Classification of Subsystems for Local Nets with Trivial Superselection Structure

Dedicated to S. Doplicher and J. E. Roberts on the occasion of their 60th birthday

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Abstract

Let $\mathcal{F}$ be a local net of von Neumann algebras in four spacetime dimensions satisfying certain natural structural assumptions. We prove that if $\mathcal{F}$ has trivial superselection structure then every covariant, Haag-dual subsystem $\mathcal{B}$ is of the form $\mathcal{F}_1^G \otimes I$ for a suitable decomposition $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ and a compact group action. Then we discuss some application of our result, including free field models and certain theories with at most countably many sectors.

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1 Introduction

In the algebraic approach to QFT [31] the main objects under investigation are (isotonous) nets of von Neumann algebras over bounded regions in the Minkowski spacetime, satisfying pertinent additional requirements. Any such correspondence is usually denoted by $\mathcal{O} \to \mathcal{F}(\mathcal{O})$.

Internal symmetries of a net $\mathcal{F}$ can be defined as those automorphisms of the C^*-inductive limit $(\cup_{\mathcal{O}\in\mathcal{K}}\mathcal{F}(\mathcal{O}))^{-\|\cdot\|}$ (the *quasi-local* C^*-algebra; it is customary to denote it in the same way as the net), that leave every element $\mathcal{F}(\mathcal{O})$ globally invariant; unbroken internal symmetries leave the vacuum state invariant.

Given a certain (compact) group G of (unbroken) internal symmetries of $\mathcal{F}$, the fixpoint net $\mathcal{F}^G$ defined by $\mathcal{F}^G(\mathcal{O}) = \mathcal{F}(\mathcal{O})^G$ is an example of *subsystem* (sometimes also called *subnet* or *subtheory* in the literature), i.e. a net of (von Neumann) subalgebras of $\mathcal{F}$. This is the typical situation allowing one to recover an observable net from a field net via a principle of gauge invariance. However, in certain situations one can easily produce examples of subsystems that can hardly be seen to arise in this way. See e.g. the discussion in [46, 1, 12].

In this work we address the problem of classifying subsystems of a given net $\mathcal{F}$. Some related work has been already done in [37, 38, 18, 15, 11, 9]. Our main result states that if $\mathcal{F}$ satisfies certain structural properties then all the reasonably well-behaved subsystems morally arise in the way explained above, namely they are fixpoints for a compact group action on $\mathcal{F}$ or on one component $\mathcal{F}_1$ in a tensor product decomposition $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$.

We confine our discussion to nets $\mathcal{F}$ satisfying usual postulates such as Poincaré covariance, Bisognano-Wichmann and the split property, plus an additional condition, the absence of nontrivial sectors, whose meaning has been recently clarified in [15]. Our assumptions are sufficiently general to cover many interesting situations, including the well-known Bosonic free field models (massive or massless). In particular in the case of (finitely many) multiplets of the massive scalar free fields we (re)obtain a classification result of Davidson [18], but with a different method of proof. Moreover our discussion applies to the massless case as well. In a different direction, we also provide a first solution to a long-standing open problem, proposed by S. Doplicher, concerning the relationship between an observable net $\mathcal{A}$ and the subsystem $\mathcal{C}$ generated by the local energy-momentum tensor [22, 12]. As to the main ingredients, now $\mathcal{A}$ is required to have the split property and at most countably many superselection sectors, all with finite statistical dimension [1] (and Bosonic).

Still our assumptions are restrictive enough to rule out the occurrence of models with undesiderable features. This allows us to overcome certain technical difficulties that cannot be handled in too general (perhaps pathological) situations.

This paper is organized in the following way. In the next section we describe our setup and collect some preliminaries. The third section contains the stated classification result. In the fourth section we present some applications. Some of the assumptions can be relaxed to some extent, at the price of much more complicated proofs and no sensible improvement. We end the article with some brief comments and suggestions for future work. An appendix is included to provide some technical results about scalar free field theories.

[1] If one can rule out the occurrence of sectors with infinite statistics for $\mathcal{A}$, the other two facts are easily implied by the split property for the canonical field net $\mathcal{F}$, that is anyhow needed from the start to define the subsystem $\mathcal{C}$.

2 Preliminaries

Throughout this article we denote $\mathcal{P}$ the connected component of the identity of the Poincaré group in four spacetime dimensions and $\mathcal{K}$ the set of open double cones of $\mathbb{R}^4$. We will denote the elements of $\mathcal{P}$ by pairs (Λ, x), where Λ is an element of the restricted Lorentz group and $x \in \mathbb{R}^4$ is a spacetime translation, or alternatively by a single letter L. Double cones and wedges will be denoted $\mathcal{O}$ and $\mathcal{W}$ respectively, with subscripts if necessary. We consider a net $\mathcal{F}$ over $\mathcal{K}$, i.e. a map $\mathcal{O} \to \mathcal{F}(\mathcal{O})$ from double cones to von Neumann algebras acting on a separable Hilbert space $\mathcal{H}$, satisfying the following assumptions.

(i) *Isotony.* If $\mathcal{O}_1 \subset \mathcal{O}_2,\ \mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$, then

$$\mathcal{F}(\mathcal{O}_1) \subset \mathcal{F}(\mathcal{O}_2). \tag{1}$$

(ii) *Locality.* If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ and $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$ then

$$\mathcal{F}(\mathcal{O}_1) \subset \mathcal{F}(\mathcal{O}_2)', \tag{2}$$

(iii) *Covariance.* There is a strongly continuous unitary representation U of $\mathcal{P}$ such that, for every $L \in \mathcal{P}$ and every $\mathcal{O} \in \mathcal{K}$, there holds

$$U(L)\mathcal{F}(\mathcal{O})U(L)^* = \mathcal{F}(L\mathcal{O}). \tag{3}$$

(iv) *Existence and uniqueness of the vacuum.* There exists a unique (up to a phase) unit vector Ω which is invariant under the restriction of U to the one-parameter subgroup of spacetime translations.

(v) *Positivity of the energy.* The joint spectrum of the generators of the spacetime translations is contained in the closure $\overline{V}_+$ of the open forward light cone V_+.

(vi) *Reeh-Schlieder property.* The vacuum vector Ω is cyclic for $\mathcal{F}(\mathcal{O})$ for every $\mathcal{O} \in \mathcal{K}$.

(vii) *Haag duality.* For every double cone $\mathcal{O} \in \mathcal{K}$ there holds

$$\mathcal{F}(\mathcal{O}') = \mathcal{F}(\mathcal{O})', \tag{4}$$

where $\mathcal{O}'$ is the interior of the spacelike complement of $\mathcal{O}$ and, for every open set $\mathcal{S} \subset \mathbb{R}^4$, $\mathcal{F}(\mathcal{S})$ denote the algebra defined by

$$\mathcal{F}(\mathcal{S}) = \vee_{\mathcal{O}\subset\mathcal{S}} \mathcal{F}(\mathcal{O}). \tag{5}$$

(viii) *TCP covariance.* There exists an antiunitary involution Θ (the TCP operator) such that:

$$\Theta U(\Lambda, x)\Theta = U(\Lambda, -x) \;\; \forall (\Lambda, x) \in \mathcal{P}; \tag{6}$$

$$\Theta\mathcal{F}(\mathcal{O})\Theta = \mathcal{F}(-\mathcal{O}). \tag{7}$$

(ix) *Bisognano-Wichmann property.* Let

$$\mathcal{W}_R = \{x \in \mathbb{R}^4 : x^1 > |x^0|\}$$

be the right wedge and let Δ and J be the modular operator and the modular conjugation of the algebra $\mathcal{F}(\mathcal{W}_R)$ with respect to Ω, respectively. Then it holds:

$$\Delta^{it} = U(\Lambda(-2\pi t), 0); \tag{8}$$

$$J = \Theta U(R_1(\pi), 0); \tag{9}$$

where $\Lambda(t)$ and $\mathcal{R}_1(\theta)$ are the one-parameter groups of Lorentz boosts in the x^1-direction and of spatial rotations around the first axis, respectively.

