

The Gumm level equals the alvin level in congruence distributive varieties

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Abstract. Congruence modular and congruence distributive varieties are characterized by the existence of sequences of Gumm and Jónsson terms, respectively. Such sequences have variable lengths, in general.

It is immediate from the above paragraph that there is a variety with Gumm terms but without Jónsson terms. We prove the quite unexpected result that, on the other hand, if some variety has both kinds of terms, then the minimal lengths of the sequences differ at most by 1.

It follows that every r -modular congruence distributive variety is $r^2 - r + 2$ -distributive.

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An *algebra* (short for *algebraic system*) is a nonempty set endowed with a family of operations. A *variety* is a class of algebras of the same type which is definable by a set of equations. A *congruence* on some algebra is the kernel of some homomorphism, equivalently, a compatible equivalence relation. The set of congruences on some algebra has a lattice structure. An algebra is *congruence modular* if its lattice of congruences is modular and a variety is congruence modular if so are all of its members. Congruence distributivity is defined in a similar manner. Congruence modular varieties include the varieties of groups, of rings, of quasigroups, as well as all congruence distributive varieties. Congruence distributive varieties include the varieties of lattices and of Boolean algebras.

Of course, there have been interactions between the general theory of algebras and more specific kinds of algebraic systems [18]. Quite unexpectedly, interesting connections recently emerged with the theory of computational

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complexity, in particular, the algebraic approach to the Constraint Satisfaction Problem. In a nutshell, well-behaved classes of algebras first studied only for their algebraic properties—and the identities such classes satisfy—turned out to correspond to algorithmic results for CSP over constraint languages. See, e. g., [1, 9] for a survey. Congruence distributivity and congruence modularity are among the first studied and most important conditions providing ‘good algebraic structure’. We show that two well-known and intensively studied characterizations of the above conditions are equivalent even as far as length is concerned, provided they are both applicable.

In more detail, a *term* is, roughly, a word obtained by composition from the basic operations of a variety. Both congruence distributivity and congruence modularity are characterized by the existence of finite sequences of terms satisfying appropriate equations. In general, the lengths of such sequences are variable. Characterizations of this kind are called Maltsev conditions. See Theorems 1, 3 and Definition 2 below for specific examples. Many problems are still open about the relationship among the relative lengths of different sequences characterizing different conditions. Such problems date back at least to [4, p. 173]. See, e. g., [3, 6, 11, 12, 14, 15, 16, 17, 21] for other problems, comments and results. The reader can find further references in the quoted works. See also Remark 7 below.

Gumm terms and alvin terms, defined below, characterize congruence modularity and congruence distributivity, respectively. In particular, there is a variety with a sequence of Gumm terms but without a sequence of alvin terms. In this note we prove the quite unexpected result that, within a variety and *as soon as we have both kinds of terms*, the minimal lengths of the sequences are identical. While it is well established that Gumm terms are relevant for the study of congruence modular varieties [5, 8, 13, 14, 21], it is quite surprising to discover that they have a deep direct influence on congruence distributive varieties. Of course, a congruence distributive variety is also congruence modular, but it should be expected that in a congruence distributive variety “all the work” is done by an alvin sequence. On the contrary, we show that the weaker notion of a Gumm sequence has applications, if the sequence is sufficiently short.

Congruence modularity can be characterized by a different set of terms formerly introduced by A. Day. The problem of the relationship between the minimal lengths of Day and Gumm sequences in a congruence modular variety is not completely solved yet. The results presented in this note stress the importance of the above problem. Anyway, we get the remarkable corollary that an r -modular congruence distributive variety is $r^2 - r + 2$ -distributive, though we do not know how far this bound can be improved. In passing, we get new proofs, usually simpler, usually providing better bounds, of results from [3, 15, 16, 21].

Let us now recall the main definitions together with some needed classical results. See, e. g., [5, 8, 11, 17, 19] for further undefined notions and full formal details.

Theorem 1. [10, 19] *A variety \mathcal{V} is congruence distributive if and only if there is some natural number n for which one of the following equivalent conditions holds.*

- (1) \mathcal{V} has a sequence t_0, \dots, t_n of alvin terms, that is, terms such that the following equations are satisfied in all algebras in \mathcal{V} .

