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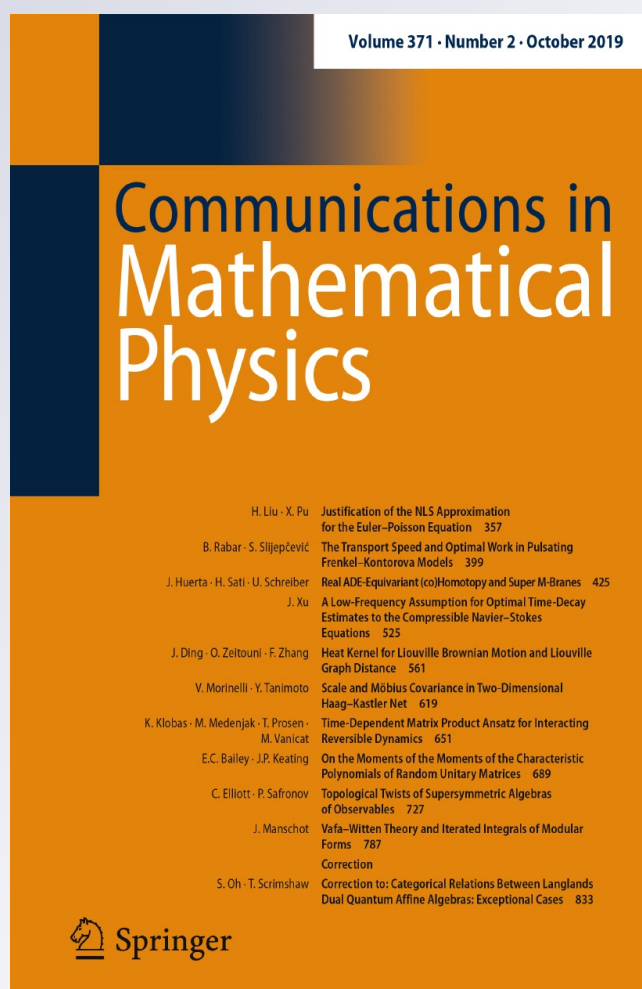
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Scale and Möbius Covariance in Two-Dimensional Haag–Kastler Net

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Abstract: Given a two-dimensional Haag–Kastler net which is Poincaré-dilation covariant with additional properties, we prove that it can be extended to a Möbius covariant net. Additional properties are either a certain condition on modular covariance, or a variant of strong additivity. The proof relies neither on the existence of stress-energy tensor nor any assumption on scaling dimensions. We exhibit some examples of Poincaré-dilation covariant net which cannot be extended to a Möbius covariant net, and discuss the obstructions.

1. Introduction

For a relativistic quantum field theory, there has been a long-standing question whether scale invariance (dilation covariance) implies conformal covariance [Nak15]. In $(1 + 1)$ -dimensions, we call the latter *Möbius covariance*¹ in order to distinguish it from diffeomorphism covariance (an action of the Virasoro algebra). This claim, of course, should not be taken literally. A simple counterexample can be given based on a generalized free field which is dilation covariant but not Möbius or conformally covariant (see Sect. 4). On the other hand, in $(3 + 1)$ -dimensions, there is no known example (even in the physical sense) of relativistic (unitary) dilation-covariant quantum field theory with certain additional conditions which is not conformally covariant, although there is currently no proof of the enhancement either. In $(1 + 1)$ -dimensions, the implication “dilation \implies Möbius” is considered a “theorem”, whose proof exploits the existence of stress-energy tensor and the discreteness of scaling dimension [Zom86, Pol88].

¹ The word “conformal” is reserved for diffeomorphism covariance, c.f. [KL04].

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Dilation covariance is believed to appear naturally in physical models. If one looks at longer and longer length scale of a physical system, the behaviour of the system should not depend on the details in the smaller scale and may obtain a low-energy effective theory which is scale invariant. Alternatively, one might look at smaller and smaller spacetime regions in the quantum chromodynamics, and should be able to see quarks which are otherwise confined and not visible. Such a limiting theory is expected to be simpler and to obtain the dilation symmetry (yet this is not automatic, see [BDM10]). Now, a dilation-covariant theory has often an additional symmetry, the conformal symmetry. Indeed, in $(1 + 1)$ -dimensions, most of important dilation-covariant theories are indeed Möbius covariant. Although not all dilation-covariant theories have Möbius covariance, it is natural to expect some additional conditions should imply the enhancement of symmetry.

In theoretical physics, the problem is considered to be solved in $(1 + 1)$ -dimensions by the arguments by Zamolodchikov [Zom86] and Polchinski [Pol88], which are based on the existence of scale current and the discreteness of scale dimensions. On the other hand, the enhancement of symmetry can be clearly stated even in terms of axiomatic/algebraic quantum field theory, hence it is natural to expect that certain additional assumptions should really imply Möbius covariance in the mathematical level. In this respect, Guido, Longo, and Wiesbrock proved that a dilation-translation covariant net of von Neumann algebras on the real line \mathbb{R} satisfying the Bisognano–Wichmann property can be extended to the compactified real line S^1 and obtain Möbius covariance [GLW98]. Remarkably, this last result does not assume any other physical requirement such as the existence of current or scaling dimensions, but the proof is based on the modular theory of von Neumann algebras. Hence one might expect a similar result for two-dimensional dilation-covariant quantum field theories.

In this paper, we present a proof of enhancement to Möbius covariance, in addition to the standard Haag–Kastler axioms, under the following operator-algebraic conditions

- The vacuum is cyclic and separating for the lightcone algebra.
- The theory is covariant under dilations and it is implemented by the modular group for the lightcone algebra, and one of the following holds.
 - (a) The modular automorphism group associated to the half-band algebra maps the double cone algebra (see Fig. 1) into a double cone algebra dilated in the direction

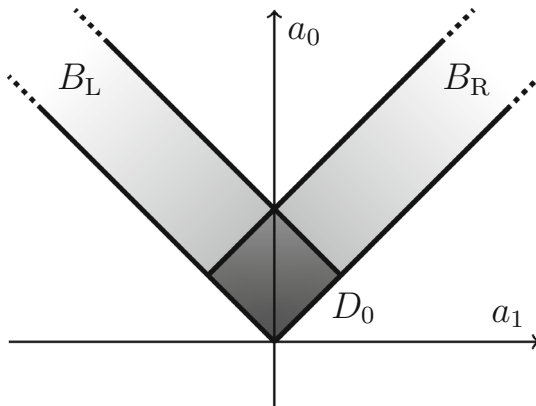


Fig. 1. Double cone D_0 and half-bands B_R, B_L

of the strip. In particular the half-band and the double cone algebras consist a half-sided modular inclusion [Wie93,AZ05].

- (b) The theory satisfies a variation of strong additivity.

Only one of the last two conditions is needed in our proof. The first one is an assumption about the modular group of a certain infinitely extended region, and might look too strong, but actually we show that it is a consequence of the second, which appears to have little to do with conformal covariance. With these conditions, which again are not concerned with either the current/stress-energy tensor or scaling dimensions, we can extend the symmetry group to the two-dimensional Möbius group by the modular theory.

We present two families of counterexamples. In one of them, we simply break the Bisognano–Wichmann property for the future lightcone V_+ which is a necessary condition for Möbius covariance [GLW98]. In the other, we take a certain representation of the two-dimensional Möbius group and apply the BGL construction [BGL93]. This itself is Möbius covariant, but its dual net is the dual net of a generalized free field which cannot be Möbius covariant. This last example provides a Möbius covariant net with the trace class property whose dual net is not Möbius covariant and does not have the split property. The reason why this dual net cannot be Möbius covariant (the vacuum is not separating for the algebra of V_+) is different from the reason why some generalized free fields cannot be Möbius covariant (wrong scaling dimension). We also examine arguments in physics literature and see to which extent they work.

This paper is organized as follows. In Sect. 2 we explain the geometric setting and the symmetry structure of two-dimensional Möbius covariant net, and state our additional assumptions on dilation covariant nets. In Sect. 3 we give a proof of Möbius covariance based on these assumptions. In Sect. 4, examples of dilation-covariant nets which do not extend to Möbius covariant nets are provided. In Sect. 5 we discuss to what extent our assumptions are necessary, some arguments in physics literature and open problems. Besides, we need the two-dimensional spin statistics theorem in the course of the proof, and we exhibit a proof in Appendix A for self-containedness. In Appendix B we provide some basic results on direct integrals of Hilbert subspaces which we were not able to find in literature, and are essential for our counterexamples.

2. Preliminaries

Here we are going to describe the operator-algebraic setting for quantum field theory and various spacetime symmetries.

2.1. One-dimensional Möbius group. The $(1 + 1)$ -dimensional Minkowski space is the product of two lightrays. The subgroup of lightlike translations and dilations of the Poincaré group acts on each lightray $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. This action can be extended to the Möbius group $\text{Möb} = \text{PSL}(2, \mathbb{R}) \cong \text{PSU}(1, 1)$, which we review here. See [Lon08, Wei05] for our notations.

Consider the Cayley transform $C : \overline{\mathbb{R}} \ni x \mapsto -\frac{x-i}{x+i} \in S^1$, where S^1 is the complex unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and $C(\infty)$ is defined to be equal to -1 by convention. With this map C , we can pass from the line to the circle picture. The Cayley transform is the inverse of the stereographic projection $C^{-1} : S^1 \ni z \mapsto -i \frac{z-1}{z+1} \in \overline{\mathbb{R}}$ and sends

S^1 onto $\overline{\mathbb{R}}$. With this convention, the upper semicircle is mapped in to the right half-line $(0, +\infty)$.

The group $SL(2, \mathbb{R})$ acts on the compactified line $\overline{\mathbb{R}}$ by linear fractional transformations. The kernel of its action is $\{\pm 1\}$ and $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$ defines the **Möbius group**, the group of orientation preserving conformal transformations of $\overline{\mathbb{R}}$. We denote it by $Möb$. On the unit circle in \mathbb{C} , the action of $Möb$ translates to that of $SU(1, 1)$ again by linear fractional transformations through the Cayley transform.

The group $Möb$ is a three-dimensional Lie group and can be generated by the following one-parameter subgroups:

- Rotations ρ_θ : for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, $\rho_\theta = e^{i\theta}z \in S^1$, in the circle picture.
- Dilations δ_s : for $s \in \mathbb{R}$, $\delta_s a = e^s a \in \mathbb{R}$, on the line picture.
- Translation τ_t : for $t \in \mathbb{R}$, $\tau_t a = a + s \in \mathbb{R}$, on the line picture.

In literature they are respectively denoted with \mathbf{K} , \mathbf{A} and \mathbf{N} , and any element $g \in Möb$ can be uniquely decomposed following the **KAN** decomposition (Iwasawa decomposition), i.e. the product of elements from each of these groups. The subgroups \mathbf{A} and \mathbf{N} generate the translation-dilation group \mathbf{P} which preserves the point ∞ in the real line picture.

In general, an element $g \in Möb$ is determined by its action on three points of the circle. Any pair of points on S^1 , hence any interval on S^1 , can be brought to another pair, respectively another interval, by a Möbius transformation. If $g \in Möb$ takes \mathbb{R}_+ (in the line picture) to a general interval I (in the circle picture), then we denote by $\Lambda_I(t) = g\delta_{-t}g^{-1}$, and call them the dilations² associated with I . Note that Λ_I does not depend on the choice of g . By this correspondence, $\Lambda_{\mathbb{R}_-}(t) = \delta_t$, and $\Lambda_{\mathbb{R}_+}(t) \cdot a = e^{-t}(a - 1) + 1$. The two subgroups $\{\Lambda_{\mathbb{R}_+}(t)\}$ and $\{\Lambda_{\mathbb{R}_+}(t)\}$ generate a two-dimensional subgroup of $Möb$ which is isomorphic to \mathbf{P} and preserves the point ∞ . Furthermore, any element of a small neighborhood of the unit element can be written as a simple product: indeed, we have

$$\Lambda_{\mathbb{R}_+}(t)\Lambda_{\mathbb{R}_+}(s) = \Lambda_{\mathbb{R}_+}(-\ln(e^{-t-s} + 1 - e^{-t}))\Lambda_{\mathbb{R}_+}\left(-\ln\left(\frac{e^{-t-s}}{e^{-t-s} + 1 - e^{-t}}\right)\right)$$

and it is immediate that any finite product of $\Lambda_{\mathbb{R}_+}$ and $\Lambda_{\mathbb{R}_+}$, as long as the parameters are sufficiently small (namely when $e^{-t-s} + 1 - e^{-t} > 0$), can be reduced to a product of two in the desired order. A similar relation holds for $\Lambda_{\mathbb{R}_+}$ and $\Lambda_{(0,1)}$:

$$\Lambda_{\mathbb{R}_+}(t)\Lambda_{(0,1)}(s) = \Lambda_{(0,1)}(\ln(e^{t+s} + 1 - e^t))\Lambda_{\mathbb{R}_+}\left(\ln\left(\frac{e^{t+s}}{e^{t+s} + 1 - e^t}\right)\right). \quad (1)$$

By bringing the three points $0, 1, \infty$ to another three points, an analogous relation holds for $\Lambda_{I_1}, \Lambda_{I_2}$, where $I_1 \supset I_2$ and there is one and only one of the endpoints shared by I_1 and I_2 . We call these relations simply the **commutation relations** of $\Lambda_{I_1}, \Lambda_{I_2}$.

The group $Möb$ can be generated by different subgroups. Consider I_1, I_2, I_3 , disjoint intervals whose union is dense in S^1 , then $\Lambda_{I_k}, k = 1, 2, 3$ generate $Möb$. This can be seen from the fact that they together can move any ordered three points to any other ordered three points. In particular, for $I_1 = (-\infty, 1), I_2 = (0, 1), I_3 = (0, \infty), \Lambda_{I_k}, k \in \mathbb{Z}_3$ generate $Möb$ and any pair $\Lambda_{I_k}, \Lambda_{I_{k+1}}$ generates a subgroup isomorphic to \mathbf{P} .

² By convention, the sign is reversed: $\Lambda_{\mathbb{R}_+}(t) = \delta_{-t}$, in accordance with [GLW98].

2.2. *(1 + 1)-dimensional Minkowski space and Einstein cylinder.* Consider the set of coordinates given by the lightrays $\left(\frac{a_0 - a_1}{\sqrt{2}}, \frac{a_0 + a_1}{\sqrt{2}}\right)$, where a_0 is the time coordinate and a_1 is the space coordinate. The following spacetime regions play important roles in our work.

- Forward and backward light-cones: $V_+ = \mathbb{R}_+ \times \mathbb{R}_+$ and $V_- = \mathbb{R}_- \times \mathbb{R}_-$
- Right and left standard wedges: $W_R = \mathbb{R}_- \times \mathbb{R}_+$ and $W_L = \mathbb{R}_+ \times \mathbb{R}_-$
- Right and left half-bands: $B_{R,(c,d)}^\pm = (a, b) \times \mathbb{R}_\pm$ and $B_{L,(a,b)}^\pm = \mathbb{R}_\pm \times (a, b)$
- Double cones: $D_{(a,b),(c,d)} = (a, b) \times (c, d)$

Then, we take also some specific regions (see Fig. 1):

- $B_R = (0, 1) \times \mathbb{R}_+$ and $B_L = \mathbb{R}_+ \times (0, 1)$
- $D_0 = (0, 1) \times (0, 1)$

Let $\widetilde{\text{Möb}}$ be the universal covering group of Möb. It is again generated by three one-parameter subgroups ρ, δ, τ (we use the same symbols for elements in $\widetilde{\text{Möb}}$, as long as no confusion arises), and ρ is now lifted from $\mathbb{R}/2\pi\mathbb{Z}$ to \mathbb{R} . Let G be the quotient group of $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ by the normal subgroup generated by $(\rho_{-2\pi}, \rho_{2\pi})$. The group G acts locally on the Minkowski space M identified with the product of two lightrays $\mathbb{R} \times \mathbb{R}$, and its action can be promoted to an action on the Einstein cylinder \widetilde{M} [BGL93].³ The Minkowski space is identified with a maximal square $(-\pi, \pi) \times (-\pi, \pi)$, where the product is intended for the lightlike decomposition parametrized by the lifted rotations. Let ι be the unit element of $\widetilde{\text{Möb}}$. For any $g \in \widetilde{\text{Möb}}$, elements of the form $g \times \iota$ (respectively $\iota \times g$) act trivially on the positive lightray $a_0 = a_1$ (respectively the negative lightray $a_0 = -a_1$)

We introduce the following elements in G in terms of the lightlike components:

- let Λ_{V_+} be the two-dimensional dilation of M : $\Lambda_{V_+}(t) = \delta(-t) \times \delta(-t)$.
- let Λ_{W_L} be the one-parameter group of Lorentz boosts associated with the left standard wedge W_L : $\Lambda_{W_L}(t) = \delta(-t) \times \delta(t)$.

