makes it clear that the constant obtained in the decay rate (in terms of degree of differentiability of the observable) cannot be improved without additional ideas. We avoid the need for Liverani's 'silly preliminary fact' [**19**, Lemma 2.14]. Additionally, we are able to obtain a spectral decomposition (2.3) of the transfer operator in a sense similar to, although weaker than, the results of Tsujii [**24–26**]. This means we obtain a precise description of the mixing, and moreover it is to be expected that further information concerning other statistical properties can be obtained from this operator-theoretic representation. The result is in a form especially amenable to the ideas of [**17**] regarding the use of operator perturbation theory in order to understand various questions in dynamical systems.

Note that in this paper we do not prove the required estimates for any particular dynamical systems with respect to any particular Banach space. Rather we isolate the abstract argument and make some improvements to this. It is an important question and a subject of ongoing research to investigate the rate of mixing (and other fine statistical properties) for a broad spectrum of flows. The method we are discussing (i.e. functional analysis applied to dynamical systems) requires as a first step the choice or design of a Banach space on which the one-parameter family of transfer operators acts 'nicely'. Moreover, at this point in time, to answer such questions for flows, no one knows a method which does not involve functional-analytic ideas to some extent. Designing appropriate Banach spaces and proving such estimates for systems of interest (including many physically relevant systems with discontinuities and singularities) remains an important subject of ongoing research (see, for example, [**5**, **9**]). In many cases the appropriate choice of Banach space is far from obvious. In this note we are able to reduce the assumptions that such a Banach space must satisfy in order to be useful and consequently simplify the search for and construction of the dynamically relevant Banach spaces. In particular, we avoid the requirement that the one-parameter semigroup is strongly continuous.

In view of potential numerical applications throughout the argument we will keep track of all the relevant constants. In §2 we present the results in two theorems, one concerning the exponentially mixing case and the other concerning the rapid mixing case. In §3 we give details of systems where the required assumptions have already been shown to be satisfied. We hope these assumptions will soon be shown to be satisfied in many more settings. Sections 4 and 5 are devoted to the proofs of the results.

2. *Results*

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be Banach spaces such that $\mathcal{A} \supset \mathcal{B}$ and $\|\cdot\|_{\mathcal{A}} \leq \|\cdot\|_{\mathcal{B}}$†. Let $\mathscr{B}(\mathcal{B}, \mathcal{B})$ denote the Banach space of bounded linear operators $T : \mathcal{B} \to \mathcal{B}$ equipped with

† In actual fact one needs only the Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ equipped with an auxiliary, weaker norm $\|\cdot\|_{\mathcal{A}}$. However, in this case one can always define $\mathcal{A}$ to be the completion of $\mathcal{B}$ with respect to $\|\cdot\|_{\mathcal{A}}$ and so without loss of generality with give the assumptions as above.

the standard operator norm which we denote $\|T\|_{\mathcal{B}}$. We consider a measurable operator-valued function $T : [0, \infty) \to \mathscr{B}(\mathcal{B}, \mathcal{B})$ denoted by $t \mapsto T_t$ such that

$$T_0 = \mathrm{id}, \quad T_s \circ T_t = T_{s+t} \quad \text{for all } t, s \geq 0,$$

and that $\|T_t\|_{\mathcal{B}} \leq C_1$ for some $C_1 > 0$. In other words, $T_t : \mathcal{B} \to \mathcal{B}$ is a bounded one-parameter semigroup†. We define a weaker operator norm

$$\|T\|_{\mathcal{B}\to\mathcal{A}} := \sup\{\|T\mu\|_{\mathcal{A}} : \mu \in \mathcal{B}, \|\mu\|_{\mathcal{B}} \leq 1\}. \tag{2.1}$$

It would be unrealistic in the intended applications to hope that the semigroup is norm continuous; often the semigroup is not even strongly continuous‡. We merely require the following, substantially weaker, continuity condition.

ASSUMPTION 1. (Weak Lipschitz) *There exists $C_2 > 0$ such that*

$$\frac{1}{t}\|T_t - \mathrm{id}\|_{\mathcal{B}\to\mathcal{A}} \leq C_2 \quad \textit{for all } t \geq 0.$$

See §3 for discussion of this assumption and how it is natural in the intended applications. For all $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, let $R(z) \in \mathscr{B}(\mathcal{B}, \mathcal{B})$ be defined by the Bochner integral

$$R(z) := \int_0^\infty e^{-zt} T_t \, dt. \tag{2.2}$$

Since the semigroup is bounded we know that $\|R(z)\|_{\mathcal{B}} \leq C_1 \mathrm{Re}(z)^{-1}$ for all $\mathrm{Re}(z) > 0$ but we need a bit more information concerning $R(z)$.

ASSUMPTION 2. *There exists $\lambda > 0$ such that the essential spectral radius of $R(z)$: $\mathcal{B} \to \mathcal{B}$ is not greater than $(\mathrm{Re}(z) + \lambda)^{-1}$ for all $\mathrm{Re}(z) > 0$.*

In all cases we will assume the both Assumptions 1 and 2 hold. In addition, we will assume that one of the two following assumptions holds. The first is an oscillatory cancellation type estimate of the form used by Dolgopyat in the study of Anosov flows [**10**].