$$x = t_h(x, y, x), \quad \text{for } 0 < h < n, \quad (\text{A1})$$

$$x = t_0(x, y, z), \quad (\text{A2})$$

$$t_h(x, z, z) = t_{h+1}(x, z, z), \quad \text{for } h \text{ even, } 0 \leq h < n, \quad (\text{A3})$$

$$t_h(x, x, z) = t_{h+1}(x, x, z), \quad \text{for } h \text{ odd, } 0 \leq h < n, \quad (\text{A4})$$

$$t_n(x, y, z) = z. \quad (\text{A5})$$

- (2) *The following inclusion holds in every algebra $\mathbf{A} \in \mathcal{V}$ for all congruences α, β and γ of \mathbf{A} .*

$$\alpha(\beta \circ \gamma) \subseteq \alpha\gamma \circ \alpha\beta \circ .^n. \quad (\text{CD})$$

The notation we use in congruence identities like (CD) above goes as follows. Juxtaposition denotes intersection, in particular, meet of congruences. Join in congruence lattices is denoted by $+$. Composition of binary relations is denoted by \circ and $R \circ S \circ .^n$ denotes $R \circ S \circ R \circ S \dots$ with n factors, that is, $n - 1$ occurrences of \circ . In the above notation factors of the form $\alpha\beta$ are always counted as *one* factor. It is formally convenient to allow the extreme cases $n = 0$ and $n = 1$. We set $R \circ S \circ .^1 = R$ and $R \circ S \circ .^0$ to be the minimal congruence of the algebra under consideration. We also let $R^m = R \circ R \circ .^m$. Exponentiation ties more than any other operator; juxtaposition comes next in tying force.

The equivalence of (1) and (2) in Theorem 1 is implicit in [10] and is an immediate application of the algorithm described in [20, 22]. See also [14, 15, 21] for further comments and related results.

The classical characterization [10] of congruence distributive varieties involves *Jónsson terms*, which are defined as in clause (1) above with even and odd exchanged in (A3) - (A4). Of course, the definitions are completely equivalent if we are not concerned with the exact value of n . However, for a fixed even n , the alvin and the Jónsson conditions are not equivalent [6], though it is obvious that the minimal lengths of the sequences differ at most by one. As mentioned, we shall show that the minimal lengths of an alvin and a Gumm sequence coincide in a congruence distributive variety. Since the minimal length of a Jónsson sequence might differ by 1, we get an exact correspondence only if we deal with the alvin condition.

Definition 2. A sequence of *Gumm terms* for a variety is a sequence t_0, \dots, t_n , for some n , of terms satisfying the equations (A2) - (A5) in Clause (1) in Theorem 1 above, as well as

$$x = t_h(x, y, x), \quad \text{for } 1 < h < n. \quad (\text{G1})$$

In other words, Gumm terms satisfy the equations (A1) - (A5), except possibly for the equation $x = t_1(x, y, x)$.

Notice that $x = t_1(x, x, x)$ still holds in the case of Gumm terms, by (A2) and (A3).

Theorem 3. [7, 8] *A variety \mathcal{V} is congruence modular if and only if \mathcal{V} has a sequence of Gumm terms.*

A variety is *n-alvin* (*n-Gumm*, *n-distributive*) if it has a sequence t_0, \dots, t_n of alvin (Gumm, Jónsson) terms. The *alvin (Gumm) level* of a congruence distributive (modular) variety \mathcal{V} is the minimal n such that \mathcal{V} is *n-alvin* (*n-Gumm*). Our main result about the above levels is the following theorem. The proof shall be given after some auxiliary results of independent interest.

Theorem 4. *If \mathcal{V} is a congruence distributive variety, then the alvin level of \mathcal{V} is equal to the Gumm level of \mathcal{V} .*

Notice that the sequence t_0, \dots, t_n actually contains $n + 1$ terms; moreover, the two “outer” terms t_0 and t_n are projections, hence the number of nontrivial terms is $n - 1$. Thus any definition of the levels has a somewhat conventional nature. Here we follow by analogy the classical and universally adopted convention concerning *n*(-Jónsson)-distributivity.

If either $n = 0$ or $n = 1$ in Theorem 1 or in Definition 2, then we get a condition which is satisfied only by trivial varieties with just one-element algebras. Indeed, say, for $n = 1$ we get $x = t_0(x, y, y) = t_1(x, y, y) = y$ from (A2), (A3) and (A5). However, it is formally convenient to consider the above trivial cases, too, otherwise the alvin and Gumm levels would be undefined in the case of trivial varieties. Of course, the levels trivially coincide (and both equal 0) in such trivial cases.

According to Definition 2, Gumm terms can be seen as a “defective” variant of alvin terms. While this idea is sometimes useful, Gumm terms have a more important and fruitful interpretation as terms which “compose” the classical Maltsev conditions for congruence permutability and distributivity. See [7, 8, 16, 21] for a more detailed discussion. In some respects, the main point of the present note is to stretch the “permutable” side of the term t_1 to the extreme limit. Notice also that the present definition is slightly different from the original definition by H.-P. Gumm from [7, 8]. The present definition allows for a finer counting of the number of terms. See [15, p. 12]. To the best of our knowledge, the present definition first appeared in [14, 21]. Notice that the indexing of terms, whatever the definition, is different from the present one in most of the quoted papers, including works by the present author. The indexing here is intended to stress the similarity between Gumm and alvin terms.