(x) *Split property.* Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ be open double cones such that the closure of $\mathcal{O}_1$ is contained in $\mathcal{O}_2$ (as usual we write $\mathcal{O}_1 \subset\subset \mathcal{O}_2$). Then there is a type I factor $\mathcal{N}(\mathcal{O}_1, \mathcal{O}_2)$ such that

$$\mathcal{F}(\mathcal{O}_1) \subset \mathcal{N}(\mathcal{O}_1, \mathcal{O}_2) \subset \mathcal{F}(\mathcal{O}_2). \tag{10}$$

Using standard arguments (cf. [16]) it can be shown that the previous assumptions imply the irreducibility of the net $\mathcal{F}$, namely the algebra $\mathcal{F}(\mathbb{R}^4)$ coincides with the algebra $\mathrm{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. Another easy consequence of the assumptions is that Ω is U-invariant. Moreover the algebra $\mathcal{F}(\mathcal{W})$ is a factor (in fact a type III_1 factor), for every wedge $\mathcal{W}$, see e.g. [5, Theorem 5.2.1]. Strictly speaking, it is also possible to deduce (viii) from the other assumptions [29, Theorem 2.10].

From Haag duality it follows that the algebra associated with a double cone coincides with intersection of the algebras associated to the wedges containing it, i.e.

$$\mathcal{F}(\mathcal{O}) = \cap_{\mathcal{O}\subset\mathcal{W}} \mathcal{F}(\mathcal{W}), \tag{11}$$

for every $\mathcal{O} \in \mathcal{K}$. Thus our net $\mathcal{F}$ corresponds to a particular case of the AB-systems described in [46], see also [45]. Moreover the Bisognano-Wichmann property implies wedge duality, i.e.

$$\mathcal{F}(\mathcal{W})' = \mathcal{F}(\mathcal{W}'), \tag{12}$$

for every wedge $\mathcal{W}$, where $\mathcal{W}'$ denotes the interior of the causal complement of $\mathcal{W}$.

Another important fact is that, due to the split property, the net $\mathcal{F}$ satisfies Property B for double cones: given $\mathcal{O} \subset\subset \tilde{\mathcal{O}}$, $\mathcal{O}, \tilde{\mathcal{O}} \in \mathcal{K}$, for each nonzero selfadjoint projection $E \in \mathcal{F}(\mathcal{O})$ there exists an isometry $W \in \mathcal{F}(\tilde{\mathcal{O}})$ with

$E = WW^*$. Moreover, for every nonempty open set $\mathcal{S} \subset \mathbb{R}^4$, the algebra $\mathcal{F}(\mathcal{S})$ is properly infinite.

Definition 2.1. A *covariant subsystem* $\mathcal{B}$ of $\mathcal{F}$ is an isotonous net of non-trivial von Neumann algebras over $\mathcal{K}$, such that:

$$\mathcal{B}(\mathcal{O}) \subset \mathcal{F}(\mathcal{O}); \tag{13}$$

$$U(L)\mathcal{B}(\mathcal{O})U(L)^* = \mathcal{B}(L\mathcal{O}), \tag{14}$$

for every $\mathcal{O} \in \mathcal{K}$ and every $L \in \mathcal{P}$.

We use the notation $\mathcal{B} \subset \mathcal{F}$ to indicate that $\mathcal{B}$ is a covariant subsystem of $\mathcal{F}$. As in the case of $\mathcal{F}$, for every open set $\mathcal{S} \subset \mathbb{R}^4$ we define $\mathcal{B}(\mathcal{S})$ by

$$\mathcal{B}(\mathcal{S}) = \vee_{\mathcal{O} \subset \mathcal{S}} \mathcal{B}(\mathcal{O}). \tag{15}$$

Definition 2.2. We say that a covariant subsystem $\mathcal{B}$ of $\mathcal{F}$ is *Haag-dual* if

$$\mathcal{B}(\mathcal{O}) = \cap_{\mathcal{O} \subset \mathcal{W}} \mathcal{B}(\mathcal{W}) \ \ \forall \mathcal{O} \in \mathcal{K}. \tag{16}$$

If a covariant subsystem $\mathcal{B}$ is not Haag-dual, one can associate to it an Haag-dual covariant subsystem $\mathcal{B}^d$ (the *dual subsystem*) defined by

$$\mathcal{B}^d(\mathcal{O}) = \cap_{\mathcal{O} \subset \mathcal{W}} \mathcal{B}(\mathcal{W}), \tag{17}$$

cf. [45, 46]. Note that $\mathcal{B}(\mathcal{W}) = \mathcal{B}^d(\mathcal{W})$ for every wedge $\mathcal{W}$.

Given a covariant subsystem $\mathcal{B}$ of $\mathcal{F}$ we denote $\mathcal{H}_\mathcal{B}$ the closure of $\mathcal{B}(\mathbb{R}^4)\Omega$ and by $E_\mathcal{B}$ the corresponding orthogonal projection. It is trivial that the algebras $\mathcal{B}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ leave $\mathcal{H}_\mathcal{B}$ stable. Hence we can consider the reduced von Neumann algebras $\hat{\mathcal{B}}(\mathcal{O}) := \mathcal{B}(\mathcal{O})_{E_\mathcal{B}}$, $\mathcal{O} \in \mathcal{K}$ acting on the Hilbert space $\mathcal{H}_\mathcal{B}$ and denote $\hat{\mathcal{B}}$ the corresponding net. It is straightforward to verify that

$$\mathcal{B}(\mathcal{S})_{E_\mathcal{B}} = \vee_{\mathcal{O} \subset \mathcal{S}} \hat{\mathcal{B}}(\mathcal{O}), \tag{18}$$

for every open set $\mathcal{S} \subset \mathbb{R}^4$. Therefore the notation $\hat{\mathcal{B}}(\mathcal{S})$ is unambiguous. Moreover, due to the Reeh-Schlieder property (for $\mathcal{F}$), the map $B \in \mathcal{B}(\mathcal{S}) \mapsto \hat{B} := B_{E_{\mathcal{B}}} \in \hat{\mathcal{B}}(\mathcal{S})$, is an isomorphism of von Neumann algebras, whenever the interior $\mathcal{S}'$ of the causal complement of $\mathcal{S}$ is nonempty.

The following result is due in large part to Wichmann [46] and Thomas and Wichmann [45].

Proposition 2.1. *Let $\mathcal{B}$ be a Haag-dual subsystem of $\mathcal{F}$. Then the following properties hold:*

(a) Θ and U commute with $E_{\mathcal{B}}$. Accordingly we can consider the reduced operators $\hat{\Theta} := \Theta_{E_{\mathcal{B}}}$ and $\hat{U} := U_{E_{\mathcal{B}}}$ on $\mathcal{H}_{\mathcal{B}}$;

(b) All the properties from (i) to (x) listed in the beginning of this section holds with $\mathcal{F}$, $\mathcal{H}$, U, Θ, replaced by $\hat{\mathcal{B}}$, $\mathcal{H}_{\mathcal{B}}$, $\hat{U}$, $\hat{\Theta}$, respectively.

Proof. For (a) and (b), properties from (i) to (ix), we refer the reader to [46] and [45, Section 5]. Proving (x) for $\hat{\mathcal{B}}$ corresponds to show that the split property is hereditary. This fact is well known (cf. e.g. [21, Section 5]) but we include here a proof for convenience of the reader.

Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ be such that $\mathcal{O}_1 \subset\subset \mathcal{O}_2$. It is sufficient to show that there is a faithful normal product state on $\hat{\mathcal{B}}(\mathcal{O}_1) \vee \hat{\mathcal{B}}(\mathcal{O}_2)'$, i.e. a faithful normal state ϕ satisfying

$$\phi(BB') = \phi(B)\phi(B') \ \ \forall B \in \hat{\mathcal{B}}(\mathcal{O}_1), \ \forall B' \in \hat{\mathcal{B}}(\mathcal{O}_2)', \tag{19}$$

see e.g. [24]. $\hat{\mathcal{B}}$ satisfies Haag duality and

$$\hat{\mathcal{B}}(\mathcal{O}_1) \vee \hat{\mathcal{B}}(\mathcal{O}_2') = [\mathcal{B}(\mathcal{O}_1) \vee \mathcal{B}(\mathcal{O}_2')]_{E_{\mathcal{B}}}$$

is isomorphic to $\mathcal{B}(\mathcal{O}_1) \vee \mathcal{B}(\mathcal{O}_2')$, being $\mathcal{H}_{\mathcal{B}}$ separating for the latter algebra. Therefore it remains to show the existence of a faithful normal product state on $\mathcal{B}(\mathcal{O}_1) \vee \mathcal{B}(\mathcal{O}_2')$. This trivially follows from the existence of a faithful normal product state for $\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2')$, which is a consequence of the split property for $\mathcal{F}$. [2] □

From the previous proposition it follows that if $\mathcal{B}$ is Haag-dual then $\hat{\mathcal{B}}$ satisfies Haag duality.[3] It is quite easy to show that also the converse is true. This remark should make it clear that considering only Haag-dual subsystems is not a too serious restriction.

If $\mathcal{B}$ is a covariant subsystem of $\mathcal{F}$, we can consider the net $\mathcal{B}^c$ defined by

$$\mathcal{B}^c(\mathcal{O}) = \mathcal{B}(\mathbb{R}^4)' \cap \mathcal{F}(\mathcal{O}), \tag{20}$$

cf. [18, 5]. If $\mathcal{B}^c$ is trivial, then we say that $\mathcal{B}$ is *full* (in $\mathcal{F}$). If $\mathcal{B}^c$ is nontrivial, then it is easy to check that it is a Haag-dual covariant subsystem of $\mathcal{F}$ (*the coset subsystem*). It follows from the definition that $\mathcal{B} \subset \mathcal{B}^{cc}$, and $\mathcal{B}^c = \mathcal{B}^{ccc}$.

For later use it is convenient to introduce the notions of tensor product and of unitary equivalence of two nets. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two nets acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and let U_1, U_2 and Ω_1, Ω_2 the corresponding representations of the Poincaré group and the vacuum vectors. By *tensor product* of nets $\mathcal{F}_1 \otimes \mathcal{F}_2$ we mean the net $\mathcal{K} \ni \mathcal{O} \mapsto \mathcal{F}_1(\mathcal{O}) \otimes \mathcal{F}_2(\mathcal{O})$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ together with the representation $U_1 \otimes U_2$ of $\mathcal{P}$ and the vacuum $\Omega_1 \otimes \Omega_2$. It follows that $\mathcal{F}_1 \otimes \mathcal{F}_2$ satisfies properties (i)–(x) if $\mathcal{F}_1$ and $\mathcal{F}_2$ do so. We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *unitarily equivalent* if there exists a unitary operator $W : \mathcal{H}_1 \to \mathcal{H}_2$ with $W\mathcal{F}_1(\mathcal{O})W^* = \mathcal{F}_2(\mathcal{O})$ ($\mathcal{O} \in \mathcal{K}$), $WU_1(L)W^* = U_2(L)$. Note that since the vacuum is unique up to a phase, one can always choose W so that $W\Omega_1 = \Omega_2$.

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[2] A similar argument shows that split for wedges (cf. [41]) is inherited by subsystems satisfying wedge duality; here the space-time dimension is not important.

[3] This is not true in two spacetime dimensions.

3 General Classification Results

In this section we consider a net $\mathcal{F}$ satisfying all the properties (i)–(x) described in the previous section. Moreover we will assume the following condition (cf. [15]):

(A) Every representation of (the quasi-local C^*-algebra) $\mathcal{F}$ satisfying the DHR selection criterion is a multiple of the vacuum representation.[4]

Let us observe that condition (A) is equivalent to the seemingly weaker condition that all the irreducible representations satisfying the selection criterion are equivalent to the vacuum representation. This is a consequence of the fact that the irreducible representations occurring in the direct integral decomposition of a localized[5] representation are localized a.e. (see [34, Appendix B]).

Now let $\mathcal{B}$ be a Haag-dual, covariant subsystem of $\mathcal{F}$ and let π be the corresponding representation of $\hat{\mathcal{B}}$ in $\mathcal{H}$, i.e. the representation defined by $\pi(\hat{B}) = B$ for $B \in \cup_{\mathcal{O}\in\mathcal{K}}\mathcal{B}(\mathcal{O})$. We denote π^0 the identical (vacuum) representation of $\mathcal{F}$ on $\mathcal{H}$ and π_0 the vacuum representation of $\hat{\mathcal{B}}$, i.e. its identical representation on $\mathcal{H}_\mathcal{B}$. The following result is already known (see e.g. [15]) but we include a proof for the sake of completeness.

Lemma 3.1. *π satisfies the DHR criterion.*

Proof. For every $\mathcal{O} \in \mathcal{K}$ the von Neumann algebras $\mathcal{B}(\mathcal{O}')$ and $\hat{\mathcal{B}}(\mathcal{O}')$ are isomorphic. Moreover, as noted in the previous section, these von Neumann algebras are properly infinite with properly infinite commutants. By [32,

[4]For the basic notions concerning the DHR theory of superselection sectors we refer the reader to [31] and references therein.

[5]In this article the word *localized* referred to representations or endomorphisms means *localized in double cones*.

Theorem 7.2.9.] and [32, Proposition 9.1.6.] we can find a unitary operator

$$U_{\mathcal{O}} : \mathcal{H}_{\mathcal{B}} \to \mathcal{H}$$

such that

$$U_{\mathcal{O}} \hat{B} U_{\mathcal{O}}{}^* = B \ \ \forall B \in \mathcal{B}(\mathcal{O}').$$

Hence if $\mathcal{O}_1 \in \mathcal{K}$ is contained in $\mathcal{O}'$ there holds

$$\pi_0(\hat{B}) = U_{\mathcal{O}}{}^* \pi(\hat{B}) U_{\mathcal{O}} \ \ \forall \hat{B} \in \hat{\mathcal{B}}(\mathcal{O}_1).$$

Actually, this is the DHR criterion. □

Proposition 3.1. *For every irreducible localized transportable morphism σ of $\hat{\mathcal{B}}$, $\pi_0 \circ \sigma$ is equivalent to a subrepresentation of π. Moreover σ is covariant with positive energy and it has finite statistical dimension.*

Proof. Since π satisfies the DHR criterion we can find a transportable localized morphism ρ of $\hat{\mathcal{B}}$ such that there holds the unitary equivalence

$$\pi \simeq \pi_0 \circ \rho, \tag{21}$$

cf. [40, Proposition 3.4.].

Let us consider the extension $\hat{\sigma}$ of σ to $\mathcal{F}$ [15], cf. [40]. Then the assumption (A) for $\mathcal{F}$ imply that

$$\pi^0 \circ \hat{\sigma} \simeq \oplus_i \pi^0, \tag{22}$$

where the index i in the direct sum on the r.h.s. runs over a set whose cardinality is at most countable. Restricting these representations to $\mathcal{B}$ we find

$$\pi \circ \sigma \simeq \oplus_i \pi \tag{23}$$

and therefore using equation 21

$$\rho\sigma \simeq \oplus_i \rho. \tag{24}$$

Since ρ contains the identity sector we have $\sigma \prec \rho\sigma$ and hence

$$\sigma \prec \oplus_i \rho. \tag{25}$$

Thus, being σ arbitrary, every irreducible representation of $\hat{\mathcal{B}}$ satisfying the DHR criterion is contained in a countable multiple of ρ. The latter multiple is a representation on a separable Hilbert space. Hence there are at most countably many irreducible sectors of $\hat{\mathcal{B}}$.