ASSUMPTION 3A. (Exponential) *There exist $\beta, \alpha, C_3 > 0$ and $\gamma \in (0, 1/\ln(1 + \lambda/\alpha))$ such that, for all $\mathrm{Re}(z) = \alpha$, $|\mathrm{Im}(z)| \geq \beta$,*

$$\|R(z)^{\tilde{n}}\|_{\mathcal{B}} \leq C_3(\mathrm{Re}(z) + \lambda)^{-\tilde{n}} \quad \textit{where } \tilde{n} = \lceil \gamma \ln|\mathrm{Im}(z)| \rceil.$$

An alternative and far weaker assumption is the following estimate of the form used by Dolgopyat in the study of the prevalence of rapid mixing among Axiom A flows [**11**].

ASSUMPTION 3B. (Rapid) *There exist $\beta, C_4, s, r > 0$ such that $R(z)$ admits a holomorphic extension to the set $\{z \in \mathbb{C} : |\mathrm{Im}(z)| \geq \beta, \mathrm{Re}(z) \geq -|\mathrm{Im}(z)|^{-r}\}$ and on this set*

$$\|R(z)\|_{\mathcal{B}} \leq C_4 |\mathrm{Im}(z)|^s.$$

† The boundedness requirement is essentially superfluous since if a one-parameter semigroup satisfies a bound of the form $\|T_t\|_{\mathcal{B}} \leq Ce^{\gamma t}$ then we may simply consider the operator $\tilde{T}_t := e^{-\gamma t} T_t$ and proceed as before.

‡ The one-parameter semigroup T_t is said to be *strongly continuous* if $T : [0, \infty) \times \mathcal{B} \to \mathcal{B}$ is jointly continuous. Given the semigroup structure, it is only required to check the continuity at 0: It is known [**8**, Theorem 6.2.1] that T_t is strongly continuous if and only if $\lim_{t\to 0} T_t\mu = \mu$ for all $\mu \in \mathcal{B}$. There are examples [**8**, Example 6.1.10] such that $T : (0, \infty) \times \mathcal{B} \to \mathcal{B}$ is jointly continuous but $T : [0, \infty) \times \mathcal{B} \to \mathcal{B}$ is not jointly continuous.

That Assumption 3A is stronger than Assumption 3B can be seen from the calculations in §4.

For the moment suppose that $T_t : \mathcal{B} \to \mathcal{B}$ is a strongly continuous one-parameter semigroup. The *generator* of the semigroup is the linear operator defined by

$$Z\mu := \lim_{t\to 0} \frac{1}{t}(T_t\mu - \mu)$$

the domain of Z, which we denote $\mathrm{Dom}(Z)$, being the set of $\mu \in \mathcal{B}$ for which the limit exists. There is no reason to expect Z to be a bounded operator. It is known [**8**, Lemma 6.1.15] that $\mathrm{Dom}(Z)$ is complete with respect to the norm

$$\|\mu\|_Z := \|Z\mu\|_{\mathcal{B}} + \|\mu\|_{\mathcal{B}},$$

and consequently Z is a closed operator [**8**, Problem 6.1.1]. According to [**8**, Lemma 6.1.11] we know that $\mathrm{Dom}(Z)$ is $\|\cdot\|_{\mathcal{B}}$-dense in $\mathcal{B}$.

In the case where $T_t : \mathcal{B} \to \mathcal{B}$ is not strongly continuous it will be convenient to have a subset of $\mathcal{B}$ on which T_t is known to be well behaved. Let, just as in [**4**],

$$\mathcal{D}_0 := \left\{ \int_0^s T_t\mu \, dt : \mu \in \mathcal{B},\, s > 0 \right\}.$$

Let $\mathcal{D} \subseteq \mathcal{B}$ denote the completion, with respect to $\|\cdot\|_{\mathcal{B}}$, of $\mathcal{D}_0$. Assumption 1 implies that $\mathcal{D}$ is $\|\cdot\|_{\mathcal{A}}$-dense in $\mathcal{B}$. (If it were known that $T_t : \mathcal{B} \to \mathcal{B}$ is strongly continuous then $\mathcal{D}$ is $\|\cdot\|_{\mathcal{B}}$-dense in $\mathcal{B}$.) It is easy to see that $T_t\mathcal{D} \subseteq \mathcal{D}$, and a simple estimate [**4**, Lemma 2.8] implies that $\|T_t\mu - \mu\|_{\mathcal{B}} \to 0$ as $t \to 0$ for all $\mu \in \mathcal{D}$. Consequently $T_t : \mathcal{D} \to \mathcal{D}$ is a strongly continuous one-parameter semigroup†. In this case we define the generator Z with $\mathrm{Dom}(Z)$ as above but for the semigroup $T_t : \mathcal{D} \to \mathcal{D}$. In this case we know that $\mathcal{D}_0 \subseteq \mathrm{Dom}(Z)$ and consequently $\mathrm{Dom}(Z)$ is $\|\cdot\|_{\mathcal{A}}$-dense in $\mathcal{B}$. From this point forward when we refer to $\mathrm{Dom}(Z)$ this should be understood to imply the Banach space $(\mathrm{Dom}(Z), \|\cdot\|_Z)$, defined as above, depending on whether the semigroup is strongly continuous or not.