We now present our main new applications of Gumm terms. The next theorem works for any congruence modular variety. A *tolerance* on some algebra \mathbf{A} is a reflexive and symmetric binary compatible relation on A . In other words, a tolerance is like a congruence, except that transitivity is not required. Recall the notational conventions established shortly after the statement of Theorem 1. For convenience, we shall frequently write $a R b$ in

place of $(a, b) \in R$ and we shall also concatenate the above notation, e. g., $a R b S c$ means both $(a, b) \in R$ and $(b, c) \in S$.

Theorem 5. *Suppose that $n \geq 1$ and \mathcal{V} is a variety with a sequence t_0, \dots, t_n of Gumm (alvin) terms. Then, for every $m \geq 1$, \mathcal{V} has a sequence s_0, \dots, s_n of Gumm (alvin) terms such that s_1 satisfies the following additional property.*

(T_m) *For every $\mathbf{A} \in \mathcal{V}$ and every tolerance Θ on \mathbf{A} , if $a, c \in A$ and $a \Theta^m c$, then $a \Theta s_1(a, a, c)$.*

Proof. The property (T_1) is trivially satisfied by the term t_1 of the original sequence, since if $a \Theta c$, then $a = t_1(a, a, a) \Theta t_1(a, a, c)$, by equations (A2) - (A3) and since Θ is reflexive and compatible.

We shall prove that if $m \geq 1$ and \mathcal{V} has a sequence s_0, \dots, s_n of Gumm (alvin) terms such that s_1 satisfies (T_m), then \mathcal{V} has a sequence s_0^*, \dots, s_n^* of Gumm (alvin) terms such that s_1^* satisfies (T_{m+1}). The theorem follows by induction on m . Define

$$s_h^*(x, y, z) = s_h(x, s_h(x, y, y), s_h(x, y, z)), \quad \text{for } h = 0, \dots, n. \quad (*)$$

If s_0, \dots, s_n is a sequence of Gumm terms, then

$$s_h^*(x, y, x) = s_h(x, s_h(x, y, y), s_h(x, y, x)) = s_h(x, s_h(x, y, y), x) = x \quad (h > 1),$$

$$s_0^*(x, y, z) = s_0(x, s_0(x, y, y), s_0(x, y, z)) = x,$$

$$\begin{aligned} s_h^*(x, z, z) &= s_h(x, s_h(x, z, z), s_h(x, z, z)) = \\ &= s_{h+1}(x, s_{h+1}(x, z, z), s_{h+1}(x, z, z)) = s_{h+1}^*(x, z, z) \quad (h \text{ even}), \end{aligned}$$

$$\begin{aligned} s_h^*(x, x, z) &= s_h(x, s_h(x, x, x), s_h(x, x, z)) = s_h(x, x, s_{h+1}(x, x, z)) = \\ &= s_{h+1}(x, x, s_{h+1}(x, x, z)) = s_{h+1}^*(x, x, z) \quad (h \text{ odd}), \end{aligned}$$

$$s_n^*(x, y, z) = s_n(x, s_n(x, y, y), s_n(x, y, z)) = s_n(x, y, z) = z,$$

thus s_0^*, \dots, s_n^* is a sequence of Gumm terms, as well. The case of alvin terms is identical, just let h be arbitrary in the first displayed line.

Suppose now that \mathbf{A} belongs to \mathcal{V} , Θ is a tolerance on \mathbf{A} , $a, c \in A$ and $a \Theta^{m+1} c$. Thus there is $b \in A$ such that $a \Theta^m b \Theta c$. Then

$$a = s_1(a, \underline{s_1(a, a, b)}, s_1(a, a, \underline{b})) \Theta s_1(a, \underline{a}, s_1(a, a, \underline{c})) = s_1^*(a, a, c)$$

by (A2) - (A3), where we have underlined the Θ -related elements. We have used the assumptions that Θ is reflexive and compatible and that s_1 satisfies (T_m), thus $s_1(a, a, c) \Theta a$, since Θ is symmetric. \square

Compare (*) with the definition shortly before [8, Observation 10.1 on p. 64]. Compare also [12, Section 3]. The position (*) sends (directed, Pixley [12]) Jónsson terms to (directed, Pixley) Jónsson terms, as well. In particular, Theorem 5 applies to *Pixley terms* in the sense of [12]. Theorem 5 applies also to *directed Gumm terms* [12], provided either we define directed Gumm terms in a specular way, or else we consider the variant of (T_m) whose conclusion asks for $s_{n-1}(a, c, c) \Theta c$.