Being π a direct integral of irreducible DHR representations [34, Appendix B] and appealing to some standard arguments (see e.g. [19, 20]) one gets that π is in fact a direct sum. From equation 25 it is not difficult to show that, being σ irreducible, we have $\sigma \prec \rho$ i.e. $\pi_0 \circ \sigma$ is unitarily equivalent to a subrepresentation of π.

Since $\mathcal{B}$ is covariant π is covariant with positive energy. We have to show that every irreducible subrepresentation has the same property, cf. [4]. Since the action induced by the representation U of the Poincaré group leaves $\mathcal{B}(\mathbb{R}^4)$ globally invariant it leaves globally invariant also its centre. Being the latter purely atomic (due to the decomposition of π into irreducibles) and $\mathcal{P}$ connected, it follows that the orthogonal projection $E_{[\sigma]} \in \mathcal{B}(\mathbb{R}^4)' \cap \mathcal{B}(\mathbb{R}^4)$ onto the isotypic subspace corresponding to σ must commute with U. Let $U_{[\sigma]}$ and $\pi_{[\sigma]}$ be the restrictions to $E_{[\sigma]}\mathcal{H}$ of U and π respectively. Then we have the unitary equivalence

$$\pi_{[\sigma]} \simeq (\pi_0 \circ \sigma) \otimes I \ . \tag{26}$$

Moreover, using the relation

$$U_{[\sigma]}(L)\pi_{[\sigma]}(\hat{B})U_{[\sigma]}(L)^* = \pi_{[\sigma]}(\hat{U}(L)\hat{B}\hat{U}(L)^*), \tag{27}$$

where $B \in \cup_{\mathcal{O}\in\mathcal{K}}\mathcal{B}(\mathcal{O}),\ L \in \mathcal{P}$, and a classical result by Wigner on projective unitary representations of $\mathcal{P}$ [47, 2], it is quite easy to show that

$$U_{[\sigma]}(L) \simeq U_\sigma(L) \otimes X_\sigma(L), \tag{28}$$

where U_σ and X_σ are unitary continuous representations of (the covering group of) $\mathcal{P}$ and U_σ is such that

$$U_\sigma(L)\sigma(\hat{B})U_\sigma(L)^* = \sigma(\hat{U}(L)\hat{B}\hat{U}(L)^*). \tag{29}$$

Since $U_{[\sigma]}$ satisfies the spectral condition, both U_σ and X_σ have to satisfy it.[6] Hence σ is covariant with positive energy.

Finally, from $\rho\sigma \simeq \sigma\rho$ and equation 24 it follows that id $\prec \sigma\rho$. Therefore, being σ covariant with positive energy, it has finite statistical dimension because of [23, prop. A.2]. □

A related result has been independently obtained by R. Longo, in the context of nets of subfactors [39].

Let $\mathcal{F}_\mathcal{B}$ be the canonical field net of $\hat{\mathcal{B}}$ as defined in [26, Section 3]. In natural way $\mathcal{F}_\mathcal{B}$ can be considered as a Haag-dual subsystem of $\mathcal{F}$ containing $\mathcal{B}$ [15, Theorem 3.5]. In fact one finds that $\mathcal{F}_\mathcal{B}(\mathcal{O})$ coincides with the von Neumann algebra generated by the family of Hilbert spaces $\mathcal{H}_{\hat{\sigma}}$ in $\mathcal{F}$, where σ runs over all the transportable morphisms of $\mathcal{B}$ which are localized in $\mathcal{O}$ and $\hat{\sigma}$ denotes the functorial extension of σ to $\mathcal{F}$. From the fact that the latter extension commutes with spacetime symmetries, namely $(\sigma_L)\hat{} = (\hat{\sigma})_L$ for every $L \in \mathcal{P}$ it is also easy to show that $\mathcal{F}_\mathcal{B}$ is a covariant subsystem. (Besides, by [13, Proposition 2.1] $\mathcal{F}_\mathcal{B}$ coincides with its covariant companion, cf. [26].)

[6] This follows from the fact that if S_1 and S_2 are two orbits of the restricted Lorentz group such that $S_1 + S_2 \subset \overline{V}_+$ then $S_1 \subset \overline{V}_+$ and $S_2 \subset \overline{V}_+$.

Theorem 3.1. *$\hat{\mathcal{F}_{\mathcal{B}}}$ has no irreducible DHR sectors other than the vacuum.*

Proof. By the previous proposition it is enough to consider sectors with finite statistical dimension. Let $\mathcal{R}$ be the canonical field algebra of $\hat{\mathcal{F}_{\mathcal{B}}}$. Then $\mathcal{R}$ is a Haag-dual covariant subsystem of $\mathcal{F}$, and as such it inherits the split property. By the results discussed in [8] this is sufficient[7] to deduce that $\mathcal{F}_{\mathcal{B}} = \mathcal{R}$.[8] In fact the group $\tilde{G}$ of the (unbroken) symmetries of $\mathcal{R}$ extending the gauge automorphisms of $\mathcal{F}_{\mathcal{B}}$ is compact in the strong operator topology by (the proof of) [24, Theorem 10.4], and obviously $\mathcal{R}^{\tilde{G}} = \mathcal{B}$. The conclusion follows by the uniqueness of the canonical field net [26]. □

Theorem 3.2. *There exists a unitary isomorphism of $\mathcal{F}$ with $\hat{\mathcal{F}_{\mathcal{B}}} \otimes \hat{\mathcal{B}}^c$. In particular $\mathcal{F}_{\mathcal{B}} = \mathcal{B}^{cc}$, and if $\mathcal{B}$ is full[9] in $\mathcal{F}$ then $\mathcal{F}_{\mathcal{B}} = \mathcal{F}$.*

Proof. Let $\tilde{\pi}$ be the representation of $\hat{\mathcal{F}_{\mathcal{B}}}$ on $\mathcal{H}$ (the vacuum Hilbert space of $\mathcal{F}$) arising from the embedding $\mathcal{F}_{\mathcal{B}} \subset \mathcal{F}$ and $\tilde{\pi}_0$ the vacuum representation of $\hat{\mathcal{F}_{\mathcal{B}}}$ on $\mathcal{H}_{\mathcal{F}_{\mathcal{B}}} \subset \mathcal{H}$. By the previous theorem $\hat{\mathcal{F}_{\mathcal{B}}}$ has no nontrivial sectors. Moreover Lemma 3.1 applied to $\mathcal{F}_{\mathcal{B}}$ instead of $\mathcal{B}$ implies that $\tilde{\pi}$ is (spatially) equivalent to a multiple of $\tilde{\pi}_0$ and therefore to $\tilde{\pi}_0 \otimes \mathrm{I}$, on $\mathcal{H}_{\mathcal{F}_{\mathcal{B}}} \otimes \mathcal{H}_1$, where $\mathcal{H}_1$ is a suitable Hilbert space. Let $W : \mathcal{H} \to \mathcal{H}_{\mathcal{F}_{\mathcal{B}}} \otimes \mathcal{H}_1$ be a unitary operator implementing this equivalence. For every double cone $\mathcal{O}$ there holds

$$\hat{\mathcal{F}_{\mathcal{B}}}(\mathcal{O}') \otimes \mathrm{I} \subset \tilde{\mathcal{F}}(\mathcal{O}') \tag{30}$$

where $\tilde{\mathcal{F}}(\mathcal{O}) = W\mathcal{F}(\mathcal{O})W^*$. Therefore, using Haag duality for $\hat{\mathcal{F}_{\mathcal{B}}}$,

$$\hat{\mathcal{F}_{\mathcal{B}}}(\mathcal{O}) \otimes \mathrm{I} \subset \tilde{\mathcal{F}}(\mathcal{O}) \subset \hat{\mathcal{F}_{\mathcal{B}}}(\mathcal{O}) \otimes \mathrm{B}(\mathcal{H}_1). \tag{31}$$

[7]This idea is not new, see e.g. [42, Section 2], however some technical difficulties are circumvented when the assumptions made in this paper are used.