The first main result of this paper is the following theorem.

THEOREM 1. *Suppose that $T_t : \mathcal{B} \to \mathcal{B}$ is a bounded one-parameter semigroup satisfying Assumptions 1, 2, and 3A. Then there exist a finite set*

$$\{z_j\}_{j=1}^N \subset \{z \in \mathbb{C} : -\lambda < \mathrm{Re}(z) \le 0,\, |\mathrm{Im}(z)| \le \beta\},$$

a set of finite rank projectors $\{\Pi_j\}_{j=1}^N \subset \mathscr{B}(\mathcal{B}, \mathcal{B})$ and an operator-valued function $t \mapsto P_t \in \mathscr{B}(\mathcal{B}, \mathcal{B})$ such that

$$T_t = P_t + \sum_{j=1}^N e^{tz_j}\Pi_j \quad \textit{for all } t \ge 0. \tag{2.3}$$

Moreover for all $\ell < \lambda$ there exists $C_\ell > 0$ such that, for all $\mu \in \mathrm{Dom}(Z)$, $t \ge 0$,

$$\|P_t\mu\|_{\mathcal{A}} \le C_\ell e^{-\ell t}\|Z\mu\|_{\mathcal{B}}. \tag{2.4}$$

The proof of the theorem is the content of §4.

† Of course $\mathcal{D}$ should be understood to mean the Banach space $(\mathcal{D}, \|\cdot\|_{\mathcal{B}})$.

Remark 2.1. The theorem is only useful if the set $\mathrm{Dom}(Z)$ is sufficiently large. As discussed immediately prior to the theorem, $\mathrm{Dom}(Z)$ is $\|\cdot\|_{\mathcal{A}}$-dense in $\mathcal{B}$. However, if T_t were a strongly continuous one-parameter semigroup, then $\mathrm{Dom}(Z)$ is $\|\cdot\|_{\mathcal{B}}$-dense in $\mathcal{B}$.

Remark 2.2. Since parts of the above result are limited to $\mathrm{Dom}(Z)$ one might suppose that it would have been convenient to work with the reduced operator semigroup $T_t : \mathcal{D} \to \mathcal{D}$ instead of $T_t : \mathcal{B} \to \mathcal{B}$ from the very beginning. Sometimes this is convenient but, as illustrated in [**4**], it can easily cause problems when studying flows and their perturbations, particular systems with discontinuities. The crucial problem being that the Banach space $\mathcal{D}$ depends on the flow, as does $\mathrm{Dom}(Z)$. Note that the part of the above result which concerns the peripheral spectrum does not depend on $\mathcal{D}$ or $\mathrm{Dom}(Z)$ allowing for the possibility of studying perturbation from an operator theory point of view.

Remark 2.3. For the purpose of this remark, denote by $Z_{\mathcal{D}}$ the generator associated to $T_t : \mathcal{D} \to \mathcal{D}$ (this is the operator denoted by Z throughout the rest of the paper). Similarly, denote by $Z_{\mathcal{B}}$ the generator associated to $T_t : \mathcal{B} \to \mathcal{B}$. Then

$$\mathcal{D}_0 \subseteq \mathrm{Dom}(Z_{\mathcal{D}}) \subseteq \mathrm{Dom}(Z_{\mathcal{B}}) \subseteq \mathcal{D}.$$

The first inclusion was discussed in the above paragraph and the second is obvious. Here we will prove the final inclusion. Let $\mu \in \mathrm{Dom}(Z_{\mathcal{B}})$. For all $s > 0$ let $\nu_s := s^{-1} \int_0^s T_t\mu \, dt \in \mathcal{D}_0$. Note that $\nu_s - \mu = s^{-1} \int_0^s (T_t\mu - \mu) \, dt$. Since $\lim_{t\to 0} (1/t)$ $(T_t\mu - \mu) = Z_{\mathcal{B}}\mu \in \mathcal{B}$ it follows that $\|\nu_s - \mu\|_{\mathcal{B}} \to 0$ as $s \to 0$ and so $\mu \in \mathcal{D}$.

Remark 2.4. With the current ideas we cannot hope for a strengthening of the theorem whereby $\|P_t\|_{\mathcal{B}} \leq Ce^{-\ell t}$. This is a subtlety of one-parameter semigroups as demonstrated by Zabczyk's example† [**8**, Theorem 8.2.9]. This is a problem that was overcome in the work of Tsujii [**24–26**], but results are limited to systems which are rather regular and it is not clear if such a strategy is possible in general.

Remark 2.5. If the one-parameter semigroup was actually a one-parameter semigroup of operators associated to an ergodic flow, as in the intended applications, then one can typically show that mixing is equivalent to $\{z_j\}_{j=1}^N \cap \{\mathrm{Re}(z) = 0\} = \{0\}$ (see, for example [**6**, **7**]).