The complexity of the terms constructed in the proof of Theorem 5 is not optimal. Moreover, using some additional arguments, it is possible to

prove the version of Theorem 5 in which the tolerance Θ is replaced by a reflexive and compatible relation. See the appendix. We shall not need the above generalizations here, hence we have favored ease over generality.

Only part (1) of the following corollary shall be used in order to prove Theorems 4 and 8.

Corollary 6. *Suppose that $\ell \geq 1$, $n \geq 2$ and \mathbf{A} is an algebra belonging to a variety with Gumm terms t_0, \dots, t_n . If α, β, γ are congruences and Θ, Ψ are tolerances on \mathbf{A} , then the following inclusions hold.*

- (1) $(\beta \circ \gamma)(\alpha\beta + \alpha\gamma) \subseteq \alpha\gamma \circ \alpha\beta \circ .^n$. (here possibly $n = 0$ or $n = 1$),
- (2) $\alpha(\beta \circ \gamma \circ .^\ell) \subseteq \alpha(\beta \circ \gamma)(\gamma \circ \beta) \circ (\alpha\beta \circ \alpha\gamma \circ .^k)$, for $k = (n-2)(\ell-1) + 1$,
- (3) $(\beta \circ \gamma \circ .^\ell)(\alpha\beta + \alpha\gamma) \subseteq \alpha\gamma \circ \alpha\beta \circ .^k$, for $k = (n-2)(\ell-1) + 2$,
- (4) $\Psi\Theta^\ell \subseteq (\Psi\Theta)^{\ell(n-2)+1}$.

Proof. (1) As we mentioned before, if $n \leq 1$, then we are in a trivial variety, hence the conclusion holds. So let us suppose $n \geq 2$.

Let Θ be the tolerance $(\alpha\beta \circ \alpha\gamma)(\alpha\gamma \circ \alpha\beta)$ so that Θ contains both $\alpha\beta$ and $\alpha\gamma$. Suppose that $(a, c) \in (\beta \circ \gamma)(\alpha\beta + \alpha\gamma)$. Using the classical characterization of the join of two congruences, from $(a, c) \in \alpha\beta + \alpha\gamma$ we get that there is some m (depending on a and c , in general) such that $(a, c) \in \alpha\beta \circ \alpha\gamma \circ .^m \subseteq \Theta^m$. By Theorem 5, we have Gumm terms s_0, \dots, s_n such that s_1 satisfies (T_m) , hence $(a, s_1(a, a, c)) \in \Theta \subseteq \alpha\gamma \circ \alpha\beta$, by (A4) and (A5). If $n = 2$, then $s_1(a, a, c) = c$ and we are done. If $n > 2$, it follows from a classical argument in [10] (or see the proof of (2) below) that $(s_2(a, a, c), c) \in \alpha\beta \circ \alpha\gamma \circ .^{n-1}$, since $(a, c) \in \alpha(\beta \circ \gamma)$. By (A4) we get $s_1(a, a, c) = s_2(a, a, c)$, hence $(a, c) \in (\alpha\gamma \circ \alpha\beta) \circ (\alpha\beta \circ \alpha\gamma \circ .^{n-1}) = \alpha\gamma \circ \alpha\beta \circ .^n$, since composition of relations is associative and $\alpha\beta$ is a congruence, hence transitive.

(2) Let Ψ be the tolerance $(\beta \circ \gamma)(\gamma \circ \beta)$, thus Ψ contains both β and γ . Suppose that $(a, c) \in \alpha(\beta \circ \gamma \circ .^\ell)$, hence $(a, c) \in \Psi^\ell$, thus $(a, s_1(a, a, c)) \in \Psi$, for the appropriate term s_1 given by Theorem 5. We have $a = s_1(a, a, a) \alpha s_1(a, a, c)$, so that $(a, s_1(a, a, c)) \in \alpha(\beta \circ \gamma)(\gamma \circ \beta)$. If $n = 2$, we are done, actually, we have $k = 0$ and we can save one factor.

Classical arguments show that $(s_2(a, a, c), c) \in \alpha\beta \circ \alpha\gamma \circ .^k$, for $k = (n-2)(\ell-1) + 1$. Say, if ℓ is even and $a \beta b_1 \gamma b_2 \beta b_3 \gamma \dots \beta b_{\ell-1} \gamma c$, then

$$s_2(a, a, c) \beta s_2(a, b_1, c) \gamma s_2(a, b_2, c) \beta s_2(a, b_3, c) \gamma \dots \\ \dots \beta s_2(a, b_{\ell-1}, c) \gamma s_2(a, c, c) = s_3(a, c, c) \gamma s_3(a, b_{\ell-1}, c) \beta \dots$$

Notice that then $s_2(a, b_{\ell-1}, c) \gamma s_3(a, b_{\ell-1}, c)$, since γ is transitive.