[8]Alternatively, the same result may be deduced combining Proposition 3.1 with [15].

[9]Irreducible subsystems, namely those satisfying $\mathcal{B}' \cap \mathcal{F} = \mathbb{C}$, are full.

It follows that

$$\hat{\mathcal{F}_{\mathcal{B}}}(\mathcal{W}) \otimes \mathrm{I} \subset \tilde{\mathcal{F}}(\mathcal{W}) \subset \hat{\mathcal{F}_{\mathcal{B}}}(\mathcal{W}) \otimes \mathrm{B}(\mathcal{H}_1). \tag{32}$$

The algebras of wedges are factors. By the results in [28] (cf. also [44]) there exists a von Neumann algebra $\mathcal{M}(\mathcal{W}) \subset \mathrm{B}(\mathcal{H}_1)$ such that

$$\tilde{\mathcal{F}}(\mathcal{W}) = \hat{\mathcal{F}_{\mathcal{B}}}(\mathcal{W}) \otimes \mathcal{M}(\mathcal{W}). \tag{33}$$

Taking on both sides of this equality the intersection over all the wedges containing a given $\mathcal{O} \in \mathcal{K}$ we find

$$\tilde{\mathcal{F}}(\mathcal{O}) = \hat{\mathcal{F}_{\mathcal{B}}}(\mathcal{O}) \otimes \mathcal{M}(\mathcal{O}), \tag{34}$$

where

$$\mathcal{M}(\mathcal{O}) = \cap_{\mathcal{O} \subset \mathcal{W}} \mathcal{M}(\mathcal{W}). \tag{35}$$

Now, using the commutant theorem for von Neumann tensor products, it is straightforward to show that

$$\mathrm{I} \otimes \mathcal{M}(\mathcal{O}) = W \mathcal{B}^c(\mathcal{O}) W^*$$

for every $\mathcal{O} \in \mathcal{K}$. The previous equation implies the existence of a representation τ of $\hat{\mathcal{B}}^c$ on $\mathcal{H}_1$ such that $WBW^* = \mathrm{I} \otimes \tau(\hat{B})$, $B \in \mathcal{B}^c(\mathcal{O})$ for every $\mathcal{O} \in \mathcal{K}$. Moreover, since $\mathcal{M}$ acts irreducibly on $\mathcal{H}_1$ and the vacuum representation π^c of $\hat{\mathcal{B}}^c$ is contained in $\mathrm{I} \otimes \tau$, τ is spatially isomorphic to π^c and thus the mapping $\mathcal{O} \to \mathcal{M}(\mathcal{O})$ gives a net unitarily equivalent to $\hat{\mathcal{B}}^c$. Therefore without loss of generality we can assume that $\mathcal{H}_1 = \mathcal{H}_{\mathcal{B}^c}$ and that $W\mathcal{F}(\mathcal{O})W^* = \hat{\mathcal{F}_{\mathcal{B}}}(\mathcal{O}) \otimes \hat{\mathcal{B}}^c(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$. The conclusion follows noticing that $WUW^* = U_{E_{\mathcal{F}_{\mathcal{B}}}} \otimes U_{E_{\mathcal{B}^c}}$. Here we omit the easy details. □

Applying the previous theorem to $\mathcal{B}^c$ in place of $\mathcal{B}$ we get that $\mathcal{B}^c$ as no nontrivial sectors, since $\mathcal{F}_{\mathcal{B}^c} = \mathcal{B}^{ccc} = \mathcal{B}^c$.

Corollary 3.1. *Let $\mathcal{B}$ be a Haag-dual covariant subsystem of $\mathcal{F}$, then the net of inclusions $\mathcal{K} \ni \mathcal{O} \mapsto \mathcal{B}(\mathcal{O}) \subset \mathcal{F}(\mathcal{O})$ is (spatially) isomorphic to $\mathcal{O} \mapsto \hat{\mathcal{F}}_{\mathcal{B}}(\mathcal{O})^G \otimes I \subset \hat{\mathcal{F}}_{\mathcal{B}}(\mathcal{O}) \otimes \hat{\mathcal{B}}^c(\mathcal{O})$, where G is the canonical gauge group of $\hat{\mathcal{B}}$.*

Corollary 3.2. *If $\mathcal{B}$ is a Haag-dual covariant subsystem of $\mathcal{F}$ and if $\mathcal{F}_{\mathcal{B}}$ is full (in particular if $\mathcal{B}$ is full) then there exists a compact group G of unbroken internal symmetries of $\mathcal{F}$ such that $\mathcal{B} = \mathcal{F}^G$.*

Now let $\mathcal{C}$ be the (local) net generated by the canonical implementations of the translations on $\mathcal{F}$ [12]. It is a covariant subsystem of $\mathcal{F}$. Since $\mathcal{C}$ is (irreducible thus) full in $\mathcal{F}$ and $\mathcal{C}^d \subset \mathcal{F}^{G_{\max}}$, where $G_{\max}$ is the (compact) group of all unbroken internal symmetries of $\mathcal{F}$, we have

Corollary 3.3. *In the situation described above it holds*

$$\mathcal{C}^d = \mathcal{F}^{G_{\max}}. \tag{36}$$

4 Applications

4.1 Free fields

Our standing assumptions are satisfied in the case where $\mathcal{F}$ is generated by a finite set of free scalar fields [27, 7] and also by suitable infinite sets of such fields [25]. They are also satisfied in other Bosonic theories, e.g. when $\mathcal{F}$ is generated by the free electromagnetic field, see [7].

Therefore from our Corollary 3.2 one can obtain all the results in [18] in the case of full subsystems, even without assuming the existence of a mass gap. Concerning subsystems that are not full, one has to study the possible decompositions

$$\hat{\mathcal{F}}_{\mathcal{B}}(\mathcal{O}) \otimes \hat{\mathcal{B}}^c(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \tag{37}$$

(up to unitary equivalence). In the case where $\mathcal{F}$ is generated by a finite set of free scalar fields, it turns out that $\mathcal{F}_{\mathcal{B}}$ and $\mathcal{B}^c$ are always free scalar theories generated by two suitable disjoint subsets of the generating fields of $\mathcal{F}$. We present a detailed proof of this fact in the appendix. [10] In particular, if $\mathcal{F}$ is generated by a single scalar free field $\varphi(x)$ of mass $m \geq 0$, no such nontrivial decomposition is possible and hence all of the subsystems of $\mathcal{F}$ are full. Accordingly, in this case, the unique Haag-dual covariant proper subsystem of $\mathcal{F}$ is the fixed point net $\mathcal{F}^{\mathbb{Z}_2}$ under the action of the group of (unbroken) internal symmetries.

Note that when $m = 0$ there are covariant subsystems which are not Haag-dual. For instance the subsystem $\mathcal{A} \subset \mathcal{F}$ generated by the derivatives $\partial_\mu \varphi(x)$ is Poincaré covariant but not Haag-dual and in fact one has $\mathcal{F} = \mathcal{A}^d$ [7]. However it can been shown that conformally covariant subsystems of $\mathcal{F}$ are always Haag-dual. Actually the latter fact still holds in a more general context.

4.2 Theories with countably many sectors

Summing up, we have shown a classification result for Haag-dual subnets of a purely Bosonic net with trivial superselection structure (including infinite statistics) and with the split property. Moreover we have exhibited an important class of examples, namely (multiplets of) the free fields, to which our results apply. This is already quite satisfactory. One can consider a more general situation in which $\mathcal{F}$ is the canonical field net of an observable net $\mathcal{A}$. A closely related problem is, of course, to look for the structural hypotheses on $\mathcal{A}$ ensuring that $\mathcal{F} = \mathcal{F}_{\mathcal{A}}$ will have the required properties. It has been known for some time that if $\mathcal{A}$ has only a finite number of irreducible

[10]Davidson obtained this result in the purely massive case [18].

