Representations of conformal nets and vertex operator algebra modules

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(joint work with Mihály Weiner and Feng Xu)

Vertex operator algebras (VOAs) and conformal nets give two different axiomatizations for chiral two-dimensional conformal field theory (chiral CFT). VOAs are mainly of algebraic nature [4, 5, 8, 10]. A VOA (over $\mathbb{C}$) is a complex vector space V together with a linear map $V \ni a \mapsto Y(a,z)$ satisfying certain assumptions related to the underlying CFT interpretation. The *vertex operators* $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ are formal power series with coefficients $a_{(n)} \in \mathrm{End}(V)$ or, equivalently, operator-valued formal distributions on $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. They should be interpreted as the quantum fields of the theory. The map $a \mapsto Y(a,z)$ is called the *state field correspondence* and among its properties we mention here the so called *Borcherds identity* which is deeply related with the locality property of quantum fields [8].

On the other hand conformal nets are defined in terms of operator algebras on Hilbert spaces and hence they are mainly functional analytic objects [6, 9]. They are the chiral CFT version of algebraic quantum field theory (AQFT) [7]. A conformal net $\mathcal{A}$ on S^1 is a map $I \mapsto \mathcal{A}(I)$ from the set $\mathcal{I}$ of open, non-dense, non-empty intervals of the unit circle S^1 into the family of von Neumann algebras acting on a fixed Hilbert space (the *vacuum Hilbert space*) satisfying certain assumptions which, also in this case, are related to the underlyng CFT interpretation. Among these assumptions we mention here *locality* which means that the von Neumann algebras associated to any pair $I_1, I_2 \in \mathcal{I}$ of disjoint intervals commute.

Despite their significant mathematical differences, these two formulations show their common CFT origin through many structural similarities. Moreover, many interesting chiral CFT unitary models can be considered from both point of view with similar outputs. However, a direct general connection between unitary VOAs and conformal nets has been studied for the first time only recently by Y. Kawahigashi, R. Longo, M. Weiner and me in [1]. For every sufficiently nice unitary VOA V we have shown how to define a corresponding conformal net $\mathcal{A}_V$ and how to recover V with its VOA structure from $\mathcal{A}_V$.

We shortly outline the construction. Let V be a simple unitary VOA [1, 3] whose vertex operators satisfy certain *energy bounds* and let $f \in C^\infty(S^1)$ be a smooth complex valued function on the circle. Then, every operator valued distribution $Y(a,z)$, $z \in S^1$, gives rise to a closed (unbounded) operator $Y(a,f)$ acting on the Hilbert space completion $\mathcal{H}_V$ of V (the *smeared vertex operator*). Moreover, for every open interval $I \in \mathcal{I}$, the family of operators $\{Y(a,f) : a \in V, f \in C_c^\infty(I)\}$ generates a von Neumann algebra $\mathcal{A}_V(I)$ on $\mathcal{H}_V$, i.e. the smallest von Neumann algebra with which every operator in the family is *affiliated*. Then, V is said to be *strongly local* if the map $I \mapsto \mathcal{A}_V(I)$ satisfies locality. It turns out that if V is strongly local then $\mathcal{A}_V$ satisfies also remaining properties and hence it is a conformal net. Many known examples of unitary VOAs such as the unitary

Virasoro VOAs, the unitary affine Lie algebras VOAs, the known $c = 1$ unitary VOAs, the moonshine VOA $V^\natural$, together with their coset and orbifold subVOAs, have been shown to be strongly local in [1] and it has been conjectured that every simple unitary VOA is strongly local.

VOAs and conformal nets have very interesting representation theories (theory of superselection sectors) but these play only a marginal role in [1]. A first important step towards the analysis of the representation theory aspects of the map $V \mapsto \mathcal{A}_V$ has been recently made by M. Weiner, F. Xu and me [2]. Let V be a strongly local VOA and let $\mathcal{A}_V$ be the corresponding conformal net. A VOA module for V is a complex vector space M together with linear map $a \mapsto Y_M(a,z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$, $a^M_{(n)} \in \mathrm{End}(M)$, from V into the set of operator valued formal distributions on M, which is compatible, in an appropriate sense, with the vertex operator algebra structure of V. In particular the *represented vertex operators* $Y_M(a,z)$ satisfy the Borcherds identity on M.

On the other hand, a representation π of $\mathcal{A}_V$ is a family $\{\pi_I : I \in \mathcal{I}\}$ where each π_I is a representation of the von Neumann algebra $\mathcal{A}_V(I)$ on a fixed Hilbert space $\mathcal{H}_\pi$. If π is locally normal, i.e. if every π_I is continuous with respect to the σ-weak topology (the natural topology of von Neumann algebras), then, each representation π_I naturally extends to the unbounded closed operators affiliated with $\mathcal{A}(I)$. In particular, if $f \in C^\infty_c(I)$ then $\pi_I(Y(a,f))$ is a well defined closed operator on $\mathcal{H}_\pi$. This is the starting point for the notion of *strongly integrable module* introduced in [2].

Let M be a VOA module for the strongly local VOA V. We assume that M is unitary, i.e. that it has a scalar product $(\cdot|\cdot)_M$ which is compatible with the unitary structure of V, see [3]. Furthermore, we assume that the represented vertex operators $Y_M(a,z)$ satisfy energy bounds similar to those of the vertex operators $Y(a,z)$. Then we can consider the *represented smeared vertex operators* $Y_M(a,f)$, $f \in C^\infty(S^1)$. In [2] M is defined to be strongly integrable if there exists a (necessarily unique) locally normal representation π^M of the conformal net $\mathcal{A}_V$ such that $\pi^M_I(Y(a,f)) = Y_M(a,f)$ for all $a \in V$, all $I \in \mathcal{I}$ and all $f \in C^\infty_c(I)$.

The corresponding map $M \mapsto \pi^M$ and its inverse preserve unitary equivalence, direct sums and irreducibility. Moreover, using free Femi field constructions, one can give many interesting examples of strongly integrable modules. In particular all the VOA modules of the type A unitary affine VOAs and the related coset VOA modules are strongly integrable. As a consequence, various results previously obtained by Feng Xu for the representation theory of type A diagonal coset conformal nets by means of subfactor theory methods [11, 12] can be transported to the VOA setting giving a solution of various long standing open problems in VOA representation theory. In this way we obtain e.g. a solution to certain purely VOA irreducibility problems for diagonal type A coset VOA modules thanks to the power of subfactor theory.

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