The distributivity spectrum of Baker's variety

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Abstract. For every n, we evaluate the smallest k such that the congruence inclusion $\alpha(\beta \circ_n \gamma) \subseteq \alpha\beta \circ_k \alpha\gamma$ holds in a variety of reducts of lattices introduced by K. Baker. We also study varieties with a near-unanimity term and discuss identities dealing with reflexive and admissible relations.

1. Introduction

Baker [1] considered the variety generated by polynomial reducts of lattices in which the only basic operation is b defined by $b(a, c, d) = a(c + d)$. Here juxtaposition denotes meet and $+$ denotes join. In a few cases, for clarity, the meet of a and c shall be denoted by $a \cdot c$. We shall denote the above variety by $\mathcal{B}$ and we shall call it *the Baker's variety*, but let us mention that [1] contains a more general study of varieties which arise as reducts of lattices; see, in particular, [1, Theorem 2]. Notice that, in every algebra in $\mathcal{B}$, the term $x \cdot y = b(x, y, y)$ provides a semilattice operation; in particular, we can consider any algebra in $\mathcal{B}$ as an ordered set in a natural way. A related variety is obtained by taking polynomial reducts of lattices in which the only basic operation is u defined by $u(a_1, a_2, a_3, a_4) = \prod_{j \neq j}(a_i + a_j)$, where the indices on the product vary on the set $\{1, 2, 3, 4\}$. We shall denote this variety by $\mathcal{N}_4$. Notice that u is a near-unanimity term in $\mathcal{N}_4$ and that the position $b(a, c, d) = u(a, a, c, d)$ provides an interpretation of $\mathcal{B}$ in $\mathcal{N}_4$.

Baker showed that $\mathcal{B}$ is 4-distributive but not 3-distributive. Recall that a variety $\mathcal{V}$ is *m-distributive*, or Δ_m, if $\mathcal{V}$ satisfies the congruence identity $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_m \alpha\gamma$. In the above formula, $\alpha, \beta, \dots$ are intended to be variables for congruences of some algebra in $\mathcal{V}$, juxtaposition denotes intersection and we have used the shorthand $\beta \circ_m \gamma$ for $\beta \circ \gamma \circ \beta \dots$ with m factors, that is, with $m-1$ occurrences of $\circ$. If, say, m is odd, we sometimes write $\beta \circ \gamma \circ \overset{m}{\dots} \circ \beta$ in place of $\beta \circ_m \gamma$ in order to make clear that β is the last factor. Conventionally, $\beta \circ_0 \gamma = 0$, the minimal congruence of the algebra under consideration; otherwise the reader might always suppose that $m \geq 1$. We refer to Baker [1], Jónsson [10] or Lipparini [15] for other unexplained notions and notations.

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The original definition of m-distributivity involves the existence of a certain number of terms introduced by Jónsson [9]; Jónsson terms are exactly the terms arising from the Maltsev condition associated to $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_m \alpha\gamma$. Here it will be more convenient to express results by means of congruence identities rather than terms. See [15] for a more detailed discussion and further references. Jónsson proved that a variety is distributive if and only if it is m-distributive, for some m. It follows from Jónsson's proof that, for every n and m, there is some k such that every m-distributive variety satisfies the congruence identity $\alpha(\beta \circ_n \gamma) \subseteq \alpha\beta \circ_k \alpha\gamma$. We initiated the study of the related "Jónsson distributivity spectra" in [15]. Here we shall evaluate exactly the distributivity spectra of $\mathcal{N}_4$ and of Baker's variety $\mathcal{B}$. We shall also show that we get exactly the same spectra if we consider the corresponding reducts of *distributive* lattices, call such reducts $\mathcal{N}_4^d$ and $\mathcal{B}^d$.

Relying heavily on Kazda, Kozik, McKenzie and Moore [12], we observed in [15] that congruence distributive varieties satisfy also identities of the form $\alpha(R \circ_n T) \subseteq \alpha R \circ_{k'} \alpha T$, where R and T denote reflexive and admissible relations. In Sections 3 and 4 we shall find the best bounds for identities of this kind in $\mathcal{B}$ and $\mathcal{N}_4$; moreover, we shall show that in the case of $\mathcal{B}$ and $\mathcal{N}_4$ it is possible to take α, too, as an admissible relation. As far as relation identities are concerned, $\mathcal{B}$ and $\mathcal{N}_4$ exhibit a subtly different behavior. This partially confirms the suggestion implicit in [10, p. 370] and explicitly advanced in [16] that the study of relation identities might provide a finer classification of varieties (in particular, congruence distributive varieties), in comparison with the study of congruence identities alone.

The relation identities found in Sections 3 and 4 solve also some earlier problems. In [14] we have showed that, under a fairly general assumption, a congruence identity is equivalent to the same identity when considered for representable tolerances, instead. In Remark 3.4 we show that the assumption of representability of tolerances is necessary in the above equivalence.

It is known [5, 18] that the identities $\alpha(\Theta \circ \Theta) \subseteq \alpha\Theta \circ_{k'} \alpha\Theta$ and $\alpha(R \circ R) \subseteq \alpha R \circ_k \alpha R$, for some k, k', both characterize congruence modularity, where Θ denotes a tolerance. Remarks 4.4 and 4.5 show that, for a variety $\mathcal{V}$, the best values of k or k' in the above identities are not determined by the Day modularity level of $\mathcal{V}$. It is an open problem to find the example of a variety for which the best values for k and k' above are distinct.

Section 5 contains a few remarks about relation identities satisfied by varieties with a near-unanimity term and by varieties with an edge term. Here we are dealing with the general case, not with specific examples such as $\mathcal{N}_4$. Further remarks are contained in Section 6. Among other, and following the lines of [1], we consider identities satisfied by arbitrary polynomial reducts of lattices. We also consider polynomial reducts of Boolean algebras.

2. The distributivity spectra of $\mathcal{B}$ and $\mathcal{N}_4$

Recall that $\mathcal{B}$ is the variety generated by polynomial reducts of lattices in which the only basic operation is $b(a,c,d) = a(c+d)$ and that $\mathcal{N}_4$ is defined similarly with respect to the operation $u(a_1,a_2,a_3,a_4) = \prod_{j\neq j}(a_i + a_j)$. The varieties obtained by considering only reducts of distributive lattices are denoted, correspondingly, by $\mathcal{B}^d$ and $\mathcal{N}_4^d$.

Theorem 2.1. *Suppose that $n \geq 2$ and $\mathcal{V}$ is either $\mathcal{B}$, $\mathcal{B}^d$, $\mathcal{N}_4$ or $\mathcal{N}_4^d$. Then $\mathcal{V}$ satisfies the following congruence identities:*

$$\alpha(\beta \circ_n \gamma) \subseteq \alpha\beta \circ_{2n} \alpha\gamma, \qquad \textit{for } n \textit{ even, and} \tag{2.1}$$

$$\alpha(\beta \circ_n \gamma) \subseteq \alpha\beta \circ_{2n-1} \alpha\gamma, \qquad \textit{for } n \textit{ odd;} \tag{2.2}$$

and the subscripts on the right-hand sides are best possible; actually, $\mathcal{V}$ does not even satisfy

$$\alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{n-2}{\dots} \circ \alpha\beta) \circ \gamma) \subseteq \alpha\beta \circ_{2n-1} \alpha\gamma, \qquad \textit{for } n \textit{ even, and} \tag{2.3}$$

$$\alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{n-2}{\dots} \circ \alpha\gamma) \circ \beta) \subseteq \alpha\beta \circ_{2n-2} \alpha\gamma, \qquad \textit{for } n \textit{ odd.} \tag{2.4}$$

Proof. The positive result that equations (2.1) and (2.2) hold in $\mathcal{B}$ is an observation in [15, Section 3], however inserted there in a quite abstract and general context. In the special case of $\mathcal{B}$ the proof is direct and is an almost immediate generalization of Baker's argument. Indeed, if n is even and $(a,d) \in \alpha(\beta \circ_n \gamma)$, then $a \ \alpha \ d$ and there are elements c_i such that $a = c_0 \ \beta \ c_1 \ \gamma \ c_2 \ \beta \dots c_{n-1} \ \gamma \ c_n = d$. Then the elements

$$\begin{gathered} a = b(a,a,d) = b(a,c_0,d), \quad b(a,c_1,d), \quad b(a,c_2,d), \quad \dots \quad b(a,c_{n-1},d), \\ b(a,c_n,d) = b(a,d,d) = a \cdot d = b(d,a,a) = b(d,c_0,a), \\ b(d,c_1,a), \quad b(d,c_2,a), \quad \dots \quad b(d,c_{n-1},a), \quad b(d,c_n,a) = b(d,d,a) = d \end{gathered} \tag{2.5}$$

witness $(a,d) \in \alpha\beta \circ_{2n} \alpha\gamma$. Notice that, say, $b(a,c_j,d) \ \alpha \ b(a,c_j,a) = a = b(a,c_{j+1},a) \ \alpha \ b(a,c_{j+1},d)$, for every j. The same chain of elements works in the case n odd, but in this case $c_{n-1} \ \beta \ c_n = d$, hence $b(a,c_{n-1},d) \ \alpha\beta \ b(a,d,d) = a \cdot d = b(d,a,a) \ \alpha\beta \ b(d,c_1,a)$, in particular, $b(a,c_{n-1},d) \ \alpha\beta \ b(d,c_1,a)$, thus one passage might be skipped and we get $(a,d) \in \alpha\beta \circ_{2n-1} \alpha\gamma$. Since $\mathcal{B}$ is interpretable in $\mathcal{N}_4$, then (2.1) and (2.2) hold in $\mathcal{N}_4$, too. The result for $\mathcal{N}_4$ can be obtained also directly from the case $m = 2$ of equations (5.1) and (5.2) in Proposition 5.1 below. Clearly, if some congruence identity holds in $\mathcal{B}$, respectively, $\mathcal{N}_4$, then it holds in $\mathcal{B}^d$, respectively, $\mathcal{N}_4^d$.

Now we show that equations (2.3) and (2.4) fail, hence the bounds in (2.1) and (2.2) are optimal. We shall present the argument for $\mathcal{N}_4$ and $\mathcal{N}_4^d$. This is enough, since, say, $\mathcal{B}$ is interpretable in $\mathcal{N}_4$. In any case, the same argument works for $\mathcal{B}$ and $\mathcal{B}^d$, too, with no essential modification.

For $h \geq 1$, let $\mathbf{C}_{h+1}$ denote the $h+1$-elements chain with underlying set $C_{h+1} = \{0,1,\dots,h\}$ and with the standard ordering, inducing the standard lattice operations of min and max. Let $\mathbf{L}$ be the lattice $\mathbf{C}_{n+1} \times \mathbf{C}_{n+1} \times \mathbf{C}_2$.