DHR sectors with finite statistical dimension (i.e. $\mathcal{A}$ is rational), all of which are Bosonic, then $\mathcal{F}$ (is local and) has no nontrivial DHR sectors with finite statistical dimension [13, 42]. This result is not sufficient for our purposes, because it does not rule out the possible presence of irreducible DHR representations of $\mathcal{F}$ with infinite statistical dimension. However, a solution to this problem can be achieved by using the stronger results given in [15].

Theorem 4.1. *Let $\mathcal{A}$ be a local net satisfying the split property and Haag duality in its (irreducible) vacuum representation. If $\mathcal{A}$ has at most countably many irreducible (DHR) sectors, all of which are Bosonic and with finite statistical dimension, then any sector of $\mathcal{A}$ is a direct sum of irreducible sectors. Moreover, the canonical field net $\mathcal{F}$ of $\mathcal{A}$ has no nontrivial sectors with any (finite or infinite) statistical dimension.*

Proof. In view of [15, Theorem 4.7] it is enough to show the first statement. But using the split property and the bound on the number of inequivalent sectors, this follows arguing as in the proof of Proposition 3.1. □

This result [11] shows that $\mathcal{F}$ satisfies the condition (A) of section 3. Moreover if $\mathcal{A}$ satisfies all of the conditions (i)-(vii) then the same is true for $\mathcal{F}$ [26]. In order to apply the above result about classification of subsystems and solve the problem about local charges, we need to know conditions on $\mathcal{A}$ implying the validity of properties (viii)-(x). Concerning (x), it would be a consequence of the split property for $\mathcal{A}$ if G were finite and abelian. In other cases one can invoke some version of nuclearity for $\mathcal{A}$, implying that $\mathcal{F}$ is split [6]. But it is also necessary to know if the existence of a TCP symmetry and the special condition of duality for $\mathcal{A}$ imply the same for its

[11] As in [13], in the case of rational theories a different argument could be given when the local algebras are factors, based on a restriction-extension argument (cf. [34, Lemma 27]).

canonical field system $\mathcal{F}$. The relationship between the validity of conditions (viii)-(ix) for $\mathcal{A}$ and its canonical field system $\mathcal{F}$ has been discussed in [35, 36] (the TCP symmetry has been also treated in [14] under milder hypotheses). The conclusion is that if $\mathcal{A}$ satisfies the usual axioms (and all its sectors are covariant), moreover it is purely Bosonic and satisfies a suitable version of nuclearity (implying, among other things, the existence of at most countably many sectors), TCP covariance and the Bisognano-Wichmann property, then we know how to classify all the subsystems of $\mathcal{F}$ satisfying Haag duality.

Corollary 4.1. *Let $\mathcal{A}$ be an observable net satisfying the properties (i)-(ix) above, without DHR sectors with infinite statistical dimension or para-Fermi statistics of any finite order, whose (Bosonic) canonical field net $\mathcal{F}$ has the split property. Then, if $\mathcal{C}$ is the net generated by the local energy-momentum tensor, one has*

$$\mathcal{C}^d = \mathcal{F}^{G_{\max}}.$$

Moreover $\mathcal{A} = \mathcal{C}^d$ if and only if $\mathcal{A}$ has no proper full Haag-dual subsystem (in which case $\mathcal{A}$ has no unbroken internal symmetries).

Proof. Since $\mathcal{A}$ satisfies the split property and has at most countably many sectors, all with finite statistics, the first statement follows by the previous result and Corollary 3.3. If G denotes the canonical gauge group of $\mathcal{A}$, so that $\mathcal{A} = \mathcal{F}^G$, the equality $\mathcal{A} = \mathcal{C}^d$ is equivalent to the equality $G = G_{\max}$, which, due to Corollary 3.2, means that there is no proper subsystem of $\mathcal{A}$ full (or irreducible) in $\mathcal{F}$. To complete the proof we only need to show that every full subsystem of $\mathcal{A}$ is full in $\mathcal{F}$, when $G = G_{\max}$. Let $\mathcal{B}$ be a (Haag-dual) subsystem of $\mathcal{A}$. Due to the results in the previous section, for every wedge $\mathcal{W}$ the inclusions

$$\mathcal{B}(\mathcal{W}) \subset \mathcal{A}(\mathcal{W}) \subset \mathcal{F}(\mathcal{W})$$

are spatially isomorphic to

$$\hat{\mathcal{B}}(\mathcal{W}) \otimes I \subset \tilde{\mathcal{A}}(\mathcal{W}) \subset \hat{\mathcal{F}}_{\mathcal{B}}(\mathcal{W}) \otimes \hat{\mathcal{B}}^c(\mathcal{W}),$$

with $\tilde{\mathcal{A}}$ isomorphic to $\mathcal{A}$. Moreover, from $G = G_{\max}$ it follows that

$$\tilde{\mathcal{A}}(\mathcal{W}) \subset \hat{\mathcal{B}}(\mathcal{W}) \otimes \hat{\mathcal{B}}^c(\mathcal{W}).$$

Arguing as in the proof of theorem 9 we find that if $\mathcal{B}$ is not full in $\mathcal{F}$ then for every $\mathcal{O} \in \mathcal{K}$, the algebra $\mathcal{B}(\mathbb{R}^4)' \cap \mathcal{A}(\mathcal{O})$ is nontrivial. It follows that $\mathcal{B}$ is not full in $\mathcal{A}$. □

5 Comments on the assumptions

Some of the results of the previous sections are in fact still true even after relaxing some conditions. We will briefly discuss some aspects here.

The hypothesis (x) is useful to derive property B (also for the subsystems), to apply the results in [34] and also to define the local charges. If we renounce to (x), and possibly (A), taking $\mathcal{F}$ as the DHR field algebra of $\mathcal{A} \supset \mathcal{B}$ in its vacuum representation on $\mathcal{H}$ (here it is not even essential to require the condition of covariance, nor the additional assumptions of the main theorem in [15]), it is still possible to deduce that $\tilde{\pi} \simeq \tilde{\pi}_0 \otimes I$ as in the proof of Theorem 9. For this purpose one needs to know that $\mathcal{A}$ and $\mathcal{B}$ both satisfy property B, and that $\tilde{\pi}$ in restriction to $\mathcal{B}$ (thought of as a representation of $\hat{\mathcal{B}}$) is quasi-contained in the canonical embedding of $\hat{\mathcal{B}}$ into its field net. By the results in [15], the latter property holds if it is possible to rule out the occurrence of representations with infinite statistics for $\hat{\mathcal{B}}$ acting on $\mathcal{H}$ (e.g. if $[\mathcal{A} : \mathcal{B}] < \infty$ in the case of nets of subfactors). In fact we don't even need to know a priori that π satisfies the DHR selection criterion. Relaxing covariance is necessary to discuss QFT on (globally hyperbolic) curved spacetimes. Possibly results resembling those presented here should hold also in that context (cf. [30]).

The Bisognano-Wichmann property for $\mathcal{F}$ and TCP covariance may also be relaxed, but, for the time being, $\mathcal{F}$ and the considered subsystems always have to satisfy Haag duality in order to deduce some nice classification result.