In some literature a different convention is used where the parameter is reversed. The sign of our convention coincides with that of the modular group (see the Bisognano–Wichmann property in Sect. 2.4).

2.3. *The modular theory of von Neumann algebras and half-sided modular inclusions.*

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a cyclic and separating vector $\Omega \in \mathcal{H}$. The associated Tomita operator $S_{\mathcal{M},\Omega}$ is an antilinear involution which is the closure of

$$\mathcal{H} \supset \mathcal{M}\Omega \ni x\Omega \longmapsto x^*\Omega \in \mathcal{M}\Omega \subset \mathcal{H}.$$

Through its polar decomposition $S_{\mathcal{M},\Omega} = J_{\mathcal{M},\Omega} \Delta_{\mathcal{M},\Omega}^{\frac{1}{2}}$ one obtains the modular conjugation $J_{\mathcal{M},\Omega}$ and the modular operator $\Delta_{\mathcal{M},\Omega}$. They satisfy the relation $J_{\mathcal{M},\Omega} \Delta_{\mathcal{M},\Omega} J_{\mathcal{M},\Omega} = \Delta_{\mathcal{M},\Omega}^{-1}$. Furthermore $\Delta_{\mathcal{M},\Omega}$ is the generator of a one-parameter group of automorphisms called the **modular automorphism group**, namely $\Delta_{\mathcal{M},\Omega}^{it} \mathcal{M} \Delta_{\mathcal{M},\Omega}^{-it} = \mathcal{M}$ (see e.g. [Tak03]). For the anti-unitary conjugation $J_{\mathcal{M},\Omega}$ we have $J_{\mathcal{M},\Omega} \mathcal{M} J_{\mathcal{M},\Omega} = \mathcal{M}'$.

³ \widetilde{M} is homeomorphic to $\mathbb{R} \times S^1$, but this product decomposition is different from the lightlike decomposition above: \mathbb{R} goes in the a_0 -direction while S^1 is the a_1 -direction. This decomposition will not be used in this paper.

Lemma 2.1. *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a cyclic and separating vector $\Omega \in \mathcal{H}$ and a U such that $U\mathcal{M}U^* = \mathcal{M}$ and $U\Omega = \Omega$. Then it holds that $US_{\mathcal{M},\Omega}U^* = S_{\mathcal{M},\Omega}$ and hence*

$$U\Delta_{\mathcal{M},\Omega}U^* = \Delta_{\mathcal{M},\Omega}, \quad UJ_{\mathcal{M},\Omega}U^* = J_{\mathcal{M},\Omega}.$$

The following theorem, due to Borchers [Bor92] (and significantly simplified by Florig [Flo98]), ensures that when there is a one-parameter semigroup of endomorphisms implemented by unitaries with positive generator, these unitaries and the modular group generate a representation of the group \mathbf{P} of dilations and translations. In this representation, they are assigned $\Delta_{\mathcal{M},\Omega}^{\frac{it}{2\pi}}$ and $U(1)\Delta_{\mathcal{M},\Omega}^{\frac{is}{2\pi}}U(1)^*$, respectively and indeed satisfy Eq. (1).

Theorem 2.2 (Borchers). *Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a cyclic and separating vector $\Omega \in \mathcal{H}$, and $t \mapsto U(t) = e^{iHt}$ be a unitary one-parameter group such that $\text{sp } H \subset \mathbb{R}_{\pm}$, $U(t)\Omega = \Omega$ and $\text{Ad } U(t)(\mathcal{M}) \subset \mathcal{M}$, $t \geq 0$. Then the following hold:*

$$\begin{aligned} \Delta_{\mathcal{M},\Omega}^{is}U(t)\Delta_{\mathcal{M},\Omega}^{-is} &= U(e^{\mp 2\pi s}t), \\ J_{\mathcal{M},\Omega}U(t)J_{\mathcal{M},\Omega} &= U(-t), \quad t, s \in \mathbb{R}. \end{aligned}$$

The first equality shows that $\Delta_{\mathcal{M},\Omega}^{is}$ and $U(t)$ provide a positive energy representation of \mathbf{P} . We note that the group \mathbf{P} can be also generated by the $\Lambda_{(0,\infty)}$ and $\Lambda_{(1,\infty)}$, namely dilations based on 0 and 1, respectively.

Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be an inclusion of von Neumann algebra with a common cyclic and separating vector $\Omega \in \mathcal{H}$, then the inclusion is said to be a **half-sided modular inclusion** (\pm -HSMI) if

$$\text{Ad } \Delta_{\mathcal{M}}^{-it}(\mathcal{N}) \subset \mathcal{N}, \quad \pm t \geq 0.$$

The following is a fundamental result on HSIMIs [Wie93, AZ05].

Lemma 2.3 (Wiesbrock, Araki-Zsido). *If $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is a $+$ - (respectively $-$ -)HSMI, then the modular groups $\Delta_{\mathcal{M}}^{it}, \Delta_{\mathcal{N}}^{is}$ satisfy the same commutation relations as those of $\Lambda_{\mathbb{R}_-}$ and $\Lambda_{\mathbb{R}_- - 1}$ (respectively those of $\Lambda_{\mathbb{R}_+}$ and $\Lambda_{\mathbb{R}_+ + 1}$).*

The following is a slight variation of [GLW98, Lemma 1.1], see also [Wie98, Theorem 6].

Lemma 2.4. *Let Υ be the universal group algebraically generated by 3 one-parameter subgroups $t_k \mapsto \Lambda_k(t_k)$, $k \in \mathbb{Z}_3$, such that Λ_k and Λ_{k+1} satisfy the same commutation relation as $\Lambda_{I_k}, \Lambda_{I_{k+1}}$, where $I_1 = (-\infty, 1)$, $I_2 = (0, 1)$, $I_3 = (1, \infty)$ for t_k in an open neighborhood of the origin. Then Υ can be made a topological group and there is a continuous isomorphism between Υ and $\widetilde{\text{Möb}}$ which intertwines Λ_{I_k} and Λ_k .*

Proof. This is essentially covered by [GLW98, Lemma 1.1] by noting that the group generated by Λ_{I_k} is isomorphic to $\widetilde{\text{Möb}}$. Indeed, $\Lambda_{\mathbb{R}_-} = \Lambda_{I_3}^{-1}$, $\Lambda_{(1,\infty)} = \Lambda_{I_1}^{-1}$ and since $(-\infty, 0), (0, 1), (1, \infty)$ is a factorization of S^1 , hence they satisfy the commutation relations of the corresponding intervals.

Here we stress the topology of the Υ . By construction, $\widetilde{\text{Möb}}$ is isomorphic to a certain quotient group of Υ , and let q be the quotient map. There exists a neighbourhood of the origin $\mathcal{U} \subset \mathbb{R}^3$ such that the map

$$\Phi : \mathbb{R}^3 \supset \mathcal{U} \ni (t_1, t_2, t_3) \mapsto \Lambda_{I_1}(t_1)\Lambda_{I_2}(t_2)\Lambda_{I_3}(t_3) \in \widetilde{\text{Möb}},$$

is a diffeomorphism onto a neighbourhood of the identity in $\widetilde{\text{Möb}}$. We also define

$$\tilde{\Phi} : \mathbb{R}^3 \supset \mathcal{U} \ni (t_1, t_2, t_3) \mapsto \Lambda_1(t_1)\Lambda_2(t_2)\Lambda_3(t_3) \in \Upsilon.$$

It is still one-to-one because $\Phi = q \circ \tilde{\Phi}$ and Φ is one-to-one on \mathcal{U} . We introduce a neighbourhood system \mathcal{N} of the identity in Υ as all sets including the image of an open sets in \mathcal{U} containing $(0, 0, 0)$. By translation, the sets $\{g\mathcal{N}\}_{g \in \Upsilon}$ define a neighbourhood system of g , and the open sets in Υ are defined as those subsets of Υ which include $g\Phi(\mathcal{V})$ for some open $\mathcal{V} \subset \mathcal{U}$, $(0, 0, 0) \in \mathcal{V}$ as above: an image $\Phi(\mathcal{V})$ of an open set $\mathcal{V} \subset \mathcal{U}$ which is small enough is a neighborhood of any point g in it, because $g^{-1}\Phi(\mathcal{V})$ is contained in $\Phi(\mathcal{U})$. This topology makes Υ a topological group. It is clear that Υ is locally connected. Then Υ is isomorphic to $\widetilde{\text{Möb}}$ by universality of $\widetilde{\text{Möb}}$, see e.g. [Pon46, Theorem 63]. \square

As pointed out in [GLW98, Lemma 1.1], Υ has a natural structure as a Lie group. The map $\Phi_{\widetilde{\text{Möb}}}$ is a diffeomorphism respecting the Lie structure, and through q , we can also introduce a manifold structure on Υ .

2.4. Haag–Kastler nets. Let \mathcal{K} be the set of all the double cones in the Minkowski spacetime \mathbb{R}^{1+1} . A two-dimensional **Haag–Kastler net** (\mathcal{A}, U, Ω) is a net of von Neumann algebras $\{\mathcal{A}(D)\}_{D \in \mathcal{K}}$ in $\mathcal{B}(\mathcal{H})$ on a fixed Hilbert space \mathcal{H} , together with a strongly continuous unitary representation U of the Poincaré group \mathcal{P}_+^\uparrow and the vacuum vector Ω satisfying the following assumptions (see e.g. [Tan12, Section 2.1])

- (HK1) **Isotony:** if $D_1 \subset D_2$, then $\mathcal{A}(D_1) \subset \mathcal{A}(D_2)$.
- (HK2) **Locality:** if D_1 and D_2 are spacelike separated, then $\mathcal{A}(D_1) \subset \mathcal{A}(D_2)'$.
- (HK3) **Poincaré covariance:** it holds that

$$U(g)\mathcal{A}(D)U(g)^* = \mathcal{A}(gD), \quad \text{for } g \in \mathcal{P}_+^\uparrow, D \in \mathcal{K}.$$

- (HK4) **Positivity of the energy:** the joint spectrum of the translation subgroup in U is contained in the closed forward light cone $\overline{V}_+ = \{(a_0, a_1) \in \mathbb{R}^{1+1} : a_0^2 - a_1^2 \geq 0, a_0 \geq 0\}$.
- (HK5) **Vacuum and the Reeh–Schlieder property:** there exists a unique (up to a phase) vector $\Omega \in \mathcal{H}$ such that $U(g)\Omega = \Omega$ for $g \in \mathcal{P}_+^\uparrow$ and is cyclic for any local algebra, namely $\overline{\mathcal{A}(D)\Omega} = \mathcal{H}$.
- (HK6) **The Bisognano–Wichmann property:** Let Λ_{W_L} be the boost one-parameter group associated with the wedge W_L (see Sect. 2.2), and $\mathcal{A}(W_L) = \bigvee_{D \subset W_L} \mathcal{A}(D)$ then

$$U(\Lambda_{W_L}(2\pi t)) = \Delta_{\mathcal{A}(W_L), \Omega}^{it}.$$

We included the Reeh–Schlieder property already in the axioms, because it follows from weak additivity which is traditionally included in the axioms. The Bisognano–Wichmann property is not automatic in general (see e.g. [Mor18]), but is a consequence of Möbius covariance (see below) [BGL93], and it is a natural necessary condition for Möbius covariance.

Let (\mathcal{A}, U, Ω) be a net satisfying (HK1)–(HK6). We can also define algebras for more general open regions X by $\mathcal{A}(X) = \bigvee_{D \subset X} \mathcal{A}(D)$ such as, for instance, wedges, forward and backward lightcones and half-bands.

A Poincaré covariant net (\mathcal{A}, U, Ω) is said to be **Möbius covariant** if the representation U extends to the two-dimensional Möbius group \tilde{G} (see Sect. 2.2) which acts covariantly on the extension of the net \mathcal{A} to the cylinder \tilde{M} . In such a case we shall say that (\mathcal{A}, U, Ω) is a Möbius covariant net.

In order to deduce Möbius covariance, we further introduce the following conditions.

(HK7) **Dilation covariance:** we assume that U extends to a representation of the group of Poincaré transformations \mathcal{P}_+^\uparrow and the dilation group Λ_{V_+} , which still acts covariantly on the net, namely $U(g)\mathcal{A}(D)U(g)^* = \mathcal{A}(gD)$ for all g in the Poincaré-dilation group.

(HK8) **Reeh–Schlieder property for V_+ :** Ω is cyclic and separating for $\mathcal{A}(V_+)$.

(HK9) **Bisognano–Wichmann property for dilations:**

$$U(\Lambda_{V_+}(2\pi t)) = \Delta_{\mathcal{A}(V_+), \Omega}^{it}$$

(HK10) One of the following conditions holds.

(a) **Modular covariance:**⁴

$$\begin{aligned} \text{Ad } \Delta_{B_L}^{it}(\mathcal{A}(D_0)) &= \text{Ad } U(\delta(-2\pi t) \times \iota)(\mathcal{A}(D_0)) \\ &= (\text{Ad } U(\Lambda_{\mathbb{R}_+}(2\pi t) \times \iota)(\mathcal{A}(D_0))), \end{aligned}$$

in particular, $\mathcal{A}(D_0) \subset \mathcal{A}(B_L)$ is a +-HSMI.

(b) **\tilde{M} -strong additivity:** Let $a < b < c$ and $d > 0$ in \mathbb{R} and $B_{L,(a,c)}$ and $B_{L,(a,b)} + (d, 0)$ two half-band with a common edge (see Fig. 2), then

$$\mathcal{A}(D_{(0,d)(b,c)}) = \mathcal{A}(B_{L,(a,c)}) \cap \mathcal{A}(B_{L,(a,b)} + (d, 0))'$$

The condition (HK10a) is concerned with the modular groups of half-bands, and might look too strong. On the other hand, (HK10b) can be considered as a variation of strong additivity: indeed, let the net be Möbius covariant and extend to \tilde{M} . We say the net is strongly additive if $\mathcal{A}(D) = \mathcal{A}(D_1) \vee \mathcal{A}(D_2)$, where D, D_1, D_2 are double cones such that D_1 and D_2 share one boundary point, are spacelike to each other and the causal completion of their union is D . It is immediate to see that this condition is equivalent to Haag duality on M (see for the latter e.g. [KL04, Section 2.1 a)], [CLTW12, Proposition 5.2, a)]. Furthermore, if \mathcal{A} is Möbius covariant, then Haag duality on \tilde{M} is automatic [BGL93, Theorem 2.3(i)]. Therefore, under Möbius covariance, Haag duality on M is equivalent to (HK10b). For this reason we call (HK10b) a (variant of) strong additivity, although \mathcal{A} does not *a priori* extend to \tilde{M} .

Differently from (HK10a), (HK10b) does not refer to modular groups which are without *a priori* Möbius covariance difficult to determine, hence is more transparent for a sufficient condition for the implication “dilation + $\alpha \implies$ Möbius”.

⁴ This requirements is different from $\text{Ad } \Delta_{B_L}^{it} = \text{Ad } U(\delta(-2\pi t) \times \iota)$. Indeed, if U extends to G , we show that $\Delta_{B_L}^{it} = U(\delta(-2\pi t) \times \Lambda_{(0,1)}(2\pi t))$.

