From vertex operator algebras to conformal nets and back

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We have two different mathematical formulations of chiral two-dimensional CFT (CFT on $S^1 \equiv$ compactified light-ray): vertex operator algebras (VOAs) and conformal nets on S^1. The VOA approach is mainly algebraic [6, 7, 11, 15]. A VOA (over $\mathbb{C}$) is a complex vector space V together with a linear map (the *state field correspondence*) $V \ni a \mapsto Y(a,z)$ satisfying certain physically motivated assumptions: locality, vacuum, conformal covariance, ... Here, for any $a \in V$, the *vertex oparator* $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ is a formal power series with coefficients in $\mathrm{End}(V)$ or, equivalently, an operator-valued formal distribution. On the other hand, the conformal net approach is mainly operator algebraic (hence functional analytic). It corresponds to the chiral CFT version of the so called algebraic quantum field theory (AQFT) [9], see [12] for a recent review on the AQFT approach to two-dimensional CFT. A conformal net $\mathcal{A}$ on S^1 is a map $I \mapsto \mathcal{A}(I)$ from the set of intervals of S^1 into the family of von Neumann algebras acting on a fixed Hilbert space (the *vacuum Hilbert space*) satisfying (again) certain physically motivated assumptions: isotony, locality, vacuum, conformal covariance, ... Despite their significant differences from a mathematical point of view, these two formulations show their common "physical root" through many structural similarities. Moreover, various interesting chiral CFT models can be considered from both point of view with similar outputs.

In a recent paper [2] we studied, for the first time, the mathematical correspondence between VOAs and conformal nets from a general point of view. We start with a unitary simple VOA V [2, 3] assumed to satisfy certain nice estimates (energy bounds). Then we follow the traditional approach to the construction of local nets of von Neumann algebras starting from quantum fields (i.e. operator valued distributions in the sense of Wightman [17]), see [9]. For every smooth function $f \in C^\infty(S^1)$, the operator valued distribution $Y(a,z)$ gives rise to a *smeared vertex operator* $Y(a,f)$ acting on the Hilbert space completion $\mathcal{H}_V$ of V. Then, for every open interval $I \subset S^1$ we consider the von Neumann algebra

$$\mathcal{A}_V(I) \equiv W^*(\{Y(a,f) : a \in V, f \in C_c^\infty(I)\})$$

generated by all the vertex operators smeared with test functions with support in I. Since the smeared vertex operators are in general unbounded it is not *a priori* clear that the locality axiom for the VOA V implies that the the map $I \mapsto \mathcal{A}_V(I)$ will satisfy locality i.e. that $[\mathcal{A}_V(I_1), \mathcal{A}_V(I_2)] = \{0\}$ whenever $I_1 \cap I_2 = \emptyset$. We say that V is *strongly local* if this is actually the case. We then prove that if V is a strongly local VOA then the map $I \mapsto \mathcal{A}_V(I)$ defines an irreducible conformal net on S^1.

The class of strongly local VOAs is closed under taking tensor products and unitary subVOAs. Moreover, for every strongly local VOA V, the map $W \mapsto \mathcal{A}_W$ gives a one-to-one correspondence between the unitary subVOAs W of V and the covariant subnets of $\mathcal{A}_V$.

Many known examples of unitary VOAs such as the unitary Virasoro VOAs, the unitary affine Lie algebras VOAs, the known $c = 1$ unitary VOAs, the moonshine VOA $V^\natural$, together with their coset and orbifold subVOAs, turn out to be strongly local. The corresponding conformal nets coincide with those previously constructed by different methods: the Virasoro nets [1, 13, 16], the loop groups nets [8, 18, 19], the coset conformal nets [20], the $c = 1$ conformal nets [21], the moonshine conformal net $\mathcal{A}^\natural$ [14] ....

The even shorter moonshine vertex operator algebra constructed by Höhn [10] also turns out to be strongly local being a subVOA of $V^\natural$. Moreover, the automorphism group of the corresponding conformal net coincides the VOA automorphism group which is known to be the baby monster group $\mathbb{B}$.