Remark 2.6. Most often Assumption 2 is proven by the combination of a compact embedding $\mathcal{B} \hookrightarrow \mathcal{A}$ and an estimate of the form $\|R(z)^n\mu\|_{\mathcal{B}} \leq C(\mathrm{Re}(z) + \lambda)^{-n}\|\mu\|_{\mathcal{B}} + C(1 + \mathrm{Im}(z))\|\mu\|_{\mathcal{A}}$. Such information is sufficient to deduce the estimate of the essential spectral radius by following Hennion's argument [**15**] based on the formula by Nussbaum [**22**] (see, for example, [**19**]). In this case Assumption 3A can be weakened: it is then sufficient to prove the estimate of Assumption 3A in the weaker norm $\|\cdot\|_{\mathcal{A}}$ rather than in the original norm $\|\cdot\|_{\mathcal{B}}$ and only for $\mu \in \mathcal{B}$ for which $\|\mu\|_{\mathcal{B}}$ is sufficiently small in comparison to $\|\mu\|_{\mathcal{A}}$.

† There exists a one-parameter group T_t acting on a Hilbert space such that the spectrum of Z is contained in $i\mathbb{R}$ but $e^{|t|}$ is in the spectrum of T_t for all $t \in \mathbb{R}$. This means that the inclusion proved in [**8**, Theorem 8.2.7] cannot be improved to an equality.

In order to state the result which corresponds to rapid mixing we must have higher-order control on the regularity in the flow direction. For any $q \in \mathbb{N}$ define the norm

$$\|\mu\|_{Z^q} := \sum_{0 \le n \le q} \|Z^n \mu\|_{\mathcal{B}},$$

for all $\mu \in \mathrm{Dom}(Z^q)$. As before, Z is understood to be the generator of the strongly continuous one-parameter semigroup $T_t : \mathcal{D} \to \mathcal{D}$. The second main result of this paper is the following theorem.

THEOREM 2. *Suppose that $T_t : \mathcal{B} \to \mathcal{B}$ is a bounded one-parameter semigroup satisfying Assumptions 1, 2, and 3B. Then there exist a finite set*

$$\{z_j\}_{j=1}^N \subset \{z \in \mathbb{C} : -\lambda < \mathrm{Re}(z) \le 0, |\mathrm{Im}(z)| \le \beta\},$$

a set of finite rank projectors $\{\Pi_j\}_{j=1}^N \subset \mathscr{B}(\mathcal{B}, \mathcal{B})$ and an operator-valued function $t \mapsto P_t \in \mathscr{B}(\mathcal{B}, \mathcal{B})$ such that

$$T_t = P_t + \sum_{j=1}^N e^{tz_j} \Pi_j \quad \textit{for all } t \ge 0. \tag{2.5}$$

Moreover, for all $p \in \mathbb{N}$ there exist $q \in \mathbb{N}$, $C_p > 0$ such that, for all $\mu \in \mathrm{Dom}(Z^q)$, $t \ge 0$,

$$\|P_t \mu\|_{\mathcal{A}} \le C_p t^{-p} \|\mu\|_{Z^q}. \tag{2.6}$$

The proof of the theorem is the content of §5.

Remark 2.7. Note that the required regularity $q = q(p)$ depends on the desired decay rate p and must be taken larger when p increases. The exact connection of the two can be seen in the calculation at the end of §5. When considering rates of mixing of a flow the above requirement of $\mu \in \mathrm{Dom}(Z^q)$ becomes the unfortunate requirement of the observables being 'rather smooth' in the flow direction.

Remark 2.8. Dolgopyat's original formulation [**11**] of rapid mixing considered $\mathscr{C}^\infty$ functions as observables. See [**20**, Definition 2.2] for a formulation closer to the above statement. Note that being *sufficiently regular* in the flow direction is crucial for this result. However, it is of some help that the notion of regularity is entirely dependent on the choice of $\|\cdot\|_{\mathcal{B}}$ and $\|\cdot\|_{\mathcal{A}}$.

Remark 2.9. Usually Assumption 3B is proven by showing the non-existence of *approximate eigenvalues* [**11–13**, **20**, **21**].

3. *Applications*

Assumptions 2 and 3A have been shown for contact Anosov flows [**19**] (in the reference the two spaces $\mathcal{B}$ and $\mathcal{A}$ are denoted $\mathcal{B}(\mathcal{M}, \mathbb{C})$ and $\mathcal{B}_w(\mathcal{M}, \mathbb{C})$ respectively). In order to show that Assumption 1 holds it is convenient to modify the stronger of the two norms by adding a term which controls (in supremum) the derivative in the flow direction. As a result Assumption 1 is simple to prove in this setting once one notices that

$\int_0^t V\eta \circ \Phi^s \, ds = \eta \circ \Phi^t - \eta$ for all $t \geq 0$ where V is the vector field associated to the flow $\Phi^t : \mathscr{M} \to \mathscr{M}$. Let $\mathcal{L}_t$ be the associated transfer operator. This means that

$$\int_{\mathscr{M}} (\mathcal{L}_t h - h) \cdot \eta \, dm = \int_{\mathscr{M}} h \cdot (\eta \circ \Phi^t - \eta) \, dm = \int_0^t \int_{\mathscr{M}} \mathcal{L}_s h \cdot V\eta \, dm \, ds.$$

This immediately implies the weak Lipschitz control required by Assumption 1. Similarly these assumptions have been shown to be satisfied in several other settings [**3**, **14**].