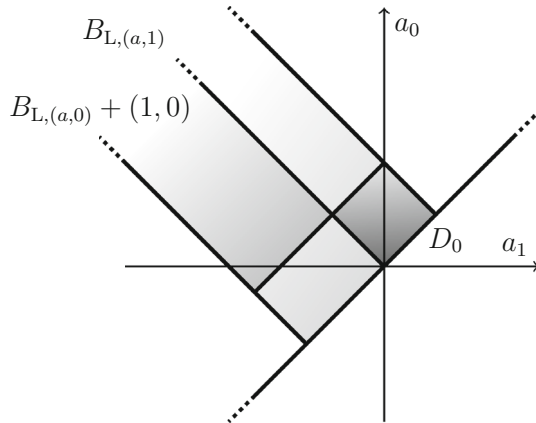


Fig. 2. Double cone D_0 as the relative causal complement of the shifted half-band $B_{L,(a,0)} + (1, 0)$ in $B_{L,(a,1)}$ with $a < 0$

In Proposition 3.2 below we show that (HK10b) implies (HK10a). Only (HK1)–(HK10a) are needed for the proof of conformal covariance in Theorem 3.3.

3. Proof of Möbius Covariance

Lemma 3.1. *Let (\mathcal{A}, U, Ω) be a von Neumann algebra net satisfying conditions (HK1)–(HK9). Then U extends to a representation $U_{\mathbb{R}}$ of the group $\mathbf{P} \times \widetilde{\text{Möb}}$ (and analogously $\widetilde{\text{Möb}} \times \mathbf{P}$) and local covariance holds for left half-band algebras: $\text{Ad } U_{\mathbb{R}}(g)(\mathcal{A}(\widetilde{B_{L,(a,b)}})) = \mathcal{A}(g \cdot B_{L,(a,b)})$ as long as g is in a neighborhood of the unit element of $\mathbf{P} \times \widetilde{\text{Möb}}$ whose action does not take $B_{L,(a,b)}$ out of the Minkowski space M . Similar covariance holds for left wedges.*

Proof. We shall denote $\mathcal{M}_1 = \mathcal{A}(W_L + (0, 1))$, $\mathcal{M}_2 = \mathcal{A}(B_L)$, $\mathcal{M}_3 = \mathcal{A}(V_+)$ and Δ_k , $k = 1, 2, 3$ the associated modular operators w.r.t the vacuum vector Ω . By (HK6) and (HK9), we have that $\mathcal{M}_2 = \mathcal{A}(B_L) \subset \mathcal{A}(V_+) = \mathcal{M}_3$ and $\mathcal{M}_2 = \mathcal{A}(B_L) \subset \mathcal{A}(W_L + (0, 1)) = \mathcal{M}_1$ are $+$ - and $-$ -HSMI, respectively (see Fig. 3). Therefore, by Lemma 2.3, their modular groups $\Delta_2^{it_1}$, $\Delta_3^{it_3}$ ($\Delta_1^{it_1}$, $\Delta_2^{it_2}$ respectively) satisfy the commutation relations of $\Lambda_{(0,1)}$ and $\Lambda_{\mathbb{R}_+}$ (respectively those of $\Lambda_{\mathbb{R}_+ + 1}$ and $\Lambda_{(0,1)}$). Let $\delta(\cdot) \times \iota$ be dilations along the line $a_0 + a_1 = 0$. They are included in the Poincaré-dilation group: indeed, $\delta(t) \times \iota = \Lambda_{V_+}(-\frac{t}{2})\Lambda_{W_L}(-\frac{t}{2})$, and hence we have $U(\delta(t) \times \iota) = U(\Lambda_{V_+}(-\frac{t}{2}))U(\Lambda_{W_L}(-\frac{t}{2}))$. By Lemma 2.1, all the above modular groups Δ_k^{it} commute with $U(\delta(t) \times \iota)$, since the latter preserves each algebra and the vacuum. Note also that

$$\begin{aligned} &U(\delta(t) \times \iota)U(\delta(s) \times \iota) \\ &= U\left(\delta\left(\ln(e^{t+s} + 1 - e^t)\right) \times \iota\right)U\left(\delta\left(\ln\left(\frac{e^{t+s}}{e^{t+s} + 1 - e^t}\right)\right) \times \iota\right) \end{aligned} \quad (2)$$

as $U(\delta(\cdot) \times \iota)$ is a one-parameter group. In particular, they satisfy the commutation relations of $\Lambda_{(0,1)}$ and $\Lambda_{\mathbb{R}_+}$. Similarly, it also satisfies the commutation relations of $\Lambda_{(0,1)}$ and $\Lambda_{\mathbb{R}_+ + 1}$. Therefore, by straightforward computations, we have the following:

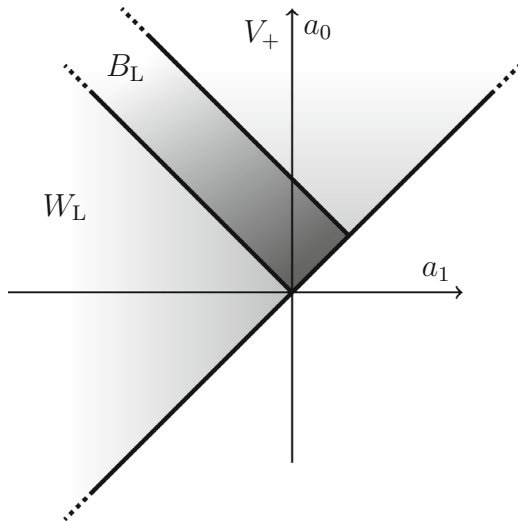


Fig. 3. Regions V_+ , B_L , $W_L + (1, 1)$ and their shadows \mathbb{R}_+ , $(0, 1)$, $\mathbb{R}_- + 1$ on the line $a_0 = a_1$

- $\Delta_2^{it_2} U(\delta(2\pi t_2) \times \iota)$ and $\Delta_3^{it_3} U(\delta(2\pi t_3) \times \iota) = U(\iota \times \delta(-2\pi t_3))$ satisfy the commutation relations of $\Lambda_{(0,1)}$ and $\Lambda_{\mathbb{R}_+}$.
- $\Delta_2^{it_2} U(\delta(2\pi t_2) \times \iota)$ and $\Delta_1^{it_1} U(\delta(2\pi t_1) \times \iota) = U(\iota \times \Lambda_{\mathbb{R}_{-+1}}(2\pi t_1))$ satisfy the commutation relations of $\Lambda_{(0,1)}$ and $\Lambda_{\mathbb{R}_{-+1}}$.

On the other hand, it is immediate that also $U(\iota \times \delta(-2\pi t_3)) = U(\iota \times \Lambda_{\mathbb{R}_+}(2\pi t_3))$ and $U(\iota \times \Lambda_{\mathbb{R}_{-+1}}(2\pi t_1))$ satisfy the commutation relations of $\Lambda_{\mathbb{R}_+}$ and $\Lambda_{\mathbb{R}_{-+1}}$. Therefore, by Lemma 2.4, we obtain a strongly continuous representation U_R of $\widetilde{\text{Möb}}$ which coincides with $U(\iota \times \cdot)$ when restricted to $\iota \times \mathbf{P}$ by construction.

We claim that this representation commutes with $U(g \times \iota)$, $g \in \mathbf{P}$. We only have to show that $\Delta_2^{it} U(\delta(2\pi t) \times \iota)$ commutes with $U(g \times \iota)$. By Lemma 2.1, Δ_2^{it} and hence $\Delta_2^{it} U(\delta(2\pi t) \times \iota)$ commute with $U(\delta(s) \times \iota)$ since $\text{Ad } U(\delta(2\pi t) \times \iota)(\mathcal{A}(B_L)) = \mathcal{A}(B_L)$ and $U(\delta(s) \times \iota)\Omega = \Omega$. Furthermore, as $\text{Ad } U(\tau(a) \times \iota)(\mathcal{A}(B_L)) \subset \mathcal{A}(B_L)$ for $a \geq 0$ and $U(\tau(a) \times \iota)$ preserves Ω , by Theorem 2.2 we have that $\text{Ad } \Delta_2^{it}(U(\tau(a) \times \iota)) = U(\tau(e^{-2\pi t} a) \times \iota)$. Moreover, it also holds that $\text{Ad } U(\delta(2\pi t_2) \times \iota)(U(\tau(a) \times \iota)) = U(\tau(e^{2\pi t} a) \times \iota)$ as it is a representation of \mathbf{P} . It follows that $\Delta_2^{it} U(\delta(2\pi t) \times \iota)$ commutes with $U(\tau(a) \times \iota)$.

Altogether, $U(g_1 \times \iota)$ and $U_R(\iota \times g_2)$ commute for $g_1 \in \mathbf{P}$, $g_2 \in \widetilde{\text{Möb}}$. We define a representation U_R of $\mathbf{P} \times \widetilde{\text{Möb}}$ by $U_R(g_1 \times g_2) := U(g_1)U_R(g_2)$.

Now we prove local covariance of $\{\mathcal{A}(B_{L,(a,b)})\}$ with respect to U_R . It is enough to check it for $\iota \times g \in \iota \times \widetilde{\text{Möb}}$, because covariance for elements in $\mathbf{P} \times \iota$ is the assumption (HK3) and (HK7). We view that $\widetilde{\text{Möb}}$ acts on the universal covering $\widetilde{S^1}$ of S^1 (which is homeomorphic to \mathbb{R} , see Fig. 4). Let us denote by $I_{(a,b)}$ the interval in $\widetilde{S^1}$ corresponding to (a, b) in \mathbb{R} . We also identify \mathbb{R} with an interval S^1 and denote it by $I_{\mathbb{R}}$. In this and the next paragraphs, we consider \mathcal{A} as a net defined on open regions in $I_{\mathbb{R}} \times I_{\mathbb{R}}$. Let $I \in I_{\mathbb{R}}$ be a bounded interval. Any $g \in \widetilde{\text{Möb}}$ such that $gI \in I_{\mathbb{R}}$ can be written as a product of three elements $g = g_1 g_2 g_3$, such that $g_1, g_3 \in \mathbf{P}$, $g_3 I = I_{(0,1)}$, $g_1 \cdot I_{(0,1)} = gI$ and

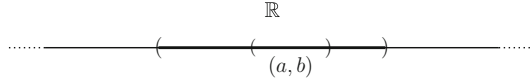


Fig. 4. The universal covering of S^1 , and \mathbb{R} as an interval on it

$g_2 = \Lambda_{(0,1)}(t_2)$ for some $t_2 \in \mathbb{R}$. For such $g \in \widetilde{\text{Möb}}$, note that $U_{\mathbb{R}}(\iota \times g_1)$ and $U_{\mathbb{R}}(\iota \times g_3)$ acts geometrically by (HK3) and (HK7), and $U_{\mathbb{R}}(\iota \times g_2) = \Delta_2^{it_2} U(\delta(2\pi t_2) \times \iota)$ preserves $\mathcal{A}(B_L) = \mathcal{A}(I_{\mathbb{R}_+} \times g_3 I) = \mathcal{A}(I_{\mathbb{R}_+} \times I_{(0,1)})$, therefore, we have

$$\begin{aligned} \text{Ad } U_{\mathbb{R}}(\iota \times g)(\mathcal{A}(I_{\mathbb{R}_+} \times I)) &= \text{Ad } U_{\mathbb{R}}(\iota \times g_1 g_2 g_3)(\mathcal{A}(I_{\mathbb{R}_+} \times I)) \\ &= \text{Ad } U_{\mathbb{R}}(\iota \times g_1 g_2)(\mathcal{A}(I_{\mathbb{R}_+} \times g_3 I)) \\ &= \text{Ad } U_{\mathbb{R}}(\iota \times g_1)(\mathcal{A}(I_{\mathbb{R}_+} \times g_3 I)) \\ &= \mathcal{A}(I_{\mathbb{R}_+} \times g I), \end{aligned}$$

which is the desired local covariance for left half-bands.

At this point, we can prove **continuity from below** (c.f. [FJ96, (24)]) for left half-bands, namely, $\mathcal{A}(I_{\mathbb{R}_+} \times I) = \bigvee_{I_\alpha \in I} \mathcal{A}(I_{\mathbb{R}_+} \times I_\alpha)$. Indeed, for each I_α we can find $g_\alpha \in \widetilde{\text{Möb}}$ such that $g_\alpha I = I_\alpha$ and $g_\alpha \rightarrow \iota$ as I_α tends to I . Now by continuity of $U_{\mathbb{R}}$ in the strong operator topology, $\bigvee_{I_\alpha \in I} \mathcal{A}(I_{\mathbb{R}_+} \times I_\alpha) = \bigvee_{\alpha} \text{Ad } U_{\mathbb{R}}(\iota \times g_\alpha) \mathcal{A}(I_{\mathbb{R}_+} \times I) \supset \mathcal{A}(I_{\mathbb{R}_+} \times I)$, and the converse inclusion is trivial.

Now we bring back the original notations and I, I_α are intervals in \mathbb{R} . As for wedges, we have by definition that $\mathcal{A}(W_L) = \bigvee_{I \in \mathbb{R}_-} \mathcal{A}(\mathbb{R}_+ \times I)$. Therefore, for g which takes any interval of \mathbb{R}_- into a compact interval in \mathbb{R} , we have $\text{Ad } U_{\mathbb{R}}(\iota \times g)(\mathcal{A}(W_L)) = \bigvee_{I \in \mathbb{R}_-} \mathcal{A}(\mathbb{R}_+ \times g I) = \mathcal{A}(\mathbb{R}_+ \times g \mathbb{R}_-)$, where the last equality follows from continuity from below. In other words, covariance holds also for W_L . By taking $g = g_1 g_2$ where $g_2 \in \mathbf{P}$, covariance holds for any $W_L + (0, a_R)$, $a_R \in \mathbb{R}$. Finally, as $U_{\mathbb{R}}(\tau(a_L) \times \iota)$ commutes with $U_{\mathbb{R}}(\iota \times g)$, the above covariance holds for any left wedge. \square

Proposition 3.2. *Let (\mathcal{A}, U, Ω) be a von Neumann algebra net satisfying conditions (HK1)–(HK9)(HK10b). Then it also satisfies (HK10a).*

Proof. By (HK10b), it holds that $\mathcal{A}(D_0) = \mathcal{A}(\mathbb{R}_+ \times (a_L, 1)) \cap \mathcal{A}((\mathbb{R}_+ + 1) \times (a_L, 0))'$ for any $a_L < 0$. For $t > 0$, $\Lambda_{(0,1)}(t)$ takes any interval in \mathbb{R}_- in \mathbb{R}_- itself, and say, $\Lambda_{(0,1)}(t) \cdot a_L = b_L < 0$, and $\Lambda_{(0,1)}(t) \cdot 1 = 1$, $\Lambda_{(0,1)}(t) \cdot 0 = 0$. By covariance of Lemma 3.1, we have

$$\text{Ad } U_{\mathbb{R}}(\iota \times \Lambda_{(0,1)}(t))(\mathcal{A}(D_0)) = \mathcal{A}(\mathbb{R}_+ \times (b_L, 1)) \cap \mathcal{A}((\mathbb{R}_+ + 1) \times (b_L, 0))' = \mathcal{A}(D_0),$$

where the last equality holds for any negative number b_L by (HK10b). Now we can invert t and the equality holds also for $t \leq 0$. Recalling the definition $U_{\mathbb{R}}(\iota \times \Lambda_{(0,1)}(t)) = \Delta_{B_L}^{it} U(\delta(2\pi t) \times \iota)$, and that $\text{Ad } U_{\mathbb{R}}(\iota \times \Lambda_{(0,1)}(t))$ preserves $\mathcal{A}(D_0)$, we conclude that $\text{Ad } U(\delta(-2\pi t) \times \iota)(\mathcal{A}(D_0)) = \text{Ad } \Delta_{B_L}^{it}(\mathcal{A}(D_0))$. \square

Theorem 3.3. *Let (\mathcal{A}, U, Ω) be a von Neumann algebra net satisfying conditions (HK1)–(HK10a). Then U extends to the two-dimensional Möbius group G and with this extension (\mathcal{A}, U, Ω) is a Möbius covariant net.*

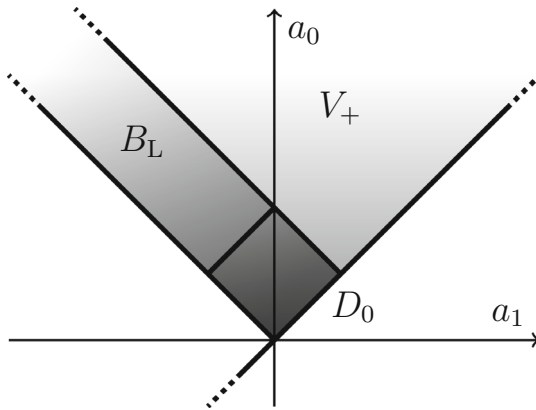


Fig. 5. Triple HSMI, $D_0 \subset B_L$, $B_L \subset V_+$, $D_0 \subset V_+$

Proof. Let us similarly define U_L (the only assumption which is not symmetric between left and right is (HK10a), but we have not used it for Lemma 3.1). Namely, $U_L(\Lambda_{(0,1)}(2\pi s) \times \iota) = \Delta_{B_R}^{is} U(\iota \times \delta(2\pi s))$. We are going to prove that $U_R(\iota \times \Lambda_{(0,1)}(2\pi t))$ and $U_L(\Lambda_{(0,1)}(2\pi s) \times \iota)$ commute, and hence we will have a representation of $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$.