Since in what follows the "last" $\mathbf{C}_2$ will play a different role with respect to the first two $\mathbf{C}_{n+1}$'s, we shall usually denote the largest elements of $\mathbf{C}_2$ by $\uparrow$ and the smallest element of $\mathbf{C}_2$ by $\downarrow$. Consider the following elements of L:

$$a = c_0 = (n, 0, \uparrow), \qquad d = c_n = (0, n, \uparrow), \qquad \text{and}$$
$$c_i = (n - i, i, \downarrow), \qquad \text{for } i = 1, \dots, n - 1.$$

Recall that we let $u(x_1, x_2, x_3, x_4) = \prod_{j \neq k}(x_j + x_k)$ and $b(x_1, x_3, x_4) = u(x_1, x_1, x_3, x_4) = x_1(x_3 + x_4)$. Let

$$B = \{a \in L \mid a \leq c_i, \text{ for some } i \leq n\}.$$

We show that B is closed under u, hence $\mathbf{B} = (B, u)$ is an algebra in $\mathcal{N}_4$, actually, in $\mathcal{N}_4^d$, since $\mathbf{L}$ is a distributive lattice. Indeed, suppose that $a_1 \leq c_{i_1}$, $\dots$, $a_4 \leq c_{i_4}$. Since u is invariant under any permutation of its arguments, it is no loss of generality to assume that $i_1 \leq i_2 \leq i_3 \leq i_4$. If $u(a_1, a_2, a_3, a_4)$ has a third $\uparrow$ component, then at least three among a_1, a_2, a_3, a_4 have a third $\uparrow$ component, hence at least two among a_1, a_2, a_3, a_4 are either $\leq c_0$ or $\leq c_n$. Say, $a_1, a_2 \leq c_0$, hence, since u is a monotone operation, $u(a_1, a_2, a_3, a_4) \leq u(c_0, c_0, c_{i_3}, c_{i_4}) = c_0(c_{i_3} + c_{i_4}) \leq c_0$. Otherwise $u(a_1, a_2, a_3, a_4)$ has a third $\downarrow$ component and $u(a_1, a_2, a_3, a_4) \leq u(c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}) = (n - i_3, i_2, \downarrow) = c_{i_2}c_{i_3} \leq c_{i_2}$. In any case, $u(a_1, a_2, a_3, a_4) \in B$.

Hence $\mathbf{B} = (B, u)$ is an algebra in $\mathcal{N}_4$; in particular, (B, b) is an algebra in $\mathcal{B}$. Let $B^{\uparrow}$ denote the set of all the elements of B with a last $\uparrow$. By the definition of B, the elements of $B^{\uparrow}$ are exactly the following:

$$e_i = a(d + c_i) = (n - i, 0, \uparrow), \quad f_i = d(a + c_i) = (0, i, \uparrow), \quad i = 0, \dots, n. \quad (B^{\uparrow})$$

Now we can show that (2.3) and (2.4) fail, in general. Let α be the kernel of the third projection, thus α is a congruence on $\mathbf{B}$.

Let β be the congruence on $\mathbf{B}$ defined in such a way that two elements (i_1, i_2, i_3) and (j_1, j_2, j_3) of B are β-related if and only if, for every $\ell = 1, 2$, their components i_ℓ and j_ℓ differ at most by 1, and:

(a_β) if $i_1 \neq j_1$, then $\sup\{i_1, j_1\}$ has the same parity of n, and

(b_β) if $i_2 \neq j_2$, then $\sup\{i_2, j_2\}$ is odd.

It can be checked directly that β is a congruence; otherwise, argue as follows. Let β_1' be the congruence on $\mathbf{C}_{n+1}$ whose blocks are $\{n, n-1\}, \{n-2, n-3\}, \dots$ If β_1'' is the counterimage in $\mathbf{L}$ of β_1' through the first projection, then β_1'' is a congruence on $\mathbf{L}$, hence a congruence on (L, u). Thus the restriction β_1 of β_1'' to B is a congruence on (B, u). Similarly, define β_2 using the counterimage through the second projection of the congruence on $\mathbf{C}_{n+1}$ whose blocks are $\{0, 1\}, \{2, 3\}, \dots$ Then $\beta = \beta_1\beta_2$, hence β is a congruence, being the meet of two congruences.

The congruence γ on $\mathbf{B}$ is defined in a similar way: two elements (i_1, i_2, i_3) and (j_1, j_2, j_3) of B are γ-related if and only if, for every $\ell = 1, 2$, their components i_ℓ and j_ℓ differ at most by 1, and:

(a_γ) if $i_1 \neq j_1$, then $\sup\{i_1, j_1\}$ has not the same parity of n, and

(b_γ) if $i_2 \neq j_2$, then $\sup\{i_2, j_2\}$ is even.

In passing, let us mention that there is an alternative construction of $\mathbf{B}$ which makes the definitions of β and γ simpler; actually, in this alternative construction β and γ turn out to be kernels of appropriate projections. See [17]. However, all the remaining arguments are much more involved in [17]; moreover, the current presentation has the advantage of being more compact.

With the above definitions of α, β and γ, we have $c_0 \; \alpha \; c_n$ and $c_j \; \alpha \; c_{j+1}$, for $j = 1, \dots, n-2$. Moreover, $c_{2i} = (n-2i, 2i, -) \; \beta \; (n-2i-1, 2i+1, -) = c_{2i+1}$, for all the appropriate values of i and where the value of the third component is not relevant. Similarly, $c_{2i+1} \; \gamma \; c_{2i+2}$, for every appropriate i. Hence $(c_0, c_n) \in \alpha(\beta \circ (\alpha\gamma \circ_{n-2} \alpha\beta) \circ \gamma)$, for n even, and $(c_0, c_n) \in \alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{n-2}{\ldots} \circ \alpha\gamma) \circ \beta)$, for n odd.

On the other hand, in view of the above description of $B^{\uparrow}$, the only elements $\alpha\beta$-connected to $c_0 = e_0 = (n, 0, \uparrow)$ are c_0 itself and $e_1 = (n-1, 0, \uparrow)$. No other element of $B^{\uparrow}$ is $\alpha\gamma$-connected to c_0, hence there is no advantage in "staying at c_0". The only other element $\alpha\gamma$-connected to $e_1 = (n-1, 0, \uparrow)$ is $e_2 = (n-2, 0, \uparrow)$ and, so on, the only element $\alpha\beta$-connected to e_{2i} is e_{2i+1} and the only element $\alpha\gamma$-connected to e_{2i+1} is e_{2i+2}, until we reach e_{n-1}, where the situation splits into two cases.

If n is even, then $(1, 0, \uparrow) = e_{n-1} \; \alpha\gamma \; e_n = (0, 0, \uparrow) = f_0 \; \alpha\beta \; f_1 = (0, 1, \uparrow)$ and no other nontrivial relation holds among these elements. Symmetrical considerations hold for the f_j's and, since $f_n = c_n$, we get that any chain from c_0 to c_n in which each pair of elements is either $\alpha\beta$ or $\alpha\gamma$-connected must involve all the $2n+1$ elements of $B^{\uparrow}$, hence any chain as above is of length at least $2n$, thus (2.3) fails in $\mathbf{B}$.

On the other hand, if n is odd, then $(1, 0, \uparrow) = e_{n-1} \; \alpha\beta \; e_n = (0, 0, \uparrow) = f_0 \; \alpha\beta \; f_1 = (0, 1, \uparrow)$, thus $e_{n-1} \; \alpha\beta \; f_1$ and we do not need all the elements of $B^{\uparrow}$ to get an $\alpha\beta$-or-$\alpha\gamma$-chain, we can skip $e_n = f_0$. However, all the rest is the same and we need $2n-1$ steps from c_0 to c_n, hence (2.4) fails. $\square$

The case $n = 2$ for $\mathcal{B}$ in Theorem 2.1 gives another proof of Baker result that $\mathcal{B}$ is 4-distributive but not 3-distributive. The proof of 4-distributivity is the same. The counterexample to 3-distributivity in [1] has 10 elements and the counterexample here can be taken to have 9 elements, since two elements can be discarded from B, still having an algebra in $\mathcal{B}$, as we shall show in the proof of Proposition 2.3 below. In the special case $n = 2$ the treatment from [17] would be slightly simpler; the classes of congruences in the example from [1] are to be computed by hand, while in [17] we have considered kernels of projections, which are automatically congruences.

Remarks 2.2. (a) There is a short and simple syntactical folklore proof that Baker's variety is not 2-distributive, that is, that $\mathcal{B}$ has no majority term. Actually, the proof shows that $\mathcal{B}$ has no near-unanimity term. If t is a term of $\mathcal{B}$, define the *relevant* variable of t inductively as follows. If t is a variable x_j, then x_j is the relevant variable of t. Otherwise, $t = b(t_1, t_2, t_3)$ and we define the relevant variable of t to be the relevant variable of t_1. If $\mathbf{B} \in \mathcal{B}$, $\mathbf{B}$ has a minimal element 0 and we substitute 0 for the relevant variable of some term t, then t is evaluated as 0, no matter what we substitute for the other variables. Thus $\mathcal{B}$ has no near-unanimity term, in particular, no majority term.

More generally, the argument shows that, for every k-ary term t, there is some "place" $i \leq k$ such that $\mathcal{B}$ satisfies no equation of the form $t(\dots, y, \dots) = x$, where y is put in place i, x is a variable distinct from y and the other arguments of t are arbitrary variables. This shows that $\mathcal{B}$ has no *cube term*, as introduced by Berman, Idziak, Marković, McKenzie, Valeriote and Willard [3].

(b) Using a different method, Mitschke [20] proved that the variety $\mathcal{I}$ of implication algebras has no near-unanimity term. Since Baker's variety $\mathcal{B}$ is interpretable in $\mathcal{I}$, then Mitschke's result furnishes another proof that $\mathcal{B}$ has no near-unanimity term. The method in (a) can be applied also to $\mathcal{I}$, providing a shorter proof of the mentioned result by Mitschke. Simply argue as in (a) above, defining the relevant variable of $t_1 \to t_2$ to be the relevant variable of t_2 and dealing with some maximal element 1 rather than with 0. Thus we also get that $\mathcal{I}$ has no cube term. Essentially, this is the argument hinted on [3, p. 1470]. In particular, 3-distributive 3-permutable varieties do not necessarily have a cube term.

(c) The argument in (a) can be extended in order to give still another proof that Baker's variety is not 3-distributive. Indeed, 3-distributivity is equivalent to the existence of ternary terms j_1 and j_2 satisfying $x = j_1(x, x, y) = j_1(x, y, x)$, $j_1(x, y, y) = j_2(x, y, y)$ and $j_2(x, x, y) = j_2(y, x, y) = y$ [9]. With the same assumptions and definitions as in (a) above, the first equations imply that the relevant variable of $j_1(x, y, z)$ is x, hence $0 = j_1(0, b, b) = j_2(0, b, b)$, for every $b \in B$. Under the order induced by the semilattice operation, we have that every term operation is monotone (this applies to $\mathcal{B}$ but *not* to the variety of implication algebras!), hence $0 = j_2(0, b, b) \geq j_2(0, 0, b) = b$, which is impossible if $\mathbf{B}$ is taken to be of cardinality ≥ 2. Thus $\mathcal{B}$ is not 3-distributive.