The observant reader will have noticed that the modification of the norm as described above has the unfortunate side effect that the Banach space is then dependent on the dynamics and therefore unsuitable to studying perturbations as outlined in Remark 2.2. The improved norms [**6**, **7**] for Anosov flows are immediately suitable in terms of satisfying Assumptions 1 and 2. Unfortunately, at present, it is not known if Assumption 3A is satisfied with respect to these norms, and for Anosov flows without contact structure all indications suggest that some new idea is required.

Assumptions 1, 2, and 3B have been shown for a prevalent set of Axiom A flows in [**11**]. However, in the reference everything is described in the *twisted transfer operator* language for suspension flows. To pass from that viewpoint to the present language, note that the calculation (see, for example, [**23**] or [**2**, Lemma 7.17]) used to relate the Laplace transform of the correlation to a sum of twisted transfer operators may equally well be used for the Laplace transform of the transfer operator of the flow for the suspension flow.

4. *The exponentially mixing case*

Throughout we suppose that Assumptions 1, 2, and 3A are satisfied. First we recall a fact which appeared in [**4**]. Note that the proof is done using the integral definition of $R(z)$ (and not by associating it to a resolvent of some operator) using the fact that Fubini also holds for Bochner integrals [**1**, Theorem 1.1.9].

LEMMA 4.1. [**4**, Lemma 2.2] *For all* $\mathrm{Re}(z) > 0$, $\mathrm{Re}(\zeta) > 0$, *we have on* $\mathscr{B}(\mathcal{B}, \mathcal{B})$ *that*

$$(z - \zeta) R(\zeta) R(z) = R(\zeta) - R(z).$$

We already know that the operator-valued function $z \mapsto R(z) \in \mathscr{B}(\mathcal{B}, \mathcal{B})$ is holomorphic on the set $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ [**1**, Theorem 1.5.1]. We now take advantage of Assumption 3A for the following result.

LEMMA 4.2. *The operator- valued function* $z \mapsto R(z) \in \mathscr{B}(\mathcal{B}, \mathcal{B})$ *admits an extension which is meromorphic on the set* $\{z \in \mathbb{C} : \mathrm{Re}(z) > -\lambda\}$ *and holomorphic on the set* $\{z \in \mathbb{C} : \mathrm{Re}(z) > -\lambda, |\mathrm{Im}(z)| \geq \beta\}$.

Proof. Consider $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$ and $\eta \in \mathbb{C}$, $|\eta| > \mathrm{Re}(z)^{-1}$. By Lemma 4.1, $\eta^{-1} R(z + \eta^{-1}) \, R(z) = R(z + \eta^{-1}) - R(z)$ since in particular $\eta \neq 0$ and $\mathrm{Re}(z - (1/\eta)) > 0$. Consequently

$$R\left(z + \frac{1}{\eta}\right) = \eta R(z)(\eta \,\mathrm{id} - R(z))^{-1}. \tag{4.1}$$

We know that $(\eta \,\mathrm{id} - R(z))$ is invertible since the spectral radius of $R(z)$ is not greater than $\mathrm{Re}(z)^{-1}$. Consequently (4.1) defines the extension of $R(z)$ into the left half of the imaginary plane. By Assumption 2 the operator-valued function $\eta \mapsto (\eta \,\mathrm{id} - R(z))^{-1}$

is meromorphic on the set $\{|\eta| > (\mathrm{Re}(z)+\lambda)^{-1}\}$. By Assumption 3A we know that the spectral radius of $R(z)$ is not greater than $(\mathrm{Re}(z)+\lambda)^{-1}$ when $\mathrm{Re}(z) > -\lambda$ and $|\mathrm{Im}(z)| \geq \beta$. This means that in this case the operator-valued function $\eta \mapsto (\eta \,\mathrm{id} - R(z))^{-1}$ is holomorphic on this set. □

Proof of the first part of Theorem 1. An immediate consequence of Lemma 4.2 is that the function $z \mapsto R(z) \in \mathscr{B}(\mathcal{B}, \mathcal{B})$ has no more than a finite number of poles on the set $\{z \in \mathbb{C} : \mathrm{Re}(z) > -\lambda\}$. We let $\{z_j\}_{j=0}^{N} \subset \mathbb{C}$ denote this finite set of poles. For each z_j let

$$\Pi_j := \frac{1}{2\pi i} \int_{\Gamma_j} R(z)\, dz$$

where Γ_j is a positively orientated small circle enclosing z_j but excluding all other singularities of $R(z)$. As is well known for spectral projectors, the resolvent equation, which was proven in Lemma 4.1, implies that the definition is independent of the choice of Γ_j subject to the above conditions. We now, for all $t \geq 0$, define $P_t : \mathcal{B} \to \mathcal{B}$ by

$$P_t := T_t - \sum_{j=1}^{N} e^{t z_j} \Pi_j.$$

To complete the proof of the theorem it remains to give the appropriate estimates on P_t. This is the substantial part of the present argument and will be postponed until the end of the section. □

The following key step is an application of the inverse of the Laplace–Stieltjes transform of an operator-valued function [**1**, Theorem 2.3.4] to the present situation.