However, let us discuss the inheritance of the split property in a slightly more general situation. We start with a subsystem $\mathcal{B} \subset \mathcal{F}$, but now both $\mathcal{F}$ and $\mathcal{B}$ are only assumed to satisfy essential duality (cf. [31]) in their respective vacuum representation, namely $\mathcal{F}^d = \mathcal{F}^{dd}$ and $(\hat{\mathcal{B}})^d = (\hat{\mathcal{B}})^{dd}$ (this is consistent with the notation adopted in the previous sections). Moreover we require the split property for $\mathcal{F}^d$. In the situation where one has an embedding of $(\hat{\mathcal{B}})^d$ inside $\mathcal{F}^d$, [12] we may deduce the split property for $(\hat{\mathcal{B}})^d$ by our previous argument. For instance if $\mathcal{F}$ satisfies the Bisognano-Wichmann property (thus in particular wedge duality, which implies essential duality), then $\hat{\mathcal{B}}$ satisfies the same property as well [46] and moreover there exists the embedding alluded above, therefore the split property for $\mathcal{F}^d$ entails the split property for $(\hat{\mathcal{B}})^d$.[13]

6 Outlooks

In this article we have not discussed graded local (Fermionic) nets. As far as we can see, it should be possible to obtain classification results also in this case, once the natural changes in the assumptions, the statements and the proofs are carried out.

In the situation described in the present paper the index of a subsystem

[12] This may be true or not and is related to the validity of the equality $(\hat{\mathcal{B}})^d = (\mathcal{B}^d)\hat{}$.

[13] As a matter of fact, the same argument goes through when we just have essential duality for $\mathcal{F}$ and wedge duality for $\hat{\mathcal{B}}$, see e.g. [15, Section 3].

is clearly always infinite, or an integer. Moreover any integer value is in fact realized[14]. In a broader context (e.g. inclusions of conformal nets on S^1), the computation of the set of possible index values for subsystems seems an interesting problem. In the case of concrete models many calculations are now available. We hope to return on these subjects in the future.

A Appendix

In this appendix we study the possible tensor product decompositions of a net generated by a finite number of scalar free fields.

We consider a net $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$, acting irreducibly on its vacuum Hilbert space $\mathcal{H}$, generated by a finite family of Hermitian scalar free fields $\varphi_1(x)$, $\varphi_2(x) \ldots, \varphi_n(x)$, where $n = n_1 + n_2 + \ldots + n_k$ and $\varphi_1(x), \ldots, \varphi_{n_1}(x)$ have mass m_1, $\varphi_{n_1+1}(x), \ldots, \varphi_{n_1+n_2}(x)$ have mass m_2, and so forth, and $0 \leq m_1 < \ldots < m_k$.

Accordingly, for each $\mathcal{O} \in \mathcal{K}$, $\mathcal{F}(\mathcal{O})$ is the von Neumann algebra generated by the Weyl unitaries $e^{i\varphi_j(f)}$ for $j = 1, \ldots, n$ and real-valued $f \in \mathcal{S}(\mathbb{R}^4)$ with support in $\mathcal{O}$.

We denote U, Θ, Ω the corresponding representation of $\mathcal{P}$, TCP operator and vacuum vector respectively.

For every i we let K_i be the closed subspace of $\mathcal{H}$ generated by the vectors $\varphi_i(f)\Omega$ with $f \in \mathcal{S}(\mathbb{R}^4)$.

Each K_i is U-invariant, and the restriction V_i of U to $\mathcal{H}_i$ is the irreducible representation of $\mathcal{P}$ with spin 0 and corresponding mass.

Moreover the generating fields are chosen so that K_i is orthogonal to K_j for $i \neq j$.

[14] To see this, consider the fixpoint net of the complex scalar free field under the subgroup $\mathbb{Z}_n$ of the gauge group S^1.

If $\mathrm{K} = \oplus_{i=1}^n \mathrm{K}_i$ and $V = \oplus_{i=1}^n V_i$, then $\mathcal{H}$ can be identified with the (symmetric) Fock space $\Gamma(\mathrm{K})$ and U with the second quantization representation $\Gamma(V)$, see e.g. [43].

If $\mathcal{F}_i$ is the covariant subsystem of $\mathcal{F}$ generated by $\varphi_i(x)$, then $\mathcal{H}_{\mathcal{F}_i}$ can be identified with $\Gamma(\mathrm{K}_i)$ and from the relation $\mathcal{F}(\mathcal{O}) = \vee_i \mathcal{F}_i(\mathcal{O})$ and the properties of the second quantization functor it follows that the net $\mathcal{F}$ is isomorphic to $\hat{\mathcal{F}}_1 \otimes \ldots \otimes \hat{\mathcal{F}}_n$ on $\otimes_i \Gamma(\mathrm{K}_i)$.

Note that there is some freedom in the choice of the generating fields, corresponding to the internal symmetry group $G = \mathrm{O}(n_1) \times \ldots \times \mathrm{O}(n_k)$.

Let E_{m_h} be the orthogonal projection from $\mathcal{H}$ onto $\mathrm{K}_{m_h} := \oplus_{i=n_{h-1}+1}^{n_{h-1}+n_h} \mathrm{K}_i$, where by convention $n_0 = 0$. For each $m \geq 0$, let P_m be the orthogonal projection onto $\mathrm{Ker}(P^2 - m^2)$, where P^2 denotes the mass operator corresponding to U. It is not difficult to see that $P_m(\mathrm{K} + \mathbb{C}\Omega)^\perp = 0$ by a direct calculation on the k-particles subspaces of $\mathcal{H}$ (note that $P_m = 0$ whenever $m \notin \{0\} \cup \{m_1, \ldots, m_k\}$). It follows that $P_{m_h} = E_{m_h}$ if $m_h > 0$, while for $m_h = 0$ we have $P_{m_h} = E_{m_h} + P_\Omega$ where $P_\Omega \in U(\mathcal{P})' \cap U(\mathcal{P})''$ is the orthogonal projection onto $\mathbb{C}\Omega$. In particular, for any $h \in \{1, \ldots, k\}$ we have $E_{m_h} \in U(\mathcal{P})' \cap U(\mathcal{P})''$.

The following simple lemma will be used to study the tensor product decomposition of $\mathcal{F}$.

Lemma A.1. *Let U_1 and U_2 be subrepresentations of U on subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{H}$ both orthogonal to $\mathbb{C}\Omega$. Then there are no eigenvectors for the mass operator corresponding to the representation $U_1 \otimes U_2$.*

Proof. We consider the net $\tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}$ and the corresponding representation $\tilde{U} = U \otimes U$ of $\mathcal{P}$. Obviously the net $\tilde{\mathcal{F}}$ is of the same type as $\mathcal{F}$, with the same masses but different multiplicities. $U_1 \otimes U_2$ is a subrepresentation of $\tilde{U}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $\tilde{P}^2$ is the mass operator corresponding to $\tilde{U}$ and $\tilde{P}_m$ is the orthogonal projection onto $\mathrm{Ker}(\tilde{P}^2 - m^2)$, we only have to show that for every $m \geq 0$ we have $\tilde{P}_m \mathcal{H}_1 \otimes \mathcal{H}_2 = 0$. But this follows by the discussion in the last paragraph before the statement, since $\mathcal{H}_1 \otimes \mathcal{H}_2$ is orthogonal to $\mathbb{C}(\Omega \otimes \Omega) + \tilde{\mathrm{K}}$ where $\tilde{\mathrm{K}} = \mathrm{K} \otimes \Omega + \Omega \otimes \mathrm{K}$ is the one-particle subspace of $\mathcal{H} \otimes \mathcal{H}$. □

We are now ready to study the possible tensor product decompositions $\mathcal{F}_A \otimes \mathcal{F}_B$ of $\mathcal{F}$. In the sequel we assume to have such a decomposition, and deduce some consequences.

Then $\mathcal{H}$ is given by $\mathcal{H}_A \otimes \mathcal{H}_B$ so that $\Omega = \Omega_A \otimes \Omega_B$ and $U = U_A \otimes U_B$.

We set $\mathcal{H}_A = \mathbb{C}\Omega_A \oplus \tilde{\mathcal{H}}_A$, and analogously for $\mathcal{B}$, so that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}\Omega \oplus (\Omega_A \otimes \tilde{\mathcal{H}}_B) \oplus (\tilde{\mathcal{H}}_A \otimes \Omega_B) \oplus (\tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B)$. We also set $F_0 = P_\Omega$, $F_A = [\Omega_A \otimes \tilde{\mathcal{H}}_B]$, $F_B = [\tilde{\mathcal{H}}_A \otimes \Omega_B]$, $F_{AB} = [\tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B]$. Notice that these orthogonal projections commute not only with U but also with Θ.