In fact, in the above argument we have not used the equation $j_2(y, x, y) = y$. This shows that $\mathcal{B}$ does not even satisfy $\alpha(\gamma \circ \beta) \subseteq \alpha\gamma \circ \beta \circ \gamma$, equivalently, taking converses, $\mathcal{B}$ does not satisfy $\alpha(\beta \circ \gamma) \subseteq \gamma \circ \beta \circ \alpha\gamma$. This negative result shall be improved in the following Proposition. Compare equation (2.8) below.

In the terminology from [15], Theorem 2.1 implies that $J_{\mathcal{B}}(n-1) = 2n-1$, for n even and that $J_{\mathcal{B}}(n-1) = 2n-2$, for n odd. In [15] we have also considered "reversed" Jónsson spectra, given by identities like $\alpha(\beta \circ_n \gamma) \subseteq \alpha\gamma \circ_k \alpha\beta$. We are going to see that the proof of Theorem 2.1 gives exact bounds for identities of the above kind both in $\mathcal{B}$ and $\mathcal{N}_4$, as well as in their distributive counterparts.

Moreover, it follows from results by Tschantz [21] that, for every congruence modular variety $\mathcal{V}$ and every n, there is some k such that $\mathcal{V}$ satisfies $\alpha(\beta \circ_n \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma \circ_k \alpha\beta)$. See, e. g., [15, Section 4] for details. Of course, in a congruence distributive variety we already know that $\alpha(\beta \circ_n \gamma) \subseteq \alpha\beta \circ_{k'} \alpha\gamma$, for some k'. However, in principle, it might happen that Tschantz-like formulae provide a value of k much smaller than k'. This is not the case for $\mathcal{B}$ and $\mathcal{N}_4$.

Proposition 2.3. *Suppose that* $n \geq 2$ *and* $\mathcal{V}$ *is either* $\mathcal{B}$, $\mathcal{B}^d$, $\mathcal{N}_4$ *or* $\mathcal{N}_4^d$. *Then* $\mathcal{V}$ *satisfies the following identities.*

$$\alpha(\beta \circ_n \gamma) \subseteq \alpha\gamma \circ_{2n+1} \alpha\beta \qquad \text{for } n \text{ even,} \tag{2.6}$$
$$\alpha(\beta \circ_n \gamma) \subseteq \alpha\gamma \circ_{2n} \alpha\beta \qquad \text{for } n \text{ odd,} \tag{2.7}$$
$$\alpha(\beta \circ_n \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma \circ_{2n-1} \alpha\beta) \qquad \text{for } n \text{ even,} \tag{2.8}$$
$$\alpha(\beta \circ_n \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma \circ_{2n-2} \alpha\beta) \qquad \text{for } n \text{ odd,} \tag{2.9}$$

and the values of the indices on the right-hand sides give the best possible bounds. Actually, $\mathcal{V}$ *fails to satisfy*

$$\alpha(\beta \circ (\alpha\gamma \circ_{n-2} \alpha\beta) \circ \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma \circ_{2n-4} \alpha\beta) \circ \alpha(\gamma \circ \beta), \ \text{for } n \text{ even,}$$
$$\alpha(\beta \circ (\alpha\gamma \circ_{n-2} \alpha\beta) \circ \beta) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma \circ_{2n-5} \alpha\beta) \circ \alpha(\beta \circ \gamma), \ \text{for } n \text{ odd.}$$

Proof. Equations (2.6) - (2.9) are immediate from (2.1) and (2.2), since, say, $\alpha\beta \circ_{2n} \alpha\gamma \subseteq \alpha\gamma \circ_{2n+1} \alpha\beta$ and $\alpha\beta \subseteq \alpha(\gamma \circ \beta)$.

The proof of Theorem 2.1 shows that the bounds on the right-hand sides of (2.6) and (2.7) are optimal. Indeed, in the proof that (2.3) and (2.4) fail we have observed that c_0 is $\alpha\gamma$-connected to no other element of $B^{\uparrow}$, hence we "lose one turn" if we want the chain to start with $\alpha\gamma$. Actually, we have that, say, for n even, already $\alpha(\beta \circ (\alpha\gamma \circ_{n-2} \alpha\beta) \circ \gamma) \subseteq \alpha\gamma \circ_{2n} \alpha\beta$ fails in $\mathcal{V}$.

In order to show that the indices in (2.8) and (2.9) are best possible, we shall modify the construction in the proof of Theorem 2.1. With the definitions and notations in the mentioned proof, let $B^- = B \setminus \{(n, 0, \downarrow), (0, n, \downarrow)\}$. We claim that B^- is (the base set for) an algebra in, say, $\mathcal{N}_4$. We shall show that if $a_1, a_2, a_3, a_4 \in B^-$, then it is not the case that $u(a_1, a_2, a_3, a_4) = (n, 0, \downarrow)$. Indeed, since c_0 is the only element of B^- with first component n, if the first component of $u(a_1, a_2, a_3, a_4)$ is n, then at least three arguments of u have n as the first component, hence at least three arguments of u are equal to c_0, thus $u(a_1, a_2, a_3, a_4)$ is itself c_0. Notice that, by construction, n is the maximum possible value for the first component. Similarly, if $a_1, a_2, a_3, a_4 \in B^-$, then $u(a_1, a_2, a_3, a_4)$ is not $(0, n, \downarrow)$, hence $\mathbf{B}^-$ is an algebra in $\mathcal{N}_4$.

We have that c_0 is γ-connected to no other element of B^-, because of clause (a_γ) in the definition of γ. Thus if $c_0 \ \alpha(\gamma \circ \beta) \ f$ in B^-, for some f, then $c_0 \ \gamma \ e \ \beta \ f$, for some e, hence necessarily $c_0 = e$ and $c_0 \ \alpha\beta \ f$. Thus if, say, n is even and we suppose by contradiction that $(c_0, c_n) \in \alpha(\gamma \circ \beta) \circ (\alpha\gamma \circ_{2n-2} \alpha\beta)$, then we would have $(c_0, c_n) \in \alpha\beta \circ_{2n-1} \alpha\gamma$, but this is impossible because of the counterexample constructed in the proof of Theorem 2.1. Hence we cannot get better bounds in (2.8) or (2.9). Performing also the symmetric argument, we have that $\mathcal{V}$ fails to satisfy the last equations in the statement. $\square$

Recall that a variety $\mathcal{V}$ is *n-modular* if $\mathcal{V}$ satisfies the identity $\alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\beta \circ_n \alpha\gamma$. Cf. Day [6]. Equations (2.2) and (2.4) in Theorem 2.1 in the case $n = 3$ provide the following corollary.

Corollary 2.4. *The varieties $\mathcal{B}$, $\mathcal{B}^d$, $\mathcal{N}_4$ and $\mathcal{N}_4^d$ are 4-distributive, 5-modular and not 4-modular.*

3. Some relation identities

We shall use R, S and T as variables for reflexive and admissible binary relations and Θ as a variable for tolerances. All the relations considered in the present note are assumed to be reflexive, hence we shall sometimes simply say *admissible* in place of *reflexive and admissible*. In [15, Proposition 3.1] we noticed that congruence distributive varieties satisfy also relation identities of the form $\Theta(R \circ_n S) \subseteq \Theta R \circ_k \Theta S$. This is a consequence of results by Kazda, Kozik, McKenzie and Moore [12]. See [15] for further details and references.

We do not know whether every congruence distributive variety satisfies $T(R \circ S) \subseteq TR \circ_k TS$, for some k. However, we showed in [16] that the above relation holds in Baker's variety with $k = 4$. In the present section we shall provide exact bounds for identities of the above kind, both in the case of $\mathcal{B}$ and in the case of $\mathcal{N}_4$. We shall exhibit a subtle difference between the two varieties, when relation identities are concerned. Compare Theorems 3.1, 4.1 and Proposition 3.2 below. On the other hand, as we showed in the previous section, $\mathcal{B}$ and $\mathcal{N}_4$ behave in the same way as far as congruence identities are concerned.

Each result in the present and in the following section holds for $\mathcal{B}$ if and only if it holds for $\mathcal{B}^d$. In fact, we shall never use lattice distributivity in the proofs; on the other hand, all the counterexamples we shall deal with are based on the construction in the proof of Theorem 2.1 and this construction is the reduct of a distributive lattice. Similarly, each result holds for $\mathcal{N}_4$ if and only if it holds for $\mathcal{N}_4^d$. For the sake of simplicity, we shall not mention $\mathcal{B}^d$ and $\mathcal{N}_4^d$ explicitly in the following statements. However, the reader might always consider $\mathcal{B}^d$ in place of $\mathcal{B}$ and $\mathcal{N}_4^d$ in place of $\mathcal{N}_4$ in what follows.

Recall that, if n is odd, we sometimes write $R \circ S \circ \overset{n}{\dots} \circ R$ in place of $R \circ_n S$, when we want to make clear that the last factor is R.

Theorem 3.1. *If $n \geq 2$, then the following identities are satisfied:*

$$T(R \circ_n S) \subseteq TR \circ_{2n} TS, \qquad \text{by } \mathcal{B}, \mathcal{N}_4, \ n \text{ even;} \tag{3.1}$$

$$T(R \circ_n S) \subseteq (TR \circ TS \circ \overset{n}{\dots} \circ TR) \circ (TR \circ TS \circ \overset{n}{\dots} \circ TR), \ \text{ by } \mathcal{B}, \ n \text{ odd;} \tag{3.2}$$

$$T(R \circ_n S) \subseteq TR \circ_{2n-1} TS, \quad \text{by } \mathcal{N}_4, \ n \text{ odd,} \tag{3.3}$$

and the bounds on the right are best possible; moreover, $\mathcal{B}$ fails to satisfy

$$\alpha(\Theta \circ_n \gamma) \subseteq \alpha\Theta \circ_{2n} \alpha\gamma, \tag{3.4}$$

for n odd, where Θ is a tolerance and α and γ are congruences.

Before proving Theorem 3.1 we shall prove Proposition 3.2 below, a more general result whose formulation, however, is more involved. The statement of Theorem 3.1 suggests that, for n even, $\mathcal{B}$ and $\mathcal{N}_4$ behave in the same way and no essential difference seems to appear in comparison with the previous section. On the other hand, for n odd, Theorem 3.1 shows that $\mathcal{N}_4$ satisfies the somewhat stronger identity (3.3). However, in a sense, $\mathcal{N}_4$ behaves better than $\mathcal{B}$ for every value of the index n, as we shall show in Proposition 3.2. The difference between $\mathcal{B}$ and $\mathcal{N}_4$ will appear in a clearer light in Theorem 4.1.

After the proof of Proposition 3.2, furnishing the positive side of Theorem 3.1, we shall present the example of a tolerance on (B, b), the algebra constructed in the proof of Theorem 2.1. The example shall then be used in order to show that the bounds in identity (3.2) are optimal.

For a binary relation R, let R^n denote the n-fold composition of R with itself, that is, $R^n = R \circ_n R$. If R and S are binary relations, let $\overline{R \cup S}$ denote the smallest admissible relation containing both R and S.