First, we show that $U(\iota \times \delta(2\pi t))$ commutes with $\Delta_{D_0}^{is} U_R(\iota \times \Lambda_{(0,1)}(-2\pi s))$. Note that

- From (HK10a) and Lemma 2.1, $U_R(\iota \times \Lambda_{(0,1)}(2\pi s)) = \Delta_2^{it} U(\delta(2\pi t) \times \iota)$ commutes with the modular group $\Delta_{D_0}^{it}$ of $\mathcal{A}(D_0)$.
- By (HK9) and (HK10a) respectively, namely $\Delta_{V_+}^{it} = U(\delta(-2\pi t) \times \delta(-2\pi t))$ and $\Delta_{B_L}^{is} = U_R(\delta(-2\pi s) \times \Lambda_{(0,1)}(2\pi s))$ (see Lemma 3.1), the following three are all +-HSMI (see Fig. 5):

$$\mathcal{A}(D_0) \subset \mathcal{A}(V_+), \quad \mathcal{A}(D_0) \subset \mathcal{A}(B_L), \quad \mathcal{A}(B_L) \subset \mathcal{A}(V_+).$$

- The same relation as (2) holds for any one-parameter group.

Therefore, by putting $s_1 = \ln(e^{t+s} + 1 - e^t)$, $t_1 = \ln\left(\frac{e^{t+s}}{e^{t+s} + 1 - e^t}\right)$ for small t we have

$$\begin{aligned} & U(\iota \times \delta(-2\pi t)) \Delta_{D_0}^{is} U_R(\iota \times \Lambda_{(0,1)}(-2\pi s)) U(\iota \times \delta(2\pi t)) \\ &= U(\delta(2\pi t) \times \iota) \Delta_{V_+}^{it} \Delta_{D_0}^{is} \Delta_{B_L}^{-is} U(\delta(-2\pi s) \times \iota) \Delta_{V_+}^{-it} U(\delta(-2\pi t) \times \iota) && \text{(def. of } U_R, \text{ (HK9))} \\ &= U(\delta(2\pi t) \times \iota) \Delta_{V_+}^{it} \Delta_{D_0}^{is} \Delta_{B_L}^{-is} \Delta_{V_+}^{-it} U(\delta(-2\pi s) \times \iota) U(\delta(-2\pi t) \times \iota) && \text{(reordering)} \\ &= U(\delta(2\pi t) \times \iota) \Delta_{D_0}^{is_1} \Delta_{V_+}^{it_1} \Delta_{V_+}^{-it_1} \Delta_{B_L}^{-is_1} U(\delta(-2\pi s) \times \iota) U(\delta(-2\pi t) \times \iota) && \text{(by Eq. (1))} \\ &= U(\delta(2\pi t) \times \iota) \Delta_{D_0}^{is_1} \Delta_{B_L}^{it_1} \Delta_{B_L}^{-it_1} \Delta_{B_L}^{-is_1} U(\delta(-2\pi s) \times \iota) U(\delta(-2\pi t) \times \iota) && \text{(cancelling factors)} \\ &= U(\delta(2\pi t) \times \iota) \Delta_{B_L}^{it} \Delta_{D_0}^{is} \Delta_{B_L}^{-is} \Delta_{B_L}^{-it} U(\delta(-2\pi s) \times \iota) U(\delta(-2\pi t) \times \iota) && \text{(by Eqs. (1)(2))} \\ &= U_R(\iota \times \Lambda_{(0,1)}(2\pi t)) \Delta_{D_0}^{is} U_R(\iota \times \Lambda_{(0,1)}(2\pi(-s-t))) && \text{(def. of } U_R) \\ &= \Delta_{D_0}^{is} U_R(\iota \times \Lambda_{(0,1)}(-2\pi s)). && ([U_R(\Lambda_2(t)), \Delta_{D_0}^{is}] = 0) \end{aligned}$$

But if this is valid for small t , it is valid also for any t by iteration.

We claim that $\Delta_{D_0}^{it} = U_R(\iota \times \Lambda_{(0,1)}(2\pi t))U_L(\Lambda_{(0,1)}(2\pi t) \times \iota)$. On one hand, we know from the previous paragraph that

$$\text{Ad}U(\iota \times \delta(2\pi s))(\Delta_{D_0}^{it} U_R(\iota \times \Lambda_{(0,1)}(-2\pi t))) = \Delta_{D_0}^{it} U_R(\iota \times \Lambda_{(0,1)}(-2\pi t)).$$

On the other hand, by Theorem 2.2 and Proposition 3.1, we have

$$\text{Ad}U(\iota \times \delta(2\pi s))(\Delta_{D_0}^{it} U_R(\iota \times \Lambda_{(0,1)}(-2\pi t))) = \Delta_{\iota \times \delta(2\pi s) \cdot D_0}^{it} U_R(\iota \times \Lambda_{(0, e^{2\pi s})}(-2\pi t)).$$

Combining these two equalities and the limit $s \rightarrow \infty$, we obtain $\Delta_{D_0}^{it} U_R(\iota \times \Lambda_{(0,1)}(-2\pi t)) = \Delta_{B_R}^{it} U(\iota \times \Lambda_{\mathbb{R}_+}(-2\pi t))$ (if $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \dots \subset \mathcal{M} = \bigvee_n \mathcal{M}_n$ and Ω is cyclic and separating for \mathcal{M}_1 and \mathcal{M} , then $\Delta_{\mathcal{M}_n}^{it}$ converges to $\Delta_{\mathcal{M}}^{it}$ [Lon78, Lemma 3]). Recall that $\Delta_{B_R}^{it} = U(\iota \times \delta(-2\pi t))U_L(\Lambda_{(0,1)}(t) \times \iota)$ by definition, and we have $U(\iota \times \delta(-2\pi t)) = U(\iota \times \Lambda_{\mathbb{R}_+}(2\pi t))$, hence, $\Delta_{D_0}^{it} = U_R(\iota \times \Lambda_{(0,1)}(2\pi t))U_L(\Lambda_{(0,1)}(2\pi t) \times \iota)$. In particular, $U_R(\iota \times \Lambda_{(0,1)}(t))$ and $U_L(\Lambda_{(0,1)}(s) \times \iota)$ commute and we obtain a representation U of $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ by $U(g_L \times g_R) = U_L(g_L \times \iota)U_R(\iota \times g_R)$.

Now we prove local covariance of double cone algebras under $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ action. By dilation and translation covariance, it is enough to consider D_0 . Let I_L and I_R be intervals on the line \mathbb{R} such that $D_0 = I_L \times I_R$. Recall that in the last step we proved $\Delta_{D_0}^{it} = U_+(\Lambda_{(0,1)}(-2\pi t))U_-(\Lambda_{(0,1)}(-2\pi t))$. Now let us take an element of the form $g \times \iota \in \widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$, such that $g_L = g_{L,1}g_{L,2}$, $g_R = g_{R,1}g_{R,2}$, where $g_{L,1}, g_{R,1} \in \mathbf{P}$, $g_{L,1} = \Lambda_{(0,1)}(t)$, $g_{R,1} = \Lambda_{(0,1)}(s)$ for some $t, s \in \mathbb{R}$. For such element, $\text{Ad}U(g_L \times g_R)(\mathcal{A}(D_0)) = \text{Ad}U(g_{L,1} \times g_{R,1})U(g_{L,2} \times g_{R,2})(\mathcal{A}(D_0)) = \text{Ad}U(g_{L,1} \times g_{R,1})(\mathcal{A}(D_0)) = \mathcal{A}((g_{L,1} \times g_{R,1}) \cdot D_0)$, because $U(g_{L,2} \times g_{R,2}) = U(\Lambda_{(0,1)}(t) \times \Lambda_{(0,1)}(s))$ preserves $\mathcal{A}(D_0)$ and covariance for $g_{L,1} \times g_{R,1} \in \mathbf{P} \times \mathbf{P}$ holds by assumptions. As any element $g \in \widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ which does not take D_0 outside the Minkowski space M can be written as above, this establishes local covariance with respect to $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$.

From here, by the conformal spin-statistic theorem (Theorem A.5), U factors through G and we conclude that (\mathcal{A}, U, Ω) is a Möbius covariant net. \square

Corollary 3.4. *Let (\mathcal{A}, U, Ω) be a conformal net satisfying (HK1)–(HK9) and (HK10b). Then U extends to the two-dimensional Möbius group G and with this extension (\mathcal{A}, U, Ω) is a Möbius covariant net.*

Proof. Immediate from Proposition 3.2 and Theorem 3.3. \square

4. Counterexamples

In this section we discuss several Haag–Kastler nets which are covariant with respect to the Poincaré-dilation group but cannot be extended to a Möbius covariant net. Constructions and results of these sections can be easily adapted to $(3+1)$ -dimensions. One should substitute the representation of $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ in Sect. 4.2.2 by “massive” representations of the conformal group in the sense of [Mac77], and note that any double cone O can be obtained as the intersection of countably many wedges.

4.1. Breaking the Bisognano–Wichmann property for future light cone. The Bisognano–Wichmann property for wedges holds in any Möbius covariant net on $(1+1)$ -dimensional the Minkowski spacetime [BGL93, Theorem 2.3]. Furthermore, since wedges are mapped to lightcones by Möbius transformations, the Bisognano–Wichmann property for V_+ holds in a Möbius covariant net (corresponding results hold for any conformally covariant net on higher-dimensional Minkowski space).

Let (\mathcal{A}, U, Ω) be a Möbius covariant net which admits a one-parameter inner symmetry, namely, there is a one-parameter unitary group $\{V(s)\}$ such that $\text{Ad } V(s)\mathcal{A}(O) = \mathcal{A}(O)$ and $[U(g), V(s)] = 0$ for $g \in G$ and $s \in \mathbb{R}$ and $V(s)\Omega = \Omega$. In this situation, we can construct a new representation of the Poincaré-dilation group by setting $U_V(g) = U(g)$ for $g \in \mathcal{P}_+^\uparrow$ and $U_V(\delta(t) \times \delta(t)) = U(\delta(t) \times \delta(t))V(t)$. It is easy to check that U_V is still a representation of the Poincaré-dilation group, \mathcal{A} is covariant under U_V and Ω is invariant under U_V . While the original net (\mathcal{A}, U, Ω) satisfies the Bisognano–Wichmann property, the new net $(\mathcal{A}, U_V, \Omega)$ violates it since the algebra $\mathcal{A}(V_+)$ and the vacuum Ω stay the same while $U_V(\delta(t) \times \delta(t))$ has been modified. Therefore, this U_V cannot be extended to G in such a way that \mathcal{A} is still covariant, because that would contradict the Bisognano–Wichmann property for V_+ [BGL93, Theorem 2.3].

It is easy to find such Möbius covariant nets. For example, if $(\mathcal{A}_0, U_0, \Omega_0)$ is a Möbius covariant net on S^1 with a one-parameter inner symmetry $V_0(s)$, one can just take the tensor product $\mathcal{A}(I_L \times I_R) := \mathcal{A}_0(I_L) \otimes \mathcal{A}_0(I_R)$, $U(g_L \times g_R) := U_0(g_L) \otimes U_0(g_R)$, $\Omega := \Omega_0 \otimes \Omega_0$, and $V(s) := V_0(s) \otimes V_0(s)$. As concrete examples, one can take a loop group net $\mathcal{A}_{G,k}$ with a compact group G at level k , and as V_0 one can just take any one-parameter group in G which acts as inner symmetry [GF93, Section III]. One can also consider the tensor product of two copies of the $U(1)$ -current net as a chiral component (see [Tan14, Section 5] for the construction of inner symmetry on the complex massive free field, and [BT15, Section 5.2] for restricting it to a lightray to obtain a net on S^1).

As for the converse, we do not know whether it is always possible to satisfy (HK9) by modifying U of a given Poincaré-dilation covariant net with (HK8), see the discussions in Sect. 5.1. The fact that it is possible to modify the representation of the Poincaré-dilation group is known in the physics literature [Nak15, Section 2.2, below (2.6)]. Finally, we remark that it is also easy in $(1+1)$ -dimension to violate the Bisognano–Wichmann property for wedges [Tan14, Section 5].

4.2. BGL construction and generalized free fields. The simplest two-dimensional quantum field theory is the massive free field. It cannot be dilation covariant because it has an isolated mass shell. Yet, if one glue together continuously many massive free fields, the mass spectrum becomes continuous and dilation may act on it. This idea can be indeed realized as generalized free field. Here we take the construction based on the one-particle representation of the Poincaré group: to a positive energy (anti-)unitary representation of the Poincaré group with the CPT transformation, Brunetti, Guido and Longo associated a net of real subspaces on wedges, and its second quantized net [BGL02].

When the given representation of the Poincaré group extends to Möbius group $G = (\widehat{\text{Möb}} \times \widehat{\text{Möb}})/\mathbb{Z}_2$, there are two choices for double cones: either one defines the real subspaces for double cones by covariance à la BGL, or by duality for wedges. We take a representation of G such that the former construction yields a Möbius covariant net without chiral components, while the latter contains a generalized free field and fails to have both Möbius covariance and the split property.

Generalized free fields can have dilation covariance but fail to be Möbius covariant, see e.g. [DR09, Section 3.1]. In this case, a generalized free field has scaling dimension which is not consistent with the unitarity condition of the Möbius group (or the conformal group in the case of $(3 + 1)$ -dimensions), and consequently, the field cannot be Möbius covariant. This is different from our counterexamples. We consider first a Möbius covariant net (which corresponds to a generalized free field with unitary scaling dimension), and show that its dual net is dilation covariant but not Möbius covariant.

4.2.1. Nets of standard subspaces and first quantization nets A general reference for this section is [Lon08]. Firstly, we recall the notion of real subspaces. A linear, real, closed subspace H of a complex Hilbert space \mathcal{H} is called **cyclic** if $H + iH$ is dense in \mathcal{H} , **separating** if $H \cap iH = \{0\}$ and **standard** if it is cyclic and separating.