All the elements in the above chain are α -connected by (G1), e. g., $s_2(a, a, c) \alpha s_2(a, a, a) = a = s_2(a, b_1, a) \alpha s_2(a, b_1, c)$, etc. Notice that we do not need the identity $x = s_1(x, y, x)$ in order to show $a = s_1(a, a, a) \alpha s_1(a, a, c) = s_2(a, a, c)$, the equations (A2) - (A4) are enough. The conclusion of (2) follows from $s_1(a, a, c) = s_2(a, a, c)$, as in (1).

The proof of (3) merges the arguments in (1) and (2). As in the proof of (1) and for the same definition of Θ , we get $(a, s_1(a, a, c)) \in \Theta \subseteq \alpha\gamma \circ \alpha\beta$, for the appropriate s_1 given by Theorem 5. The rest goes as in the last part of

the proof of (2). Finally, notice that here, as in (1), two adjacent occurrences of $\alpha\beta$ join into one, by transitivity.

(4) As in the above arguments, if $(a, c) \in \Psi\Theta^\ell$, we have $a \Theta s_1(a, a, c)$, for some appropriate s_1 , and $a = s_1(a, a, a) \Psi s_1(a, a, c)$, thus $a \Psi\Theta s_1(a, a, c)$. If Ψ is a congruence, the arguments in (2) give $(s_2(a, a, c), c) \in (\Psi\Theta)^{\ell(n-2)}$ (here we are not allowed to use transitivity in order to get a better value). The conclusion follows again from $s_1(a, a, c) = s_2(a, a, c)$. In order to deal with the case when Ψ is a tolerance, it is enough to use an additional argument from [2]. E. g.,

$$s_2(a, a, c) = s_2(s_2(a, a, \underline{a}), b_1, s_2(\underline{a}, a, c)) \Psi s_2(s_2(a, a, \underline{a}), b_1, s_2(\underline{a}, a, c)) = s_2(a, b_1, c),$$

by (G1). See [2] or the proof of [15, Proposition 3.1] for full details. \square

Letting ℓ vary in (2) above and using [7], we get another proof of the result from [21] that a variety \mathcal{V} is congruence modular if and only if the congruence identity $\alpha(\beta + \gamma) = \alpha(\beta \circ \gamma) \circ (\alpha\beta + \alpha\gamma)$ holds in \mathcal{V} . Clause (2) generally gives the best known bound for $\alpha(\beta \circ \gamma \circ \cdot^\ell)$, to date. For example, it follows from [15, Corollary 2.2] that an $m + 1$ -distributive variety satisfies $\alpha(\beta \circ \gamma \circ \cdot^{\ell+1}) \subseteq \alpha\beta \circ \alpha\gamma \circ \cdot^{m\ell+1}$. In case m is even (thus $m+1$ is odd) the property of being $m + 1$ -distributive is equivalent to $m + 1$ -alvin, hence Corollary 6(3) provides the improved inclusion $\alpha(\beta \circ \gamma \circ \cdot^{\ell+1}) \subseteq \alpha\gamma \circ \alpha\beta \circ \cdot^{(m+1)\ell+2}$

Moreover, using (4), we get bounds for expressions of the form

$$(\beta_{11} \circ \beta_{12} \circ \beta_{13} \dots)(\beta_{21} \circ \beta_{22} \circ \beta_{23} \dots)(\beta_{31} \circ \beta_{32} \circ \beta_{33} \dots) \dots,$$

thus getting a proof for [21, Condition C] on p. 281]. See the appendix. In this respect, compare also [3, Theorem 5]. The examples in [17] show that in some cases the bounds given by Corollary 6 are optimal or close to be optimal.

Applying Clause (4) above twice, we get a slightly different proof, in comparison with [2], that congruence modular varieties satisfy the *Tolerance Intersection Property* (TIP) $\Psi^*\Theta^* = (\Psi\Theta)^*$, where $*$ denotes transitive closure. Again, Clause (4) seems to give the best known bound for $\Psi\Theta^\ell$. Notice that TIP has many important applications to congruence modular varieties. See, e. g., [3] for some examples and history.

Proof of Theorem 4. If \mathcal{V} is congruence distributive, then \mathcal{V} has indeed an alvin level $a(\mathcal{V})$, by Theorem 1. Since every congruence distributive variety is congruence modular, then \mathcal{V} has also a Gumm level $g(\mathcal{V})$, by Theorem 3.