Note also that the (still hypothetical) Haagerup VOA with $c = 8$ considered by Evans and Gannon in [4] has been suggested to be a unitary subVOA of a unitary affine Lie algebra VOA and hence it should be strongly local.

Furthermore, a construction of Fredenhagen and Jörß [5] gives back the strongly local VOA V from the irreducible conformal net $\mathcal{A}_V$. More generally, in [2] we give conditions on an irreducible conformal net $\mathcal{A}$ implying that $\mathcal{A} = \mathcal{A}_V$ for some strongly local vertex operator algebra V.

We conjecture that every unitary VOA is strongly local and that every irreducible conformal net comes from a unitary VOA in the way described above.

The representation theory aspects of the correspondence $V \mapsto \mathcal{A}_V$ will be considered in future research.

References

[1] S. Carpi, *On the representation theory of Virasoro nets*, Commun. Math. Phys. **244** (2004), no. 2, 261-284.

[2] S. Carpi, Y. Kawahigashi, R. Longo, M. Weiner, *From vertex operator algebras to conformal nets and back*, arXiv:1503.01260 [math.OA]

[3] C. Dong, X. Lin, *Unitary vertex operator algebras*, J. Algebra **397** (2014), 252–277.

[4] D.E. Evans, T. Gannon, *Exoticness and realizability of twisted Haagerup-Izumi modular data*, Commun Math. Phys. **307** (2011), no. 2, 463–512.

[5] K. Fredenhagen, M. Jörß, *Conformal Haag-Kastler nets, pointlike localized fields and the existence of operator product expansions*, Commun. Math. Phys. **176** (1996), no. 3, 541–554.

[6] I.B. Frenkel, Y.-Z. Huang, J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. **104** (1993), no. 494, viii + 64 pp.

[7] I.B. Frenkel, J. Lepowsky, A. Meurman, *Vertex operator algebras and the monster*, Academic Press, Boston, 1989.

[8] F. Gabbiani, J. Fröhlich, *Operator algebras and conformal field theory*, Commun. Math. Phys. **155** (1993), no. 3, 569–640.

[9] R. Haag: *Local quantum physics*, 2nd ed. Springer-Verlag, Berlin-Heidelberg-New York, 1996.

[10] G. Höhn, *The group of symmetries of the shorter Moonshine module*, Abh. Math. Semin. Univ. Hambg. **80** (2010), no. 2, 275–283.

[11] V.G. Kac, *Vertex algebras for beginners.* 2nd ed. Providence, RI: AMS 1998.

[12] Y. Kawahigashi, *Conformal Field Theory, Tensor Categories and Operator Algebras*, arXiv:1503.05675 [math-ph].

[13] Y. Kawahigashi, R. Longo, *Classification of local conformal nets. Case* $c < 1$, Ann of Math. **60** (2004), no. 2, 493–522.

[14] Y. Kawahigashi, R. Longo, *Local conformal nets arising from framed vertex operator algebras* Adv. Math. **206** (2006), no. 2, 729–751.

[15] J. Lepowsky and H. Li, *Introduction to vertex operator algebras and their representations*, Birkäuser, Boston, 2004.

[16] T. Loke, *Operator algebras and conformal field theory of the discrete series representation of* $\mathrm{Diff}^+(S^1)$, PhD Thesis, University of Cambridge, 1994.

[17] R.F. Streater, A.S. Wightman, *PCT, spin and statistics and all that.* Addison-Wesley, 1989.

[18] V. Toledano Laredo, *Fusion of positive energy representations of* $LSpin_{2n}$, PhD Thesis, University of Cambridge, 1997.

[19] A. Wassermann, *Operator algebras and conformal field theory III: Fusion of positive energy representations of* $SU(N)$ *using bounded operators*, Invent. Math. **133** (1998), no. 3, 467–538.

[20] F. Xu, *Algebraic coset conformal field theory*, Commun. Math. Phys. **211** (2000), no. 1, 1–43.

[21] F. Xu, *Strong additivity and conformal nets*, Pacific J. Math. **221** (2005), no. 1, 167-199.