Proposition 3.2. *For every $n \geq 2$, the following identities are satisfied:*

$$\textit{by } \mathcal{B}\textit{:} \quad T(R_1 \circ R_2 \circ \ldots \circ R_n) \subseteq (TR_1 \circ TR_2 \circ \ldots \circ TR_n)^2; \tag{3.5}$$

$$\begin{aligned}\textit{by } \mathcal{N}_4\textit{:} \quad T(R_1 \circ R_2 \circ \ldots \circ R_{n-1} \circ R_n) &\subseteq TR_1 \circ TR_2 \circ \ldots \circ TR_{n-1} \circ \\ &T(TR_n \circ TR_1)(\overline{R_n \cup R_1}) \circ TR_2 \circ \ldots \circ TR_{n-1} \circ TR_n.\end{aligned} \tag{3.6}$$

Proof. Were T a congruence, the proof of (2.1) in Theorem 2.1 would give a proof of (3.5), since in the proof of Theorem 2.1 we have not used the assumption that β and γ are congruences, we have only used that β and γ are admissible relations. Dealing with many relations instead of just two relations presents no new difficulty. At certain places in Theorem 2.1 we need transitivity of β, but this is not necessary here, according the formulations of Theorem 3.1 and Proposition 3.2. For example, were both T and R transitive in equation (3.2), then the two adjacent occurrences of TR would absorb into one. This is the reason why the indices in Theorem 2.1 can be improved by one in the case n odd, since α and β there are congruences, hence transitive. But we are not assuming transitivity in Theorem 3.1 and in Proposition 3.2; correspondingly, we are not asking for the stronger conclusions.

We have to give a proof of Proposition 3.2 in the case when T is just an admissible relation. This involves considering a different sequence of elements in comparison with the sequence described in (2.5). We shall first give the proof of (3.5) for $\mathcal{B}$; then we shall improve (3.5) to (3.6) for $\mathcal{N}_4$ by an additional and somewhat delicate argument.

Suppose that $(a, d) \in T(R_1 \circ R_2 \circ \ldots \circ R_n)$ in some algebra in $\mathcal{B}$. This means that $a\ T\ d$ and that there are elements c_i such that $a = c_0\ R_1\ c_1\ R_2\ c_2 \ldots R_{n-1}$ $c_{n-1}\ R_n\ c_n = d$. In what follows, we shall write, say, $a(d + c_1)$ for $b(a, d, c_1)$, and $a(d + c_1)(d + c_2)$ for $b(b(a, d, c_1), d, c_2)$. Equations like $a(d + c_1)(d + c_1) = a(d+c_1)$ or $a(d+c_1)(a+c_2) = a(d+c_1)$ hold in $\mathcal{B}$ since corresponding equations hold in lattices. Consider the following elements.

$$\begin{aligned}
g_0 &= a & g_1 &= a(d+c_1)\\
g_2 &= a(d+c_1)(d+c_2) & &\dots\\
g_{n-1} &= a(d+c_1)(d+c_2)\dots(d+c_{n-1}) & g_n &= h_0 = ad\\
h_1 &= d(a+c_1)(a+c_2)\dots(a+c_{n-1}) & h_2 &= d(a+c_2)\dots(a+c_{n-1})\\
\dots \quad h_{n-1} &= d(a+c_{n-1}) & h_n &= d
\end{aligned}$$

We have $g_i \ TR_{i+1} \ g_{i+1}$ for $i < n$ and similarly for the h_i's. Indeed, for example,

$$g_1 = a(d+c_1) = a(d+c_1)(a+c_2) \ \ T \ \ a(d+c_1)(d+c_2) = g_2, \text{ and}$$
$$g_1 = a(d+c_1) = a(d+c_1)(d+c_1) \ R_2 \ a(d+c_1)(d+c_2) = g_2.$$

Notice that, in the definition of the g_i's, when going from g_{n-1} to g_n we follow the preceding pattern; indeed, according to the pattern, g_n would be $a(d+c_1)\dots(d+c_{n-1})(d+c_n) = a(d+c_1)\dots(d+c_{n-1})(d+d)$ which in fact is equal to ad. Thus we have $(a, g_n) \in TR_1 \circ TR_2 \circ \dots \circ TR_n$ and $(h_0, d) \in TR_1 \circ TR_2 \circ \dots \circ TR_n$, hence $(a, d) \in (TR_1 \circ TR_2 \circ \dots \circ TR_n)^2$, since $g_n = h_0$.

We have proved (3.5).

In passing, let us remark that the above elements $g_0, g_1, \dots, g_n = h_0$, $\dots, h_{n-1}, h_n$ satisfy some additional relation identities. We have that each element in the above chain is T-related with every element which follows. For example, $g_{n-1} \ T \ h_1$, since

$$g_{n-1} = a(d+c_1)\dots(d+c_{n-1}) = a(d+c_1)\dots(d+c_{n-1})(a+c_1)\dots(a+c_{n-1}) \ T$$
$$d(d+c_1)\dots(d+c_{n-1})(a+c_1)\dots(a+c_{n-1}) = d(a+c_1)\dots(a+c_{n-1}) = h_1.$$

All the other relations are proved in a similar way.

The proof of equation (3.6) involves the same chain of elements, this time working in $\mathcal{N}_4$. Notice that the above elements are expressible in $\mathcal{N}_4$, since $b(x_1, x_3, x_4) = u(x_1, x_1, x_3, x_4) = x_1(x_3 + x_4)$. In the above-displayed formula we have showed $g_{n-1} \ T \ h_1$. It remains to show that, in addition, $g_{n-1} \ \overline{R_n \cup R_1} \ h_1$. In order to prove this relation, we shall write $g_{n-1} = a(d+c_1)\dots(d+c_{n-1})$ as $u(a, a, d, c_{n-1}) \cdot (d+c_1)\dots(d+c_{n-2})$. This formula should be interpreted in the sense that, say, $u(a, a, d, c_{n-1}) \cdot (d+c_1)$ is an abbreviation for $b(u(a, a, d, c_{n-1}), d, c_1)$, and we can add further factors of the form $(d + c_j)$ by iterated applications of b, as we did in the first part of the proof. The identities we shall use will all follow from corresponding identities holding in lattices. Now we compute

$$\begin{gathered}
g_{n-1} = a(d+c_1)\dots(d+c_{n-2})(d+c_{n-1}) = \\
a(d+c_{n-1})(d+c_1)\dots(d+c_{n-2})(a+c_2)\dots(a+c_{n-1}) = \\
u(a, a, d, c_{n-1}) \cdot (d+c_1)\dots(d+c_{n-2})(a+c_2)\dots(a+c_{n-1}) \ \overline{R_n \cup R_1} \\
u(a, c_1, d, d) \cdot (d+c_1)\dots(d+c_{n-2})(a+c_2)\dots(a+c_{n-1}) =
\end{gathered}$$

$$d(a+c_1)(d+c_1)\dots(d+c_{n-2})(a+c_2)\dots(a+c_{n-1}) = \\ d(a+c_1)(a+c_2)\dots(a+c_{n-1}) = h_1.$$

Thus $g_{n-1}\ \overline{R_n \cup R_1}\ h_1$ and the proof of Proposition 3.2 is complete. □

Recall the definitions of α, β, γ, B, $B^{\uparrow}$ from the proof of Theorem 2.1. Let $B^{\downarrow} = B \setminus B^{\uparrow}$ be the set of those elements of B with a third $\downarrow$ component. Let $E = \{e_0, \dots e_n\}$ and $F = \{f_0, \dots f_n\}$, where $e_0, \dots, f_n$ are the elements in the displayed list $(B^{\uparrow})$ in the proof of Theorem 2.1. We now present the example of a tolerance on (B, b). Recall that (B, b) is an algebra in $\mathcal{B}$.

Example 3.3. Let Λ be the binary relation on B defined as follows. Two elements x and y of B are Λ-related if and only if either

(a) both x and y belong to E, or

(b) both x and y belong to F, or

(c) at least one of x and y belongs to $B^{\downarrow}$.

We are going to show that Λ is a tolerance on (B, b).

Indeed, Λ is clearly symmetric and reflexive, since $B^{\uparrow} = E \cup F$, hence $B^{\downarrow} = B \setminus (E \cup F)$. We have to check that Λ is admissible. Suppose that $x_1\ \Lambda\ y_1$, $x_2\ \Lambda\ y_2$ and $x_3\ \Lambda\ y_3$ are elements of B. Letting $x = b(x_1, x_2, x_3)$ and $y = b(y_1, y_2, y_3)$, we have to show that $x\ \Lambda\ y$. If either $x \in B^{\downarrow}$ or $y \in B^{\downarrow}$, there is nothing to prove. If both $x = b(x_1, x_2, x_3)$ and $y = b(y_1, y_2, y_3)$ belong to $B^{\uparrow}$, that is, they have a last $\uparrow$ component, then both x_1 and y_1 have a last $\uparrow$ component, that is, $x_1, y_1 \in B^{\uparrow}$. By the definition of Λ, either $x_1, y_1 \in E$ or $x_1, y_1 \in F$, and, correspondingly, x_1 and y_1 either both have a null second component or have a null first component; hence this applies to x and y, too. By the description of $B^{\uparrow}$ in the proof of Theorem 2.1, if x and y have a null second component, then they both belong to E, and if x and y have a null first component, then they both belong to F. We have showed that $x\ \Lambda\ y$, thus Λ is admissible (we have not used the assumption that $x_2\ \Lambda\ y_2$ and $x_3\ \Lambda\ y_3$).

On the other hand, Λ is not admissible on (B, u); see Remark 4.2.

Proof of Theorem 3.1. The positive result that equations (3.1) - (3.3) hold is an immediate consequence of Proposition 3.2, taking $R_1 = R_3 = \dots = R$ and $R_2 = R_4 = \dots = S$. Notice that if n is odd, then $R_n = R_1 = R$, hence the factor $T(TR_n \circ TR_1)(\overline{R_n \cup R_1})$ in (3.6) becomes TR, thus we get (3.3) from (3.6).

The bounds given by equations (3.1) and (3.3) are optimal even in the case of congruences, as shown by equations (2.2) and (2.4) in Theorem 2.1. As soon as we show that (3.4) fails, we get that (3.2) is the best possible result; in particular, the two adjacent occurrences of TR in the middle of the right-hand side of (3.2) do not always "absorb into one", even in the case when T is a congruence and R is a tolerance.