If H is a real linear subspace of \mathcal{H} , the *symplectic complement* of H is defined by

$$H' \equiv \{\xi \in \mathcal{H} ; \operatorname{Im} \langle \xi, \eta \rangle = 0, \text{ for } \eta \in H\} = (iH)^{\perp_{\mathbb{R}}},$$

where $\perp_{\mathbb{R}}$ denotes the orthogonal in \mathcal{H} with respect to the real part of the scalar product on \mathcal{H} . H' is a closed, real linear subspace of \mathcal{H} . If H is standard, then $H = H''$. H is cyclic (respectively separating) if and only if H' is separating (respectively cyclic), thus H is standard if and only if H' is standard. The **Tomita operator** S_H associated to a standard subspace $H \subset \mathcal{H}$ is the densely defined closed anti-linear involution $H + iH \ni \xi + i\eta \mapsto \xi - i\eta \in H + iH$. The polar decomposition $S_H = J_H \Delta_H^{1/2}$ defines the positive self-adjoint **modular operator** Δ_H and the anti-unitary **modular conjugation** J_H . In particular, Δ_H is invertible and

$$J_H \Delta_H J_H = \Delta_H^{-1}. \tag{3}$$

We further have that $S_{H'} = S_H^*$, $J_H H = H'$ and $\Delta_H^{it} H = H$ for every $t \in \mathbb{R}$. The one-parameter, strongly continuous group $t \mapsto \Delta_H^{it}$ is the **modular group** of H . There is a 1–1 correspondence between Tomita operators and standard subspaces, namely between standard subspaces $H \subset \mathcal{H}$, operators S which are closed, densely defined anti-linear involutions on \mathcal{H} and pairs (J, Δ) of an anti-unitary involution J and a positive self-adjoint operator Δ on \mathcal{H} satisfying (3). We recall the following analogue of Takesaki's theorem [Lon08, Proposition 2.1.10].

Lemma 4.1. *Let $K \subset H \subset \mathcal{H}$ be an inclusion of standard subspaces in \mathcal{H} . If $\Delta_H^{it} K = K$ for every $t \in \mathbb{R}$, then $K = H$.*

Let us denote the set of (left and right) wedges by \mathcal{W} in the two-dimensional Minkowski space M . The one-parameter group of Lorentz boosts $\Lambda_{W_L}(t) = \delta(-t) \times \delta(t)$ is associated to W_L . We associate to a general left wedge $W = gW_L$ the one-parameter group $\Lambda_W(t) = g\Lambda_{W_L}(t)g^{-1}$, and to a right wedge $W = gW_R$ the reversed one-parameter group $\Lambda_W(t) = g\Lambda_{W_L}(-t)g^{-1}$.

Let \mathcal{P}_+ be the proper Poincaré group, namely, it is generated by the connected component \mathcal{P}_+^\uparrow of the full Poincaré group and $j : (a_L, a_R) \mapsto (-a_L, -a_R)$. Let α be the action of j on \mathcal{P}_+^\uparrow , then $\mathcal{P}_+ = \mathcal{P}_+^\uparrow \rtimes_{\alpha} \mathbb{Z}_2$. We introduce also $j_W := gjg^{-1}$, where $g \in \mathcal{P}_+^\uparrow$ is such that $W = gW_L$ or gW_R , depending on whether W is a left or right wedge (this does not depend on the choice of g , hence is well-defined). Let U be a (anti-)unitary, positive energy representation of \mathcal{P}_+ , namely unitary on \mathcal{P}_+^\uparrow and anti-unitary on $j_W \mathcal{P}_+^\uparrow$. In particular j_W is represented by an anti-unitary operator $U(j_W)$. Define Δ_W by the

equation $U(\Lambda_W(t)) = \Delta_W^{i2\pi t}$. Brunetti, Guido and Longo associated a standard real subspace $H(W) = \ker(\mathbb{1} - U(j_W)\Delta_W^{\frac{1}{2}})$. They form a $(\mathcal{P}_+$ -covariant) **net of standard subspaces for wedges** in the following sense [BGL02, Theorem 4.7]:

- (SS1) **Isotony:** If W_1, W_2 and $W_1 \subset W_2$ then $H(W_1) \subset H(W_2)$;
- (SS2) **Poincaré Covariance:** $U(g)H(W) = H(gW)$ for $g \in \mathcal{P}_+, W \in \mathcal{W}$;
- (SS3) **Positivity of energy:** the joint spectrum of translations in U is contained in \overline{V}_+ ;
- (SS4) **Reeh–Schlieder property:** $H(W)$ is standard in \mathcal{H} for any $W \in \mathcal{W}$
- (SS5) **Locality:** for any $W_1 \subset W_2'$ then $H(W_1) \subset H(W_2)'$, where W' denotes the causal complement of W in the Minkowski space M
- (SS6) **Bisognano–Wichmann property:** $U(\Lambda_W(t)) = \Delta_W^{i2\pi t}$ for every $W \in \mathcal{W}$ and $t \in \mathbb{R}$.

Given a net of standard subspaces $\{H(W)\}$ for wedges, the **dual net** H^d of real subspaces is defined for a double cone O by

$$H^d(O) \doteq H(O')', \tag{4}$$

where the causal complement O' of O in the Minkowski space M consists of two wedges W_1, W_2 , and $H(O')$ is the real closed subspace spanned by $H(W_1)$ and $H(W_2)$. We do not know for which class of U $H(O)$ is standard, but we can prove it for a concrete class in Sect. 4.2.2. In that case, the net of standard subspaces $\{H^d(O)\}$ satisfies isotony, \mathcal{P}_+ -covariance, positivity of energy, the Reeh–Schlieder property and locality in the natural sense.

If U extends to the group $G \rtimes_{\alpha} \mathbb{Z}_2$ (where G is the two-dimensional Möbius group), there is another choice to assign a standard subspace to double cones. Let us define $H(O) = \ker(\mathbb{1} - U(j_O)\Delta_O^{\frac{1}{2}})$, where $U(j_O) = U(g)U(J_W)U(g)^*$ with g such that $O = gW$ and $\Delta_O = U(g)\Delta_W U(g)^*$. With this definition, the net of standard subspaces $\{H(O)\}$ satisfies isotony, $G \rtimes_{\alpha} \mathbb{Z}_2$ -covariance, positivity of energy, the Reeh–Schlieder property and locality in the natural sense where O is a double cone on \tilde{M} . We refer to this the **BGL construction** of standard subspaces associated with U .

Bosonic second quantization. Let \mathcal{H} be a (complex) Hilbert space and $\mathcal{F}_+(\mathcal{H})$ be the associated bosonic Fock space. Given a real subspace $H \subset \mathcal{H}$, we shall denote with $\mathcal{R}_+(H)$ the second quantization von Neumann algebra

$$\mathcal{R}_+(H) = \{W_+(f) : f \in H\}'' \subset \mathcal{B}(\mathcal{F}_+(\mathcal{H}))$$

where $W_+(f)$ are the Weyl operator⁵ on the Fock space, satisfying the CCR

$$W_+(f)W_+(g) = e^{\text{Im}(f,g)} W_+(f+g), \quad f, g \in H$$

and $\omega(W_+(f)) = (\Omega, W_+(f)\Omega) = e^{-\frac{1}{2}\|f\|^2}$. The second quantization construction respects the lattice structure and the modular theory. Let $\Gamma_+(A)$ be the multiplicative Bose second quantization of a one-particle operator A on \mathcal{H} .

Proposition 4.2 [Ara63, LRT78, LMR16]. *Let H and $\{H_{\kappa}\}$ be closed, real linear subspaces of \mathcal{H} . We have*

$$(a) \mathcal{R}_+(H)' = \mathcal{R}_+(H)';$$

⁵ The symbol W_+ should not be confused with wedges W . Weyl operators appear only in this section.

- (b) $\mathcal{R}_+(\sum_{\kappa} H_{\kappa}) = \bigvee_{\kappa} \mathcal{R}_+(H_{\kappa})$;
- (c) $\mathcal{R}_+(\bigcap_{\kappa} H_{\kappa}) = \bigcap_{\kappa} \mathcal{R}_+(H_{\kappa})$.
- (d) If H is standard, then $S_{\mathcal{R}_+(H),\Omega} = \Gamma_+(S_H)$, $J_{\mathcal{R}_+(H),\Omega} = \Gamma_+(J_H)$, $\Delta_{\mathcal{R}_+(H),\Omega} = \Gamma_+(\Delta_H)$.

In particular if $O \mapsto H(O)$ is a Poincaré covariant net of standard subspaces satisfying (SS1)–(SS6), then its second quantization $O \mapsto \mathcal{A}(O) = \mathcal{R}_+(H(O))$ is a Poincaré covariant net of von Neumann algebras satisfying (HK1)–(HK6) in Sect. 2.4. Furthermore, it follows from Proposition 4.2, that $\mathcal{A}^d(O) = \mathcal{R}_+(H^d(O))$, where $\mathcal{A}^d(O) := \mathcal{A}(O)'$ is the dual net of \mathcal{A} .

4.2.2. Möbius covariant nets without chiral components We apply the BGL construction to a particular class of representations of G . To be specific, let us take the irreducible positive energy representation U_0 of Möb with lowest weight 1. Let j_0 be the map $z \mapsto \bar{z}$ on S^1 and it acts on Möb through the identification of Möb as $SU(1, 1)$, which we call α_0 . U_0 extends to an (anti-)unitary representation of $\text{Möb} \rtimes_{\alpha_0} \mathbb{Z}_2$ which we denote again by U_0 , namely U_0 is unitary on Möb and anti-unitary on j_0 Möb. (see e.g. [Lon08, Section 1.6.2]). We can define a positive-energy (anti-)unitary representation of $G \rtimes_{\alpha} \mathbb{Z}_2$ by $U(g_L \times g_R) := U_0(g_L) \otimes U_0(g_R)$ and $U(j) := U(j_0) \otimes U(j_0)$ where, on the real line picture, $j_0 : \mathbb{R} \ni a \mapsto -a$. Note further that the joint spectrum of U restricted to the translation group \mathbb{R}^{1+1} has trivial spectral projections corresponding to the sets $\mathbb{R}_+ \times \{0\}$ or $\{0\} \times \mathbb{R}_+$. In the second quantization $\Gamma_+(U)$, these spectral projections contain only $\mathbb{C}\Omega$.

By the BGL construction, we obtain a Möbius covariant net $(\mathcal{A}_U, \Gamma_+(U), \Omega)$. This net does not have chiral components, i.e. $\mathcal{A}_{U,L}^{\max}(I_L) := \mathcal{A}(I_L \times I_R) \cap \left(\Gamma_+(U)(\iota \times \widetilde{\text{Möb}}) \right)' = \mathbb{C}\mathbb{1} = \mathcal{A}(I_L \times I_R) \cap \left(\Gamma_+(U)(\widetilde{\text{Möb}} \times \iota) \right)' =: \mathcal{A}_{U,R}^{\max}(I_R)$ (see [Reh00, Definition 2.1]). Indeed, if these algebras were nontrivial, the representation $\Gamma_+(U)(\tau \times \tau)$ of \mathbb{R}^{1+1} would have nontrivial spectral projections (properly larger than $\mathbb{C}\Omega$) corresponding to the sets $\mathbb{R}_+ \times \{0\}$ or $\{0\} \times \mathbb{R}_+$, which is a contradiction.

A Möbius covariant net \mathcal{A} is said to have the **split property** if for each $O \Subset \tilde{O}$ (namely, $\bar{O} \subset \tilde{O}$) there is an intermediate type I factor $\mathcal{R}_{O,\tilde{O}}$ such that $\mathcal{A}(O) \subset \mathcal{R}_{O,\tilde{O}} \subset \mathcal{A}(\tilde{O})$. To show that \mathcal{A}_U has the split property, let us consider the theory on restricted to the timelike line $a_R = a_L$, namely $\mathbb{R} \supset I \mapsto \mathcal{A}_U(O_I)$ where O_I is the minimal double cone including I . The diagonal action $\Gamma_+(U)(g \times g)$ acts on $\{\mathcal{A}(O_I)\}$ covariantly, where $g \in \text{Möb}$ and the one-particle conformal Hamiltonian is $L_0 \otimes 1 + 1 \otimes L_0$, where L_0 is the rotation generator in U_0 . As we took U_0 as an irreducible representation of Möb, the one-particle conformal Hamiltonian satisfies the trace class property:

$$\text{Tr}(e^{-\beta(L_0 \otimes 1 + 1 \otimes L_0)}) = \text{Tr}(e^{-\beta L_0})^2 < \infty, \quad \text{for any } \beta > 0$$

since $\text{Tr}(e^{-\beta L_0}) < \infty$. Now the trace class property is preserved through second quantization [Lon08, Corollary 7.4.2] and it ensures the split property for any inclusion $\mathcal{A}_U(O_I) \subset \mathcal{A}_U(O_{\tilde{I}})$, with $I \Subset \tilde{I}$ [BDL07, Corollary 6.4]. For any inclusion of double cones $O \Subset \tilde{O}$, we can find O_1 such that $O \Subset O_1 \Subset \tilde{O}$ and $O_1 \Subset \tilde{O}$ is conformally equivalent to some $O_I \Subset O_{\tilde{I}}$ for which the split property holds, therefore, $\mathcal{A}_U(O) \subset \mathcal{A}_U(\tilde{O})$ with $O \Subset \tilde{O} \subset \mathbb{R}^{1+1}$ satisfies the split property.

4.2.3. Dual net without Möbius covariance As $(\mathcal{A}, \Gamma_+(U), \Omega)$ is Möbius covariant, and especially Poincaré-dilation covariant, the dual net $(\mathcal{A}_U^d, \Gamma_+(U), \Omega)$, where $\mathcal{A}(O) = \mathcal{A}(O')' = \mathcal{A}(W_1) \cap \mathcal{A}(W_2)$, where W_1, W_2 are wedges such that $O = W_1 \cap W_2$, remains to be Poincaré-dilation covariant, because the set of wedges is closed under Poincaré and dilation transformations. Here we show that it cannot be extended to a Möbius covariant net because $\mathcal{A}_U^d(V_+) = \mathcal{B}(\mathcal{H})$. We identify U with the direct integral of massive representations U_m of \mathcal{P}_+^\uparrow , for which the property $H_{U_m}(V_+) = \mathcal{H}_m$ is well known. This provides an example of a Möbius covariant net $(\mathcal{A}_U, \Gamma_+(U), \Omega)$ whose dual net $(\mathcal{A}_U^d, \Gamma_+(U), \Omega)$ neither is Möbius covariant nor satisfies the split property.

The massive free field. Let U_m be the scalar representation of the Poincaré group \mathcal{P}_+ with mass m . It has the form

$$\begin{aligned} (U_m(a, \lambda)\xi)(p_1) &= e^{ia \cdot p_m} \xi(\lambda^{-1}(p_1)), & (a, \lambda) \in \mathcal{P}_+^\uparrow \\ (U_m(j)\xi)(p_1) &= \overline{\xi(p_1)}, \end{aligned}$$

where $p_m(p_1) = (\omega_m(p_1), p_1) \in \mathbb{R}^{1+1}$ (in (p_0, p_1) -coordinate, not in (p_L, p_R) -coordinate), $\omega_m(p_1) = \sqrt{m^2 + p_1^2}$, $\lambda(p_1) = -\sinh(\lambda) \omega_m(p_1) + \cosh(\lambda) p_1$ (where we identify $\lambda \in \mathbb{R}$ and an element of the Lorentz group) and $\xi \in \mathcal{H}_m = L^2(\mathbb{R}, \frac{dp_1}{2\omega_m(p_1)})$. Let $\{H_m(W)\}$ be the net of standard subspaces for wedges associated to U_m as in Sect. 4.2.1. We define subspaces relatively to double cones by duality as in Eq. (4). Actually, the more traditional construction of the free massive field net satisfies Haag duality, c.f. [Ost73], hence the one-particle local subspaces can be explicitly described as

$$\begin{aligned} H_m(O) &= \overline{\{\hat{f}^+(p_m) \in \mathcal{H}_m : f \in \mathcal{S}(\mathbb{R}^{1+1}, \mathbb{R}), \text{supp } f \subset O\}}, & \hat{f}^+(p_1) &= \frac{1}{2\pi} \int d^2a f(a) e^{-ia \cdot p_m}. \\ H_m(W) &= \overline{\{\hat{f}^+(p_m) \in \mathcal{H}_m : f \in \mathcal{S}(\mathbb{R}^{1+1}, \mathbb{R}), \text{supp } f \subset W\}}, \end{aligned}$$

We associate the following real subspaces to the forward and the backward light cones V_\pm :

$$H_m(V_\pm) = \overline{\sum_{O \subset V_\pm} H_m(O)}.$$

The following proposition is partly an adaptation of the arguments in [SW71].