LEMMA 4.3. *Suppose that $t \geq 0$, $a > 0$. Then, in $\mathscr{B}(\mathcal{B}, \mathcal{A})$, we have that*

$$T_t = \lim_{k\to\infty} \frac{1}{2\pi i} \int_{-k}^{k} e^{(a+ib)t} R(a+ib)\, db.$$

Details for passing from the formulation in [**1**] and the present setting can be found in [**4**, Theorem 1] (using, crucially, Assumption 1).

The whole idea of the present argument is to obtain better information on $R(z)$ and then use the formula given by the above lemma whilst shifting the contour.

LEMMA 4.4. *Suppose that $\ell \in (0, \lambda)$. For all $b \in \mathbb{R}$, $|b| \geq \beta$, on $\mathscr{B}(\mathcal{B}, \mathcal{B})$,*

$$R(-\ell + ib) = R(\alpha + ib)\left(\sum_{n=0}^{\infty} (\alpha+\ell)^n R(\alpha+ib)^n\right).$$

Moreover, there exists $C_5 > 0$ such that for all $|b| \geq \beta$,

$$\left\| \sum_{n=0}^{\infty} (\alpha+\ell)^n R(\alpha+ib)^n \right\|_{\mathcal{B}} \leq C_5 |b|^{\gamma_0}$$

where $\gamma_0 := \gamma \ln(1 + \ell\alpha^{-1}) \in (0, 1)$.

Proof. Since the extension of $R(z)$ was defined in Lemma 4.2 by the resolvent equation, we have

$$R(-\ell + ib) = R(\alpha + ib)[\mathrm{id} - (\alpha + \ell) R(\alpha + ib)]^{-1}$$
$$= R(\alpha + ib)\left(\sum_{n=0}^{\infty} (\alpha + \ell)^n R(\alpha + ib)^n\right).$$

It is convenient to split the sum as

$$\sum_{n=0}^{\infty} (\alpha + \ell)^n R(\alpha + ib)^n = \sum_{k=0}^{\infty} (\alpha + \ell)^{k\tilde{n}(b)} R(\alpha + ib)^{k\tilde{n}} \sum_{m=0}^{\tilde{n}(b)-1} (\alpha + \ell)^m R(\alpha + ib)^m,$$

where $\tilde{n}(b) = \lceil \gamma \ln|b| \rceil$. We use the estimate $\|R(\alpha + ib)\|_{\mathcal{B}} \leq C_1 \alpha^{-1}$ and the estimate $\|R(\alpha + ib)^{\tilde{n}(b)}\|_{\mathcal{B}} \leq C_3(\alpha + \lambda)^{-\tilde{n}(b)}$ of Assumption 3A. The norm of the first sum decreases as $|b|$ increases and so we have

$$\sum_{k=0}^{\infty} (\alpha + \ell)^{k\tilde{n}(b)} \|R(\alpha + ib)^{k\tilde{n}(b)}\|_{\mathcal{B}} \leq C_6,$$

where $C_6 := C_3[1 - (\alpha + \ell/\alpha + \lambda)^{\gamma \ln \beta}]^{-1}$. The norm of the second sum is increasing as $|b|$ increases. We have

$$\sum_{m=0}^{\tilde{n}(b)-1} (\alpha + \ell)^m \|R(\alpha + ib)^m\|_{\mathcal{B}} \leq C_1 \sum_{m=0}^{\tilde{n}(b)-1} \left(\frac{\alpha + \ell}{\alpha}\right)^m$$
$$\leq C_1 \alpha \ell^{-1} |b|^{\gamma_0},$$

recalling that $\gamma_0 = \gamma \ln(1 + \ell\alpha^{-1})$. We let $C_5 := C_1 C_6 \alpha \ell^{-1}$. The above two estimates complete the proof of the lemma. □

LEMMA 4.5. *There exists $C_7 > 0$ such that for all $|b| \geq \beta$,*

$$\|R(\alpha + ib)\|_{\mathcal{B} \to \mathcal{A}} \leq C_7 |b|^{-1}.$$

Proof. This lemma is a consequence of Assumption 1. Fix $b \in \mathbb{R}$. For all $n \in \mathbb{N}$ let $t_n := 2\pi n |b|^{-1}$ and hence

$$R(\alpha + ib) = \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} e^{-(\alpha + ib)t} T_t \, dt$$
$$= \sum_{n=0}^{\infty} e^{-\alpha t_n} \int_{t_n}^{t_{n+1}} e^{-ibt} (e^{-\alpha(t - t_n)} T_t - T_{t_n}) \, dt,$$

since $\int_{t_n}^{t_{n+1}} e^{-ibt} \, dt = 0$. We have that $|e^{-\alpha(t-t_n)} - 1| \leq \alpha(t - t_n) \leq 2\pi\alpha|b|^{-1}$. Using Assumption 1, we have that $\|T_t - T_{t_n}\|_{\mathcal{B} \to \mathcal{A}} \leq (t - t_n) C_2 C_1 \leq 2\pi C_2 C_1 |b|^{-1}$ for all $t \in (t_n, t_{n+1})$. This means that

$$\|e^{-\alpha(t-t_n)} T_t - T_{t_n}\|_{\mathcal{B} \to \mathcal{A}} \leq 2\pi C_1 (\alpha + C_2) |b|^{-1}.$$