To show that (3.4) can fail, consider the construction in the proof of Theorem 2.1 in the case n odd, this time taking $\mathbf{B} = (B, b)$ in place of $\mathbf{B} = (B, u)$. Let $\Theta = \Lambda\beta$, where Λ is the tolerance constructed in Example 3.3. Proceed as in the proof of Theorem 2.1 and notice that $(c_0, c_n) \in \alpha(\Theta \circ_n \gamma)$, as witnessed, again, by $c_1, c_2, \dots$ In the case at hand $e_{n-1} = (1, 0, \uparrow)$ and $f_1 = (0, 1, \uparrow)$ are not Θ-related, since they are not Λ-related. Hence here we cannot skip the passage from e_{n-1} to e_n, as we did in the case n odd in the proof of Theorem 2.1. Of course, we do have $(c_0, c_n) \in (\alpha\Theta \circ \alpha\gamma \circ \overset{n}{\dots} \circ \alpha\Theta) \circ (\alpha\Theta \circ \alpha\gamma \circ \overset{n}{\dots} \circ \alpha\Theta)$, as shown be equation (3.2); however, we lose one more turn if we want that $\alpha\Theta$ and $\alpha\gamma$ strictly alternate on the right-hand side of (3.4), hence we cannot have $(c_0, c_n) \in \alpha\Theta \circ_{2n} \alpha\gamma$. ☐

Notice that the above argument shows that even the identity $\alpha(\Theta \circ (\alpha\gamma \circ \alpha\Theta \circ \overset{n-2}{\dots} \circ \alpha\gamma) \circ \Theta) \subseteq \alpha\Theta \circ_{2n} \alpha\gamma$ fails in $\mathcal{B}$, for n odd.

Remark 3.4. In [14] we have showed that, under a fairly general hypothesis, any congruence identity is equivalent to the corresponding tolerance identity, provided that only tolerances *representable* as $R \circ R^\smile$ are considered in the latter identity. Here $R^\smile$ denotes the *converse* of R.

Equations (2.2) in Theorem 2.1 and (3.4) in Theorem 3.1 show that, in general, the assumption of representability is necessary in the results from [14]. A similar counterexample has been presented in [19].

It follows from [14] that the tolerance Θ used in the proof of Theorem 3.1 is not representable. It can be checked directly that in (B, b) neither Θ for n odd, nor Λ for every n are representable, where Λ is the tolerance constructed in Example 3.3. In fact, if R is reflexive and admissible, $c_0 \ R \circ R^\smile \ c_1$ and $c_{n-1} \ R \circ R^\smile \ c_n$, then $e_{n-1} \ R \circ R^\smile \ f_1$. Indeed, if $c_0 \ R \ g \ R^\smile \ c_1$ and $c_{n-1} \ R \ h \ R^\smile \ c_n$, then $c_n \ R \ h$ and $g \ R^\smile \ c_0$, hence $e_{n-1} = c_0(c_{n-1} + c_n) \ R \ g(h + h) = gh = h(g + g) \ R^\smile \ c_n(c_0 + c_1) = f_1$.

4. Identities with just two relations

If R is a congruence, or just a transitive relation, then then there is no point in considering identities of the form $T(R \circ R) \subseteq$ *something*, since $R \circ R = R$. In passing, let us point out that this latter identity $R \circ R = R$ is equivalent to congruence permutability, as noticed independently by Hutchinson [8] and Werner [22]. If we only suppose that R is admissible, many more identities of the form $T(R \circ R) \subseteq$ *something* become interesting. For example, we showed in [18] that a variety $\mathcal{V}$ is congruence modular if and only if there is some k such that $\mathcal{V}$ satisfies the identity $\Theta(R \circ R) \subseteq (\Theta R)^k$.

In the present section we evaluate the best possible bounds for identities of the above kind both in $\mathcal{B}$ (equivalently, $\mathcal{B}^d$) and $\mathcal{N}_4$ (equivalently $\mathcal{N}_4^d$). In a certain respect, here the situation is simpler than in the preceding sections, since we do not need the division into the two cases n odd and n even; moreover, the bounds for $\mathcal{N}_4$ are always better than the bounds for $\mathcal{B}$. Notice that all the identities considered in the present section are strictly weaker than congruence distributivity, since they might hold in non distributive congruence modular varieties; actually, all the identities below hold in congruence permutable varieties.

Theorem 4.1. *For $n \geq 2$, the following identities are satisfied:*

$$\text{by } \mathcal{B}: \quad TR^n \subseteq (TR)^{2n} \tag{4.1}$$

$$\text{by } \mathcal{N}_4: \quad TR^n \subseteq (TR)^{2n-1} \tag{4.2}$$

and the exponents on the right are best possible; actually,

$$\mathcal{B} \text{ fails to satisfy} \quad \alpha(\Theta \circ (\alpha\Theta)^{n-2} \circ \Theta) \subseteq (\alpha\Theta)^{2n-1}, \text{ and} \tag{4.3}$$

$$\mathcal{N}_4 \text{ fails to satisfy} \quad \alpha(\Theta \circ (\alpha\Theta)^{n-2} \circ \Theta) \subseteq (\alpha\Theta)^{2n-2}. \tag{4.4}$$

Proof. Identities (4.1) and (4.2) are immediate from identities (3.1) - (3.3) in Theorem 3.1, taking $S = R$; they can also be obtained directly from Proposition 3.2.

We first show that (4.4) can fail, hence the bound in (4.2) is best possible. Consider again the counterexample (B, u) constructed in the proof of Theorem 2.1. Let Ψ be the binary relation on B defined in such a way that two elements x and y in B are Ψ-related if and only if

(d) for each $\ell = 1, 2$, the components x_ℓ and y_ℓ differ at most by 1.

We claim that Ψ is a tolerance on $\mathbf{B}$. Indeed, condition (d) defines a tolerance Ψ_L on the lattice $\mathbf{L} = \mathbf{C}_{n+1} \times \mathbf{C}_{n+1} \times \mathbf{C}_2$, since Ψ_L is a product of tolerances on the factors. Hence Ψ_L is a tolerance on the polynomial reduct (L, u) and Ψ, being the restriction of Ψ_L to B, is a tolerance, too.

Now we have $(c_0, c_n) \in \alpha(\Psi \circ (\alpha\Psi)^{n-2} \circ \Psi)$, again, as witnessed by $c_1, c_2, \dots$ On the other hand, as in the proof of Theorem 2.1, the only other element $\alpha\Psi$-connected to $c_0 = e_0$ is e_1, the only other element $\alpha\Psi$-connected to e_1 is e_2 and so on, until we reach e_{n-1}, which is $\alpha\Psi$-connected only to e_{n-2} (but this has no use), to $e_n = f_0$ and to f_1. We get the fastest path going directly through f_1; in any case, we need $2n-1$ steps, thus $(c_0, c_n) \in (\alpha\Psi)^{2n-2}$ fails in $\mathcal{N}_4$. Hence (4.4) fails with Ψ in place of Θ.

In order to disprove the identity in (4.3), let us work in (B, b) instead. Recall that $(B, b) \in \mathcal{B}$. The relation Ψ defined above is a tolerance on (B, b), being a tolerance on (B, u). Let $\Theta = \Lambda\Psi$, where Λ is the tolerance defined in Example 3.3. As above, we have $(c_0, c_n) \in \alpha(\Theta \circ (\alpha\Theta)^{n-2} \circ \Theta)$. We have that $(c_0, c_n) \in (\alpha\Theta)^{2n-1}$ fails, since any chain of $\alpha\Theta$-related elements from c_0 to c_n must contain all the elements of $B^{\uparrow}$. The difference with the previous case dealing with $\mathcal{N}_4$ is that here e_{n-1} and f_1 are not Θ-related, being not Λ-related, hence one more step is necessary. $\square$

Remark 4.2. In Example 3.3 we have seen that Λ is a tolerance on (B, b). It follows from the above proof that Λ is not a tolerance on (B, u). It is easy to see directly that Λ is not even compatible in (B, u); otherwise, we would get $e_{n-1} = u(c_0, c_0, c_{n-1}, c_n) \mathrel{\Lambda} u(c_0, c_1, c_n, c_n) = f_1$, a contradiction.

As a small improvement on some results in this and in the previous section, notice that in the identities in (3.1), (3.2), (4.1) and (3.5) it is enough to assume that T, R and S are set-theoretical unions of admissible relations. Indeed, in the proofs only one element is moved at a time. Cf. [16].

The following lemma provides a simpler argument to show that the exponent on the right in the identity in (4.2) cannot be improved. Recall that if T is a binary relation on some algebra, $\overline{T}$ denotes the smallest reflexive and admissible relation containing T. The definition of n-modularity has been recalled shortly before Corollary 2.4.

Lemma 4.3. *Let $\mathcal{V}$ be any variety.*

(1) *If $\mathcal{V}$ satisfies $\alpha(R \circ R) \subseteq (\alpha R)^k$, then $\mathcal{V}$ is $2k$-modular.*
(2) *More generally, if $\mathcal{V}$ satisfies $\alpha(R \circ_n R) \subseteq (\alpha R)^k$, for some $n \geq 2$, then $\mathcal{V}$ satisfies $\alpha(\beta \circ_{2n} \alpha\gamma) \subseteq \alpha\beta \circ_{2k} \alpha\gamma$.*
(3) *If $k \geq 2$ and $\mathcal{V}$ satisfies $\alpha(R \circ S) \subseteq \alpha R \circ (\alpha(\overline{R \cup S}))^{k-1}$, then $\mathcal{V}$ is $2k-1$-modular.*

Proof. (1) Taking $R = \beta \circ \alpha\gamma$, we have $\alpha(\beta \circ \alpha\gamma) = \alpha\beta \circ \alpha\gamma$, hence

$$\alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha(\beta \circ \alpha\gamma \circ \beta \circ \alpha\gamma) = \alpha(R \circ R) \subseteq$$
$$(\alpha R)^k = (\alpha(\beta \circ \alpha\gamma))^k = (\alpha\beta \circ \alpha\gamma)^k = \alpha\beta \circ_{2k} \alpha\gamma.$$

(2) is proved in the same way.

(3) Take $R = \beta$, $S = \alpha\gamma \circ \beta$ and observe that $R \cup S = \alpha\gamma \circ \beta$ is admissible. $\square$

Thus $\mathcal{N}_4$ fails to satisfy $TR^2 \subseteq (TR)^2$, since otherwise $\mathcal{N}_4$ would be 4-modular, by 4.3(1), contradicting Corollary 2.4.

More generally, equation (4.2) cannot be improved to $TR^n \subseteq (TR)^{2n-2}$, since otherwise Lemma 4.3(2) would give

$$\alpha(\beta \circ (\alpha\gamma \circ \alpha\beta \circ \overset{2n-3}{\dots} \circ \alpha\gamma) \circ \beta) \subseteq \alpha(\beta \circ_{2n} \alpha\gamma) \subseteq^{4.3(2)} \alpha\beta \circ_{4n-4} \alpha\gamma, \quad (4.5)$$

where the superscript in $\subseteq^{4.3(2)}$ means that we are applying item (2) in Lemma 4.3. Taking $m = 2n-1$ in (4.5), we get $2n-3 = m-2$ and $4n-4 = 2m-2$, thus equation (4.5) becomes equation (2.4) in Theorem 2.1 with m in place of n and Theorem 2.1 shows that this equation fails for $\mathcal{N}_4$.