Lemma A.2. *For each $h = 1, \dots, k$ it holds $E_{m_h} F_{AB} = 0$.*

Proof. It is an immediate consequence of Lemma A.1. □

Since $E_{m_h} F_0 = 0$, the previous lemma implies that $E_{m_h}(F_A + F_B) = E_{m_h}$, for $h = 1, \dots, k$. This amounts to say that $\mathrm{K} \subset \tilde{\mathcal{H}}_A \otimes \Omega_B \oplus \Omega_A \otimes \tilde{\mathcal{H}}_B$. As a consequence, with the aid of some linear algebra and the fact that F_A and F_B commute with Θ, it is not difficult to show that there is a partition in two disjoint sets $\{1, \dots, n\} = \alpha_A \cup \alpha_B$ along with a suitable choice of the generating fields such that, for every $f \in \mathcal{S}(\mathbb{R}^4)$,

$$\varphi_i(f)\Omega \in \tilde{\mathcal{H}}_A \otimes \Omega_B \text{ for } i \in \alpha_A, \quad \varphi_i(f)\Omega \in \Omega_A \otimes \tilde{\mathcal{H}}_B \text{ for } i \in \alpha_B. \tag{38}$$

Because of equations 38, for every $f \in \mathcal{S}(\mathbb{R}^4)$ and $i \in \alpha_A$ one can define a vector $T_i(f) \in \tilde{\mathcal{H}}_A$ by

$$\varphi_i(f)(\Omega_A \otimes \Omega_B) =: T_i(f) \otimes \Omega_B. \tag{39}$$

It follows that if $\mathrm{supp}(f) \subset \mathcal{O}$, f real, and $X_A \in \mathcal{F}_A(\mathcal{O}')$, $X_B \in \mathcal{F}_B(\mathcal{O}')$, we get that

$$\begin{aligned}\varphi_i(f)(X_A\Omega_A \otimes X_B\Omega_B) &= \varphi_i(f)(X_A \otimes X_B)(\Omega_A \otimes \Omega_B) \\ &= (X_A \otimes X_B)\varphi_i(f)(\Omega_A \otimes \Omega_B) \\ &= X_A T_i(f) \otimes X_B\Omega_B,\ i \in \alpha_A. \end{aligned} \tag{40}$$

By a continuity argument (we are assuming $\varphi_i(f)$ to be closed),

$$\varphi_i(f)(X_A\Omega_A \otimes \xi) = X_A T_i(f) \otimes \xi \quad \forall\, \xi \in \mathcal{H}_B.$$

Therefore, for every $T \in \mathrm{B}(\mathcal{H}_B)$, $(I \otimes T)(X_A\Omega_A \otimes X_B\Omega_B)$ belongs to the domain of $\varphi_i(f)$ and

$$(I \otimes T)\varphi_i(f)(X_A\Omega_A \otimes X_B\Omega_B) = \varphi_i(f)(I \otimes T)(X_A\Omega_A \otimes X_B\Omega_B). \tag{41}$$

Hence again by continuity we find that, for every $X \in \mathcal{F}(\mathcal{O}')$,

$$(I \otimes T)\varphi_i(f)X\Omega = \varphi_i(f)(I \otimes T)X\Omega,\ i \in \alpha_A. \tag{42}$$

Similarly, for each $T \in \mathrm{B}(\mathcal{H}_A)$,

$$(T \otimes I)\varphi_i(f)X\Omega = \varphi_i(f)(T \otimes I)X\Omega,\ i \in \alpha_B. \tag{43}$$

Our next goal is to show that $\mathcal{F}(\mathcal{O}')\Omega$ is a core for $\varphi_i(f)$ for any real f as above and $i = 1, \ldots, n$. This will entail that $e^{i\varphi_i(f)} \in (I \otimes \mathrm{B}(\mathcal{H}_B))' = \mathrm{B}(\mathcal{H}_A) \otimes I$ for every real-valued test function f with compact support (by arbitrariness of $\mathcal{O}$ in the argument above) and $i \in \alpha_A$, and similarly $e^{i\varphi_i(f)} \in I \otimes \mathrm{B}(\mathcal{H}_B)$ for $i \in \alpha_B$, from which it is easy to see that $\vee_{i\in\alpha_A}\mathcal{F}_i(\mathcal{O}) = \mathcal{F}_A(\mathcal{O}) \otimes I$ and $\vee_{i\in\alpha_B}\mathcal{F}_i(\mathcal{O}) = I \otimes \mathcal{F}_B(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$.

Proposition A.1. *For any $f \in \mathcal{S}(\mathbb{R}^4)$ real, $\mathcal{O} \in \mathcal{K}$ and $i = 1, \ldots, n$, $\mathcal{F}(\mathcal{O})\Omega$ contains a core for $\varphi_i(f)$. In particular if* $\mathrm{supp}(f) \subset \mathcal{O}$ *then $\mathcal{F}(\mathcal{O}')\Omega$ is a core for $\varphi_i(f)$.*

Proof. We use some techniques concerning energy-bounds, cf. [3, Section 13.1.3]. Let N be the total number operator acting on $\mathcal{H} = \Gamma(\mathrm{K})$. Then N is the closure of $\sum_i N_i$ with N_i the number operator on $\Gamma(\mathrm{K}_i)$. Using well known enstimates about free fields (see [43, Section X.7]) for every real f and ψ in the domain of N we have

$$\|\varphi_i(f)\psi\| \leq c(f)\|\sqrt{N+I}\psi\| \leq c(f)\|(N+I)\psi\| \tag{44}$$

for some constant $c(f)$ depending only on f. Moreover $\varphi_i(f)$ is essentially self-adjoint on any core for N.

We define a self-adjoint operator H as (the closure of) the sum of the H_i, where H_i on $\Gamma(\mathrm{K}_i)$ is the conformal Hamiltonian if $\varphi_i(x)$ has vanishing mass and the generator of time translations otherwise. Note that $N_i^2 \leq c_i^2 H_i^2$, where c_i is the inverse of the mass corresponding to $\varphi_i(x)$ if that is different from 0, and equal to 1 otherwise.

It follows that, for ψ in the domain of H,

$$\|\varphi_i(f)\psi\| \leq b(f)\|(H+I)\psi\| \tag{45}$$

for some constant $b(f)$.

Thus, since N is essentially self-adjoint on the domain of H, $\varphi_i(f)$ is essentially self-adjoint on any core for H.

To complete the proof we only need to show that, for each $\mathcal{O} \in \mathcal{K}$, $\mathcal{F}(\mathcal{O})\Omega$ contains a core for H. But this follows from [10, Appendix], after noticing that given $\mathcal{O}_1 \subset\subset \mathcal{O}$ then $e^{itH}\mathcal{F}(\mathcal{O}_1)e^{-itH} \subset \mathcal{F}(\mathcal{O})$ for $|t|$ small enough. □

Summing up, we have thus proved the following result.

Theorem A.1. *Let $\mathcal{F}$ be the net generated by a finite family of free Hermitian scalar fields and let $\mathcal{F} = \mathcal{F}_A \otimes \mathcal{F}_B$ be a tensor product decomposition, then, for a suitable choice $\varphi_1(x), \ldots, \varphi_n(x)$ of the generating fields for $\mathcal{F}$*

and a $k \in \{1, \dots, n\}$, $\mathcal{F}_A \otimes I$ *is generated by* $\varphi_1(x), \dots, \varphi_k(x)$ *and* $I \otimes \mathcal{F}_B$ *by* $\varphi_{k+1}(x), \dots, \varphi_n(x)$.

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