Proposition 4.3. $H_m(V_\pm) = \mathcal{H}_m$

Proof. We prove the claim for V_- . It can be proved analogously for V_+ .

Let $O \subset V_-$ and take the vectors $\xi \in H_m(O)$ and $\eta \in H_m(V_-)'$. Consider the function

$$f(a) = \text{Im} \langle \eta, U(a)\xi \rangle.$$

It follows that f is real, it vanishes for any $a \in V_-$ and, since $(\square + m^2)f = 0$, $\text{supp } \hat{f} \subset (-\Omega_m \cup \Omega_m)$ as a distribution.

We claim that $f \equiv 0$. Let $d\mu(p) = \hat{f}(p) \delta(p^2 - m^2) d^2p$ be the measure associated to the Fourier transform of f , namely

$$f(a_0) = \int e^{ia_0 p_0} d\mu_{a_1}(p_0), \quad a = (a_0, a_1), p = (p_0, p_1)$$

where

$$\begin{aligned} d\mu_{a_1}(p_0) &= \int e^{-ia_1 p_1} d\mu(p_0, p_1) \\ &= \left(e^{-ia_1 \sqrt{p_0^2 - m^2}} f\left(p_0, \sqrt{p_0^2 - m^2}\right) + e^{-ia_1 \sqrt{p_0^2 - m^2}} f\left(p_0, -\sqrt{p_0^2 - m^2}\right) \right) dp_0. \end{aligned}$$

Now, note that the support of $a_0 \mapsto f(a_0, a_1)$ is contained in $\mathbb{R}^+ - |a_1|$, thus the Fourier transform $d\mu_{a_1}(p_0)$ extends to an analytic function on the upper complex half-plane. Furthermore, $p_0 \mapsto d\mu_{a_1}(p_0)$ is null on the interval $(-m, m)$ and hence for any test function h , the analytic function $h * d\mu_{a_1}(p_0)$ is 0 by the reflection principle. Therefore, $f(\cdot, a_1) \equiv 0$ for every $a_1 \in \mathbb{R}$, and hence $f \equiv 0$.

Now let η be in the symplectic complement of any $H_m(O)$: $\eta \in \left(\overline{\sum_{O \subset M} H_m(O)}\right)'$, hence in particular it belongs to $H_m(W')' \cap H_m(W)' = H_m(W) \cap H_m(W')$. Indeed,

$$\eta \in \bigcap_{O \subset W'} H_m(O)' = \left(\overline{\sum_{O \subset W'} H_m(O)}\right)' = H_m(W')' = H_m(W),$$

and similarly for W' .

We show that $\eta = 0$. Firstly we observe that $\eta \in H_m(W) \cap H_m(W')$ hence $\Delta_{H_m(W)}^{it} \eta = \eta$. Indeed,

$$\begin{aligned} \eta &\in \ker(\mathbb{1} - S_{H_m(W)}) \cap \ker(\mathbb{1} - S_{H_m(W')}) \Leftrightarrow \\ \eta &\in \ker(\Delta_{H_m(W)}^{1/2} - \Delta_{H_m(W)}^{-1/2}) \Leftrightarrow \\ &\Delta_{H_m(W)} \eta = \eta. \end{aligned}$$

In particular, by the Bisognano–Wichmann property, we have that $U(\Lambda_W(t))\eta = \eta$ for any wedge W and $t \in \mathbb{R}$. Since U_m is an irreducible representation of \mathcal{P}_+^\uparrow , boosts does not have proper invariant vectors, and we conclude that $\eta = 0$. \square

This proof can be adapted in any Minkowski space \mathbb{R}^{1+s} with $s \geq 1$.

The product representation as the direct integral of massive representations. We have seen in [BT15, Section 5.2] that the representation U_0 , restricted to the translation-dilation group, can be realized on $L^2(\mathbb{R}_+, pdp)$ as follows:

$$\begin{aligned} (U_0(\tau(t))\xi)(p) &= e^{itp} \xi(p) \\ (U_0(\delta(s))\xi)(p) &= e^{-s} \xi(e^{-s} p) \\ (U_0(j_0)\xi)(p) &= \overline{\xi(p)} \end{aligned}$$

and accordingly the product representation U restricted to the Poincaré-dilation group on $L^2(\mathbb{R}_+, p_L dp_L) \otimes L^2(\mathbb{R}_+, p_R dp_R)$ is given by

$$\begin{aligned} (U(\tau(t_L) \times \tau(t_R))\xi)(p_L, p_R) &= e^{i(t_L p_L + t_R p_R)} \xi(p_L, p_R) \\ (U(\delta(s_L) \times \delta(s_R))\xi)(p_L, p_R) &= e^{-s_L - s_R} \xi(e^{-s_L} p_L, e^{-s_R} p_R) \\ (U(j_0 \times j_0)\xi)(p_L, p_R) &= \overline{\xi(p_L, p_R)}. \end{aligned}$$

With this realization and the correspondence $2p_L p_R = p_0^2 - p_1^2 = m^2$, $p_1 = \frac{p_R - p_L}{\sqrt{2}}$, we have the natural identification

$$\begin{aligned} L^2(\mathbb{R}_+, p_L dp_L) \otimes L^2(\mathbb{R}_+, p_R dp_R) &\cong L^2(\mathbb{R}_+^2, p_L p_R dp_L dp_R) \\ &\cong L^2(V_+, \frac{m^3 dmdp_1}{2\omega_m(p_1)}) \\ &= \int_{\mathbb{R}_+}^{\oplus} d\mu(m) L^2(\mathbb{R}, \frac{m^3 dp_1}{2\omega_m(p_1)}). \end{aligned}$$

This identification is given by the map $\xi(p_L, p_R) \mapsto \xi'(m, p_1) = 2\xi(\frac{\omega_m(p_1)+p_1}{\sqrt{2}}, \frac{\omega_m(p_1)-p_1}{\sqrt{2}})$, where the factor 2 is needed to make it unitary. It is straightforward to check that this intertwines the representation U above and

$$(U'(a, \lambda)\xi')(m, p_1) = e^{ia \cdot (\omega_m(p_1), p_1)} \xi'(m, \lambda^{-1}(p_1)) \quad (a, \lambda) \in \mathcal{P}_+^\uparrow,$$

acting on $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R}, d\mu(m) dp \frac{m^3}{2\omega_m(p_1)}) = \int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m) L^2(\mathbb{R}, \frac{dp_1}{2\omega_m(p_1)})$, where the boost λ corresponds to $\delta(\lambda) \times \delta(-\lambda)$. Then, U decomposes into the direct integral

$$U = \int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m) U_m.$$

A generalized free field can act on the second quantization of this Hilbert space, and it is covariant with respect to $\Gamma_+(U)$. It is well known that the net of von Neumann algebras for generalized free fields does not always satisfy Haag duality [Lan74], depending on the measure on the space of m : it is proven that if the measure decays exponentially, Haag duality holds [Lan74]. We will show that the measure $m^3 d\mu(m)$ is associated with a Möbius covariant net while the dual net cannot be made Möbius covariant. Accordingly, we conjecture that the generalized free field corresponding to the measure $m^3 d\mu$ fails to have Haag duality.

The dual net \mathcal{A}^d is not Möbius covariant. Here we show that the dual net \mathcal{A}^d does not satisfy (HK8), therefore, it cannot be made Möbius covariant. It turns out that, since U is a direct integral of massive representations, \mathcal{A}^d does not satisfy the split property either.

Let $\{H_U(W)\}$ (respectively $\{H_U^d(O)\}$) be the covariant (respectively dual) BGL net of standard subspaces for wedges (respectively double cones) associated with the representation $U = \int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m) U_m$.

We show that $H_U^d(O) = \int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m) H_m(O)$, where the direct integral is taken with \mathcal{H}_U as a real Hilbert space. Note also that $H_m(O)$ is a μ -measurable family of (real) subspaces in the sense of [Dix81, Section II.1.7] (see Appendix B), because one can take a sequence of real test functions supported in O which separate the points in O (see Proposition B.2). Now, we have that

$$\begin{aligned} H_U^d(O) &= \bigcap_{W \supset O} H_U(W) \stackrel{(\bullet)}{=} \bigcap_{W \supset O} \int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m) H_m(W) \\ &\stackrel{(\star)}{=} \int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m) \bigcap_{W \supset O} H_m(W) = \int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m) H_m(O). \end{aligned}$$

Since U disintegrates as $\int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m)U_m$ it is easy to see that $H_U(W) = \ker(\mathbb{1} - U(j_W)\Delta_W^{\frac{1}{2}}) \supset \int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m)H_m(W)$. The converse inclusion follows since the Lorentz boosts $U(\Lambda_W(t)) = \int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m)U_m(\Lambda_W(t))$ fix the subspace $\int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m)H_m(W)$ for any $t \in \mathbb{R}$. Since the latter subspace is standard and (SS6) holds on H_U , we conclude (\bullet) by Lemma 4.1. To justify (\star) , we refer to the Appendix B, Lemma B.3(b), and note that we only need two wedges whose intersection is O . Therefore, by Lemma B.3(c),

$$H_U^d(V_+) = \overline{\sum_{O \supset V_+} H_U^d(O)} = \int_{\mathbb{R}_+}^{\oplus} m^3 d\mu(m)H_m(V_+).$$

We have seen in Proposition 4.3 that $H_m(V_+) = \mathcal{H}_m$, hence it follows that $H_U^d(V_+) = \mathcal{H}$. In particular, (HK8) fails and the net \mathcal{A}^d cannot be made Möbius covariant by replacing $\Gamma_+(U)$ by any other representation.

It is known that the dual net of the generalized free field does not satisfy the split property if the measure is not atomic [DL84, Theorem 10.2]. Our measure is $m^3 d\mu(m)$, therefore, the dual net \mathcal{A}^d fails to have the split property.⁶

5. Comments on Assumptions and Examples

5.1. Operator-algebraic assumptions. As we discuss the question whether dilation covariance can be promoted to Möbius covariance, (HK7) is a natural assumption. The physical meaning of (HK8) and (HK9) are not very clear, but they are necessary conditions for a Haag–Kastler net to be Möbius covariant. Indeed, if a Möbius covariant net extends to the cylinder \tilde{M} , any double cone is conformally equivalent to a lightcone, hence (HK8), and the Bisognano–Wichmann property holds automatically for wedges, double cones and lightcones, hence (HK9). We showed in Sect. 4.2 that (HK8) excludes the dual net of certain generalized free fields which are counterexamples to the implication “dilation \implies Möbius” without any additional assumption. Furthermore, even if (HK8) is satisfied, (HK9) may fail and in this case Möbius covariance cannot be expected, as shown in Sect. 4.1. Conversely, if (\mathcal{A}, U, Ω) satisfies (HK1)–(HK6) and (HK8), the modular group of $\mathcal{A}(V_+)$ commutes with Lorentz boosts $U(\delta(t) \times \delta(-t))$ (because it preserves $\mathcal{A}(V_+)$ and Ω , hence Theorem 2.2 applies), however, it is unclear whether it acts covariantly on \mathcal{A} .

(HK10a) is certainly a necessary condition for Möbius covariance. It requires that $\Delta_{B_L}^{it}$ acts as a certain Möbius transformation up to an inner symmetry. On the other hand, we proved the implication (HK10b) \implies (HK10a) (under (HK1)–(HK9)) and (HK10b) does not refer to any Möbius transformation, therefore, Corollary 3.4 is rather satisfactory. Furthermore, let us stress that our proofs do not rely on either stress-energy

⁶ This can be seen at the one-particle level. The second quantized net \mathcal{A}_U has the split property if and only if the operator

$$\int_0^{+\infty} \Delta_{F_m^{O, \tilde{O}}} |_{[0, 1]} d\mu(m)$$

is trace class, where $F_m^{O, \tilde{O}}$ is the intermediate type I factor subspace between $H_m(O) \subset H_m(\tilde{O})$, cf. [FG94, DL84]. In particular it is a necessary condition for the split property that $d\mu$ has to be purely atomic, concentrated on isolated points, cf. [Mor18].

tensor or current, c.f. Sect. 5.2. Indeed, there are Möbius covariant nets without stress-energy tensor. Therefore, at least from the mathematical point of view, the modular theory is more essential for Möbius covariance.

Yet, there are examples of Möbius covariant net not satisfying (HK10b). The simplest examples are two-dimensional nets $\text{Vir}_c \otimes \text{Vir}_c$, where Vir_c is the Virasoro net with central charge $c > 1$. As the chiral components Vir_c does not satisfy strong additivity [BSM90, Section 4, P.122], the two-dimensional net fails to satisfy (HK10b). Another family of examples is the derivatives of the U(1)-current [GLW98, Corollary 2.11], and one can again construct two-dimensional nets by tensor product which do not satisfy (HK10b). These examples satisfy (HK10a), hence are covered by Theorem 3.3 (although Möbius covariance for these examples is trivial because each chiral component is Möb-covariant). Interestingly, in both examples the current is missing, and there is even no stress-energy tensor in the latter case [Koe03, Proposition 3], hence it is not covered by the physics arguments (see below).

In the $d = 1$ case, Guido, Longo and Wiesbrock have shown that any local net which is covariant with respect to the translation-dilation group and satisfies the Bisognano–Wichmann property extends to a Möbius covariant net on S^1 [GLW98, Theorem 1.4]. This does not straightforwardly generalize to $d = 1 + 1$ because the inclusion $\mathcal{A}(W_R + (0, a_R)) \subset \mathcal{A}(W_R)$ is in general not a standard inclusion, and hence, even if one assumes the Bisognano–Wichmann property for wedges, it is not enough to construct a representation of the group $\overline{\text{Möb}} \times \overline{\text{Möb}}$. This is why we needed the assumption (HK10a).

5.2. *Physics literature.* The arguments by Zamolodchikov and Polchinski [Zom86, Pol88] are considered a proof in physics literature (e.g. [Nak15]). They are expressed in terms of Wightman-type assumptions on pointlike observables $O_k(x)$, i.e. [Nak15, Section 3.1]:

- unitarity
- Poincaré covariance
- (unbroken) dilation covariance
- discrete spectrum in scaling dimension: each of the pointlike observables $O_k(x)$ has a definite “scaling dimension” $\text{Ad } U(\delta(t))(O_k(x)) = t^{\Delta_k} O_k(tx)$.
- existence of scale current: there are a stress-energy tensor $T_{\mu\nu}(x)$ and a current $J_\mu(x)$ such that $\int d^{d-1}x [x^\rho T_{\rho 0}(x) - J_0(x)]$ is the generator of dilations.

Note that, apart from implicit assumptions such as $\{O_k(x)\}$ are Wightman fields and the existence of the integral for the generator of dilations, the well-definedness of scaling dimension is also a quite strong assumption: it implicitly says that there is a family of fields $\{O_k(x)\}$ which *linearly* generate the whole Hilbert space from the vacuum. This is different from the usual Wightman situation where the whole Hilbert space is generated *algebraically*, namely, one is allowed to use polynomial of such (smeared) fields. Once a complete set of fields $\{O_k(x)\}$ is obtained, one defines the “scaling dimension matrix” $\gamma_{k\ell}$ which is defined by $[D, O_k(x)] = -i(\sum_\ell \gamma_{k\ell} O_\ell(x) + x^\mu \partial_\mu O_k(x))$. Actually it appears that it is not always diagonalizable only from dilation covariance [Nak15, Section 2.2, below (2.6)]. Therefore, the discreteness of scaling dimension, and invertibility of $\gamma_{k\ell}$, strongly anticipates the Möbius covariance.

Under these assumptions, the main concern of [Pol88] is whether one can take a stress-energy tensor $T_{\mu\nu}$ which has the “canonical scaling dimension”, namely

$\text{Ad}U(\delta(t))T_{\mu\nu}(x) = t^2T_{\mu\nu}(tx)$. Such a new stress-energy tensor is obtained by the inverse of $\gamma_{k\ell}$. Once $T_{\mu\nu}$ acquires the canonical scaling dimension, the Lüscher–Mack theorem [FST89, Theorem 3.1] gives a clear explanation of why it satisfies the Lie algebra of vector fields.