Remark 4.4. Recall that $R^{\smile}$ denotes the *converse* of the relation R. It is not difficult to show that a variety $\mathcal{V}$ is $k+1$-modular if and only if $\mathcal{V}$ satisfies the identity

$$\alpha(R \circ R^{\smile}) \subseteq \alpha R \circ_k \alpha R^{\smile}. \quad (4.6)$$

See [18], where it is also shown that α can be equivalently taken to vary among tolerances. If we let $R = \Theta$ be a tolerance in (4.6), we get

$$\alpha(\Theta \circ \Theta) \subseteq (\alpha\Theta)^k. \quad (4.7)$$

Clearly, in turn, (4.7) implies back congruence modularity; actually, (4.7) implies $2k+2$-modularity (perhaps the bound $2k+2$ can be improved). Indeed, taking $\Theta = \alpha\gamma \circ \beta \circ \alpha\gamma$ in (4.7) we get

$$\alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha(\alpha\gamma \circ \beta \circ \alpha\gamma \circ \beta \circ \alpha\gamma) = \alpha(\Theta \circ \Theta) \subseteq^{(4.7)}$$
$$(\alpha\Theta)^k = (\alpha\gamma \circ \alpha\beta \circ \alpha\gamma)^k = \alpha\gamma \circ_{2k+1} \alpha\beta \subseteq \alpha\beta \circ_{2k+2} \alpha\gamma.$$

Theorem 4.1 shows that the best value of k in (4.7) for a congruence modular variety $\mathcal{V}$ is not determined by the Day modularity level of $\mathcal{V}$ (and it is not determined by the Jónsson distributivity level, either, for congruence distributive varieties). Indeed, both $\mathcal{B}$ and $\mathcal{N}_4$ are 5-modular, not 4-modular, 4-distributive and not 3-distributive, but the best value for k in (4.7) is 4 for $\mathcal{B}$ and 3 for $\mathcal{N}_4$: just take $n = 2$ in Theorem 4.1. In particular, the variety $\mathcal{N}_4$ shows that (4.7) for some given k does not imply $k + 1$-modularity.

Remark 4.5. Since the identity

$$\alpha(R \circ R) \subseteq (\alpha R)^k \tag{4.8}$$

implies (4.7), then, by the above remark, the identity (4.8) implies congruence modularity. This follows also directly from Lemma 4.3. As we mentioned in the introduction, it is shown in [18] that the converse holds, that is, every congruence modular variety satisfies (4.8), for some k. Again, this argument from [18] relies heavily on [12].

As in Remark 4.4, we get that Theorem 4.1 shows that the best value of k in (4.8) for a variety $\mathcal{V}$ is not determined by the Day modularity level of $\mathcal{V}$. It is likely that there is some variety $\mathcal{V}$ such that the best value of k in (4.7) is strictly smaller than the best value of k in (4.8). The varieties considered here do not furnish a (counter-)example for this possible inequality.

5. Near-unanimity and edge terms

Of course, it is an interesting problem to evaluate the distributivity spectra of other congruence distributive varieties. Varieties with a near-unanimity term appear to be a significant test case. Recall that R, S, $T, \dots$ denote reflexive and admissible relations, Θ denotes a tolerance and $\overline{R \cup S}$ denotes the smallest admissible relation containing both R and S.

Proposition 5.1. *Suppose that $m \geq 1$ and that the variety $\mathcal{V}$ has an $m+2$-ary near-unanimity term. Then, for every $n \geq 2$, the variety $\mathcal{V}$ satisfies*

$$\alpha(\beta \circ_n \gamma) \subseteq \alpha\beta \circ_{mn} \alpha\gamma \qquad \textit{for } n \textit{ even,} \tag{5.1}$$
$$\alpha(\beta \circ_n \gamma) \subseteq \alpha\beta \circ_{1+m(n-1)} \alpha\gamma \qquad \textit{for } n \textit{ odd,} \tag{5.2}$$
$$\Theta(R \circ_n R) \subseteq (\Theta R)^{1+m(n-1)} \qquad \textit{for every } n. \tag{5.3}$$

Taking $n = 2$ in equation (5.1) *we get that $\mathcal{V}$ is $2m$-distributive* [20]*. Taking $n = 3$ in equation* (5.2) *with $\alpha\gamma$ in place of γ, we get that $\mathcal{V}$ is $2m+1$-modular.*

In fact, we shall prove a more general result.

Proposition 5.2. *Under the assumptions in Proposition 5.1, $\mathcal{V}$ satisfies*

$$\begin{aligned} \Theta(R_1 \circ R_2 \circ \ldots \circ R_n) \subseteq \Theta R_1 \circ \Theta R_2 \circ \ldots \circ \Theta R_{n-1} \circ \Theta(\overline{R_n \cup R_1}) \circ \\ \Theta R_2 \circ \ldots \circ \Theta R_{n-1} \circ \Theta(\overline{R_n \cup R_1}) \circ \\ \ldots \Theta R_2 \circ \ldots \circ \Theta R_{n-1} \circ \Theta R_n, \end{aligned} \tag{5.4}$$

with m lines, that is, with a total of $1 + m(n-1)$ factors (a total of $m(n-1)$ occurrences of $\circ$) on the right-hand side.

Equation (5.4) should read $\Theta(R_1 \circ \ldots \circ R_n) \subseteq \Theta R_1 \circ \Theta R_2 \circ \ldots \circ \Theta R_{n-1} \circ \Theta R_n$ when $m = 1$ and $\Theta(R_1 \circ \ldots \circ R_n) \subseteq \Theta R_1 \circ \ldots \circ \Theta R_{n-1} \circ \Theta(\overline{R_n \cup R_1}) \circ\ \Theta R_2 \circ \ldots \circ \Theta R_n$ when $m = 2$.

Proofs. Suppose that u is an $m+2$-ary near-unanimity term, $a \ \Theta \ d$ and $a \ R_1 \ c_1 \ R_2 \ c_2 \ldots c_{n-2} \ R_{n-1} \ c_{n-1} \ R_n \ d$. Then

$$\begin{gathered} a = u(a, \ldots, a, a, a, d) \ R_1 \ u(a, \ldots, a, a, c_1, d) \ R_2 \ u(a, \ldots, a, a, c_2, d) \ldots \\ R_{n-1} \ u(a, \ldots, a, a, c_{n-1}, d) \ \overline{R_n \cup R_1} \ u(a, \ldots, a, c_1, d, d) \ R_2 \ u(a, \ldots, a, c_2, d, d) \\ \ldots \\ R_{n-1} \ u(a, a, c_{n-1}, d, \ldots, d) \ \overline{R_n \cup R_1} \ u(a, c_1, d, d, \ldots, d) \ R_2 \ u(a, c_2, d, d, \ldots, d) \\ \ldots R_{n-1} \ u(a, c_{n-1}, d, d, \ldots, d) \ R_n \ u(a, d, d, d, \ldots, d) = d. \end{gathered}$$

In the above chain of relations we have only used a minimal part of the assumption that u is a near-unanimity term: we have used only the two special cases in which the "dissenter" is either the first or the last element. The full assumption will be used in order to show that all the above elements are Θ-related. Were $\Theta = \alpha$ a congruence, this would be trivial, since

$$\begin{aligned} u(a, \ldots, a, c_j, d, \ldots, d) \ \alpha \ u(a, \ldots, a, c_j, a, \ldots, a) = a = \\ u(a, \ldots, a, c_k, a, \ldots, a) \ \alpha \ u(a, \ldots, a, c_k, d, \ldots, d), \end{aligned}$$

for all pairs of indices j and k and where the c_j's and the c_k's can occur in any pair of possibly distinct positions.

Notice that the case in which Θ is a congruence is enough in order to prove (5.1) and (5.2) in Proposition 5.1. Formally, (5.1) does not follow from (5.4); however in (5.4) we can replace each occurrence of $\Theta(\overline{R_n \cup R_1})$ by $\Theta R_n \circ \Theta R_1$. Indeed, say, $u(a, \ldots, a, a, c_{n-1}, d) \ \Theta R_n \ u(a, \ldots, a, a, d, d) \ \Theta R_1$ $u(a, \ldots, a, c_1, d, d)$.

It remains to consider the case in which Θ is only supposed to be a tolerance. The argument resembles a proof in Czédli and Horváth [4]. As above, we shall show that any two elements in the above chain (disregarding their ordering, that is, slightly more than required) are Θ-related. Indeed,

$$\begin{aligned} & u(a, \ldots, a, c_j, d, \ldots, d) = \\ & u(u(\ldots a, c_k, a \ldots), \ldots u(\ldots a, c_k, a \ldots), c_j, u(\ldots d, c_k, d \ldots), \ldots u(\ldots d, c_k, d \ldots)) \ \Theta \\ & u(u(\ldots a, c_k, d \ldots), \ldots u(\ldots a, c_k, d \ldots), c_j, u(\ldots a, c_k, d \ldots), \ldots u(\ldots a, c_k, d \ldots)) = \\ & u(a, \ldots, a, c_k, d, \ldots, d), \end{aligned}$$

again, for all pairs of indices j and k and where the c_j's and the c_k's can occur in any pair of possibly distinct positions. In the above-displayed formula the vertical bars link distinct Θ-related elements and, in order to keep the formula within a reasonable length, we have written, say, $u(\dots a, c_k, d \dots)$ in place of $u(a, a, \dots, a, a, c_k, d, d, \dots, d, d)$. In conclusion, we get that (a, d) belongs to the right-hand side of (5.4). □

Equations (5.1) and (5.2) show that a variety with a near-unanimity term is congruence distributive, a result originally due to Mitschke [20]. The above proof seems simpler than the one from [20] and uses folklore ideas. Cf., e. g., Kaarli and Pixley [11, Lemma 1.2.12], whose proof is credited to E. Fried. Notice that here and in [11, Lemma 1.2.12], as well, it is not necessary to use Jónsson's characterization [9] of congruence distributive varieties.

From a more recent point of view, (5.4) might be seen as a combination of two observations. First, the fact that a near-unanimity term easily yields a set of directed Jónsson terms; see, e. g., Barto and Kozik [2, Section 5.3.1]. Second, the observation in [15] that directed Jónsson terms not only imply congruence distributivity, but also imply certain similar relation identities. The technical idea of merging R_n and R_1, so as to obtain a smaller number of factors in (5.4) and (5.3) seems new, at least in the present context.

We do not know whether we can replace the tolerance Θ by an admissible relation T in (5.4) and (5.3) (with or without the same number of factors on the right). Apart from this, the variety of lattices shows that the results in Proposition 5.1 are best possible when $m = 1$. Since u is a 4-ary near-unanimity term in $\mathcal{N}_4$, then Theorems 2.1, 3.1 and 4.1 show that the values of the indices on the right-hand sides of Proposition 5.1 are best possible when $m = 2$.

We now turn to edge terms, an important generalization of near-unanimity terms. Berman, Idziak, Marković, McKenzie, Valeriote and Willard [3] have introduced edge terms in [3], providing equivalent characterizations for their existence. Further characterizations have been found by Kearnes and Szendrei in [13].