A sequence of alvin terms is obviously also a sequence of Gumm terms, hence $g(\mathcal{V}) \leq a(\mathcal{V})$. On the other hand, suppose that $g(\mathcal{V}) = n$. We will show that the inclusion (CD) holds, hence $a(\mathcal{V}) \leq n$, by the equivalence of (1) and (2) in Theorem 1. Let $(a, c) \in \alpha(\beta \circ \gamma)$. Since $\alpha(\beta \circ \gamma) \subseteq \alpha(\beta + \gamma)$, then, by congruence distributivity, $(a, c) \in \alpha\beta + \alpha\gamma$. By Corollary 6(1), $(a, c) \in \alpha\beta \circ \alpha\gamma \circ \cdot^n$. hence clause (2) in Theorem 1 holds. \square

Theorem 4 can be proved by constructing a sequence of alvin terms u_1, \dots, u_n starting from a sequence of Gumm terms t_1, \dots, t_n and some other

sequence of alvin terms of length not prescribed in advance. The procedure involves some deep nesting, naturally leads to the introduction of terms of large a -arity and might find further applications. However, the simplest way to prove Theorem 4 seems the way we have presented here, using condition (2) in Theorem 1.

A nontrivial 2-Gumm term is a *Maltsev term*, and it characterizes congruence permutable varieties. A nontrivial 2-alvin term is a *Pixley term*. Thus Theorem 4 is a generalization of the classical result that a congruence distributive variety \mathcal{V} has a Pixley term if and only if \mathcal{V} is congruence permutable.

Remark 7. An earlier characterization [4] of congruence modularity involves a sequence u_0, \dots, u_r of quaternary *Day terms*. A variety with such a sequence is said to be *r-modular*. We shall not need the explicit definition of Day terms here. See [4, 5, 11, 16, 17, 21] for full details. From [4] and [7] we get that a variety \mathcal{V} has a sequence of Day terms if and only if \mathcal{V} has a sequence of Gumm terms. However, the relationship about the lengths of the above sequences is far from being clear [14].

Let the *Day-to-Gumm function* DG be defined by setting $DG(r)$ to be the smallest n such that every variety with a sequence u_0, \dots, u_r of Day terms has a sequence t_0, \dots, t_n of Gumm terms. Let DG_{dist} be defined in the same way, but restricting ourselves to congruence distributive varieties. In principle, it is possible that DG and DG_{dist} are different functions. The *Gumm-to-Day functions* GD and GD_{dist} are defined symmetrically.

In [8, 14] it is shown that an argument from [4] carries over with minimal modifications in order to show that $GD(n) \leq 2n - 2$, for $n \geq 2$. It is proved in [17] that this bound is optimal for n even, also when we restrict ourselves to congruence distributive varieties. Namely, we have $GD(n) = GD_{\text{dist}}(n) = 2n - 2$, for n even, $n \geq 2$. Less is known about $DG(r)$. The arguments in [14] show that $DG(r) \leq r^2 - r + 1$ and in some special cases the value can be slightly improved. However, the general problem of evaluating $DG(r)$ seems completely open. Provisional results about this and related problems appear in [16]. Notice that, as we mentioned, some results from [16] are improved here, so that [16] should be updated.

Except for the evaluation of $DG(r)$, the relationships among the Jónsson, alvin, Day and Gumm levels have been almost completely settled in [6, 17] and the present work. In any case, using the above definitions and a result from [14], we get the following corollary, showing that the Day level of a congruence distributive variety \mathcal{V} affects the distributivity levels of \mathcal{V} .

Theorem 8. *If \mathcal{V} is an r -modular congruence distributive variety, then \mathcal{V} is $r^2 - r + 2$ -distributive. More generally, if $n = DG_{\text{dist}}(r)$, then \mathcal{V} is n -alvin and $n + 1$ -distributive.*

Proof. By definition, an r -modular congruence distributive variety \mathcal{V} is n -Gumm, for $n = DG_{\text{dist}}(r)$. By Theorem 4, \mathcal{V} is n -alvin, hence $n+1$ -distributive. We have proved the second statement. The first statement is then immediate from the mentioned result from [14]. \square

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Appendix

In this appendix we present the proofs of some results only stated in the main body of the manuscript.

Remark 9. As already mentioned, the terms constructed in the proof of Theorem 5 have not minimal complexity. For example, given any sequence t_0, \dots, t_n of Gumm terms, the term t_1 satisfies (T_2) . Indeed, if $a \Theta b \Theta c$, then $a = t_1(a, b, b) \Theta t_1(a, a, c)$.