By smearing the stress-energy tensor with a slightly singular function (whose existence is implicitly assumed, see below), one obtains the generators of the Lie algebra of $\widetilde{\text{Möb}}$. However, we are unable to find arguments or explanation on the following points:

- Once a stress-energy tensor with canonical dimension is found, how can one show that the rest of the fields $\{O_k(x)\}$ are Möbius covariant? Indeed, if one of $\{O_k(x)\}$ are “descendant”, one is forced to introduce the primary field [Koe03, ESNR11]. Namely, there is no guarantee that the original set of fields $\{O_k(x)\}$ is sufficient. It is not always possible to find such an extended family solely from the representation of $\widetilde{\text{Möb}}$. Indeed, we have the example from Sect. 4.2, where the representation U extends to $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$, but the observables do not extend to \widetilde{M} .
- Does the stress-energy really give the generator of rotation? Here we have two examples in which stress-energy tensor is not directly connected with rotation.
 - The $U(1)$ -current net. One can take a new stress-energy tensor $T_{\mu\nu}^c$ with central charge $c > 1$ which are relatively local to the $U(1)$ -current J and generate the same translation-dilation group, but does not give the correct rotation [BSM90].
 - The dual net of the Virasoro net Vir_c with $c > 1$. The stress-energy tensor $T_{\mu\nu}^c$ generate the correct translation-dilation group, but not the rotations for the dual net.

In addition, in order to obtain the generator of rotation, one has to smear the stress-energy tensor with the function $x^2 + 1$ (c.f. [Nak15, below (2.14)]), which is more singular than other generators (1 for translations and x for dilations). Indeed, this is exactly why T^c fails to extend to S^1 on the whole Hilbert space of the $U(1)$ -current. It appears to be known that in this case the conformal covariance cannot be obtained [Nak18].

- Stress-energy tensor is not uniquely determined, even if it is assumed to have the canonical dimension, as in the $U(1)$ -current net. Which one is the “physically” correct stress-energy tensor?

Generalized free fields (c.f. Sect. 4.2) are referred to as “fake counterexamples” in [DR09, Section 3.1], because they do not possess stress-energy tensor. In fact, one can construct a stress-energy tensor which generate the Poincaré symmetry, but it turns out very singular [DR03, Section 3]. Yet, the smeared stress-energy tensor gives a quadratic form on the Wightman domain, which means that the Wightman-type assumptions are crucial. Besides, the existence of stress-energy tensor in this sense does not depend on the scaling dimension Δ , while the extension of U to the Möbius group (or the conformal group in the case of $(3 + 1)$ -dimensions) depends on Δ . Therefore, one might conclude that the absence of conformal covariance in generalized free fields for certain Δ is due to the representation theory of the Möbius/conformal group, and not to the absence of stress-energy tensor.

5.3. Open problems. We do not know whether (HK10a) or (HK10b) can be dropped or weakened: (HK10a) is surely a necessary condition for Möbius covariance, but we do not have an example where (HK1)–(HK9) hold but (HK10a) fails. A natural candidate for a counterexample is the generalized free field with the measure $m^s d\mu$ where $s < 1$.

It is clear that the *field* cannot be $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ covariant because such a field should possess negative scaling dimension, which violates unitarity. However, to show that the *net* generated by it cannot be extended to a Möbius covariant net, one has to show that there is no extension of U which makes the net covariant. The latter is more difficult than the (non-)covariance of the field, because the possible extension is determined by the modular group, which is difficult to compute if it is not *a priori* Möbius covariant.

One would naturally expect that a similar result should hold in $d = 3 + 1$. However, this problem is widely open. Differently from $d = 1 + 1$ where the group of symmetry is a product $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$, the conformal group in four dimensions is locally isomorphic to $SU(1, 1)$, which is a simple Lie group [Mac77]. In order to extend the representation U of the Poincaré-dilation group, one might use the modular groups of double cones, but they must act in a compatible way. To us it is unclear how this can be obtained from Poincaré-dilation covariance, even with some additivity property or any other natural condition.

Another important problem raised by the examples in Sect. 4.2 is which nice properties are expected to be inherited by the dual net, under which conditions. As we have shown that the dual net of a Möbius covariant net in Sect. 4.2 fails to have the split property, it is a natural question whether the dual nets of Vir_c have the split property (if not, they could not be conformally covariant [MTW18]).

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A. The Two-Dimensional Conformal Spin-Statistics Theorem

We saw in the proof of Theorem 3.3 that for a Haag–Kastler net (\mathcal{A}, U, Ω) satisfying (HK1)–(HK10a) U extends to $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ and the net \mathcal{A} remains locally covariant with respect to U . To conclude it, we have to show that U factors through G , where G is the quotient of $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ by the normal subgroup generated by $\{\rho_{-2\pi} \times \rho_{2\pi}\}$ (see Sect. 2.2). A similar statement for nets on \mathbb{R} is known as the conformal spin-statistic theorem [BGL93, GL96], and the two-dimensional version is known to experts (see e.g. [KL04, Proposition 2.1]), but proof is missing in literature. For the reader’s convenience, we present a self-contained proof.

Let j denote the spacetime reflection, namely $j(a_L, a_R) = (-a_L, -a_R)$. We denote by $j_W := gjg^{-1}$ the spacetime reflection with respect to the wedge W , where $W = gW_R$, $g \in \mathcal{P}_+^\uparrow$.

Lemma A.1. *Let (\mathcal{A}, U, Ω) a Poincaré covariant net of von Neumann algebras on wedges satisfying (HK1)–(HK6) of Sect. 2.4. Then U extends to the group $\mathcal{P}_+^\uparrow \rtimes \mathbb{Z}_2$ by the modular conjugation:*

$$U(j) := J_{\mathcal{A}(W_R), \Omega} = J_{\mathcal{A}(W_L), \Omega}$$

and wedge algebras are covariant with respect to this extension, namely, $\text{Ad } U(j)\mathcal{A}(W) = \mathcal{A}(jW)$.

Proof. From the Bisognano–Wichmann property (HK6), wedge duality $\mathcal{A}(W_R) = \mathcal{A}(W_L)'$ follows (see e.g. [Tan12, Proposition A.2]). Locality (HK2) and wedge duality properties together imply that $J_{\mathcal{A}(W_R), \Omega} \mathcal{A}(W_R) J_{\mathcal{A}(W_R), \Omega} = \mathcal{A}(W_R)' = \mathcal{A}(W_L)$. Again by (HK6), $J_{\mathcal{A}(W_R)}$ commutes with the boosts and by Theorem 2.2 it satisfies the right commutation relations with translations. For a general wedge $W = \tau(a)W_R$, $a \in \mathbb{R}^{1+1}$, we have

$$\begin{aligned} \text{Ad } J_{\mathcal{A}(W_R)}(\mathcal{A}(W)) &= \text{Ad } [J_{\mathcal{A}(W_R)} U(\tau(a))] (\mathcal{A}(W_R)) = \text{Ad } [U(\tau(-a)) J_{\mathcal{A}(W_R)}] (\mathcal{A}(W_R)) \\ &= \text{Ad } U(\tau(-a)) \mathcal{A}(W_L) = \mathcal{A}(\tau(-a)W_L), \end{aligned}$$

which is the covariance of wedge algebras. \square

Remark A.2. We do not know whether covariance holds for the whole net \mathcal{A} including double cones. It does if we assume Haag duality for double cones, but that assumption might be too strong for Möbius covariant nets, as we saw in Sect. 4.2.3 that going to the dual net may break Möbius covariance.

Consider the local action of $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ on the Minkowski space given by $(g_L \times g_R) \cdot (a_L, a_R) = (g_L a_L, g_R a_R)$. We identify \mathbb{R} with the universal covering of S^1 , and with this identification, $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ acts on \mathbb{R}^2 . The Minkowski space can be identified with $(-\pi, \pi) \times (-\pi, \pi) \subset \mathbb{R} \times \mathbb{R}$ and we denote it by M_0 . See [BGL93]. Any diamond with center in $(a, b) \in \mathbb{R}^2$ of the form $(a - \pi, a + \pi) \times (b - \pi, b + \pi)$ is a copy of the Minkowski spacetime and is denoted by $M_{(a,b)}$. Let $\theta \mapsto \rho_\theta \in \widetilde{\text{Möb}}$ and $t \mapsto \tau_t \in \widetilde{\text{Möb}}$ be the lifts of the rotation and translation groups, respectively. Copies $\{M_{(a,b)}\}$ of the Minkowski spacetime are transformed to each other by some element in $\mathbb{R} \times \mathbb{R} \ni (\theta_1, \theta_2) \mapsto \rho_{\theta_1} \times \rho_{\theta_2} \in \widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$. Let $t \mapsto \Lambda_W(t)$ be the lift of the boosts associated with the wedge W . Wedges and double cones are diamonds with sides shorter than 2π , and in \mathbb{R}^2 they are indistinguishable. We call then double cones (in \mathbb{R}^2) (Fig. 6).

Let (\mathcal{A}, U, Ω) satisfy (HK1)–(HK6) and assume the local $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ -covariance:

(LM) U extends to $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ and $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$, for g in a small neighborhood \mathcal{U}_O of $\iota \times \iota \in \widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ such that $\mathcal{U}_O O$ stays in M_0 .

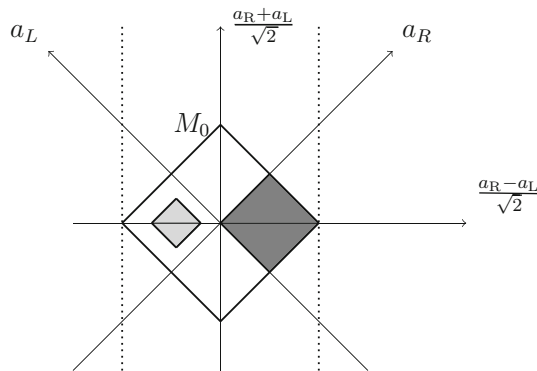


Fig. 6. The Minkowski space M_0 in \mathbb{R}^2 . The cylinder is obtained by identifying the dotted lines. The dark grey region is the right wedge $W_R \subset M_0$ and the light grey region is a doublecone $O \subset M_0$

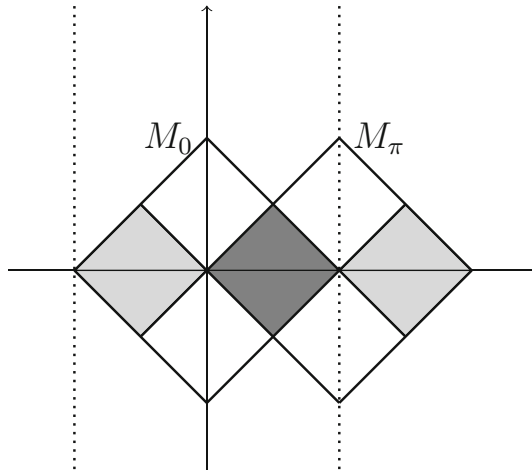


Fig. 7. Light grey areas are wedge regions in the two copies of the Minkowski spacetimes M_0 and M_π . Equation (5) shows that algebras associated with these regions are the equal

The net \mathcal{A} of von Neumann algebras can be extended to any double cone in \mathbb{R}^2 (diamonds with sides shorter than 2π) by covariance. Isotony (HK1) and locality (HK2) continue to hold for double cones included in one copy M of Minkowski space. For a double cone which is a wedge W in M_0 , there are a priori two definitions: $\mathcal{A}(W) = \bigvee_{D \subset W} \mathcal{A}(D)$ and by covariance. But actually they coincide by the continuity of U , hence we only have to deal with $\{\mathcal{A}(D)\}$ where D is a “double cone” in \mathbb{R}^2 .

If $\text{Ad}U(\rho_{-2\pi} \times \rho_{2\pi})\mathcal{A}(O) = \mathcal{A}(O)$ holds for any double cone in \mathbb{R}^2 , we can identify points connected by $\rho_{-2\pi} \times \rho_{2\pi}$ and obtain a net on \widetilde{M} . In this case, we say that (\mathcal{A}, U, Ω) reduces to the cylinder \widetilde{M} .

Lemma A.3. *Let $(\mathcal{A}_2, U, \Omega)$ be a $\widetilde{\text{Möb}}$ covariant net satisfying (HK1)–(HK6) and (LM). Then, it reduces to \widetilde{M} .*

Proof. We show that for any double cone $O \subset M_0$ it holds that $\text{Ad}U(\rho_{-2\pi} \times \rho_{2\pi})\mathcal{A}(O) = \mathcal{A}(O)$. As any double cone on \mathbb{R}^2 are contained in a copy M of the Minkowski space, and the net $\{\mathcal{A}(O)\}$ is defined by covariance with respect to $\rho \times \rho$, this suffices to conclude that $\text{Ad}U(\rho_{-2\pi} \times \rho_{2\pi})\mathcal{A}(O) = \mathcal{A}(O)$ holds for any double cone.

Let $W_{R/L}$ and $W_{R/L}^\pi$ be the right and left wedges in M_0 and $M_{(-\pi, \pi)} = (\rho_{-\pi} \times \rho_\pi)M_0$, respectively. Note that W_R coincides with the left wedge W_L^π in $M_{(-\pi, \pi)}$, see Fig. 7. Therefore, by wedge duality, $\mathcal{A}(W_R)' = \mathcal{A}(W_L)$ and $\mathcal{A}(W_R)' = \mathcal{A}(W_L^\pi)' = \mathcal{A}(W_R^\pi)$. Altogether,

$$\mathcal{A}(W_L) = \text{Ad}U(\rho_{-2\pi} \times \rho_{2\pi})(\mathcal{A}(W_L)) = \mathcal{A}(W_R^\pi). \tag{5}$$

By composing with an appropriate $\tau_t \times \tau_s$, (5) holds for any wedge W in M_0 . It also holds for double cones. Indeed, W_R is a double cone in $M_{(-\epsilon, \epsilon)}$ with $\epsilon > 0$ (see Fig. 8) and by covariance one can infer that for any double cone $O \subset M_0$

$$\mathcal{A}(O) = \text{Ad}U(\rho_{-2\pi} \times \rho_{2\pi})(\mathcal{A}(O)) = \mathcal{A}((\rho_{-2\pi} \times \rho_{2\pi})O).$$

□

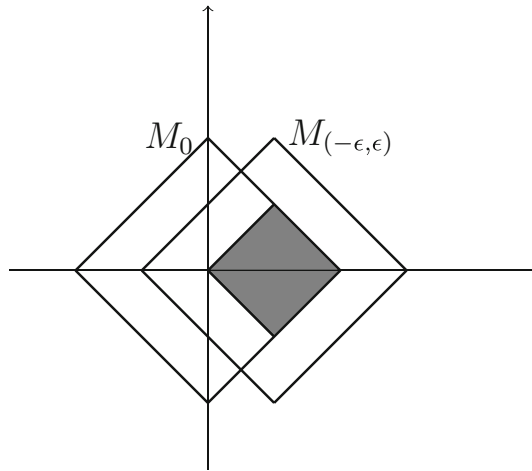


Fig. 8. The shaded region represents the right wedge in M_0 and a double cone in $M_{(-\epsilon, \epsilon)}$

Lemma A.4. *Let (\mathcal{A}, U, Ω) be a $\widetilde{\text{Möb}}$ covariant net satisfying (HK1)–(HK6) and (LM). Then, for any wedge W in a copy of Minkowski space M , the wedge modular conjugation J_W implements j_W on M .*

Proof. The net can be extended to the cylinder by Lemma A.3. Let $W_L = (0, \pi) \times (-\pi, 0)$ and $W_{L,1} = (1, \pi) \times (-\pi, -1) = (\tau(1) \times \tau(-1))W_L \subset W_L$ be wedges in M_0 . The restriction of the net \mathcal{A} on M_0 satisfies the assumptions of Lemma A.1, hence we have

$$\text{Ad } U(j)(\mathcal{A}(W_{L,1})) = \mathcal{A}(W_{R,1}), \tag{6}$$

where $W_{R,1} = (-\pi, -1) \times (1, \pi)$. Now, note that $U(j) = J_{W_L} = J_{W_R}$ and that on $M_{(-\pi, \pi)}$ the region $W_{R,1}$ is a double cone (see Fig. 9). Thus, by (6),

$$\text{Ad } U(j)(\mathcal{A}(W_{R,1})) = \mathcal{A}(W_{L,1}) = \mathcal{A}((\rho_{-2\pi} \times \rho_{2\pi})W_{L,1}),$$

where the last equality follows because the net is defined on \widetilde{M} . Note that $(\rho_{-2\pi} \times \rho_{2\pi})W_{L,1}$ is the double cone which is obtained by reflecting $W_{R,1}$ by $j_{W_R}^\pi$ in $M_{(-\pi, \pi)}$, namely, this is covariance of double cone algebras in $M_{(-\pi, \pi)}$. We obtain covariance for any other double cone by \mathcal{P}_+^\uparrow -covariance, and this result can be brought back to M_0 by $\text{Ad } U(\rho_{-2\pi} \times \rho_{2\pi})$. \square

Now we can prove the conformal spin-statistics theorem.