If $k \geq 2$, a $k+1$-ary term t is a *k-edge term* for some variety $\mathcal{V}$ if the equations $x = t(y, y, x, x, \dots, x) = t(x, y, y, x, \dots, x)$ hold and, moreover, all the equations of the form $x = t(x, x, x, \dots, y, \dots)$ hold in $\mathcal{V}$, where a single occurrence of y appears in any place after the third place, surrounded by x's elsewhere. We have used here the formulation from [13] in which the first two places are exchanged. Notice that a k-ary near-unanimity term becomes a k-edge term by adding a dumb variable at the second place. The following proposition provides, among other, still another proof that varieties with an edge term are congruence modular.

Proposition 5.3. *If $k \geq 3$ and $\mathcal{V}$ has a k-edge term, then $\mathcal{V}$ satisfies*

$$\Theta(R \circ R) \subseteq (\Theta R)^{k-1} \qquad \textit{and, more generally,} \qquad (5.5)$$

$$\Theta(R \circ S) \subseteq \Theta R \circ (\Theta(\overline{R \cup S}))^{k-2}. \tag{5.6}$$

Thus $\mathcal{V}$ is $2k-3$-modular by Lemma 4.3(3).

Proof. The proof is similar to the proof of Proposition 5.1, with a variation "near the edge". Equation (5.5) is the particular case $R = S$ of equation (5.6), hence it is enough to prove the latter. If $a \ \Theta \ d$ and $a \ R \ c \ S \ d$, then

$$\begin{aligned}
a = {} & t(a, a, a, a, a, a, \dots, a, a, a, d) \ R \\
& t(a, a, a, a, a, a, \dots, a, a, c, d) \ \overline{R \cup S} \\
& t(a, a, a, a, a, a, \dots, a, c, d, d) \ \overline{R \cup S} \ \dots \\
\dots \ & t(a, a, a, a, c, d, \dots, d, d, d, d) \ \overline{R \cup S} \\
& t(a, a, a, c, d, d, \dots, d, d, d, d) \ \overline{R \cup S} \\
& t(a, c, c, d, d, d, \dots, d, d, d, d) \ \overline{R \cup S} \\
& t(c, c, d, d, d, d, \dots, d, d, d, d) = d
\end{aligned}$$

The proof that all the above elements are Θ-related is similar to the corresponding proof in 5.1. For example,

$$\begin{aligned}
t(a, c, c, d ...) = {} & t(t(a, a, a, c, a ...), c, c, t(d, d, d, c, d ...) ...) \ \Theta \\
& t(t(a, a, a, c, d ...), c, c, t(a, a, a, c, d ...) ...) = t(a, a, a, c, d ...). \ \square
\end{aligned}$$

Remark 5.4. Merging the proofs of Propositions 5.2 and 5.3 we get that if $k \geq 4$ and $\mathcal{V}$ has a k-edge term, then, for every $n \geq 2$, the variety $\mathcal{V}$ satisfies

$$\begin{aligned}
\Theta(R \circ S)(R_1 \circ \dots \circ R_n) \subseteq {} & \Theta R_1 \circ \Theta R_2 \circ \dots \circ \Theta R_{n-1} \circ \Theta(\overline{R_n \cup R_1}) \circ \\
& \Theta R_2 \circ \dots \circ \Theta R_{n-1} \circ \Theta(\overline{R_n \cup R_1}) \circ \\
& \dots \Theta R_2 \circ \dots \circ \Theta R_{n-1} \circ \Theta(\overline{R_n \cup R}) \circ \Theta(\overline{R \cup S}),
\end{aligned}$$

with $k - 3$ lines.

6. Further remarks

Recall that α, β, $\gamma, \dots$ denote congruences, Θ denotes a tolerance and R, S, $T, \dots$ denote reflexive and admissible relations.

Remark 6.1. It is standard and easy to show that varieties with a majority term satisfy $T(R \circ S) \subseteq TR \circ TS$. See, e. g., [15, 16]. Since the composition of two admissible relations is still admissible, then, by substitution and an easy induction, we get that if $\mathcal{V}$ has a majority term, then, for every $n \geq 2$, the variety $\mathcal{V}$ satisfies $T(R_1 \circ R_2 \circ \dots \circ R_n) \subseteq TR_1 \circ TR_2 \circ \dots \circ TR_n$.

Moreover, by taking $T = (R_3 \circ R_2)(R_1 \circ R_3)$, we get that a variety with a majority term satisfies $(R_1 \circ R_2)(R_3 \circ R_2)(R_1 \circ R_3) \subseteq R_1 R_3 \circ R_1 R_2 \circ R_1 R_2 \circ R_2 R_3$.

Remark 6.2. In passing, we notice the curious fact that while, of course, the identity $\alpha(\beta \circ \gamma) = \alpha\beta \circ \alpha\gamma$ for congruences is equivalent to the existence of a majority term, on the other hand, the identity

$$(\alpha \circ \delta)(\beta \circ \gamma) = \alpha\beta \circ \alpha\gamma \circ \delta\beta \circ \delta\gamma \tag{6.1}$$

is equivalent to arithmeticity. Notice that we are assuming equality, not just inclusion.

To prove the claim, assume equation (6.1) and expand the product in two ways, getting $\alpha\beta \circ \alpha\gamma \circ \delta\beta \circ \delta\gamma = \alpha\beta \circ \delta\beta \circ \alpha\gamma \circ \delta\gamma$. Then taking $\gamma = \alpha$ and $\delta = \beta$ we get $\alpha \circ \beta = \alpha\beta \circ \alpha \circ \beta \circ \alpha\beta = \alpha\beta \circ \beta \circ \alpha \circ \alpha\beta = \beta \circ \alpha$, that is, congruence permutability. On the other hand, by taking $\beta = \gamma$ in (6.1), we get 2-distributivity, that is, a majority term. It is well-known that arithmeticity is equivalent to congruence permutability together with the existence of a majority term, hence our claim follows.

Remark 6.3. (a) It follows from [9] and is by now standard that, for every m, some variety $\mathcal{V}$ satisfies the congruence identity

$$\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_m \alpha\gamma \tag{6.2}$$

if and only if (6.2) holds in $\mathbf{F}_{\mathcal{V}}(3)$, the free algebra in $\mathcal{V}$ generated by three elements. Actually, what is relevant in the above sentence is the left-hand side of (6.2); the sentence is true whenever we replace the right-hand side of (6.2) with any expression in function of α, β, γ constructed by using intersection, composition and even transitive closure.

Henceforth, a possible way to check whether some variety $\mathcal{V}$ satisfies (6.2), or even many related identities, is to check (6.2) in $\mathbf{F}_{\mathcal{V}}(3)$. Actually, if x, y and z are the generators of $\mathbf{F}_{\mathcal{V}}(3)$, it is enough to check (6.2) in the special case when α, β and γ are the congruences generated, respectively, by the pairs (x, z), (x, y), and (y, z). This is classical, by now. As we shall mention in (e) below, the above procedure is not the simplest way to check (6.2), or to check congruence distributivity; however, it is the one relevant to the following discussion.

(b) Let us compute, for example, $\mathbf{F}_{\mathcal{B}^d}(3)$. Since $\mathbf{F}_{\mathcal{B}^d}(3)$ is naturally embedded into $\mathbf{F}_{\mathcal{D}}(3)$, where $\mathcal{D}$ is the variety of distributive lattices, it is easy to see that the elements of $\mathbf{F}_{\mathcal{B}^d}(3)$ are

$$x, \qquad x(y+z), \qquad xy, \qquad xyz,$$

together with the elements arising from all the possible permutations of x, y and z. Cf. [1]. The elements in the above list are exactly also the elements of $\mathbf{F}_{\mathcal{N}_4^d}(3)$, since $F_{\mathcal{B}^d}(3)$ is closed under u. Indeed, if $a \in F_{\mathcal{B}^d}(3)$, then, by the above description, either $a \leq x$, or $a \leq y$, or $a \leq z$. Hence if $a_1, a_2, a_3, a_4 \in F_{\mathcal{B}^d}(3)$, then at least two elements are less than or equal to some generator, say, $a_1, a_2 \leq x$, thus $u(a_1, a_2, a_3, a_4) \leq x$, hence $u(a_1, a_2, a_3, a_4) \in F_{\mathcal{B}^d}(3)$. Of course, the above argument would fail, were we considering $\mathbf{F}_{\mathcal{V}}(4)$ in place of $\mathbf{F}_{\mathcal{V}}(3)$.

In a sense, computing $\mathbf{F}_{\mathcal{V}}(3)$ is a way to see that $\mathcal{N}_4^d$ and $\mathcal{B}^d$ satisfy exactly the same identities of the form $\alpha(\beta \circ \gamma) \subseteq$ *something* for congruences, for many possible variations on the expression on the right. However, there is a subtle related issue we are going to discuss soon.

(c) Consider now relation identities of the form

$$T(R \circ S) \subseteq TR \circ_m TS. \tag{6.3}$$

Again, many variations are possible, including letting some variable be a congruence or a tolerance. It is still true that some variety $\mathcal{V}$ satisfies (6.3) if and only if (6.3) holds in $\mathbf{F}_{\mathcal{V}}(3)$.

The argument is standard but not very usual. To prove the non trivial implication, let $\mathbf{F}_{\mathcal{V}}(3)$ be generated by x, y and z and R, S and T be the smallest reflexive and admissible relations containing, respectively, the pairs (x, z), (x, y) and (y, z). Since $(x, z) \in T(R \circ S)$, then, if (6.3) holds, there are elements $t_0, \dots, t_m \in F_{\mathcal{V}}(3)$ witnessing $(x, z) \in TR \circ_m TS$. Since we are working in $\mathbf{F}_{\mathcal{V}}(3)$, the t_i's correspond to ternary terms of $\mathcal{V}$. What does it mean, say, that $t_i \ R \ t_{i+1}$? It is easy to see that $R = \{(w(x, y, z, x), w(x, y, z, z)) \mid w \text{ a 4-ary term of } \mathcal{V}\}$. Hence $t_i(x, y, z) = w_i(x, y, z, x)$ and $w_i(x, y, z, z) = t_{i+1}(x, y, z)$, for some 4-ary term w_i. Similarly, the S- and T-relations are witnessed by certain other 4-ary terms. Once we have found appropriate terms and appropriate equations, it is standard to see that they witness that (6.3) holds in $\mathcal{V}$. See [16, Proposition 3.7] for full details.

(d) We now see an essential difference between the observations in (a) and (c) above. By (b), we have $F_{\mathcal{N}_4^d}(3) = F_{\mathcal{B}^d}(3)$; nevertheless, we have seen in Theorem 3.1 that $\mathcal{N}_4^d$ and $\mathcal{B}^d$ *do not* satisfy the same identities of the form (6.3), in the sense that the best possible indices on the right are not the same. At first, this might generate some perplexity, but in the end the explanation is easy. The point is that, when considering congruence identities of the form (6.2), only ternary terms are relevant; in other words, only the elements of $\mathbf{F}_{\mathcal{V}}(3)$ are relevant. On the other hand, as shown in remark (c) above, though the validity of (6.3) is checked in $\mathbf{F}_{\mathcal{V}}(3)$, the relevant terms, in this case, are the 4-ary ones. Thus we have to deal with the algebraic structure of $\mathbf{F}_{\mathcal{V}}(3)$, not simply with the set of its elements. Notice that, were we considering tolerance identities, rather than relation identities, we should deal with 5-ary terms.