On the other hand,

$$\sum_{n=0}^{\infty} e^{-\alpha t_n} \int_{t_n}^{t_{n+1}} dt = \sum_{n=0}^{\infty} \frac{2\pi}{|b|} e^{-\alpha 2\pi n |b|^{-1}}$$
$$= \frac{2\pi |b|^{-1}}{1 - e^{-\alpha 2\pi |b|^{-1}}} \leq \frac{2\pi \beta^{-1}}{1 - e^{-\alpha 2\pi \beta^{-1}}} =: C_8.$$

We have shown that $\|R(\alpha + ib)\|_{\mathcal{B}\to\mathcal{A}} \leq C_7 |b|^{-1}$ where $C_7 := 2\pi C_1 C_8 (\alpha + C_2)$. □

LEMMA 4.6. *For all $z \in \mathbb{C}$ in the holomorphic domain of $R(z)$ and $z \neq 0$, on $\mathscr{B}(\mathrm{Dom}(Z), \mathcal{B})$,*

$$R(z) - \frac{1}{z}\,\mathrm{id} = \frac{1}{z} R(z) Z.$$

Proof. The claimed result concerns only $\mathrm{Dom}(Z)$ and so it suffices to consider $T_t : \mathcal{D} \to \mathcal{D}$, which, as discussed in the paragraph proceeding Theorem 1, is a strongly continuous one-parameter semigroup (see also Remark 2.3). Consequently, by standard [**8**, Theorem 8.2.1] semigroup theory $R(z) = (z\,\mathrm{id} - Z)^{-1}$. This means that $R(z)(z\,\mathrm{id} - Z) = \mathrm{id} = z R(z) - R(z) Z$. □

Proof of the second part of Theorem 1. Let $\mathrm{Re}(a) > 0$ and let $\ell < \lambda$ such that $\mathrm{Re}(z_j) > -\ell$ for all j. By Lemma 4.3 and shifting the contour of integration, remembering that $R(z)$ has a pole at each $\{z_j\}_{j=1}^N$, we have, on $\mathscr{B}(\mathcal{B}, \mathcal{A})$, for all $t \geq 0$,

$$T_t = \lim_{k\to\infty} \frac{1}{2\pi i} \int_{-k}^{k} e^{(a+ib)t} R(a + ib)\, db$$
$$= \lim_{k\to\infty} \frac{1}{2\pi i} \int_{-k}^{k} e^{(-\ell+ib)t} R(-\ell + ib)\, db + \sum_{j=1}^{N} \frac{e^{t z_j}}{2\pi i} \int_{\Gamma} R(z)\, dz.$$

This means that, for all $t \geq 0$,

$$P_t = \lim_{k\to\infty} \frac{e^{-\ell t}}{2\pi i} \int_{-k}^{k} e^{ibt} R(-\ell + ib)\, db. \tag{4.2}$$

Since $\int_{-\infty}^{\infty} (e^{(\ell+ib)t}/\ell + ib)\, db = 0$, we have

$$P_t = \lim_{k\to\infty} \frac{e^{-\ell t}}{2\pi i} \int_{-k}^{k} e^{ibt} \left(R(-\ell + ib) - \frac{\mathrm{id}}{-\ell + ib} \right) db.$$

By Lemma 4.6 we have that, on $\mathcal{A}$, for every $\mu \in \mathrm{Dom}(Z)$,

$$P_t \mu = \lim_{k\to\infty} \frac{e^{-\ell t}}{2\pi i} \int_{-k}^{k} e^{ibt} \frac{R(-\ell + ib) Z\mu}{-\ell + ib}\, db.$$

We must estimate $\|P_t \mu\|_{\mathcal{A}}$. Note that $\|R(-\ell + ib) Z\mu\|_{\mathcal{A}} \leq \|R(-\ell + ib)\|_{\mathcal{B}\to\mathcal{A}} \|Z\mu\|_{\mathcal{B}}$. Let

$$C_9 := \frac{1}{2\pi} \int_{-\beta}^{\beta} \frac{\|R(-\ell + ib)\|_{\mathcal{B}\to\mathcal{A}}}{|-\ell + ib|}\, db.$$

Since the contour $\{z \in \mathbb{C}, \operatorname{Re}(z) = -\ell, |\operatorname{Im} z| \leq \beta\}$ was chosen to avoid all the singularities of $R(z)$, we have that $C_9 < \infty$. By Lemmas 4.5 and 4.4, $\|R(-\ell + ib)\|_{\mathcal{B}\to\mathcal{A}} \leq C_7C_5|b|^{-(1-\gamma_0)}$ for all $|b| \geq \beta$. Since $(1-\gamma_0) \in (0, 1)$,

$$C_{10} := \frac{1}{2\pi} \int_{\beta}^{\infty} |b|^{-(2-\gamma_0)}\, db < \infty.$$

We have shown that $\|P_t\mu\|_{\mathcal{A}} \leq C_\ell e^{-\ell t}\|Z\mu\|_{\mathcal{B}}$ where $C_\ell := (C_9 + 2C_{10}C_7C_5)$. □

5. *The rapid mixing case*

Throughout we suppose that Assumptions 1, 2, and 3B are satisfied. The argument closely follows [**11**] but instead of using a Taylor expansion we take advantage of the generator Z.