We now present a strengthening of Theorem 5. Not only we can replace the tolerance Θ in Theorem 5 by a reflexive and compatible relation R . We are allowed to replace *each* occurrence of Θ by *either* R or R^\smile , where R^\smile denotes the converse of R .

Theorem 10. *Suppose that $n \geq 1$ and \mathcal{V} is a variety with a sequence t_0, \dots, t_n of Gumm (alvin) terms. Then, for every $m \geq 1$, \mathcal{V} has a sequence s_0, \dots, s_n of Gumm (alvin) terms such that s_1 satisfies the following additional property.*

(A_m) For every $\mathbf{A} \in \mathcal{V}$, every reflexive compatible relation R on \mathbf{A} and binary relations X_1, \dots, X_m such that, for every $j = 1, \dots, m$, either $X_j = R$ or $X_j = R^\smile$, the following holds.

If $a, c \in A$ and $(a, c) \in X_1 \circ X_2 \circ \dots \circ X_m$, then $a R s_1(a, a, c)$.

In particular, if (A_m) holds, then, by taking $X_j = R$ for all j 's, we have that $a R^m c$ implies $a R s_1(a, a, c)$. By taking $X_j = R^\smile$ for all j 's, we have that $a (R^\smile)^m c$ implies $a R s_1(a, a, c)$. In general, (A_m) asserts that

$a R s_1(a, a, c)$ follows from the assumption that a and c are related by a chain of factors of length m , when each factor is either R or R^\sim .

Notice also that the term s_1 depends only on m , it does not depend on R , nor on the choice of the X_j 's. In particular, using the fact that $R^{\sim\sim} = R$ and considering the converse of R in place of R in (A_m) we also get $a R^\sim s_1(a, a, c)$, that is, $s_1(a, a, c) R a$.

Proof. By induction on m . If $m = 1$, then the original term t_1 works, since if $a R c$, then $a = t_1(a, a, a) R t_1(a, a, c)$. This is the argument already presented in the proof of Theorem 5. If $a R^\sim c$, that is, $c R a$, then $a = t_1(a, c, c) R t_1(a, a, c)$.

In order to carry over the induction we shall need two claims. Let (A_m^-) be defined like (A_m) , but restricted to the cases when the last factor X_m is R . We first claim that if s_1 satisfies (A_m) , then s_1^* satisfies (A_{m+1}^-) , where s_1^* is defined in (*) in the proof of Theorem 5. Again, this is essentially the same argument. Choose X_1, \dots, X_m in some arbitrary way, let $X_{m+1} = R$ and set $X = X_1 \circ X_2 \circ \dots \circ X_m$. If $(a, c) \in X_1 \circ \dots \circ X_{m+1}$, then $a X b R c$, for some b . Then

$$a = s_1(a, \underline{s_1(a, a, b)}, s_1(a, a, \underline{b})) R s_1(a, \underline{a}, s_1(a, a, \underline{c})) = s_1^*(a, a, c).$$

We have $s_1(a, a, b) R a$ by (A_m) and the remark shortly after the statement of the present theorem.

Our second claim is that if (A_m^-) is satisfied by some term r_1 , then (A_m) is satisfied by r_1^* . Indeed, choose X_1, \dots, X_m in an arbitrary way and, as above, set $X = X_1 \circ X_2 \circ \dots \circ X_m$. Suppose that $a X c$. If $X_m = R$, then $a R r_1(a, a, c)$ by (A_m^-) , hence

$$a = r_1(a, a, \underline{a}) R r_1(a, a, \underline{r_1(a, a, c)}) = r_1^*(a, a, c).$$

On the other hand, if $X_m = R^\sim$, then we get $a R^\sim r_1(a, a, c)$ by applying (A_m^-) with R^\sim in place of R . Equivalently, $r_1(a, a, c) R a$. Then

$$a = r_1(a, \underline{r_1(a, a, c)}, r_1(a, a, c)) R r_1(a, \underline{a}, r_1(a, a, c)) = r_1^*(a, a, c)$$

and our second claim follows.

Now suppose that s_0, \dots, s_n is a sequence of Gumm terms and s_1 satisfies (A_m) . By the arguments in the proof of Theorem 5, both s_0^*, \dots, s_n^* and $s_0^{**}, \dots, s_n^{**}$ are sequences of Gumm terms. By our first claim, s_1^* satisfies (A_{m+1}^-) . By our second claim with $m+1$ in place of m and s_1^* in place of r_1 , we get that s_1^{**} satisfies (A_{m+1}) , thus the induction is complete. \square

A less general version of the following corollary has been stated as an open problem in [21, Condition C) on p. 281]. On the other hand, a more general result is proved in [3, Theorem 5]. However, the present proof can be used to provide an explicit bound for $\prod_{i \leq h} (\beta_{i1} \circ \beta_{i2} \circ \dots \circ \beta_{ig})$, with possible repetitions among the β_{ij} .