Theorem A.5. *Let (\mathcal{A}, U, Ω) satisfy (HK1)–(HK6) and (LM). Then the representation U of $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$ factors through a representation of G .*

Proof. We have to show that $U(\rho_{-2\pi} \times \rho_{2\pi}) = \mathbb{1}$. Let $O_1 = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ be a double cone in $M_0 = (-\pi, \pi) \times (-\pi, \pi)$. We saw in Lemma A.4 that $\text{Ad } J_{\mathcal{A}(W_R), \Omega}$ implements the reflection j , hence it is an antilinear automorphism of $\mathcal{A}(O_1)$. In particular, $J_{\mathcal{A}(W_R), \Omega}$ commutes with $J_{\mathcal{A}(O_1), \Omega}$ and the unitary $J_{\mathcal{A}(W_R), \Omega} J_{\mathcal{A}(O_1), \Omega}$ is self-adjoint.

We are going to show that

$$J_{\mathcal{A}(W_R), \Omega} J_{\mathcal{A}(O_1), \Omega} = U(\rho_{-\pi} \times \rho_\pi). \tag{7}$$

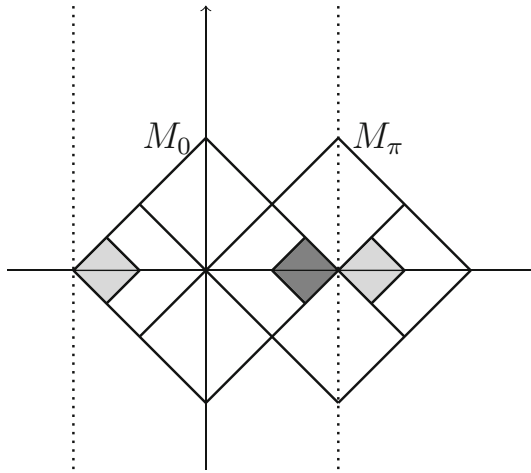


Fig. 9. $U(j)$ transforms the algebras in light grey to dark grey regions (and vice versa). They are wedge and doublecone algebras in the M_0 and M_π pictures, respectively

Let $L_{0,L}$ be the generator of $\theta \mapsto U(\rho_\theta \times \iota)$, P_L be the generator of $t \mapsto U(\tau_t \times \iota)$ and \bar{P}_L be the generator of anti-translations $U((\rho_\pi \circ \tau_t \circ \rho_{-\pi}) \times \iota)$. Analogously we introduce $L_{0,R}$, P_R and \bar{P}_R . As operators on an appropriate domain,

$$L_{0,R} = \frac{P_R + \bar{P}_R}{2}, \quad L_{0,L} = \frac{P_L + \bar{P}_L}{2},$$

see e.g. [Kos02, Proposition 1]. By applying Theorem 2.2 to $\mathcal{A}(W_R)$ and translations or anti-translations, $J_{\mathcal{A}(W_R),\Omega}$ commutes with each of $P_R, \bar{P}_R, P_L, \bar{P}_L$, thus by antilinearity of $J_{\mathcal{A}(W_R)}$,

$$\begin{aligned} J_{\mathcal{A}(W_R),\Omega} U(\rho_{\theta_1} \times \rho_{\theta_2}) J_{\mathcal{A}(W_R),\Omega} &= J_{\mathcal{A}(W_R),\Omega} \cdot e^{i\theta_1 L_{0,L} + i\theta_2 L_{0,R}} \cdot J_{\mathcal{A}(W_R),\Omega} \\ &= e^{-i\theta_1 L_{0,L} - i\theta_2 L_{0,R}} = U(\rho_{-\theta_1} \times \rho_{-\theta_2}). \end{aligned}$$

The claimed Eq. (7) follows since $\mathcal{A}(W_R) = \text{Ad } U(\rho_{-\frac{\pi}{2}} \times \rho_{\frac{\pi}{2}}) \mathcal{A}(O_1)$, therefore,

$$J_{\mathcal{A}(W_R),\Omega} = U(\rho_{-\frac{\pi}{2}} \times \rho_{\frac{\pi}{2}}) J_{\mathcal{A}(O_1),\Omega} U(\rho_{-\frac{\pi}{2}} \times \rho_{\frac{\pi}{2}})^*.$$

From Eq. (7) we conclude that $U(\rho_{-2\pi} \times \rho_{2\pi}) = (J_{\mathcal{A}(W_R),\Omega} J_{\mathcal{A}(O_1),\Omega})^2 = \mathbb{1}$, hence the Lemma. \square

B. Some Basic Properties of Direct Integrals

Here we follow [Dix81, Section II.1] and supply some additional results concerning direct integral of real and complex Hilbert spaces.

Given a field of Hilbert spaces $m \mapsto \mathcal{H}_m$ on a standard measure space (X, μ) we can construct the direct integral Hilbert space $\int_X^\oplus \mathcal{H}_m d\mu(m)$ if the field is μ -measurable: This definition requires and depends on the choice of a linear subspace \mathcal{S} of $\prod_{m \in X} \mathcal{H}_m$ which are by definition μ -measurable vector fields (\mathcal{S} must (i) consist of fields whose norm is measurable, (ii) be complete in the sense that it contains any vector field whose

pointwise inner product with any other field in \mathcal{S} is measurable (iii) and contain a sequence which is total at any point $m \in X$, see [Dix81, Section II.1.3, Definition 1]). Note that given a sequence of measurable vector field ξ_n μ -a.e. pointwise converging to ξ , namely $\|\xi_n(m) - \xi(m)\| \xrightarrow{n \rightarrow \infty} 0$ for μ -a.e. $m \in X$, then ξ is a μ -measurable vector field. We also recall that a vector field of bounded operators $m \mapsto T_m \in \mathcal{B}(\mathcal{H}_m)$ is μ -measurable if for any μ -measurable field $m \mapsto \xi(m) \in \mathcal{H}_m$ then $m \mapsto T_m \xi(m) \in \mathcal{H}_m$ is μ -measurable. In our concrete case, we take $X = \mathbb{R}$, $\mathcal{H}_m = L^2(\mathbb{R}, \frac{dp}{2\omega_m(p)})$ and \mathcal{S} consists of Lebesgue-measurable functions in $\mathbb{R} \times \mathbb{R}$.

Most of the results [Dix81, Section II.1] which we need are written for complex Hilbert spaces, but actually one can consider direct integral of real Hilbert spaces and similar results hold.⁷ If $\mathcal{H} = \int_X^\oplus \mathcal{H}_m d\mu(m)$ with the scalar product $\langle \cdot, \cdot \rangle$ is the direct integral of complex Hilbert spaces \mathcal{H}_m with the scalar product $\langle \cdot, \cdot \rangle_m$, \mathcal{H} as a real Hilbert space with $\text{Re} \langle \cdot, \cdot \rangle$ can be seen as the direct integral of real Hilbert spaces \mathcal{H}_m with $\text{Re} \langle \cdot, \cdot \rangle_m$: the μ -measurable set \mathcal{S} can be regarded as the set of real μ -measurable vector fields (i) The norm does not change. (ii) If $\xi \in \mathcal{S}$, then also $i\xi \in \mathcal{S}$ and $\langle \eta, \xi \rangle_m$ is measurable if and only if both of $\text{Re} \langle \eta, \xi \rangle_m$ and $\text{Re} \langle \eta, i\xi \rangle_m = -\text{Im} \langle \eta, \xi \rangle_m$ are measurable. (iii) The metric does not change, hence from a total sequence $\{\xi_n\}$ with respect to the complex scalar product we can make a sequence $\{\xi_n, i\xi_n\}$ which is total with respect to the real scalar product).

In order to properly describe the BGL-net associated to a direct integral representation in Sect. 4.2.3, we need the following two propositions, see [Hal62] and [Dix81, Section II.1.7, Proposition 9], respectively. The proofs work for \mathbb{C} as well as for \mathbb{R} .

Proposition B.1. *Let $\mathcal{K}_1, \dots, \mathcal{K}_k$ be closed subspaces of a Hilbert space \mathcal{H} and $\mathcal{K} = \bigcap_k \mathcal{K}_k$. Let $P_{\mathcal{K}_j}$ and $P_{\mathcal{K}}$ be the associated orthogonal projections, then $(P_{\mathcal{K}_1}, \dots, P_{\mathcal{K}_k})^n$ strongly converge to $P_{\mathcal{K}}$, as $n \rightarrow +\infty$.*

Proposition B.2. *Let $m \mapsto \mathcal{H}_m$ be a μ -measurable field of Hilbert spaces over a measure space (X, μ) . Let H_m be a closed linear subspace of \mathcal{H}_m and $E_H(m)$ be the projection onto H_m . Let \mathcal{S}_H be the set of all measurable vector fields $m \mapsto \xi(m)$ such that $\xi(m) \in H_m$. Then the following are equivalent:*

- (i) *the field of subspaces $m \mapsto H_m$, endowed with \mathcal{S}_H , is μ -measurable;*
- (ii) *There exists a sequence $\{\xi_n\}_{n \in \mathbb{N}}$ of μ -measurable vector fields ($m \mapsto \xi_n(m) \in \mathcal{H}_m$) such that $\{\xi_n(m)\}_{n \in \mathbb{N}}$ is a total sequence in each H_m ;*
- (iii) *for any measurable vector field ξ with respect to \mathcal{S} , the field $m \mapsto E_H(m)\xi(m)$ is measurable.*

This proposition allows us to consider the direct integral $\int_X^\oplus H_m d\mu(m)$ of standard subspaces. A vector ξ in this subspace is a L^2 -measurable field of vectors $\{\xi(m)\}$ such that $\xi(m) \in H_m$ for almost all m . In our application, H_m is the closure of the Fourier transforms of real compactly supported continuous functions on the Minkowski space \mathbb{R}^{1+1} , restricted to the mass shell with mass m .

Lemma B.3. *With the definitions in Lemma B.2, let H_m be a μ -measurable field of subspaces, and consider $H = \int_X^\oplus H_m d\mu(m)$,*

⁷ We mostly use calligraphic letters (such as \mathcal{H}_m) for complex Hilbert spaces and roman letters (such as H_m) for real Hilbert spaces. However, a complex Hilbert space can be seen as a real Hilbert space by taking the real part of the scalar product, and we do this in this paragraph.

- (a) $m \mapsto H_m^\perp$ $m \mapsto H_m^\perp$ is a μ -measurable field and $H^\perp = \int_X^\oplus H_m^\perp d\mu(m)$. If \mathcal{H} is a real Hilbert space obtained from a complex Hilbert space and $\text{Re} \langle \cdot, \cdot \rangle$, then $m \mapsto H'_m$ $m \mapsto H'_m$ is a μ -measurable field and $H' = \int_X^\oplus H'_m d\mu(m)$.
- (b) let $\{H_k\}_{k \in \mathbb{N}}$ be a countable family of μ -measurable fields, then $\bigcap_{k \in \mathbb{N}} (H_k)_m$ is a μ -measurable field and $\bigcap_{k \in \mathbb{N}} H_k = \int_X^\oplus \bigcap_{k \in \mathbb{N}} (H_k)_m d\mu(m)$.
- (c) let $\{H_k\}_{k \in \mathbb{N}}$ be a countable family of μ -measurable fields, then $\overline{\sum_{k \in \mathbb{N}} (H_k)_m}$ is a μ -measurable field and $\overline{\sum_{k \in \mathbb{N}} H_k} = \int_X^\oplus \overline{\sum_{k \in \mathbb{N}} (H_k)_m} d\mu(m)$.

Proof. (a) By definition $H^\perp = \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle = 0, \eta \in H\}$. Since H_m is a measurable field, by Proposition B.2 (iii) for any $\xi \in \mathcal{H}$, $m \mapsto E_H(m)\xi(m) \in S_H$ is a measurable map. Then for every $\xi \in \mathcal{H}$, the vector field $m \mapsto (1 - E_H(m))\xi(m)$ is clearly μ -measurable, hence by implication (iii) \Rightarrow (i) in Proposition B.2, we have that $m \mapsto H_m^\perp$ is a μ -measurable vector field of subspaces. We denote their direct integral by $\int_X^\oplus d\mu(m)H_m^\perp$. The inclusion $H^\perp \supset \int_X^\oplus d\mu(m)H_m^\perp$ is obvious. Any ξ can be decomposed as $E_H\xi + (1 - E_H)\xi$, hence $\int_X^\oplus d\mu(m)H_m^\perp$ must coincide with H^\perp .

Now consider $m \mapsto H_m$ μ -measurable field of real subspaces, then $m \mapsto iH_m$ is also a measurable of real subspaces. Indeed, let E_m be the projection on H_m for any $x \in \mathcal{H}$, $m \mapsto iE(m)x(m)$ is a measurable vector field, as S is closed under the multiplication by i . Therefore, (i) in Proposition B.2 holds and we obtain $iH = \int_X^\oplus iH_m d\mu(m)$. We conclude by recalling that $H' = (iH)^\perp$, and combining the last comments, that $H' = \int_X^\oplus d\mu(m)H'_m$.

(b) Let $\{H_k\}_{k \in \mathbb{N}}$ be a family of measurable fields of subspaces. We need to show that $m \mapsto \bigcap_k (H_m)_k$ is a measurable field of real spaces and $\int_X^\oplus d\mu(m) (\bigcap_k (H_m)_k) \subset \bigcap_k H_n$. Firstly, if the family is finite, namely if we have H_1, \dots, H_K , then for any $\xi \in \mathcal{H}$, then $((P_{H_1})_m \dots (P_{H_K})_m)^n \xi(m) \xrightarrow{n \rightarrow +\infty} (P_{\bigcap_k H_k})_m \xi(m)$ for each m by Proposition B.1, and this is measurable. Hence by implication (iii) \Rightarrow (i) in Proposition B.2 we conclude that $m \mapsto \bigcap_{k=1}^K (H_m)_k$ is a measurable field of real spaces. If the family is countably infinite, we take $\tilde{P}_K : m \mapsto (\tilde{P}_K)_m$ as the projection on $\bigcap_{k=0}^K (H_k)_m$. This is a μ -measurable family of decreasing projections, thus for any $\xi \in H$ the limit $(\tilde{P}_K)_m \xi(m) \xrightarrow{K \rightarrow +\infty} (P_{\bigcap_{k=1}^\infty H_k})_m \xi(m)$ is still measurable. Thus if H_k is a countable family of real subspaces, $m \mapsto \bigcap_k (H_k)(m)$ gives a measurable family of real subspaces. The same sequence of projections shows that, if $\xi \in \bigcap_k H_k$, then $\xi = P_{\bigcap_k H_k} \xi$ and hence $\xi(m) \in H_m$, which concludes the claim.

(c) This follows by combining (a) and (b). \square

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