Proof of the first part of Theorem 2. As before, we use Lemma 4.3 to write

$$T_t = \lim_{k\to\infty} \frac{1}{2\pi i} \int_{-k}^{k} e^{(a+ib)t} R(a+ib)\, db. \tag{5.1}$$

Identically to the proof of the first part of Theorem 1, we deal with the part of the integral from $-\beta$ to β by selecting a finite set of projectors $\{\Pi_j\}_j$ corresponding to the poles $\{z_j\}_j$ of $R(z)$ in the region $\{z \in \mathbb{C} : \operatorname{Re}(z) > -\ell, |\operatorname{Im}(z)| \leq \beta\}$. We define (as before)

$$P_t := T_t - \sum_{j=1}^{N} e^{tz_j}\Pi_j.$$

It now remains to estimate $\|P_t\mu\|_{\mathcal{A}}$ in terms of $\|\mu\|_{Z^q}$ (for some $q \in \mathbb{N}$) crucially using Assumption 3B. It is convenient to shift the contour of integration to $\{ib - \min(\epsilon, |b|^{-r}), b \in \mathbb{R}\}$ where $\epsilon \in (0, \ell)$ is chosen such that the new contour avoids all the singularities of $R(z)$. The central part of this integral ((5.1) after the shift of the contour) gives an exponentially bounded term as per (4.2) with a constant which depends on $\sup_{|b|\leq\beta}\|R(-\epsilon, b)\|_{\mathcal{B}} < \infty$. This means that we merely need to estimate the norm of

$$\lim_{k\to\infty} \int_{\beta}^{k} \exp(-t|b|^{-r})e^{ibt} R(ib - |b|^{-r})\, db, \tag{5.2}$$

and the similar integral from $-k$ to $-\beta$. This will be postponed until the end of this section. □

Now we will need the following higher-order version of Lemma 4.6.

LEMMA 5.1. *Let $n \in \mathbb{N}$. For all $z \in \mathbb{C}$ in the holomorphic domain of $R(z)$ and $z \neq 0$, on $\mathscr{B}(\operatorname{Dom}(Z^n), \mathcal{B})$,*

$$R(z) = \frac{1}{z^n}R(z)Z^n + \sum_{j=0}^{n-1} \frac{1}{z^{j+1}}Z^j.$$

Proof. The case $n = 1$ is Lemma 4.6. That is, $R(z) = (1/z)R(z)Z + (1/z)\,\mathrm{id}$. Simply iterating this formula proves the result for all $n \in \mathbb{N}$. □

For the following it is essential that Assumption 3B is satisfied.

LEMMA 5.2. *Let $n \in \mathbb{N}$. There exists $C_{11} > 0$ such that*

$$\|R(ib - |b|^{-r})\mu\|_{\mathcal{B}} \le C_{11}|b|^{s-n}\|\mu\|_{Z^n}$$

for all $\mu \in \mathrm{Dom}(Z^n)$, $b \in \mathbb{R}$, $|b| \ge \beta$.

Proof. Using Lemma 5.1, we have

$$\|R(z)\mu\| \le \frac{1}{|z|^n}\|R(z)\|_{\mathcal{B}}\|\mu\|_{Z^n} + \sum_{j=0}^{n-1}\frac{1}{|z|^{j+1}}\|\mu\|_{Z^j}.$$

We now substitute $z = ib - |b|^{-r}$. Since $\|R(z)\|_{\mathcal{B}} \le C_4|\mathrm{Im}(z)|^s$ by Assumption 3B there exists some $C_{11} > 0$ such that the lemma holds. □

Proof of the second part of Theorem 2. We now use the above lemma to estimate the norm of the integral of (5.2) and so complete the proof of Theorem 2:

$$\lim_{k\to\infty}\left\|\int_\beta^k \exp(-t|b|^{-r})e^{ibt}R(ib - |b|^{-r})\mu\, db\right\|_{\mathcal{B}}$$
$$\le C_{11}\left(\int_\beta^\infty \exp(-t|b|^{-r})|b|^{s-q}\, db\right)\|\mu\|_{Z^q}.$$

This holds for any $q \in \mathbb{N}$, but for our purposes we must choose q large, in particular larger than s. Estimating the integral† and choosing q even larger depending also on the required rate of polynomial decay (denoted p in the statement of the theorem) concludes the estimate. □

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† Suppose that $a > 0$, $n, k \in \mathbb{N}$. Let $I(n) := \int_0^a e^{-tx}x^n\, dx$. Integrating by parts $I(n) \le (n/t)I(n-1)$, and $I(0) \le (1/t)$. Consequently $I(n) \le n!t^{-n}$. By a change of variables $x = y^{-k}$ the integral $I(n)$ is equal to $k\int_{a^k}^\infty \exp(-ty^{-k})y^{-(nk+k+1)}\, dy$.

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