Corollary 11. *For $(\beta_{ij})_{i \leq h, j \leq g}$ congruences in an algebra belonging to a congruence modular variety, we have*

$$\prod_{i \leq h} (\beta_{i1} + \beta_{i2} + \cdots + \beta_{ig}) = \prod_{i \leq h} (\beta_{i1} \circ \beta_{i2} \circ \cdots \circ \beta_{ig}) \circ \sum_{f: \{1, \dots, h\} \rightarrow \{1, \dots, g\}} \beta_{1f(1)} \beta_{2f(2)} \cdots \beta_{hf(i)}$$

where \prod denotes intersection.

Proof. Let $\Theta_i = \overline{\beta_{i1} \cup \beta_{i2} \cup \cdots \cup \beta_{ig}}$, where \overline{R} denotes the smallest compatible relation containing R . In particular, $\Theta_i^* = \beta_{i1} + \beta_{i2} + \cdots + \beta_{ig}$. By iterating Corollary 6(4), we get $\prod_{i \leq h} (\beta_{i1} + \beta_{i2} + \cdots + \beta_{ig}) = \Theta_1^* \Theta_2^* \cdots \Theta_h^* = (\Theta_1 \Theta_2 \cdots \Theta_h)^*$. Letting $\Psi = \Theta_1 \Theta_2 \cdots \Theta_h$, we have $\Psi \subseteq \prod_{i \leq h} (\beta_{i1} \circ \beta_{i2} \circ \cdots \circ \beta_{ig})$. If (a, c) belongs to the left-hand side of the displayed formula, then we get $a \Psi s_1(a, a, c)$ by Theorem 5. Moreover

$$(s_2(a, a, c), c) \in \sum_{f: \{1, \dots, h\} \rightarrow \{1, \dots, g\}} \beta_{1f(1)} \beta_{2f(2)} \cdots \beta_{hf(i)},$$

arguing as in the proof of Corollary 6(3). \square

We now prove a more general version of Theorem 4. For simplicity, we state the version dealing with congruence identities. We also get a ‘‘bilateral’’ version.

Theorem 12. *Suppose that $r \geq 1$ and, for each i with $1 < i \leq r$, either $A_i = \alpha(\gamma \circ \beta)$, or $A_i = \alpha\gamma \circ \alpha\beta$. If \mathcal{V} is a congruence distributive variety and \mathcal{V} satisfies*

$$\alpha(\beta \circ \gamma) \subseteq \alpha(\gamma \circ \beta) \circ A_2 \circ A_3 \circ \cdots \circ A_{r-1} \circ A_r, \quad (\text{S})$$

then \mathcal{V} satisfies

$$\alpha(\beta \circ \gamma) \subseteq \alpha\gamma \circ \alpha\beta \circ A_2 \circ A_3 \circ \cdots \circ A_{r-1} \circ A_r. \quad (\text{S1})$$

If in addition $r \geq 2$ and $A_r = \alpha(\gamma \circ \beta)$, then \mathcal{V} satisfies

$$\alpha(\beta \circ \gamma) \subseteq \alpha\gamma \circ \alpha\beta \circ A_2 \circ A_3 \circ \cdots \circ A_{r-1} \circ \alpha\gamma \circ \alpha\beta. \quad (\text{S}^+)$$

Proof. Assuming (S), the standard arguments used to prove the equivalence of (1) and (2) in Theorem 1 show that we have terms t_1, \dots, t_n ($n = 2r$) satisfying (A2) - (A5), as well as $x = t_h(x, y, x)$, for a set of indices depending on the actual forms of the A_i 's. The arguments in the proofs of Theorem 5 show that, for every m , we can have t_1 satisfying (T_m) . Then the proof of Corollary 6 shows that (S1) holds.

If $A_r = \alpha(\gamma \circ \beta)$, then, taking converses, (S1) is equivalent to $\alpha(\gamma \circ \beta) \subseteq \alpha(\beta \circ \gamma) \circ A_{r-1} \circ \cdots \circ A_3 \circ A_2 \circ \alpha\beta \circ \alpha\gamma$. By applying the first statement with β and γ exchanged, we get $\alpha(\gamma \circ \beta) \subseteq \alpha\beta \circ \alpha\gamma \circ A_{r-1} \circ \cdots \circ A_3 \circ A_2 \circ \alpha\beta \circ \alpha\gamma$, hence (S⁺) follows by taking converses again. \square

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