(e) In spite of the considerations in (a) above, working in $\mathbf{F}_{\mathcal{V}}(3)$ is not the simplest way in order to check the validity of some *congruence* identity of the form (6.2). Since Jónsson's equations [9] are essentially two-variable equations, a variety $\mathcal{V}$ is m-distributive if and only if $\mathbf{F}_{\mathcal{V}}(2)$ generates an m-distributive variety (warning: it might happen that $\mathbf{F}_{\mathcal{V}}(2)$ is congruence distributive, but the variety it *generates* is not!) In fact, it is enough to check m-distributivity in an appropriate subalgebra of $(\mathbf{F}_{\mathcal{V}}(2))^3$ and, for finite idempotent algebras, there are even computationally more effective methods to check congruence distributivity. Cf. Freese and Valeriote [7].

Now consider, in general, a variety $\mathcal{R}$ which is obtained by taking polynomial reducts of lattices. Then $\mathbf{F}_{\mathcal{R}}(2)$ is a polynomial reduct of $\mathbf{F}_{\mathcal{L}}(2)$, where $\mathcal{L}$ is the variety of lattices. But $\mathbf{F}_{\mathcal{L}}(2) = \mathbf{F}_{\mathcal{D}}(2)$, where $\mathcal{D}$ is the variety of distributive lattices, hence the variety generated by $\mathbf{F}_{\mathcal{R}}(2)$ is a polynomial reduct of *distributive* lattices. This is one of the main arguments in the proof of [1, Theorem 2] (warning: it is not necessarily the case that $\mathcal{R}$ itself is a reduct of the variety of distributive lattices: this applies only to the subvariety of $\mathcal{R}$ generated by $\mathbf{F}_{\mathcal{R}}(2)$). Exactly as in [1], some results provable for $\mathcal{B}$ extend to *every* variety which is a congruence distributive polynomial reduct of some variety of lattices. This is the content of the next theorem, where we shall also show that there are limitations to the counterexamples which can be furnished by polynomial reducts of Boolean algebras.

If P is a set of lattice terms and $\mathbf{L}$ is a lattice, we denote by $\mathbf{L}_P$ the algebra with base set L and with, as basic operations, those induced by the terms of P. If $\mathcal{V}$ is a variety of lattices, we let $\mathcal{V}_P$ be the variety generated by all algebras $\mathbf{L}_P$, with $\mathbf{L}$ varying in $\mathcal{V}$. If $\mathcal{W} = \mathcal{V}_P$, for some P, we shall say that $\mathcal{W}$ is a *polynomial reduct* of $\mathcal{V}$.

Theorem 6.4. *(1) If the variety $\mathcal{W}$ is a congruence distributive polynomial reduct of some variety of lattices, then $\mathcal{W}$ satisfies*

$$\begin{aligned} &\Theta(R \circ_n S) \subseteq \Theta R \circ_{2n} \Theta S && \text{for } n \text{ even, and} \\ &\Theta(R \circ_n S) \subseteq (\Theta R \circ \Theta S \circ \overset{n}{\ldots} \circ \Theta R) \circ (\Theta R \circ \Theta S \circ \overset{n}{\ldots} \circ \Theta R) && \text{for } n \text{ odd.} \end{aligned}$$

(2) If the variety $\mathcal{W}$ is a congruence distributive polynomial reduct of the variety of distributive *lattices, then $\mathcal{W}$ satisfies the equations* (3.1) *and* (3.2) *in Theorem 3.1. In other words, we may allow Θ to be a reflexive and admissible relation in the identities in (1) above.*

(3) If $\mathcal{W}$ is a congruence modular polynomial reduct of the variety of Boolean algebras, then either $\mathcal{W}$ has a majority term, or $\mathcal{W}$ has a Maltsev term, or $\mathcal{W}$ interprets $\mathcal{B}$. If in addition $\mathcal{W}$ is congruence distributive, then $\mathcal{W}$ satisfies the equations (3.1) *and* (3.2) *in Theorem 3.1.*

Proof. (1) First, notice that in the proof that Baker's variety $\mathcal{B}$ satisfies the identities (2.1) and (2.2) in Theorem 2.1 we have only used the equations

$$x = b(x, x, y) = b(x, y, x) \quad \text{and} \quad b(x, y, y) = b(y, x, x) \tag{6.4}$$

and that, as we mentioned, the proof works also when β and γ are admissible relations. Using the idea from Czédli and Horváth [4], an idea we have already used above, we can replace α in (2.1) and (2.2) by a tolerance Θ. Indeed, if Θ is a tolerance and $a \mathrel{\Theta} d$, then from the equation $x = b(x, y, x)$ we get

$$\begin{aligned} b(a, c_j, d) = {} & b(b(a, c_k, a), c_j, b(d, c_k, d)) \mathrel{\Theta} \\ & b(b(a, c_k, d), c_j, b(a, c_k, d)) = b(a, c_k, d), \end{aligned}$$

for all pairs of indices j and k. Notice that we have showed a little more than requested, namely, that all the elements from the list in equation (2.5) in the proof of 2.1 are Θ-related.

Hence if a variety has a term satisfying (6.4), then the conclusion in (1) holds. We shall show that a variety satisfying the assumptions in (1) either has a majority term, or a term satisfying (6.4). The argument goes exactly as in the proof of [1, Theorem 2], as we shall see. Since the equations (6.4) depend only on two variables, then, for some given term t, they hold in $\mathcal{W}$ if and only if they hold in the free algebra $\mathbf{F}_{\mathcal{W}}(2)$ in $\mathcal{W}$ generated by two elements. Suppose that $\mathcal{W} = \mathcal{V}_P$. Since the free lattice generated by two elements is $\mathbf{C}_2 \times \mathbf{C}_2$, where $\mathbf{C}_2 = \{0, 1\}$ is the chain with two elements, then $\mathbf{F}_{\mathcal{W}}(2)$ is a (possibly improper) subalgebra of $(\mathbf{C}_2)_P \times (\mathbf{C}_2)_P$. Hence the equations (6.4) hold in $\mathcal{W}$ if and only if they hold in $(\mathbf{C}_2)_P$, if and only if they hold in the variety $\mathcal{W}'$ generated by $(\mathbf{C}_2)_P$. If $\mathcal{W}$ is congruence distributive, then $\mathcal{W}'$ is congruence distributive, too. Now the free algebra $\mathbf{F}_{\mathcal{W}'}(3)$ in $\mathcal{W}'$ generated by three elements x, y and z can be seen as a subalgebra of the free distributive lattice generated by x, y and z. Baker [1, proof of Theorem 2] shows that $\mathbf{F}_{\mathcal{W}'}(3)$ must contain either the median $xy + xz + yz$, or the Baker element $x(y + z)$ or its dual. These element are given by a 3-ary term t of $\mathcal{W}'$ and the above arguments show that in the former case t is a majority term for $\mathcal{W}$, while in the latter case t satisfies the equations (6.4). Notice that it is not necessarily the case that t is interpreted as $xy+xz+yz$ or $x(y+z)$ throughout $\mathcal{W}$, we only get that t is either a majority term or satisfies (6.4). However this is enough, by Fact 6.1 in the former case, and by the comment in the first paragraph of the proof in the latter case. Hence (1) is proved.

(2) is a particular case of the last statement of (3), however a direct proof along the lines of (1) is easy. Under the additional assumption, we can argue directly in $\mathcal{W}$, rather than in $\mathcal{W}'$, hence in the present case t can be actually interpreted as $xy+xz+yz$ or $x(y+z)$ or the dual throughout $\mathcal{W}$. In the former case Fact 6.1 is enough and in the latter case the arguments in the proof of (3.5) in Proposition 3.2 carry over.

(3) Let us prove the first statement. If $\mathcal{W}$ is congruence modular, then $\mathcal{W}$ has ternary *directed Gumm terms*, as introduced in Kazda, Kozik, McKenzie, Moore [12, p. 205]. See [12, Theorem 1.1 (3)]. We shall recall the equations that directed Gumm terms satisfy as soon as needed. Obviously, a ternary term of $\mathcal{W}$ corresponds to a ternary Boolean term t, hence it is no loss of generality to assume that $t(x, y, z) = a_1xyz + a_2xyz' + a_3xy'z + \dots$, where $'$ denotes complement and each a_1, a_2, $\dots$ is either 0 or 1. The first term d_1 in the set of directed Gumm terms satisfies the equations $d_1(x, x, y) = x = d_1(x, y, x)$. Represent d_1 by a Boolean expression as above. By the first equation, the coefficients of xyz and xyz' must be 1 and the coefficients of $x'y'z$ and $x'y'z'$ must be 0. By the second equation, the coefficients of xyz and $xy'z$ must be 1 and the coefficients of $x'yz'$ and $x'y'z'$ must be 0. Considering all the possibilities, one easily sees that d_1 is either the majority term $xy + xz + yz$,

or the Baker term, or the dual of the Baker term, or the first projection. In all but the last case we are done. If d_1 is the first projection, then the equations for directed Gumm terms give $d_2(x, x, y) = d_1(x, y, y) = x$ and $x = d_2(x, y, x)$, hence we can repeat the above argument for d_2. Going on, if we either get a majority term, or a Baker term, or its dual, we are done as above. Otherwise, all the d_j's are first projections. Then the remaining term q in the set of directed Gumm terms satisfies $q(x, x, y) = y$ and $q(x, y, y) = d_n(x, y, y) = x$, hence q is a Maltsev term for permutability.

To prove the last statement, if $\mathcal{W}$ is congruence distributive, then we have *directed Jónsson terms*, to the effect that q as above is the third projection (or, simply, discard q and ask for $d_n(x, y, y) = y$). Arguing as above, we get that some d_j satisfies all the equations satisfied in distributive lattices by either the majority term or by the Baker term, hence, again, either Fact 6.1 or the proof of (3.5) apply. □

We expect that 6.4(2) might fail if $\mathcal{W} = \mathcal{V}_P$, when $\mathcal{V}$ is not the variety of distributive lattices. In other words, we expect that (at least, without affecting the subscripts) 6.4(1) cannot be improved in such a way that admissible relations are taken into account everywhere. However, we notice that in the proof of Theorem 2.1 all the lattices we have considered are indeed distributive. Hence, in view of 6.4(2), in order to provide a counterexample that 6.4(1) cannot be improved in the above sense, one should start with a different and more complicated example, i. e., it is not enough to consider different lattice term operations on the same set B considered in the proof of 2.1.

The author considers that it is highly inappropriate, and strongly discourages, the use of indicators extracted from the list below (even in aggregate forms in combination with similar lists) in decisions about individuals (job opportunities, career progressions etc.), attributions of funds and selections or evaluations of research projects.

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